The University of Sydney

MATH2022 Linear and Abstract Algebra

Semester 1 **Exercises for Week 10**

Important Ideas and Useful Facts:

(i) Matrix exponentials: If M is a real square matrix then we may form the *matrix exponential*

$$e^M = I + M + \frac{M^2}{2!} + \frac{M^3}{3!} + \dots .$$

It is a theorem that the series always converges. If M is a diagonal $n \times n$ matrix with diagonal entries $\lambda_1, \dots, \lambda_n$ then e^M is also diagonal with diagonal entries $e^{\lambda_1}, \dots, e^{\lambda_n}$. If A, B and P are real square matrices of the same size, P invertible, and $B = P^{-1}AP$ then

$$e^B = P^{-1}e^A P .$$

If A and B commute, that is, $AB = BA$, then $e^{A+B} = e^A e^B$.

(ii) Solving systems of differential equations: Suppose that we have n differentiable functions $x_1 = x_1(t), x_2 = x_2(t), \dots, x_n = x_n(t)$ of a real variable t that satisfy the following system of differential equations with constant coefficients:

$$\begin{array}{ccccccccc} x_1' & = & a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n \\ x_2' & = & a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ x_n' & = & a_{n1}x_1 & + & a_{n2}x_2 & + & \cdots & + & a_{nn}x_n \end{array}$$

Put $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $\mathbf{x}' = \begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix}$ and $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$, so that the system may be expressed in matrix form $\mathbf{x}' = A\mathbf{x}$. The solution to this system is

$$\mathbf{x} = e^{tA}\mathbf{c}$$

where $\mathbf{c} = \mathbf{x}(0)$ is a column vector of constants.

(iii) Linear transformations (general case): Let V and W be vector spaces over a field F. A function $T : V \to W$ is called a *linear transformation* if T respects vector addition and scalar multiplication, that is, for all $\mathbf{v}, \mathbf{w} \in V$ and $\lambda \in F$,

$$T(\mathbf{v} + \mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w}) \quad \text{and} \quad T(\lambda\mathbf{v}) = \lambda T(\mathbf{v}) ,$$

or, equivalently, T preserves linear combinations, that is for all $\mathbf{v}_1, \mathbf{v}_2 \in V$ and $\lambda_1, \lambda_2 \in F$,

$$T(\lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2) = \lambda_1 T(\mathbf{v}_1) + \lambda_2 T(\mathbf{v}_2) .$$

If $V = W$ then T is called a *linear operator*. If T is bijective (one-one and onto) then T is called a *vector space isomorphism*. The composite of linear transformations, when defined, is also a linear transformation.

(iv) Matrix of a linear transformation with respect to choice of bases: Let $T : V \to W$ be a linear transformation, and let $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ and $D = \{\mathbf{d}_1, \ldots, \mathbf{d}_m\}$ be ordered bases for V and W respectively. Define the *matrix of T with respect to B and D* to be

$$[T]_D^B \;=\; \left[\; [T(\mathbf{b}_1)]_D \quad \ldots \quad [T(\mathbf{b}_n)]_D \;\right] ,$$

by which we mean that we write down, in order, columns of coordinates, in W with respect to D, of the images under T of successive basis elements from B. Note that $[T]_D^B$ is an $m \times n$ matrix. It follows from the definitions that, for all $\mathbf{v} \in V$,

$$[T(\mathbf{v})]_D \;=\; [T]_D^B [\mathbf{v}]_B ,$$

enabling the effect of the linear transformation T to be described in terms of matrix multiplication between coordinates of vectors. If $S : U \to V$ is another linear transformation, where A is an ordered basis for U, so that $T \circ S : U \to W$ is also a linear transformation, then

$$[T \circ S]_D^A \;=\; [T]_D^B [S]_B^A .$$

(v) The identity linear operator: Given any vector space V the mapping $\mathrm{id} = \mathrm{id}_V : V \to V$ where $\mathrm{id}(\mathbf{v}) = \mathbf{v}$, fixing all vectors in V, is called the *identity linear transformation* or *identity operator*. If V is n-dimensional and B is any basis for V then $[\mathrm{id}]_B^B = I_n$, the $n \times n$ identity matrix. If $T : V \to W$ is a linear transformations then

$$T \circ \mathrm{id}_V \;=\; T \qquad \text{and} \qquad \mathrm{id}_W \circ T \;=\; T .$$

Further, if T is a vector space isomorphism, so that T is invertible and $T^{-1} : W \to V$, then

$$T^{-1} \circ T \;=\; \mathrm{id}_V \qquad \text{and} \qquad T \circ T^{-1} \;=\; \mathrm{id}_W .$$

(vi) Change of basis matrix: Let B and D be any bases for an n-dimensional vector space V. The matrix $[\mathrm{id}]_D^B$ is called a *change of basis matrix* and has the effect of converting coordinates of vectors with respect to B into coordinates with respect to D, in the following sense, for any vector $\mathbf{v} \in V$:

$$[\mathrm{id}]_D^B [\mathbf{v}]_B \;=\; [\mathbf{v}]_D .$$

Furthermore, the change of basis matrices $[\mathrm{id}]_D^B$ and $[\mathrm{id}]_B^D$ are mutually inverse, that is,

$$[\mathrm{id}]_D^B [\mathrm{id}]_B^D \;=\; [\mathrm{id}]_B^D [\mathrm{id}]_D^B \;=\; I_n .$$

Tutorial Exercises:

1. Find the exponential matrix e^{tA} where A is each of the following matrices:

(a) $\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$ (d) $\begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix}$

