

Important Ideas and Useful Facts:

- (i) **Inner product spaces:** Let V be a vector space over \mathbb{R} . We call V an *inner product space* if it is equipped with an *inner product*, that is, a mapping $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{R}$ such that

- (a) $(\forall \mathbf{u}, \mathbf{v} \in V) \quad \langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle,$
- (b) $(\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V) \quad \langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle,$
- (c) $(\forall \mathbf{u}, \mathbf{v} \in V)(\forall \lambda \in \mathbb{R}) \quad \langle \lambda \mathbf{u}, \mathbf{v} \rangle = \lambda \langle \mathbf{u}, \mathbf{v} \rangle,$
- (d) $(\forall \mathbf{v} \in V) \quad \langle \mathbf{v}, \mathbf{v} \rangle \geq 0$ and $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ if and only if $\mathbf{v} = \mathbf{0}$.

Common examples are $V = \mathbb{R}^n$ with respect to the usual dot product, and V the vector space of continuous real functions on a closed interval $[a, b]$ with inner product defined by, for $f, g \in V$,

$$\langle f, g \rangle = \int_a^b f(x)g(x) dx .$$

- (ii) **Simple consequences of the inner product definition:** Let V be an inner product space. Then

- (e) $(\forall \mathbf{v} \in V) \quad \langle \mathbf{0}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{0} \rangle = 0,$
- (f) $(\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V) \quad \langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle,$
- (g) $(\forall \mathbf{u}, \mathbf{v} \in V)(\forall \lambda \in \mathbb{R}) \quad \langle \mathbf{u}, \lambda \mathbf{v} \rangle = \lambda \langle \mathbf{u}, \mathbf{v} \rangle.$

- (iii) **Length or norm of a vector:** Let V be an inner product space. Define the *length* or *norm* of a vector $\mathbf{v} \in V$ to be

$$\|\mathbf{v}\| = \langle \mathbf{v}, \mathbf{v} \rangle^{1/2} .$$

Length has the following properties:

- (a) $(\forall \mathbf{v} \in V) \quad \|\mathbf{v}\| \geq 0$ and $\|\mathbf{v}\| = 0$ if and only if $\mathbf{v} = \mathbf{0}$,
- (b) $(\forall \lambda \in \mathbb{R})(\forall \mathbf{v} \in V) \quad \|\lambda \mathbf{v}\| = |\lambda| \|\mathbf{v}\|,$
- (c) $(\forall \mathbf{u}, \mathbf{v} \in V) \quad \|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|.$

The last property is known as the *triangle inequality*.

- (iv) **Distance between vectors:** If \mathbf{v} and \mathbf{w} are vectors in an inner product space V then the *distance* between \mathbf{v} and \mathbf{w} is $\|\mathbf{v} - \mathbf{w}\| = \|\mathbf{w} - \mathbf{v}\|$. Thus, from the triangle inequality, for any $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$,

$$\|\mathbf{u} - \mathbf{w}\| \leq \|\mathbf{u} - \mathbf{v}\| + \|\mathbf{v} - \mathbf{w}\| .$$

- (v) **The Cauchy-Schwarz inequality:** If $\mathbf{u}, \mathbf{v} \in V$, where V is an inner product space, then

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \|\mathbf{v}\| .$$

- (vi) **Normalising a vector:** Call a vector \mathbf{v} from an inner product space *normal* if $\|\mathbf{v}\| = 1$. If $\mathbf{v} \neq \mathbf{0}$ then we *normalise* \mathbf{v} by forming the normal vector

$$\hat{\mathbf{v}} = \frac{1}{\|\mathbf{v}\|} \mathbf{v} .$$

- (vii) **Orthogonal vectors:** Vectors \mathbf{u} and \mathbf{v} from an inner product space are said to be *orthogonal* or *mutually perpendicular* if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.
- (viii) **Orthogonal and orthonormal sets of vectors:** A set of vectors from an inner product space is said to be *orthogonal* if every pair of distinct vectors from the set is orthogonal. An orthogonal set in which every vector is normal is said to be *orthonormal*. It is an important fact that any orthogonal set of nonzero vectors is linearly independent.
- (ix) **Utility of an orthonormal basis in finding coordinates:** If $B = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ is an orthonormal basis for an inner product space V then, for all $\mathbf{v} \in V$,

$$\mathbf{v} = \langle \mathbf{v}, \mathbf{b}_1 \rangle \mathbf{b}_1 + \langle \mathbf{v}, \mathbf{b}_2 \rangle \mathbf{b}_2 + \dots + \langle \mathbf{v}, \mathbf{b}_n \rangle \mathbf{b}_n .$$

that is, $[\mathbf{v}]_B = (\langle \mathbf{v}, \mathbf{b}_1 \rangle, \langle \mathbf{v}, \mathbf{b}_2 \rangle, \dots, \langle \mathbf{v}, \mathbf{b}_n \rangle)^T$.

Tutorial Exercises:

1. Let $\mathbf{u} = (1, -3, 2)$, $\mathbf{v} = (1, 1, 0)$, $\mathbf{w} = (2, 2, -4)$. Find

$$\begin{array}{lll} \text{(a)} & -2\mathbf{w} & \text{(b)} \quad \|\mathbf{w}\| \\ \text{(c)} & \left\| \frac{-2}{\|\mathbf{w}\|} \mathbf{w} \right\| & \\ \text{(d)} & \mathbf{u} + \mathbf{v} & \text{(e)} \quad \|\mathbf{u} + \mathbf{v}\| \\ \text{(f)} & \|\mathbf{u}\| + \|\mathbf{v}\| & \end{array}$$

Verify that the triangle inequality is holding in parts (e) and (f).

