1. (a) Since A is diagonal, we have immediately that $e^{tA} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{bmatrix}$.

(b) The characteristic polynomial of A is

$$(\lambda - 1)(\lambda - 1) - 1 \;=\; \lambda^2 - 2\lambda \;=\; \lambda(\lambda - 2)\,,$$

so that the eigenvalues are 0 and 2. Observe that

$$A \;=\; \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad 2I - A \;=\; \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

from which it follows that $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ are eigenvectors for A corresponding to 0 and 2 respectively. Thus $A = PDP^{-1}$ where

$$P \;=\; \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad D \;=\; \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}.$$

Hence

$$\begin{aligned} e^{tA} \;&=\; Pe^{tD}P^{-1} \;=\; \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{2t} \end{bmatrix} \frac{1}{2} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \\ &=\; \frac{1}{2} \begin{bmatrix} -1 & e^{2t} \\ 1 & e^{2t} \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \;=\; \frac{1}{2} \begin{bmatrix} 1 + e^{2t} & -1 + e^{2t} \\ -1 + e^{2t} & 1 + e^{2t} \end{bmatrix}. \end{aligned}$$

(c) The characteristic polynomial of A is

$$(\lambda - 1)(\lambda - 2) - 6 \;=\; \lambda^2 - 3\lambda - 4 \;=\; (\lambda - 4)(\lambda + 1)\,,$$

so that the eigenvalues are 4 and -1. Observe that

$$4I - A \;=\; \begin{bmatrix} 3 & -3 \\ -2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad -I - A \;=\; \begin{bmatrix} -2 & -3 \\ -2 & -3 \end{bmatrix} \sim \begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix}$$

from which it follows that $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -3 \\ 2 \end{bmatrix}$ are eigenvectors for A corresponding to 4 and -1 respectively. Thus $A = PDP^{-1}$ where

$$P \;=\; \begin{bmatrix} 1 & -3 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad D \;=\; \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}.$$

Hence

$$\begin{aligned} e^{tA} \;&=\; Pe^{tD}P^{-1} \;=\; \begin{bmatrix} 1 & -3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} e^{4t} & 0 \\ 0 & e^{-t} \end{bmatrix} \frac{1}{5} \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix} \\ &=\; \frac{1}{5} \begin{bmatrix} e^{4t} & -3e^{-t} \\ e^{4t} & 2e^{-t} \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix} \;=\; \frac{1}{5} \begin{bmatrix} 2e^{4t} + 3e^{-t} & 3e^{4t} - 3e^{-t} \\ 2e^{4t} - 2e^{-t} & 3e^{4t} + 2e^{-t} \end{bmatrix}. \end{aligned}$$

(d) The characteristic polynomial of A is

$$(\lambda-5)(\lambda+4)+18 \;=\; \lambda^2-\lambda-2 \;=\; (\lambda-2)(\lambda+1)\,,$$

so that the eigenvalues are 2 and -1. Observe that

$$2I-A \;=\; \begin{bmatrix} -3 & 6 \\ -3 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad -I-A \;=\; \begin{bmatrix} -6 & 6 \\ -3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

from which it follows that $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ are eigenvectors for A corresponding to 2 and -1 respectively. Thus $A = PDP^{-1}$ where

$$P \;=\; \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad D \;=\; \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}.$$

Hence

$$\begin{aligned} e^{tA} \;&=\; Pe^{tD}P^{-1} \;=\; \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^{2t} & 0 \\ 0 & e^{-t} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \\ &=\; \begin{bmatrix} 2e^{2t} & e^{-t} \\ e^{2t} & e^{-t} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \;=\; \begin{bmatrix} 2e^{2t}-e^{-t} & -2e^{2t}+2e^{-t} \\ e^{2t}-e^{-t} & -e^{2t}+2e^{-t} \end{bmatrix}. \end{aligned}$$

2. (a) We have, using the answer to part (a) of the previous exercise,

$$\begin{bmatrix} x \\ y \end{bmatrix} \;=\; e^{tA}\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} \;=\; \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} \;=\; \begin{bmatrix} e^{-t} \\ 2e^{2t} \end{bmatrix},$$

so that $x = e^{-t}$ and $y = 2e^{2t}$.

