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Fast approximation of betweenness centrality through sampling

Matteo Riondato[1] · **Evgenios M. Kornaropoulos**[1]

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Abstract Betweenness centrality is a fundamental measure in social network analysis, expressing the importance or influence of individual vertices (or edges) in a network in terms of the fraction of shortest paths that pass through them. Since exact computation in large networks is prohibitively expensive, we present two efficient randomized algorithms for betweenness estimation. The algorithms are based on random sampling of shortest paths and offer probabilistic guarantees on the quality of the approximation. The first algorithm estimates the betweenness of all vertices (or edges): all approximate values are within an additive factor $\varepsilon \in (0, 1)$ from the real values, with probability at least $1 - \delta$. The second algorithm focuses on the top-K vertices (or edges) with highest betweenness and estimate their betweenness value to within a multiplicative factor ε, with probability at least $1 - \delta$. This is the first algorithm that can compute such approximation for the top-K vertices (or edges). By proving upper and lower bounds to the VC-dimension of a range set associated with the problem at hand, we can bound the sample size needed to achieve the desired approximations. We obtain sample sizes that are independent from the number of vertices in the network and only depend on a characteristic quantity that we call the vertex-diameter, that is the maximum number of vertices in a shortest path. In some cases, the sample size is

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✉ Matteo Riondato
matteo@cs.brown.edu

Evgenios M. Kornaropoulos
evgenios@cs.brown.edu

[1] Department of Computer Science, Brown University, Providence, RI, USA

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completely independent from any quantitative property of the graph. An extensive experimental evaluation on real and artificial networks shows that our algorithms are significantly faster and much more scalable as the number of vertices grows than other algorithms with similar approximation guarantees.

Keywords Social network analysis · Betweenness centrality · VC-dimension · Sampling · Approximation algorithms

1 Introduction

Centrality indices are fundamental metrics for network analysis. They express the relative importance of a vertex or an edge in the network. Some of them, e.g., degree centrality, reflect local properties of the underlying graph, while others, like betweenness centrality, give information about the global network structure, as they are based on counting shortest paths (Newman 2010). In this work we are interested in *betweenness centrality* (Anthonisse 1971; Freeman 1977), that is, for every vertex or edge in the graph, the fraction of shortest paths that goes through that vertex or edge (see Sect. 3 for formal definitions) , and on some variants of it (Opsahl et al. 2010; Borgatti and Everett 2006; Brandes 2008). Betweenness centrality has been used to analyze social and protein interaction networks, to evaluate traffic in communication networks, and to identify important intersections in road networks (Newman 2010; Geisberger et al. 2008). There exist polynomial-time algorithms to compute the exact betweenness centrality (Brandes 2001), but they are not practical for the analysis of the very large networks that are of interest these days. Graphs representing online social networks, communication networks, and the web graph have millions of nodes and billions of edges, making a polynomial-time algorithm too expensive in practice. Given that data mining is exploratory in nature, approximate results are usually sufficient, especially if the approximation error is guaranteed to be within user-specified limits. In practice, the user is interested in the relative ranking of the vertices according to their betweenness, rather than the actual value of the betweenness, so a very good estimation of the value of each vertex (or edge) is sufficiently informative for most purposes. It is therefore natural to develop algorithms that trade off accuracy for speed and efficiently compute high-quality approximations of the betweenness values. Nevertheless, in order for these algorithms to be practical, they must scale well and have a low runtime dependency on the size of the network (number of vertices and/or edges).

Our contributions We present two randomized algorithms to approximate the betweenness centrality (and some of its variants) of the vertices (or edges) of a graph. The first algorithm guarantees that the estimated betweenness values for all vertices (or edges) are within an *additive* factor ε from the real values, with probability at least $1 - \delta$. The second algorithm focuses on the top-K vertices (or edges) with highest betweenness and returns a *superset* of the top-K, while ensuring that the estimated betweenness for all returned vertices is within a *multiplicative* factor ε from the real value, with probability at least $1 - \delta$. This is the first algorithm to reach such a high-quality approximation for the set of top-K vertices (or edges). The algorithms are based on random sampling of shortest paths. The analysis to derive the sufficient sample size is novel and uses notions and results from VC-dimension theory. We define a range set associated with the problem at hand and prove strict bounds to its VC-dimension. The resulting sample size *does not depend on the size of the graph*, but only on the maximum number of vertices in a shortest path, a *characteristic quantity* of the graph that we call the *vertex-diameter*. For some networks, we show that the VC-dimension is actually at most a constant and so the sample size depends *only on the approximation parameters* and not on any quantitative property of the graph, a somewhat surprising fact that points out interesting insights. Thanks to the lower runtime dependency on the size of the network, our algorithms are *much faster and more scalable* than previous contributions (Jacob et al. 2005; Brandes and Pich 2007; Geisberger et al. 2008), while offering the same approximation guarantees. Moreover, the amount of work performed by our algorithms per sample is also less than that of others algorithms. We extensively evaluated our methods on real graphs and compared their performances to the exact algorithm for betweenness centrality (Brandes 2001) and to other sampling-based approximation algorithms (Jacob et al. 2005; Brandes and Pich 2007; Geisberger et al. 2008), showing that our methods achieve a huge speedup (3–4 times faster) and scale much better as the number of vertices in the network grows.

Outline We present related work in Sect. 2. Section 3 introduces all the basic definitions and results that we use throughout the paper. A range set for the problem at hand and the bounds to its VC-dimension are presented in Sect. 4. Based on these results we develop and analyze algorithms for betweenness estimation that we present in Sect. 5. Extensions of our methods to various variants of the problem (including edge betweenness) are presented in Sect. 6. Section 7 reports the methodology and the results of our extensive experimental evaluation.

2 Related work

Over the years, a number of centrality measures have been defined (Newman 2010). In this work we focus on betweenness centrality and some of its variants.

Betweenness centrality was introduced in the sociology literature (Anthonisse 1971; Freeman 1977) and many variants of it have been developed over the year (Brandes 2008). A particularly interesting variant called “k-bounded-distance betweenness” limits the length of the shortest paths considered when computing the centrality (Borgatti and Everett 2006; Brandes 2008; Pfeffer and Carley 2012). This is not to be confused with “k-path betweenness centrality” (Kourtellis et al. 2012), which considers simple random walks that are not necessarily shortest paths. Dolev et al. (2010) present a generalization of betweenness centrality which takes into account routing policies in the network. Opsahl et al. (2010) define a new distance function between pair of vertices in order to penalize paths with a high number of hops in weighted network. This function induces a generalized and parametrized definition of betweenness.

The need of fast algorithms to compute the betweenness of vertices in a graph arose as large online social networks started to appear. Brandes (2001) presents the first efficient algorithm for the task, running in time $O(nm)$ on unweighted graphs and $O(nm + n^2 \log n)$ on weighted ones. The algorithm computes, for each vertex v, the shortest path to every other vertex and then traverses these paths backwards to efficiently compute the contribution of the shortest paths from v to the betweenness of other vertices. For very large networks, the cost of this algorithm would still be prohibitive in practice, so many approximation algorithms were developed (Jacob et al. 2005; Brandes and Pich 2007; Bader et al. 2007; Geisberger et al. 2008; Maiya and Berger-Wolf 2010; Lim et al. 2011). The use of random sampling was one of the more natural approaches to speed up the computation of betweenness. Inspired by the work of Eppstein and Wang (2004), Jacob et al. (2005) and independently Brandes and Pich (2007) present an algorithm that mimics the exact one, with the difference that, instead of computing the contribution of all vertices to the betweenness of the others, it only considers the contributions of some vertices sampled uniformly at random. To guarantee that all estimates are within ε from their real value with probability at least $1 - \delta$, the algorithm from Jacob et al. (2005) and Brandes and Pich (2007) needs $O(\log(n/\delta)/\varepsilon^2)$ samples. The analysis for the derivation of the sample size uses Hoeffding bounds (Hoeffding 1963) and the union bound (Mitzenmacher and Upfal 2005). Geisberger et al. (2008) noticed that this can lead to an overestimation of the betweenness of vertices that are close to the sampled ones and introduced different unbiased estimators that are experimentally shown to have smaller variance and do not suffer from this overestimation issue. Our algorithm takes a different approach from the above algorithms. Specifically it sample each time a single random shortest path. This leads to a much smaller sample size and less work done for each sample, resulting in a much faster way to compute approximations of the betweenness with the same probabilistic guarantees. For certain applications it is sufficient to obtain a high-quality approximation of the centrality of the top-K vertices. Although existing algorithms (Brandes and Pich 2007; Jacob et al. 2005; Geisberger et al. 2008) can be extended to return a superset of the top-K vertices with highest betweenness, they only offer an *additive* approximation guarantee, while our algorithm for the top-K vertices offers a *multiplicative* factor guarantee, which is much stricter. We delve more in the comparisons with these algorithms in Sects. 5.3 and 7.

A number of works explored the use of adaptive sampling, in contrast with the previous algorithms (and ours) which use a fixed sample size. Bader et al. (2007) present an adaptive sampling algorithm which computes good estimations for the betweenness of high-centrality vertices, by keeping track of the partial contribution of each sampled vertex, obtained by performing a single-source shortest paths computation to all other vertices. Maiya and Berger-Wolf (2010) use concepts from expander graphs to select a connected sample of vertices. They estimate the betweenness from the sample, which includes the vertices with high centrality. They build the connected sample by adding the vertex which maximizes the number of connections with vertices not already in the sample. Lim et al. (2011) present modified versions of this algorithm and an extensive experimental evaluation. The algorithm does not offer any guarantee on the quality of the approximations. Compared to these adaptive sampling approaches, our methods ensure that the betweenness of all (or top-K) vertices is well approximated, while using a fixed, predetermined amount of samples. Sarıyüce et al. (2013) present an algorithm that pre-processes the network in multiple ways by removing degree-1 vertices and identical vertices and splitting the network" in separate components where the computation of betweenness can be performed independently and then aggregated. They do not present an analysis of the complexity of the algorithm.

In the analysis of our algorithm we use results from VC-dimension theory (Vapnik and Chervonenkis 1971), a key component of statistical learning theory. We compute an upper bound to the VC-dimension of a range set defined on shortest paths. Kranakis et al. (1997) present a number of results on the VC-dimension of various range sets for graphs (stars, connected sets of vertices, sets of edges), but do not study the case of shortest paths. Abraham et al. (2011) use VC-dimension to speed up shortest path computation but their range set is different from the one we use: their ground set is the set of vertices while ours is defined on shortest paths.

The use of sampling in algorithms for social network analysis is widespread, although the word may assume different meanings. For example, Tang et al. (2015) are interested in *extracting* ("sampling") sets of users (vertices) that are statistically representative of entire set of vertices. This concept of "sampling" is different from ours. Papagelis et al. (2013) focus on algorithms to obtain random samples of the neighborhood of a vertex as fast as possible. This is a different setting than ours, as we assume that the entire network is accessible. For additional information, both basic and advanced, on the use of sampling in graph analysis algorithms, we refer the reader to the tutorial by Cormode and Duffield (2014), with the caveat that it does not cover VC-dimension and related techniques that we use in this work.

The present work extends substantially the preliminary version (Riondato and Kornaropoulos 2014). The proof of all the theorems and lemmas are presented here for the first time. Examples of betweenness centrality and of the concept of VC-dimension have been added to Sect. 3. A major new contribution is a tighter analysis of the algorithm by Brandes and Pich (2007) and a thorough comparison with it, presented in Sect. 5.3. We also discuss many other variants of betweenness, including edge betweenness in Sect. 6, where we also present a new tighter analysis of the algorithm for k-path betweenness centrality by Kourtellis et al. (2012).

3 Preliminaries

In this section we introduce the definitions and lemmas that we use throughout the paper to develop and analyze our results.

3.1 Graphs and betweenness centrality

Let $G = (V, E)$ be a graph, where $E \subseteq V \times V$, with $n = |V|$ vertices and $m = |E|$ edges. The graph G can be directed or undirected. Each edge $e \in E$ has a non-negative weight $\mathsf{w}(e)$. Given a pair of distinct vertices $(u, v) \in V \times V$, $u \neq v$, a *path* $p_{uv} \subseteq V$ *from* u *to* v is an ordered sequence of vertices $p_{uv} = (w_1, \ldots c, w_{|p_{uv}|})$ such that $w_1 = u$, $w_{|p_{uv}|} = v$ and for each $1 \leq i < |p_{uv}|$, $(w_i, w_{i+1}) \in E$. The vertices u and v are called the *end points* of p_{uv} and the vertices in $\mathsf{Int}(p_{uv}) = p_{uv} \setminus \{u, v\}$ are the *internal vertices of* p_{uv}. The (edge) *weight* $\mathsf{w}(p_{uv})$ of a path $p_{uv} = (u = w_1, w_2, \ldots, w_{p_{|uv|}} = v)$ from u to v is the sum of the weights of the edges composing the path: $\mathsf{w}(p_{uv}) = \sum_{i=1}^{|p_{uv}|-1} \mathsf{w}((w_i, w_{i+1}))$. We denote with $|p_{uv}|$ the number of vertices composing the path and call this the *size of the path* p_{uv}. Note that if the weights are not all unitary, it is not necessarily true that $\mathsf{w}(p_{uv}) = |p_{uv}| - 1$. A special and degenerate path is the *empty path* $p_\emptyset = \emptyset$, which by definition has weight $\mathsf{w}(p_\emptyset) = \infty$, no end points, and $\mathsf{Int}(p_\emptyset) = \emptyset$.

Given two distinct vertices $(u, v) \in V \times V$, the *shortest path distance* d_{uv} between u and v is the weight of a path with minimum weight between u and v among all paths between u and v. If there is no path between u and v, $d_{uv} = \infty$. We call a path between u and v with weight d_{uv} a *shortest path between u and v*. There can be multiple shortest paths between u and v and we denote the set of these paths as $\mathcal{S}_{uv}$ and the number of these paths as $\sigma_{uv} = |\mathcal{S}_{uv}|$. If there is no path between u and v, then $\mathcal{S}_{uv} = \{p_\emptyset\}$.[1] We denote with $\mathbb{S}_G$ the union of all the $\mathcal{S}_{uv}$'s, for all pairs $(u, v) \in V \times V$ of distinct nodes $u \neq v$:

$$\mathbb{S}_G = \bigcup_{\substack{(u,v)\in V\times V\\ u\neq v}} \mathcal{S}_{uv} \ .$$

We now define a characteristic quantity of a graph that we will use throughout the paper.

