The University of Sydney
MATH2022 Linear and Abstract Algebra

Semester 1 **Third Quiz Sample Exercises** 2024

The second quiz is on May 23 on Canvas. See the announcement on Ed for relevant information. Below are some sample problems for extra practice. This need not be a perfect replica of our quiz, however!

1. Let M be the following real matrix:

$$M = \begin{bmatrix} 2 & 3 & 0 & 1 & 4 \\ 2 & -1 & 2 & 3 & 2 \\ 4 & -6 & 6 & 8 & 2 \end{bmatrix}$$

Which one of the following is correct:

(a) $\text{rank}(M) = 3$ and $\text{nullity}(M) = 2$.
(b) $\text{rank}(M) = 2$ and $\text{nullity}(M) = 3$.
(c) $\text{rank}(M) = 4$ and $\text{nullity}(M) = 1$.
(d) $\text{rank}(M) = 3$ and $\text{nullity}(M) = 3$.
(e) $\text{rank}(M) = 1$ and $\text{nullity}(M) = 4$.

2. Let M be the following matrix over $\mathbb{Z}_2$:

$$M = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

Which one of the following is correct:

(a) $\text{rank}(M) = 3$ and $\text{nullity}(M) = 1$.
(b) $\text{rank}(M) = 2$ and $\text{nullity}(M) = 2$.
(c) $\text{rank}(M) = 3$ and $\text{nullity}(M) = 2$.
(d) $\text{rank}(M) = 4$ and $\text{nullity}(M) = 1$.
(e) $\text{rank}(M) = 4$ and $\text{nullity}(M) = 0$.

3. Which one of the following subsets does not form a basis for the vector space $\mathbb{Z}_3^3$?

(a) $\{(1, 1, 0), (1, 0, 1), (0, 2, 2)\}$
(b) $\{(1, 2, 0), (1, 0, 1), (0, 2, 1)\}$
(c) $\{(1, 1, 1), (1, 0, 1), (0, 2, 2)\}$
(d) $\{(1, 0, 2), (1, 0, 1), (0, 2, 1)\}$
(e) $\{(1, 0, 0), (1, 1, 2), (0, 2, 1)\}$

4. Which one of the following subsets B is a basis for the real vector space $\mathbb{P}_4$?

(a) $\{x, x^2, x^3, x^4\}$
(b) $\{1-x^2, x+x^2, -x-x^3, 1-2x^2, x^4-x\}$
(c) $\{1, x, x^2, x^3\}$
(d) $\{1, x+x^2, -x-x^3, 1+x^2+x^3\}$
(e) $\{1-x, x+x^2, x^2+x^3, 1-x^3, x^4+1\}$

5. Let $M = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 1 & 2 & -3 & -1 \end{bmatrix}$ with entries from $\mathbb{R}$. Then a basis B for the null space $M^\perp$ of M is which one of the following?

(a) $B = \left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 1 \\ 1 \end{bmatrix} \right\}$

(b) $B = \left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix} \right\}$

(c) $B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ -1 \end{bmatrix} \right\}$

(d) $B = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \end{bmatrix} \right\}$

(e) $B = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \end{bmatrix} \right\}$

6. Let $M = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ with entries from $\mathbb{Z}_3$. Which of the following is a basis B for the null space $M^\perp$ of M?

(a) $B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$

(b) $B = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

(c) $B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

(d) $B = \left\{ \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$

(e) $B = \emptyset$, the empty set

7. Let $B = \{(2,1), (-1,1)\}$ be an ordered basis for $\mathbb{R}^2$.
Find the coordinate vector of $\mathbf{v} = (8,1)$ with respect to B.

(a) $[\mathbf{v}]_B = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$

(b) $[\mathbf{v}]_B = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$

(c) $[\mathbf{v}]_B = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$

(d) $[\mathbf{v}]_B = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$

(e) $[\mathbf{v}]_B = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$

8. Let $B = \{(1,0), (0,1)\}$ and $D = \{(1,1), (-1,1)\}$, both of which are ordered bases for $\mathbb{R}^2$. Let $M = [\mathrm{id}]_D^B$ be a change of basis matrix. Which one of the following is true?

