THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Solutions to Calculus Tutorial 5 (Week 6)

MATH1062/MATH1023: Mathematics 1B (Calculus) Semester 2, 2024

Questions marked with * are harder questions.

Material covered

(1) Second-order linear differential equations

Summary of essential material

Recall that second-order linear homogeneous differential equations with constant coefficients can be solved using the associated quadratic equation. There are three solutions types.

Questions to complete during the tutorial

1. Find the particular solution of $\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} - 8y = 0$ which satisfies $y(0) = 0$ and $y'(0) = 3$.

Solution: The associated quadratic equation $m^2 + 2m - 8 = 0$ so that $m_1 = 2$ or $m_2 = -4$. Therefore, the general solution is $y = Ae^{2t} + Be^{-4t}$. If $y(0) = 0$, then $A + B = 0$ (and $B = -A$). Now, $y'(t) = 2Ae^{2t} - 4Be^{-4t}$, so $y'(0) = 3 = 2A - 4B = 6A$. Therefore, $A = \dfrac{1}{2}$ and $B = -\dfrac{1}{2}$, and the particular solution is $y = \dfrac{1}{2}e^{2t} - \dfrac{1}{2}e^{-4t}$.

2. Find the general solution of $\dfrac{d^2y}{dt^2} - 2\dfrac{dy}{dt} + 5y = 0$, expressing your answer in terms of real functions. What is the particular solution satisfying $y(0) = 1$ and $y(\pi/4) = 2$?

Solution: The associated quadratic equation is $m^2 - 2m + 5 = 0$. The roots of this equation are $m = \dfrac{2 \pm \sqrt{4 - 20}}{2} = 1 \pm 2i$. The general solution of the differential equation is therefore

$$y = Ae^{(1+2i)t} + Be^{(1-2i)t} = e^t(Ae^{2it} + Be^{-2it}).$$

In terms of real functions, the general solution is

$$y = e^t\,(C\cos 2t + D\sin 2t)\,.$$

If $y(0) = 1$, then $C = 1$. If $y(\pi/4) = 2$, then $2 = e^{\pi/4}D$, and $D = 2e^{-\pi/4}$. The required particular solution is therefore $y = e^t(\cos 2t + 2e^{-\pi/4}\sin 2t)$.

3. Find the particular solution of $\dfrac{d^2x}{dt^2} + 2\dfrac{dx}{dt} + x = 0$ which satisfies $x(0) = 1$ and $x'(0) = 2$.

Solution: The associated quadratic equation is $m^2 + 2m + 1 = 0$ and this has two equal roots $m = -1$. Since the associated quadratic equation has equal roots, the general solution to the differential equation is

$$x = Ate^{-t} + Be^{-t}.$$

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When $t = 0$, $x = B$, and so $B = 1$ and $x = Ate^{-t} + e^{-t}$. Now, $x'(t) = A(-te^{-t} + e^{-t}) - e^{-t}$, so $x'(0) = A - 1 = 2$ and $A = 3$. The particular solution is therefore $x = 3te^{-t} + e^{-t}$.

***4.** Recall that the values for the gravitational constant $G = 6.67 \times 10^{-11}\frac{m^3}{kgs^2}$, the mass of the Earth $M = 5.972 \times 10^{24}kg$, and the radius of the Earth $R = 6.371 \times 10^6 m$.

Suppose a hole is drilled through the Earth along a diameter. If a particle p of mass m is dropped from rest at the surface, into the hole, then the distance $y = y(t)$ of the particle from the center of the earth at time t is given by

$$\frac{d^2y}{dt^2} = -k^2 y$$

where $k = \sqrt{GM/R} \cdot R^{-1}$.

(a) Conclude that the particle p undergoes simple harmonic motion. Find the period T.