Definition 1 Given a graph $G = (V, E)$, the *vertex-diameter* $\mathsf{VD}(G)$ *of* G is the size of the shortest path in G with maximum size:

$$\mathsf{VD}(G) = \max\{|p| \ : \ p \in \mathbb{S}_G\} \ .$$

If all the edge weights are unitary, then $\mathsf{VD}(G)$ is equal to $\mathsf{diam}(G) + 1$, where $\mathsf{diam}(G)$ is the number of edges composing the longest shortest path in G.

Given a vertex v, let $\mathcal{T}_v \subseteq \mathbb{S}_G$ be the set of all shortest paths that v is *internal* to:

$$\mathcal{T}_v = \{p \in \mathbb{S}_G \ : \ v \in \mathsf{Int}(p)\} \ .$$

In this work we are interested in the *betweenness centrality* of the vertices and of the edges of a graph. From now until Sect. 6.3 we only refer to the betweenness of vertices, and present the extension to edges in Sect. 6.3.

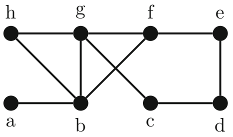
Definition 2 (Anthonisse 1971; Freeman 1977) Given a graph $G = (V, E)$, the *betweenness centrality of a vertex* $v \in V$ is defined as[2]

$$\mathsf{b}(v) = \frac{1}{n(n-1)} \sum_{p_{uw}\in\mathbb{S}_G} \frac{\mathbb{1}_{\mathcal{T}_v}(p)}{\sigma_{uw}} \ .$$

It is easy to see that $\mathsf{b}(v) \in [0, 1]$.

[1] Note that even if $p_\emptyset = \emptyset$, the set $\{p_\emptyset\}$ is not empty. It contains one element.

[2] We use the normalized version of betweenness as we believe it to be more suitable for presenting approximation results.



(a) Example graph

(b) Betweenness values

Vertex v	a	b	c	d	e	f	g	h
$\mathsf{b}(v)$	0	0.250	0.125	0.036	0.054	0.080	0.268	0

Fig. 1 Example of betweenness values

Figure 1 shows an example of betweenness values for the vertices of a (undirected, unweighted) graph. Just by looking at the graph, one expects that vertices b and g should have higher betweenness than the others, given that they somehow act as bridges between two sides of the network, and indeed that is the case.

Brandes (2001) presented an algorithm to compute the betweenness centrality for all $v \in V$ in time $O(nm)$ for unweighted graphs and $O(nm + n^2 \log n)$ for weighted graphs.

We present many variants of betweenness in Sect. 6.

3.2 Vapnik–Chervonenkis dimension

The Vapnik–Chernovenkis (VC) dimension of a class of subsets defined on a set of points is a measure of the complexity or expressiveness of such class (Vapnik and Chervonenkis 1971). Given a probability distribution on the set of points, a finite bound on the VC-dimension of the class of subsets implies a bound on the number of random samples required to approximate the probability of each subset in the class with its empirical average. We outline here some basic definitions and results and refer the reader to the book by Shalev-Shwartz and Ben-David (2014) for an in-depth presentation.

Let D be a domain and $\mathcal{R}$ be a collection of subsets from D. We call $\mathcal{R}$ a *range set on* D. Given $B \subseteq D$, the *projection of* $\mathcal{R}$ *on* B is the set $P_{\mathcal{R}}(B) = \{B \cap A \ : \ A \in \mathcal{R}\}$. We say that the set B is *shattered* by $\mathcal{R}$ if $P_{\mathcal{R}}(B) = 2^B$, where 2^B denotes the powerset of B, i.e., all subsets of B.

Definition 3 The *Vapnik–Chervonenkis (VC) dimension of* $\mathcal{R}$, denoted as $\mathsf{VC}(\mathcal{R})$, is the cardinality of the largest subset of D that is shattered by $\mathcal{R}$.

Note that a range space (X, R) with an arbitrary large set of points X and an arbitrary large family of ranges R can have a bounded VC-dimension. A simple example is the family of intervals in [0, 1] (i.e. X is all the points in [0, 1] and R all the intervals $[a, b]$, such that $0 \leq a \leq b \leq 1$). Let $A = \{x, y, z\}$ be the set of three points $0 < x < y < z < 1$. No interval in R can define the subset $\{x, z\}$ so the VC-dimension of this range space is less than three (Matoušek 2002, Lemma 10.3.1). Another example is shown in Fig. 2.

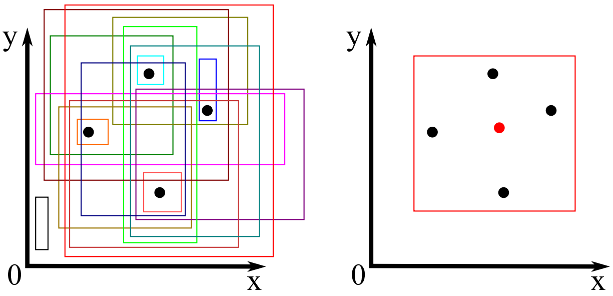


Fig. 2 Example of range space and VC-dimension. The space of points is the plane $\mathbb{R}^2$ and the set R of ranges is the set of all *axis-aligned rectangles*. The figure on the *left* shows graphically that it is possible to shatter a set of four *points* using 16 *rectangles*. On the *right* instead, one can see that it is impossible to shatter five *points*, as, for any choice of the five *points*, there will always be one (the *red* point in the figure) that is internal to the convex hull of the other four, so it would be impossible to find an axis-aligned rectangle containing the four point*s* but not the internal one. Hence $\mathsf{VC}((\mathbb{R}^2, R)) = 4$. This figure and caption are taken from Riondato and Upfal (2014)

The main application of VC-dimension in statistics and learning theory is in computing the number of samples needed to approximate the probabilities of the ranges using their empirical averages as unbiased estimators. Formally, let $X_1^k = (X_1, \ldots c, X_k)$ be a collection of independent identically distributed random variables taking values in D, sampled according to some distribution ϕ defined on the elements of D. For a set $A \subseteq D$, let $\phi(A)$ be the probability that a sample from ϕ belongs to the set A, and let the *empirical average* of $\phi(A)$ on X_1^k be

$$\phi_{X_1^k}(A) = \frac{1}{k} \sum_{j=1}^{k} \mathbb{1}_A(X_j) \ .$$

where $\mathbb{1}_A$ is the indicator function for the set A ($\mathbb{1}_A(X)$ is 1 if $X \in A$, and 0 otherwise). The empirical average of $\phi(A)$ can be used as an *unbiased* estimator for $\phi(A)$.

Definition 4 Let $\mathcal{R}$ be a range set on D and ϕ be a probability distribution on D. For $\varepsilon \in (0, 1)$, an *ε-approximation to* $(\mathcal{R}, \phi)$ is a multiset (i.e., a bag) S of elements of D such that

$$\sup_{A \in \mathcal{R}} |\phi(A) - \phi_S(A)| \leq \varepsilon \ .$$

When an upper bound to the VC-dimension of $\mathcal{R}$ is available, it is possible to build an ε-approximation by sampling points of the domain according to the distribution ϕ.

Theorem 1 [Theorem 2.12 (Har-Peled and Sharir 2011) (see also Li et al. 2001)] *Let $\mathcal{R}$ be a range set on a domain D with $\mathsf{VC}(\mathcal{R}) \leq d$, and let ϕ be a distribution on D. Given $\varepsilon, \delta \in (0, 1)$ let S be a collection of $|S|$ points from D sampled according to ϕ, with*

$$|S| = \frac{c}{\varepsilon^2}\left(d + \ln \frac{1}{\delta}\right) \tag{1}$$

where c is an universal positive constant. Then S is an ε-approximation to $(\mathcal{R}, \phi)$ with probability at least $1-\delta$.

The constant c is estimated to be approximately 0.5 (Löffler and Phillips 2009). It is possible to obtain *relative* guarantees on the approximation.

Definition 5 Let $\mathcal{R}$ be a range set on D and ϕ be a probability distribution on D. For $p, \varepsilon \in (0,1)$, a *relative (p, ε)-approximation to $(\mathcal{R}, \phi)$* is a bag S of elements from D such that

- For any $A \in \mathcal{R}$ such that $\phi(A) \geq p$, we have

$$|\phi(A) - \phi_S(A)| \leq \varepsilon\phi(A) \ .$$

- For any $B \in \mathcal{R}$ such that $\phi(B) < p$, we have $\phi_S(B) \leq (1+\varepsilon)p$.

Theorem 2 [Theorem 2.11 (Har-Peled and Sharir 2011)] *Let $\mathcal{R}$ be a range set on a domain D with $\mathsf{VC}(\mathcal{R}) \leq d$, and let ϕ be a distribution on D. Given $\varepsilon, \delta, p \in (0,1)$ let S be a collection of $|S|$ points from D sampled according to ϕ, with*

$$|S| \geq \frac{c'}{\varepsilon^2 p}\left(d \log\frac{1}{p} + \log\frac{1}{\delta}\right) \tag{2}$$

where c' is an absolute positive constant. Then S is a relative (p, ε)-approximation to $(\mathcal{R}, \phi)$ with probability at least $1-\delta$.

It is important to mention that if $\mathsf{VC}(\mathcal{R})$ and/or the upper bound d do not depend on $|D|$ or on $|\mathcal{R}|$ neither do the sample sizes presented in Theorem 1 and 2. This will make our algorithms scale well as the size of the network increases.

4 A range set on shortest paths

We now define a range set on the set of shortest paths of a graph $G = (V, E)$, and present a tight upper bound to its VC-dimension. We use the range set and the bound in the analysis of our algorithms for estimating the betweenness centrality of vertices of G.

The range set $\mathcal{R}_G$ is defined on the set $\mathbb{S}_G$ of all shortest paths between vertices of G. It contains, for each vertex $v \in V$, the set $\mathcal{T}_v$ of shortest paths that v is internal to:

$$\mathcal{R}_G = \{\mathcal{T}_v \ : \ v \in V\} \ .$$

Given a graph $G = (V, E)$, assume, for technical reasons, that $|E| \geq 2$ (the case $|E| = 1$ is trivial). Consider the set

$$I = \{\mathsf{Int}(p) \ : \ p \in \mathbb{S}_G\} \ .$$

The proper-subset relation between sets defines a partial order on I. Let AI be the collection of *anti-chains* in I according to this partial order (AI is a set of subsets of I). Let $\mathsf{H}(G)$ be the maximum integer h such that there is an anti-chain $A \in AI$, $A = \{\mathsf{Int}(p_1), \dots c, \mathsf{Int}(p_d)\}$ of size $d = \lfloor \log_2(h-2) \rfloor + 1$ such that $|p_i| \geq h$, for each $1 \leq i \leq d$. Note that since $|E| \geq 2$ and the size of a path is always at least 2, then the formula for d is always well defined.

Lemma 1 $\mathsf{VC}(\mathcal{R}_G) \leq \lfloor \log_2(\mathsf{H}(G) - 2) \rfloor + 1$.

Proof Let $\ell > \lfloor \log_2(\mathsf{H}(G) - 2) \rfloor + 1$ and assume for the sake of contradiction that $\mathsf{VC}(\mathcal{R}_G) = \ell$. From the definition of VC-dimension there is a set $Q \subseteq \mathbb{S}_G$ of size ℓ that is shattered by $\mathcal{R}_G$. For any two shortest paths p', p'' in Q we must have neither $\mathsf{Int}(p') \subseteq \mathsf{Int}(p'')$ nor $\mathsf{Int}(p'') \subseteq \mathsf{Int}(p')$, otherwise one of the two paths would appear in all ranges where the other one appears, and so it would be impossible to shatter Q. Then Q must be such that the collection of the sets of internal vertices of the paths in Q form an anti-chain. From this and from the definition of $\mathsf{H}(G)$ we have that Q must contain a path p of size $|p| \leq \mathsf{H}(G)$. There are $2^{\ell-1}$ non-empty subsets of Q containing the path p. Let us label these non-empty subsets of Q containing p as $S_1, \dots c, S_{2^{\ell-1}}$, where the labelling is arbitrary. Given that Q is shattered, for each set S_i there must be a range R_i in $\mathcal{R}_G$ such that $S_i = Q \cap R_i$. Since all the S_i's are different from each other, then all the R_i's must be different from each other. Given that p is a member of every S_i, p must also belong to each R_i, that is, there are $2^{\ell-1}$ distinct ranges in $\mathcal{R}_G$ containing p. But p belongs only to the ranges corresponding to internal vertices of p, i.e., to vertices in $\mathsf{Int}(p)$. This means that the number of ranges in $\mathcal{R}_G$ that p belongs to is equal to $|p| - 2$. But $|p| \leq \mathsf{H}(G)$ by definition of $\mathsf{H}(G)$, so p can belong to at most $\mathsf{H}(G) - 2$ ranges from $\mathcal{R}_G$. Given that $2^{\ell-1} > \mathsf{H}(G) - 2$, we reached a contradiction and there cannot be $2^{\ell-1}$ distinct ranges containing p, hence not all the sets S_i can be expressed as $Q \cap R_i$ for some $R_i \in \mathcal{R}_G$. Then Q cannot be shattered and $\mathsf{VC}(\mathcal{R}_G) \leq \lfloor \log_2(\mathsf{H}(G) - 2) \rfloor + 1$. □

Computing $\mathsf{H}(G)$ is not a viable option, but given that $\mathsf{H}(G) \leq \mathsf{VD}(G)$, we have the following corollary.

Corollary 1 $\mathsf{VC}(\mathcal{R}_G) \leq \lfloor \log_2(\mathsf{VD}(G) - 2) \rfloor + 1$.

