Istituzioni di Algebra 2022/2023

Francesco Minnocci

20 dicembre 2022

Homework 4

Exercise 4.1.

(a) Let F be the functor from $D^0(\mathcal{A})$ to $\mathcal{A}$ which sends $A^\bullet$ to its 0-th homology; we construct a quasi-inverse G as the inclusion functor which sends an object A to a complex concentrated in degree 0.

Clearly, $G \circ F = 1_{D^0(A)}$, as in a complex where the maps are all zero the i-th homology picks out precisely the i-th term. On the other hand, $F \circ G(A^\bullet) \cong A^\bullet$ in $D(A)$ by the following isomorphism:

$$\begin{tikzcd}
\dots \arrow[r] & A^{-1} \arrow[r, "d^{-1}"] \arrow[d] & A^0 \arrow[r, "d^0"] \arrow[d, "\pi"] & A^1 \arrow[r] \arrow[d, equal] & \dots \\
& 0 \arrow[r] \arrow[d, equal] & A^0/B^0 \arrow[r, "\overline{d^0}"] & A^1 \arrow[r] & \dots \\
& 0 \arrow[r] & H^0(A^\bullet) \arrow[r] \arrow[u, "i"] & 0, \arrow[u] &
\end{tikzcd} \tag{1}$$

where both morphisms of complexes are quasi-isomorphisms by hypothesis, as they are the natural morphisms associated to the truncations of respectively the first row in degrees ≥ 0, and of the second row in degrees ≤ 0.

(b) We claim that there cannot be such an equivalence: let $D^{0,1}(\mathcal{A})$ be the full subcategory of $D(\mathcal{A})$ spanned by objects with $H^i(A^\bullet) = 0$ for $i \neq 0$, and let F be the inclusion functor from $C^{0,1}(\mathcal{A})$ to $D^{0,1}(\mathcal{A})$.

Now, we observe that F cannot be a faithful functor, which will imply it cannot lead to an equivalence of categories: indeed, let $\mathcal{A}$ be the category of abelian groups, and consider the complexes

$$[\mathbb{Z}/2\mathbb{Z} \xrightarrow{0} \mathbb{Z}/2\mathbb{Z}]$$

and

$$[\mathbb{Z}/2\mathbb{Z} = \mathbb{Z}/2\mathbb{Z}].$$

Then, the group of morphisms between such complexes in $C^{0,1}(\mathcal{A})$ is not zero, since the morphism of complexes given vertically by respectively the zero map and the identity is not the zero morphism. However, $[\mathbb{Z}/2\mathbb{Z} = \mathbb{Z}/2\mathbb{Z}]$ is exact, and thus isomorphic to $[0]$ in $D(\mathcal{A})$, so the only morphism between the two complexes in $D^{0,1}(\mathcal{A})$ is the zero morphism. Thus, F induces a non-injective map between the respective hom-sets, and is as such not faithful.

Exercise 4.2. In the following, we indicated with *Ab* the category of abelian groups.

(a) Consider the following morphism between two instances of the same complex in $C(Ab)$:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r, "\cdot 2"] \arrow[d, equal] & \mathbb{Z}/4\mathbb{Z} \arrow[r, "\pi"] \arrow[d, equal] \arrow[dl, dashed, "\nexists"'] & \mathbb{Z}/2\mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r, "\cdot 2"] & \mathbb{Z}/4\mathbb{Z} \arrow[r, "\pi"] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0.
\end{tikzcd}$$

Evidently, it induces the zero morphism in the derived category as the complex is acyclic; however, it cannot be homotopic to 0 as the vertical identity on the first term cannot factor through multiplication by two on the first row, as indicated visually in the diagram (as 1 goes to 2, and all group homomorphisms from $\mathbb{Z}/4\mathbb{Z}$ to $\mathbb{Z}/2\mathbb{Z}$ send 2 to 0).

(b) We consider the following morphism of complexes concentrated in degrees $0, 1$ in $C(Ab)$:

$$\begin{tikzcd}
\mathbb{Z}/2\mathbb{Z} \arrow[r, "\cdot 2"] \arrow[d, equal] & \mathbb{Z}/4\mathbb{Z} \arrow[d, "0"] \\
\mathbb{Z}/2\mathbb{Z} \arrow[r, "0"] & 0
\end{tikzcd}$$

This induces the zero map in homology since the homology of the first complex is 0 in degree 0 and that of the second complex is 0 in degree 1.

