The University of Sydney

MATH2022 Linear and Abstract Algebra

| Semester 1 | **Exercises for Week 5** |
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**Important Ideas and Useful Facts:**

(i) Eigenvalues and eigenvectors: Let $M$ be a square matrix, $\mathbf{x}$ a nonzero column vector and $\lambda$ a scalar such that
$$M\mathbf{x} \;=\; \lambda\mathbf{x}\,.$$
Then $\lambda$ is called an *eigenvalue* of $M$ and $\mathbf{x}$ is called an *eigenvector* of $M$ associated with or corresponding to the eigenvalue $\lambda$.

(ii) The eigenspace of a matrix: The *eigenspace* of a square matrix $M$ associated with an eigenvalue $\lambda$ is the collection
$$\left\{\, \mathbf{v} \;\middle|\; M\mathbf{v} = \lambda\mathbf{v} \,\right\} \;=\; \left\{\, \mathbf{v} \;\middle|\; (\lambda I - M)\mathbf{v} = \mathbf{0} \,\right\}$$
comprising all of the eigenvectors of $M$ associated with $\lambda$ and the zero vector (which is never an eigenvector).

(iii) Description of eigenvalues in terms of determinants: A scalar $\lambda$ is an eigenvalue of a square matrix $M$ if and only if
$$\det(\lambda I - M) \;=\; 0\,.$$

(iv) The characteristic polynomial of a square matrix: The expression $\det(\lambda I - M)$ is always a polynomial in $\lambda$ and is called the *characteristic polynomial* of $M$. Thus the eigenvalues of a square matrix are precisely the roots of its characteristic polynomial.

(v) Finding eigenspaces: Finding the eigenspace corresponding to the eigenvalue $\lambda$ of a matrix $M$ is equivalent to solving the homogeneous system with coefficient matrix $\lambda I - M$. After the eigenspace has been found, substituting particular values of the parameters yields particular eigenvectors.

(vi) Eigenvalues of a triangular matrix: The eigenvalues of a triangular matrix are simply the diagonal entries.

(vii) The Cayley-Hamilton Theorem: Every square matrix $M$ is the root of its own characteristic polynomial, that is,
$$\chi(M) \;=\; 0\,,$$
where $\chi(\lambda) = \det(\lambda I - M)$ denotes the characteristic polynomial, $\chi(M)$ is the result of evaluating the matrix expression obtained from $\chi(\lambda)$ by substituting $M$ for the indeterminate $\lambda$ and $I$ for the constant 1, and 0 denotes the zero matrix.

(viii) Reflection matrices: A *reflection matrix* in the real plane has the form

$$M = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

for some real $\theta$, and corresponds to reflection in the plane through a line through the origin making an angle $\theta$ with the positive $x$-axis. The eigenvalues of $M$ are $\pm 1$. The eigenspace corresponding to 1 is the line of reflection. The eigenspace corresponding to $-1$ is the line through the origin perpendicular to the line of reflection.

(ix) Rotation matrices: A *rotation matrix* in the real plane has the form

$$M = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

for some real $\theta$, and corresponds to rotation in the plane anticlockwise through an angle $\theta$ about the origin. The eigenvalues of $M$ are the complex numbers

$$e^{i\theta} = \cos\theta \pm i\sin\theta$$

where $i = \sqrt{-1}$. The eigenvalues are real if and only if $\theta$ is an integer multiple of $\pi$, in which case all nonzero vectors are eigenvectors, corresponding to eigenvalue 1 if the integer multiple is even, and corresponding to $-1$ if the integer multiple is odd.

*Questions labelled with an asterisk are suitable for students aiming for a distinction or higher.*

**Tutorial Exercises:**

**1.** Find eigenvalues and eigenspaces for the following matrices, working over $\mathbb{R}$:

(a) $A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$ (b) $B = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$ (c) $C = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$

**2.** Find eigenvalues and eigenspaces for the following matrices, working over $\mathbb{R}$:

(a) $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix}$ (b) $B = \begin{bmatrix} 0 & 1 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ (c)* $C = \begin{bmatrix} -7 & -2 & 6 \\ -2 & 1 & 2 \\ -10 & -2 & 9 \end{bmatrix}$

**3.** Let $A$ be a square matrix over a field $F$ with eigenvalue $\lambda$. Prove the following implications:

(a) $A^2 = 0 \quad \Rightarrow \quad \lambda = 0$ (b) $A^2 = A \quad \Rightarrow \quad (\lambda = 0 \text{ or } \lambda = 1)$

(c) $A^2 = I \quad \Rightarrow \quad (\lambda = 1 \text{ or } \lambda = -1)$

**4.** Consider the group $G$ of symmetries of a regular hexagon, generated by a rotation $\alpha$ one sixth of a turn and a reflection $\beta$.