Solution: First part follows directly from the form of the equation $\frac{d^2y}{dt^2} + k^2y = 0$. The frequency is given by $T = 2\pi/k$ and thus $T = \frac{2\pi R^{3/2}}{\sqrt{GM}}$. Hence, with $k = \sqrt{GM/R}\frac{1}{R} = \sqrt{\frac{6.67 \times 5.972}{6.371}10^{24-11-6}}\frac{1}{6.371 \times 10^6} \approx 1.24 \times 10^{-3}$ this yields $T = 2\pi/0.00124 = 5062.55s$ - or a bit under one and a half hours.

(b) With what speed does the particle pass through the centre of the Earth?

Solution: General solution of this equation is given by $y(t) = A\cos(kt + \phi)$ for $A \in \mathbb{R}$ and $\phi \in [0, 2\pi)$. We have initial solution $y(0) = R$ (the particle starts falling at the earths surface) and $dy/dt(0) = 0$ (initially, the particle is at rest).

Hence, we obtain $A = R$ and $\phi = 0$ and thus the equation for the location of p depedning on t is given by $y(t) = 6.371 \times 10^6 \cos(1.24 \times 10^{-3}t)$. Particle p is at the centre of the Earth for $1.24 \times 10^{-3}t = \pi/2$ which is a quarter of the period and hence at $t^* = 5062.55/4 = 1265.64s$ (see part (a)). To obtain the speed at t^*, we calculate $dy/dt = -6.371\times1.24\times10^3 \sin(1.24\times10^{-3}t)$ and by construction $\sin(1.24\times10^{-3}t^*) = 1$ and thus the speed at the centre of the Earth is given by $-6.371\times1.24\times10^3 = -7900m/s$. The minus sign tells us that we are falling, hence moving into the negative direction.

Further reading: http://www.physicscentral.com/explore/poster-earth.cfm

5. Two species struggling to compete against each other in the same environment have populations at time t of $x(t)$ and $y(t)$, satisfying the equations

$$x'(t) = 3x(t) - 4y(t)\,, \qquad y'(t) = -2x(t) + y(t)\,.$$

Find the second-order differential equation satisfied by $x(t)$. Hence find $x(t)$ and $y(t)$.

Solution: Differentiating $x' = 3x - 4y$ gives $x'' = 3x' - 4y'$. But $y' = -2x + y$, so

$$\begin{aligned} x'' &= 3x' - 4(-2x + y) \\ &= 3x' + 8x - 4y \\ &= 3x' + 8x + (x' - 3x) \quad \text{(since } -4y = x' - 3x\text{)} \\ &= 4x' + 5x\,. \end{aligned}$$

That is, $x'' - 4x' - 5x = 0$.

For $x'' - 4x' - 5x = 0$, the auxiliary equation is $m^2 - 4m - 5 = 0$ with roots $m = 5$ or $m = -1$. So $x = Ae^{5t} + Be^{-t}$. Hence $x' = 5Ae^{5t} - Be^{-t}$, and from $x' = 3x - 4y$ we obtain

$$\begin{aligned} y &= \frac{1}{4}(3x - x') \\ &= \frac{1}{4}[3(Ae^{5t} + Be^{-t}) - (5Ae^{5t} - Be^{-t})] \\ &= -\frac{1}{2}Ae^{5t} + Be^{-t} . \end{aligned}$$

*6. Two species are in a predator-prey relationship. One species, which numbers Y individuals, eats the other, which numbers X individuals. Historically the numbers of these species have been constant with $X = 3000$ and $Y = 1500$. After a severe environmental disturbance the populations cease to be constant and start to change with time.

Let $x(t)$ and $y(t)$ be the difference between the historically constant population numbers and the new, changing population numbers $X(t)$ and $Y(t)$. Then $x(t) = X(t) - 3000$ and $y(t) = Y(t) - 1500$ are the sizes of the perturbations from the historically steady states. The sizes of the perturbations are described by the following:

$$\begin{aligned} x'(t) &= 3x(t) - 2y(t) \\ y'(t) &= 4x(t) - y(t). \end{aligned}$$

(a) Show that $x''(t) - 2x'(t) + 5x(t) = 0$.

