Extended Answer Section

There are **three** *questions in this section, each with a number of parts. Write your answers in the space provided below each part. There is extra space at the end of the paper.*

1. (a) The number x in a population satisfies the logistic equation

$$\frac{dx}{dt} = 2x(10 - x),$$

where t is the time in years. If the population is initially 2, find the time it takes to increase to 4. Leave your answer in exact form.

$$\int \frac{1}{x(10-x)}\,dx = \int 2\,dt$$

$$\frac{1}{10}\int \frac{1}{10-x} + \frac{1}{x}\,dx = \int 2\,dt$$

$$-\ln|10-x| + \ln|x| = 20t + C$$

$$\ln\left|\frac{x}{10-x}\right| = 20t + C$$

$$\frac{x}{10-x} = Ae^{20t}$$

$$x = 10Ae^{20t} - Axe^{20t}$$

$$x(1 + Ae^{20t}) = 10Ae^{20t}$$

$$x(t) = \frac{10Ae^{20t}}{1 + Ae^{20t}}$$

$$x(0) = 2 \Rightarrow A = \frac{1}{4}$$

$$x(t) = \frac{10e^{20t}}{4 + e^{20t}}$$

$$x(t_0) = 4$$

$$\Rightarrow 10e^{20t_0} = 16 + 4e^{20t_0}$$

$$e^{20t_0} = \frac{8}{3}$$

$$t_0 = \frac{\ln 8 - \ln 3}{20}$$

(b) The velocity v of a particle follows the first-order differential equation

$$\frac{dv}{dt} = v + e^t,$$

where t is time.

(i) If at $t = 0$ the velocity is $v = -3$ determine whether the particle is experiencing a positive or negative acceleration (change in velocity $v(t)$) at $t = 0$? Justify your answer.

$$\frac{dv}{dt} = -3 + e^0 = -2 \implies \underline{\text{negative acceleration}}$$

$\underline{\text{at } t=0}$

(ii) Compute how the velocity v depends on time t by finding the general solution of the equation.

$$\frac{dv}{dt} - v = e^t \implies \text{linear first order eq.}$$

$p(t) = -1$; $q(t) = e^t$

$$r(t) = e^{\int -1\,dt} = e^{-t}$$

$$v(t) = e^t \left(\int 1\,dt + C \right)$$

$$\underline{\underline{v(t) = e^t (t + C)}}$$

(*iii*) Find the particular solution of the equation with initial condition $v(0) = -3$.

$v(0) = e^0(0 + C) = -3 \implies C = -3$

$\underline{\underline{v(t) = e^t(t - 3)}}$

(*iv*) In an experiment designed to track the particle described in part (*iii*), two detectors are placed at a distance 10 of the particle's initial position (one in the positive direction and one in the negative direction). Determine the detectors, if any, at which the particle will be tracked. Detectors track particles that cross their position but do not affect their movement.

Position x :

$$x(t) = \int e^t (t-3)\,dt$$

$$= e^t (t-3) - \int e^t \, dt$$

$$= e^t (t-4) + C$$

$x(0) = 0 \quad \Rightarrow \quad C = 4$

Critical point at $v(t) \overset{!}{=} 0 \quad \Rightarrow \quad t = 3$

$$x(3) = -e^3 + 4 < -10$$

But $\lim_{t \to \infty} x(t) = +\infty$

$\Rightarrow \quad x(0) = 0 \ ; \quad x(3) < -10 \ ; \quad \lim_{t \to \infty} x(t) = +\infty$

$\Rightarrow$ Both detectors will track the particles

2. (a) A tank contains 1000 litres of water with 20 kg of dissolved salt. Pure water flows into the tank at a rate of 20 L/min, the solution drains at the same rate. You can assume the solution in the tank is well mixed.

(i) Write down the differential equation for the mass m of salt in the tank after t minutes.

$$\frac{dm}{dt} = -\frac{1}{50}m$$

The tank loses $20/1000 = \frac{1}{50}$ of its overall salt content per minute (no salt is added).

