The University of Sydney
MATH2022 Linear and Abstract Algebra

Semester 1 **Second Quiz Sample Exercises** 2024

The Second Quiz is on April 24 on Canvas. See the announcement on Ed for relevant information. Below are some sample problems for extra practice. This need not be a perfect replica of our quiz, however!

1. Working over $\mathbb{Z}_5$, the eigenvalues of the matrix $\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$ are

(a) 1 and 2. (b) 2 and 3. (c) 4 only.
(d) 2 and 4. (e) 1 only.

2. Which one of the following is a true statement about the real matrix $M = \begin{bmatrix} 0 & 2 \\ -3 & -5 \end{bmatrix}$?

(a) -2 is an eigenvalue of M with corresponding eigenspace $\left\{ \begin{bmatrix} t \\ t \end{bmatrix} \,\middle|\, t \in \mathbb{R} \right\}$.

(b) 2 is an eigenvalue of M with corresponding eigenspace $\left\{ \begin{bmatrix} t \\ t \end{bmatrix} \,\middle|\, t \in \mathbb{R} \right\}$.

(c) 3 is an eigenvalue of M with corresponding eigenspace $\left\{ \begin{bmatrix} -t \\ t \end{bmatrix} \,\middle|\, t \in \mathbb{R} \right\}$.

(d) -2 is an eigenvalue of M with corresponding eigenspace $\left\{ \begin{bmatrix} \frac{2t}{3} \\ t \end{bmatrix} \,\middle|\, t \in \mathbb{R} \right\}$.

(e) -3 is an eigenvalue of M with corresponding eigenspace $\left\{ \begin{bmatrix} -\frac{2t}{3} \\ t \end{bmatrix} \,\middle|\, t \in \mathbb{R} \right\}$.

3. Which one of the following is a true statement about the matrix M that represents the reflection of $\mathbb{R}^2$ in the line given by the equation $y - 2x = 0$?

(a) 1 is an eigenvalue of M with corresponding eigenvector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

(b) 1 is an eigenvalue of M with corresponding eigenvector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

(c) -1 is an eigenvalue of M with corresponding eigenvector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

(d) -1 is an eigenvalue of M with corresponding eigenvector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

(e) 2 is an eigenvalue of M with corresponding eigenvector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

4. Let $M = \begin{bmatrix} 1 & 1 \\ 4 & 5 \end{bmatrix}$ with entries from $\mathbb{Z}_7$. Then $M = PDP^{-1}$ where

(a) $D = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$ and $P = \begin{bmatrix} 6 & 1 \\ 4 & 1 \end{bmatrix}$ (b) $D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & 6 \\ 1 & 4 \end{bmatrix}$

(c) $D = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & 1 \\ 4 & 6 \end{bmatrix}$ (d) $D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ and $P = \begin{bmatrix} 6 & 1 \\ 4 & 1 \end{bmatrix}$

(e) $D = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & 6 \\ 1 & 4 \end{bmatrix}$

5. Working over $\mathbb{R}$, suppose that $M = PDP^{-1}$ where $P = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$.

Then, for any positive integer k, we have that M^k is

(a) $\begin{bmatrix} -3^k & 2^k - 3^k \\ 0 & -2^k \end{bmatrix}$ (b) $\begin{bmatrix} 3^k & 2^k - 3^k \\ 0 & 2^k \end{bmatrix}$ (c) $\begin{bmatrix} 2^k & 3^k - 2^k \\ 0 & 3^k \end{bmatrix}$

(d) $\begin{bmatrix} 2^k & 1 \\ 0 & 3^k \end{bmatrix}$ (e) $\begin{bmatrix} 2k & k \\ 0 & 3k \end{bmatrix}$

6. The characteristic polynomial of the real matrix $M = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ -2 & 0 & 3 \end{bmatrix}$ is

(a) $\lambda^3 - 5\lambda^2 + 6\lambda$. (b) $\lambda^3 - 5\lambda^2 + 8\lambda - 4$. (c) $\lambda^3 + 5\lambda^2 + 8\lambda + 4$.

(d) $\lambda^3 - \lambda^2 - 4\lambda + 4$. (e) $\lambda^3 - 5\lambda^2 + 4\lambda + 4$.

7. Which of the following expressions describes M^{-1} where $M = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ -2 & 0 & 3 \end{bmatrix}$ and I is the 3×3 identity matrix, working over $\mathbb{R}$?

(a) $\frac{1}{4}(M^2 - 5M + 8I)$ (b) $M^2 - 5M + 6I$ (c) $-\frac{1}{4}(M^2 - 5M + 4I)$

(d) $-\frac{1}{4}(M^2 + 5M + 8I)$ (e) $\frac{1}{4}(M^2 - M - 4I)$

8. Find the steady state probability vector of the following 3×3 stochastic matrix:

$$\begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{2}{3} \\ \frac{1}{2} & \frac{2}{3} & 0 \end{bmatrix}$$

(a) $\begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$ (b) $\begin{bmatrix} \frac{1}{4} \\ \frac{3}{8} \\ \frac{3}{8} \end{bmatrix}$ (c) $\begin{bmatrix} \frac{3}{8} \\ \frac{3}{8} \\ \frac{1}{4} \end{bmatrix}$ (d) $\begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix}$ (e) $\begin{bmatrix} 0 \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$

