THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Calculus Tutorial 8 (Week 9)

MATH1062/MATH1023: Mathematics 1B (Calculus) Semester 2, 2024

Questions marked with * are harder questions.

Material covered

(1) Partial derivatives and tangent planes

Summary of essential material

The equation of the tangent plane to the surface $z = f(x, y)$ at $(x, y) = (a, b)$ is given by

$$z - f(a, b) = f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

Questions to complete during the tutorial

1. For the given function f, find the indicated derivative (ordinary or partial). (Recall that to compute $f_x(x, y)$, regard y as a constant and differentiate $f(x, y)$ with respect to x. Similarly, to compute $f_y(x, y)$ regard x as a constant and differentiate $f(x, y)$ with respect to y.)

(a) $f(x) = 3x^2 - 16$, find $\dfrac{df}{dx}$.

(b) $f(x, y) = 3x^2 - y^4$, find $\dfrac{\partial f}{\partial x}$, $\dfrac{\partial f}{\partial y}$.

(c) $f(y) = 5e^{3y}$, find $\dfrac{df}{dy}$.

(d) $f(x, y) = xe^{3y}$, find $\dfrac{\partial f}{\partial x}$, $\dfrac{\partial f}{\partial y}$.

(e) $f(x) = 4 \ln x$, find $\dfrac{df}{dx}$.

(f) $f(x, y) = y \ln x$, find $\dfrac{\partial f}{\partial x}$, $\dfrac{\partial f}{\partial y}$.

(g) $f(x) = \dfrac{2x}{x^2 + 4}$, find $\dfrac{df}{dx}$.

(h) $f(x, y) = \dfrac{xy}{x^2 + y^2}$, find $\dfrac{\partial f}{\partial x}$, $\dfrac{\partial f}{\partial y}$

2. Let $f(x, y) = x^3 + x^2y^3 - 2y^2$. Calculate the partial derivatives $f_x(x, y)$ and $f_y(x, y)$ and evaluate each at the point $(1, 2)$.

3. Find the equation of the tangent plane to the paraboloid $z = x^2 + 4y^2$ at the point $(2, 1, 8)$.

4. Let $f(x, y) = 2x - 3y + 2$. Find the equation of the tangent plane to the surface $z = f(x, y)$ at the point $(x, y) = (3, 1)$. In this example there is a striking relationship between the given surface and its tangent plane! Explain it.

***5.** The Ideal Gas Law $PV = kT$ (where k is a constant) determines each of P, V, T (pressure, volume and temperature, respectively) as functions of the other two. Show that

$$\frac{\partial P}{\partial V}\frac{\partial V}{\partial T}\frac{\partial T}{\partial P} = -1.$$

6. Find the two first-order partial derivatives of the following function which gives the volume V of a cylinder, radius r and height a:

$$V = \pi r^2 a.$$

Explain what information these partial derivatives give about the effect on the volume of the cylinder of changing either only its radius or only its height.

***7.** Show the ellipsoid $3x^2 + 2y^2 + z^2 = 9$ and sphere $x^2 + y^2 + z^2 - 8x - 6y - 8z + 24 = 0$ are tangential to each other at $(1, 1, 2)$. That is, show that these two surfaces have a common tangent plane at the point $(1, 1, 2)$.

Short answers to selected exercises

1. (a) $f'(x) = 6x$

(b) $f_x(x, y) = 6x$, $f_y(x, y) = -4y^3$

(c) $f'(y) = 15e^{3y}$

(d) $f_x(x, y) = e^{3y}$, $f_y(x, y) = 3xe^{3y}$

(e) $f'(x) = 4/x$

(f) $f_x(x, y) = y/x$, $f_y(x, y) = \ln x$

(g) $f'(x) = (8 - 2x^2)/(x^2 + 4)^2$

(h) $f_x(x, y) = y(y^2 - x^2)/(x^2 + y^2)^2$, $f_y(x, y) = x(x^2 - y^2)/(x^2 + y^2)^2$

2. $f_x(1, 2) = 19$, $f_y(1, 2) = 4$

3. $z = 4x + 8y - 8$