4.1 Unique shortest paths

In the restricted case when the graph is undirected and every pair of distinct vertices has either none or a unique shortest path between them, the VC-dimension of $\mathcal{R}_G$ reduces to a *constant*. This is a somewhat surprising result with interesting consequences. From a theoretical point of view, it suggests that there should be other characteristic quantities of the graph different from the vertex diameter that control the VC-dimension of the range set of shortest paths, and these quantities are constant on graph with unique shortest paths between vertices. From a more practical point of view, we show in Sect. 5 that this result has an impact on the sample size needed to approximate the betweenness centrality of networks where the unique-shortest-path property is satisfied or even enforced, like road networks (Geisberger et al. 2008). In particular, the resulting sample size will be *completely independent* from any characteristic of the network, and will only be a function of the parameters controlling the desired approximation guarantees.

Lemma 2 *Let $G = (V, E)$ be an undirected graph with $|\mathcal{S}_{uv}| \leq 1$ for all pairs $(u, v) \in V \times V$. Then* $\mathsf{VC}(\mathcal{R}_G) \leq 3$.

Proof In this restricted setting, if two different shortest paths p_1 and p_2 "meet" at a vertex u, then they either continue together to a vertex $v \neq u$ or they separate never to "meet again" at any other vertex $v \neq u$. More formally if the node coming after u in p_1 and p_2 is different (or exists only for one path but not for the other), then the subpaths of p_1 and p_2 starting from u have an empty intersection. This is easy to see: if they could separate at u and then meet again at some v, then there would be two distinct shortest paths between u and v, which is a contradiction of the hypothesis. Let us denote this fact as F.

Assume now that $\mathsf{VC}(\mathcal{R}_G) > 3$, then there must be a set $Q = \{p_1, p_2, p_3, p_4\}$ of four shortest paths that can be shattered by $\mathcal{R}_G$. Then there is a vertex w such that $\mathcal{T}_w \cap Q = Q$, i.e., all paths in Q go through w. Let x be the farthest predecessor of w along p_1 that p_1 shares with some other path from Q, and let y be the farthest successor of w along p_1 that p_1 shares with some other path from Q. It is easy to see that if either x or y (or both) do not exist, then Q cannot be shattered, as we would incur in a contradiction of fact F.

Let us then assume that both x and y exist. Let $Q_x = \mathcal{T}_x \cap Q$ and $Q_y = \mathcal{T}_y \cap Q$. From fact F, we have that all paths in Q_x must go through the same vertices between x and w and all paths in Q_y must go through the same vertices between w and y. This also means that all paths in $Q_x \cap Q_y$ must go through the same vertices between x and y. If $Q_x \cup Q_y \subsetneq Q$, let $p^* \in Q \setminus (Q_x \cup Q_y)$. Since p^* and p_1 are distinct, then from this, from the definition of x and y, and from fact F we have that there is no vertex v such that $\mathcal{T}_v \cap Q = \{p_1, p^*\}$, which implies that Q can not be shattered. Indeed if there was a vertex v such that $\mathcal{T}_v \cap Q = \{p_1, p^*\}$, the paths p_1 and p^* would meet in w and v; which contradicts fact F.

From now we can therefore consider only the case $Q_x \cup Q_y = Q$. If $Q_x \cap Q_y = Q$, then all the paths in Q go through the same vertices between x and y. From this and the definition of x and y we have that there is no vertex v such that, for example, $\mathcal{T}_v \cap Q = \{p_1, p_2\}$, hence Q cannot be shattered. Suppose instead that $Q_x \cap Q_y \subsetneq Q$ and let $S = (Q_x \cap Q_y) \setminus \{p_1\}$. If $S \neq \emptyset$ then there is at least a path $p' \in S$ which, from the definition of S and fact F, must go through all the same vertices as p_1 between x and y. Moreover, given that $Q_x \cap Q_y \neq Q$, there must be a path $p'' \in Q \setminus \{p_1\}$ different from p_1 such that $p'' \notin S$. Then, from the definition of x, y, and S, and from the existence of p', there can be no vertex v such that $\mathcal{T}_v \cap Q = \{p_1, p''\}$, hence Q cannot be shattered. Assume now that $S = \emptyset$ and consider the case $Q_x = \{p_1, p_2, p_3\}$, $Q_y = \{p_1, p_4\}$ (all other cases follow by symmetry with this case). Consider the set $\{p_1, p_3\}$. From the definition of x and Q_x, and from fact F we have that there can not be a vertex v between the end point of p_1 before x (according to p_1) and w such that $\mathcal{T}_v \cap Q = \{p_1, p_3\}$. At the same time, from the definition of y and from fact F, we have that such a v can not be between w and the end point of p_1 after y. This implies that Q can not be shattered.

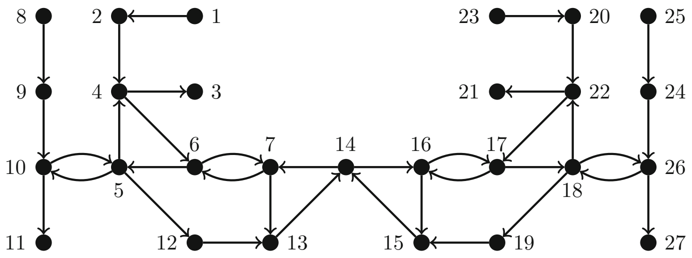


Fig. 3 Directed graph $G = (V, E)$ with $|\mathcal{S}_{uv}| \leq 1$ for all pairs $(u, v) \in V \times V$ and such that it is possible to shatter a set of four paths

We showed that in all possible cases we reached a contradiction: there is no set Q of 4 shortest paths that can be shattered by $\mathcal{R}_G$. Hence $\mathsf{VC}(\mathcal{R}_G) \leq 3$, which concludes our proof. □

The case of directed graphs It is natural to ask whether the above lemma or a similar result also holds for *directed* graphs. Fact F does not hold for directed graphs so the above proof does not extend immediately. In Fig. 3 we show a directed graph for which there is a set of four shortest paths that can be shattered. For any pair of vertices in the graph, there is at most one shortest path connecting them. Consider now the following four directed shortest paths:

- $p_A = \{1, 2, 4, 6, 7, 13, 14, 16, 17, 18, 22, 21\}$
- $p_B = \{8, 9, 10, 5, 12, 13, 14, 16, 17, 18, 26, 27\}$
- $p_C = \{25, 24, 26, 18, 19, 15, 14, 7, 6, 5, 4, 3\}$
- $p_D = \{23, 20, 22, 17, 16, 15, 14, 7, 6, 5, 10, 11\}$

It is easy to check that the set $Q = \{p_A, p_B, p_C, p_D\}$ is shattered and Table 1 shows, for each subset $R \subseteq Q$, the vertex v such that $R = Q \cap \mathcal{T}_v$. This means that Lemma 2 is not true for directed graphs. It is an open question whether it is true for a different constant.

4.2 Tightness

The bound presented in Corollary 1 (and therefore Lemma 1) is strict in the sense that for each $d \geq 1$ we can build a graph G_d with vertex-diameter $\mathsf{VD}(G_d) = 2^d + 1$ and such that the range set $\mathcal{R}_{G_d}$ associated to the set of shortest paths of G_d has VC-dimension exactly $d = \lfloor \log_2(\mathsf{VD}(G_d) - 2) \rfloor + 1$.

We now introduce a class $\mathcal{G} = (G_d)_{d \geq 1}$ of graphs indexed by d. The graphs in $\mathcal{G}$ are the ones for which we can show the tightness of the bound to the VC-dimension of the associated range set. We call the graph $G_d \in \mathcal{G}$ the dth *concertina graph*. Figure 4 shows G_1, G_2, G_3, and G_4. The generalization to higher values of d is straightforward. By construction, $\mathsf{VD}(G_d) = 2^d + 1$, so that $\lfloor \log_2(\mathsf{VD}(G_d) - 2) \rfloor + 1 = d$. The G_d concertina graph has $3(2^{d-1})$ vertices and they can be partitioned into three classes, *top*, *bottom*, and *middle*, according to their location in a drawing of the graph similar to those in Fig. 4. G_d has $2^{d-1} - 1$ top vertices, $2^{d-1} - 1$ bottom vertices, and $2^{d-1} + 2$

Table 1 How to shatter $Q = \{p_A, p_B, p_C, p_D\}$

$P \subseteq Q$	Vertex v such that $P = Q \cap \mathcal{T}_v$
$\emptyset$	1
$\{p_A\}$	2
$\{p_B\}$	9
$\{p_C\}$	24
$\{p_D\}$	20
$\{p_A, p_B\}$	13
$\{p_A, p_C\}$	4
$\{p_A, p_D\}$	22
$\{p_B, p_C\}$	26
$\{p_B, p_D\}$	10
$\{p_C, p_D\}$	15
$\{p_A, p_B, p_C\}$	18
$\{p_A, p_B, p_D\}$	16
$\{p_A, p_C, p_D\}$	7
$\{p_B, p_C, p_D\}$	5
$\{p_A, p_B, p_C, p_D\}$	14

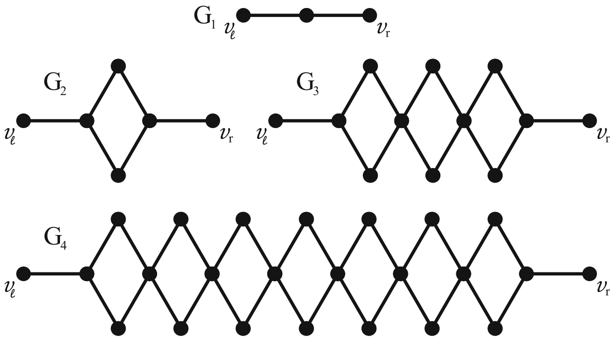


Fig. 4 Examples of concertina graphs G_d for $d = 1, 2, 3, 4$

middle vertices. For each top vertex v, let $\mathsf{f}(v)$ be the *corresponding bottom vertex*, i.e., the bottom vertex u whose two neighbors are the same two middle vertices that are neighbors of v. Analogously, the *corresponding top vertex* of a bottom vertex w is the top vertex v such that $(v) = w$. Among the middle vertices, the two with degree 1 are special and are called the *end vertices* of G_d and denoted as v_ℓ and v_r, where the labels can be arbitrarily assigned.

In Lemma 3 we build a set Q of d shortest paths from v_ℓ to v_r and show that it is shattered by $\mathcal{R}_{G_d}$, therefore proving that $\mathsf{VC}(\mathcal{R}_{G_d}) \geq d$. This fact, together with Corollary 1, allows us to conclude that $\mathsf{VC}(\mathcal{R}_{G_d}) = d$.

Lemma 3 $\mathsf{VC}(\mathcal{R}_{G_d}) = d$.

Proof Let $D = \{1, 2, 3, \ldots c, d\}$. The proof proceeds as follows:

(1) Define a map r from $\mathcal{S} = 2^D \setminus \{\emptyset, D\}$ to set of of top and bottom vertices of G_d.
(2) Use the map r to build the set Q of d shortest paths from v_ℓ to v_r.
(3) Show that Q is shattered by $\mathcal{R}_{G_d}$, implying $\mathsf{VC}(\mathcal{R}_{G_d} \geq d$.
(4) Conclude from the above and from Corollary 1 that $\mathsf{VC}(\mathcal{R}_{G_d} = d$.

1. Defining the map r. For any set (of subsets of D) $s' \in \mathcal{S}$, let $\mathsf{c}(s') = D \setminus s'$. The set (of subsets of D) $\mathsf{c}(s')$ is the *unique* set (of subsets of D) $s'' \in \mathcal{S}$ such that $s' \cap s'' = \emptyset$ and $s' \cup s'' = D$.

We can partition $\mathcal{S}$ in two sets $\mathcal{A}$ and $\mathcal{B}$ ($\mathcal{A} \cup \mathcal{B} = \mathcal{S}$, $\mathcal{A} \cap \mathcal{B} = \emptyset$) as follows: for any *unordered pair* $(s', \mathsf{c}(s'))$, we arbitrarily put s' in $\mathcal{A}$ and $\mathsf{c}(s')$ in $\mathcal{B}$. It is easy to see that $|\mathcal{A}| = |\mathcal{B}| = 2^{d-1} - 1$, which is the number of top vertices of G_d (and the number of bottom vertices of G_d)

We now build a mapping $\mathsf{r}_\mathcal{A}$ (resp. $\mathsf{r}_\mathcal{B}$) that will map each element of $\mathcal{A}$ (resp. of $\mathcal{B}$) to a *top* (resp. *bottom*) vertex of G_d. We will then use these mappings to build the set of paths to be shattered.

Let $\mathsf{r}_\mathcal{A}$ be an arbitrary one-to-one map from each element of $\mathcal{A}$ to a top vertex of G_d (i.e., it maps a set of subsets of D to a top vertex). We now build the one-to-one map $\mathsf{r}_\mathcal{B}$ from each element of $\mathcal{B}$ to a bottom vertex of G_d. Consider the inverse map $\mathsf{r}_\mathcal{A}^{-1}$ from the top vertices of G_d to the elements of $\mathcal{A}$ and let v be any top vertex of G_d. For any top vertex v, $\mathsf{r}_\mathcal{A}^{-1}(v)$ is a set of subsets of D, and a element of $\mathcal{A}$ (and of $\mathcal{S}$), and therefore there is an element s'' of $\mathcal{B}$ such that $s'' = \mathsf{c}(\mathsf{r}_\mathcal{A}^{-1}(v))$.

The bijection $\mathsf{r}_\mathcal{B}$ maps the element $\mathsf{c}(\mathsf{r}_\mathcal{A}^{-1}(v))$ of $\mathcal{B}$ to the bottom vertex $\mathsf{f}(v)$ corresponding to v. In other words, if a set s' of subsets of D is mapped by $r_\mathcal{A}$ to a top vertex v, then $r_\mathcal{B}$ maps to the bottom vertex $\mathsf{f}(v)$ the unique set s'' of subsets of D such that $s'' = \mathsf{c}(s')$.

It is easy to see that, if we take the union of $\mathsf{r}_\mathcal{A}$ and $\mathsf{r}_\mathcal{B}$, we obtain a map r from $\mathcal{S}$ to the set of top and bottom vertices of G_d. An example of a possible r for G_3 is presented in Fig. 5.

