THE UNIVERSITY OF SYDNEY

MATH2022 LINEAR AND ABSTRACT ALGEBRA

Semester 1 **Week 6 Longer Solutions**

1. (a) From Week 5 solutions, we may choose eigenvectors $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ corresponding to eigenvalues 1 and 2 respectively. Put $P = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, so that

$$P^{-1}MP = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \qquad \text{and} \qquad M = P\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}P^{-1}.$$

Observe that $P^{-1} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$, so that, for all positive k,

$$\begin{aligned} M^k &= P\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^k P^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2^k \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 2^k \\ 1 & 2^k \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2^k & 0 \\ 2^k-1 & 1 \end{bmatrix}. \end{aligned}$$

(b) From Week 5 solutions, we may choose eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$ corresponding to eigenvalues 2 and 3 respectively. Put $P = \begin{bmatrix} 1 & \frac{1}{2} \\ 1 & 1 \end{bmatrix}$, so that

$$P^{-1}MP = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \qquad \text{and} \qquad M = P\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}P^{-1}.$$

Observe that $P^{-1} = \begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix}$, so that, for all positive k,

$$\begin{aligned} M^k &= P\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}^k P^{-1} = \begin{bmatrix} 1 & \frac{1}{2} \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2^k & 0 \\ 0 & 3^k \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & \frac{1}{2} \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2^{k+1} & -2^k \\ -2(3^k) & 2(3^k) \end{bmatrix} = \begin{bmatrix} 2^{k+1}-3^k & -2^k+3^k \\ 2^{k+1}-2(3^k) & -2^k+2(3^k) \end{bmatrix}. \end{aligned}$$

(c) Observe that $\det(\lambda I - M) = \begin{vmatrix} \lambda & 1 \\ 1 & \lambda \end{vmatrix} = \lambda^2 - 1 = (\lambda-1)(\lambda+1)$, yielding eigenvalues 1 and -1. To find the eigenspace corresponding to 1:

$$I - M = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \qquad \text{yielding} \qquad \left\{ \begin{bmatrix} -t \\ t \end{bmatrix} \,\middle|\, t \in \mathbb{R} \right\}.$$

To find the eigenspace corresponding to -1:

$$-I - M = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \qquad \text{yielding} \qquad \left\{ \begin{bmatrix} t \\ t \end{bmatrix} \,\middle|\, t \in \mathbb{R} \right\}.$$

Thus we may choose eigenvectors $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ corresponding to eigenvalues 1 and -1 respectively. Put $P = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$, so that

$$P^{-1}MP \;=\; \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \text{and} \qquad M \;=\; P\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}P^{-1}\,.$$

Observe that $P^{-1} = \frac{1}{2}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$, so that, for all positive k,

$$\begin{aligned} M^k \;&=\; P\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^k P^{-1} \;=\; \frac{1}{2}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & (-1)^k \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \\ &=\; \frac{1}{2}\begin{bmatrix} -1 & (-1)^k \\ 1 & (-1)^k \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \;=\; \frac{1}{2}\begin{bmatrix} 1+(-1)^k & -1+(-1)^k \\ -1+(-1)^k & 1+(-1)^k \end{bmatrix}\,. \end{aligned}$$

As expected, this simplifies to the following:

$$M^k \;=\; \begin{cases} I & \text{if } k \text{ is even} \\ M & \text{if } k \text{ is odd.} \end{cases}$$

(d) From Week 5 solutions, we may choose eigenvectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ corresponding to eigenvalues 1, 2 and 3 respectively. Put $P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$, so that

$$P^{-1}MP \;=\; \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \qquad \text{and} \qquad M \;=\; P\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}P^{-1}\,.$$

Observe that

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & -1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right],$$

so that $P^{-1} = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. Hence, for all positive k,

$$\begin{aligned} M^k \;&=\; P\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}^k P^{-1} \;=\; \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^k & 0 \\ 0 & 0 & 3^k \end{bmatrix}\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \\ &=\; \begin{bmatrix} 1 & 2^k & 0 \\ 0 & 2^k & -3^k \\ 0 & 0 & 3^k \end{bmatrix}\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \;=\; \begin{bmatrix} 1 & 2^k-1 & 2^k-1 \\ 0 & 2^k & 2^k-3^k \\ 0 & 0 & 3^k \end{bmatrix}\,. \end{aligned}$$

