MATH2022

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1 Week1

1.1 Arithmetics

- Addition
  Operations Used: $+, \times$
  Limits: $-, /$

- Integers
  Operations Used: $+, \times, -$
  Limits: $/$

- The Rational Numbers
  $\mathbb{Q} = \{\frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0\}$
  Operations Used: $+, -, \times, /$
  Limits:

- The Real Numbers
  Operations Used: $+, -, \times, /$
  Limits: $i = \sqrt{-1}$

- The Complex Number
  $\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R} where i = \sqrt{-1}\}$ Operations Used: $+, -, \times, /$
  Limits:

- Modular Arithmetic
  Let $n \in \mathbb{Z}^*$ and let Z_n be the set of remainders after dividing by n.
  So $Z_n = \{0, 1, 2, 3 ... n - 1\}$

1.2 Fields

A **field** $(F, +, \cdot)$ is a set F equipped with two operations: addition $(+)$ and multiplication $(\cdot)$, satisfying the following axioms:

1. *Closure under Addition and Multiplication*

$$\forall a, b \in F, \quad a + b \in F$$
$$\forall a, b \in F, \quad a \cdot b \in F$$

2. *Associativity of Addition and Multiplication*

$$\forall a, b, c \in F, \quad (a + b) + c = a + (b + c)$$
$$\forall a, b, c \in F, \quad (a \cdot b) \cdot c = a \cdot (b \cdot c)$$

3. *Commutativity of Addition and Multiplication*

$$\forall a, b \in F, \quad a + b = b + a$$
$$\forall a, b \in F, \quad a \cdot b = b \cdot a$$

4. *Identity Elements*

$$\exists 0 \in F \text{ such that } \forall a \in F, \quad a + 0 = a$$
$$\exists 1 \in F \text{ with } 1 \neq 0, \text{ such that } \forall a \in F, \quad a \cdot 1 = a$$

5. *Additive and Multiplicative Inverses*

$$\forall a \in F, \quad \exists - a \in F \text{ such that } a + (-a) = 0$$
$$\forall a \in F \text{ with } a \neq 0, \quad \exists a^{-1} \in F \text{ such that } a \cdot a^{-1} = 1$$

6. *Distributivity of Multiplication over Addition*

$$\forall a, b, c \in F, \quad a \cdot (b + c) = (a \cdot b) + (a \cdot c)$$

1.3 Group Definition

A **group** $(G, *)$ is a set G together with a binary operation $*$ that combines any two elements a and b to form another element $a * b$. The binary operation satisfies the following four properties:

1. *Closure*: For every $a, b \in G$, the result of the operation $a * b$ is also in G.

$$\forall a, b \in G, \quad a * b \in G$$

2. *Associativity*: For every $a, b, and c \in G$, the equation $(a * b) * c = a * (b * c)$ holds.

$$\forall a, b, c \in G, \quad (a * b) * c = a * (b * c)$$

3. *Identity Element*: There exists an element $e \in G$, called the identity element, such that for every element $a \in G$, the equation $e * a = a * e = a$ holds.

$$\exists e \in G \text{ such that } \forall a \in G, \quad e * a = a * e = a$$

4. *Inverse Element*: For each $a \in G$, there exists an element $b \in G$ such that $a * b = b * a = e$, where e is the identity element.

$$\forall a \in G, \quad \exists b \in G \text{ such that } a * b = b * a = e$$

A group is called **abelian** (or **commutative**) if, in addition, the binary operation is commutative, that is, $a * b = b * a$ for all $a, b \in G$.

Notes:

1. As in the case of fields, the identity element and the inverse can be shown to be unique.

2. Our notation might imply that this operation is multiplication, but it could just as easily be addition or another operation.

1.4 Cyclic Groups

A **cyclic group** G is a special type of group that can be entirely generated by a single element $g \in G$. This element g is called a generator of the group. The main characteristic that distinguishes cyclic groups from general groups is the ability to generate all elements of the group by repeatedly applying the group operation to the generator.

1. *Generator*

$$\begin{aligned} \exists g \in G \,\text{such that}\, G &= \{g^n | n \in \mathbb{Z}\} \,\text{(for multiplicative groups)} \\ \text{or} \quad G &= \{ng | n \in \mathbb{Z}\} \,\text{(for additive groups)} \end{aligned}$$

2. *Uniqueness*

Every element of G can be uniquely expressed as g^n for some $n \in \mathbb{Z}$.