(b) We have, using the answer to part (b) of the previous exercise,

$$\begin{bmatrix} x \\ y \end{bmatrix} \;=\; e^{tA}\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} \;=\; \frac{1}{2}\begin{bmatrix} 1+e^{2t} & -1+e^{2t} \\ -1+e^{2t} & 1+e^{2t} \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} \;=\; \frac{1}{2}\begin{bmatrix} -1+3e^{2t} \\ 1+3e^{2t} \end{bmatrix},$$

so that $x = \dfrac{3e^{2t}-1}{2}$ and $y = \dfrac{3e^{2t}+1}{2}$.

(c) We have, using the answer to part (c) of the previous exercise,

$$\begin{bmatrix} x \\ y \end{bmatrix} \;=\; e^{tA}\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} \;=\; \frac{1}{5}\begin{bmatrix} 2e^{4t}+3e^{-t} & 3e^{4t}-3e^{-t} \\ 2e^{4t}-2e^{-t} & 3e^{4t}+2e^{-t} \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} \;=\; \frac{1}{5}\begin{bmatrix} 8e^{4t}-3e^{-t} \\ 8e^{4t}+2e^{-t} \end{bmatrix},$$

so that $x = \dfrac{8e^{4t}-3e^{-t}}{5}$ and $y = \dfrac{8e^{4t}+2e^{-t}}{5}$.

(d) We have, using the answer to part (d) of the previous exercise,

$$\begin{bmatrix} x \\ y \end{bmatrix} \;=\; e^{tA}\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} \;=\; \begin{bmatrix} 2e^{2t}-e^{-t} & -2e^{2t}+2e^{-t} \\ e^{2t}-e^{-t} & -e^{2t}+2e^{-t} \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} \;=\; \begin{bmatrix} -2e^{2t}+3e^{-t} \\ -e^{2t}+3e^{-t} \end{bmatrix},$$

so that $x = 3e^{-t}-2e^{2t}$ and $y = 3e^{-t}-e^{2t}$.

3. Since D consists of two vectors and neither is a scalar multiple of the other, they are linearly independent. But $\mathbb{R}^2$ is two-dimensional, so D is a basis. Clearly $A = C = I$, the identity matrix. Observe that $[(1,1)]_B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $[(-1,0)]_B = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$, so we have

$$E = [\mathrm{id}]_B^D = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} .$$

Using the formula for the inverse of a 2×2 matrix, we have

$$E^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} .$$

Observe that

$$0(1,1) + (-1)(-1,0) = (1,0) \qquad \text{and} \qquad 1(1,1) + 1(-1,0) = (0,1) ,$$

recovering the elements of B, in order, so that $[(1,0)]_D = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ and $[(0,1)]_D = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, verifying indeed that $E^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} [(1,0)]_D & [(0,1)]_D \end{bmatrix} = [\mathrm{id}]_D^B$.

4. (a) Observe, taking images of elements of B, in order, that

$$f(1,0) = (1,3) , \quad f(0,1) = (2,-4) , \quad g(1,0) = (3,0) , \quad g(0,1) = (-1,2) ,$$

so we have, immediately, that

$$[f]_B^B = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix} \qquad \text{and} \qquad [g]_B^B = \begin{bmatrix} 3 & -1 \\ 0 & 2 \end{bmatrix} .$$

Observe, taking images of elements of D, in order, and then finding coefficients of linear combinations, by inspection, that

$$f(1,1) = (3,-1) = (-1)(1,1) + (-4)(-1,0) , \quad \text{so that} \quad [f(1,1)]_D = \begin{bmatrix} -1 \\ -4 \end{bmatrix} ,$$

$$f(-1,0) = (-1,-3) = (-3)(1,1) + (-2)(-1,0) , \quad \text{so that} \quad [f(-1,0)]_D = \begin{bmatrix} -3 \\ -2 \end{bmatrix} ,$$

$$g(1,1) = (2,2) = 2(1,1) + (0)(-1,0) , \quad \text{so that} \quad [g(1,1)]_D = \begin{bmatrix} 2 \\ 0 \end{bmatrix} ,$$

$$g(-1,0) = (-3,0) = (0)(1,1) + 3(-1,0) , \quad \text{so that} \quad [g(-1,0)]_D = \begin{bmatrix} 0 \\ 3 \end{bmatrix} .$$

Thus, we have

$$[f]_D^D = \begin{bmatrix} [f(1,1)]_D & [f(-1,0)]_D \end{bmatrix} = \begin{bmatrix} -1 & -3 \\ -4 & -2 \end{bmatrix}$$

and

$$[g]_D^D = \begin{bmatrix} [g(1,1)]_D & [g(-1,0)]_D \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} .$$

(b) Using the change of basis matrices calculated in the previous exercise, we have

$$\begin{aligned}[\mathrm{id}]^B_D[f]^B_B[\mathrm{id}]^D_B &= \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 3 & -4 \\ 2 & -6 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -3 \\ -4 & -2 \end{bmatrix} = [f]^D_D \ ,\end{aligned}$$

and

$$\begin{aligned}[\mathrm{id}]^B_D[g]^B_B[\mathrm{id}]^D_B &= \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 2 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = [g]^D_D \ ,\end{aligned}$$

as expected.