(e) From Week 5 solutions, we may choose eigenvectors $\begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ corresponding to eigenvalues 2, 2 and 1 respectively. Put $P = \begin{bmatrix} \frac{1}{2} & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, so that

$$P^{-1}MP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{and} \qquad M = P \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} P^{-1}.$$

Observe that

$$\left[\begin{array}{ccc|ccc} \frac{1}{2} & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -2 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array}\right],$$

so that $P^{-1} = \begin{bmatrix} -2 & 2 & 0 \\ 0 & 0 & 1 \\ 2 & -1 & 0 \end{bmatrix}$. Hence, for all positive k,

$$\begin{aligned} M^k &= P \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}^k P^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2^k & 0 & 0 \\ 0 & 2^k & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 2 & 0 \\ 0 & 0 & 1 \\ 2 & -1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2} & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -2(2^k) & 2(2^k) & 0 \\ 0 & 0 & 2^k \\ 2 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 2-2^k & 2^k-1 & 0 \\ 2-2^{k+1} & 2^{k+1}-1 & 0 \\ 0 & 0 & 2^k \end{bmatrix}. \end{aligned}$$

(f) From Week 5 solutions, we may choose eigenvectors $\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ corresponding to eigenvalues 3, 1 and -1 respectively. Put $P = \begin{bmatrix} \frac{1}{2} & 1 & 1 \\ \frac{1}{2} & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$, so that

$$P^{-1}MP = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad \text{and} \qquad M = P \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} P^{-1}.$$

Observe that

$$\begin{aligned} &\left[\begin{array}{ccc|ccc} \frac{1}{2} & 1 & 1 & 1 & 0 & 0 \\ \frac{1}{2} & -1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 2 & 0 & 0 \\ 0 & -2 & -1 & -1 & 1 & 0 \\ 0 & -1 & -1 & -2 & 0 & 1 \end{array}\right] \\ &\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 3 & 1 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 0 & 2 \\ 0 & 1 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 3 & 1 & -2 \end{array}\right], \end{aligned}$$

so that $P^{-1} = \begin{bmatrix} -2 & 0 & 2 \\ -1 & -1 & 1 \\ 3 & 1 & -2 \end{bmatrix}$. Hence, for all positive k,

$$\begin{aligned}
M^k &= P \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}^k P^{-1} \\
&= \begin{bmatrix} \frac{1}{2} & 1 & 1 \\ \frac{1}{2} & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3^k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (-1)^k \end{bmatrix} \begin{bmatrix} -2 & 0 & 2 \\ -1 & -1 & 1 \\ 3 & 1 & -2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2} & 1 & 1 \\ \frac{1}{2} & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -2(3^k) & 0 & 2(3^k) \\ -1 & -1 & 1 \\ 3(-1)^k & (-1)^k & -2(-1)^k \end{bmatrix} \\
&= \begin{bmatrix} 3(-1)^k - 3^k - 1 & (-1)^k - 1 & 3^k + 1 - 2(-1)^k \\ 1 - 3^k & 1 & 3^k - 1 \\ 3(-1)^k - 2(3^k) - 1 & (-1)^k - 1 & 2(3^k) + 1 - 2(-1)^k \end{bmatrix}.
\end{aligned}$$

2. The entries of M are all positive and the columns sum to 1, so M is regular stochastic. The unique steady state vector is a probability vector that is also an eigenvector corresponding to eigenvalue 1. To find the eigenspace corresponding to eigenvalue 1:

$$I - M = \begin{bmatrix} 1/2 & -2/5 \\ -1/2 & 2/5 \end{bmatrix} \sim \begin{bmatrix} 1 & -4/5 \\ 0 & 0 \end{bmatrix},$$

yielding $\left\{ \begin{bmatrix} 4t/5 \\ t \end{bmatrix} \,\middle|\, t \in \mathbb{R} \right\}$. Thus, we want nonnegative t such that $1 = \frac{4t}{5} + t = \frac{9t}{5}$, so that $t = 5/9$, yielding the unique steady state vector