The cyclic nature of G implies that it possesses a structure that can be systematically described by the powers (or multiples) of a single element, making cyclic groups particularly simple to understand and work with.

1.5 Symmetric Groups

A **symmetric group** S_n on a set of n symbols is the group consisting of all possible permutations of these symbols, with group operation being the composition of these permutations. The symmetric group on n symbols is denoted as S_n and plays a crucial role in various areas of mathematics due to its fundamental nature in the study of permutations.

1. **Recap**
   definitions: A bijection is a mapping that is injective (one to one) and surjective (onto).

   Important Convention
   We write the action of $f : X \to Y$ on the right:
   $$x \mapsto xf \ (\forall x \in X)$$
   and we compose from left to right:
   $$fg : X \to Z \text{ (where } f : X \to Y,\ g : Y \to Z)$$
   Instead of using $f \circ g(x) = f(g(x))$, we use $x(fg) := (xf)g$
   We apply f on x first, then g

2. Proof that: *The composite of two bijection is also a bijection*
   Let $f : X \to Y, g : Y \to Z$

   - Injective: Need to show that if $x(fg) = y(fg)$ then $x = y$
     Suppose $x(fg) = y(fg)$.Then $(xf)g = (yf)g$ by definition,So $xf = yf$ since g is injective,samething agagin, x = y.

   - Surjective: Need to show that the range of $x(fg)$ equals to Z

3. **Permutations**

   - For a finite set of size n, there are n! permutations (write $|X| = n$), The set of permutations of X is denoted $Sym(X)$ or S_x,So,$|Sym(X)| = n!$

4. THEOREM:
   The set of permutations on a set X, Sym(X), is a group under composition of permutations. We call it the Symmetric Group on X:

   - **closure** The composite of two bijections is also a bijection.
   - **associative** Composition of bijections is associative.
   - **Identity element** The identity map is a bijection.
   - **Inverse element** The inverse of a bijection is a bijection

2 Week2

3 Week 3

Matrix Recap

- **Elementary Matrices and Invertibility**
- **Determinants, Properties**

Odd and Even Permutations

- **Transposition**
  Definition: A transposition is a permutation $\phi : \mathbb{X} \to \mathbb{X}$, which interchanges two distinct elements $a, b \in \mathbb{X}$ leaving all other elements unchanged. Thus,

  $$\phi = (ab).$$

  Fact: All cycles are a product (composition) of transpositions.

  $$(a_1, a_2, a_3, \dots, a_n) = (a_1a_2)(a_1a_3)\dots(a_1a_n).$$

  Corollary: Every permutation of a finite set is a product (composition) of transpositions.

- **Even/Odd**
  Definition: We call a permutation even (or odd) if it is a product of an even (or odd, respectively) number of transpositions.

  - $(123) = (12)(13)$ is even.
  - $(1234) = (12)(13)(14)$ is odd.
  - $(1) = (12)(12) = (13)(13)$ is even.

  Properties:

  - Single transpositions are self-inverse: $(ab)(ab) = 1$.
  - A permutation and its inverse have the same parity.

- **Permutation Matrix**
  Definition: A permutation matrix is the result of applying a permutation to the rows of the identity matrix.

  $$\phi = (132) = (13)(12) = R_1 \leftrightarrow R_3, R_1 \leftrightarrow R_2.$$

  Hence, $\mathrm{Det}(M) = \mathrm{Det}(E_1E_2) = \mathrm{Det}(E_1)\mathrm{Det}(E_2) = (-1)^2 = 1$.

The Alternating Subgroup, Alt(n), of Sym(n)

- **Subgroup**
  Definition: A subgroup, H, of a group, G, is a subset of G which is also a group under the same operation. We write $H \leqslant G$ or $H < G$ if H is a proper subset of G.

  Proof a subgroup:

  - It is non-empty
  - It is closed under the operation.
    Associativity: Inherited from G
    Identity: $\exists a \in G, a^k = e$
    Inverse: $a^k = e \to aa^{k-1} = e = a^{k-1}a$

- **Alternating Group**
  Definition: The alternating group (on n letters) is the set of even permutations in $Sym(n)$. That is, $Alt(n) = \{\text{even permutations of } \{1, 2, \ldots, n\}\}$.