(c) We have

$$[h]^B_B = [f]^D_B = [f]^B_B[\mathrm{id}]^D_B = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -1 & -3 \end{bmatrix} ,$$

from which it follows that the rule for h is

$$h(x, y) = (3x - y, -x - 3y) \ .$$

Alternatively, and as a check, one may calculate

$$[h]^B_B = [f]^D_B = [\mathrm{id}]^D_B[f]^D_D = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & -3 \\ -4 & -2 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -1 & -3 \end{bmatrix} ,$$

confirming the above rule for h. We have

$$[k]^B_B = [f]^B_D = [\mathrm{id}]^B_D[f]^B_B = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 2 & -6 \end{bmatrix} ,$$

from which it follows that the rule for k is

$$k(x, y) = (3x - 4y, 2x - 6y) \ .$$

Alternatively, and as a check, one may calculate

$$[k]^B_B = [f]^B_D = [f]^D_D[\mathrm{id}]^B_D = \begin{bmatrix} -1 & -3 \\ -4 & -2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 2 & -6 \end{bmatrix} ,$$

confirming the above rule for k.

5. We place the coefficients of the polynomials in D as rows of a matrix and row reduce:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -2 \end{bmatrix}$$

We see that the matrix has rank 3, so the polynomials in D are linearly independent. Since the size of D coincides with the dimension of $\mathbb{P}$, which is 3, we know that D is a basis. We have

$$[1+x^2]_B \;=\; \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} , \quad [x+2x^2]_B \;=\; \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} , \quad [1+2x+3x^2]_B \;=\; \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} ,$$

so that

$$E \;=\; [\mathrm{id}]_B^D \;=\; \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix} .$$

Finding E^{-1} in the usual way:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 1 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \end{array}\right] \sim \left[\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -2 \end{bmatrix}\right| \left.\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -2 & 1 \end{matrix}\right]$$

$$\sim \left[\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right| \left.\begin{matrix} \frac{1}{2} & -1 & \frac{1}{2} \\ -1 & -1 & 1 \\ \frac{1}{2} & 1 & -\frac{1}{2} \end{matrix}\right] ,$$

so that $E^{-1} = \begin{bmatrix} \frac{1}{2} & -1 & \frac{1}{2} \\ -1 & -1 & 1 \\ \frac{1}{2} & 1 & -\frac{1}{2} \end{bmatrix}$. Observe that the following linear combinations simplify:

$$\begin{aligned} \frac{1}{2}(1+x^2) + (-1)(x+2x^2) + \frac{1}{2}(1+2x+3x^2) \;&=\; 1 , \\ (-1)(1+x^2) + (-1)(x+2x^2) + (1)(1+2x+3x^2) \;&=\; x , \\ \frac{1}{2}(1+x^2) + (1)(x+2x^2) + \Big(-\frac{1}{2}\Big)(1+2x+3x^2) \;&=\; x^2 , \end{aligned}$$

recovering the elements of B, in order, so that $[1]_D = \begin{bmatrix} \frac{1}{2} \\ -1 \\ \frac{1}{2} \end{bmatrix}$, $[x]_D = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ and $[x^2]_D = \begin{bmatrix} \frac{1}{2} \\ 1 \\ -\frac{1}{2} \end{bmatrix}$, verifying indeed that

$$E^{-1} \;=\; \begin{bmatrix} \frac{1}{2} & -1 & \frac{1}{2} \\ -1 & -1 & 1 \\ \frac{1}{2} & 1 & -\frac{1}{2} \end{bmatrix} \;=\; \big[\; [1]_D \quad [x]_D \quad [x^2]_D \;\big] \;=\; [\mathrm{id}]_D^B .$$

6. By writing the elements of D as rows of a matrix and row reducing,

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} ,$$

we get a matrix of full rank, so the elements of D are linearly independent. Thus D forms a basis, as its size coincides with the dimension of $\mathbb{R}^3$, which is 3. Observe that $[(1,0,1)]_B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $[(1,1,0)]_B = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $[(1,1,1)]_B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ so we have

$$E = [\mathrm{id}]_B^D = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} .$$