$$\mathbf{v} = \begin{bmatrix} \frac{4}{9} \\ \frac{5}{9} \end{bmatrix}.$$

To diagonalise M, observe that

$$\begin{aligned}
\det(\lambda I - M) &= \begin{vmatrix} \lambda - 1/2 & -2/5 \\ -1/2 & \lambda - 3/5 \end{vmatrix} = \left(\lambda - \frac{1}{2}\right)\left(\lambda - \frac{3}{5}\right) - \frac{2}{10} \\
&= \lambda^2 - \frac{11}{10}\lambda + \frac{1}{10} = \left(\lambda - 1\right)\left(\lambda - \frac{1}{10}\right),
\end{aligned}$$

yielding eigenvalues 1 and 1/10. To find the eigenspace corresponding to 1/10:

$$\frac{1}{10}I - M = \begin{bmatrix} -2/5 & -2/5 \\ -1/2 & -1/2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},$$

yielding $\left\{ \begin{bmatrix} -t \\ t \end{bmatrix} \,\middle|\, t \in \mathbb{R} \right\}$. Taking $t = 1$ we may use the eigenvector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and put

$$P = \begin{bmatrix} \frac{4}{9} & -1 \\ \frac{5}{9} & 1 \end{bmatrix},$$

so that

$$P^{-1}MP = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{10} \end{bmatrix} \qquad \text{and} \qquad M = P\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{10} \end{bmatrix} P^{-1} .$$

Observe that $P^{-1} = \begin{bmatrix} 1 & 1 \\ -\frac{5}{9} & \frac{4}{9} \end{bmatrix}$, so that, for all positive k,

$$\begin{aligned} M^k &= P\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{10} \end{bmatrix}^k P^{-1} = \begin{bmatrix} \frac{4}{9} & -1 \\ \frac{5}{9} & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 10^{-k} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -\frac{5}{9} & \frac{4}{9} \end{bmatrix} \\ &= \begin{bmatrix} \frac{4}{9} & -10^{-k} \\ \frac{5}{9} & 10^{-k} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -\frac{5}{9} & \frac{4}{9} \end{bmatrix} = \begin{bmatrix} \frac{4}{9}+\frac{5}{9}10^{-k} & \frac{4}{9}-\frac{4}{9}10^{-k} \\ \frac{5}{9}-\frac{5}{9}10^{-k} & \frac{5}{9}+\frac{4}{9}10^{-k} \end{bmatrix} \\ &\to \begin{bmatrix} \frac{4}{9} & \frac{4}{9} \\ \frac{5}{9} & \frac{5}{9} \end{bmatrix} \end{aligned}$$

as $k \to \infty$. This confirms the original calculation of the steady state vector, because, for any probability vector $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ where $x + y = 1$, we have

$$\lim_{k\to\infty} M^k\mathbf{x} = \begin{bmatrix} \frac{4}{9} & \frac{4}{9} \\ \frac{5}{9} & \frac{5}{9} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{4}{9}(x+y) \\ \frac{5}{9}(x+y) \end{bmatrix} = \begin{bmatrix} \frac{4}{9} \\ \frac{5}{9} \end{bmatrix} = \mathbf{v} .$$

3. (a) We have

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 13 \\ 18 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 80 \\ 111 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 493 \\ 684 \end{bmatrix}, \quad \mathbf{v}_5 = \begin{bmatrix} 3038 \\ 4215 \end{bmatrix}.$$

(b) The characteristic polynomial of A is

$$\chi(\lambda) = \det(\lambda I - A) = \begin{vmatrix} \lambda-2 & -3 \\ -3 & \lambda-4 \end{vmatrix} = (\lambda-2)(\lambda-4)-9 = \lambda^2-6\lambda-1 ,$$

with roots $\lambda = \frac{6\pm\sqrt{40}}{2} = 3 \pm \sqrt{10}$. Thus the eigenvalues of A are

$$\lambda_1 = 3+\sqrt{10} = 6.16228 \qquad \text{and} \qquad \lambda_2 = 3-\sqrt{10} = -0.16228$$

to 5 decimal places.