Examples:

Sym(2) $= \{1, (12)\}$ $\quad A_2 = \{1\}$

Sym(3) $= \{1, (12), (13), (23), (123), (132)\}$

$A_n = \{1, (123), (132)\}$

Sym(4) $= \{1, (12), (13), (14), (23), (24), (34), (123), (132), (124), (142), (134), (143), (234), (243), (1234), (1243), (1324), (1342), (1423), (1432), (12)(34), (13)(24), (23)(14)\}$

4 Week4

1. Elementary matrices: An $n \times n$ matrix is called elementary if it is the result of applying a single elementary row operation to the identity matrix I_n.

2. Effect of multiplication by an elementary matrix: If E is the elementary matrix obtained by applying the elementary row operation ρ to I_n, and A is any matrix with n rows, then the matrix product EA is the matrix obtained by applying ρ to A.

3. Invertibility criterion and algorithm: A square matrix A is invertible if and only if A is a product of elementary matrices, which occurs if and only if the augmented matrix $[A \mid I]$ can be row reduced to $[I \mid B]$, in which case $A^{-1} = B$.

4. Half of the definition of invertibility suffices for square matrices: If A is a square matrix and $AB = I$ or $BA = I$ then $AB = BA = I$, in which case the inverse A^{-1} exists and equals B.

5. Determinants of matrices of dimensions 1,2 and 3: The determinant of a 1×1 matrix $[a]$ is simply the entry a. The determinant of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$. The determinant of a 3×3 matrix $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}$ is

$$\det A = |A| = a \begin{vmatrix} e & f \\ h & k \end{vmatrix} - b \begin{vmatrix} d & f \\ g & k \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

   called the expansion along the first row, where the smaller determinant arises by ignoring the row and column of the entry being used as a coefficient.

6. Determinants in general: Following the pattern for 3×3 matrices, we may expand along any row or down any column of a given square matrix A of any size, producing the same number, called the determinant of A, denoted by $\det A$ or $|A|$, provided one uses adjustment factors given by the chequerboard patterns

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}, \quad \begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

   and so on to higher dimensions.

7. Multiplicative property of determinants: If A and B are square matrices of the same size then $\det(AB) = (\det A)(\det B)$.

8. Invertibility criterion using determinants: A square matrix is invertible if and only if its determinant is nonzero.

9. Effects of elementary row and columns operations on determinants: Let A be a square matrix. If B is obtained from A by swapping two rows or swapping two columns then

$$\det B = -\det A.$$

If B is obtained from A by multiplying a row or column by a scalar λ then

$$\det B = \lambda \det A.$$

If B is obtained from A by adding a multiple of one row [column] to another row [column] then

$$\det B = \det A.$$

10. Determinant of the transpose: If A is a square matrix then $\det\left(A^T\right) = \det A$.

11. Transpositions: A permutation that interchanges two letters and fixes all other letters is called a transposition.

12. Even and odd permutations: A permutation of a finite set is called even if it is a product of an even number of transpositions (and by default the identity permutation is even), and called odd if it is a product of an odd number of transpositions.

13. No permutation can be both even and odd: If n is a positive integer then the symmetric group S_n is the disjoint union of A_n, the subset of even permutations (and called the alternating group), and $S_n \backslash A_n$, the complement of A_n, which comprises exactly the subset of all odd permutations.

14. Cosest:For a Symmetric Group $S_n, n \geqslant 2$,

- **Symmetric Group and Cosets:** The symmetric group, denoted as S_n, represents the group of all permutations of n elements. Within this group, a *coset* is a form of subset created by multiplying a group element by every element within a subgroup. Specifically, for a subgroup H in a group G, and any element $g \in G$, we define:
  - A *left coset* of H in G with respect to g as $gH = \{gh : h \in H\}$.
  - A *right coset* of H in G with respect to g as $Hg = \{hg : h \in H\}$.

  Important properties of cosets include:
  - Cosets of a subgroup partition the group.
  - All left (or right) cosets of a subgroup have the same cardinality as the subgroup itself.

Suppose $n \geqslant 2$.

