THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Calculus Tutorial 4 (Week 5)

MATH1062/MATH1023: Mathematics 1B (Calculus) Semester 2, 2024

Questions marked with * are harder questions.

Material covered

(1) Linear first order differential equations

Summary of essential material

Recall that when we multiply a *linear differential equation* by the integrating factor $r(x)$ we obtain

$$\frac{d}{dx}(r(x)y) = r(x)q(x), \quad \text{and thus} \quad r(x)y = \int r(x)q(x)\,dx.$$

Questions to complete during the tutorial

1. For each of the first-order differential equations below, determine whether it is separable, linear, or neither of those.

If an equation is linear, write it in standard form $\frac{dy}{dx} + p(x)y = q(x)$ (with a suitable renaming of variables where necessary) and identify the functions p and q.

(a) $\frac{dy}{dx} + 3y = x$

(b) $\frac{x}{2}\frac{dy}{dx} = x^2 - y$

(c) $\left(\cos(t) + t^2\right)\frac{dx}{dt} + 3x = 1$

(d) $\frac{dy}{dx} = y^2 - x$

(e) $\frac{dy}{dx} = \frac{xy^2 + xy + y^2 + y}{y+1}$

(f) $\frac{dy}{dx} = e^{x-y} + e^x + e^{-y} + 1$

2. (a) For the equation $\frac{dy}{dx} + 3y = x$, write down the integrating factor $r(x) = e^{\int p(x)\,dx}$. Hence find the general solution of this equation.

(b) Find the particular solution of $\frac{x}{2}\frac{dy}{dx} = x^2 - y$ for which $y = 1$ when $x = 1$.

3. The size of a fish varies in time according to the law

$$\frac{dV}{dt} = -V + \frac{1}{10}S\,,$$

where V is the volume of the fish and S is its surface area. For a particular species, the volume and surface area are related to the length of the fish L (in metres) according to

$$V = \frac{L^3}{10} \qquad \text{and} \qquad S = L^2\,.$$

(a) Show that L satisfies the differential equation

$$\frac{dL}{dt} = \frac{1}{3}(1 - L)\,.$$

(b) Solve this equation as a linear differential equation to find $L(t)$ given that $L = 0$ when $t = 0$.

(c) What is the maximum size to which such a fish can grow?

(d) If t is measured in years, how long does it take for a fish to grow to 50 cm in length?

*__4.__ Given the differential equation $\dfrac{d^2y}{dx^2} + \dfrac{2}{x}\dfrac{dy}{dx} = 0$.

(a) What is the order of the given differential equation? Is it linear?

(b) Solve the equation using the substitution $w = \dfrac{dy}{dx}$.

Short answers to selected exercises

1. (a) linear (b) linear (c) separable and linear

2. (a) $y = \frac{1}{3}x - \frac{1}{9} + Ce^{-3x}$. (b) $y = \frac{x^2}{2} + \frac{1}{2x^2}$.

3. (b) $L = 1 - e^{-t/3}$ (d) $t = 3\ln(2) \approx 2.08$ years

(c) 1 metre

4. (a) second-order linear (b) $y(x) = -\frac{A}{x} + B$

5. (a) linear, $\dfrac{d^3y}{dx^3} + x^2\sin(x)\dfrac{dy}{dx} - x^2y = e^x$.