

THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

MATH1002
LINEAR ALGEBRA

June 2019 LECTURERS: Z. Afsar, A. Aksamit, D. Badziahin, A. Fish, A. Kersch, B. Pauwels.

TIME ALLOWED: **Writing - one and a half hours; Reading - 10 minutes**

EXAM CONDITIONS: This is a closed-book examination — no material permitted. Writing is not permitted at all during reading time.

Family Name: SID:

Other Names: Seat Number:

Please check that your examination paper is complete (21 pages) and indicate by signing below.
I have checked the examination paper and affirm that it is complete.

Signature: Date:

This examination has two sections: Multiple Choice and Extended Answer.

The Multiple Choice Section is worth 50% of the total examination.
There are 20 questions. The questions are of equal value.
All questions should be attempted.

Answers to the Multiple Choice questions must be entered on
the Multiple Choice Answer Sheet before the end of the examination.

The Extended Answer Section is worth 50% of the total examination.
There are 3 questions. The questions are of equal value.
All questions should be attempted. Working must be shown.

**THE QUESTION PAPER MUST NOT BE REMOVED FROM THE
EXAMINATION ROOM.**

MARKER'S USE
ONLY

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Multiple Choice Section

For each question, choose exactly one option.

Your answers must be entered on the Multiple Choice Answer Sheet.

1. Let $\mathbf{u} = [1, \alpha, 0]$, $\mathbf{v} = [5, -4, 15]$ and $\mathbf{w} = [-1, \beta, -3]$. Find the values of α and β so that \mathbf{u} and \mathbf{v} are orthogonal and \mathbf{v} and \mathbf{w} are parallel.

- (a) $\alpha = \frac{5}{4}, \beta = \frac{4}{5}$ (b) $\alpha = \frac{4}{5}, \beta = \frac{5}{4}$ (c) $\alpha = \frac{5}{4}, \beta = -\frac{4}{5}$
(d) $\alpha = -\frac{5}{4}, \beta = \frac{4}{5}$ (e) $\alpha = -\frac{4}{5}, \beta = \frac{5}{4}$

2. What is the area of the triangle inscribed by the vectors $[2, -1, 3]$ and $[-1, 1, 0]$?

- (a) $\sqrt{19}$ (b) $\frac{19}{2}$ (c) $\frac{\sqrt{7}}{2}$ (d) $\frac{\sqrt{19}}{2}$ (e) $\sqrt{7}$

3. Which one of the following augmented matrices is in row echelon form?

- (a) $\left[\begin{array}{ccc|c} 1 & -3 & 0 & -1 \\ 1 & 6 & 1 & 6 \\ 0 & 0 & 0 & 2 \end{array} \right]$ (b) $\left[\begin{array}{cc|c} 2 & 4 & 3 \\ 0 & 0 & 1 \end{array} \right]$ (c) $\left[\begin{array}{ccc|c} 0 & 0 & 1 & 2 \\ 1 & 1 & 0 & 1 \end{array} \right]$
(d) $\left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right]$ (e) None of the above.

4. The determinant of the matrix $\begin{bmatrix} 1 & 1 & 0 \\ 3 & -1 & 2 \\ 2 & 4 & -2 \end{bmatrix}$ is equal to

- (a) 16 (b) -16 (c) 0 (d) 4 (e) -4

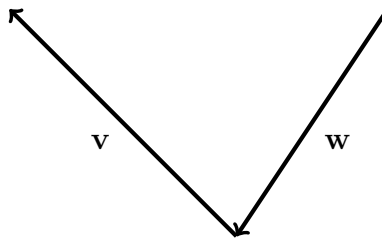
5. Consider the following system of equations:

$$\begin{array}{rcccccl} x & + & y & + & z & - & w & = & 0 \\ & & -y & + & 2z & - & w & = & 0 \\ & & & & 3z & - & 2w & = & 0 \end{array}$$

Which one of the following statements about this system is true?