(a) $M = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

(b) $M = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

(c) $M = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

(d) $M = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$

(e) $M = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$

9. Let $B = \{(1,1),(2,3)\}$ and $D = \{2,1),(3,4)\}$, both of which are ordered bases for $\mathbb{Z}_7^2$. Let $M = [\mathrm{id}]_D^B$ be a change of basis matrix. Which one of the following is true?

(a) $M = \begin{bmatrix} 3 & 3 \\ 4 & 5 \end{bmatrix}$ (b) $M = \begin{bmatrix} 3 & 4 \\ 3 & 5 \end{bmatrix}$ (c) $M = \begin{bmatrix} 2 & 1 \\ 3 & 6 \end{bmatrix}$

(d) $M = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ (e) $M = \begin{bmatrix} 5 & 6 \\ 2 & 4 \end{bmatrix}$

10. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear operator given by the rule

$$L(x,y) \;=\; (2x+y, x-y)\;.$$

and let B be the following ordered basis for $\mathbb{R}^2$:

$$B \;=\; \{(2,1),(-1,1)\}\;.$$

Then the matrix $[L]_B^B$ of L with respect to B is which one of the following?

(a) $[L]_B^B = \begin{bmatrix} 2 & -1 \\ -1 & -1 \end{bmatrix}$ (b) $[L]_B^B = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$ (c) $[L]_B^B = \begin{bmatrix} 4 & -1 \\ -2 & -1 \end{bmatrix}$

(d) $[L]_B^B = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix}$ (e) $[L]_B^B = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$

11. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear operator given by the rule

$$L(x,y) \;=\; (-x-2y, 2y-2x)\;.$$

Find an ordered basis B for $\mathbb{R}^2$ such that

$$[L]_B^B \;=\; \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}\;.$$

(a) $B = \{(2,-1),(1,2)\}$ (b) $B = \{(3,1),(1,-1)\}$ (c) $B = \{(2,1),(-1,2)\}$

(d) $B = \{(-1,-2),(1,-2)\}$ (e) $B = \{(-2,0),(0,3)\}$

12. Let V be the vector space of differentiable functions spanned by $B = \{e^{-x}, xe^{-x}, x^2e^{-x}\}$. Find the matrix $[D]_B^B$ of the linear operator $D : V \to V$ that maps a function to its derivative.

(a) $[D]_B^B = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 2 & -1 \end{bmatrix}$ (b) $[D]_B^B = \begin{bmatrix} -1 & 1 & 2 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ (c) $[D]_B^B = \begin{bmatrix} -1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$

(d) $[D]_B^B = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & 0 & -1 \end{bmatrix}$ (e) $[D]_B^B = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & -1 \end{bmatrix}$

13. Let V be the vector space of differentiable functions spanned by $B = \{\sinh x, \cosh x\}$. Find the matrix $[D]_B^B$ of the linear operator $D : V \to V$ that maps a function to its derivative.

(a) $[D]_B^B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ (b) $[D]_B^B = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ (c) $[D]_B^B = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$

(d) $[D]_B^B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (e) $[D]_B^B = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$

14. Consider the real vector space $V = \langle e^x, e^{-x} \rangle$ with ordered bases

$$B = \{e^x, e^{-x}\} \quad \text{and} \quad C = \{\cosh x, \sinh x\}.$$

Find the matrix $[D]_B^C$, where D is the linear operator $D : V \to V$ that maps a function to its derivative.

(a) $[D]_B^C = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$ (b) $[D]_B^C = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$ (c) $[D]_B^C = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

(d) $[D]_B^C = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$ (e) $[D]_B^C = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

15. Consider the real vector space

$$V = \left\langle \frac{\sin x + \cos x}{2}, \frac{\sin x - \cos x}{2} \right\rangle$$

of real-valued functions of a real variable, with ordered basis

$$B = \left\{ \frac{\sin x + \cos x}{2}, \frac{\sin x - \cos x}{2} \right\}.$$

Find the coordinate vector $[-\sin x]_B$.

