MATH1023/MATH1062 Calculas

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1 Week 9

Approximating values of functions using tangents

- **Differential:** The differential of a differentiable function $y = f(x)$ is
$$dy = f'(x)dx$$
In Leibniz notation $dy = \frac{dy}{dx}dx$
- **Differential:** The differential of a differentiable function $z = f(x, y)$ is
$$dz = f_x(x, y)dx + f_y(x, y)dy$$
- **Approximation:** If (x, y) is near (a, b) then we have
$$\begin{gathered} f(x, y) \approx f(a, b) + dz \\ = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \\ = \text{z value of equation of tangent plane} \end{gathered}$$

The total derivative

- **Definition:** If $z = f(x, y), x = g(t), y = h(t)$ are differentiable functions, then the total derivative of z with respect to t at $t = a$ is:
$$\frac{dz}{dt} = \lim_{k \to 0} \frac{f(g(a + k), h(a + k)) - f(g(a), h(a))}{k}$$
- To calculate the total derivative, we user the total derivative rule:
$$\frac{dz}{dt} = \frac{\partial z}{\partial dx}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$
- **Chain Rule**: If $z = f(x, y), x = g(s, t), y = h(s, t)$ are differentiable functions, we have:
$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$
$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Implicit Differentiation

- **Implicit function theorem:(IFT):** Let $C \subseteq \mathbb{R}^2$ be a curve defined by $f(x, y) = k$ for some differentiable function $f : D \to \mathbb{R}, D \subseteq \mathbb{R}^2$ and $k \in \mathbb{R}$. If $(a, b) \in D, f(a, b) = k$ and $f_y(a.b) \neq 0$ then C can be described around (a, b) by a function
$$y = g(x)$$
- **Application of the IFT:** If we can apply the IFT then we can find $\frac{dy}{dx}$ using the following method:
  1. Start with $f(x, y) = k$
  2. Use the IFT to express y locally as a function of x and substitute into the formula for the curve
  $$f(x, g(x)) = k$$
  3. Use the chain rule to differentiate with respect to x
  $$f_x \frac{dx}{dx} + f_y \frac{dg}{dx} = 0$$
  4. Solve for $\frac{dg}{dx}$
  $$\frac{dg}{dx} = -\frac{fx}{fy}$$
- **A formula for $\frac{dy}{dx}$:**
$$\left.\frac{dy}{dx}\right|_{x=a} = -\frac{f_x(a, b)}{f_y(a, b)}$$