Linear Transformations on Cartesian Products

Recall: For a field F and natural number $n \geq 0$, the Cartesian power F^n is given by

$$F^n : \left\{ (a_1, a_2, \ldots, a_n) \,\middle|\, a_i \in F \right\}$$

We defined a linear map $L : F^n \to F^m$ by

① $L(\underline{v} + \underline{w}) = L(\underline{v}) + L(\underline{w})$

② $L(\lambda \underline{v}) = \lambda L(\underline{v})$.

for all $\underline{v}, \underline{w} \in F^n$, $\lambda \in F$.

• Examples: ① $L_\theta : \mathbb{R}^2 \to \mathbb{R}^2$

$$L_\theta(x, y) = (x\cos\theta - y\sin\theta,\ x\sin\theta + y\cos\theta)$$

– Rotation anticlockwise by angle θ

② $S_\theta : \mathbb{R}^2 \to \mathbb{R}^2$

$$S_\theta(x,y) = (x\cos\theta + y\sin\theta,\ x\sin\theta - y\cos\theta)$$

– Reflection

- Importantly, we can represent these maps via matrices:

① $L_\theta(x,y) = (x', y') \iff \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$

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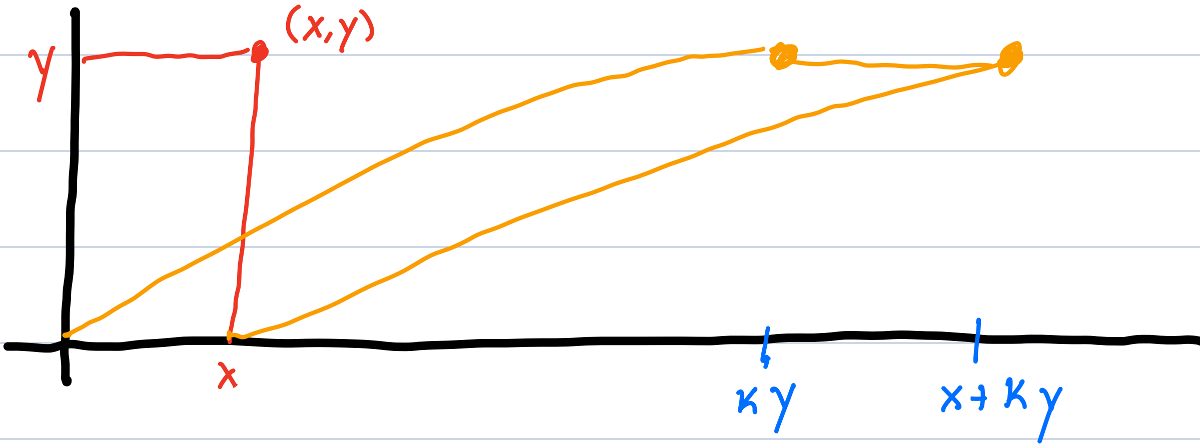
Example: Shear Transformations

Fix $k \in \mathbb{R}$. Define $L : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$L(x,y) = (x + ky,\ y)$$

- Geometrically: L fixes the y-coordinate, and shifts points horizontally by factor k.



• Put $M = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$, shear matrix

$$\Rightarrow M\begin{bmatrix} x \\ y \end{bmatrix} = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+ky \\ y \end{bmatrix}$$

So, M represents L.

Example: Let $L: \mathbb{R}^3 \to \mathbb{R}^4$, where

$$L(x,y,z) = (x+y+z,\ x-y-z,\ y+z,\ x-z)$$

Can compute:

$$L(1,1,2) = (4, -2, 3, -1)$$
$$L(0,0,1) = (1, -1, 1, -1)$$
$$L(0,0,0) = (0,0,0,0)$$

In fact, L is a linear transformation (check this!)

Put $M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$. Then

$$M \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+y+z \\ x-y-z \\ y+z \\ x-z \end{bmatrix}$$

- So, M represents L.

- This pattern of representing linear transformations by matrices is universal.