However, we will show that it is not 0 in the derived category: indeed, suppose there is a complex $D^\bullet$ and a quasi-isomorphism to it coming from the second complex in the above diagram with a homotopy between the morphism given by vertical composition and 0:

$$\begin{tikzcd}
 & & \mathbb{Z}/2\mathbb{Z} \arrow[r, "\cdot 2"] \arrow[d, equal] \arrow[ddl, dashed, "k^{-1}"'] & \mathbb{Z}/4\mathbb{Z} \arrow[d, "0"] \arrow[ddl, dashed, "k^0"'] & \\
 & & \mathbb{Z}/2\mathbb{Z} \arrow[r] \arrow[d, "s"] & 0 \arrow[d, "0"] & \\
\cdots \arrow[r] & D^{-1} \arrow[r, "d^{-1}"] & D^0 \arrow[r, "d^0"] & D^1 \arrow[r] & \cdots.
\end{tikzcd}$$

Since s needs to induce an isomorphism in homology, the image of 1 through it should generate $H^0(D^\bullet)$ modulo the cycles $B^0_{D^\bullet}$.

Moreover, since we assumed there is an homotopy between $s \circ \mathrm{id}$ and 0, we would have

$$s(1) = d^{-1}_{D^\bullet} \circ k^{-1}(1) + k^0(1 \cdot 2).$$

However, the above implies that in homology the image of 1 through the map induced by s is congruent to 0, which is a contradiction since it should generate $H^0(D^\bullet) \cong \mathbb{Z}/2\mathbb{Z}$.

Exercise 4.3. (Done in collaboration with Marco Sanna)

(a) We can assume by shifting that $a = 0$, and by applying the canonical truncation in degrees ≥ 0 (which here induces an isomorphism in $D(\mathcal{A})$ by hypothesis) we can also assume that $A^\bullet$ is concentrated in degrees ≥ 0.

Now, using the natural morphisms of complexes from $A^\bullet$ to its truncation in degrees ≥ 1 defined in Construction 10.13 of notes, we have that the following diagram commutes:

$$\begin{tikzcd}
A^\bullet \arrow[r, "\psi"] \arrow[d, "\phi"] & \tau_{\geq 1}(A^\bullet) \arrow[d, "\tau_{\geq 1}(\phi)=0"] \\
A^\bullet \arrow[r, "\psi"] & \tau_{\geq 1}(A^\bullet)
\end{tikzcd}$$

Therefore, we have that $\psi \circ \phi$ factors through zero, and we can use the exact triangle associated with the cone of the identity map on $A^\bullet$ to get the desired factorization: indeed, we can place it above the exact sequence from construction 10.13 and shift it to obtain the following diagram (using Lemma 9.10 (2))

$$\begin{tikzcd}
A^\bullet \arrow[r] \arrow[d, "\phi"] & 0 \arrow[r] \arrow[d] & A^\bullet[1] \arrow[r, equal] \arrow[d, dashed, "\exists\chi[1]"] & (A^\bullet)[1] \arrow[d, "\phi[1]"] \\
A^\bullet \arrow[r, "\psi"] & \tau_{\geq 1}(A^\bullet) \arrow[r] & H^0(A^\bullet)[1] \arrow[r, "i[1]"] & A^\bullet[1]
\end{tikzcd}$$

Now, we shift back to get the desired factorization:

$$\begin{tikzcd}
A^\bullet \arrow[r, equal] \arrow[d, dashed, "\chi"] & (A^\bullet) \arrow[r] \arrow[d, "\phi"] & 0 \arrow[r] \arrow[d] & A^\bullet[1] \arrow[d, dashed, "\chi[1]"] \\
H^0(A^\bullet) \arrow[r, "i"] & A^\bullet \arrow[r, "\psi"] & \tau_{\geq 1}(A^\bullet) \arrow[r] & H^0(A^\bullet)[1]
\end{tikzcd}$$

(b) As in part (a), suppose without loss of generality that $a = 0$ (by shifting), and also that $A^\bullet$ is concentrated in degrees $0 \leq i \leq b$ (since in such degrees the two canonical truncations induce isomorphisms in $D(\mathcal{A})$ by hypothesis).

