Solution 1.

(a)

We need to find $\lambda_1, \lambda_2, \lambda_3$ such that:

$$\lambda_1(1,0,3) + \lambda_2(2,1,8) + \lambda_3(1,-1,2) = (3,-5,4)$$

written in augument matrix:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & -1 & -5 \\ 3 & 8 & 2 & 4 \end{array}\right] \underset{\sim}{R_3=3R_1-R_3} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & -1 & -5 \\ 0 & -2 & 1 & 5 \end{array}\right] \underset{\sim}{R_3=2R_1+R_3} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & -1 & -5 \\ 0 & 0 & -1 & -5 \end{array}\right]$$

So that, we have:

$$\begin{cases} \lambda_1 = -2 \\ \lambda_2 = 0 \\ \lambda_3 = 5 \end{cases}$$

(b)

We need to find $\lambda_1, \lambda_2, \lambda_3$ such that:

$$\lambda_1(1+t^2) + \lambda_2(t+t^2) + \lambda_3(1+2t+t^2) = (1+4t+7t^2)$$

which is

$$(\lambda_1+\lambda_3) + (\lambda_2+2\lambda_3)t + (\lambda_1+\lambda_2+\lambda_3)t^2 = (1+4t+7t^2)$$

written in augument matrix:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 0 & 1 & 2 & 4 \\ 1 & 0 & 1 & 1 \end{array}\right] \underset{\sim}{R_3=R_3-R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 0 & 1 & 2 & 4 \\ 0 & -1 & 0 & -6 \end{array}\right] \underset{\sim}{R_3=R_3+R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 2 & -2 \end{array}\right] \underset{\sim}{R_2=R_2-R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 2 & -2 \end{array}\right] \underset{\sim}{R_3=\frac{1}{2}R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & -1 \end{array}\right] \underset{\sim}{R_1=R_1-R_2, R_1=R_1-R_3}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

So that, we have:

$$\begin{cases} \lambda_1 = 2 \\ \lambda_2 = 6 \\ \lambda_3 = -1 \end{cases}$$

Solution 2.

$$\begin{bmatrix} 1 & 1 & 6 & 2 & 6 \\ 4 & 1 & 4 & 2 & 5 \\ 5 & 2 & 3 & 5 & 0 \\ 3 & 4 & 6 & 2 & 4 \\ 1 & 2 & 1 & 4 & 3 \end{bmatrix} \underset{\sim}{R_2=R_2-R_4-R_5,R_3=R_3-R_4-2R_5} \begin{bmatrix} 1 & 1 & 6 & 2 & 6 \\ 0 & -5 & -3 & -4 & -2 \\ 0 & -6 & -5 & -5 & -7 \\ 3 & 4 & 6 & 2 & 4 \\ 1 & 2 & 1 & 4 & 3 \end{bmatrix}$$

$$\underset{\sim}{R_4=R_4-3R_5,R_5=R_1-R_5} \begin{bmatrix} 1 & 1 & 6 & 2 & 6 \\ 0 & -5 & -3 & -4 & -2 \\ 0 & -6 & -5 & -5 & -7 \\ 0 & -2 & 3 & -10 & -2 \\ 0 & -1 & 5 & -4 & 3 \end{bmatrix} \underset{\sim}{R_4=R_2+R_4,R_5=R_3+R_5} \begin{bmatrix} 1 & 1 & 6 & 2 & 6 \\ 0 & 2 & -3 & -4 & 2 \\ 0 & -6 & -5 & -5 & -7 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 3 \end{bmatrix}$$

$$\underset{\sim}{R_4=R_2+R_4,R_5=R_3+R_5} \begin{bmatrix} 1 & 1 & 6 & 2 & 6 \\ 0 & 2 & -3 & -4 & 2 \\ 0 & -6 & -5 & -5 & -7 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 3 \end{bmatrix}$$