Solution 1.

(a)

We need to find $\lambda_1, \lambda_2, \lambda_3$ such that:

$$\lambda_1(1,0,3) + \lambda_2(2,1,8) + \lambda_3(1,-1,2) = (3,-5,4)$$

written in augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & -1 & -5 \\ 3 & 8 & 2 & 4 \end{array}\right] \underset{\sim}{R_3=3R_1-R_3} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & -1 & -5 \\ 0 & -2 & 1 & 5 \end{array}\right] \underset{\sim}{R_3=2R_1+R_3} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & -1 & -5 \\ 0 & 0 & -1 & -5 \end{array}\right]$$
$$\underset{\sim}{R_2=R_2-R_3} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -5 \end{array}\right] \underset{\sim}{R_1=R_1-2R_2+R_3, R_3=-R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

So that, we have:

$$\begin{cases} \lambda_1 = -2 \\ \lambda_2 = 0 \\ \lambda_3 = 5 \end{cases}$$

Thus,$[V]_B = (-2,0,5)$

(b)

We need to find $\lambda_1, \lambda_2, \lambda_3$ such that:

$$\lambda_1(1+t^2) + \lambda_2(t+t^2) + \lambda_3(1+2t+t^2) = (1+4t+7t^2)$$

which is

$$(\lambda_1+\lambda_3) + (\lambda_2+2\lambda_3)t + (\lambda_1+\lambda_2+\lambda_3)t^2 = (1+4t+7t^2)$$

written in augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 0 & 1 & 2 & 4 \\ 1 & 0 & 1 & 1 \end{array}\right] \underset{\sim}{R_3=R_3-R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 0 & 1 & 2 & 4 \\ 0 & -1 & 0 & -6 \end{array}\right] \underset{\sim}{R_3=R_3+R_2}$$
$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 2 & -2 \end{array}\right] \underset{\sim}{R_2=R_2-R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 2 & -2 \end{array}\right] \underset{\sim}{R_3=\frac{1}{2}R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & -1 \end{array}\right] \underset{\sim}{R_1=R_1-R_2, R_1=R_1-R_3}$$
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

So that, we have:

$$\begin{cases} \lambda_1 = 2 \\ \lambda_2 = 6 \\ \lambda_3 = -1 \end{cases}$$

Thus,$[V]_B = (2,6,-1)$

Solution 2.

$$\begin{bmatrix} 1 & 1 & 6 & 2 & 6 \\ 4 & 1 & 4 & 2 & 5 \\ 5 & 2 & 3 & 5 & 0 \\ 3 & 4 & 6 & 2 & 4 \\ 1 & 2 & 1 & 4 & 3 \end{bmatrix} \underset{\sim}{R_2{=}R_2{-}R_4{-}R_5, R_3{=}R_3{-}R_4{-}2R_5} \begin{bmatrix} 1 & 1 & 6 & 2 & 6 \\ 0 & -5 & -3 & -4 & -2 \\ 0 & -6 & -5 & -5 & -3 \\ 3 & 4 & 6 & 2 & 4 \\ 1 & 2 & 1 & 4 & 3 \end{bmatrix}$$

$$\underset{\sim}{R_4{=}R_4{-}3R_5, R_5{=}R_5{-}R_1} \begin{bmatrix} 1 & 1 & 6 & 2 & 6 \\ 0 & -5 & -3 & -4 & -2 \\ 0 & -6 & -5 & -5 & -3 \\ 0 & -2 & 3 & -10 & 2 \\ 0 & 1 & -5 & 2 & -3 \end{bmatrix} \underset{\sim}{R_4{=}R_2{+}R_4, R_5{=}R_3{-}R_5} \begin{bmatrix} 1 & 1 & 6 & 2 & 6 \\ 0 & 2 & -3 & -4 & -2 \\ 0 & -6 & -5 & -5 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\underset{\sim}{R_3{=}3R_2{+}R_3} \begin{bmatrix} 1 & 1 & 6 & 2 & 6 \\ 0 & 2 & -3 & -4 & -2 \\ 0 & 0 & 0 & -3 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \underset{\sim}{R_2{=}4R_2, R_3{=}2R_3} \begin{bmatrix} 1 & 1 & 6 & 2 & 6 \\ 0 & 1 & 2 & 5 & 6 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \underset{\sim}{R_2{=}R_2{-}5R_3} \begin{bmatrix} 1 & 1 & 6 & 2 & 6 \\ 0 & 1 & 2 & 0 & 5 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\underset{\sim}{R_1{=}R_1{-}R_2{-}2R_3} \begin{bmatrix} 1 & 0 & 4 & 0 & 2 \\ 0 & 1 & 2 & 0 & 5 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a)

According to "Rank-Nullity Theorem", the rank of a matrix is equal to the number of its non-zero rows.So that the rank of this Matrix over $\mathbf{Z}_7$ is 3, and the nullity is 2.