The single transposition $(1\,2)$ is odd, so

$$(1\,2) \notin A_n .$$

coset decomposition

$$\boxed{\textbf{Claim}: \quad S_n = \underbrace{A_n}_{\text{even permutations}} \,\dot{\cup}\, \underbrace{(1\,2)\,A_n}_{\text{odd permutations}}}$$

even permutations: form a subgroup

odd permutations: form a coset, translation of a subgroup

15. Conjugates of permutations: If α and β are any permutations of a given set then the conjugate of α by β is the permutation $\beta^{-1}\alpha\beta$, denoted by α^β, using exponential notation. We denote by $\alpha^{-\beta}$ the inverse of α^β, so that $\alpha^{-\beta} = \beta^{-1}\alpha^{-1}\beta$, the conjugate of α^{-1} by β (not to be confused with $\beta\alpha\beta^{-1}$, which is $\alpha^{\beta^{-1}}$, the conjugate of α by β^{-1}.

16. ***Properties on Conjugation:*** Let $(S, *)$ be a set with an associative binary operation, $*$, and an identity element, e.

    - *multiplicative* $(ab)^c = a^c b^c$

    $$(ab)^c = c^{-1}(ab)c = (c^{-1}ac)(c^{-1}bc) = a^c b^c$$

    - $(a^b)^{-1} = (a^{-1})^b$
      Note that: $a^{b^{-1}}$ is the conjugate of a by b^{-1}

17. Effect of conjugation on the cycle decomposition of a permutation: If α and β are permutations and $\begin{pmatrix} a_1 & a_2 & \dots & a_k \end{pmatrix}$ is a cycle in the cycle decomposition of α then $(a_1\beta a_2\beta \dots a_k\beta)$ is a cycle in the cycle decomposition of the conjugate α^β.

18. ***More on Conjugation***:
    Suppose $\alpha, \beta \in Sym(n)$,we want to write decomposition of α^β from the cycle decomposition of α.

    - Lemma:If $(a_1, a_2, a_3......a_k)$ is cycle in α, then $(a_1\beta a_2\beta a_3\beta a_4\beta...a_k\beta)$ is a cycle in α^β.($a_k\beta$ means $\beta(a_k)$)

      Proof

      $$\text{Suppose } i \in \{1, 2, 3...k-1\}, \text{Then}$$
      $$(a_i\beta) \to (a_i\beta)(\beta^{-1}a\beta) = a_i\beta\beta^{-1}a\beta = a_i a\beta = a_{i+1}\beta$$

      Similarly

      $$(a_k\beta) \to (a_k\beta)(\beta^{-1}a\beta) = a_k\beta\beta^{-1}a\beta = a_k a\beta = a_1\beta$$

      $\to$ stands for "maps to".
      This proof only shows that the conjugate of a cycle is in(one of) α^β.
      But in fact:

      $$(a_1a_2a_3...a_k)^\beta = (a_1\beta a_1\beta...a_k\beta)(\star)$$

      Which means the conjugate of a cycle a_i is exactly a $a_i\beta$.

    - ***Corollary***: To find α^β, just replace every a_i, in each cycle with $a_i\beta$

      Proof: If $\alpha = a_1a_2a_3...a_k$ for cycles a_i, the conjugate
      $$\alpha^\beta = a_1^\beta a_2^\beta a_2^\beta...a_k^\beta \text{ —— } \textit{multiplicativity}$$
      $$= (a_1\beta a_1\beta...a_k\beta) \text{ —— } \textit{see} \star \textit{ above}$$

5 Week 5

1. Eigenvalues and eigenvectors: Let M be a square matrix, $\mathbf{x}$ a nonzero column vector and λ a scalar such that

$$M\mathbf{x} = \lambda \mathrm{x}$$

Then λ is called an eigenvalue of M and $\mathbf{x}$ is called an eigenvector of M associated with or corresponding to the eigenvalue λ.

2. The eigenspace of a matrix: The eigenspace of a square matrix M associated with an eigenvalue λ is the collection

$$\{\mathbf{v} \mid M\mathbf{v} = \lambda\mathbf{v}\} = \{\mathbf{v} \mid (\lambda I - M)\mathbf{v} = \mathbf{0}\}$$

comprising all of the eigenvectors of M associated with λ and the zero vector (which is never an eigenvector).

