Semester 1 Main, 2018

The University of Sydney
School of Mathematics and Statistics

MATH1002
Linear Algebra

June 2018 Lecturers: A. Aksamit, D. Badziahin, N. Brownlowe, A. Fish, D. Tran.

Time Allowed: **Writing - one and a half hours; Reading - 10 minutes**

Exam Conditions: This is a closed-book examination — no material permitted. Writing is not permitted at all during reading time.

Family Name: .... SID: ....

Other Names: .... Seat Number: ....

Please check that your examination paper is complete (20 pages) and indicate by signing below.
I have checked the examination paper and affirm that it is complete.

Signature: .... Date: ....

This examination has two sections: Multiple Choice and Extended Answer.

The Multiple Choice Section is worth 50% of the total examination.
There are 20 questions. The questions are of equal value.
All questions should be attempted.

Answers to the Multiple Choice questions must be entered on the Multiple Choice Answer Sheet before the end of the examination.

The Extended Answer Section is worth 50% of the total examination.
There are 3 questions. The questions are of equal value.
All questions should be attempted. Working must be shown.

THE QUESTION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM.

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Multiple Choice Section

For each question, choose exactly one option.
Your answers must be entered on the Multiple Choice Answer Sheet.

1. Let $\mathbf{u} = [1, \alpha, 0]$, $\mathbf{v} = [3, -2, 9]$ and $\mathbf{w} = [-1, \beta, -3]$. Find the values of α and β so that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal and $\mathbf{v}$ and $\mathbf{w}$ are parallel.

(a) $\alpha = \frac{3}{2}, \beta = \frac{2}{3}$ (b) $\alpha = \frac{2}{3}, \beta = \frac{3}{2}$ (c) $\alpha = \frac{3}{2}, \beta = -\frac{2}{3}$

(d) $\alpha = -\frac{3}{2}, \beta = \frac{2}{3}$ (e) $\alpha = -\frac{2}{3}, \beta = \frac{3}{2}$

2. Which one of the following augmented matrices is in row echelon form?

(a) $\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$ (b) $\left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & 0 & 1 \end{array}\right]$ (c) $\left[\begin{array}{ccc|c} 1 & -3 & 0 & -1 \\ 1 & 6 & 1 & 4 \\ 0 & 0 & 0 & 2 \end{array}\right]$

(d) $\left[\begin{array}{ccc|c} 0 & 0 & 1 & 2 \\ 1 & 1 & 0 & 1 \end{array}\right]$ (e) None of the above.

3. What is the area of the parallelogram inscribed by the vectors $[2, -1, 3]$ and $[-1, 1, 0]$?

(a) 19 (b) 7 (c) $\sqrt{7}$ (d) $\sqrt{19}$ (e) $2\sqrt{7}$

4. What is the distance of the point $P = (-1, -2, -3)$ from the plane $x + 2y + 3z = -7$?

(a) 7 (b) 21 (c) $\sqrt{7}$ (d) $\frac{\sqrt{14}}{2}$ (e) $\frac{\sqrt{21}}{3}$

5. Consider the following system of equations:

$$\begin{array}{rcrcrcrcl} x & + & y & + & z & - & w & = & 0 \\ & & -y & + & 2z & - & w & = & 0 \\ 2x & & & + & 6z & - & 4w & = & 0 \end{array}$$

Which one of the following statements about this system is true?

(a) There is a unique solution.

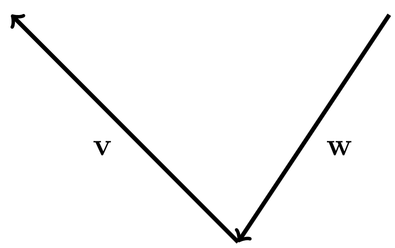
(b) The general solution is expressed using exactly 1 parameter.

(c) The general solution is expressed using exactly 2 parameters.

(d) The general solution is expressed using 3 or more parameters.

(e) There is no solution.

6. Suppose $\mathbf{v}$ and $\mathbf{w}$ are two non-zero vectors lying in this page:



Which of the following is true?

(a) $\mathbf{v}$ and $\mathbf{v} \times \mathbf{w}$ are parallel.

(b) $(\mathbf{v} \times \mathbf{w}) \cdot \mathbf{v}$ is a non-zero scalar.

(c) $(\mathbf{v} \times \mathbf{w}) \times \mathbf{v}$ is perpendicular to both $\mathbf{v}$ and $\mathbf{w}$.

(d) $\mathbf{v} \times \mathbf{w}$ points upwards, towards the ceiling.

(e) $(\mathbf{w} \times \mathbf{v}) \times (\mathbf{v} \times \mathbf{w})$ is parallel to $\mathbf{v}$ but not $\mathbf{w}$.