Finding E^{-1} in the usual way, we have

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] ,$$

so that $E^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -1 & 1 & 1 \end{bmatrix}$. Observe that

$$\begin{aligned} 1(1,0,1) + 1(1,1,0) + (-1)(1,1,1) &= (1,0,0) , \\ (-1)(1,0,1) + 0(1,1,0) + 1(1,1,1) &= (0,1,0) , \\ 0(1,0,1) + (-1)(1,1,0) + 1(1,1,1) &= (0,0,1) , \end{aligned}$$

recovering the elements of B, giving $[(1,0,0)]_D = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, $[(0,1,0)]_D = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ and $[(0,0,1)]_D = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$, verifying that

$$E^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -1 & 1 & 1 \end{bmatrix} = \left[\ [(1,0,0)]_D \quad [(0,1,0)]_D \quad [(0,0,1)]_D\ \right] = [\mathrm{id}]_D^B .$$

7. (a) Since A is diagonal, we have immediately that $e^{tA} = \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{3t} \end{bmatrix}$.

(b) The characteristic polynomial of A is

$$\begin{vmatrix} \lambda & -1 & 1 \\ -1 & \lambda & -1 \\ -1 & -1 & \lambda \end{vmatrix} = \begin{vmatrix} \lambda & 0 & 1 \\ -1 & \lambda-1 & -1 \\ -1 & \lambda-1 & \lambda \end{vmatrix} = \begin{vmatrix} \lambda & 0 & 1 \\ -1 & \lambda-1 & -1 \\ 0 & 0 & \lambda+1 \end{vmatrix}$$
$$= (\lambda+1)\begin{vmatrix} \lambda & 0 \\ -1 & \lambda-1 \end{vmatrix} = \lambda(\lambda+1)(\lambda-1) ,$$

so that the eigenvalues are 0, 1 and -1. Observe that

$$A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} , \quad I-A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} ,$$

$$-I - A = \begin{bmatrix} -1 & -1 & 1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

from which it follows that $\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ are eigenvectors for A corresponding to 0, 1 and -1 respectively. Thus $A = PDP^{-1}$ where

$$P = \begin{bmatrix} -1 & 0 & -1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Observe that

$$\left[\begin{array}{ccc|ccc} -1 & 0 & -1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{array}\right]$$

so that $P^{-1} = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$ and we have

$$\begin{aligned} e^{tA} &= Pe^{tD}P^{-1} = \begin{bmatrix} -1 & 0 & -1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{-t} \end{bmatrix} \begin{bmatrix} -1 & -1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 0 & -e^{-t} \\ 1 & e^t & e^{-t} \\ 1 & e^t & 0 \end{bmatrix} \begin{bmatrix} -1 & -1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 - e^{-t} & -1 + e^{-t} \\ -1 + e^t & -1 + e^t + e^{-t} & 1 - e^{-t} \\ -1 + e^t & -1 + e^t & 1 \end{bmatrix}. \end{aligned}$$

(c) The characteristic polynomial of A is

$$\begin{aligned} \begin{vmatrix} \lambda - 1 & -1 & -2 \\ 0 & \lambda + 1 & 0 \\ -2 & -1 & \lambda - 1 \end{vmatrix} &= (\lambda + 1) \begin{vmatrix} \lambda - 1 & -2 \\ -2 & \lambda - 1 \end{vmatrix} = (\lambda + 1)\big((\lambda - 1)^2 - 4\big) \\ &= (\lambda + 1)(\lambda^2 - 2\lambda - 3) = (\lambda + 1)^2(\lambda - 3), \end{aligned}$$

so that the eigenvalues are -1 and 3. Observe that

$$-I - A = \begin{bmatrix} -2 & -1 & -2 \\ 0 & 0 & 0 \\ -2 & -1 & -2 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

so the corresponding eigenspace is spanned by, for example, $\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ and

$$3I - A = \begin{bmatrix} 2 & -1 & -2 \\ 0 & 4 & 0 \\ -2 & -1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

so the corresponding eigenspace is spanned by, for example, $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. Thus we have $A = PDP^{-1}$ where

$$P = \begin{bmatrix} -1 & -1 & 1 \\ 2 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Observe that

$$\left[\begin{array}{rrr|rrr} -1 & -1 & 1 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 2 & 1 & \frac{1}{2} & 1 \end{array}\right] \sim \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & -\frac{1}{2} & -\frac{1}{4} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{2} \end{array}\right]$$

