THE UNIVERSITY OF SYDNEY

MATH2022 LINEAR AND ABSTRACT ALGEBRA

Semester 1 **Exercises for Week 9**

Important Ideas and Useful Facts:

(i) Linear dependence and independence: Let V be a vector space over a field F, and $\mathbf{v}_1, \ldots, \mathbf{v}_k \in V$ for some $k \geq 1$. We call the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ and the set $X = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ *linearly independent* if, for all $\lambda_1, \ldots, \lambda_k \in F$,

$$\lambda_1\mathbf{v}_1 + \ldots + \lambda_k\mathbf{v}_k \ = \ \mathbf{0} \qquad \text{implies} \qquad \lambda_1 = \ldots = \lambda_k = 0 \ ,$$

equivalently, in the case $k > 1$, no vector from X can be expressed as a linear combination of other vectors from X. We say that they are *linearly dependent* otherwise, that is, if $X = \{\mathbf{0}\}$ or at least one vector from X can be expressed as a linear combination of other vectors from X. In particular if $\mathbf{0} \in X$, then X is linearly dependent. If $k = 1$ then X is linearly independent if and only if $\mathbf{v}_1$ is nonzero. If $k = 2$ then X is linearly independent if and only if neither of $\mathbf{v}_1$ nor $\mathbf{v}_2$ is a scalar multiple of the other. The emptyset $\emptyset$ is declared by definition to be *linearly independent*. If Y is an infinite subset of V then we say that Y is *linearly independent* if every finite subset is linearly independent, and otherwise *linearly dependent*.

(ii) Basis and dimension of a vector space: A *basis* for a vector space V is a linearly independent subset B that spans V. In particular, the empty set is a basis for the trivial vector space. If follows, when B is nonempty, that every vector in V can be expressed uniquely (up to the order of the vectors) as a linear combination of elements of B. In applications, a basis B is typically a nonempty finite ordered list of vectors (and order is important with respect to building matrices, see later). It is an important theorem that every vector space V has a basis and every basis for V has the same size (even when the size is infinite). The size of any basis for V is called the *dimension* of the vector space and denoted by $\dim(V)$. It is another important theorem that every linearly independent subset can be extended to a basis, and every spanning set contains a basis. It follows that, if V is known to be finite dimensional of dimension n, then any linearly independent set or any spanning set of size n is automatically a basis for V.

(iii) Standard bases: Let F be any field. If $n \geq 1$ then the *standard basis* for F^n is

$$B \ = \ \{\mathbf{e}_1 \, , \ \ldots \, , \ \mathbf{e}_n\}$$

where $\mathbf{e}_i = (0, \ldots, 0, 1, 0 \ldots, 0)$ with i in the *ith* place, for $i = 1, \ldots, n$. In particular, F^n has dimension n. The empty set $\emptyset$ is the basis for any trivial vector space (such as F^0), so the dimension of any trivial vector space is zero. Let $\mathbb{P}_n$ denote the vector space of polynomials in x over F of degree at most n, where $n \geq 0$. Then the *standard basis* for $\mathbb{P}_n$ is

$$B \ = \ \{1, x, \ldots, x^n\} \ .$$

In particular, $\mathbb{P}_n$ has dimension $n + 1$.

(iv) Coordinates of a vector with respect to a basis: Let V be a vector space over a field F of dimension n and let $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be an ordered basis for V. Let $\mathbf{v} \in V$. Then there are unique scalars $\lambda_1, \ldots, \lambda_n \in F$ such that

$$\mathbf{v} \;=\; \lambda_1 \mathbf{b}_1 + \ldots + \lambda_n \mathbf{b}_n \,.$$

We define the *coordinate vector (coordinates) of* $\mathbf{v}$ *with respect to* B to be the following column vector:

$$[\mathbf{v}]_B \;=\; \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix}$$

If $V = F^n$ and B is the standard basis for V then $[\mathbf{v}]_B = \mathbf{v}^\top$, for all $\mathbf{v} \in V$.

(v) Vector spaces with the same dimension are isomorphic: If V is a vector space over a field F having a basis B with $n \geq 1$ elements, so has dimension n, then V is isomorphic to F^n under the mapping $\mathbf{v} \mapsto [\mathbf{v}]_B^\top$ (for $\mathbf{v} \in V$), where the row vector $[\mathbf{v}]_B^\top$ is, as usual, identified with the n-tuple in F^n. Obviously, all trivial vector spaces, that is, vector spaces of dimension zero, are isomorphic to F^0.