2. Building the set Q *of shortest paths* We now build the set $Q = \{p_1, \ldots c, p_d\}$ which contains d shortest paths from v_ℓ to v_r. Given the definition of G_d, it should be clear that all these shortest paths must go through all other middle vertices of G_d. Let M be the set of all middle vertices in G_d, and consider the set $E = V \setminus M$ (i.e., E is the set containing all the top and bottom vertices of G_d. For any $i \in D$, the *set* of vertices that the path $p_i \in Q$ goes through is the set $P_i = M \cup \{v \in E\ :\ i \in r^{-1}(v)\}$. We build the path p_i by sorting the vertices in P_i in increasing order by their shortest

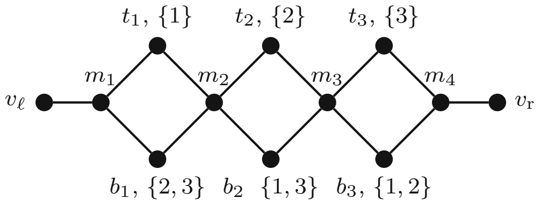


Fig. 5 An example of the r map for G_3. The set next to each *top* and *bottom* vertex u is the set s such that $\mathsf{r}(s) = u$

path distance from v_ℓ. There is a unique such ordering of the vertices in P_i, as they all have different shortest path distances from v_ℓ. It is easy to see that p_i is a shortest path. Note that a path $p_i \in Q$ goes through a top vertex v if and only if $i \in \mathsf{r}_{\mathcal{A}}^{-1}(v)$. Analogously, p_i goes through a bottom vertex u if and only if $i \in \mathsf{r}_{\mathcal{B}}^{-1}(u)$. Taking the map r from Fig. 5 as an example, the set $Q = \{p_1, p_2, p_3\}$ contains the following shortest paths:

- $p_1 = \{v_\ell, m_1, t_1, m_2, b_2, m_3, b_3, m_4, v_\mathrm{r}\}$
- $p_2 = \{v_\ell, m_1, b_1, m_2, t_2, m_3, b_3, m_4, v_\mathrm{r}\}$
- $p_3 = \{v_\ell, m_1, b_1, m_2, b_2, m_3, t_3, m_4, v_\mathrm{r}\}$

3. Showing that Q is shattered by $\mathcal{R}_{G_d}$ We now show that the set Q of shortest paths is shattered by $\mathcal{R}_{G_d}$. We want to show that each subset s of Q can be expressed as the intersection between Q and a range $\mathcal{T}_v \in \mathcal{R}_{G_d}$, for some vertex v in G_d. Consider first the case $s = Q$ (as Q is a subset of itself). All paths in Q go through all the middle vertices that are not v_ℓ or v_r, so if we let v_Q be any arbitrary middle vertex different from v_ℓ or v_r, we have $Q = Q \cap \mathcal{T}_{v_Q}$. Also, given that v_ℓ is not *internal* to any path in Q, we have $\emptyset = Q \cap \mathcal{T}_{v_\ell}$.

Let now $\mathcal{Q} = 2^Q \setminus \{\emptyset, Q\}$ and denote with I_s the set of indexes $i \in D$ such that $p_i \in s$. We want to show that $s = Q \cap \mathcal{T}_{\mathsf{r}(I_s)}$. It is easy to see that $s \subseteq Q \cap \mathcal{T}_{\mathsf{r}(I_s)}$, as the vertex $\mathsf{r}(I_s)$ is internal to all paths in s, by definition of $\mathsf{r}(I_s)$ and of the paths in Q. On the other end, no path in $Q \setminus s$ belongs to $Q \cap \mathcal{T}_{\mathsf{r}(I_s)}$ as no path in $Q \setminus s$ goes through $\mathsf{r}(I_s)$ because it goes through the corresponding vertex of $\mathsf{r}(I_s)$ (this corresponding vertex is a top vertex if $\mathsf{r}(I_s)$ is a bottom vertex, and a bottom vertex otherwise). Hence we showed that $s = Q \cap \mathcal{T}_{\mathsf{r}(I_s)}$ for any $s \in \mathcal{Q}$.

We showed that all subsets of Q can be expressed as the intersection between Q and a range from $\mathcal{R}_{G_d}$, which means that Q is shattered and therefore $\mathsf{VC}(\mathcal{R}_{G_d}) \geq d$.

4. Concluding the proof From Corollary 1 we know that $\mathsf{VC}(\mathcal{R}_{G_d}) \leq d$, so it must be $\mathsf{VC}(\mathcal{R}_{G_d}) = d$, which concludes our proof. □

The upper bound presented in Lemma 2 for the case of unique shortest paths is also strict in the same sense.

Lemma 4 *There is a graph $G = (V, E)$ with $|\mathcal{S}_{uv}| \leq 1$ for all pairs $(u, v) \in V \times V$ such that the range set $\mathcal{R}_G$ associated to the shortest paths in G has VC-Dimension exactly 3.*

Proof Consider the graph G in Fig. 6. Let $p_1 = (a, b, c, e, i, j)$, $p_2 = (m, f, e, i, k, l)$, $p_3 = (d, c, e, f, g, h)$ be three paths. We now show that $Q = \{p_1, p_2, p_3\}$ can be shattered by $\mathcal{R}_G$, which implies $\mathsf{VC}(\mathcal{R}_G) \geq 3$. We have $\emptyset = Q \cap \mathcal{T}_a$, $\{p_1\} = Q \cap \mathcal{T}_b$, $\{p_2\} = Q \cap \mathcal{T}_k$, $\{p_3\} = Q \cap \mathcal{T}_g$, $\{p_1, p_2\} = Q \cap \mathcal{T}_i$, $\{p_1, p_3\} = Q \cap \mathcal{T}_c$, $\{p_2, p_3\} = Q \cap \mathcal{T}_f$, $\{p_1, p_2, p_3\} = Q \cap \mathcal{T}_e$. Hence all subsets of Q can be expressed as the intersection between Q and some range in $\mathcal{R}_G$ which means that Q can be shattered and $\mathsf{VC}(\mathcal{R}_G) \geq 3$. Lemma 2 gives us an upper bound $\mathsf{VC}(\mathcal{R}_G) \leq 3$, so we can conclude that $\mathsf{VC}(\mathcal{R}_G) = 3$. □

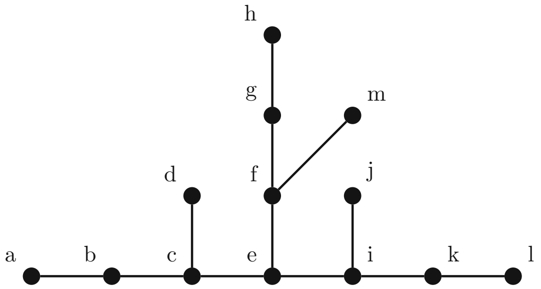


Fig. 6 Graph G with $\mathsf{VC}(\mathcal{R}_G) = 3$

Although the example in Fig. 6 is a tree, this is not a requirement for either Lemma 2 or Lemma 4: in a weighted graph with cycles (i.e., not a tree) the weights may be such that there is a unique shortest path between any pair of connected vertices.

5 Algorithms

In this section we present our algorithms to compute a set of approximations for the betweenness centrality of the (top-K) vertices in a graph through sampling, with probabilistic guarantees on the quality of the approximations.

5.1 Approximation for all the vertices

The intuition behind the algorithm to approximate the betweenness values of all vertices is the following. Given a graph $G = (V, E)$ with vertex-diameter $\mathsf{VD}(G)$ and two parameters $\varepsilon, \delta \in (0, 1)$ we first compute a sample size r using (1) with

$$d = \lfloor \log_2(\mathsf{VD}(G) - 2) \rfloor + 1 \ .$$

The resulting sample size is

$$r = \frac{c}{\varepsilon^2} \left(\lfloor \log_2(\mathsf{VD}(G) - 2) \rfloor + 1 + \ln \frac{1}{\delta} \right) \ . \tag{3}$$

This is sufficient to achieve the desired accuracy (expressed through ε) with the desired confidence (expressed through $1-\delta$). The algorithm repeats the following steps r times: (1) it samples a pair u, v of distinct vertices uniformly at random, (2) it computes the set $\mathcal{S}_{uv}$ of all shortest paths between u and v, (3) it selects a path p from $\mathcal{S}_{uv}$ uniformly at random, (4) it increases by $1/r$ the betweenness estimation of each vertex in $\mathsf{Int}(p)$. Note that if the sampled vertices u and v are not connected, we can skip steps 3 and 4 because we defined $\mathcal{S}_{uv} = \{p_\emptyset\}$. Denoting with S the set of the sampled shortest paths, the *unbiased* estimator $\tilde{\mathsf{b}}(w)$ for the betweenness $\mathsf{b}(w)$ of a vertex w is the sample average

$$\tilde{\mathsf{b}}(w) = \frac{1}{r}\sum_{p\in S} \mathbb{1}_{\mathsf{Int}(p)}(w) = \frac{1}{r}\sum_{p\in S} \mathbb{1}_{\mathcal{T}_w}(p) \ .$$

There are two crucial steps in this algorithm: the computation of $\mathsf{VD}(G)$ and the sampling of a path uniformly at random from $\mathcal{S}_{uv}$. We first deal with the latter, and then present a linear-time constant-factor approximation algorithm for $\mathsf{VD}(G)$. Algorithm 1 presents the pseudocode of the algorithm, including the steps to select a random path. The `computeAllShortestPaths(`u, v`)` on line 8 is a call to a modified Dijkstra's (or BFS) algorithm to compute the set $\mathcal{S}_{uv}$, with the same modifications as the exact betweenness algorithm by Brandes (2001). The `getDiameterApprox()` procedure computes an approximation for $\mathsf{VD}(G)$.

Sampling a shortest path Our procedure to select a random shortest path from $\mathcal{S}_{uv}$ is inspired by the dependencies accumulation procedure used in Brandes' exact algorithm (Brandes 2001). Let u and v be the vertices sampled by our algorithm (Step 1 of Algorithm 1). We assume that u and v are connected otherwise the only possibility is to select the empty path $p_\emptyset$. Let y be any vertex belonging to at least one shortest path from u to v. Following Brandes (2001), we can compute σ_{uy} and $\mathcal{S}_{uy}$ while we compute the set $\mathcal{S}_{uv}$ of all the shortest paths from u to v. We can then use this information to select a shortest path p uniformly at random from $\mathcal{S}_{uv}$ as follows. For each vertex w let $P_u(w)$ be the subset of neighbors of w that are *predecessors* of w along the shortest paths from u to w. Let p^* be the sampled shortest path that we build *backwards* starting from the endpoint v and adding the sampled predecessors before v. Initially we have $p^* = \{v\}$. Starting from v, we select one of its predecessors $z \in P_u(v)$ using weighted random sampling: each $z \in P_u(v)$ has probability $\sigma_{uz}/\sum_{w\in P_u(v)} \sigma_{uw} = \sigma_{uz}/\sigma_{uv}$ of being sampled. We add z to p^* and then repeat the procedure for z. That is, we select one of z's predecessors (denote it with ℓ) from $P_u(z)$ using weighted sampling with weight $\sigma_{u\ell}/\sigma_{uz}$, and add it to p^* before v (i.e., p^* is now $\{\ell, v\}$, and so on until we reach u. Note that we can update the estimation of the betweenness of the internal vertices along p^* (the only ones for which the estimation is updated) as we compute p^*.

Lemma 5 *The path p^* built according to the above procedure is selected uniformly at random among the paths in $\mathcal{S}_{uv}$.*

Proof The probability of sampling $p^* = (u, z_1, \ldots c, z_{|p^*|-2}, v)$ equals to the product of the probabilities of sampling the vertices internal to p^*, hence

$$\Pr(p^*) = \frac{\sigma_{uz_{|p^*|-2}}}{\sigma_{uv}} \frac{\sigma_{uz_{|p^*|-3}}}{\sigma_{uz_{|p^*|-2}}} \ldots b\frac{1}{\sigma_{uz_2}} = \frac{1}{\sigma_{uv}}$$

where we used a result by Brandes (2001, Lemma 3) which gives us the following expression about the number of shortest paths for $w \neq u$,

$$\sigma_{uw} = \sum_{j\in P_u(w)} \sigma_{uj}$$

and the fact that for z_1, which is a neighbor of u, $\sigma_{uz_1} = 1$. □

Algorithm 1: Computes approximations $\tilde{\mathsf{b}}(v)$ of the betweenness centrality $\mathsf{b}(v)$ for all vertices $v \in V$.

```
    Input  : Graph $G = (V, E)$ with $|V| = n$, $\varepsilon, \delta \in (0, 1)$
    Output: A set of approximations of the betweenness centrality of the vertices in $V$
 1  foreach $w \in V$ do
 2      $\tilde{\mathsf{b}}(v) \leftarrow 0$
 3  end
 4  $\mathsf{VD}(G) \leftarrow$getVertexDiameter($G$);
 5  $r \leftarrow (c/\varepsilon^2)(\lfloor \log_2(\mathsf{VD}(G) - 2)\rfloor + \ln(1/\delta))$;
 6  for $i \leftarrow 1$ to $r$ do
 7      $(u, v) \leftarrow$sampleUniformVertexPair($V$);
 8      $\mathcal{S}_{uv} \leftarrow$computeAllShortestPaths($u, v$);
 9      if $\mathcal{S}_{uv} \neq \{p_\emptyset\}$ then
            //Random path sampling and estimation update
10          $t \leftarrow v$;
11          while $t \neq u$ do
12              sample $z \in P_u(t)$ with probability $\sigma_{uz}/\sigma_{ut}$;
13              if $z \neq u$ then
14                  $\tilde{\mathsf{b}}(z) \leftarrow \tilde{\mathsf{b}}(z) + 1/r$;
15              end
16              $t \leftarrow z$;
17          end
18      end
19  end
20  return $\{(v, \tilde{\mathsf{b}}(v)), v \in V\}$
```

Approximating the vertex-diameter The algorithm presented in the previous section requires the value of the vertex-diameter $\mathsf{VD}(G)$ of the graph G (line 4 of Algorithm 1). Computing the exact value of $\mathsf{VD}(G)$ could be done by solving the All Pair Shortest Paths (APSP) problem, and taking the shortest path with the maximum size. Algorithms for exactly solving APSP problem such as Johnson's which runs in $O(V^2 \log V + VE)$ or Floyd–Warshall's ($\Theta(V^3)$), would defeat our purposes: that is, once we have all the shortest paths for the computation of the diameter, we may as well calculate the betweenness of all the vertices exactly, given that the most expensive part of this latter computation (i.e., obtaining the shortest paths) is already done. Given that Theorem 1 (and Theorem 2) only requires an upper bound to the VC-dimension of the range set, an approximation of the vertex-diameter would be sufficient for our purposes. Several refined algorithms for approximating the diameter are known (Aingworth et al. 1999; Boitmanis et al. 2006; Roditty and Williams 2012), with various running times and quality of approximations. We briefly present a well-known and simple approximation algorithm that has the right balance of accuracy and speed for our purposes.