We then prove the claim by induction on b : if $b = 0$, then $A^\bullet = [0 \to A^0 \to 0]$, but then $A^\bullet$ is a complex concentrated in degree 0, and by the equivalence functor F of abelian categories constructed in Exercise 4.1 (a) above, the fact that $H^0(\phi) = 0$ from $H^0(A^\bullet) = F(A^\bullet)$ to itself implies that $\phi = 0$ as well.

If $b > 0$, we first apply the truncation $\tau_{\geq 1}$ to obtain a shorter complex, concentrated in degrees $1 \leq i \leq b$, and since $H^i(\tau_{\geq 1}(\phi)) = 0$ holds a fortiori we can apply the inductive hypothesis to get $(\tau_{\geq 1}(\phi))^b = 0$, which by functoriality means that

$$\tau_{\geq 1}(\phi^b) = 0.$$

Then, by part (a) of this exercise ϕ^b factors through $H^0(A^\bullet)$, and now we can use the exact triangle (21) from Construction 10.13 in the course notes to get the commutative diagram

$$\begin{tikzcd}
H^0(A^\bullet) \arrow[r, "i"] \arrow[d, "H^0(\phi)"] & A^\bullet \arrow[d, "\phi"] \arrow[l, dashed, bend right, "\overline{\phi}"'] \\
H^0(A^\bullet) \arrow[r, "i"] & A^\bullet,
\end{tikzcd}$$

where $\overline{\phi}$ is the factoring map (such that $\phi^b = i \circ \overline{\phi}$). Then, by commutativity of the diagram ϕ^{b+1} factors through $H^0(\phi) = 0$, so it is the zero map.

Exercise 4.4.

(a) The "only if" part of the statement follows from the fact that the connecting map of the exact triangle associated with the split exact sequence $0 \to A^\bullet \to A^\bullet \oplus C^\bullet \to C^\bullet \to 0$ is 0: indeed, consider the following diagram

$$\begin{tikzcd}
A^\bullet \arrow[r] \arrow[d, "\cong"] & B^\bullet \arrow[r] \arrow[d, "\cong"] & C^\bullet \arrow[r, "f"] \arrow[d, "\cong"] & (A^\bullet)[1] \arrow[d, "\cong"] \\
A^\bullet \arrow[r] & A^\bullet \oplus C^\bullet \arrow[r, "\pi_C"] & C^\bullet \arrow[r, "\omega"] \arrow[l, dashed, bend left, "i_C"] & A^\bullet[1].
\end{tikzcd}$$

If we retract $C^\bullet$ with the inclusion in the direct sum, then project back to it (which yields the identity), and then go forward again with the connecting map, by part (1) of **Lemma** 9.9 in the course notes (which says that composition of two successive maps in an exact triangle is zero) we get that the connecting map ω is 0. Finally, the isomorphism between the two exact triangles tells us exactly that $f = 0$.

For the "if" part, we take the identity maps on the first, third and fourth terms of the two exact triangles in question, and since $f = 0$ we have commutativity of the right-most square (as the connecting map is still 0 as above). Therefore, we can shift and use **Lemma** 9.10 (2) as usual to get a vertical map which completes the diagram:

$$\begin{tikzcd}
A^\bullet \arrow[r] \arrow[d, equal] & B^\bullet \arrow[r] \arrow[d, dashed, "\varphi"] & C^\bullet \arrow[r, "0"] \arrow[d, equal] & (A^\bullet)[1] \arrow[d, equal] \\
A^\bullet \arrow[r] & A^\bullet \oplus C^\bullet \arrow[r] & C^\bullet \arrow[r, "0"] & A^\bullet[1].
\end{tikzcd}$$

By **Corollary** 9.11 (2), φ also an isomorphism (since the identities are), which shows the claim.

(b) Let $A^\bullet$ be an object in $D^b(\mathcal{A})$. Then, by truncating above and below and shifting we can assume $A^\bullet$ to be a complex concentrated in degrees $0 \leq i \leq b$ for some $b \in \mathbb{N}$. Let us proceed by induction on b: if $b = 0$, then $A = H^0(A^\bullet) = [0 \to A^0 \to 0]$ which concludes.