Solution:

$$\begin{aligned} x' &= 3x - 2y \\ x'' &= 3x' - 2y' \\ &= 3x' - 2(4x - y) \\ &= 3x' - 8x + 2y \\ &= 3x' - 8x + (3x - x') \\ &= 2x' - 5x . \end{aligned}$$

That is, $x'' - 2x' + 5x = 0$.

(b) Find $x(t)$ if $x(0) = 100$ and $x'(0) = 100$. (Take $t = 0$ to be the time at which monitoring of the population sizes begins.)

Solution: The auxiliary equation is $m^2 - 2m + 5 = 0$, with roots $m = 1 + 2i$ and $m = 1 - 2i$. Therefore $x = e^t (A\cos 2t + B\sin 2t)$. If $x = 100$ when $t = 0$, then $A = 100$ and $x = e^t(100\cos 2t + B\sin 2t)$. Hence, $x' = e^t(-200\sin 2t + 2B\cos 2t) + e^t(100\cos 2t + B\sin 2t)$. When $t = 0$, $x' = 2B + 100 = 100$, and so $B = 0$. Therefore $x = 100e^t\cos 2t$.

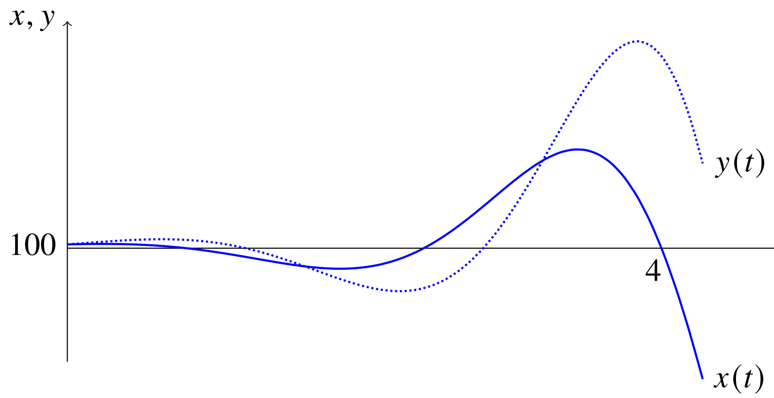
(c) Hence find $y(t)$.

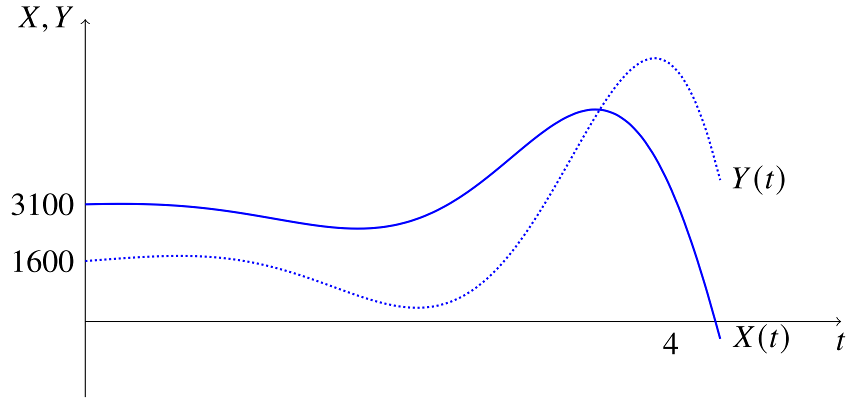
Solution:

$$\begin{aligned} y(t) &= \frac{1}{2}(3x - x') \\ &= \frac{1}{2}(300e^t\cos 2t - 100e^t\cos 2t + 200e^t\sin 2t) \\ &= 100e^t(\cos 2t + \sin 2t) . \end{aligned}$$

(d) Sketch $x(t)$ and $y(t)$ and then $X(t)$ and $Y(t)$ as a function of t. What will happen to the original populations?

Solution: The graphs look as follows:





The populations will oscillate with increasingly large swings. Eventually the swings will get so big that $X(t) = 3000 + x(t)$ will be zero. (Once this happens the predictions of the model will no longer be valid.) The prey will then become extinct. The predator will then also become extinct unless it has an alternative source of food.