Istituzioni di Algebra 2022/2023

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Homework 2

Exercise 2.1. Since A, B are Noetherian, by the **Theorem** on Dimension of Fibres we have the inequality

$$\operatorname{ht}(Q) \leq \operatorname{ht}(P) + \dim\left(\frac{B_Q}{PB_Q}\right).$$

For the other inequality, we first replace A, B by A_P, B_Q and P, Q by P_Q, B_Q since by correspondance of prime ideals in localizations the heights of P, Q don't change in the localizations. Then, set $r := \operatorname{ht}(P)$ and $s := \operatorname{ht}(Q \bmod PB)$, and let

$$\begin{tikzcd} & PB \arrow[d, dash, dashed] & \subsetneq Q_1 \subsetneq Q_2 \subsetneq \dots Q_s \subsetneq Q \\ P_1 \subsetneq P_2 \subsetneq \dots P_r \subsetneq & P & \end{tikzcd}$$

be maximal chains of prime (except eventually PB) ideals realizing the heights r, s. Now, by the **Remark** 5.23 of the course notes, we can use the Going Down property for flat extensions starting from Q_1 (which contracts to P) and inductively go down to obtain a prime chain of length $r + s$ ending in Q, which lets us conclude.

Exercise 2.2. First notice that $(f, f') = 1$ implies $(\overline{f}, \overline{f}') = 1$ in the residue field, as by Bezout's theorem there is a non-zero combination

$$f \cdot g + f' \cdot h = 1$$

with $g, h \in A[x]$, which projects to a non-zero combination in $\mathbb{K}[x]$ (as if $\overline{f}'$ was zero, f would project to a invertible element), where $\mathbb{K}$ is the residue field of A. Now, $\overline{f}$ factors completely in $\mathbb{K}[x]$ because $\mathbb{K}$ is algebraically closed), and since f is monic we have that $n = \deg(f) = \deg(\overline{f})$, so that $\overline{f}$ factors as $\prod_{i=1}^{n} (x - \overline{a}_i)$ for some $\overline{a}_i \in \mathbb{K}$. Moreover, we showed that the "derivative criterion" holds for $\overline{f}$, so that the $\overline{a}_i$s must all be distinct. Thus, by the Hensel Lemma it follows that we can lift this factorization to

$$f = \prod_{i=1}^{n} (x - a_i),$$

and the ideals $(x - a_i)$ are comaximal since the ideal generated by two distinct ideals them contains the element $a_i - a_j \in A$, which is invertible in A as it projects to the non-zero element $\overline{a}_i - \overline{a}_j \in \mathbb{K}$. Finally, we apply the **Chinese Remainder Theorem** to obtain

$$A[x]/(f) = \frac{A[x]}{\prod_{i=1}^{n} (x - a_i)} \cong \prod_{i=1}^{n} A[x]/(x - a_i) \cong \prod_{i=1}^{n} A = A^n.$$

Exercise 2.3.

(a) We claim that the unique maximal ideal of B is

$$\frac{(P[x],x)}{(f)},$$

where $P[x]$ indicates the extension of P in $A[x]$; by correspondence, this is equivalent to show that there is an unique maximal ideal of $A[x]$ containing f.

We claim it is enough to show that

$$\text{Every maximal ideal of } A[x] \text{ which contains } f \text{ also contains } P[x]. \tag{1}$$

Indeed, notice that this claim concludes: with the above, any maximal ideal M of $A[x]$ containing f contains $(P[x],f) = (P[x],x^n)$, where $n = \deg(f)$ and the equality follows from the fact that every coefficient of f excluding the leading coefficient is in P. With M being prime, containing a power of an element means containing such element, so we have that M must contain the maximal ideal $(P[x],x)$, and by maximality they are actually equal.

We now show (1), which by corrispondence is equivalent to showing that every maximal ideal of B contains PB, and we do so by contradiction: suppose that PB was not contained in some maximal ideal M of B, then we would have

$$B = M + PB.$$

Now, since B is a finitely generated module over A and P is equal to the Jacobson radical of A, we can apply **Nakayama**'s Lemma (its third form, from the course notes for *Algebra 2*) to obtain $M = B$ $\lightning$.