(a) Write out all of the elements of $G$ as simply as possible. What happens if you reverse the order of the generators in each of the expressions that you found?

(b) Evaluate $\beta\alpha^5\beta^5\alpha^{-3}\beta^{-3}\alpha^8\beta$ as simply as possible.

(c)* Find a rotation matrix $A$ and a reflection matrix $B$ that together generate a group $H$ in a one-one correspondence to $G$. Explain why matrix multiplication in $H$ corresponds to composition of symmetries in $G$ under this correspondence (thus exhibiting an *isomorphism* between $H$ and $G$).

**5.*** Consider the permutations $\alpha = (1\ 2\ 3\ 4\ 5\ 6)$ and $\beta = (1\ 6)(2\ 5)(3\ 4)$. Verify that $\alpha^6 = \beta^2 = 1$ and $\beta^{-1}\alpha\beta = \alpha^{-1}$. Write out all permutations in $\langle \alpha, \beta \rangle$ using cycle notation. Find all $\gamma \in \mathrm{Sym}(6)$ such that $\gamma^{-1}\alpha\gamma = \alpha^{-1}$. What do you notice?

**6.*** Consider the matrix $M = \begin{bmatrix} 2 & 2 & 4 \\ 2 & 5 & 8 \\ -1 & -2 & -3 \end{bmatrix}$.

(a) Verify directly that $M^2 = 3M - 2I$ and that

$$\chi(\lambda) \;=\; \det(\lambda I - M) \;=\; (\lambda - 1)^2(\lambda - 2) \;=\; \lambda^3 - 4\lambda^2 + 5\lambda - 2\,.$$

(b) What conclusion follows from the second part of (a) and the Cayley-Hamilton Theorem? How is this conclusion reconciled with the first part of (a)?

(c) Prove by induction that, for all positive integers $k$,

$$M^k \;=\; (2^k - 1)M + (2 - 2^k)I\,.$$

Now deduce this formula for all integers $k$.

(d) Evaluate $M^5$, $M^{-1}$ and $M^{-5}$.

**Further Exercises:**

**7.** Consider the matrices

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad B = \begin{bmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{bmatrix}, \quad C = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

for some $\theta, \phi \in \mathbb{R}$. Describe each of the following as simply as possible:

(a) $A^2$ (b) $B^2$ (c) $A^{-3}$ (d) $AB$ (e) $AC$

(f) $BA$ (g) $BC$ (h) $ABA$ (i) $BA^2B$ (j) $BAC$

**8.** Suppose that $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Verify that the characteristic polynomial of $M$ is

$$\chi(\lambda) \;=\; \lambda^2 - (a+d)\lambda + ad - bc\,.$$

Now also verify that

$$M^2 - (a+d)M + (ad-bc)I \;=\; 0\,.$$

This verifies the Cayley-Hamilton Theorem directly for $2 \times 2$ matrices.

**9.** Let $\lambda$ be an eigenvalue of a matrix $M$.

(a) Show that $\lambda^k$ is an eigenvalue of $M^k$ for all positive integers $k$.

(b) Show that if $M$ is invertible then $\lambda$ is nonzero and $\lambda^{-1}$ is an eigenvalue of $M^{-1}$.

**10.*** Working over $\mathbb{C}$, find the eigenspaces of a rotation matrix.

**11.*** Let $A$ be a square matrix. Denote by $A_{ij}$ the square matrix that results by deleting the $i$th row and $j$th column of $A$. Define the *adjugate matrix* adj$A$ to be the square matrix, of the same size as $A$, whose $(j,i)$-entry is $(-1)^{i+j}\det A_{ij}$. Quoting results about determinants, verify that

$$A(\mathrm{adj}A) \;=\; (\mathrm{adj}A)A \;=\; (\det A)I\,.$$

This proves that $A$ is invertible if and only if $\det A \neq 0$, in which case $A^{-1} = \frac{1}{\det A}\mathrm{adj}A$.

**12.*** Use the adjugate matrix to verify the following fact, known as *Cramer's Rule*: if $M$ is an invertible $n \times n$ matrix and $\mathbf{c}$ a column vector, then the equation $M\mathbf{x} = \mathbf{c}$ has a unique solution $\mathbf{x}$ whose $i$th entry is

$$x_i \;=\; \frac{\det M_i}{\det M}$$

where $M_i$ is the matrix obtained by replacing the $i$th column of $M$ by $\mathbf{c}$. Use Cramer's Rule to solve the following system of equations:

$$\begin{array}{rcrcrcr} 2x & + & 3y & + & 4z & = & -4 \\ 5x & + & 5y & + & 6z & = & -3 \\ 3x & + & y & + & 2z & = & -1 \end{array}$$