The University of Sydney

MATH2022 Linear and Abstract Algebra

Semester 1 **Exercises for Week 6**

Important Ideas and Useful Facts:

(i) Diagonal matrices: A square matrix D is *diagonal* if all entries off the diagonal are zero. If D and E are diagonal then $DE = ED$ is also diagonal, and its diagonal entries are simply the products of corresponding diagonal entries of D and E. Thus the diagonal elements of D^n are just the nth powers of the diagonal elements of D. A *scalar* matrix is a diagonal matrix in which all elements along the diagonal are equal. The scalar matrices commute with all square matrices of the same size.

(ii) Diagonalisation: Let M be a square $n \times n$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ and corresponding eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$. Then

$$MP \;=\; PD$$

where D is the diagonal matrix with eigenvalues down the diagonal and P the matrix with corresponding eigenvectors as columns. If P is invertible then

$$M \;=\; PDP^{-1} \qquad \text{and} \qquad D \;=\; P^{-1}MP\ .$$

In this case we say that M is *diagonalisable*, in which case powers of M can be found easily by the formula

$$M^k \;=\; PD^kP^{-1}\ .$$

If the eigenvalues are all different then P is invertible and M is diagonalisable.

(iii) Similar matrices: Two matrices A and B are said to be *similar* or *conjugate* if there is an invertible matrix P such that $B = P^{-1}AP$. Similarity is an equivalence relation (that is, similarity is reflexive, symmetric and transitive). In particular, a matrix is diagonalisable if and only if it is similar to a diagonal matrix.

(iv) Stochastic matrices: A square matrix M is *stochastic* if all the entries are nonnegative and the columns add to 1, and *regular* if, further, some positive power of M has all positive entries. A column matrix $\mathbf{v}$ is a *probability vector* if all of its entries are nonnegative and add to 1, and, further, becomes a *steady state vector* for M if $M\mathbf{v} = \mathbf{v}$.

(v) Existence and uniqueness of a steady state vector: If M is a regular stochastic matrix then there exists a unique steady state vector $\mathbf{v}$ for M, in which case, for any probability vector $\mathbf{x}$,

$$\lim_{k\to\infty} M^k\mathbf{x} \;=\; \mathbf{v}\ .$$

(vi) Perron's Theorem and existence of dominant eigenvalues: If M is a square matrix all of whose entries are positive then M has a positive real eigenvalue λ such that $|\mu| \le \lambda$ for all eigenvalues μ of M, and, furthermore, there exists an eigenvector corresponding to λ, all of whose entries are positive.

Questions labelled with an asterisk are suitable for students aiming for a distinction or higher.

Tutorial Exercises:

1. Working over $\mathbb{R}$, diagonalise M and find M^k for any positive integer k where M is each of the following:

(a) $\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix}$ (e) $\begin{bmatrix} 0 & 1 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ (f)* $\begin{bmatrix} -7 & -2 & 6 \\ -2 & 1 & 2 \\ -10 & -2 & 9 \end{bmatrix}$

2. Verify that the matrix

$$M = \begin{bmatrix} 1/2 & 2/5 \\ 1/2 & 3/5 \end{bmatrix}$$

is regular stochastic and find its unique steady state vector. Find an expression for M^n and check the limiting behaviour as $n \to \infty$.

3. Consider the real matrix $A = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$. Put $\mathbf{v}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_k = A\mathbf{v}_{k-1}$ for $k \geq 1$.

(a) Compute $\mathbf{v}_1, \dots, \mathbf{v}_5$ exactly.

(b) Find the characteristic polynomial of A. Find its two roots exactly, and also write down their decimal expansions correct to five decimal places.

(c) Let $\mathbf{v}$ be the unique scalar multiple of $\mathbf{v}_5$ with 1 in the first position. Verify that, correct to five decimal places, $\mathbf{v}$ is an eigenvector of A corresponding to its largest eigenvalue.

4. Consider the matrix $B = \begin{bmatrix} -4 & 3 \\ 3 & -2 \end{bmatrix}$. Put $\mathbf{w}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{w}_k = B\mathbf{w}_{k-1}$ for $k \geq 1$.

(a) Compute $\mathbf{w}_1, \dots, \mathbf{w}_5$ exactly.

(b) Let $\mathbf{w}$ be the unique scalar multiple of $\mathbf{w}_5$ with 1 in the first position. Verify that, correct to five decimal places, $\mathbf{w}$ is an eigenvector of B.

(c)* Check that $B = A^{-1}$ from the previous exercise. Explain how the phenomenon in part (b) is connected to the phenomenon in part (c) of the previous exercise.

5.* The sequence of *Fibonacci numbers* is obtained by writing down 1 twice, and obtaining each successive number by adding the previous two numbers together:

$$1\,,\ 1\,,\ 2\,,\ 3\,,\ 5\,,\ 8\,,\ 13\,,\ 21\,,\ 34\,,\ 55\,,\ 89\,,\ 144\,,\ \dots$$

If we let x_n denote the nth Fibonacci number then

$$x_1 = x_2 = 1\,, \quad x_n = x_{n-1} + x_{n-2} \quad \text{for } n \geq 3\,,$$

so that, by a simple induction,

$$\begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_{n-1} \\ x_{n-2} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{n-2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Diagonalise $M = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ to find a general formula for the nth Fibonacci number.

Further Exercises:

6. Consider the matrix $M = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 1/2 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix}$.

(a) Verify that M is stochastic and regular.

(b) Find the unique steady state vector for M.

(c)* Find an expression for M^n and check the limiting behaviour observed in part (b) as $n \to \infty$.

7. Verify that if A and B are similar matrices such that B is a scalar matrix then $A = B$.

8.* Prove that if A is a real square matrix with exactly one eigenvalue $\lambda \in \mathbb{C}$ then in fact $\lambda \in \mathbb{R}$ and, further, A is diagonalisable if and only if $A = \lambda I$.

9.* Consider the matrices

$$A \;=\; \begin{bmatrix} 3 & -1 \\ 4 & -1 \end{bmatrix} \qquad \text{and} \qquad B \;=\; \begin{bmatrix} 34 & 99 & 0 \\ -11 & -32 & 0 \\ -4 & -12 & 1 \end{bmatrix}$$

Find the characteristic polynomials of A and B. Deduce, from the previous exercise, that neither A nor B is diagonalisable.

10.* Let A be a square matrix over a field F. Prove that $A\mathbf{v} = \mathbf{0}$ for some nonzero column vector $\mathbf{v}$ if and only if A is not invertible.

11. Let M be an upper or lower triangular $n \times n$ matrix with diagonal entries $\lambda_1, \ldots, \lambda_n$. Verify that the characteristic polynomial of M is

$$\chi(\lambda) \;=\; \det(\lambda I - M) \;=\; (\lambda - \lambda_1) \ldots (\lambda - \lambda_n)\,.$$

Thus the eigenvalues of M are just its diagonal entries.

12.* Let M be a square matrix over $\mathbb{C}$. Define the *trace* of M, denoted by $\mathrm{tr}(M)$ to be the sum of the diagonal elements of M.

(a) Prove that if A and B are square matrices of the same size then $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.

(b) Deduce that if A and B are similar matrices then $\mathrm{tr}(A) = \mathrm{tr}(B)$.

(c) It is a theorem that M is similar to a triangular matrix. Deduce from this and the previous exercise that $\mathrm{tr}(M)$ is the sum of the eigenvalues of M (with multiplicity).