2. Let $\mathbf{u} = (2, -1, 1)$ and $\mathbf{v} = (1, 1, 2)$. Find $\mathbf{u} \cdot \mathbf{v}$ and the angle θ between \mathbf{u} and \mathbf{v} .
3. Let $P_0(0, 0, 0)$, $P_1(1, 1, 0)$, $P_2(1, 0, 1)$, $P_3(0, 1, 1)$ be the vertices of a tetrahedron in \mathbb{R}^3 .
- (a) Verify that the tetrahedron is regular (all faces are equilateral triangles).
- (b) Find the angle θ between two rays joining the centre to two vertices (the “bond angle” of a methane molecule).
4. Use the dot product to verify that the angle inscribed in a semicircle is a right angle.
5. Let $\mathbf{u} = (2, 0, -1, 3)$ and $\mathbf{v} = (5, 4, 7, -1)$. Find $\|\mathbf{u}\|$, $\|\mathbf{v}\|$, $\|\mathbf{u} + \mathbf{v}\|$, $\mathbf{u} \cdot \mathbf{v}$ and the angle θ between \mathbf{u} and \mathbf{v} . Verify (as expected) that $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$. Verify, however, that

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2.$$

Is this to be expected? (See Exercise 13 below.)

6. (a) Write down a vector \mathbf{v} , as a linear combination of \mathbf{i} and \mathbf{j} , pointing in the direction of the line $y = 2x$ in the xy -plane.
- (b) Find $\hat{\mathbf{v}}$, the unit vector pointing in the direction of \mathbf{v} .
- (c) Write down the position vector \mathbf{u} of the point $(-1, 5)$ as a linear combination of \mathbf{i} and \mathbf{j} .
- (d) Find $\text{proj}_{\mathbf{v}} \mathbf{u} = (\mathbf{u} \cdot \hat{\mathbf{v}}) \hat{\mathbf{v}}$, the projection of \mathbf{u} in the direction of \mathbf{v} .
- (e) Now find the distance from the point $(-1, 5)$ to the line $y = 2x$ and the nearest point on this line.

- 7.* Let W be the plane defined by the equation

$$x + 2y - z = 0.$$

Let $\mathbf{b}_1 = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$ and $\mathbf{b}_2 = (-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$.

- (a) Check that $\{\mathbf{b}_1, \mathbf{b}_2\}$ is an orthonormal basis for the subspace W .
- (b) Find $\text{proj}_W \mathbf{v} = \langle \mathbf{v}, \mathbf{b}_1 \rangle \mathbf{b}_1 + \langle \mathbf{v}, \mathbf{b}_2 \rangle \mathbf{b}_2$ where $\mathbf{v} = (4, 2, -5)$.
- (c) Find the distance from the point $(4, 2, -5)$ to W and the nearest point on W .
- 8.* Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ where $\mathbf{u} \neq \mathbf{0}$.

- (a) Put $\lambda = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|^2}$ and expand and simplify $(\lambda \mathbf{u} - \mathbf{v}) \cdot (\lambda \mathbf{u} - \mathbf{v})$.
- (b) Deduce that $|\mathbf{u} \cdot \mathbf{v}| = \|\mathbf{u}\| \|\mathbf{v}\|$ if and only if \mathbf{v} is a scalar multiple of \mathbf{u} .

Further Exercises:

9. Verify from the definition of dot product that, for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$,

$$(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w} \quad \text{and} \quad \lambda(\mathbf{u} \cdot \mathbf{v}) = (\lambda\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (\lambda\mathbf{v}) .$$

10. Verify the identity

$$\|\mathbf{u} + \mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2 = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2$$

in \mathbb{R}^n , and interpret this geometrically in \mathbb{R}^2 .

11. Use the Cauchy-Schwarz inequality to verify the triangle inequality in \mathbb{R}^n .

12. If $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, recall that the *distance* from \mathbf{u} to \mathbf{v} is

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| .$$

Deduce the following triangle inequality from the usual one:

$$d(\mathbf{u}, \mathbf{w}) \leq d(\mathbf{u}, \mathbf{v}) + d(\mathbf{v}, \mathbf{w})$$

for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$.

13. Let \mathbf{v}, \mathbf{w} be orthogonal elements of an inner product space. Verify the so-called *Generalised Theorem of Pythagoras*:

$$\|\mathbf{v} + \mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 .$$

14. Let V be an inner product space and W be the subspace spanned by $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n \in V$, that is,

$$W = \{ \lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 + \dots + \lambda_n \mathbf{v}_n \mid \lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{R} \} .$$

Verify that an arbitrary vector $\mathbf{v} \in V$ is orthogonal to every vector in W if and only if \mathbf{v} is orthogonal to each of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$.

- 15.* Let V be the vector space of all continuous functions: $[a, b] \rightarrow \mathbb{R}$ and define, for $f, g \in V$,

$$\langle f, g \rangle = \int_a^b f(x)g(x) dx .$$

Verify that V becomes an inner product space.

- 16.* In the previous exercise, take $a = -1$ and $b = 1$. Let $f, g, h \in V$ where $f(x) = 1$, $g(x) = x$ and $h(x) = x^3$ for $x \in [-1, 1]$.

(a) Find $\|f\|$, $\|g\|$, $\|h\|$, $\langle f, g \rangle$, $\langle f, h \rangle$, $\langle g, h \rangle$ and the distance between f and g . Which pairs of functions are orthogonal?

(b) More generally, let $p(x) = x^m$ and $q(x) = x^n$, where m, n are nonnegative integers. Find a simple condition on m and n characterising orthogonality of p and q in V .