9. Which one of the following matrices is not diagonalisable, working over $\mathbb{C}$?

(a) $\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}$

(d) $\begin{bmatrix} 0 & -1 \\ -1 & -2 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix}$

10. Which one of the following rules for $f : \mathbb{R}^2 \to \mathbb{R}^2$ defines a linear transformation?

(a) $f(x, y) = (x^2, y^2)$ (b) $f(x, y) = (y+x, x-4y)$ (c) $f(x, y) = (y + 1, x + y)$

(d) $f(x, y) = (xy, y)$ (e) $f(x, y) = (2x, 3y + 4)$

11. Find the matrix corresponding to the linear transformation $f : \mathbb{R}^2 \to \mathbb{R}^3$ with the following rule:

$$f(x, y) = (6x - y, x + 2y, y - x) \ .$$

(a) $\begin{bmatrix} 6 & -1 \\ 1 & 2 \\ -1 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 6 & -1 \\ 1 & 2 \\ 1 & -1 \end{bmatrix}$ (c) $\begin{bmatrix} -1 & 6 \\ 2 & 1 \\ 1 & -1 \end{bmatrix}$

(d) $\begin{bmatrix} 6 & 1 & 1 \\ -1 & 2 & -1 \end{bmatrix}$ (e) $\begin{bmatrix} 6 & 1 & -1 \\ -1 & 2 & -1 \end{bmatrix}$

12. Suppose $f : \mathbb{R}^3 \to \mathbb{R}^2$ and $g : \mathbb{R}^2 \to \mathbb{R}^4$ are linear transformations represented by

$$M_f = \begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 3 \end{bmatrix} \qquad \text{and} \qquad M_g = \begin{bmatrix} 0 & 1 \\ 4 & 1 \\ 1 & 0 \\ -1 & -1 \end{bmatrix}$$

respectively. Find the rule for the linear transformation $gf : \mathbb{R}^3 \to \mathbb{R}^4$.

(a) $(gf)(x, y, z) = (2x - 3y + 2z\,,\ x - y + 8z\,,\ -y - 4z\,,\ 3x + 2y)$
(b) $(gf)(x, y, z) = (x + 9y + 2z\,,\ -x - y\,,\ 3x + 7y + z\,,\ 3x - 2y)$
(c) $(gf)(x, y, z) = (x + y - 3z\,,\ 9x + y - 7z\,,\ 3x - y + 4z, -3x + y - 4z)$
(d) $(gf)(x, y, z) = (x - y + 3z\,,\ 9x - y + 7z\,,\ 2x + z\,,\ -3x + y - 4z)$
(e) $(gf)(x, y, z) = (x - y + 3z\,,\ 9x - y + 7z\,,\ -3x + y - 4z\,,\ 2x + z)$

13. Define the linear transformation $L : \mathbb{Z}_3^3 \to \mathbb{Z}_3^3$ by the following rule:

$$L(x, y, z) = (x + 2y, y + 2z, x).$$

Find the rule for the inverse linear transformation.

(a) $L^{-1}(x, y, z) = (x + y\,,\ z\,,\ y + z)$
(b) $L^{-1}(x, y, z) = (z\,,\ 2x + z\,,\ 2x + 2y + z)$
(c) $L^{-1}(x, y, z) = (z\,,\ x + y + z\,,\ y + z)$
(d) $L^{-1}(x, y, z) = (x + 2y\,,\ z\,,\ y + 2z)$
(e) $L^{-1}(x, y, z) = (x + y\,,\ y + z\,,\ 2x)$

14. Which of the following subsets is a subspace of the vector space $\mathbb{R}^3$?

(a) $\{(x, y, z) \in \mathbb{R}^3 \mid x + 2y - 3z = 1\}$
(b) $\{(x, y, z) \in \mathbb{R}^3 \mid x + y + z \leq 0\}$
(c) $\{(x, y, z) \in \mathbb{R}^3 \mid x + y = 0\}$
(d) $\{(x, y, z) \in \mathbb{R}^3 \mid x = 2,\ y + z = 0\}$
(e) $\{(x, y, z) \in \mathbb{R}^3 \mid x + 2y = z - 1\}$

15. Consider the subspace $W = \{a + bx + ax^2 \mid a,\ b \in \mathbb{R}\}$ of $\mathbb{P}_2$. Which of the following is a spanning set for W?

(a) $\{x,\ 1+x^2\}$ (b) $\{1,\ x,\ x^2\}$ (c) $\{1+x+x^2\}$ (d) $\{1+x,\ x^2\}$ (e) $\{x,\ x^2\}$

16. Which of the following groups, under addition, is cyclic?

(a) $\mathbb{Z}_3 \times \mathbb{Z}_3$ (b) $\mathbb{Z}_2 \times \mathbb{Z}_4$ (c) $\mathbb{Z}_2 \times \mathbb{Z}_3$ (d) $\mathbb{Z}_2 \times \mathbb{Z}_2$ (e) $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

17. Which of the following elements of the multiplicative group $\mathbb{Z}_7 \setminus \{0\}$ is a generator?

(a) 1 (b) 2 (c) 3 (d) 4 (e) 6

18. Which of the following elements of the additive group $\mathbb{Z}_8$ is a generator?

(a) 0 (b) 2 (c) 3 (d) 4 (e) 6