(c) Put

$$\mathbf{v} = \frac{1}{3038}\mathbf{v}_5 = \begin{bmatrix} 1 \\ \frac{4215}{3038} \end{bmatrix}.$$

Then

$$A\mathbf{v} = \begin{bmatrix} 2+3\left(\frac{4215}{3038}\right) \\ 3+4\left(\frac{4215}{3038}\right) \end{bmatrix} = \begin{bmatrix} 6.16228 \\ 8.54970 \end{bmatrix}$$

to 5 decimal places. Also

$$(3+\sqrt{10})\mathbf{v} = \begin{bmatrix} 3+\sqrt{10} \\ (3+\sqrt{10})\left(\frac{4215}{3038}\right) \end{bmatrix} = \begin{bmatrix} 6.16228 \\ 8.54970 \end{bmatrix} = A\mathbf{v}$$

to 5 decimal places, so that $\mathbf{v}$ is a close approximation to an eigenvector of A corresponding to its largest eigenvalue $\lambda_1 = 3+\sqrt{10}$.

4. (a) We have

$$\mathbf{w}_1 = \begin{bmatrix} -4 \\ 3 \end{bmatrix}, \mathbf{w}_2 = \begin{bmatrix} 25 \\ -18 \end{bmatrix}, \mathbf{w}_3 = \begin{bmatrix} -154 \\ 111 \end{bmatrix}, \mathbf{w}_4 = \begin{bmatrix} 949 \\ -684 \end{bmatrix}, \mathbf{w}_5 = \begin{bmatrix} -5848 \\ 4215 \end{bmatrix}.$$

(b) Put

$$\mathbf{w} \;=\; \frac{1}{-5848}\mathbf{w}_5 \;=\; \begin{bmatrix} 1 \\ -\frac{4215}{5848} \end{bmatrix}.$$

Then

$$B\mathbf{w} \;=\; \begin{bmatrix} -4-3\left(\frac{4215}{5848}\right) \\ 3+2\left(\frac{4215}{5848}\right) \end{bmatrix} \;=\; \begin{bmatrix} -6.16228 \\ 4.44152 \end{bmatrix} \;=\; -6.16228\,\mathbf{w}\,,$$

correct to 5 decimal places. Thus $\mathbf{w}$ is a close approximation to an eigenvector for B corresponding to an eigenvalue of approximately -6.16228.

(c) Put $A = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$, the matrix from the previous exercise. Then

$$AB \;=\; \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -4 & 3 \\ 3 & -2 \end{bmatrix} \;=\; \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \;=\; I$$

so that $B = A^{-1}$. Thus A and B share the same eigenvectors, and the eigenvalues of B are the reciprocals of the eigenvalues of A, namely,

$$\frac{1}{3+\sqrt{10}} \;=\; \frac{3-\sqrt{10}}{9-10} \;=\; -3+\sqrt{10} \;=\; 0.162278$$

and

$$\frac{1}{3-\sqrt{10}} \;=\; \frac{3+\sqrt{10}}{9-10} \;=\; -3-\sqrt{10} \;=\; -6.16228$$

to 5 decimal places. The negative eigenvalue of B is the largest in magnitude and dominates in the approximation that arises in part (b), just as the largest eigenvalue of A dominated in the approximation in part (c) of the previous exercise. This illustrates how applying the numerical technique to the inverse of A, namely B, finds an approximate eigenvector of A corresponding to the eigenvalue of A that is smaller in magnitude, namely $\lambda_2 = 3-\sqrt{10}$.

5. Observe that

$$\det(\lambda I - M) \;=\; \begin{vmatrix} \lambda-1 & -1 \\ -1 & \lambda \end{vmatrix} \;=\; (\lambda-1)\lambda - 1 \;=\; \lambda^2-\lambda-1\,,$$

whose roots

$$\lambda_1 = \frac{1+\sqrt{5}}{2} \qquad \text{and} \qquad \lambda_2 = \frac{1-\sqrt{5}}{2}$$

are the eigenvalues of M. To find the eigenspace corresponding to λ_1:

$$\lambda_1 I - M \;=\; \begin{bmatrix} \lambda_1-1 & -1 \\ -1 & \lambda_1 \end{bmatrix} \sim \begin{bmatrix} 1 & -\lambda_1 \\ 0 & 0 \end{bmatrix},$$

yielding $\left\{ \begin{bmatrix} \lambda_1 t \\ t \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\}$. To find the eigenspace corresponding to λ_2:

$$\lambda_2 I - M \;=\; \begin{bmatrix} \lambda_2-1 & -1 \\ -1 & \lambda_2 \end{bmatrix} \sim \begin{bmatrix} 1 & -\lambda_2 \\ 0 & 0 \end{bmatrix},$$

yielding $\left\{\begin{bmatrix} \lambda_2 t \\ t \end{bmatrix} \,\middle|\, t \in \mathbb{R}\right\}$. Thus we may take eigenvectors $\begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$ with respect to λ_1 and λ_2 respectively. We may put $P = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}$ so that

$$P^{-1}MP = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \qquad \text{and} \qquad M = P\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} P^{-1}.$$

