We also put $F^0 = \{0\}$ (with trivial arithmetic).

Note the similarity of $*$ and $**$ to addition and scalar multiplication of row vectors:

$$[a_1, \ldots, a_n] + [b_1, \ldots, b_n] = [a_1 + b_1, \ldots, a_n + b_n]$$
$$\lambda [a_1, \ldots, a_n] = [\lambda a_1, \ldots, \lambda a_n]$$

So we in fact identify (think of as equal), for $n \geqslant 1$,

$$F^n \equiv \{[a_1, a_2, .., a_n] \mid a_1, a_2, \ldots, a_n \in F\}$$

$$(a_1, \ldots, a_n) \leftrightarrow [a_1, \ldots, a_n]$$

Column vectors also have similar addition and scalar multiplication. So you may want to identify column vectors with F^n also. However, in this course, we will only identify n-tuples with row vectors.

How do we then think about the column vectors?

Suppose

$$V = \left\{ \begin{bmatrix} a \\ \vdots \\ a_n \end{bmatrix} \middle| a_1, \ldots, a_n \in F \right\}.$$

← n-tuples/Cartesian Power

← column vectors.

We define a function $f : F^n \longrightarrow V$

$$(a_1, a_2, .., a_n) = [a_1, a_2, .., a_n] \longmapsto [a_1, a_2, .., a_n]^T = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

This function is a bijection which respects addition and scalar multiplication. Because of this we can say that F^n and V are isomorphic vector spaces. More about this later.

Section 8.2 Linear Transformations on Cartesian Powers

Cartesian Products (and so Cartesian Powers) are examples of Vector Spaces, which we will cover in more detail later. For now, we define linear transformations over Cartesian Powers only.

Let $L: F^m \to F^n$ be a function where F is a field and $m, n \geq 0$.

We call L a linear transformation (over Cartesian powers) if the following conditions hold

$$L(\underline{v} + \underline{w}) = L(\underline{v}) + L(\underline{w}) \leftarrow \text{addition is preserved.}$$

$$L(\lambda \underline{v}) = \lambda L(\underline{v}) \leftarrow \text{scalar multiplication is preserved.}$$

for all $\underline{v}, \underline{w} \in F^m$ and $\lambda \in F$.

Theorem: A function $L: F^m \to F^n$ is a linear transformation if and only if

$$L(\lambda_1 \underline{v}_1 + \lambda_2 \underline{v}_2) = \lambda_1 L(\underline{v}_1) + \lambda_2 L(\underline{v}_2).$$

for all $\underline{v}_1, \underline{v}_2 \in F^m$ and $\lambda_1, \lambda_2 \in F$.

linear combinations are preserved

Proof: exercise.

Note: to decongest our notation, we write $L(\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n)$ for $L((\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n))$.

Example: Let $L: \mathbb{R}^2 \to \mathbb{R}^2$ where $L(x,y) = (2x+3y, -x+5y)$.

Show that L is a linear transformation.

First, let's see what it does.

$L(1,2) = (2+3(2), -1+5(2)) = (8,9)$

$L(0,0) = (0, 0)$

We will show that L preserves addition and scalar multiplication.

Let $\underline{v}, \underline{w} \in \mathbb{R}^2$ where $\underline{v} = (v_1, v_2)$ and $\underline{w} = (w_1, w_2)$ for some $v_1, v_2, w_1, w_2 \in \mathbb{R}$ and let $\lambda \in \mathbb{R}$.

Addition

$$\begin{aligned} L(\underline{v} + \underline{w}) &= L\big((v_1, v_2) + (w_1, w_2)\big) \\ &= L(v_1 + w_1, v_2 + w_2) \\ &= \big(2(v_1+w_1) + 3(v_2+w_2), -(v_1+w_1) + 5(v_2+w_2)\big) \\ &= (2v_1 + 3v_2 + 2w_1 + 3w_2, -v_1 + 5v_2 - w_1 + 5w_2) \\ &= (2v_1 + 3v_2, -v_1 + 5v_2) + (2w_1 + 3w_2, -w_1 + 5w_2) \\ &= L(\underline{v}) + L(\underline{w}) \end{aligned}$$

Scalar Multiplication

$$\begin{aligned} L(\lambda \underline{v}) &= L\big(\lambda(v_1, v_2)\big) \\ &= L(\lambda v_1, \lambda v_2) \\ &= \big(2(\lambda v_1) + 3(\lambda v_2), -(\lambda v_1) + 5(\lambda v_2)\big) \\ &= \big(\lambda(2v_1 + 3v_2), \lambda(-v_1 + 5v_2)\big) \\ &= \lambda(2v_1 + 3v_2, -v_1 + 5v_2) \\ &= \lambda L(\underline{v}) \end{aligned}$$

We can actually use a matrix to "represent" this linear transformation. Let $M = \begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix}$.