Let $G = (V, E)$ be an *undirected* graph where *all the edge weights are equal*. It is a well-known result that one can obtain a 2-approximation $\widetilde{\mathsf{VD}}(G)$ of the vertex-diameter $\mathsf{VD}(G)$ of G in time $O(V + E)$ in the following way:

(1) select a vertex $v \in V$ uniformly at random;

(2) compute the shortest paths from v to all other vertices in V;
(3) finally take $\widetilde{\mathsf{VD}}(G)$ to be the sum of the lengths of the two shortest paths with maximum size (which equals to the two longest shortest paths) from v to two distinct other nodes u and w.

Lemma 6 deals with the approximation guarantees of this algorithm.

Lemma 6 $\mathsf{VD}(G) \leq \widetilde{\mathsf{VD}}(G) \leq 2\mathsf{VD}(G)$.

Proof Let $v \in V$ be a vertex that we choose uniformly at random from the set V. Let also $u, w \in V$ be the two vertices such that the sum of the sizes of the shortest paths p_{vu} and p_{vw} is maximized among all the shortest paths that have v as a source. We have $\widetilde{\mathsf{VD}}(G) \leq 2\mathsf{VD}(G)$ because $|p_{vu}|, |p_{vw}| \leq \mathsf{VD}(G)$, so $|p_{vu}| + |p_{vw}| \leq 2\mathsf{VD}(G)$. To see that $\widetilde{\mathsf{VD}}(G) \geq \mathsf{VD}(G)$, consider a pair of vertices x and z such that the length of a shortest path between x and z is equal to $\mathsf{VD}(G)$. Let p_{xv} be a shortest path between x and v and let p_{vz} be a shortest path between v and z. From the properties of the shortest paths p_{vu}, p_{vw} we have $|p_{vu}| + |p_{vw}| \geq |p_{vx}| + |p_{vz}|$. Since the graph is undirected $|p_{s,t}| = |p_{t,s}|$ for every $s, t \in V$. Therefore:

$$\widetilde{\mathsf{VD}}(G) = |p_{vu}| + |p_{vw}| \geq |p_{vx}| + |p_{vz}| = |p_{xv}| + |p_{vz}| \geq \mathsf{VD}(G)\ .$$

For the last inequality we used the fact that since $\mathsf{VD}(G)$ is the size of the shortest path from x to z, then every other path (in this case p'_{xz} which is the merge of p_{xv} and p_{vz}) has greater or equal length from $p_{x,z}$. □

In case we have multiple connected components in G, we compute an upper bound to the vertex diameter of each component separately by running the above algorithm on each component, and then taking the maximum. The connected components can be computed in $O(n+m)$ by traversing the graph in a Breadth-First-Search (BFS) fashion starting from a random v. The time complexity of the approximation algorithm in the case of multiple connected components is again $O(n+m)$ since the sum of the vertices of individual components is n and the sum of edges is m.

The use of the above 2-approximation in the computation of the sample size from line 6 of Algorithm 1 results in at most c/ε^2 additional samples than if we used the exact value $\mathsf{VD}(G)$. The computation of $\widetilde{\mathsf{VD}}(G)$ does not affect the running time of our algorithm: for the construction of the first sample we can reuse the shortest paths from the sampled vertex v that we used to obtain the approximation. Specifically, we can sample a new vertex $u \neq v$ and then choose with uniform probability one of the (already computed) shortest paths between v and u.

If the graph is directed and/or not all edge weights are equal, the computation of a good approximation to $\mathsf{VD}(G)$ becomes more involved. In particular, notice that there is no relationship between $\mathsf{VD}(G)$ and $\mathsf{diam}(G)$ when G is weighted, as the shortest path with maximum size may not be the shortest path with maximum weight. In these cases, one can use the size (number of vertices) of the largest Weakly Connected Component (WCC), as a loose upper bound to $\mathsf{VD}(G)$. The WCC's can again be computed in $O(n+m)$ using BFS. This quantity can be as high as n but for the computation of the sample size we use its logarithm, mitigating the crudeness of the bound. In this case our sample size is comparable to that proposed by Brandes and Pich (2007).[3] Nevertheless the amount of work done per sample by our algorithm is still much smaller (see Sects. 5.3 and 7 for more details). In practice, it is possible that the nature of the network suggests a much better upper bound to the vertex-diameter of the graph, resulting in a smaller sample size.

Analysis Algorithm 1 offers probabilistic guarantees on the quality of all approximations of the betweenness centrality.

Lemma 7 *With probability at least* $1-\delta$*, all the approximations computed by the algorithm are within* ε *from their real value:*

$$\Pr\left(\exists v \in V \ s.t.\ |\mathsf{b}(v) - \tilde{\mathsf{b}}(v)| > \varepsilon\right) < \delta \ .$$

Proof For each $p_{uv} \in \mathbb{S}_G$ let

$$\pi_G(p_{uv}) = \frac{1}{n(n-1)}\frac{1}{\sigma_{uv}} \ .$$

It is easy to see that π_G is a probability distribution and $\pi_G(p_{uv})$ is the probability of sampling the path p_{uv} during an execution of the loop on line 6 in Algorithm 1, given the way that the vertices u and v are selected and Lemma 5. Given a set A of shortest paths in G, we slightly abuse the notation and denote $\pi_G(A) = \sum_{p\in A}\pi_G(p)$.

Consider the range set $\mathcal{R}_G$ and the probability distribution π_G. Let S be the set of paths sampled during the execution of the algorithm. For r as in (3), Theorem 1 tells us that the sample S is a ε-approximation to $(\mathcal{R}_G, \pi_G)$ with probability at least $1-\delta$. Suppose that this is indeed the case, then from Definition 4 and the definition of $\mathcal{R}_G$ we have that

$$\left|\pi_G(\mathcal{T}_v) - \frac{1}{r}\sum_{p\in S}\mathbb{1}_{\mathcal{T}_v}(p)\right| = \left|\pi_G(\mathcal{T}_v) - \tilde{\mathsf{b}}(v)\right| \le \varepsilon, \forall v \in V \ .$$

From the definition of π_G we have

$$\pi_G(\mathcal{T}_v) = \frac{1}{n(n-1)}\sum_{p_{uw}\in\mathcal{T}_v}\frac{1}{\sigma_{uw}} = \mathsf{b}(v),$$

which concludes the proof. □

Time and space complexity Clearly the runtime of the algorithm is dominated by the computation of the shortest path at each step, which takes time $O(|V| + |M|)$ if the graph is unweighted (BFS algorithm) and time $O(|E| + |V|\log|V|)$ otherwise

[3] After this work was accepted for publication, Bergamini and Meyerhenke (2015) presented an improved upper bound for VD(G) for undirected weighted graphs.

(Dijkstra's algorithm with Fibonacci heap). This time must then be multiplied by r as in (3) to obtain the final time complexity. The space requirements are dominated by the amount of memory needed to store the graph, so they are either $O(|V|^2)$ if using an adjacency matrix, or $O(|V| + |E|)$ if using $|V|$ adjacency lists.

Unique shortest paths When, for each pair (u, v) of vertices of G, either there is a unique shortest path from u to v or v is unreachable from u, then one can apply Lemma 2 and obtain a smaller sample size

$$r = \frac{c}{\varepsilon^2}\left(3 + \ln \frac{1}{\delta}\right)$$

to approximate the betweenness values of all the vertices. This is an interesting result: the number of samples needed to compute a good approximation to all vertices is a *constant* and completely *independent from* G. Intuitively, this means that the algorithm is extremely fast on graphs with this property. Unique shortest paths are common or even enforced in road networks by slightly perturbing the edge weights or having a deterministic tie breaking policy (Geisberger et al. 2008).

5.2 High-quality approximation of the top-*K* betweenness vertices

Very often in practice one is interested only in identifying the vertices with the highest betweenness centrality, as they are the "primary actors" in the network. We present here an algorithm to compute a very high-quality approximation of the set $\mathsf{TOP}(K, G)$ of the top-K betweenness vertices in a graph $G = (V, E)$. Formally, let $v_1, \dots c, v_n$ be a labelling of the vertices in V such that $\mathsf{b}(v_i) \geq \mathsf{b}(v_j)$ for $1 \leq i < j \leq n$. Then $\mathsf{TOP}(K, G)$ is defined as the set of vertices with betweenness at least $\mathsf{b}(v_K)$:

$$\mathsf{TOP}(K, G) = \{(v, \mathsf{b}(v)) \ : \ v \in V \text{ and } \mathsf{b}(v) \geq \mathsf{b}(v_K)\} \ .$$

Note that $\mathsf{TOP}(K, G)$ may contain more than K vertices.

Our algorithm works in two phases. Each phase is basically a run of the algorithm for approximating the betweenness of all vertices. The two phases differ in the way they compute the number of paths to sample and the additional operations at the end of each phase. In the first phase, we compute a lower bound ℓ' to $\mathsf{b}(v_K)$. In the second phase we use ℓ' to compute the number of samples r needed to obtain a relative (ℓ', ε)-approximation to $(\mathcal{R}_G, \pi_G)$. We use r samples to approximate the betweenness of all vertices again, and return a collection of vertices that is, with high probability, a superset of $\mathsf{TOP}(K, G)$.

Let $\widetilde{\mathsf{VD}}(G)$ be an upper bound to the vertex-diameter of G. Given $\varepsilon, \delta \in (0, 1)$, let δ', δ'' be two positive reals such that $(1 - \delta')(1 - \delta'') \geq (1 - \delta)$. Let

$$r' = \frac{c}{\varepsilon^2}\left(\lfloor \log_2(\widetilde{\mathsf{VD}}(G) - 2)\rfloor + 1 + \log \frac{1}{\delta'}\right) \ .$$

In the first phase we compute $\tilde{\mathsf{b}}'_k$, which is the Kth highest estimated betweenness obtained using Algorithm 1 where $r = r'$, and let $\ell' = \tilde{\mathsf{b}}_K - \varepsilon$, and

$$r'' = \frac{c'}{\varepsilon^2 \ell'} \left((\lfloor \log_2(\widetilde{\mathsf{VD}}(G) - 2) \rfloor + 1) \log \frac{1}{\ell'} + \log \frac{1}{\delta''} \right) .$$

In the second phase, we run Algorithm 1 with $r = r''$. Let $\tilde{\mathsf{b}}''_K$ be the so-obtained K-th highest estimated betweenness and let $\ell'' = \tilde{\mathsf{b}}''_K/(1+\varepsilon)$. We return the collection

$$\widetilde{\mathsf{TOP}}(K, G) = \left\{ v \in V \ : \ \frac{\tilde{\mathsf{b}}''(v)}{1-\varepsilon} \geq \ell'' \right\} .$$

The pseudocode of the algorithm is presented in Algorithm 2.

Algorithm 2: High-quality approximation of the top-K betweenness vertices

```
Input  : a graph $G = (V, E)$ with $|V| = n$, a positive integer $K \leq n$, real values $\varepsilon, \delta \in (0, 1)$
Output: a superset of $\mathsf{TOP}(K, G)$, with high-quality estimation of the betweenness for the vertices
        in the returned set.
$\delta', \delta'' \leftarrow$ two positive reals such that $(1 - \delta')(1 - \delta'') \geq (1 - \delta)$;
$\widetilde{\mathsf{VD}}(G) \leftarrow$ upper bound to $\mathsf{VD}(G)$;
//First phase
$r' \leftarrow \frac{c}{\varepsilon^2} \left( \lfloor \log_2(\widetilde{\mathsf{VD}}(G) - 2) \rfloor + 1 + \log \frac{1}{\delta''} \right)$;
$B' = \{(v, \tilde{\mathsf{b}}'(v)) \ : \ v \in V\} \leftarrow$ output of Algorithm 1 with $r = r'$;
$\tilde{\mathsf{b}}'_K \leftarrow K$th highest betweenness value from $B'$, ties broken arbitrarily;
$\ell' \leftarrow \tilde{\mathsf{b}}'_K - \varepsilon$;
$r'' \leftarrow \frac{c'}{\varepsilon^2 \ell'} \left( (\lfloor \log_2(\widetilde{\mathsf{VD}}(G) - 2) \rfloor + 1) \log \frac{1}{\ell'} + \log \frac{1}{\delta''} \right)$;
//Second phase
$B'' = \{(v, \tilde{\mathsf{b}}''(v)) \ : \ v \in V\} \leftarrow$ output of Algorithm 1 with $r = r''$;
$\tilde{\mathsf{b}}''_K \leftarrow K$th highest betweenness value from $B''$, ties broken arbitrarily;
$\ell'' \leftarrow \mathsf{b}''_K/(1 + \varepsilon)$;
return $\{(v, \tilde{\mathsf{b}}''(v)) \ : \ v \in V$ s.t. $\frac{\tilde{\mathsf{b}}''(v)}{1-\varepsilon} \geq \ell''\}$;
```

Analysis The following lemma shows the properties of the collection $\widetilde{\mathsf{TOP}}(K, G)$.

Lemma 8 *With probability at least* $1 - \delta$*,*

(1) $\mathsf{TOP}(K, G) \subseteq \widetilde{\mathsf{TOP}}(K, G)$*, and*
(2) *for all* $v \in \mathsf{TOP}(K, G)$ *we have* $|\tilde{\mathsf{b}}''(v) - \mathsf{b}(v)| \leq \varepsilon \mathsf{b}(v)$*, and*
(3) *no vertex* $u \in \widetilde{\mathsf{TOP}}(K, G) \setminus \mathsf{TOP}(K, G)$ *has an estimated betweenness greater than* $\ell'(1 + \varepsilon)$.