- **Theorem** Let $L: F^m \to F^n$ be any map. Then L is a linear transformation <u>iff</u> there is an $n \times m$ matrix M such that, for all $(a_1, \dots, a_m) \in F^m$, $(b_1, \dots, b_n) \in F^n$,

$$L(a_1, \dots, a_m) = (b_1, \dots, b_n) \iff M \cdot \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

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$$L(a_1, \dots, a_m) = (b_1, \dots, b_n) \text{ iff}$$

$$M \cdot \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

- We say M represents L.

<u>Proof</u>

$\Leftarrow$ Suppose we have an $n \times m$ matrix M such that

(*) $L(a_1, \dots, a_m) = (b_1, \dots, b_n)$ iff $M \cdot \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$.

We need to show that L is linear:

If $L(\lambda(a_1, \dots, a_m) + \lambda'(a_1', \dots, a_m')) = (b_1, \dots, b_n)$,

then want to show: $\lambda L(a_1, \dots, a_m) + \lambda' L(a_1', \dots, a_m') = (b_1, \dots, b_n)$

By ⊛, $L(\lambda(a_1,..,a_m)+\lambda'(a_1',..,a_m'))=(b_1,..,b_n)$

$$\Rightarrow \quad M\left(\lambda\begin{bmatrix}a_1\\ \vdots\\ a_m\end{bmatrix}+\lambda'\begin{bmatrix}a_1'\\ \vdots\\ a_m'\end{bmatrix}\right)=\begin{bmatrix}b_1\\ \vdots\\ b_n\end{bmatrix}.$$

$$\Rightarrow \quad = \lambda \underbrace{M\begin{bmatrix}a_1\\ \vdots\\ a_m\end{bmatrix}}_{=L(a_1,\dots,a_m)} + \lambda' \underbrace{M\begin{bmatrix}a_1'\\ \vdots\\ a_m'\end{bmatrix}}_{=L(a_1',\dots,a_m')} = \begin{bmatrix}b_1\\ \vdots\\ b_n\end{bmatrix}$$

By ⊛,

$$\Rightarrow \lambda L(a_1,..,a_m)+\lambda' L(a_1',..,a_m')=(b_1,..,b_n).$$

$\Rightarrow$ L is linear.

$\Rightarrow$ Suppose that L is linear. We need to construct a matrix M that represents L.

Write $\underline{e}_i := (0, 0, \ldots, 0, \overset{i^{th}\text{ place}}{\overset{\downarrow}{1}}, 0, 0, \ldots, 0)$

So, for example

$$\underline{e}_1 = (1, 0, 0, \ldots, 0)$$
$$\underline{e}_2 = (0, 1, 0, \ldots, 0)$$
$$\underline{e}_3 = (0, 0, 1, 0, \ldots, 0).$$

etc.

- We call $\underline{e}_i$ a standard basis vector, and the collection $\{\underline{e}_1, \underline{e}_2, \ldots, \underline{e}_m\}$ the standard basis of F^m.

(We will discuss general notion of basis later...)

- For $j = 1, \ldots, m$, we know $L(\underline{e}_j) \in F^n$.

Write

$$L(\underline{e}_j) = (c_{1j}, c_{2j}, c_{3j}, \ldots, c_{nj})$$

for some $c_{1j}, c_{2j}, \ldots, c_{nj} \in F$.

↑ Notice: putting index j in second position!

- Define a matrix with this data.

Put

$$M_{n\times m} = \begin{pmatrix} c_{11} & c_{12} & & c_{1m} \\ c_{21} & c_{22} & & c_{2m} \\ c_{31} & c_{32} & \cdots & c_{3m} \\ \vdots & \vdots & & \vdots \\ c_{n1} & c_{n2} & & c_{nm} \end{pmatrix}$$

so the j^{th} column of M is $\begin{bmatrix} c_{1j} \\ c_{2j} \\ \vdots \\ c_{nj} \end{bmatrix} \leftarrow\text{-}\text{-}\text{-}\rightarrow L(\underline{e}_j)$