- (a) There is no solution.
- (b) The general solution is expressed using exactly 1 parameter.
- (c) The general solution is expressed using exactly 2 parameters.
- (d) The general solution is expressed using 3 or more parameters.
- (e) There is a unique solution.

6. Suppose \mathbf{v} and \mathbf{w} are two non-zero vectors lying in this page:



Which of the following is true?

- (a) $(\mathbf{v} \times \mathbf{w}) \times \mathbf{v}$ is perpendicular to both \mathbf{v} and \mathbf{w} .
- (b) $(\mathbf{w} \times \mathbf{v}) \times (\mathbf{v} \times \mathbf{w})$ is parallel to \mathbf{w} but not \mathbf{v} .
- (c) $(\mathbf{v} \times \mathbf{w}) \cdot \mathbf{w}$ is a non-zero scalar.
- (d) $\mathbf{v} \times \mathbf{w}$ points upwards, towards the ceiling.
- (e) \mathbf{w} and $\mathbf{v} \times \mathbf{w}$ are parallel.

7. The two lines given by the respective parametric equations

$$\left. \begin{array}{l} x = 3 + t \\ y = -5 + 2t \\ z = -5 - t \end{array} \right\} t \in \mathbb{R} \quad \text{and} \quad \left. \begin{array}{l} x = -3 - 2s \\ y = -2 - 4s \\ z = 1 + 2s \end{array} \right\} s \in \mathbb{R}$$

- (a) do not intersect.
- (b) intersect at the point $(-3, -2, 1)$.
- (c) intersect at the point $(7, 3, -9)$.
- (d) intersect at the point $(-2, -15, 0)$.
- (e) coincide.

8. The cosine of the angle between the vectors $[1, 3, -5]$ and $[3, 2, 1]$ is equal to

- (a) $\frac{7\sqrt{10}}{4}$ (b) $\frac{11}{15}$ (c) $\frac{2}{7\sqrt{10}}$ (d) $\frac{4}{7\sqrt{10}}$ (e) $\frac{2}{245}$

9. Consider the three planes with equations

$$\mathcal{P}_1 : x + 2y - z = 1,$$

$$\mathcal{P}_2 : 2x - 5y + 3z = 1,$$

$$\mathcal{P}_3 : 2x - 4y + z = 10.$$

Which of the following is true?

- (a) \mathcal{P}_1 and \mathcal{P}_2 are parallel to each other but not parallel to \mathcal{P}_3 .
(b) \mathcal{P}_1 and \mathcal{P}_3 are parallel to each other but not parallel to \mathcal{P}_2 .
(c) \mathcal{P}_2 and \mathcal{P}_3 are parallel to each other but not parallel to \mathcal{P}_1 .
(d) None of the planes are parallel to each other.
(e) All of the planes are parallel to each other.

10. Which one of the following sequences of row operations, when applied to the matrix $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$, produces the matrix $\begin{bmatrix} d-a & e-b & f-c \\ 2a & 2b & 2c \end{bmatrix}$?

- (a) First $R_1 \rightarrow R_1 - R_2$, then $R_2 \rightarrow 2R_2$, then $R_1 \leftrightarrow R_2$.
(b) First $R_1 \rightarrow 2R_1$, then $R_1 \leftrightarrow R_2$, then $R_1 \rightarrow R_1 - R_2$.
(c) First $R_2 \rightarrow R_2 - R_1$, then $R_1 \leftrightarrow R_2$, then $R_1 \rightarrow 2R_1$.
(d) First $R_1 \leftrightarrow R_2$, then $R_1 \rightarrow 2R_1$, then $R_1 \rightarrow R_1 - R_2$.
(e) First $R_1 \leftrightarrow R_2$, then $R_1 \rightarrow R_1 - R_2$, then $R_2 \rightarrow 2R_2$.

11. Suppose that \mathbf{v} and \mathbf{w} are two linearly independent vectors in \mathbb{R}^n . Which one of the following statements is true?