2. Solve the following systems of differential equations, where $x = x(t)$ and $y = y(t)$ are differentiable functions of a real variable t, with the same initial conditions

$$x(0) = 1 \quad \text{and} \quad y(0) = 2$$

in each case:

(a) $\begin{aligned} x' &= -x \\ y' &= \phantom{-x+{}} 2y \end{aligned}$ (b) $\begin{aligned} x' &= x + y \\ y' &= x + y \end{aligned}$

(c) $\begin{aligned} x' &= x + 3y \\ y' &= 2x + 2y \end{aligned}$ (d) $\begin{aligned} x' &= 5x - 6y \\ y' &= 3x - 4y \end{aligned}$

3. Let $B = \{(1,0),(0,1)\}$ be the standard basis for $\mathbb{R}^2$. Put

$$D = \{(1,1),(-1,0)\} .$$

Explain why D is a basis for $\mathbb{R}^2$ and then write down the following matrices:

$$A = [\mathrm{id}]_B^B , \quad C = [\mathrm{id}]_D^D \quad \text{and} \quad E = [\mathrm{id}]_B^D .$$

Now find E^{-1} in the usual way and check that indeed

$$E^{-1} = \big[\ [(1,0)]_D \quad [(0,1)]_D\ \big] = [\mathrm{id}]_D^B .$$

4. Let $f, g : \mathbb{R}^2 \to \mathbb{R}^2$ be linear transformations given by the following rules:

$$f(x,y) = (x+2y, 3x-4y) \qquad \text{and} \qquad g(x,y) = (3x-y, 2y) .$$

(a) Find each of the following, by direct calculation, where B and D are the bases for $\mathbb{R}^2$ in the previous exercise:

$$[f]_B^B , \quad [f]_D^D , \quad [g]_B^B , \quad [g]_D^D .$$

(If you have done this correctly, you should have produced a diagonal matrix representation for g.)

(b) Check, as the theory predicts, that the following equations hold:

$$[f]_D^D = [\mathrm{id}]_D^B [f]_B^B [\mathrm{id}]_B^D \qquad \text{and} \qquad [g]_D^D = [\mathrm{id}]_D^B [g]_B^B [\mathrm{id}]_B^D .$$

(c)* Find rules for linear operators $h, k : \mathbb{R}^2 \to \mathbb{R}^2$ such that $[h]_B^B = [f]_B^D$ and $[k]_B^B = [f]_D^B$.

5.* Working over $\mathbb{R}$, let $B = \{1, x, x^2\}$ be the standard basis for the vector space $\mathbb{P}_2$ of polynomials of degree at most 2. Put

$$D = \{1 + x^2, x + 2x^2, 1 + 2x + 3x^2\} .$$

Explain why D is a basis for $\mathbb{P}_2$ and then write down the matrix $E = [\mathrm{id}]_B^D$. Now find E^{-1} in the usual way and check that indeed

$$E^{-1} = \big[\ [1]_D \quad [x]_D \quad [x^2]_D\ \big] = [\mathrm{id}]_D^B .$$

Further Exercises:

6. Let $B = \{(1,0,0),(0,1,0),(0,0,1)\}$ be the standard basis for $\mathbb{R}^3$. Put

$$D \;=\; \{(1,0,1),(1,1,0),(1,1,1)\}\;.$$

Explain why D is a basis for $\mathbb{R}^3$ and then write down the matrix $E = [\mathrm{id}]_B^D$. Now find E^{-1} in the usual way and check that indeed

$$E^{-1} \;=\; \big[\; [(1,0,0)]_D \;\; [(0,1,0)]_D \;\; [(0,0,1)]_D \;\big] \;=\; [\mathrm{id}]_D^B\;.$$

7. Find the exponential matrix e^{tA} where A is each of the following matrices:

(a) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & 0 \\ 2 & 1 & 1 \end{bmatrix}$

8. Solve the following systems of differential equations, where $x = x(t)$, $y = y(t)$ and $z = z(t)$ are differentiable functions of a real variable t, with the same initial conditions

$$x(0) = -1\,,\quad y(0) = -4 \quad \text{and} \quad z(0) = 2$$

in each case:

(a) $\begin{aligned} x' &= -x \\ y' &= 2y \\ z' &= 3z \end{aligned}$ (b) $\begin{aligned} x' &= y - z \\ y' &= x + z \\ z' &= x + y \end{aligned}$

(c) $\begin{aligned} x' &= x + y + 2z \\ y' &= -y \\ z' &= 2x + y + z \end{aligned}$

9. Consider the real matrix $M = \begin{bmatrix} 2 & 5 & -3 \\ 1 & -4 & 7 \end{bmatrix}$.

(a) Write down the rule for the linear transformation $f : \mathbb{R}^3 \to \mathbb{R}^2$ such that the matrix of f with respect to the standard bases is M.

(b) Explain briefly why $B = \{(1,1,1),(1,1,0),(1,0,0)\}$ and $D = \{(1,3),(2,5)\}$ are bases for $\mathbb{R}^3$ and $\mathbb{R}^2$ respectively.

(c)* Find the matrix $[f]_D^B$ of f with respect to B and D.

10.* Let D be the differential operator that takes a differentiable function to its derivative. Explain why each of the following sets is a basis of the subspace of $\mathbb{R}^{\mathbb{R}}$ that it generates:

$$B_1 = \{1, x, x^2, x^3\}\,,\quad B_2 = \{\sin x, \cos x\}\,,\quad B_3 = \{e^x, e^{2x}, xe^{2x}\}\,.$$

Each of these subspaces consists of differentiable functions on which D acts as an operator. Find $[D]_{B_i}^{B_i}$ for $i = 1,2,3$ and calculate the rank and nullity of D in each case.