- We now check that M represents L: first, note

$$(a_1, \ldots, a_m) = a_1\underline{e}_1 + a_2\underline{e}_2 + \cdots + a_m\underline{e}_m.$$

so that by linearity

$$L(a_1,\ldots,a_m) = L(a_1\underline{e}_1 + a_2\underline{e}_2 + \cdots + a_m\underline{e}_m)$$
$$= a_1L(\underline{e}_1) + a_2L(\underline{e}_2) + \cdots + a_mL(\underline{e}_m).$$

$$= a_1(c_{11}, c_{21}, \ldots, c_{n1}) + a_2(c_{12}, c_{22}, \ldots, c_{n2})$$

$$+ \cdots + a_m(c_{1m}, c_{2m}, \ldots, c_{nm})$$

$$= \Big(a_1c_{11} + a_2c_{12} + \cdots + a_mc_{1m},\ a_1c_{21} + a_2c_{22} + \cdots + a_mc_{2m},$$
$$\cdots\ a_1c_{n1} + a_2c_{n2} + \cdots a_mc_{nm}\Big).$$

So, $L(a_1, \dots, a_m) = (b_1, \dots, b_n)$ iff

$$\begin{cases} c_{11}a_1 + c_{12}a_2 + \cdots + c_{1m}a_m = b_1 \\ c_{21}a_1 + c_{22}a_2 + \cdots + c_{2m}a_m = b_2 \\ \vdots \\ c_{n1}a_1 + c_{n2}a_2 + \cdots + c_{nm}a_m = b_n \end{cases}$$

But this system of equations is precisely

$$\begin{pmatrix} c_{11} & \cdots & c_{1m} \\ c_{21} & & \vdots \\ \vdots & & \vdots \\ c_{n1} & \cdots & c_{nm} \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

$$\Rightarrow M \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}.$$

$\blacksquare$

- **Observation:** This proof gives a recipe for finding the matrix M which represents a linear transformation $L: F^m \to F^n$

- $L: F^m \to F^n$ then the j^{th} column of the $n \times m$ matrix M corresponds to $L(\underline{e}_j)$.

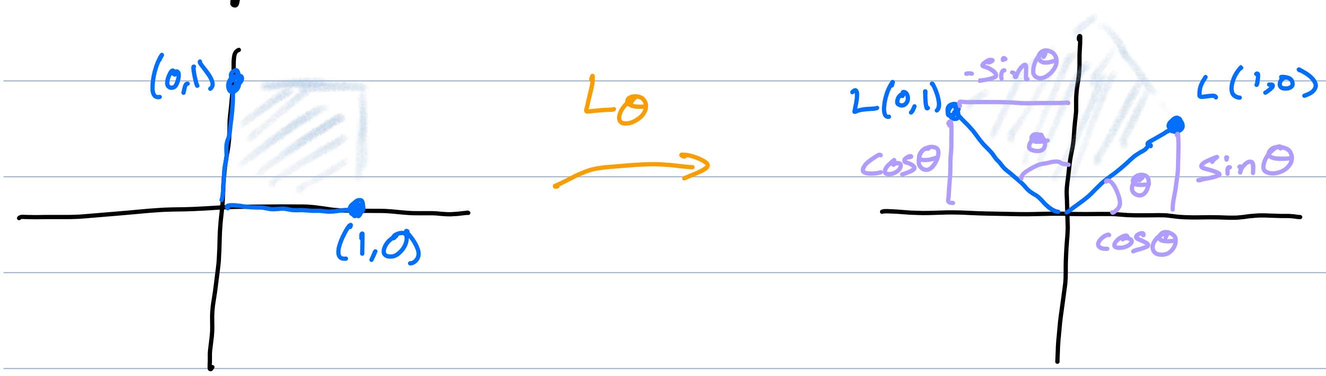
- **Example:** $L: \mathbb{R}^4 \to \mathbb{R}^3$ where
$$L(x,y,z,w) = (5x+2y-z+3w,\ x-y+2z,\ x+y-4w).$$
Find M.