so that $P^{-1} = \frac{1}{4}\begin{bmatrix} 0 & 2 & 0 \\ -2 & -1 & 2 \\ 2 & 1 & 2 \end{bmatrix}$ and we have

$$\begin{aligned} e^{tA} &= Pe^{tD}P^{-1} = \begin{bmatrix} -1 & -1 & 1 \\ 2 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{3t} \end{bmatrix}\frac{1}{4}\begin{bmatrix} 0 & 2 & 0 \\ -2 & -1 & 2 \\ 2 & 1 & 2 \end{bmatrix} \\ &= \frac{1}{4}\begin{bmatrix} -e^{-t} & -e^{-t} & e^{3t} \\ 2e^{-t} & 0 & 0 \\ 0 & e^{-t} & e^{3t} \end{bmatrix}\begin{bmatrix} 0 & 2 & 0 \\ -2 & -1 & 2 \\ 2 & 1 & 2 \end{bmatrix} \\ &= \frac{1}{4}\begin{bmatrix} 2e^{-t}+2e^{3t} & -e^{-t}+e^{3t} & -2e^{-t}+2e^{3t} \\ 0 & 4e^{-t} & 0 \\ -2e^{-t}+2e^{3t} & -e^{-t}+e^{3t} & 2e^{-t}+2e^{3t} \end{bmatrix}. \end{aligned}$$

8. (a) We have, using the answer to part (a) of the previous exercise,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = e^{tA}\begin{bmatrix} x(0) \\ y(0) \\ z(0) \end{bmatrix} = \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{3t} \end{bmatrix}\begin{bmatrix} -1 \\ -4 \\ 2 \end{bmatrix} = \begin{bmatrix} -e^{-t} \\ -4e^{2t} \\ 2e^{3t} \end{bmatrix},$$

so that $x = -e^{-t}$, $y = -4e^{2t}$ and $z = 2e^{3t}$.

(b) We have, using the answer to part (b) of the previous exercise,

$$\begin{aligned} \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= e^{tA}\begin{bmatrix} x(0) \\ y(0) \\ z(0) \end{bmatrix} = \begin{bmatrix} 1 & 1-e^{-t} & -1+e^{-t} \\ -1+e^{t} & -1+e^{t}+e^{-t} & 1-e^{-t} \\ -1+e^{t} & -1+e^{t} & 1 \end{bmatrix}\begin{bmatrix} -1 \\ -4 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} -7+6e^{-t} \\ 7-5e^{t}-6e^{-t} \\ 7-5e^{t} \end{bmatrix}, \end{aligned}$$

so that $x = -7+6e^{-t}$, $y = 7-5e^{t}-6e^{-t}$ and $z = 7-5e^{t}$.

(c) We have, using the answer to part (c) of the previous exercise,

$$\begin{aligned} \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= e^{tA}\begin{bmatrix} x(0) \\ y(0) \\ z(0) \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 2e^{-t}+2e^{3t} & -e^{-t}+e^{3t} & -2e^{-t}+2e^{3t} \\ 0 & 4e^{-t} & 0 \\ -2e^{-t}+2e^{3t} & -e^{-t}+e^{3t} & 2e^{-t}+2e^{3t} \end{bmatrix}\begin{bmatrix} -1 \\ -4 \\ 2 \end{bmatrix} \\ &= \frac{1}{4}\begin{bmatrix} -2e^{-t}-2e^{3t} \\ -16e^{-t} \\ 10e^{-t}-2e^{3t} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -e^{-t}-e^{3t} \\ -8e^{-t} \\ 5e^{-t}-e^{3t} \end{bmatrix}, \end{aligned}$$

so that $x = \dfrac{-e^{-t} - e^{3t}}{2}$, $y = -4e^{-t}$ and $z = \dfrac{5e^{-t} - e^{3t}}{2}$.

9. (a) The rule for f is

$$f(x, y, z) \;=\; (2x + 5y - 3z, x - 4y + 7z)\ .$$

(b) Since $\mathbb{R}^2$ is two-dimensional, and neither vector in D is a scalar multiple of the other, so that D is linearly independent, it follows that D is a basis for $\mathbb{R}^2$. By writing the elements of B as rows of a matrix and row reducing,

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

we get a matrix of full rank, so the elements of B are linearly independent. Thus B forms a basis, as its size coincides with the dimension of $\mathbb{R}^3$, which is 3.