7. The two lines given by the respective parametric equations

$$\begin{array}{rcl} x &=& 3 + t \\ y &=& -5 + 2t \quad t \in \mathbb{R} \\ z &=& -5 - t \end{array} \qquad \text{and} \qquad \begin{array}{rcl} x &=& -3 - 2s \\ y &=& -2 - 4s \quad s \in \mathbb{R} \\ z &=& 1 + 2s \end{array}$$

(a) do not intersect.

(b) intersect at the point $(7, 3, -9)$.

(c) intersect at the point $(-2, -15, 0)$.

(d) intersect at the point $(-3, -2, 1)$.

(e) coincide.

8. The cosine of the angle between the vectors $[1, 2, 2]$ and $[3, 0, 4]$ is equal to

(a) $\dfrac{14}{225}$ (b) $\dfrac{225}{11}$ (c) $\dfrac{11}{\sqrt{15}}$ (d) $\dfrac{11}{15}$ (e) $\dfrac{15}{11}$

9. Consider the three planes with equations

$$\begin{array}{rrcrcrcl} \mathcal{P}_1 : & x & + & 2y & - & z & = & 1, \\ \mathcal{P}_2 : & -2x & - & 5y & + & 2z & = & 1, \\ \mathcal{P}_3 : & -4x & - & 8y & + & 4z & = & 10. \end{array}$$

Which of the following is true?

(a) None of the planes are parallel to each other.

(b) $\mathcal{P}_1$ and $\mathcal{P}_2$ are parallel to each other but not parallel to $\mathcal{P}_3$.

(c) $\mathcal{P}_1$ and $\mathcal{P}_3$ are parallel to each other but not parallel to $\mathcal{P}_2$.

(d) $\mathcal{P}_2$ and $\mathcal{P}_3$ are parallel to each other but not parallel to $\mathcal{P}_1$.

(e) All of the planes are parallel to each other.

10. Which one of the following sequences of row operations, when applied to the matrix $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$, produces the matrix $\begin{bmatrix} d-a & e-b & f-c \\ 3a & 3b & 3c \end{bmatrix}$?

(a) First $R_1 \to R_1 - R_2$, then $R_2 \to 3R_2$, then $R_1 \leftrightarrow R_2$.

(b) First $R_1 \leftrightarrow R_2$, then $R_1 \to 3R_1$, then $R_1 \to R_1 - R_2$.

(c) First $R_2 \to R_2 - R_1$, then $R_1 \leftrightarrow R_2$, then $R_1 \to 3R_1$.

(d) First $R_1 \to 3R_1$, then $R_1 \leftrightarrow R_2$, then $R_1 \to R_1 - R_2$.

(e) First $R_1 \leftrightarrow R_2$, then $R_1 \to R_1 - R_2$, then $R_2 \to 3R_2$.

11. Which one of the following is true for all linearly independent vectors $\mathbf{v}$ and $\mathbf{w}$?

(a) $\mathbf{v}$ and $\mathbf{w}$ are orthogonal.

(b) If $a\mathbf{v} + b\mathbf{w} = c\mathbf{v} + d\mathbf{w}$, where $a, b, c, d \in \mathbb{R}$, then $a = c$ and $b = d$.

(c) $\mathbf{v}$ and $\mathbf{w}$ are parallel.

(d) The vectors $\mathbf{v} + \mathbf{w}$ and $\mathbf{v} - \mathbf{w}$ are linearly dependent.

(e) None of the above.

12. Which one of the following statements is true?

(a) There are three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ such that $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent.

(b) Any set of three vectors from $\mathbb{R}^2$ must span $\mathbb{R}^2$.

(c) If $\text{span}(\mathbf{u}, \mathbf{v}) = \mathbb{R}^2$, then $\{\mathbf{u}, \mathbf{v}\}$ is a basis for $\mathbb{R}^2$.

(d) The set $\{\mathbf{u}, \mathbf{v}, \mathbf{0}\}$ is a basis for $\mathbb{R}^2$ only if $\{\mathbf{u}, \mathbf{v}\}$ is a basis for $\mathbb{R}^2$.

(e) For any three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$, there is a subset of $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ that is a basis for $\mathbb{R}^2$.

13. Which one of the following statements is true, given that A is a matrix of size 4×4, B is a matrix of size 3×4, and C is a matrix of size 1×3?

(a) $A^3B^T - B^TBA$ is a 4×4 matrix.

(b) $BA + B^2$ is a 3×4 matrix.

(c) CB is a column vector.

(d) BAB is defined.

(e) $(CBA)^T$ is a 4×1 matrix.

14. If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$, which one of the following is true?

(a) A is not invertible.

(b) A is invertible and $A^{-1} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix}$.

(c) A is invertible and $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$.

(d) A is invertible and $A^{-1} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 0 \end{bmatrix}$.

(e) None of the above.