3. Description of eigenvalues in terms of determinants: A scalar λ is an eigenvalue of a square matrix M if and only if

$$\det(\lambda I - M) = 0.$$

4. The characteristic polynomial of a square matrix: The expression $\det(\lambda I - M)$ is always a polynomial in λ and is called the characteristic polynomial of M. Thus the eigenvalues of a square matrix are precisely the roots of its characteristic polynomial.

5. Finding eigenspaces: Finding the eigenspace corresponding to the eigenvalue λ of a matrix M is equivalent to solving the homogeneous system with coefficient matrix $\lambda I - M$. After the eigenspace has been found, substituting particular values of the parameters yields particular eigenvectors.

6. Eigenvalues of a triangular matrix: The eigenvalues of a triangular matrix are simply the diagonal entries.

7. *Notes*:

   - Not all real matrices have real Eigenvalues (example later). However, all complex matrices have complex Eigenvalues.

8. The Cayley-Hamilton Theorem: Every square matrix M is the root of its own characteristic polynomial, that is,

$$\chi(M) = 0$$

where $\chi(\lambda) = \det(\lambda I - M)$ denotes the characteristic polynomial, $\chi(M)$ is the result of evaluating the matrix expression obtained from $\chi(\lambda)$ by substituting M for the indeterminate λ and I for the constant 1 , and 0 denotes the zero matrix.

9. Reflection matrices: A reflection matrix in the real plane has the form

$$M = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

for some real θ, and corresponds to reflection in the plane through a line through the origin making an angle θ with the positive x-axis. The eigenvalues of M are ± 1. The eigenspace corresponding to 1 is the line of reflection. The eigenspace corresponding to -1 is the line through the origin perpendicular to the line of reflection.

10. Rotation matrices: A rotation matrix in the real plane has the form

$$M = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

for some real θ, and corresponds to rotation in the plane anticlockwise through an angle θ about the origin. The eigenvalues of M are the complex numbers

$$e^{\pm i\theta} = \cos\theta \pm i\sin\theta$$

where $i = \sqrt{-1}$. The eigenvalues are real if and only if θ is an integer multiple of $\pi(\sin\theta = 0)$, in which case all nonzero vectors are eigenvectors, corresponding to eigenvalue 1 if the integer multiple is even, and corresponding to -1 if the integer multiple is odd.

11. **Diagonalization**

$$M = PDP^{-1}$$

Let M be an $n \times n$ Matrix with eigenvalues: $\lambda_1, \lambda_2, \lambda_3 ... \lambda_n$ (all solutions to chracteristic polynomial $\chi(\lambda) = det(\lambda)I - M)$),with corresponding eigenvectors v_1, v_2, v_3, v_4......(respectively).So that:

$$Mv_i = \lambda_i v_i \text{ for } i = 1, ...n$$

Let P be the $n \times n$ matrix whose columns are the eigenvectores.So

$$P = v_1, v_2, v_3, v_4......v_n$$

Then $MP = M[v_1, v_2, v_3 ... v_n] = PD$
If P is invertible then we can rearrange to $M = PDP^{-1}$ We say that M has been diagonalised. Also, M is the conjugate of D by the inverse of P Rearranging again, we see that D is the conjugate of M by P: $D = P^{-1}MP$

Corollary

$$M^n = PD^nP^{-1}$$

12. Perron's Paradox: Let N be the largest positive integer.If $N > 1$, then $N^2 > N$, contradicting the definition of N. Hence $N = 1$.

13. **Perron's Theorem**: If M is a square matrix with all positive real entries then M has a positive real eigenvalue,λ, such that $|\mu| < \lambda$ for all eigenvalues $|\mu|$ of M. Furthermore, a corresponding eigenvector with only positive entries exists for λ.We call λ the dominant eigenvalue.

    Method:We can estimate the dominant eigenvalue by iterating powers of M multiplied by a random non-zero vector.

    (a) choose a starting vector v_0
    (b) calculate $v_1 = Mv_0$, $v_2 = Mv_1 ...$, In general $v_{k+1} = Mv_k$
    (c) At each step in the process we will "normalise" the vector to make comparisons easier. We repeat until the answers converge.

6 Week 6

1. Diagonal matrices: A square matrix D is diagonal if all entries off the diagonal are zero. If D and E are diagonal then $DE = ED$ is also diagonal, and its diagonal entries are simply the products of corresponding diagonal entries of D and E. Thus the diagonal elements of D^n are just the nth powers of the diagonal elements of D. A scalar matrix is a diagonal matrix in which all elements along the diagonal are equal. The scalar matrices commute with all square matrices of the same size.