Moving on to completeness, we start from the *Hint*, namely that B is complete with respect to the filtration $(PB)^k$: first notice that

$$(PB)^k = (P[\overline{x}])^k = P^k[\overline{x}];$$

then, by the **Third isomorphism theorem**, we have

$$B/(PB)^k = \frac{A[x]/(f)}{(P^k[\overline{x}],f)/(f)} \cong \frac{A[x]}{(P^k[x],f)} = \frac{(A/P^k)[x]}{(f)}.$$

Now, we want to prove that

$$\varprojlim_k \frac{A/P^k[x]}{\left(\overline{f}^{(k)}\right)} \cong \frac{\varprojlim_k (A/P^k)[x]}{(f)} = \frac{\left(\varprojlim_k A/P^k\right)[x]}{(f)} \cong \frac{A[x]}{(f)},$$

where the last equality follows from the completeness of A. It therefore remains to check the first isomorphism: this is a consequence of the **Lemma** 5.10 of the course notes, since if we consider the exact sequence

$$0 \longrightarrow \left(\overline{f}^{(k)}\right) \longrightarrow \left(A/P^k\right)[x] \longrightarrow \frac{\left(A/P^k\right)[x]}{\left(\overline{f}^{(k)}\right)} \longrightarrow 0$$

then condition *a)* of the aforementioned lemma holds as the maps $\left(\overline{f}^{(k+1)}\right) \to \left(\overline{f}^{(k)}\right)$ are surjective, so we have that the induced sequence of inverse limits is exact, which concludes as

$$\varprojlim_k \left(\overline{f}^{(k)}\right) = (f)\,.$$

To prove completeness of B as a local ring, we will apply **Proposition** 5.3 of the course notes with respect to the filtrations $(PB)^k = P^k[\overline{x}]$ (with respect to which we proved B to be complete) and $(P[\overline{x}], x)^k$ (from which would follow completeness of B, as it is its unique maximal ideal): on one hand, we notice that

$$P^k[\overline{x}] \subseteq (P[\overline{x}], x)^k.$$

On the other hand, if we fix k it is enough to show that, in $A[x]$, there is some t such that

$$((P[\overline{x}], x)^t, f) \subseteq (P^k[\overline{x}], f).$$

Indeed, since $P^j \subseteq P^j[\overline{x}]$ for any j and $x^{nk} \in (P^k[\overline{x}], f)$ by definition of f (with $n = \deg(f)$), if $t \geq k \cdot (n+1) - 1$ then

$$(P[\overline{x}], x)^t, f) = (P^t, P^{t-1} \cdot \overline{x}, \dots, P^k \cdot \overline{x}^{t-k}, P^{k-1} \cdot \overline{x}^{t-k+1}, \dots, \overline{x}^t, f) \subseteq (P^k[\overline{x}], f).$$

(b) Since $a_0 \in P \setminus P^2$, in particular it is non-zero, so we can extend its projection to a basis of P/P^2. Then, any lifting of the other basis elements will give rise to a regular system of parameters $(a_0, c_1, \dots, c_d)$ for A, where $\dim A = d + 1$.

Let $Q = (P[\overline{x}], \overline{x})$ be the unique maximal ideal of B. Since

$$a_0 = \overline{x}^n + \dots + a_1 \cdot \overline{x} \tag{2}$$

in B, we claim that a regular sequence which forms a system of parameters for B is given by $(\overline{x}, c_1, \dots, c_d)$: indeed, $P[\overline{x}] \subseteq (\overline{x}, c_1, \dots, c_d)B$ by (2) and obviously $\overline{x} \in (\overline{x}, c_1, \dots, c_d)B$, so they are a system of generators for Q; the reverse inclusion $(\overline{x}, c_1, \dots, c_d)B \subseteq (P[\overline{x}], x)$ is also clear.

We now show that they are minimal with such property, and that they are a regular sequence: the former follows from the fact that we started with a regular sequence (notice that A Noetherian $\implies$ B Noetherian, so **Remark** 4.8 applies and we just need to show that $\overline{x}$ is not a zero-divisor modulo $(c_1, \dots, c_d)$, but if it was then so would be a_0 by (2), which would be a contradiction); then, to conclude we just need

$$\dim B = \dim A = d + 1, \tag{3}$$

from which (by Noetherianness of B) **Theorem** 4.9 of the course notes would yield the regularity of the aforementioned sequence of parameters (and consequently of B), as we would have identified a regular sequence of parameters which also form a minimal system of generators for the maximal ideal of B, as *Corollary* 2.9 of the course notes says exactly that Q cannot be generated by less than $d + 1$ parameters.

Finally, we show (3): since A regular local $\implies$ A domain, Eisenstein's criterion holds with the given hypothesis, and f is thus irreducible in $A[x]$. As A regular local $\implies$ A UFD (by the Auslander-Buchsbaum Theorem), f is prime in $A[x]$, so the second exercise of the first Homework (with $A \hookrightarrow B$ being an integral extension of integral domains) tells us that $\dim B = \dim A$.

Exercise 2.4.