15. Let $A = P\,D_1\,P^{-1}$, and $B = P\,D_2\,P^{-1}$ where $D_1 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $D_2 = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$, and $P = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. Then $(AB)^5$ is

(a) $\begin{bmatrix} 2^5 & 3^5 - 2^5 \\ 0 & 3^5 \end{bmatrix}$. (b) $\begin{bmatrix} -3^5 & 2^5 - 3^5 \\ 0 & -2^5 \end{bmatrix}$. (c) $\begin{bmatrix} 3^5 & 0 \\ 0 & 2^5 \end{bmatrix}$.

(d) $\begin{bmatrix} 2^5 & 1 \\ 0 & 3^5 \end{bmatrix}$. (e) $\begin{bmatrix} -2^5 & -3^5 + 2^5 \\ 0 & -3^5 \end{bmatrix}$.

16. Let A and B be $n \times n$ diagonalisable matrices. Which one of the following is always true?

(a) $A + B$ is diagonalisable.

(b) AB is diagonalisable.

(c) $AB = BA$.

(d) A^2 and B^2 are diagonalisable.

(e) none of the above.

17. Let B be a 4×4 matrix and suppose that $\det(B) = 2$. Then $\det\left(-\frac{1}{\sqrt{2}}B\right)$ is

(a) $1/2$ (b) $-1/\sqrt{2}$ (c) -1 (d) $-\sqrt{2}$ (e) -2

18. Which one of the following is true for all $n \times n$ matrices A and B?

(a) $(A + B)^2 = A^2 + 2AB + B^2$

(b) if $A^2 = B^2$ then $A = \pm B$

(c) if $AB = 0$ then $A = 0$ or $B = 0$

(d) if A is invertible then AB is invertible

(e) if A is invertible and AB is invertible then B is invertible

19. Let P be a transition matrix of a Markov chain on n states. Which of the following is NOT necessarily true.

(a) P is an $n \times n$ matrix.

(b) P^2 is a transition matrix for a Markov chain.

(c) If P is invertible, then P^{-1} is a transition matrix for a Markov chain

(d) If Q is another transition matrix for a Markov chain on n states, then $\frac{1}{2}(P+Q)$ is a transition matrix for a Markov chain

(e) If Q is another transition matrix for a Markov chain on n states, then PQ is a transition matrix for a Markov chain.

20. Suppose a 3×3 matrix A has 3 distinct eigenvalues λ_1, λ_2 and λ_3. Which one of the following is NOT necessarily true?

(a) The characteristic polynomial of A has 3 distinct roots.

(b) $\det(A) = \lambda_1\lambda_2\lambda_3$.

(c) A is invertible.

(d) There is a 3×3 invertible matrix P so that $PAP^{-1} = \begin{bmatrix} \lambda_2 & 0 & 0 \\ 0 & \lambda_3 & 0 \\ 0 & 0 & \lambda_1 \end{bmatrix}$.

(e) If B is any 3×3 invertible matrix then BAB^{-1} has eigenvalues λ_1, λ_2 and λ_3.

This blank page may be used for rough working; it will not be marked.

End of Multiple Choice Section.

Make sure that your answers are entered on the Multiple Choice Answer Sheet.

The Extended Answer Section begins on the next page.

Extended Answer Section

There are **three** *questions in this section, each with a number of parts. Write your answers in the space provided. If you need more space there are extra pages at the end of the examination paper.*

1. (a) Let $a, b \in \mathbb{R}$. Find all the values of a and b such that the following system of linear equations is inconsistent.

$$\begin{array}{rcrcrcl} x & - & y & + & 2z & = & 4 \\ x & + & ay & - & 5z & = & b \\ 3x & - & 3y & - & z & = & 5 \end{array}$$

(b) Let V be the subspace of $\mathbb{R}^5$ given by

$$V = \operatorname{span}\big([1, 2, -1, 1, 2], [1, 3, 0, 0, 0], [2, 5, -1, 1, 2]\big).$$

Compute the dimension of V, and find a basis for V.

(c) (*i*) Find three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ such that $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent, and each pair $\{\mathbf{u}, \mathbf{v}\}$, $\{\mathbf{u}, \mathbf{w}\}$ and $\{\mathbf{v}, \mathbf{w}\}$ is linearly independent. Justify your answer.

(*ii*) For $n \geq 3$, does there exist three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ such that $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent, and each pair $\{\mathbf{u}, \mathbf{v}\}$, $\{\mathbf{u}, \mathbf{w}\}$ and $\{\mathbf{v}, \mathbf{w}\}$ is linearly independent? Justify your answer.

2. (a) Find the inverse of the matrix $\begin{bmatrix} 4 & 2 & 3 \\ -1 & -1 & -1 \\ -1 & 0 & -1 \end{bmatrix}$.