Then
$$M\underline{v} = \begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$
$$= \begin{bmatrix} 2v_1 + 3v_2 \\ -v_1 + 5v_2 \end{bmatrix}$$

So, in this example,

iff

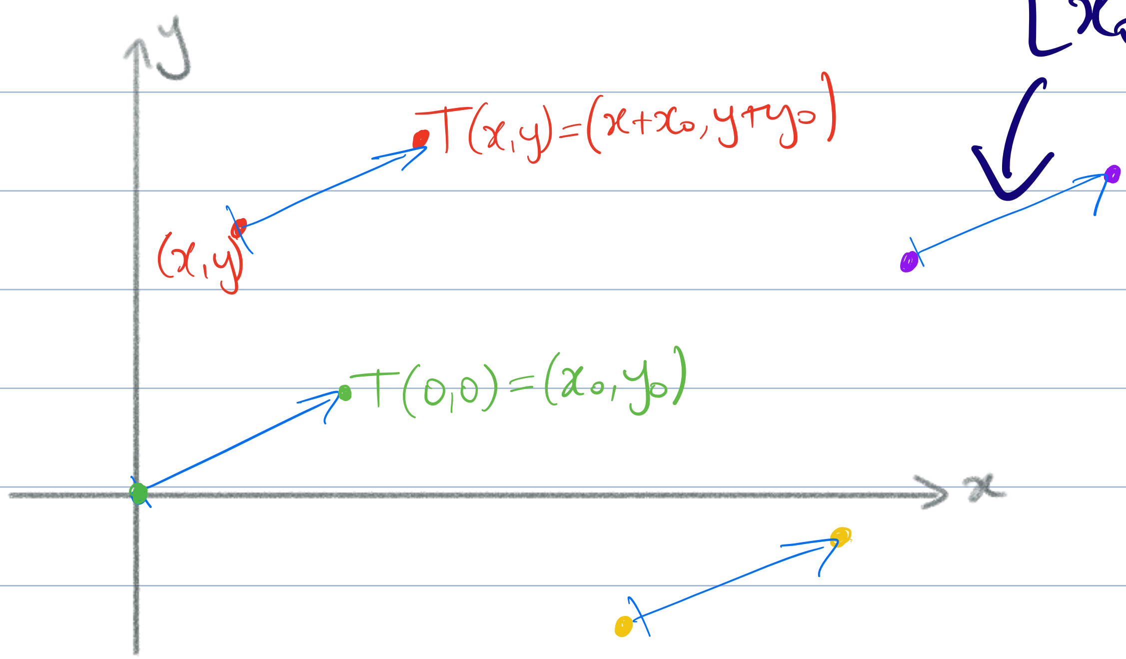
$$L(x, y) = (x', y') \Leftrightarrow M \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

Example: Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ where $T(x, y) = (x + x_0, y + y_0)$

for some fixed point $(x_0, y_0) \in \mathbb{R}^2$.

T is a translation of the plane, taking the origin (0,0) to (x_0, y_0).

$[x_0, y_0]$



Case 1 $(x_0, y_0) = (0, 0)$

In this case, $T(x,y) = (x,y)$, is the identity mapping, which is a linear transformation (trivially).