- (a) \mathbf{v} and \mathbf{w} are orthogonal.
(b) If $a\mathbf{v} + b\mathbf{w} = c\mathbf{v} + d\mathbf{w}$, where $a, b, c, d \in \mathbb{R}$, then $a = c$ and $b = d$.
(c) \mathbf{v} and \mathbf{w} are parallel.
(d) The vectors $\mathbf{v} + \mathbf{w}$ and $\mathbf{v} - \mathbf{w}$ are linearly dependent.
(e) None of the above.

12. Which one of the following statements is true?

- (a) Any set of three vectors from \mathbb{R}^2 must span \mathbb{R}^2 .
- (b) There are three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ such that $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent.
- (c) If $\text{span}\{\mathbf{u}, \mathbf{v}\} = \mathbb{R}^2$, then $\{\mathbf{u}, \mathbf{v}\}$ is a basis for \mathbb{R}^2 .
- (d) For any three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$, there is a subset of $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ that is a basis for \mathbb{R}^2 .
- (e) The set $\{\mathbf{u}, \mathbf{v}, \mathbf{0}\}$ is a basis for \mathbb{R}^2 only if $\{\mathbf{u}, \mathbf{v}\}$ is a basis for \mathbb{R}^2 .

13. Which one of the following statements is true, given that A is a matrix of size 3×3 , B is a matrix of size 3×2 , and C is a matrix of size 2×3 ?

- (a) ACB is defined.
- (b) $2A + CB$ is defined.
- (c) $(BC)^2$ is a 2×2 matrix.
- (d) $B(A - BC)$ is defined.
- (e) $A^2 + BC$ is a 3×3 matrix.

14. If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$, which one of the following is true?

- (a) A is not invertible.
- (b) A is invertible and $A^{-1} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & 1 \\ -3 & 1 & 0 \end{bmatrix}$.
- (c) A is invertible and $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$.
- (d) A is invertible and $A^{-1} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ -3 & 1 & 0 \end{bmatrix}$.
- (e) None of the above.

15. Let $A = P D_1 P^{-1}$, and $B = P D_2 P^{-1}$ where $D_1 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $D_2 = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$, and

$P = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. Then $(AB)^5$ is

- (a) $\begin{bmatrix} 2^5 & 3^5 - 2^5 \\ 0 & 3^5 \end{bmatrix}$. (b) $\begin{bmatrix} -3^5 & 2^5 - 3^5 \\ 0 & -2^5 \end{bmatrix}$. (c) $\begin{bmatrix} 3^5 & 0 \\ 0 & 2^5 \end{bmatrix}$.
 (d) $\begin{bmatrix} 2^5 & 1 \\ 0 & 3^5 \end{bmatrix}$. (e) $\begin{bmatrix} -2^5 & -3^5 + 2^5 \\ 0 & -3^5 \end{bmatrix}$.

16. Let A and B be $n \times n$ diagonalisable matrices. Which one of the following is always true?

- (a) $A + B$ is diagonalisable.
 (b) AB is diagonalisable.
 (c) A^2 and B^2 are diagonalisable.
 (d) $AB = BA$.
 (e) none of the above.

17. Let B be a 4×4 matrix and suppose that $\det(B) = 3$. Then $\det\left(-\frac{1}{\sqrt{3}}B\right)$ is

- (a) $1/3$ (b) 3 (c) $-1/\sqrt{3}$ (d) $-\sqrt{3}$ (e) -1

18. Suppose a 3×3 matrix A has 3 distinct eigenvalues λ_1 , λ_2 and λ_3 . Which one of the following is NOT necessarily true?

- (a) $\det(A) = \lambda_1 \lambda_2 \lambda_3$.
 (b) The characteristic polynomial of A has 3 distinct roots.
 (c) A is invertible.
 (d) If B is any 3×3 invertible matrix then BAB^{-1} has eigenvalues λ_1 , λ_2 and λ_3 .
 (e) There is a 3×3 invertible matrix P so that $PAP^{-1} = \begin{bmatrix} \lambda_2 & 0 & 0 \\ 0 & \lambda_3 & 0 \\ 0 & 0 & \lambda_1 \end{bmatrix}$.