(b) Use your answer to (a), or another valid method, to solve the system of equations

$$\begin{array}{rcrcrcrcr} 4x & + & 2y & + & 3w & & & = & 1 \\ -x & - & y & - & w & & & = & 2 \\ -x & & & - & w & & & = & -1 \\ 5x & + & y & + & w & + & 2z & = & 3. \end{array}$$

(c) Let $A = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}$.

(i) Find the eigenvalues of A and their corresponding eigenspaces.

(*ii*) Use *(i)*, to find a formula for $A^n \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, for an integer $n \geq 1$.

3. (a) Let

$$P = \begin{bmatrix} 1/2 & 1 & 0 \\ 0 & 0 & 1/3 \\ 1/2 & 0 & 2/3 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_0 = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix},$$

where P is the transition matrix for a Markov chain with three states, and $\mathbf{x}_0$ is the initial state vector for the population.

(i) Compute $\mathbf{x_1}$ and $\mathbf{x_2}$.

(ii) What is the probability of moving from state 2 to state 3 in 2 transitions?

(*iii*) Find the steady state vector.

(b) Let A be an $m \times n$ matrix.

(*i*) Show that $\mathrm{null}(A) \subseteq \mathrm{null}(A^T A)$.

(*ii*) Show that $\mathrm{null}(A) = \mathrm{null}(A^T A)$ by showing that $\mathrm{null}(A^T A) \subseteq \mathrm{null}(A)$, and using part *(i)*.

(Hint: To show that $\mathrm{null}(A^T A) \subseteq \mathrm{null}(A)$, recall that for any vector $\mathbf{x} \in \mathbb{R}^n$, we have $(A\mathbf{x})^T A\mathbf{x} = (A\mathbf{x}) \cdot (A\mathbf{x})$.)

There are no more questions.

More space is available on the next page.

This blank page may be used if you need more space for your answers.

End of Extended Answer Section.

End of Examination

Write your SID here ⟶

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MATH1002 Linear Algebra

Code your SID into the columns below each digit, by filling in the appropriate oval.

0	⬭	⬭	⬭	⬭	⬭	⬭	⬭	⬭	⬭	**0**
1	⬭	⬭	⬭	⬭	⬭	⬭	⬭	⬭	⬭	**1**
2	⬭	⬭	⬭	⬭	⬭	⬭	⬭	⬭	⬭	**2**
3	⬭	⬭	⬭	⬭	⬭	⬭	⬭	⬭	⬭	**3**
4	⬭	⬭	⬭	⬭	⬭	⬭	⬭	⬭	⬭	**4**
5	⬭	⬭	⬭	⬭	⬭	⬭	⬭	⬭	⬭	**5**
6	⬭	⬭	⬭	⬭	⬭	⬭	⬭	⬭	⬭	**6**
7	⬭	⬭	⬭	⬭	⬭	⬭	⬭	⬭	⬭	**7**
8	⬭	⬭	⬭	⬭	⬭	⬭	⬭	⬭	⬭	**8**
9	⬭	⬭	⬭	⬭	⬭	⬭	⬭	⬭	⬭	**9**

Family Name: ....

Other Names: ....

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Answers ⟶

Attempt every question. You will **not** be awarded negative marks for incorrect answers.

Fill in exactly one oval per question.

If you make a mistake, draw a cross (X) through any mistakenly filled in oval(s) and then fill in your intended oval.

An answer which contains two or more filled in (and uncrossed) ovals will be awarded no marks.

	a	**b**	**c**	**d**	**e**		**a**	**b**	**c**	**d**	**e**
Q1	⬭	⬭	⬭	⬭	⬭	**Q11**	⬭	⬭	⬭	⬭	⬭
Q2	⬭	⬭	⬭	⬭	⬭	**Q12**	⬭	⬭	⬭	⬭	⬭
Q3	⬭	⬭	⬭	⬭	⬭	**Q13**	⬭	⬭	⬭	⬭	⬭
Q4	⬭	⬭	⬭	⬭	⬭	**Q14**	⬭	⬭	⬭	⬭	⬭
Q5	⬭	⬭	⬭	⬭	⬭	**Q15**	⬭	⬭	⬭	⬭	⬭
Q6	⬭	⬭	⬭	⬭	⬭	**Q16**	⬭	⬭	⬭	⬭	⬭
Q7	⬭	⬭	⬭	⬭	⬭	**Q17**	⬭	⬭	⬭	⬭	⬭
Q8	⬭	⬭	⬭	⬭	⬭	**Q18**	⬭	⬭	⬭	⬭	⬭
Q9	⬭	⬭	⬭	⬭	⬭	**Q19**	⬭	⬭	⬭	⬭	⬭
Q10	⬭	⬭	⬭	⬭	⬭	**Q20**	⬭	⬭	⬭	⬭	⬭