2. Diagonalisation: Let M be a square $n \times n$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ and corresponding eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$. Then

$$MP = PD$$

   where D is the diagonal matrix with eigenvalues down the diagonal and P the matrix with corresponding eigenvectors as columns. If P is invertible then

$$M = PDP^{-1} \quad \text{and} \quad D = P^{-1}MP$$

   In this case we say that M is diagonalisable, in which case powers of M can be found easily by the formula

$$M^k = PD^kP^{-1}.$$

   If the eigenvalues are all different then P is invertible and M is diagonalisable.

3. **Terminologies**
   (a) Doubly stochastic matrix: has both rows and columns adding to 1
   (b) Idempotent matrix, is a square matrix satisfying: $A^2 = A$
      - Idempotent matrices have eigenvalues 0 and 1 only.
      - If A is an idempotent matrix, $A^n = A$ for all $n \geqslant 1$

4. Similar matrices: Two matrices A and B are said to be similar or conjugate if there is an invertible matrix P such that $B = P^{-1}AP$. Similarity is an equivalence relation (that is, similarity is reflexive, symmetric and transitive). **In particular, a matrix is diagonalisable if and only if it is similar to a diagonal matrix**.

5. Stochastic matrices: A square matrix M is stochastic if all the entries are nonnegative and the columns add to 1, and regular if, further, some positive power of M has all positive entries. A column matrix $\mathbf{v}$ is a probability vector if all of its entries are nonnegative and add to 1, and, further, becomes a steady state vector for M if $M\mathbf{v} = \mathbf{v}$.

6. Existence and uniqueness of a steady state vector: If M is a regular stochastic matrix then there exists a unique steady state vector $\mathbf{v}$ for M, in which case, for any probability vector $\mathbf{x}$,

$$\lim_{k\to\infty} M^k\mathbf{x} = \mathbf{v}$$

7. More on stochastic matrix: Let A be a stochastic matrix:

   (a) 1 is an eigenvalue of A,and so A has at least one steady-state probabiliry vector $\mathbf{v}$, which is fixed by A.

   $$A\mathbf{v} = 1\mathbf{v}$$

   (b) All eigenvalues of A are less then or equal to 1 in magnitude

   (c) If A is regular (some power of A has all positive entries), then:

   - $\mathbf{v}$ is unique
   - $$\lim_{n\to\infty} A^n = \begin{bmatrix}\mathbf{v} & \mathbf{v} & \mathbf{v} & \cdots\end{bmatrix}$$
   - for any probability vector $\mathbf{x}$,

     $$\lim_{n\to\infty} A^n\underline{x} = \underline{v}$$

   - 1 is the dominant eigenvalue.

8. Perron's Theorem and existence of dominant eigenvalues: If M is a square matrix all of whose entries are positive then M has a positive real eigenvalue λ such that $|\mu| \leqslant \lambda$ for all eigenvalues μ of M, and, furthermore, there exists an eigenvector corresponding to λ, all of whose entries are positive.

7 Week 7

1. Cartesian products of sets and inherited coordinatewise operations: If $A_1, A_2, \ldots, A_n$ are sets then we may form the Cartesian product

$$A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\}$$

which has coordinate-wise operations inherited from $A_1, \ldots, A_n$, in the case that these have arithmetic operations of the same type (such as addition or multiplication). In particular, if F is a field and $n \geqslant 1$ then we may form the Cartesian power

$$F^n = A_1 \times \ldots \times A_n = \{(a_1, \ldots, a_n) \mid a_1, \ldots, a_n \in F\}$$

where $A_1 = \ldots = A_n = F$, with coordinatewise addition, multiplication and scalar multiplication. In this case the mapping

$$(a_1, \ldots, a_n) \mapsto \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

is a bijection between F^n and the set V of all column vectors over F of length n preserving addition and scalar multiplication. Both F^n and V are examples of vector spaces (see later), and the previous statement says that F^n and V are vector space isomorphic. We also define $F^0 = \{0\}$, called the trivial vector space.