Observe that $P^{-1} = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix}$, so that, for all positive k,

$$\begin{aligned} M^k &= P\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}^k P^{-1} = \frac{1}{\sqrt{5}}\begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix}\begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix} \\ &= \frac{1}{\sqrt{5}}\begin{bmatrix} \lambda_1^{k+1} & \lambda_2^{k+1} \\ \lambda_1^k & \lambda_2^k \end{bmatrix}\begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} \lambda_1^{k+1} - \lambda_2^{k+1} & \lambda_1\lambda_2^{k+1} - \lambda_2\lambda_1^{k+1} \\ \lambda_1^k - \lambda_2^k & \lambda_1\lambda_2^k - \lambda_2\lambda_1^k \end{bmatrix}. \end{aligned}$$

But, for $n \geq 2$, we have

$$\begin{aligned} \begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix} &= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{n-2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = M^{n-2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \frac{1}{\sqrt{5}}\begin{bmatrix} \lambda_1^{n-1} - \lambda_2^{n-1} & \lambda_1\lambda_2^{n-1} - \lambda_2\lambda_1^{n-1} \\ \lambda_1^{n-2} - \lambda_2^{n-2} & \lambda_1\lambda_2^{n-2} - \lambda_2\lambda_1^{n-2} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \frac{1}{\sqrt{5}}\begin{bmatrix} \lambda_1^{n-1} - \lambda_2^{n-1} + \lambda_1\lambda_2^{n-1} - \lambda_2\lambda_1^{n-1} \\ \lambda_1^{n-2} - \lambda_2^{n-2} + \lambda_1\lambda_2^{n-2} - \lambda_2\lambda_1^{n-2} \end{bmatrix} \\ &= \frac{1}{\sqrt{5}}\begin{bmatrix} \lambda_1^{n-1}(1-\lambda_2) - \lambda_2^{n-1}(1-\lambda_1) \\ \lambda_1^{n-2}(1-\lambda_2) - \lambda_2^{n-2}(1-\lambda_1) \end{bmatrix} \\ &= \frac{1}{\sqrt{5}}\begin{bmatrix} \lambda_1^n - \lambda_2^n \\ \lambda_1^{n-1} - \lambda_2^{n-1} \end{bmatrix}, \end{aligned}$$

using the fact that $1 - \lambda_2 = \lambda_1$, so that, also, $1 - \lambda_1 = \lambda_2$. Thus, finally,

$$x_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

6. (a) Observe that the columns of M add up to 1 and

$$M^2 = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 1/2 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix}\begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 1/2 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/4 & 1/4 \\ 1/4 & 1/2 & 1/4 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$$

has all positive entries, which verifies that M is regular stochastic.

(b) The unique steady state vector is a probability vector that is an eigenvector corresponding to eigenvalue 1. To find the eigenspace corresponding to 1:

$$I - M = \begin{bmatrix} 1 & -1/2 & -1/2 \\ -1/2 & 1/2 & 0 \\ -1/2 & 0 & 1/2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & 1/4 & -1/4 \\ 0 & -1/4 & 1/4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix},$$

yielding $\left\{ \begin{bmatrix} t \\ t \\ t \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\}$. To get a probability vector, take $t = 1/3$, yielding the unique steady state vector $\mathbf{v} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$.

(c) To diagonalise M, observe that

$$\begin{aligned}
\det(\lambda I - M) &= \begin{vmatrix} \lambda & -1/2 & -1/2 \\ -1/2 & \lambda - 1/2 & 0 \\ -1/2 & 0 & \lambda - 1/2 \end{vmatrix} = \begin{vmatrix} \lambda & -1/2 & 0 \\ -1/2 & \lambda - 1/2 & 1/2 - \lambda \\ -1/2 & 0 & \lambda - 1/2 \end{vmatrix} \\
&= \begin{vmatrix} \lambda & -1/2 & 0 \\ -1 & \lambda - 1/2 & 0 \\ -1/2 & 0 & \lambda - 1/2 \end{vmatrix} = \left(\lambda - \frac{1}{2}\right) \begin{vmatrix} \lambda & -1/2 \\ -1 & \lambda - 1/2 \end{vmatrix} \\
&= \left(\lambda - \frac{1}{2}\right)\left(\lambda^2 - \frac{1}{2}\lambda - \frac{1}{2}\right) = \left(\lambda - \frac{1}{2}\right)\left(\lambda + \frac{1}{2}\right)\left(\lambda - 1\right),
\end{aligned}$$