(ii) Solve the equation and find the amount of salt after 10 minutes.

$$\Rightarrow \int \frac{1}{m}\,dm = \int -\frac{1}{50}\,dt$$

$$\ln|m| = -\frac{t}{50} + C$$

$$m(t) = Ae^{-t/50} \qquad \text{where} \quad A = e^C$$

Initial condition: $m(0) = 20$; $A = 20$

$$m(t) = 20e^{-t/50}$$

After 10 minutes:

$$m(10) = 2e^{-\frac{1}{5}}$$

(*iii*) Now suppose the pure water inflow is cut to 10 L/min, but the tank is still drained at the rate of 20 L/min. Set up the differential equation for the mass m of salt at time t.

$$\frac{dm}{dt} = -\frac{m}{1000-10t}\cdot 20$$

(*iv*) Solve the equation from the part (*iii*) with the initial condition $m(0) = 20$ kg and find the concentration of salt in the water at the time when the tank is drained completely.

$$\int \frac{1}{m}\,dm = \int -\frac{2}{100-t}\,dt$$

$$\ln|m| = 2\ln|100-t| + C$$

$$m(t) = A(100-t)^2$$

$$m(0) = A\,10.000 = 20 \quad ; \quad A = \frac{2}{1000}$$

$$m(t) = \frac{2}{1000}(100-t)^2$$

Tank is drained at $t = 100$ $\Rightarrow$ $m(100) = 0$

$\Rightarrow$ concentration is 0.

(b) Let $F(x, y) = \ln(2x^2 - y)$.

(i) Calculate the first order partial derivatives of $F(x, y)$ at the point $(1, 0)$.

$$F_x(x,y) = \frac{4x}{2x^2-y} \quad ; \quad F_y(x,y) = \frac{-1}{2x^2-y}$$

$$\underline{\underline{F_x(1,0) = 2}} \quad ; \quad \underline{\underline{F_y(1,0) = -\frac{1}{2}}}$$

(ii) Find the equation of the tangent plane to the surface $z = F(x, y)$ at the point $P = (1, 0, \ln 2)$.

$$z - F(1,0) = F_x(1,0)(x-1) + F_y(1,0)(y-0)$$

$$\Rightarrow \quad z - \ln(2) = 2(x-1) - \frac{1}{2}y$$

$$\underline{\underline{-2x + \frac{1}{2}y + z = \ln(2) - 2}}$$

3. (a) Let $g(x, y) = ye^{-x}$.

(*i*) Calculate the gradient vector $\nabla g(x, y)$. Hence find $\nabla g(0, 2)$.

$g_x(x,y) = -ye^{-x}$; $g_y(x,y) = e^{-x}$

$\underline{\underline{\nabla g(x,y) = -ye^{-x}\mathbf{i} + e^{-x}\mathbf{j}}}$; $\underline{\underline{\nabla g(0,2) = -2\mathbf{i} + \mathbf{j}}}$

(*ii*) Find the directional derivative of $g(x, y)$ at the point $(0, 2)$ in the direction $\dfrac{\mathbf{i}}{\sqrt{2}} + \dfrac{\mathbf{j}}{\sqrt{2}}$.

$D_{\frac{i}{\sqrt{2}}+\frac{j}{\sqrt{2}}}\, g(0,2) = (-2i + j)\left(\frac{i}{\sqrt{2}} + \frac{j}{\sqrt{2}}\right) = \underline{\underline{-\frac{1}{\sqrt{2}}}}$

(b) (*i*) Write down the differential of $h(x, y) = x^{1/3}y$. Find the expression for this differential when $x = 8$ and $y = 3$.

$h_x(x,y) = \frac{1}{3}x^{-2/3}y \quad ; \quad h_y(x,y) = x^{1/3}$

$df = \frac{1}{3}x^{-2/3}y\,dx + x^{1/3}\,dy$

(*ii*) Use part (*i*) to find an approximation for $2.7 \times \sqrt[3]{8.2}$.

We know: $h(8,3) = 8^{1/3} \cdot 3 = 6$

$h_x(8,3) = \frac{1}{3} \frac{3}{8^{2/3}} = \frac{1}{4}$

$h_y(8,3) = 8^{1/3} = 2$

$dx = 0.2 \quad ; \quad dy = -0.3$

$\Rightarrow \quad df = \frac{1}{4} \cdot 0.2 - 2 \cdot 0.3 = 0.05 - 0.6 = -0.55$

$\Rightarrow \quad 2.7 \cdot \sqrt[3]{8.2} = f(2.7, 8.2) \approx 6 - 0.55 = 5.45$

(c) Let $f(x,y) = \dfrac{1}{x\,y}$. Find the point (x,y) such that $0 \le x \le 2$, $0 \le y \le 2$ and the distance from the origin $(0,0,0)$ to the point $(x,y,f(x,y))$ on the surface $z = f(x,y)$ is minimal.