2. Identification of n-tuples with row vectors: It is common to make the following identification of an n-tuple with the row vector obtained by deleting commas and replacing round brackets with square brackets:

$$(a_1, a_2, \ldots, a_n) \equiv [a_1 a_2 \ldots a_n]$$

and then coordinatewise addition and scalar multiplication of n-tuples becomes addition and scalar multiplication of row vectors as $1 \times n$ matrices.

3. Linear transformations (special case): The Cartesian power F^n is the prototype structure for an n-dimensional vector space (see later for definitions). A function $L : F^m \to F^n$, where m and n are nonnegative integers, is called a linear transformation if L respects coordinatewise addition and scalar multiplication, that is, for all $\mathbf{v}, \mathbf{w} \in F^m$ and $\lambda \in F$,

$$L(\mathbf{v} + \mathbf{w}) = L(\mathbf{v}) + L(\mathbf{w}) \text{ and } L(\lambda\mathbf{v}) = \lambda L(\mathbf{v})$$

equivalently, for all $\mathbf{v}, \mathbf{w} \in F^m$ and $\lambda, \mu \in F$,

$$L(\lambda\mathbf{v} + \mu\mathbf{w}) = \lambda L(\mathbf{v}) + \mu L(\mathbf{w})$$

and we say that L respects or preserves linear combinations. If $m = n$ then L is called a linear operator. It is traditional to use functional notation for linear transformations and operators and to compose them in the reverse order to which they are written down (from right to left, by contrast with the left to right convention commonly used by algebraists). The composite of linear transformations, when defined, is also a linear transformation.

4. Standard basis: For $1 \leqslant i \leqslant n$, let $\mathbf{e}_i$ be the n-tuple with 0 in each place except for 1 in the i th place. Put $B = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$, called the standard basis for F^n. If $\mathbf{v} = (v_1, \ldots, v_n) \in F^n$ then

$$\mathbf{v} = v_1\mathbf{e}_1 + \ldots + v_n\mathbf{e}_n$$

so that B has the so-called spanning property. Also B is linearly independent (see later).

5. Matrix of a linear transformation: Let $L : F^m \to F^n$ be a linear transformation and let B_m be the standard basis for F^m. Form the $n \times m$ matrix M_L where, for $1 \leqslant i \leqslant m$, the i th column of M_L is $(L(\mathbf{e}_i))^\top$, the transpose of the row vector $L(\mathbf{e}_i)$. The action of L on row vectors corresponds to matrix multiplication of column vectors by M_L in the following sense:

$$L(\mathbf{v}) = \mathbf{w} \text{ if and only if } \quad M_L\mathbf{v}^\top = \mathbf{w}^\top.$$

6. Matrix multiplication corresponds to composition of linear transformations: Let $L_1 : F^m \to F^n$ and $L_2 : F^n \to F^q$ be linear transformations, so that $L_2L_1 = L_2 \circ L_1 : F^m \to F^q$. Then

$$M_{L_2L_1} = M_{L_2}M_{L_1}$$

7. Isomorphisms of groups: If G and H are groups then a bijection $\phi : G \to H$ is called an isomorphism if ϕ preserves the group operation, that is, $(g_1g_2)\,\phi = (g_1\phi)\,(g_2\phi)$ for all $g_1, g_2 \in G$. If there exists an isomorphism between two groups G and H then we say that G and H are isomorphic and write $G \cong H$. Then $\cong$ is an equivalence relation on the class of groups, that is, $\cong$ is reflexive, symmetric and transitive.

8. Cyclic groups: A group G is called cyclic if there exists an element g, called the generator of G such that every element is a power (possibly negative) of g (expressing the group operation multiplicatively). If a cyclic group is finite then every element is a positive power of the chosen generator. Two cyclic groups are isomorphic if and only if they have the same number of elements.

9. Shear matrices: A (standard) shear matrix is an elementary matrix of the form

$$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

for some real number k, which corresponds to the shear transformation of the xy-plane that fixes the x-axis and shifts points sideways proportional to their y-coordinates (with factor k of proportionality).

10. Invertible linear transformations of the plane: Every invertible linear transformation of the plane decomposes as the composite of a shear, a pair of dilations in the x and y-directions respectively and a rotation. Equivalently, every invertible 2×2 real matrix decomposes as a product of a shear matrix, a diagonal matrix and a rotation matrix.