(b)

To find the basis of the Null space for the given system of linear equations, we express all other variables as linear combinations of free variables.

$$\begin{bmatrix} 1 & 0 & 4 & 0 & 2 \\ 0 & 1 & 2 & 0 & 5 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

From the matrix, we have:

- $x_1 = 3x_3 + 5x_5$
- $x_2 = 5x_3 + 2x_5$
- $x_4 = 4x_5$

Since x_3 and x_5 are free variables, we set them to 1 in turn, with all other free variables at 0, to find the basis vectors of the Null space.

- Setting $x_3 = 1$ and $x_5 = 0$, we substitute into the above equations to obtain one basis vector:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

- Setting $x_3 = 0$ and $x_5 = 1$, we substitute into the above equations to obtain another basis vector:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 0 \\ 4 \\ 1 \end{bmatrix}$$

Therefore, the basis of the Null space is:

$$\left\{ \begin{bmatrix} 3 \\ 5 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ 0 \\ 4 \\ 1 \end{bmatrix} \right\}$$

Solution 3. Assume that there is a linearly dependent subset $Y \subseteq X$. Since $Y = \{y_1, y_2, \ldots, y_m\}$ is a subset of $X = \{x_1, x_2, \ldots, x_k\}$, there exist scalars $c_1, c_2, \ldots, c_m$, not all zero, such that:

$$c_1y_1 + c_2y_2 + \cdots + c_my_m = 0$$

Each y_i in Y corresponds to some x_j in X. We extend this linear combination to include all elements of X by assigning a coefficient of 0 to the vectors in X that are not in Y. Hence, we have:

$$c_1x_{i_1} + c_2x_{i_2} + \cdots + c_mx_{i_m} + 0 \cdot x_{j_1} + 0 \cdot x_{j_2} + \cdots + 0 \cdot x_{j_{k-m}} = 0$$

where $x_{i1}, x_{i2}, \ldots, x_{im}$ are the vectors in Y, and $x_{j1}, x_{j2}, \ldots, x_{jk-m}$ are the vectors in X not included in Y.

Since X is linearly independent, the only solution to this equation is for all coefficients $c_1, c_2, \ldots, c_m$ and all zero coefficients to be exactly zero. This contradicts our assumption that the coefficients $c_1, c_2, \ldots, c_m$ are not all zero.

Therefore, the assumption is false, and any non-empty subset $Y \subseteq X$ must be linearly independent.

Solution 4.

(a)

- $S(p(x,y)) = 3 - 4y + yx + 2x^2$
- $S(p(x,y)) = y^2 - 4y + 3yx - 4x + x^2 - 1$

(b)

To prove that V_S is a subspace, we need to prove that it is non-empty, and closed under addition and scalar multiplication.

- non-emptiness

  The subspace V_S is non-empty because it contains at least the zero polynomial. The zero polynomial is symmetric under any variable swap since $S(0) = 0$. Therefore, $0 \in V_S$, confirming that V_S is non-empty.

- closure under Addition

  Let $p(x, y)$ and $q(x, y)$ be two polynomials in V_S. By definition, $S(p(x, y)) = p(x, y)$ and $S(q(x, y)) = q(x, y)$. Consider the polynomial $p(x, y) + q(x, y)$:

  $$S(p(x, y) + q(x, y)) = S(p(x, y)) + S(q(x, y)) = p(x, y) + q(x, y).$$

  Since $S(p(x, y) + q(x, y)) = p(x, y) + q(x, y)$, we conclude that $p(x, y) + q(x, y) \in V_S$. Hence, V_S is closed under addition.

- closure under Scalar Multiplication

  Let $p(x, y)$ be a polynomial in V_S and let c be any real scalar. Since $S(p(x, y)) = p(x, y)$, consider the scalar multiple $c \cdot p(x, y)$:

  $$S(c \cdot p(x, y)) = c \cdot S(p(x, y)) = c \cdot p(x, y).$$

  Since $S(c \cdot p(x, y)) = c \cdot p(x, y)$, it follows that $c \cdot p(x, y) \in V_S$. Thus, V_S is closed under scalar multiplication.

Therefore, V_S is a subspace of V as it is non-empty, and closed under both addition and scalar multiplication.

(c)

We first apply the variable swap S to $p(x, y)$:

$$S(p(x, y)) = a_0 + a_1 y + a_2 x + a_3 y^2 + a_4 yx + a_5 x^2$$

To ensure that $p(x, y)$ is symmetric, we require:

$$a_1 = a_2,$$
$$a_3 = a_5.$$

The symmetric polynomials can then be expressed as:

$$p(x, y) = a_0 + a_1(x + y) + a_3(x^2 + y^2) + a_4 xy$$

The basis for V_S therefore consists of the following polynomials:

$$\{1, x + y, x^2 + y^2, xy\}$$

Considering the equation $\lambda_1 + \lambda_2(x + y) + \lambda_3(x^2 + y^2) + \lambda_4 xy = 0$, where $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R}$, the only solution is $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$. This implies that the set is linearly independent.

Thus, the basis for V_S is $\{1, x + y, x^2 + y^2, xy\}$, and the dimension of this space is 4.