yielding eigenvalues 1, 1/2 and $-1/2$. To find the eigenspace corresponding to 1/2:

$$\frac{1}{2}I - M = \begin{bmatrix} 1/2 & -1/2 & -1/2 \\ -1/2 & 0 & 0 \\ -1/2 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

yielding $\left\{ \begin{bmatrix} 0 \\ -t \\ t \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\}$. Taking $t = 1$ we may use the eigenvector $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$.
To find the eigenspace corresponding to $-1/2$:

$$-\frac{1}{2}I - M = \begin{bmatrix} -1/2 & -1/2 & -1/2 \\ -1/2 & -1 & 0 \\ -1/2 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1/2 & 1/2 \\ 0 & 1/2 & -1/2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix},$$

yielding $\left\{ \begin{bmatrix} -2t \\ t \\ t \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\}$. Taking $t = 1$ we may use the eigenvector $\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$.
Now we may put

$$P = \begin{bmatrix} 1/3 & 0 & -2 \\ 1/3 & -1 & 1 \\ 1/3 & 1 & 1 \end{bmatrix},$$

so that

$$P^{-1}MP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & -1/2 \end{bmatrix} \quad \text{and} \quad M = P \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & -1/2 \end{bmatrix} P^{-1}.$$

Observe that

$$\left[\begin{array}{ccc|ccc} 1/3 & 0 & -2 & 1 & 0 & 0 \\ 1/3 & -1 & 1 & 0 & 1 & 0 \\ 1/3 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & -6 & 3 & 0 & 0 \\ 0 & -1 & 3 & -1 & 1 & 0 \\ 0 & 1 & 3 & -1 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -6 & 3 & 0 & 0 \\ 0 & -1 & 3 & -1 & 1 & 0 \\ 0 & 0 & 6 & -2 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & -1/2 & 1/2 \\ 0 & 0 & 1 & -1/3 & 1/6 & 1/6 \end{array}\right],$$

so that

$$P^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1/2 & 1/2 \\ -1/3 & 1/6 & 1/6 \end{bmatrix}.$$

Hence, for all positive k,

$$\begin{aligned} M^k &= P \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & -1/2 \end{bmatrix}^k P^{-1} \\ &= \begin{bmatrix} 1/3 & 0 & -2 \\ 1/3 & -1 & 1 \\ 1/3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^{-k} & 0 \\ 0 & 0 & (-1)^k 2^{-k} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1/2 & 1/2 \\ -1/3 & 1/6 & 1/6 \end{bmatrix} \\ &\to \begin{bmatrix} 1/3 & 0 & -2 \\ 1/3 & -1 & 1 \\ 1/3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1/2 & 1/2 \\ -1/3 & 1/6 & 1/6 \end{bmatrix} \\ &= \begin{bmatrix} 1/3 & 0 & -2 \\ 1/3 & -1 & 1 \\ 1/3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} \end{aligned}$$

as $k \to \infty$. This confirms the original calculation of the steady state vector, because, for any probability vector $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ where $x + y + z = 1$, we have

$$\lim_{k\to\infty} M^k \mathbf{x} = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1/3(x+y+z) \\ 1/3(x+y+z) \\ 1/3(x+y+z) \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} = \mathbf{v}\,.$$

7. Suppose that A and B are similar and $B = \lambda I$ for some scalar λ. Then there exists an invertible matrix P such that $A = P^{-1}BP$, so that

$$A = P^{-1}BP = P^{-1}\big(\lambda I\big)P = \lambda P^{-1} I P = \lambda P^{-1} P = \lambda I = B\,.$$

8. Suppose that M is a real matrix with exactly one eigenvalue $\lambda \in \mathbb{C}$. Then λ is a root of the characteristic polynomial of M, which is a real polynomial, whose roots come in complex conjugate pairs. Hence λ coincides with its complex conjugate, so $\lambda \in \mathbb{R}$. If $A = \lambda I$ then A is already diagonal, so trivially diagonalisable. Suppose conversely that A is diagonalisable, so that A is similar to a diagonal matrix with its eigenvalues down the diagonal. But λ is the only eigenvalue, so that A is similar to λI. By the previous exercise, $A = \lambda I$, and we are done.