(c) We first calculate the images of the elements of B under f:

$$f(1,1,1) \;=\; (4,4)\,, \qquad f(1,1,0) \;=\; (7,-3)\,, \qquad f(1,0,0) \;=\; (2,1)\ .$$

To express these images as linear combinations of elements of B, we can simply write everything as columns and row reduce the following augmented matrix:

$$\left[\begin{array}{cc|ccc} 1 & 2 & 4 & 7 & 2 \\ 3 & 5 & 4 & -3 & 1 \end{array}\right] \sim \left[\begin{array}{cc|ccc} 1 & 2 & 4 & 7 & 2 \\ 0 & -1 & -8 & -24 & -5 \end{array}\right] \sim \left[\begin{array}{cc|ccc} 1 & 0 & -12 & -41 & -8 \\ 0 & 1 & 8 & 24 & 5 \end{array}\right].$$

Thus, we have

$$[(4,4)]_D \;=\; \begin{bmatrix} -12 \\ 8 \end{bmatrix}, \quad [(7,-3)]_D \;=\; \begin{bmatrix} -41 \\ 24 \end{bmatrix}, \quad [(2,-5)]_D \;=\; \begin{bmatrix} -8 \\ 5 \end{bmatrix},$$

so that

$$[f]_D^B \;=\; \begin{bmatrix} -12 & -41 & -8 \\ 8 & 24 & 5 \end{bmatrix}.$$

10. Let $V_i = \langle B_i \rangle$ for $i = 1, 2, 3$. Consider first B_1, which consists of powers of x, so that V_1 is the subspace of polynomials of degree at most 3. If B_1 were linearly dependent then there would exist a nonzero polynomial that evaluates to zero for all real numbers substituted for x. This is impossible, as the number of roots of a real polynomial is bounded by its degree, so is finite. Hence B_1 is linearly independent, so is a basis for V_1. Then $D : V_1 \to V_1$ where

$$[D(1)]_{B_1} = [0]_{B_1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad [D(x)]_{B_1} = [1]_{B_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$[D(x^2)]_{B_1} = [2x]_{B_1} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \qquad [D(x^3)]_{B_1} = [3x^2]_{B_1} = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix},$$

so that

$$[D]_{B_1}^{B_1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Clearly this matrix, and hence D restricted to V_1, has rank 3 and nullity 1.

Consider now B_2. It is clearly linearly independent, since B_2 comprises only two elements and neither is a scalar multiple of the other. Hence B_2 is a basis for V_2. Then $D : V_2 \to V_2$ where

$$[D(\sin x)]_{B_2} = [\cos x]_{B_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad [D(\cos x)]_{B_2} = [-\sin x]_{B_2} = \begin{bmatrix} -1 \\ 0 \end{bmatrix},$$

so that

$$[D]_{B_2}^{B_2} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Clearly this matrix, and hence D restricted to V_2, has rank 2 and nullity 0.

Consider now B_3, and suppose it is linearly dependent, so there are scalars α, β, γ such that

$$\alpha e^x + \beta e^{2x} + \gamma x e^{2x} = 0$$

for all $x \in \mathbb{R}$. In particular, putting $x = 0, 1, 2$, we get the following system

$$\begin{array}{rcrcrcl} \alpha & + & \beta & & & = & 0 \\ e\alpha & + & e^2\beta & + & e^2\gamma & = & 0 \\ e^2\alpha & + & e^4\beta & + & 2e^4\gamma & = & 0 \end{array}$$

with the following matrix of coefficients:

$$\begin{bmatrix} 1 & 1 & 0 \\ e & e^2 & e^2 \\ e^2 & e^4 & 2e^4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 1 & e & e \\ 1 & e^2 & 2e^2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & e-1 & e \\ 0 & e^2-1 & 2e^2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & e-1 & e \\ 0 & 0 & e^2-e \end{bmatrix}$$

which has rank 3 (since $e^2 - e \neq 0$), so that $\alpha = \beta = \gamma = 0$. This proves B_3 is linearly independent and therefore a basis for V_3. Then $D : V_3 \to V_3$ where

$$[D(e^x)]_{B_2} = [e^x]_{B_2} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad [D(e^{2x})]_{B_2} = [2e^{2x}]_{B_2} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix},$$

and

$$[D(xe^{2x})]_{B_2} = [e^{2x} + 2xe^{2x}]_{B_2} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix},$$

so that

$$[D]_{B_2}^{B_2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

Clearly this matrix, and therefore D restricted to V_3, has rank 3 and nullity 0.