Proof We start by proving 1. From Theorem 1 we know that, with probability at least $1 - \delta'$, a sample of size r' is a ε-approximation to $(\mathcal{R}_G, \pi_G)$ and from Theorem 2 we have that with probability at least $1 - \delta''$ a sample of size r'' is a relative (ℓ', ε)-approximation to $(\mathcal{R}_G, \pi_G)$. Suppose both these events occur, which happens with probability at least $1-\delta$. Then it is easy to see that $\ell' \leq \mathsf{b}(v_K)$, as there must be at least K vertices with exact betweenness greater or equal to ℓ'. Consider now ℓ''. Following the same reasoning as for ℓ', it should be clear that $\ell'' \leq \mathsf{b}(v_K)$. The vertices included in $\widetilde{\mathsf{TOP}}(K, G)$ are all and only the vertices that *may* have exact betweenness at least ℓ'', which implies that all vertices that have exact betweenness at least $\mathsf{b}(v_K)$ are included in $\widetilde{\mathsf{TOP}}(K, G)$. Points 2 and 3 in the thesis follow from the properties of the relative (ℓ', ε)-approximation (Definition 5), and this concludes our proof. □

The advantage of using our algorithm to approximate the collection of top-K betweenness vertices is the very high-quality of the approximation of the betweenness values for the returned set of vertices: all estimations within a multiplicative factor ε from their exact values. Previous algorithms were only able to approximate the betweenness values to within an additive error ε. The cost of computing the high quality approximation for the top-K vertices is the cost of an additional run of our algorithm to compute good approximations for all the vertices.

5.3 Discussion

Jacob et al. (2005) and independently Brandes and Pich (2007) present a sampling-based algorithm to approximate the betweenness centrality of all the vertices of the graph. The algorithm (which we call BP) creates a sample $S = \{v_1, \ldots c, v_r\}$ of r vertices drawn uniformly at random and computes all the shortest paths between each v_i to all other vertices in the graph. Their estimator $\tilde{\mathsf{b}}_{\mathsf{BP}}(u)$ for $\mathsf{b}(u)$ is

$$\tilde{\mathsf{b}}_{\mathsf{BP}}(u) = \frac{1}{(n-1)r} \sum_{v_i \in S} \sum_{\substack{w \neq v_i \\ w \neq u}} \sum_{p \in \mathcal{S}_{v_i w}} \frac{\mathbb{1}_{\mathsf{Int}(p)}(u)}{|\mathcal{S}_{v_i w}|} .$$

As it was for Algorithm 1, the key ingredient to ensure a correct approximation for the betweenness centrality is the computation of the sample size r. Inspired by the work of Eppstein and Wang (2004) and Brandes and Pich (2007) prove that, to obtain good additive (within ε) estimations for the betweenness of all vertices with probability at least $1-\delta$, it must be

$$r \geq \frac{1}{2\varepsilon^2}\left(\ln n + \ln 2 + \ln \frac{1}{\delta}\right) .$$

From this expression it should be clear that this sample size is usually much larger than ours, as in practice $\mathsf{VD}(G) \ll n$. For the same reason, this algorithm would not scale well as the network size increases (see also Sect. 7).

Another interesting aspect in which our algorithm and BP differ is the *amount of work done per sample*. Our algorithm computes a single set $\mathcal{S}_{uv}$ for the sampled pair of vertices (u, v): it performs a run of Dijkstra's algorithm (or of BFS) from u, stopping when v is reached. BP instead computes *all* sets $\mathcal{S}_{uw}$ from the sampled vertex u to *all* other vertices $w \in V$, again with a single run of Dijkstra or BFS, but without the “early-stopping condition” approach that our algorithm takes when reaching v. Although in the worst case the two computations have the same time complexity,[4] in practice we perform many fewer operations, as we can expect v not to always be very far from u and therefore we can terminate early. This fact has a huge impact on the running time. Our algorithm also touches *many fewer* edges than BP. The latter may touch all the edges in the graph *at every sample*, while our computation exhibits a much higher *locality*, exploring only a neighborhood of u until v is reached. The results of our experimental evaluation presented in Sect. 7 highlights this and other advantages of our method over the one by Jacob et al. (2005) and Brandes and Pich (2007). Using *bidirectional A^* search* (Pohl 1969; Kaindl and Kainz 1997) can further speed up the computation for each sample of our algorithms.

The analysis of Algorithm 1 allow us to obtain a tighter analysis for the algorithm by Brandes and Pich (2007) and Jacob et al. (2005).

Lemma 9 *Fix $r > 0$ and let $w \in V$. Let $\tilde{\mathsf{b}}_{\mathsf{BP}}(w)$ be the estimation of the betweenness $\mathsf{b}(w)$ as computed by BP using r samples, and let $\tilde{\mathsf{b}}(w)$ be the estimation computed by Algorithm 1 using r samples. For any value of $\mathsf{b}(w)$ and r, we have*

$$\mathrm{Var}[\tilde{\mathsf{b}}_{\mathsf{BP}}(w)] \leq \mathrm{Var}[\tilde{\mathsf{b}}(w)]\ .$$

Proof Consider the quantity

$$\mathrm{Var}[\tilde{\mathsf{b}}(w)] - \mathrm{Var}[\tilde{\mathsf{b}}_{\mathsf{BP}}(w)]\ .$$

We will show that the above quantity is greater than or equal to 0, proving the thesis. Since $\mathbb{E}[\tilde{\mathsf{b}}(w)] = \mathbb{E}[\tilde{\mathsf{b}}_{\mathsf{BF}}(w)] = \mathsf{b}(w)$ and using the definition of variance, we only need to show

$$\mathbb{E}[(\tilde{\mathsf{b}}(w))^2] - \mathbb{E}[(\tilde{\mathsf{b}}_{\mathsf{BF}}(w))^2] \geq 0\ . \tag{4}$$

Let start from computing $\mathbb{E}[(\tilde{\mathsf{b}}_{\mathsf{BF}}(w))^2]$. For every vertex v, we define α_v as

$$\alpha_v = \frac{1}{n-1} \sum_{\substack{u \in V \\ u \neq v}} \sum_{p \in \mathcal{S}_{vu}} \frac{\mathbb{1}_{\mathsf{Int}(p)}(w)}{\sigma_{vu}}\ .$$

Note that

$$\mathsf{b}(w) = \frac{1}{n} \sum_{v \in V} \alpha_v\ . \tag{5}$$

Let X_i, for $1 \leq i \leq r$, be the contribution to $\tilde{\mathsf{b}}_{\mathsf{BP}}(w)$ of the paths computed from the i^{th} sampled vertex. X_i is a random variable that takes value α_v with probability $1/n$, for all $v \in V$. We have

[4] It is a well-known open problem whether there is an algorithm to perform a single s, t-shortest path computation between a pair of vertices with smaller worst-case time complexity than the Single Source Shortest Path computation.

$$\mathbb{E}[X_i] = \frac{1}{n}\sum_{v\in V}\alpha_v = \mathsf{b}(w) \text{ and } \mathbb{E}\left[X_i^2\right] = \frac{1}{n}\sum_{v\in V}\alpha_v^2, \forall 1 \le i \le r \ . \tag{6}$$

Clearly

$$\tilde{\mathsf{b}}_{\mathsf{BP}}(w) = \frac{1}{r}\sum_{i=1}^{r} X_i \ .$$

The variables X_i's are independent and identically distributed so

$$\begin{aligned}\mathbb{E}[(\tilde{\mathsf{b}}_{\mathsf{BP}}(w))^2] &= \frac{1}{r^2}\mathbb{E}\left[\left(\sum_{i=1}^{r} X_i\right)^2\right] = \frac{1}{r^2}\sum_{i=1}^{r}\left(\mathbb{E}[X_i^2] + 2\sum_{j=i+1}^{r}(\mathbb{E}[X_i]\mathbb{E}[X_j])\right)\\ &= \frac{1}{r}\frac{1}{n}\sum_{v\in V}\alpha_v^2 + \frac{r-1}{r}(\mathsf{b}(w))^2, \end{aligned} \tag{7}$$

where we used (6).

We now compute $\mathbb{E}[(\tilde{\mathsf{b}}(w))^2]$. Consider the contribution Y_i to $\tilde{\mathsf{b}}(w)$ of the i^{th} sampled path by Algorithm 1, for $1 \le i \le r$. Y_i is a random variable that takes value $\mathbb{1}_{\mathsf{Int}(p)}(w)$ with probability $\pi(p)$, for every shortest path $p \in \mathbb{S}_G$. We have

$$\mathbb{E}[Y_i] = \mathbb{E}\left[Y_i^2\right] = \mathsf{b}(w), \forall 1 \le i \le r \ . \tag{8}$$

By definition,

$$\tilde{\mathsf{b}}(w) = \frac{1}{r}\sum_{i=1}^{r} Y_i \ .$$

The random variables Y_i are independent and identically distributed so

$$\begin{aligned}\mathbb{E}[(\tilde{\mathsf{b}}(w))^2] &= \frac{1}{r^2}\mathbb{E}\left[\left(\sum_{i=1}^{r} Y_i\right)^2\right] = \frac{1}{r^2}\sum_{i=1}^{r}\left(\mathbb{E}[Y_i^2] + \sum_{j=i+1}^{r}\mathbb{E}[Y_i]\mathbb{E}[Y_j]\right)\\ &= \frac{1}{r}\mathsf{b}(w) + \frac{r-1}{r}(\mathsf{b}(w))^2, \end{aligned} \tag{9}$$

where we used (8).

We can now rewrite the left side of (4) using (5), (7), and (9):

$$\begin{aligned}\mathbb{E}\left[(\tilde{\mathsf{b}}(w))^2\right] - \mathbb{E}\left[(\tilde{\mathsf{b}}_{\mathsf{BF}}(w))^2\right] &= \frac{1}{r}\mathsf{b}(w) + \frac{r-1}{r}(\mathsf{b}(w))^2\\ &\quad - \frac{1}{r}\frac{1}{n}\sum_{v\in V}\alpha_v^2 - \frac{r-1}{r}(\mathsf{b}(w))^2\\ &= \frac{1}{r}\frac{1}{n}\sum_{v\in V}\left(\alpha_v - \alpha_v^2\right). \end{aligned}$$

Since $\alpha_v \in [0, 1]$, we have $\alpha_v - \alpha_v^2 \geq 0$ for all v, and our proof is complete. □

The following lemma is an easy consequence of the above.

Lemma 10 *BP has lower expected Mean Squared Error than Algorithm 1:*

$$\mathbb{E}[\mathrm{MSE}_{\mathsf{BP}}] \leq \mathbb{E}[\mathrm{MSE}] \ .$$

Proof We have

$$\begin{aligned}\mathbb{E}[\mathrm{MSE}_{\mathsf{BP}}] &= \mathbb{E}\left[\frac{1}{n}\sum_{w\in V}\left(\tilde{\mathsf{b}}_{\mathsf{BP}}(w) - \mathsf{b}(w)\right)^2\right] = \frac{1}{n}\sum_{w\in V}\mathbb{E}\left[\left(\tilde{\mathsf{b}}_{\mathsf{BP}}(w) - \mathsf{b}(w)\right)^2\right]\\ &= \frac{1}{n}\sum_{v\in V}\mathrm{Var}\left[\tilde{\mathsf{b}}_{\mathsf{BP}}(w)\right],\end{aligned}$$

where we used the linearity of expectation and the fact that the estimator $\tilde{\mathsf{b}}_{\mathsf{BP}}(w)$ is unbiased for $\mathsf{b}(w)$ ($\mathbb{E}[\tilde{\mathsf{b}}_{\mathsf{BP}}(w)] = \mathsf{b}(w)$]). Analogously for Algorithm 1:

$$\mathbb{E}[\mathrm{MSE}] = \frac{1}{n}\sum_{v\in V}\mathrm{Var}[\tilde{\mathsf{b}}(w)] \ .$$

Hence

$$\mathbb{E}[\mathrm{MSE}] - \mathbb{E}\left[\mathrm{MSE}_{\mathsf{BP}}\right] = \frac{1}{n}\sum_{v\in V}\left(\mathrm{Var}\left[\tilde{\mathsf{b}}(w)\right] - \mathrm{Var}\left[\tilde{\mathsf{b}}_{\mathsf{BP}}(w)\right]\right),$$

and from Lemma 9 we have that each addend of the sum is non-negative and so is the above expression, concluding our proof. □

The estimator $\tilde{\mathsf{b}}_{\mathsf{BP}}(w)$ has lower variance and MSE than $\tilde{\mathsf{b}}(w)$, which means that it can give better estimations of the betweenness values in practice using the same number of samples. Before always opting for BP (or for the algorithm by Geisberger et al. (2008), whose estimators have even lower variance than those of BP) with a number of samples equal to the one in (3), one should nevertheless take into account two facts. Firstly, we do not currently have a proof that for a number of samples as in (3), algorithm BP (or the algorithm by Geisberger et al. (2008)) computes an high-quality (within ±ε) approximation of the betweenness of all vertices with probability at least $1 - \delta$. We conjecture this fact could be proven using pseudodimension (Anthony and Bartlett 1999, Chap. 11). Secondly we already argued that per sample, the computation of $\tilde{\mathsf{b}}_{\mathsf{BP}}(w)$ requires more time than the one for $\tilde{\mathsf{b}}(w)$. The difference could be even larger when Algorithm 1 uses bidirectional search (Kaindl and Kainz 1997; Pohl 1969).

We can conclude this discussion stating that Algorithm 1, BP, and the algorithm by Geisberger et al. (2008) share the same design principles, but choose different trade-offs between accuracy and speed.

6 Variants of betweenness centrality

It is possible to extend our results to a number of variants of betweenness centrality. For some of them, like the parametric variant for weighted networks defined by Opsahl et al. (2010), the extension is straightforward and immediate. For others, some caution is necessary. We report here some of the extensions that we consider more interesting, to give an intuition of how to extend our results to other variants.