- 19.** Let P be a transition matrix of a Markov chain on n states. Which of the following is NOT necessarily true.
- (a) P^2 is a transition matrix for a Markov chain.
 - (b) P is an $n \times n$ matrix.
 - (c) If P is invertible, then P^{-1} is a transition matrix for a Markov chain
 - (d) If Q is another transition matrix for a Markov chain on n states, then PQ is a transition matrix for a Markov chain.
 - (e) If Q is another transition matrix for a Markov chain on n states, then $\frac{1}{2}(P + Q)$ is a transition matrix for a Markov chain
- 20.** Which one of the following is true for all $n \times n$ matrices A and B ?
- (a) if $A^2 = B^2$ then $A = \pm B$
 - (b) if $AB = 0$ then $A = 0$ or $B = 0$
 - (c) $(A + B)^2 = A^2 + 2AB + B^2$
 - (d) if A is invertible then AB is invertible
 - (e) if AB is invertible then A and B are invertible

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End of Multiple Choice Section.

Make sure that your answers are entered on the Multiple Choice Answer Sheet.

THE EXTENDED ANSWER SECTION BEGINS ON THE NEXT PAGE.

Extended Answer Section

There are **three** questions in this section, each with a number of parts. Write your answers in the space provided. If you need more space there are extra pages at the end of the examination paper.

1. (a) Let $\mathcal{P}_1, \mathcal{P}_2$ and \mathcal{P}_3 be three planes in three dimensional space given by the equations
 $\mathcal{P}_1 : x + 3y + 4z = 5$; $\mathcal{P}_2 : 2x + 5y + z = b$; $\mathcal{P}_3 : -x + ay + 3z = 0$,
where $a, b \in \mathbb{R}$ are parameters.

- (i) Find $\mathbf{n}_1 \times \mathbf{n}_2$, where \mathbf{n}_1 is a normal vector for \mathcal{P}_1 , and \mathbf{n}_2 is a normal vector for \mathcal{P}_2 .

- (ii) Find the direction vector of the line of intersection of \mathcal{P}_1 and \mathcal{P}_2 .

- (b) (i) Find the projection of the vector $[3, 2, 1]$ onto the line spanned by the vector $[1, 2, 3]$.

- (ii) By use of part (i) or otherwise, find the distance between the point $P(3, 2, 1)$ and the line with the parametric equations

$$\left. \begin{array}{l} x = t \\ y = 2t \\ z = 3t \end{array} \right\} t \in \mathbb{R}.$$

(c) Show that any three non-zero vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in \mathbb{R}^3 which are orthogonal one to each other are linearly independent.

3. (a) Let

$$P = \begin{bmatrix} 0 & 0 & 1/2 \\ 1/2 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_0 = \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \end{bmatrix},$$

where P is the transition matrix for a Markov chain with three states, and \mathbf{x}_0 is the initial state vector for the population.

(i) Compute \mathbf{x}_1 and \mathbf{x}_2 .

(ii) What is the probability of moving from state 3 to state 1 in 2 transitions?

(i) Let $\lambda \neq 0$. Show that λ is an eigenvalue of AB if and only if it is also an eigenvalue of BA .

- (ii) Show that $I_n + AB$ is invertible if and only if $I_n + BA$ is invertible, where I_n is the identity $n \times n$ matrix.

THERE ARE NO MORE QUESTIONS.

More space is available on the next page.

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End of Extended Answer Section.

End of Examination

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MATH1002 Linear Algebra

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Family Name:

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Seat Number:

Answers →

Attempt every question.
You will **not** be awarded
negative marks for
incorrect answers.

Fill in exactly one oval per
question.

If you make a mistake,
draw a cross (X) through
any mistakenly filled in
oval(s) and then fill in your
intended oval.

An answer which contains
two or more filled in (and
uncrossed) ovals will be
awarded no marks.

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