It suffices to look at the square of the distance.

$\text{dist}(x,y) = x^2 + y^2 + \dfrac{1}{x^2y^2}$

$\text{dist}_x(x,y) = 2x - \dfrac{2}{x^3y^2}$

$\text{dist}_y(x,y) = 2y - \dfrac{2}{x^2y^3}$

$\text{dist}_x(x,y) = 0 \iff x^4y^2 = 1$

$\text{dist}_y(x,y) = 0 \iff x^2y^4 = 1$

$\Rightarrow \quad x,y \in \{\pm 1\}$; but $0 \le x \le 2$; $0 \le y \le 2$

$\Rightarrow \quad x = y = 1$ is the only critical point in $0 \le x \le 2$; $0 \le y \le 2$

Higher order derivatives:

$\text{dist}_{xx}(x,y) = 2 + \dfrac{6}{x^4y^2}$; $\text{dist}_{xy}(x,y) = \dfrac{4}{x^3y^3}$

$\text{dist}_{yx}(x,y) = \dfrac{4}{x^3y^3}$; $\text{dist}_{yy}(x,y) = 2 + \dfrac{6}{x^2y^4}$

$\text{dist}_{xx}(1,1) = 8 = \text{dist}_{yy}(1,1)$; $\text{dist}_{xy}(1,1) = 4$

$D(1,1) = 8 \cdot 8 - 4^2 = 64 - 16 = \underline{\underline{48 > 0}}$

$\Rightarrow$ $(1,1)$ is a local minimum (dist. to $(0,0,0)$ is $\sqrt{3}$)

Table of Standard Integrals

1. $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$

2. $\displaystyle\int \frac{dx}{x} = \ln|x| + C$

3. $\displaystyle\int e^x\,dx = e^x + C$

4. $\displaystyle\int \sin x\,dx = -\cos x + C$

5. $\displaystyle\int \cos x\,dx = \sin x + C$

6. $\displaystyle\int \tan x\,dx = -\ln\bigl|\cos x\bigr| + C$

7. $\displaystyle\int \cot x\,dx = \ln\bigl|\sin x\bigr| + C$

8. $\displaystyle\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\Bigl(\frac{x}{a}\Bigr) + C$

9. $\displaystyle\int \sec^2 x\,dx = \tan x + C$

10. $\displaystyle\int \operatorname{cosec}^2 x\,dx = -\cot x + C$

11. $\displaystyle\int \sec x\,dx = \ln\bigl|\sec x + \tan x\bigr| + C$

12. $\displaystyle\int \operatorname{cosec} x\,dx = \ln\bigl|\operatorname{cosec} x - \cot x\bigr| + C$

13. $\displaystyle\int \sinh x\,dx = \cosh x + C$

14. $\displaystyle\int \cosh x\,dx = \sinh x + C$

15. $\displaystyle\int \tanh x\,dx = \ln\cosh x + C$

16. $\displaystyle\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\Bigl(\frac{x}{a}\Bigr) + C \quad (|x| < a)$

17. $\displaystyle\int \frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1}\Bigl(\frac{x}{a}\Bigr) + C = \ln\Bigl(x + \sqrt{x^2+a^2}\Bigr) + C'$

18. $\displaystyle\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\Bigl(\frac{x}{a}\Bigr) + C = \ln\Bigl(x + \sqrt{x^2-a^2}\Bigr) + C' \quad (x > a)$

Linearity: $\displaystyle\int \bigl(\lambda f(x) + \mu g(x)\bigr)\,dx = \lambda\int f(x)\,dx + \mu\int g(x)\,dx$

Integration by substitution: $\displaystyle\int f(u(x))\frac{du}{dx}\,dx = \int f(u)\,du$

Integration by parts: $\displaystyle\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$

End of Extended Answer Section

End of Examination