(a) $[-\sin x]_B = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ (b) $[-\sin x]_B = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$ (c) $[-\sin x]_B = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

(d) $[-\sin x]_B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (e) $[-\sin x]_B = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$

16. Find the real matrix e^A where $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$.

(a) $e^A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$ (b) $e^A = \begin{bmatrix} e & e^{-1} \\ e & e^{-1} \end{bmatrix}$ (c) $e^A = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$

(d) $e^A = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$ (e) $e^A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

17. Find the real matrix e^A where $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$.

(a) $e^A = \begin{bmatrix} 2e^2 - e^3 & -2e^2 - e^3 \\ e^2 + e^3 & -e^2 + 2e^3 \end{bmatrix}$

(b) $e^A = \begin{bmatrix} 2e^2 - e^3 & -2e^2 + 2e^3 \\ e^2 - e^3 & -e^2 + 2e^3 \end{bmatrix}$

(c) $e^A = \begin{bmatrix} -e^2 - e^3 & -2e^2 - e^3 \\ e^2 + e^3 & -e^2 + 2e^3 \end{bmatrix}$

(d) $e^A = \begin{bmatrix} 2e^2 & e^3 \\ e^2 & e^3 \end{bmatrix}$

(e) $e^A = \begin{bmatrix} -e^2 - e^3 & -2e^2 - e^3 \\ 2e^2 + e^3 & -e^2 + 2e^3 \end{bmatrix}$

18. Solve the following system of differential equations, where $x = x(t)$ and $y = y(t)$ are differentiable functions of a real variable t:

$$\begin{array}{rcrcr} x' & = & x & + & 2y \\ y' & = & -x & + & 4y \end{array}$$

such that $x(0) = 5$ and $y(0) = 6$.

(a) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2e^{2t} + 7e^{3t} \\ -e^{2t} + 7e^{3t} \end{bmatrix}$

(b) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2e^{2t} + 7e^{3t} \\ e^{2t} + 5e^{3t} \end{bmatrix}$

(c) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2e^{2t} - 7e^{3t} \\ -2e^{2t} + e^{3t} \end{bmatrix}$

(d) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7e^{2t} - 2e^{3t} \\ -e^{2t} + 7e^{3t} \end{bmatrix}$

(e) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7e^{2t} - 2e^{3t} \\ e^{2t} + 5e^{3t} \end{bmatrix}$

19. Which one of the following is an orthonormal basis for the hyperplane in $\mathbb{R}^4$ with equation

$$x + 2y - z - 2w = 0 \ ?$$

(a) $\left\{ \frac{1}{\sqrt{5}}(-2,1,0,0), \frac{1}{\sqrt{15}}(1,2,5,0), \frac{1}{\sqrt{15}}(1,2,-1,3) \right\}$

(b) $\left\{ \frac{1}{\sqrt{5}}(2,-1,0,0), \frac{1}{\sqrt{30}}(-1,2,5,0), \frac{1}{\sqrt{15}}(1,2,-1,3) \right\}$

(c) $\left\{ \frac{1}{\sqrt{5}}(-2,1,0,0), \frac{1}{\sqrt{30}}(1,2,5,0), \frac{1}{\sqrt{15}}(1,2,-1,3) \right\}$

(d) $\left\{ \frac{1}{\sqrt{5}}(-2,1,0,0), \frac{1}{\sqrt{30}}(1,2,5,0), \frac{1}{\sqrt{15}}(-1,-2,1,3) \right\}$

(e) $\left\{ \frac{1}{\sqrt{5}}(-2,1,0,0), \frac{1}{\sqrt{30}}(1,2,5,0), \frac{1}{\sqrt{15}}(1,2,1,-3) \right\}$

20. Find the distance from the point $(1,0,0,0)$ to the hyperplane in $\mathbb{R}^4$ with equation

$$x + 2y - z - 2w = 0.$$

(a) $\frac{10}{\sqrt{5}}$ (b) $\frac{5}{\sqrt{10}}$ (c) $\frac{1}{\sqrt{5}}$ (d) $\sqrt{10}$ (e) $\frac{1}{\sqrt{10}}$