  - $L(1,0,0,0) = (5,1,1)$
  - $L(0,1,0,0) = (2,-1,1)$
  - $L(0,0,1,0) = (-1,2,0)$
  - $L(0,0,0,1) = (3,0,-4)$

$$\Rightarrow M = \begin{pmatrix} 5 & 2 & -1 & 3 \\ 1 & -1 & 2 & 0 \\ 1 & 1 & 0 & -4 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 5x+2y-z+3w \\ \vdots \\ \vdots \end{pmatrix}$$

- **Example:** Rotation matrices in $\mathbb{R}^2$

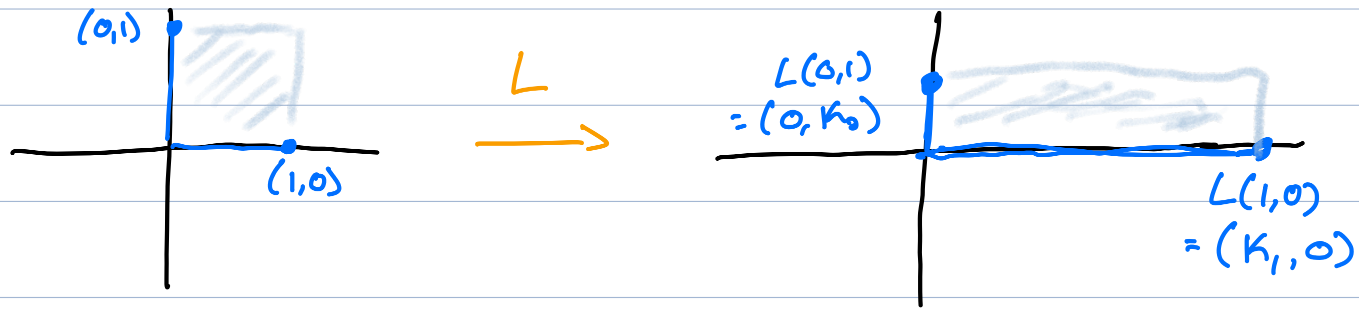


$$L_\theta(1,0) = (\cos\theta, \sin\theta)$$
$$L_\theta(0,1) = (-\sin\theta, \cos\theta)$$

$$\Rightarrow M = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \text{ as expected.}$$

- **Example:** Scaling in $\mathbb{R}^2$:

$L: \mathbb{R}^2 \to \mathbb{R}^2$ by scaling the x-axis by factor k_1, and the y-axis by factor k_2:



$$L(1,0) = (k_1, 0)$$
$$L(0,1) = (0, k_2)$$

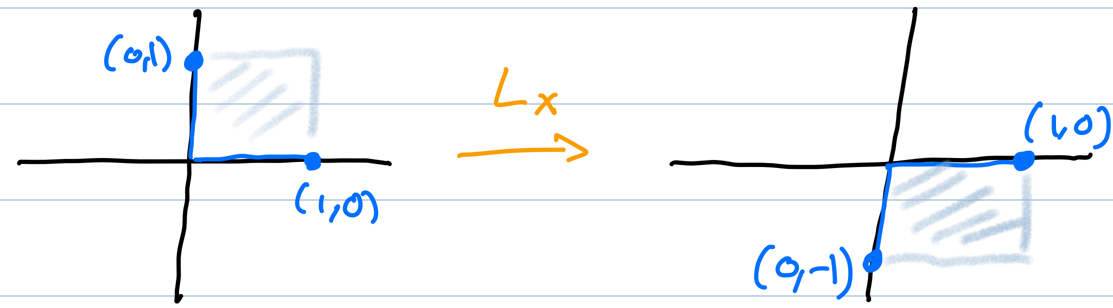
$$\Rightarrow M = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} \text{ represents } L.$$

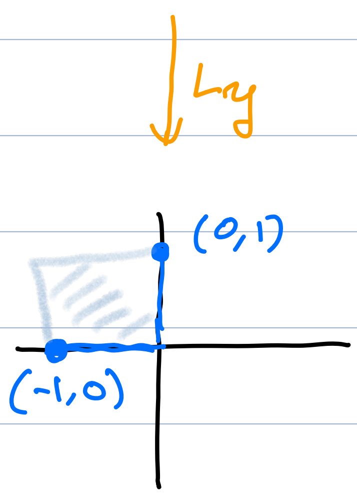
- **Exercise:** Axis Reflection in $\mathbb{R}^2$:

  Find matrices representing

  L_x: reflect across x-axis

  L_y: reflect across y-axis





- L_x is represented by

$$M_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- L_y is represented by

$$M_y = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

• Notice: "negative scaling" $\leftrightarrow$ reflect + scale.