(vi) Isomorphic vector spaces have the same dimension: If V and W are isomorphic vector spaces over a field F and B is a basis for V, then it follows that the image of B under the isomorphism is a basis for W, and so V and W have the same dimension.

(vii) Nonzero rows of a matrix in row echelon form are linearly independent: The nonzero rows of a matrix M (over a field F) in row echelon form are linearly independent and therefore form a basis for the row space of any matrix over F that can be row reduced to yield the same nonzero rows as M.

(viii) Rank of a matrix: It is an important theorem that the row and column spaces of a matrix M have the same dimension, called the *rank* of M, denoted by $\mathrm{rank}(M)$. The rank is the number of nonzero rows when M or $M^\top$ is row reduced to row echelon form.

(ix) Nullity of a matrix: Let M be an $m \times n$ matrix over a field F. Recall that the *null space* of M may refer either to the vector space

$$\{\text{column vectors } \mathbf{v} \text{ of length } n \mid M\mathbf{v} = \mathbf{0}\} \,,$$

or the solution space of the associated homogeneous system of m equations in n variables:

$$\{\mathbf{x} \in F^n \mid M\mathbf{x}^\top = \mathbf{0}\} \,.$$

The dimension of the null space is called the *nullity* of M, denoted by $\mathrm{nullity}(M)$. The nullity of M is the number of parameters that need to be introduced to yield the solution of the associated homogeneous system of equations.

(x) Rank-Nullity Theorem for matrices: If M is an $m \times n$ matrix then $\mathrm{rank}(M) + \mathrm{nullity}(M) = n$.

Tutorial Exercises:

1. Explain why $\{1, i\}$ is a basis for $\mathbb{C}$ as a vector space over $\mathbb{R}$ where $i = \sqrt{-1}$ (so that $\mathbb{C}$ becomes two dimensional).

2. Explain why $B = \{(1,1,1), (0,1,1), (0,0,1)\}$ is basis for $\mathbb{R}^3$ and find the coordinates of $\mathbf{v}$ with respect to B in the following cases:

(a) $\mathbf{v} = (3,1,-4)$ (b) $\mathbf{v} = (1,0,0)$ (c) $\mathbf{v} = (2,1,0)$

3. Consider the following real matrices:

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 4 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 3 & -4 \\ 6 & 5 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & 8 & -11 \\ 16 & 10 & 9 \end{bmatrix},$$

$$M = \begin{bmatrix} 1 & 2 & -3 & 4 & 0 & 1 \\ 1 & 3 & -4 & 6 & 5 & 4 \\ 3 & 8 & -11 & 16 & 10 & 9 \end{bmatrix}.$$

Row reduce M and $M^\top$ and observe that they have the same rank. Explain why A, B and C are linearly dependent. Express one of A, B, C as a linear combination of the other two.

4. Find a basis for the row space and a basis for the column space of the following real matrix:

$$M = \begin{bmatrix} 2 & 1 & 0 & -4 \\ 3 & 0 & -1 & 2 \\ 6 & -3 & -4 & 20 \end{bmatrix}$$

Verify that the row space and column space of M have the same dimension. Now find a basis for the null space of M. Verify that the Rank-Nullity Theorem holds in this case.

5. Decide whether the following sets of vectors from $\mathbb{R}^\mathbb{R}$ (denoted by the rule for their outputs given inputs $x \in \mathbb{R}$) are linearly independent:

(a) $\{1 + x + x^2\,,\ 1 - x\,,\ 2 + x^2\}$ (b) $\{1-x-x^2, 1+x^2, 1+x+x^2+x^3, 1-x^3\}$

(c) $\{\sin x\,,\ \cos x\}$ (d) $\{1\,,\ \cos 2x\,,\ \sin^2 x\}$

6.* Recall that $\mathbb{Q}$ is the field of rational numbers and that $\sqrt{2} \notin \mathbb{Q}$. Put

$$\mathbb{Q}(\sqrt{2}) \;=\; \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}\,.$$

Prove that $\mathbb{Q}(\sqrt{2})$ is closed under addition and multiplication and taking inverses of nonzero elements. It follows that $\mathbb{Q}(\sqrt{2})$ is a field, and becomes a vector space over $\mathbb{Q}$ by restricting scalar multiplication. Explain why $\{1, \sqrt{2}\}$ is a basis for $\mathbb{Q}(\sqrt{2})$ (so that $\mathbb{Q}(\sqrt{2})$ becomes two dimensional as a vector space over $\mathbb{Q}$).