9. We have

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - 3 & 1 \\ -4 & \lambda + 1 \end{vmatrix} = (\lambda - 3)(\lambda + 1) + 4 = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2 \,,$$

with exactly one root, $\lambda = 1$, so that A has exactly one eigenvalue. We also have

$$\begin{aligned} \det(\lambda I - B) &= \begin{vmatrix} \lambda - 34 & -99 & 0 \\ 11 & \lambda + 32 & 0 \\ 4 & 12 & \lambda - 1 \end{vmatrix} = (\lambda - 1) \begin{vmatrix} \lambda - 34 & -99 \\ 11 & \lambda + 32 \end{vmatrix} \\ &= (\lambda - 1)\big((\lambda - 34)(\lambda + 32) + 1089\big) = (\lambda - 1)(\lambda^2 - 2\lambda - 1088 + 1089) \\ &= (\lambda - 1)(\lambda^2 - 2\lambda + 1) = (\lambda - 1)^3 \,, \end{aligned}$$

again with exactly one root. If A or B is diagonalisable then at least one of A or B is a scalar matrix, by the previous exercise, which is obviously not the case. Hence neither A nor B is diagonalisable.

10. Suppose first that $A\mathbf{v} = \mathbf{0}$. We show that A is not invertible by contradiction. Suppose to the contrary that A is invertible. Then

$$\mathbf{0} = A^{-1}\mathbf{0} = A^{-1}A\mathbf{v} = I\mathbf{v} = \mathbf{v} \,,$$

which contradicts that $\mathbf{v}$ is nonzero. Hence A is not invertible.

Suppose now that A is not invertible and let M be the result of row reducing A to reduced row echelon form. Then M is not the identity matrix, so has at least one row of zeros. Hence the associated homogeneous system has a solution involving at least one parameter, so there exists a nonzero vector $\mathbf{v}$ such that $A\mathbf{v} = \mathbf{0}$.

11. Note that λI is both upper and lower triangular. Hence $\lambda I - M$ is also triangular, so its determinant is the product of the diagonal entries, that is,

$$\chi(\lambda) = \det(\lambda I - M) = (\lambda - \lambda_1) \dots (\lambda - \lambda_n) \,.$$

12. (a) Suppose that A and B are both $n \times n$ matrices, with typical entries a_{ij} and b_{ij} respectively, where $1 \le i, j \le n$. Then the kth elements on the diagonals of AB and BA are

$$\sum_{j=1}^{n} a_{kj} b_{jk} \quad \text{and} \quad \sum_{j=1}^{n} b_{kj} a_{jk}$$

respectively. Hence

$$\begin{aligned} \operatorname{tr}(AB) &= \sum_{k=1}^{n} \sum_{j=1}^{n} a_{kj} b_{jk} = \sum_{k=1}^{n} \sum_{j=1}^{n} b_{jk} a_{kj} \\ &= \sum_{j=1}^{n} \sum_{k=1}^{n} b_{jk} a_{kj} = \sum_{k=1}^{n} \sum_{j=1}^{n} b_{kj} a_{jk} = \operatorname{tr}(BA) \,. \end{aligned}$$

(b) If A and B are similar then $A = P^{-1}BP$ for some invertible matrix P, so that, by part (a),

$$\operatorname{tr}(A) = \operatorname{tr}(P^{-1}BP) = \operatorname{tr}(PP^{-1}B) = \operatorname{tr}(B) \,.$$

(c) Suppose that M is $n \times n$. By the theorem quoted, M is similar to a triangular matrix T, so that M and T have the same eigenvalues. But, by the previous exercise, the eigenvalues of T are just its diagonal entries $\lambda_1, \ldots, \lambda_n$, say. Hence, by part (b),

$$\operatorname{tr}(M) \;=\; \operatorname{tr}(T) \;=\; \lambda_1 + \ldots + \lambda_n \;,$$

so that $\operatorname{tr}(M)$ is the sum of its eigenvalues (recorded with multiplicity).