Case 2 $(x_0, y_0) \neq (0, 0)$ $\lambda = 0$ $(x,y) = (0,0)$

Here, $T(0(0,0)) = T(0,0) = (x_0, y_0)$

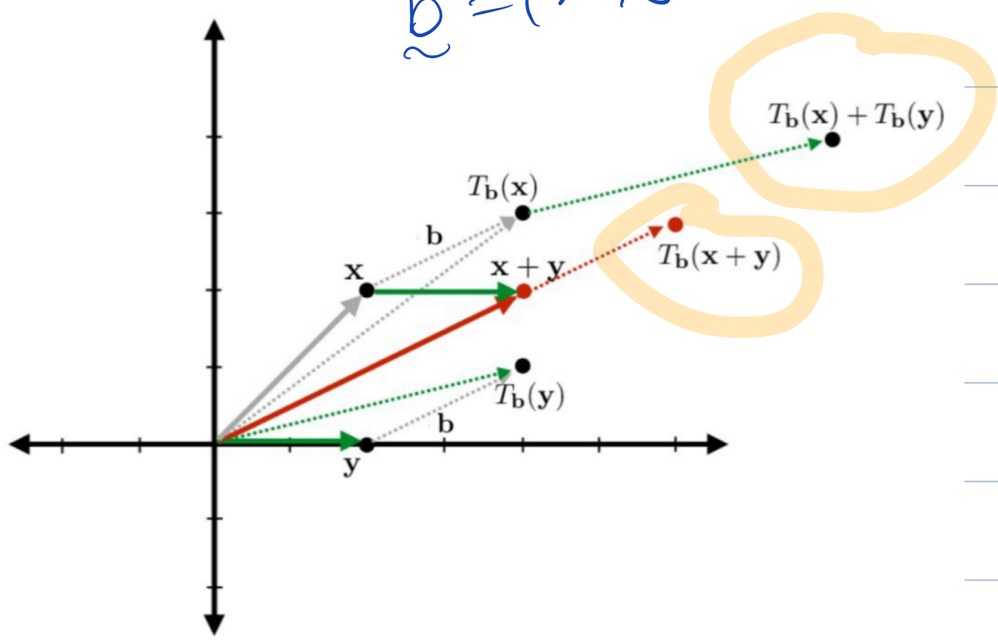
However, $0T(0,0) = 0(x_0, y_0)$
$= (0,0)$

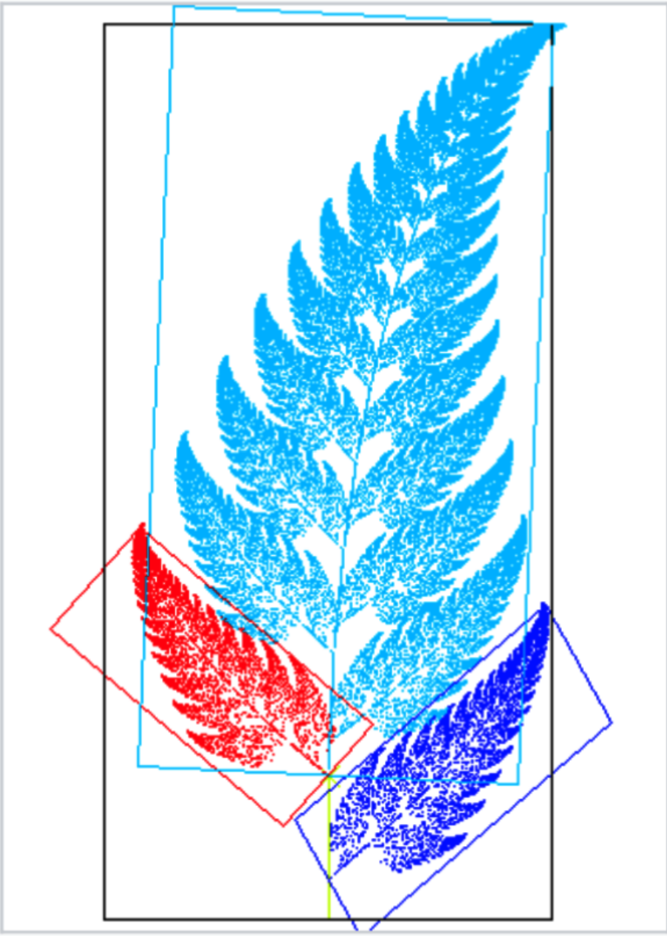
So $T(0(0,0)) \neq 0T(0,0)$.

Hence T is not linear in general.

The above function is an example of an affine transformation, which we will not cover in this course. It preserves lines, but shifts the origin.