Further Exercises:

7. Explain why $B = \{1, x-1, (x-1)^2\}$ is a basis for the vector space $\mathbb{P}_2$ of real polynomials of degree at most 2. Find the coordinates of $p(x)$ with respect to B in the following cases:

(a) $p(x) = 2x^2 - 5x + 6$ (b) $p(x) = x^2 + 1$ (c) $p(x) = x^2 - 1$

8. Let F be any field. Find a basis for $\mathrm{Mat}_{2,3}$, the set of 2×3 matrices over F, regarded as a vector space over F with respect to usual matrix addition and scalar multiplication. More generally, explain why $\mathrm{Mat}_{m,n}$ becomes an mn-dimensional vector space over F, for any $m, n \geq 1$.

9. Find the rank and nullity of the following matrices, and a basis for the null space in each case:

(a) $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ over $\mathbb{R}$, $\mathbb{Z}_2$ and $\mathbb{Z}_3$. (b) $B = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 3 & 4 \\ 1 & 0 & 2 \end{bmatrix}$ over $\mathbb{R}$ and $\mathbb{Z}_5$.

(c) $C = \begin{bmatrix} -1 & 0 & 3 & -2 \\ -1 & 1 & 0 & 3 \\ -1 & 0 & -2 & 3 \end{bmatrix}$ over $\mathbb{R}$ and $\mathbb{Z}_5$.

10. Use the previous exercise, or otherwise, to decide which of the following sets of vectors are linearly independent, as subsets of F^n for appropriate F and n:

(a) $X = \{(0,1,1)\,,\ (1,0,1)\,,\ (0,0,1)\}$ over $\mathbb{R}$, $\mathbb{Z}_2$ and $\mathbb{Z}_3$.

(b) $X = \{(1,-1,-1)\,,\ (0,3,4)\,,\ (1,0,2)\}$ over $\mathbb{R}$ and $\mathbb{Z}_5$.

(c) $X = \{(1,0,1)\,,\ (-1,3,0)\,,\ (-1,4,2)\}$ over $\mathbb{R}$ and $\mathbb{Z}_5$.

(d) $X = \{(-1,0,3,-2)\,,\ (-1,1,0,3)\,,\ (-1,0,-2,3)\}$ over $\mathbb{R}$ and $\mathbb{Z}_5$.

(e) $X = \{(-1,-1,-1)\,,\ (0,1,0)\,,\ (3,0,-2)\,,\ (-2,3,3))\}$ over $\mathbb{R}$ and $\mathbb{Z}_5$.

11. Verify carefully, from the definition, that if $\mathbf{v}$ and $\mathbf{w}$ are vectors from a vector space V over a field F then $\mathbf{v}$ and $\mathbf{w}$ are linearly independent if and only if neither $\mathbf{v}$ nor $\mathbf{w}$ can be expressed as a scalar multiple of the other.

12. Suppose that $k > 1$ and $\mathbf{v}_1, \dots, \mathbf{v}_k$ are vectors from a vector space. Verify carefully from the definition that $\mathbf{v}_1, \dots, \mathbf{v}_k$ are linearly independent if and only if no vector from this list can be expressed as a linear combination of other vectors from the list.

13.* Prove carefully that isomorphic vector spaces over the same field have the same dimension.

14.* Let $\mathbf{v}$ and $\mathbf{w}$ be eigenvectors for a square matrix M with respect to eigenvalues λ and μ respectively. Prove that is $\lambda \neq \mu$ then neither $\mathbf{v}$ nor $\mathbf{w}$ can be expressed as a scalar multiple of the other, and hence $\mathbf{v}$ and $\mathbf{w}$ are linearly independent.

15.* Let $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v_3}$ be eigenvectors of a square matrix M with respect to eigenvalues λ_1, λ_2 and λ_3 respectively, where λ_1, λ_2 and λ_3 are distinct. Prove that $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are linearly independent. (This exercise generalises to prove the theorem that any set of eigenvectors corresponding to distinct eigenvalues of a square matrix M is linearly independent.)