(a) By the third exercise of this Homework, the maximal ideal of B is

$$Q = ((p,t)[x], x)/\left(x^2 + tx + p\right),$$

since (p,t) is the maximal ideal of $\mathbb{Z}_p[[t]]$. Now, $\bar{t},\ \overline{x} \in Q \implies \bar{t} \cdot \overline{x},\ \overline{x}^2 \in Q^2 \implies p \in Q^2$.

(b) We first prove the *Hint*: this is equivalent to the claim that the nilradical of ${}^{C[[u]]}/_{(p)}$ contains a prime ideal (since by Zorn's lemma it is always the intersection of all minimal prime ideals). Since C is a DVR, let $P = (\pi)$ be its maximal ideal; then, since $p \in P$ (C has residue characteristic p and hence $\mathrm{char}(C) = 0$), we have $(p) = (\pi^n)$ for some $n \geq 1$, and if we consider an element a in the ideal ${}^{PC[[u]]}/_{(\pi^n)}$ (which is generated by $\overline{\pi}$), it must satisfy

$$a^n \in (\overline{\pi}^n) = 0.$$

Now,

$$B/_{(p)} \cong \mathbb{F}_p[[t]][x]/_{(x \cdot (x+t))},$$

which is not a domain, and we consider the two ideals $(\overline{x}), (\overline{x} + \bar{t})$: these are prime since

$$\left(B/_{(p)}\right)/_{(\overline{x})} \cong \mathbb{F}_p[[t]]$$

and also

$$\left(B/_{(p)}\right)/_{(\overline{x} + \bar{t})} \cong \mathbb{F}_p[[t]],$$

which is clearly a domain. Since (0) is not a prime ideal, these are minimal prime ideals in ${}^{B}/_{(p)}$, and they are distinct, as if a polynomial is divided by both $\overline{x}$ and $\overline{x} + \bar{t}$ then it is identically 0. In conclusion, they are distinct minimal prime ideals in ${}^{B}/_{(p)}$, which by (a) proves that B cannot be isomorphic to $C[[u]]$.

Exercise 2.5. Let t be an uniformizer for A, such that $P = (t)$. Then, by **Fact** 6.15 of the course notes we have that, if $\mathbb{K}$ is the residue field of A, there is a Cohen ring contained in A, which is the ring of Witt vectors $W(\mathbb{K}) \subset A$ such that $P \cap W(\mathbb{K}) = (p)$.

Now, as in the previous exercise $p \in P$, so $n := \nu(p) \geq 1$. Since A is a DVR and thus a domain, it is torsion-free as a $W(\mathbb{K})$-module, and since $\mathbb{K}$ is perfect by **Theorem** 7.4 $W(\mathbb{K})$ is also a DVR, and thus a PID. So, as suggested by the hint, we use the fact that any torsion-free module over a PID is free.

Moreover, we show that $\left\{1, \bar{t}, \ldots, \bar{t}^{n-1}\right\}$ is a basis for the ${}^{W(\mathbb{K})}/_{(t^n)} = \mathbb{K}$-vector space

$$A/_{t^n A},$$

which by a known (*Algebra 2* course notes) corollary of **Nakayama**'s Lemma will imply that A is finitely generated as a $W(\mathbb{K})$-module and of rank n.

Indeed, take any non-zero element $b \in A$, then modulo $P = (t)$ this is equivalent to some element $w^{(1)}$ of $W(\mathbb{K})$, which means that there is some $c^{(1)} \in A$ such that

$$b = w^{(1)} + c^{(1)} t.$$

Iterating such construction, we get that

$$b = \sum_{i=1}^{n-1} w^{(i)} t^i + c^{(n)} t^n,$$

and modulo t^n this becomes

$$\bar{b} = \sum_{i=1}^{n-1} \overline{w}^{(i)} \bar{t}^i,$$

which tells us that $\left\{1, \bar{t}, \ldots, \bar{t}^{n-1}\right\}$ generate ${}^{A}/_{t^n A}$ as a $\mathbb{K}$-vector space. We now need to show that they are linearly independent: let us take a non-trivial linear combination

$$\sum_{i=0}^{n-1} \overline{a}_i \bar{t}^i = 0,$$

then a_0 would be a combination of elements of $P = (t)$, and thus be in $W(\mathbb{K}) \cap P = (t^n)$; so $\overline{a}_0 = 0$ in the quotient. Now, lifting the remaining relation yields

$$t(a_1 + \cdots + a_{n-1} t^{n-2}) = d \cdot t^n$$

for some $d \in A$, and since A is a domain we can divide by t and then project again to get that $\overline{a}_1 \in W(\mathbb{K}) \cap P = (t^n) \implies \overline{a}_1 = 0$. Now, we can iterate the above reasoning for the rest of the coefficients to obtain the desired linear independence.