Vector Spaces

- Common mathematical theme:

  {properties something has} $\rightsquigarrow$ {general definition}

- "Vector spaces" are precisely the general notions of vectors, linear maps, ... that we've seen so far

- EVERYWHERE in maths and science.

The abstract definitions

- Fix a field F (eg, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}_p$, ...)

  We call elements $c \in F$ scalars

- A vector space V over a field F is

  ① An abelian group with respect to $+$, with

  ② A "compatible" scalar multiplication

That is,

① $\underset{\sim}{v}, \underset{\sim}{w} \in V \Rightarrow \underset{\sim}{v} + \underset{\sim}{w} \in V.$

② $\underset{\sim}{v} \in V, c \in F \Rightarrow c\underset{\sim}{v} \in V.$

- These must satisfy the following rules:

(a) Associativity of scalar multiplication:

$$c, d \in F \quad \underset{\sim}{v} \in V \Rightarrow c(d\underset{\sim}{v}) = (\underbrace{cd}_{\text{mult. in } F})\underbrace{\underset{\sim}{v}}_{\text{scalar-vector product}}$$

(b) Distributivity Properties

$$c, d \in F \quad \underset{\sim}{v}, \underset{\sim}{w} \in V \Rightarrow c(\underset{\sim}{v} + \underset{\sim}{w}) = c\underset{\sim}{v} + c\underset{\sim}{w}$$

$$(c+d)\underset{\sim}{v} = c\underset{\sim}{v} + d\underset{\sim}{v}$$

(c) Scalar identity

$$1 \in F \Rightarrow 1\underset{\sim}{v} = \underset{\sim}{v} \quad \forall \underset{\sim}{v} \in V.$$

(d) Abelian group axioms for $(V, +)$

- Commutativity: $\underset{\sim}{v} + \underset{\sim}{w} = \underset{\sim}{w} + \underset{\sim}{v}$
- Identity: $\exists \underset{\sim}{0} \in V: \underset{\sim}{0} + \underset{\sim}{v} = \underset{\sim}{v} = \underset{\sim}{v} + \underset{\sim}{0}$
- Inverses: $\forall \underset{\sim}{v} \ \exists -\underset{\sim}{v}: \underset{\sim}{v} + (-\underset{\sim}{v}) = \underset{\sim}{0}.$

↳ the "negative" of $\underset{\sim}{v} := -\underset{\sim}{v}$

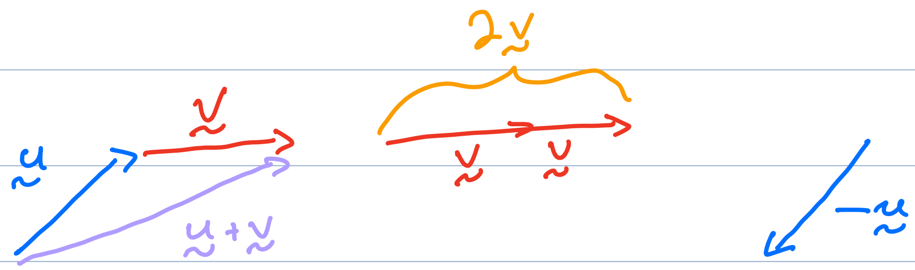
↳ Allows us to define subtraction by

$$\underset{\sim}{v} - \underset{\sim}{w} := \underset{\sim}{v} + (-\underset{\sim}{w}).$$

Examples

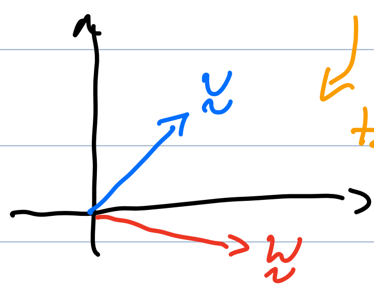
- Directed segments in plane or space, $F = \mathbb{R}$
  ("physics vectors")

"tip-to-tail" addition.



- $V = F^n = \{ (a_1, \dots, a_n) \mid a_i \in F \}$
  - These, or the column versions $\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$, are what we've worked with all along and called "vectors"
  - Component-wise addition and scaling
  - Exercise: check the axioms!

"position vectors" to get back to Ex 1



- More interesting examples ??...

- $V = \mathrm{Mat}_{n\times m}(F) := \{ n\times m \text{ matrices with entries in } F \}$

  - Operations: $+ :=$ entry-wise addition

    scalar mult: $:=$ entry-wise scaling.