If $b > 0$, then applying the inductive hypothesis to the truncation in degrees $\leq b-1$ gives

$$\tau_{\leq b-1}(A^\bullet) \cong \bigoplus_{i<b} H^i(A^\bullet)[-i].$$

Now, using the exact triangle (20) from Construction 10.13 yields the exact triangle

$$\tau_{\leq b-1}(A^\bullet) \to \tau_{\leq b}(A^\bullet) = A^\bullet \to H^b(A^\bullet)[-b] \to \tau_{\leq b-1}(A^\bullet)[1],$$

and if we prove that the connecting map is 0, we can apply part (a) of the exercise to conclude that there is a splitting, meaning

$$A^\bullet \cong \bigoplus_{i\leq b} H^i(A^\bullet)[-i],$$

and we are done.

Finally, observe that by commutativity of Hom and finite direct sums in additive categories, and by **Proposition** 11.12 of the notes we have

$$\begin{aligned}
\mathrm{Hom}(H^b(A^\bullet)[-b], \tau_{\leq b-1}(A^\bullet)[1]) &\cong \mathrm{Hom}\left(H^b(A^\bullet)[-b], \bigoplus_{i<b} H^i(A^\bullet)[-i+1]\right) \\
&\cong \bigoplus_{i<b} \mathrm{Hom}\left(H^b(A^\bullet)[-b], H^i(A^\bullet)[-i+1]\right) \\
&\cong \bigoplus_{i<b} \mathrm{Hom}\left(H^b(A^\bullet), H^i(A^\bullet)[b-i+1]\right) \\
&\overset{11.12}{\cong} \bigoplus_{i<b} \mathrm{Ext}^{b-i+1}\left(H^b(A^\bullet), H^i(A^\bullet)\right) \\
&\overset{b-1+1\geq 2}{=} 0,
\end{aligned}$$

which means that the connecting map is necessarily 0.

Exercise 4.5. In order to compute $A^\bullet \otimes^{\mathbf{L}} B^\bullet$ and $B^\bullet \otimes^{\mathbf{L}} A^\bullet$, we choose complexes with projective terms $P_A^\bullet$ and $P_B^\bullet$ quasi-isomorphic to respectively $A^\bullet$ and $B^\bullet$, which exist by **Lemma** 10.19 (1). Then, in $D^-(\mathcal{A})$,

$$\begin{aligned}
A^\bullet \otimes^{\mathbf{L}} B^\bullet &\cong P_A^\bullet \otimes^{\mathbf{L}} P_B^\bullet \\
B^\bullet \otimes^{\mathbf{L}} A^\bullet &\cong P_B^\bullet \otimes^{\mathbf{L}} P_A^\bullet.
\end{aligned}$$

Now, fix $n \in \mathbb{Z}$, and define the following for $i, j \in \mathbb{Z}$ such that $i + j = n$:

$$\begin{aligned}
\varphi_n^{i,j} : P_A^i \otimes P_B^j &\to P_B^j \otimes P_A^i \\
p \otimes q &\mapsto (-1)^{ij}(p \otimes q).
\end{aligned}$$

Then, this gives a bijection in each summand of the n-th component, and we only need to show that it is a morphism of complexes, which is the following computation:

- $d_n^{j,i}(\varphi_n^{i,j}(p \otimes q)) =$
  $d_n^{j,i}((-1)^{ij}(q \otimes p)) = (-1)^{ij}(d_B^j(q) \otimes p) + (-1)^{j(i+1)}(q \otimes d_A^i(p)),$
- $\varphi_n(d_n^{i,j}(p \otimes q)) =$
  $\varphi_n^{i+1,j}(d_A^i(p) \otimes q) + (-i)^i \varphi_n^{i,j+1}(p \otimes d_B^j(q)) = (-1)^{j(i+1)}(q \otimes d_A^i(p)) + (-1)^{i(j+2)}(d_B^j(q) \otimes p).$

This shows that φ commutes with the differentials in each degree, as the signs in the above agree:

$$ij \equiv i(j+2) \mod 2.$$