6.1 *k*-bounded-distance betweenness

A "local" variant of betweenness, called *k-bounded-distance betweenness*[5] considers the contribution only of the shortest paths that have size up to $k+1$ (Borgatti and Everett 2006; Brandes 2008). For $k > 1$ and any pair of distinct vertices $u, v \in V$, $u \neq V$, let $\mathcal{S}_{uv}^{(k)} \subseteq \mathcal{S}_{uv}$ be the set of shortest paths from u to v that have size at most $k+1$:

$$\mathcal{S}_{uv}^{(k)} = \{p \in \mathcal{S}_{uv} \ : \ |p| \leq k+1 \text{ and } p \text{ goes from } u \text{ to } v\} \ .$$

Let $\sigma_{uv}^{(k)} = |\mathcal{S}_{uv}^{(k)}|$, and let $\mathbb{S}_G^{(k)}$ be the union of all the $\mathcal{S}_{uv}^{(k)}$. Let $\mathcal{T}_v^{(k)} \subseteq \mathcal{T}_v$ be the set of all shortest paths *that have size up to k* and that v is internal to, for each $v \in V$:

$$\mathcal{T}_v^{(k)} = \{p \in \mathcal{T}_v \ : \ |p| \leq k \text{ and } v \in \mathsf{Int}(p)\} \ .$$

Definition 6 (Borgatti and Everett 2006; Brandes 2008) Given a graph $G = (V, E)$ and an integer $k > 1$, the *k-bounded-distance betweenness centrality of a vertex* $v \in V$ is defined as

$$\mathsf{bb}^{(k)}(v) = \frac{1}{n(n-1)} \sum_{p_{uw} \in \mathbb{S}_G^{(k)}} \frac{\mathbb{1}_{\mathcal{T}_v^{(k)}}(p)}{\sigma_{uw}^{(k)}} \ .$$

For the case of k-bounded-distance betweenness, if we let $\mathcal{R}_G^{(k)} = \{\mathcal{T}_v^{(k)} \ : \ v \in V\}$, it is easy to bound $\mathsf{VC}(\mathcal{R}_G^{(k)})$ following the same reasoning as in Lemma 1.

Lemma 11 $\mathsf{VC}(\mathcal{R}_G^{(k)}) \leq \lfloor \log_2(k-1) \rfloor + 1$.

Given this result, the sample size on line 6 of Algorithm 1 can be reduced to

$$r = \frac{c}{\varepsilon^2}\left(\lfloor \log_2(k-1) \rfloor + 1 + \ln \frac{1}{\delta}\right)$$

and the computation of the shortest paths on line 8 can be stopped after having reached the vertices that are k "hops" far from u.

[5] Bounded-distance betweenness is also known as k-betweenness. We prefer the former denomination to avoid confusion with k-path betweenness as defined by Kourtellis et al. (2012).

6.2 *k*-path betweenness

Another interesting variant of betweenness centrality does not consider *shortest* paths, but rather *simple random walks* of size up to $k + 1$, expressing the intuition that information in a network does not necessarily spread across shortest paths but has a high probability of "fading out" after having touched at most a fixed number of vertices.

Definition 7 (Kourtellis et al. 2012) Given a graph $G = (V, E)$ and a positive integer k, the *k-path centrality* $\mathsf{pb}^{(k)}(v)$ of $v \in V$ is defined as the average,[6] over all possible source vertices s, of the probability that a simple random walk originating from s and stopping after having touched (at most) $k + 1$ vertices (s included) goes through v.

We can define another range set $\mathcal{R}_G^{\mathrm{p},k}$ if we are interested in k-path betweenness. The domain B of the range set now is the set of all simple random walks starting from any vertex of and of size up to $k+1$. For each vertex $v \in V$, the range R_v is the subset of B containing only the random walks from B that have v as internal vertex. It is easy to see that $\mathsf{VC}(\mathcal{R}_G^{\mathrm{p},k})$ is at most $\lfloor \log_2(k-1) \rfloor + 1$, following the same reasoning as in Lemmas 1 and 11.

For the case of k-path betweenness, the algorithm is slightly different: for

$$r = \frac{c}{\varepsilon^2}\left(\lfloor \log_2(k-1) \rfloor + 1 + \ln \frac{1}{\delta}\right)$$

iterations, rather than sampling a pair of vertices (uv), computing $\mathcal{S}_{uv}$, and then sampling a shortest path between them, we first sample a single vertex and an integer ℓ uniformly at random from $[1, k + 1]$. We then sample a simple random walk touching ℓ vertices, while updating the estimated betweenness of each touched vertex by adding $1/r$ to its estimation. The method to sample a simple random walk is described by Kourtellis et al. (2012).

Kourtellis et al. (2012) show that this algorithm can estimate, with probability at least $1 - 1/n^2$, all the k-path betweenness values to within an additive error $n^{-1/2+\alpha}/(n-1)$ using $2n^{1-2\alpha}k^2 \ln n$ samples, for $\alpha \in [-1/2, 1/2]$. We can obtain the same guarantees with a *much lower* number r of samples, specifically

$$r = 2n^{1-2\alpha}\left(\ln n + \frac{\lfloor \log_2(k-1) \rfloor + 1}{2}\right) .$$

In summary we presented a tighter analysis of the algorithm by Kourtellis et al. (2012).

6.3 Edge betweenness

Until now we focused on computing the betweenness centrality of vertices. It is also possible to define a betweenness centrality index for *edges* of a graph $G =$

[6] We take the average, rather than the sum, as defined in the original work by Kourtellis et al. (2012), to normalize the value so that it belongs to the interval [0, 1]. This has no consequences on the results.

Table 2 Graph characteristics and running time ratios

Graph	Graph properties			$\frac{\mathsf{Time}_{\mathsf{BP}}}{\mathsf{Time}_{\mathsf{VC}}}$	
				diam-2approx	
	$\lvert V\rvert$	$\lvert E\rvert$	$\mathsf{VD}(G)$	Min	Max
Undirected graphs					
oregon1-010331	10,670	22,002	9	4.39	4.75
oregon1-010526	11,174	23,409	10	4.26	4.73
ca-HepPh	12,008	237,010	13	3.06	3.33
ca-AstroPh	18,772	396,160	14	3.26	3.76
ca-CondMat	23,133	186,936	15	3.75	4.08
email-Enron	36,692	421,578	12	3.60	4.16

Graph	Graph properties			$\frac{\mathsf{Time}_{\mathsf{BP}}}{\mathsf{Time}_{\mathsf{VC}}}$			
				diam-exact		diam-UB	
	$\lvert V\rvert$	$\lvert E\rvert$	$\mathsf{VD}(G)$	Min	Max	Min	Max
Directed graphs							
wiki-Vote	7,115	103,689	7	3.35	3.69	1.05	1.27
p2p-Gnutella25	22,687	54,705	11	5.45	5.78	1.94	2.09
cit-HepTh	27,770	352,807	14	3.58	3.83	1.39	1.61
cit-HepPh	34,546	421,578	12	4.91	5.01	1.60	1.71
p2p-Gnutella30	36,682	88,328	10	5.02	5.46	2.08	2.22
soc-Epinions1	75,879	508,837	13	4.20	4.25	1.35	1.38

(V, E) (Anthonisse 1971; Brandes 2008). Edge betweenness is useful, for example, to develop heuristics for community detection (Newman and Girvan 2004). Given $e \in E$, the edge betweenness $\mathsf{eb}(e)$ of e is defined as the fraction of shortest paths that *contain* e, meaning that if $e = (u, v)$, then a path p contains e if u and v appear consecutively in p. Formally,

$$\mathsf{eb}(e) = \frac{1}{n(n-1)} \sum_{p_{uv} \in \mathbb{S}_G} \frac{\mathbb{1}_{p_{uv}}(e)}{\sigma_{uv}}, \forall e \in E\ .$$

We can then define a range space $\mathcal{ER}_G$ that contains $|E|$ ranges R_e, one for each $e \in E$, where R_e is the set of shortest paths containing the edge e. By following the same reasoning as Lemma 1 we have that $\mathsf{VC}(\mathcal{ER}_G) \leq \lfloor(\log_2(\mathsf{VD}(G) - 1)\rfloor + 1$. Using this result we can adapt the sample sizes for Algorithms 1 and 2 to compute good approximations of betweenness for the (top-k) edges.

7 Experimental evaluation

We conducted an experimental evaluation of our algorithms, with two major driving goals in mind: study the behavior of the algorithms presented in this paper and compare it with that of other related algorithms (Brandes 2001; Brandes and Pich 2007; Jacob et al. 2005; Geisberger et al. 2008), in terms of accuracy of the estimation, execution time, work performed, and scalability as function of the network size.

Implementation and environment We implemented our algorithms,[7] the one presented by Brandes and Pich (2007) and Jacob et al. (2005) and the linear scaling version by Geisberger et al. (2008) in C, by extending the implementation of the exact algorithm (Brandes 2001) contained in igraph (Csárdi and Nepusz 2006). The implementations are similarly engineered, given that they are based on the same subroutines for the computation of the shortest path (Dijkstra's algorithm for weighted graphs, BFS for unweighted ones), and they received similar amounts of optimization. We exposed our implementations through Python 3.3.1, which was used for running the simulations. We run the experiments on a quad-core AMD Phenom$^{\text{TM}}$ II X4 955 Processor with 16GB of RAM, running Debian *wheezy* with a Linux kernel version 3.2.0.

Datasets In our evaluation we used a number of graphs from the Stanford Large Network Dataset Collection.[8] These are all real world datasets including online social networks, communication (email) networks, scientific citation and academic collaboration networks, road networks, Amazon frequent co-purchased product networks, and more. Basic information about the graphs we used are reported in the two leftmost columns of Table 2. We refer the reader to the SLNDC website for additional details about each dataset. To evaluate scalability we also created a number of artificial graphs of different sizes (1000 to 100,000 vertices) using the Barabási–Albert model (Barabási and Albert 1999) as implemented by igraph (Csárdi and Nepusz 2006).

Diameter approximation As we discussed in the previous sections the number of samples that the proposed algorithm requires depends on the vertex-diameter of the graph. For the computation of the vertex-diameter in case of undirected graphs we used the 2-approximation algorithm that we briefly described in Sect. 5. We denote this as "diam-2-approx" when reporting results in this section. For directed graphs, we computed the number of samples using both the exact value of the vertex-diameter (indicated as diam-exact) as well as the trivial upper bound $|V| - 2$ (indicated as diam-UB).

[7] The implementations are available at http://cs.brown.edu/~matteo/centrsampl.tar.bz2. An independent implementation of our algorithms is available in NetworKit Staudt et al. (2014).

[8] http://snap.stanford.edu/data/index.html

7.1 Accuracy

Our theoretical results from Sect. 5 guarantee that, with probability at least $1-\delta$, all estimations of the betweenness values for all vertices in the graph are within ε for their real value. We run Algorithm 1 five times for each graph and each value of ε in $\{0.01, 0.015, 0.02, 0.04, 0.06, 0.08, 0.1\}$. The parameter δ was fixed to 0.1 and we used $c = 0.5$ in (1) to compute the sample size, as suggested by Löffler and Phillips (2009). As far as the *confidence* is concerned, we report that in all the hundreds of runs we performed, the guarantee on the quality of approximation was *always* satisfied, not just with probability $1-\delta$ $(= 0.9)$. We evaluated how good the estimated values are by computing the average *estimation error* $(\sum_{v\in V} |\mathsf{b}(v) - \tilde{\mathsf{b}}(v)|)/|V|$ across five runs of our algorithm and taking the average and the standard deviation of this measure, for different values of ε. We also compute the maximum error $|\mathsf{b}(v) - \tilde{\mathsf{b}}(v)|$ overall. The results are reported in Fig. 7a for the directed graph p2p-Gnutella30, in Fig. 7c for the undirected graph ca-HepPh, in Fig. 7b for the undirected graph soc-Epinions1, and in Fig. 7d for the undirected graph email-Enron. It is evident that the maximum error is an error of magnitude smaller than the guaranteed value of ε and that the average error is almost two orders of magnitude smaller than the guarantees, and the `Avg+Stddev` points show that the estimation are quite concentrated around the average. We can conclude that in practice the algorithm performs even *better than guaranteed*, achieving higher accuracy and confidence than what the theoretical analysis indicates. This is due to a number of factors, for example the fact that we use an *upper bound* to the VC-dimension of the range set and that the sample size is therefore an upper bound to the one *necessary* to obtain the desired approximation guarantees.

We also report in the figures the error for the algorithm by Jacob et al. (2005), Brandes and Pich (2007), Geisberger et al. (2008) (denoted as BP). We report this results for the original version of the algorithm, using a sample size that depends on the number of nodes in the graph. As expected, the error is much lower than that for our algorithm, but that is because BP uses an unnecessary large number of samples because its analysis is extremely loose. Moreover, as we explain in the following section, its runtime is also much higher than that of our algorithm.