$\underset{\sim}{v} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix} \in \mathrm{Mat}_{3\times 2}(F), \quad a_i, b_i \in F$

$c\underset{\sim}{v} = \begin{bmatrix} ca_1 & cb_1 \\ ca_2 & cb_2 \\ ca_3 & cb_3 \end{bmatrix}$

- Note: $\mathrm{Mat}_{1\times n}(F) = \{ (a_1, \dots, a_n) \mid a_i \in F \}$ ← row vectors from before

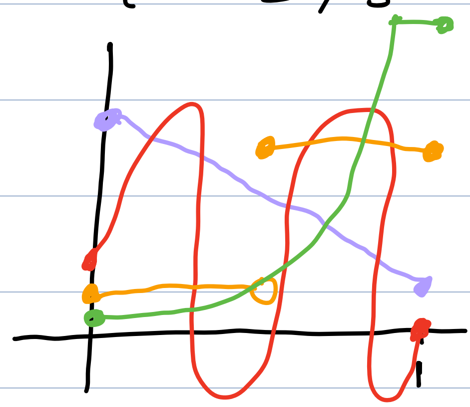
  $\mathrm{Mat}_{n\times 1}(F) = \left\{ \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \,\middle|\, a_i \in F \right\}$ ← column vectors from before.

- Let F be a field, X any set, and define

$$F^X := \{ \text{functions } f : X \to F \}$$

- Note: <u>any</u> function!

Eg: $\mathbb{R}^{[0,1]} = \{ f : [0,1] \to \mathbb{R} \}$



Each function is a <u>vector</u> in $\mathbb{R}^{[0,1]}$.

- F^X is a vector space with pointwise operations:
  - $f \in F^X$, $\lambda \in F$, define $\lambda f \in F^X$ by $(\lambda f)(x) := \lambda f(x)$

    scalar mult. definition

    multiplication in F
  - $f, g \in F^X$, define $f+g \in F^X$ by $(f+g)(x) := f(x) + g(x)$

    addition in F
  - Zero vector: Zero function: $0(x) = 0$

    ie, $0(x) = 0$ for all x : "identically zero"

- Exercise: How can we associate vectors $\underset{\sim}{v} \in F^n$ with functions $f \in F^{\{1,2,\dots,n\}}$?

- Example: $\mathbb{R}^{\mathbb{R}} = \{$ functions $\mathbb{R} \to \mathbb{R}\}$ includes important subsets like

  idea of subspaces

  $\mathcal{C}(\mathbb{R}) := \{$continuous functions $\mathbb{R} \to \mathbb{R}\}$

  $\mathcal{C}^1(\mathbb{R}) := \{$differentiable functions $\mathbb{R} \to \mathbb{R}\}$

  $\mathcal{C}^2(\mathbb{R}) := \{$twice diff'l functions$\}$

  $\mathcal{C}^\infty(\mathbb{R}) := \{$infinitely diff'l functions $\mathbb{R} \to \mathbb{R}\}$

  $e^x \in$, $\sin x \in$ $= \{$ "smooth" functions$\}$

• Polynomial spaces

Fix $k \in \mathbb{N} = \{0,1,2,\dots\}$

Let

$$\mathbb{P}_k = \{ a_0 + a_1 x + a_2 x^2 + \dots + a_k x^k \mid a_i \in F \}$$

– polynomials with degree $\leq k$ over F

– pointwise operations: $(ax_1 + bx^3) + (c + dx^2)$
$= c + ax + dx^2 + bx^3$

: $\lambda(ax^2 + bx) = \lambda a x^2 + \lambda b x$.

• Ex: $\mathbb{P}_0 = \{a_0\} = F$

$\mathbb{P}_1 = \{a_0 + a_1 x\}$

$\mathbb{P}_2 = \{a_0 + a_1 x + a_2 x^2\}$

$\mathbb{P}_3 = \{a_0 + a_1 x + a_2 x^2 + a_3 x^3\}$

• Let $\mathbb{P} := \bigcup_{k=0}^{\infty} \mathbb{P}_k$ $\leftarrow$ no restriction on degree... but must be a finite sum

$\hookrightarrow p(x) \in \mathbb{P} \Rightarrow \exists k : p(x) \in \mathbb{P}_k$.

– all polynomials over F

Then we have inclusions

$$\mathbb{P}_0 \subsetneq \mathbb{P}_1 \subsetneq \mathbb{P}_2 \subsetneq \cdots \subsetneq \mathbb{P}_k \subsetneq \cdots \subseteq \mathbb{P}$$

- Example: Formal power series over F.

$$F[\![x]\!] = \{ a_0 + a_1 x_1 + a_2 x^2 + \cdots + a_k x^k + \cdots \}$$

"formal" power series

$a_i \in F$.

- Again, pointwise operations

$$\mathbb{P} \subsetneq F[\![x]\!]$$

finite power series: $a_i = 0 \ \forall i > k$ for some k.