THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Calculus Exercises 3 (Week 3)

MATH1062/MATH1023: Mathematics 1B (Calculus) Semester 2, 2024

Material covered

(1) Applications of separable equations

Assumed Knowledge

Integration techniques, partial fractions.

Objectives

1. Solve problems involving exponential growth or decay.
2. Solve a separable differential equation using partial fractions.
3. Have an idea about how to model basic flow problems.

Exercises

Questions marked with * are harder questions.

1. Suppose that an archaeologist excaves a bone and measures its content of radioactive carbon C^{14}.

(a) If the results is 25% of the content present in bones of a living organisim, how old is the bone?

Note: The half-life of C^{14} is 5730 years.

(b) What about 5%?

(c) Very hypothetically, if we had to examine the remains of Sir Isaac Newton today, what would be the percentage of C^{14} we would find compared to a living organism?

2. (a) An algal bloom will occur in water polluted by excess nutrients when the number of algae cells x increases dramatically. The rate of increase of the number of algae cells in a sample of such water during a bloom is proportional to the number of cells present at any instant. Write down a differential equation that models this phenomenon.

(b) If the water is low in oxygen then the rate of increase is proportional to $e^{-at}x$ at any time t ($t \geq 0$) where a is a positive constant, instead of being proportional only to the total number of algae cells. [That is, the rate of increase is proportional to a fraction of the total number of cells, and that fraction decreases exponentially with time.]

(i) Write down a differential equation that expresses these statements.

(ii) Solve it to obtain the number of cells in a sample as a function of time, given the initial number of algae cells.

(iii) Is there a limit to the number of cells as $t \to \infty$?

3. Given a cylindrical container of height H and area A with an outlet at the bottom of area a. The volume of water lost through the outlet per unit time is modeled by the equation

$$\frac{dV}{dt} = -k\sqrt{h}.$$

Here, V denotes the amount (volume) of water in the container, h denotes the current water level. Moreover, $k = C_d a\sqrt{2g}$ is a constant depending on a, the gravitational constant g, and the socalled *discharge coefficient* C_d, taking into account the shape of the outlet and the type of fluid in the container.

(a) Formulate the above differential equation as a differential equation of the form $\frac{dh}{dt} = f(h)$ for some function f.

(b) Find a general solution for your differential equation.

*(c) Assume that the cylindrical container is of height $3m$, of diameter $2m$, and has an outlet of area $0.2m^2$. The container is filled with water with an initial water level of $2m$. The outlet has a sharp edge and the container is filled with water. This leads to a discharge coefficient of approximately $C_d \approx 0.6$.

Make a sketch of the set-up and compute the particular solution using these specifications. Sketch a plot of the particular solution.

*(d) Using the above particular solution, at what time will the container be half-empty? At what time is only 10% of the water left? When will the container be empty? Use your calculations to make a more precise plot of the particular solution.

Short answers to selected exercises

1. (a) 11460 years

(c) 96.5%

3. (c) $h(t) = (-0.0846t + \sqrt{2})^2$

(d) $t = 2.2$, $t = 11.4$ and $t = 16.72$