Accuracy for the top-K algorithm We also evaluated our algorithm for a higher-quality approximation of $\mathsf{TOP}(K, G)$ (Sect. 5.2). In Fig. 8 we report the accuracy results for the graph as function of k for the graph ca-HepTh and fixed values of $\varepsilon = 0.05$ and $\delta = 0.1$ (the behavior for other values of ε is similar to that we already discussed for the absolute approximation algorithm). We measured the *relative* confidence error, defined as $|\tilde{\mathsf{b}}(v) - \mathsf{b}(v)|/\mathsf{b}(v)$. While the average of this quantity is almost an order of magnitude smaller than ε, the maximum is quite close to the maximum allowed error 0.05, meaning that our sample size is not excessively large, but rather quite close to the smallest sufficient size. Nevertheless, in all our runs, the output always respected the requirements of Lemma 8, not just with probability $1-\delta$, for reasons similar to those mentioned for the absolute case. The size of the output set was always at most one error of magnitude greater than k. Notice that we can not give guarantees neither on the size of the output, nor on the ranking of the vertices, as these properties depend on

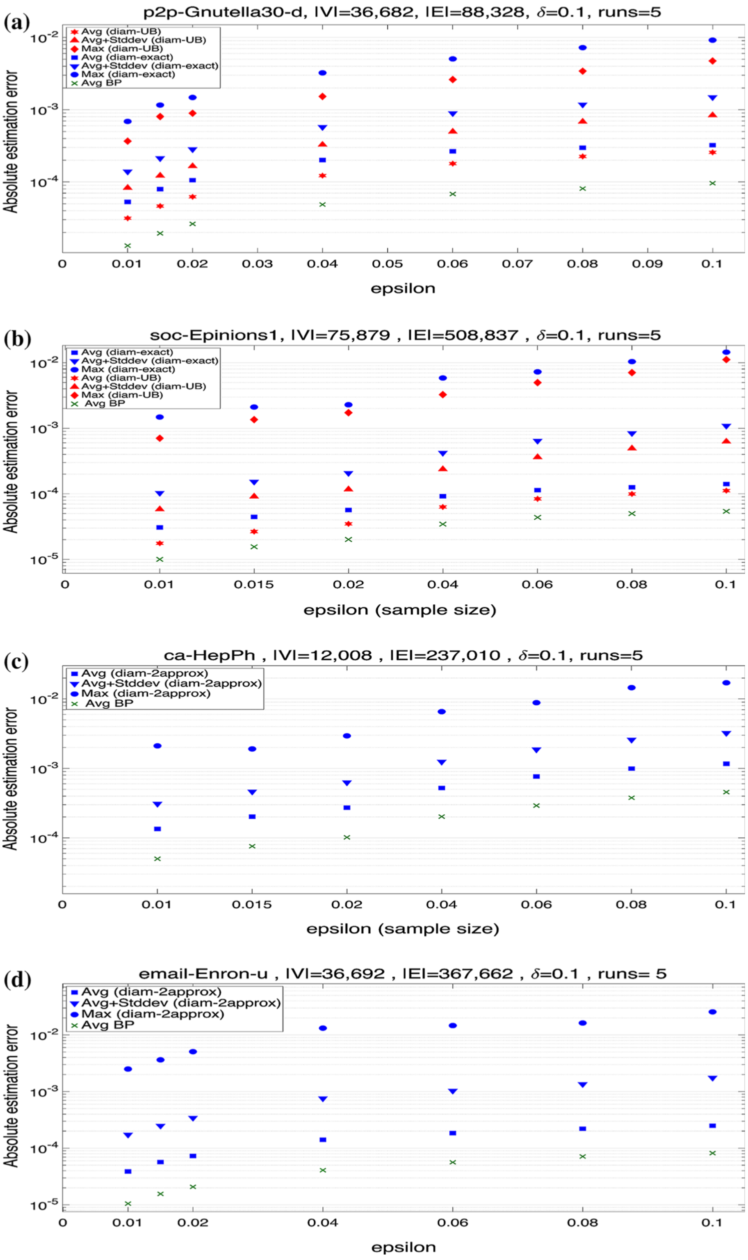


Fig. 7 Betweenness estimation absolute error $|\tilde{\mathsf{b}}(v) - \mathsf{b}(v)|$ evaluation for directed and undirected graphs as function of ε. The reported quantities are: average error, maximum error, and the sum between average and standard deviation. For directed graphs, two versions of these quantities are reported, one for runs of the algorithm using the exact vertex diameter, and one for runs using an upper bound to it. For undirected graph we report the quantities for runs that used the 2-approximation to the vertex diameter. For the BP algorithm, we only report the average. **a** p2p-Gnutella30 (directed). **b** soc-Epinions1 (directed). **c** ca-HepPh (undirected). **d** email-Enron (undirected)

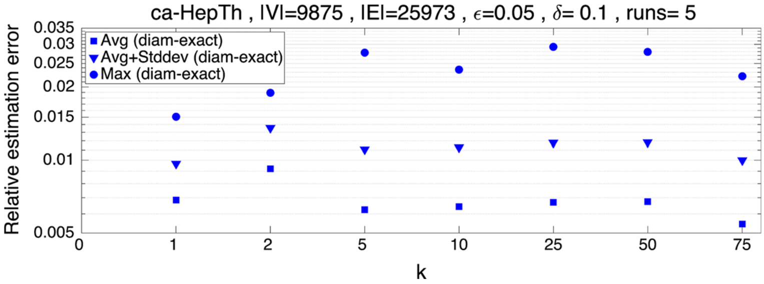


Fig. 8 Betweenness estimation relative error $|\tilde{\mathsf{b}}(v) - \mathsf{b}(v)|/\mathsf{b}(v)$ evaluation for the top-k algorithm as function of k. The reported quantities are: average error, maximum error, and the sum between average and standard deviation. We report the quantities for runs that used the exact vertex diameter

the distribution of the betweenness values: if many vertices have betweenness value close to each other and/or close to the value of the top-kth, then their relative order may be inverted and/or they may be included in the output set. Indeed it is for this reason that the behavior of the error showed in Fig. 8 does not behave monotonically, as the aggregates are computed on different sets for different values of k.

7.2 Runtime

We compared the running time of Algorithm 1 (denoted in the following as VC to that of BP, and to that of the exact algorithm by Brandes (2001). As VC and BP give the same guarantees on the accuracy and confidence of the computed estimations, it makes sense to compare their running times to evaluate which is faster in achieving the goal. The algorithm proposed by Geisberger et al. (2008) takes the same time as BP, because it follows the same sampling approach and only differs in the definiton of the estimator for the betweenness, so we do not report those. The algorithms VC and BP take parameters ε and δ and compute the sample size accordingly. We run each experiments five times for each value of ε, and measured the average running time across the runs. The results are presented in Table 2 and Fig. 9. In Table 2 we report the minimum and the maximum ratio of the running time of BP over VC, taken over the ratios obtained by running the algorithms with the different values of ε. As it can be seen from this table our algorithm performs significantly faster, more than 300%. Similar results are reported for directed graphs in Table 2. The diam-UB and the diam-exact values can be seen as the two extremes for the performance of Algorithm 1 in terms of runtime. In the case of the diam-exact we have as few samples as possible (for VC) since we use the exact value of the vertex-diameter, whereas in the case of diam-UB we have as many samples as possibles because we use the worst case estimation for the vertex-diameter of the graph. From Table 2 we can see that the value for the vertex-diameter that we consider in the case of diam-UB ($|V| - 2$) is many orders of magnitudes greater than the actual value, which translates in a significant increase of the number of samples. But even in the case of this crude vertex-diameter approximation (diam-UB), the VC algorithm performs uniformly faster than BP. In the case where the exact value of the diameter was used, we can see that our algorithm computes an estimation of the betweenness that satisfies the desired accuracy and

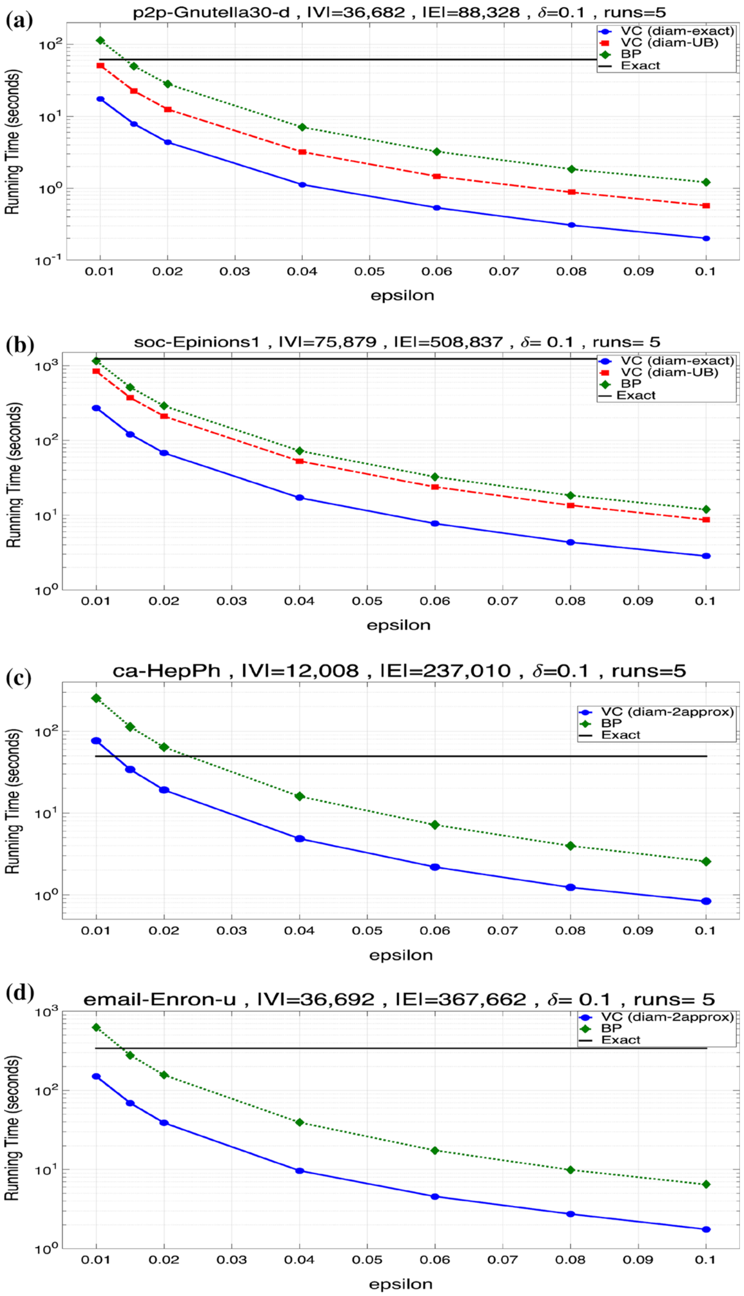


Fig. 9 Comparison of the running time (in s) between VC, BP, and the exact algorithm, as functino of ε. For directed graphs, the running time of VC is reported twice: one for runs using the exact vertex diameter, and one for runs using an upper bound to this quantity. For undirected graphs we report the running time of VC using the 2-approximation to the diameter. **a** p2p-Gnutella30 (directed). **b** soc-Epinions1 (directed). **c** ca-HepPh (undirected). **d** email-Enron (undirected)

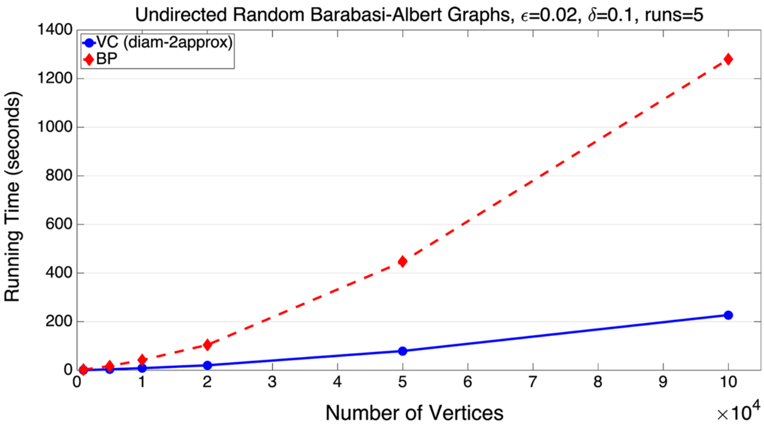


Fig. 10 Comparison of scalability (running time in seconds) between VC and BP on random undirected Barabási–Albert 1999 graphs, as function of the number of vertices in the graph. We report the running time for runs of VC using the 2-approximation to the vertex diameter

confidence guarantees *3 to 5 times faster* than BP. In Fig. 9a we study the directed graph p2p-Gnutella30 and we present the measurements of the average running time of the algorithms for different values of ε, using the exact algorithm by Brandes (2001) as baseline. The VC algorithm requires significantly less time than the BP algorithm. The figure also shows that there are values of ε for which BP takes more time than the exact algorithm, because the resulting sample size is larger than the graph size. Given that VC uses fewer samples and does fewer operations per sample, it can be used with lower ε than BP, while still saving time compared to the exact computation. Figure 9d shows the average running time of the algorithms for the undirected graph email-Enron. The behavior is similar to that for the undirected case. Algorithm 1 is faster than BP for two reasons, both originating from from our use of results from the VC-dimension theory: (1) we use a significantly smaller amount of samples anda (2) VC performs the same amount of computations *per sample* as BP only in the worst case. Indeed our algorithm needs only to find the shortest path between a sampled pair of vertices, whereas the algorithms from Geisberger et al. (2008), Brandes and Pich (2007) need to compute the shortest paths between a sampled source and all the other vertices. Since the running time of the algorithms is directly proportional to the number of edges touched during the shortest path computation, the use of bidirectional A* search (Pohl 1969; Kaindl and Kainz 1997) can help in lowering the number of touched edges for VC and therefore the runtime of our algorithm (BP would not benefit from this improvement).

7.3 Scalability

In Sect. 5.3 we argued about the reasons why Algorithm 1 is more scalable than BP, while still offering the same approximation guarantees. To evaluate our argument in practice, we created a number of graphs of increasing size (1000 to 100,000 vertices) using the Barabási–Albert (1999) and run the algorithms on them, measuring their running time. We report the results in Fig. 10. The most-scalable algorithm would be completely independent from the size (number of vertices) of the graph, corresponding to a flat (horizontal) line in the plot. Therefore, the less steep the line, the more independent from the network size would be the corresponding algorithm. From the figure, we can confirm that this is the case for VC, which is much more scalable and independent from the size of the sample than BP. This is very important, as today's networks are not only huge, but they also grow rapidly, and algorithms to mine them must scale well with graph size.

8 Conclusions

In this work we presented two random-sampling-based algorithms for accurately and efficiently estimate the betweenness centrality of the (top-K) vertices in a graph, with high probability. Our algorithms are based on a novel application of VC-dimension theory, and therefore take a different approach than previous ones achieving the same guarantees (Brandes and Pich 2007; Geisberger et al. 2008; Jacob et al. 2005). The number of samples needed to approximate the betweenness with the desired accuracy and confidence does not depend on the number of vertices in the graph, but rather on a characteristic quantity of the network that we call *vertex-diameter*. In some cases, the sample size is completely independent from any property of the graph. Our methods can be applied to many variants of betweenness, including edge betweenness. Our algorithms perform much less work than previously presented methods. As a consequence, they are much faster and scalable, as verified in the extensive experimental evaluation using many real and artificial graphs.

A number of recent works focus on algorithms for computing and keeping track of betweenness centrality on evolving graphs (Kourtellis et al. 2014; Yoshida et al. 2014; Bergamini and Meyerhenke 2015; Bergamini et al. 2015). This is an important question as networks evolve in time. Our algorithm has been used as the fundamental building block of one of these algorithms (Bergamini and Meyerhenke 2015; Bergamini et al. 2015), suggesting that our work is applicable beyond its original setting.

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