Rank-Nullity Theorem

- Recall: Given a matrix M, the rank of M was defined by

$$\underbrace{\dim(\mathrm{Row}(M))}_{\text{row rank}} = \underbrace{\dim(\mathrm{Col}(M))}_{\text{column rank}}$$

: This invariant of M approximately measures the complexity of data "stored in" M.

: We introduce a second invariant, the nullity of M, and a related subspace, which gives complementary information.

The Null Space

- Definition: Given an $n \times m$ matrix M over a field F, the null space of M, denoted $Nul(M)$, is given by

$$Nul(M) = \{ \underset{\sim}{v} \in F^m \mid M \underset{\sim}{v}^T = \underset{\sim}{0} \} \rightarrow \text{"row vector"}$$

or sometimes denoted

$$M^{\perp} = \{ \underset{\sim}{v} \in Mat_{m \times 1}(F) \mid M \underset{\sim}{v} = \underset{\sim}{0} \} \rightarrow \text{"column vector"}$$

- Said another way, $Nul(M)$ is the space of solutions to the homogueous system of equations

$\underbrace{}_{\text{zero vector.}}$

$$M \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

- $Nul(M) \subseteq F^m$ is a subspace: $\underset{\sim}{v}, \underset{\sim}{w} \in Nul(M)$, $\lambda \in F$, $\ni \mu$
  - Nonempty: $\underset{\sim}{0} \in Nul(M)$: $M \underset{\sim}{0} = \underset{\sim}{0}$.
  - Closed: $M(\lambda \underset{\sim}{v} + \mu \underset{\sim}{w}) = \lambda (M \underset{\sim}{v}) + \mu (M \underset{\sim}{w})$
    $= \lambda \underset{\sim}{0} + \mu \underset{\sim}{0} = \underset{\sim}{0}$.

• Example: Let $\underset{3\times 4}{M} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix}$ over $\mathbb{R}$.

• Check: $M \sim \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \text{rank}(M) = 2.$

$\Rightarrow$ system of equations: $\begin{cases} x_1 - x_3 - 2x_4 = 0 \\ x_2 + 2x_3 + 3x_4 = 0 \end{cases}$

(*) $(4 - \text{nullity}\, M) = 4 - 2 = 2$
$= \text{rank}(M)$

$\Rightarrow x_1 = x_3 + 2x_4, \quad x_2 = -2x_3 - 3x_4$

$$\Rightarrow \text{Nul}(M) = \{ (x_3 + 2x_4, -2x_3 - 3x_4, x_3, x_4) \mid x_3, x_4 \in \mathbb{R} \}$$

$$= \langle (1,-2,1,0), (2,-3,0,1) \rangle \subseteq \mathbb{R}^4.$$

$\rightarrow x_3(1,-2,1,0) + x_4(2,-3,0,1).$

• Notice: $\text{rank}(M) = 2 \qquad \dim(\text{Nul}(M)) = 2$

- Definition: The nullity of a matrix M is given by

"perp"

$$\text{nullity}(M) = \dim(\text{Nul}(M)) = \dim(M^{\perp})$$

- Fact: If $A \sim B$, then $A^{\perp} = B^{\perp}$, so

$$\text{nullity}(A) = \text{nullity}(B)$$

  - The solutions to the homogueous system is not affected by row reductions.

$$\left(A \,\middle|\, \begin{matrix}0\\ \vdots \\ 0\end{matrix}\right) \sim \left(B \,\middle|\, \begin{matrix}0\\ \vdots \\ 0\end{matrix}\right) \Rightarrow \text{have same solutions.}$$

$\Rightarrow$ To compute nullity(M), we

- reduce M to rref.
- count "free variables" = parameters in the solution set.

- Recall: $\text{rank}(M) = \dim(\text{Row}(M))$

$$\left.M_{n \times m}\right\}$$

$= \#$(non zero rows in echelon form)

$= \#$ "fixed"/"leading" variables

$= m - \#$ free variables $= m - \text{nullity}(M)$.

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• Theorem: "Rank-Nullity Theorem", matrix version.

Let M be an $n \times m$ matrix over a field F. Then

$$m = \text{rank}(M) + \text{nullity}(M).$$

<u>Example</u>

$$M = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \text{ over } \mathbb{Z}_2.$$

Then

free variable column.

$$M \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \text{rank}(M) = 2$$

$$\Rightarrow \text{nullity}(M) = 3 - 2 = 1$$

$$\text{Nul}(M): \begin{cases} x_1 + x_3 = 0 \\ x_2 + x_3 = 0 \end{cases} \Rightarrow x_1 = x_2 = x_3$$

$$\Rightarrow \text{Nul}(M) = \{(x_3, x_3, x_3) \mid x_3 \in \mathbb{Z}_2\}$$
$$= \langle (1,1,1) \rangle$$

or $M^\perp = \left\langle \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\rangle$.

• Example: Let M be 5×7 matrix.

Column space has dimension 4.

What is $\dim(\mathrm{Nul}(M))$?

$\Rightarrow \operatorname{rank}(M) = 4 \Rightarrow \operatorname{nullity}(M) = 7 - 4 = 3.$

• Example: M is 5×5 matrix, $\operatorname{rank}(M) = 2$.

Find $\dim(\underbrace{\text{0-eigenspace of } M}_{\text{Span of the eigenvectors of eigenvalue } \lambda = 0})$.

$\Rightarrow M\underset{\sim}{v} = \lambda \underset{\sim}{v} = 0\underset{\sim}{v} = \underset{\sim}{0}$

$\Rightarrow \mathrm{Nul}(M)$!!

$\Rightarrow \dim(\text{0-eigenspace}) = 5 - 2 = 3.$

Another Perspective

- Let M be an $n \times m$ matrix, and let

$$L : F^m \to F^n \qquad L(x_1, \ldots, x_m) = M \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}$$

be the linear map M represents.

- Question: What do rank and nullity mean in this context?

- $\mathrm{Nul}(M) = \{ \underset{\sim}{v} \in F^m \mid M \underset{\sim}{v}^T = \underset{\sim}{0} \}$

$$= \{ (v_1, v_2, \ldots, v_m) \in F^m \mid L(v_1, v_2, \ldots, v_m) = (0, \ldots, 0) \}.$$

$\Rightarrow$ $\mathrm{Nul}(M) \rightsquigarrow$ collection of vectors in F^m $\underbrace{\text{that } L \text{ sends to } \underset{\sim}{0} \in F^n}_{\text{Terminology: the kernel of } L.}$.

$= \ker(L)$.

$\Rightarrow$ $\mathrm{nullity}(M) = \dim(\ker(L))$

- $\mathrm{Col}(M) = \langle C_1, C_2, \ldots, C_m \rangle$, C_j = columns of M.

$$\subseteq F^n \cong \mathrm{Mat}_{n\times 1}(F).$$

$$\underset{\sim}{v} \in \mathrm{Col}(M) \Rightarrow \underset{\sim}{v} = x_1 C_1 + x_2 C_2 + \cdots + x_m C_m, \quad x_j \in F.$$

$$\Rightarrow \underset{\sim}{v} = M \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}$$

$$\Rightarrow \underset{\sim}{v} = L(x_1, \ldots, x_m)$$