$\underset{\sim}{b} = (x_0, y_0)$





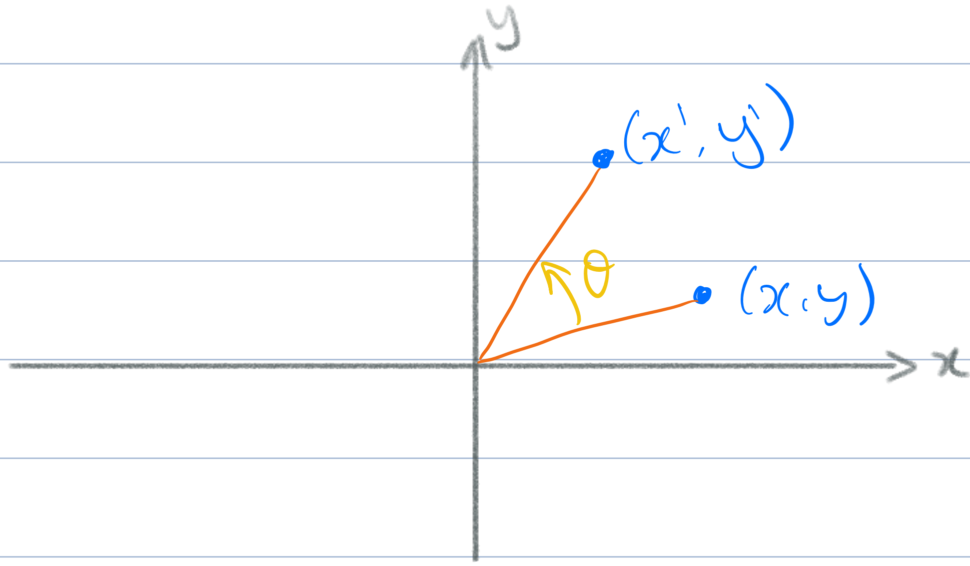
An image of a fern-like fractal (Barnsley's fern) that exhibits affine self-similarity. Each of the leaves of the fern is related to each other leaf by an affine transformation. For instance, the red leaf can be transformed into both the dark blue leaf and any of the light blue leaves by a combination of reflection, rotation, scaling, and translation.

from wikipedia.

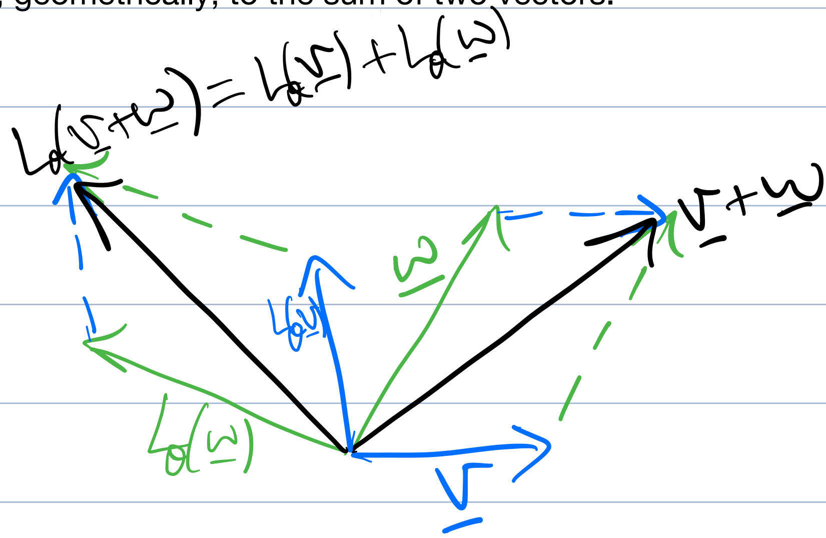
Example: Let $\theta \in \mathbb{R}$ and $L_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ where

$$L_\theta(x, y) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)\,.$$

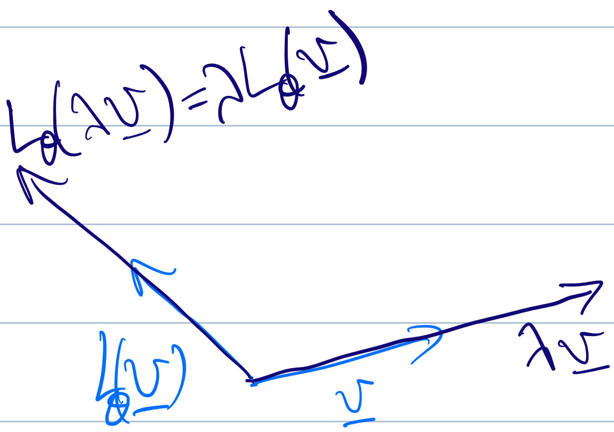
This function is a rotation of the plane, θ radians anticlockwise about the origin.



Let’s see what happens, geometrically, to the sum of two vectors.



What about scalar multiplication?



So, geometrically, we can see that a L_θ is a linear transformation.

Exercise: prove this algebraically.

Note: If we let $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

then
$$R_\theta \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$
$$= \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta - y\cos\theta \end{bmatrix}.$$

So that
$$L_\theta(x, y) = (x', y') \Leftrightarrow R_\theta \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

That is, L_θ is represented by the matrix R_θ.