# The University of Sydney
## School of Mathematics and Statistics

# MATH1062
## Mathematics 1B (Calculus)

June 2024 — Lecturers: Joseph Baine, Jonathan Spreer

Time Allowed: **Reading time — 10 minutes; Writing time — 2 hours**

Exam Conditions: This is a closed-book examination — no material permitted. Writing is not permitted at all during reading time.

Family Name: .............................................. SID: ...........................

Other Names: .............................................. Seat Number: .................

Please check that your examination paper is complete (15 pages) and indicate by signing below.
I have checked the examination paper and affirm it is complete.

Signature: .................................................. Date: ..........................

**This examination has two sections: Multiple Choice and Extended Answer.**

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The Multiple Choice Section is worth 50% of the total examination.
There are 20 questions. The questions are of equal value.
All questions may be attempted.

Answers to the Multiple Choice questions must be entered on the Multiple Choice Answer Sheet before the end of the examination.

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The Extended Answer Section is worth 50% of the total examination.
There are 4 questions. The questions are of equal value.
All questions may be attempted. Working must be shown.

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**There is a table of integrals after the last question in this booklet. Non-programmable calculators may be used, as long as they have a University of Sydney approval sticker on them.**

**THE QUESTION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM.**

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## Multiple Choice Section

*In each question, choose at most one option.*

**Your answers must be entered on the Multiple Choice Answer Sheet.**

**See end of exam for answers to multiple choice questions.**
**The exam will have 20 multiple-choice questions. 10 of them will be about calculus.**
**You will have approximately 3 minutes per multiple-choice question.**

**1.** The solution of the differential equation $\dfrac{dx}{dt} = t\sin(t^2)$ with initial condition $x(0) = 1$ is:

(a) $x(t) = \cos(t^2)$

(b) $x(t) = \dfrac{1}{2}\cos(t^2) + 0.5$

(c) $x(t) = t\cos(t^2) + 1$

(d) $x(t) = -\dfrac{1}{2}\cos(t^2) + 1.5$

**2.** Observations show that trees in a mountain range grow according to the following law: Their rate of change in height $H$ with respect to time $t$ is proportional to their current height, inversely proportional to time, and inversly proportional to the altitude $A$ in which they grow. Which differential equation models this observation?

(a) $\dfrac{dH}{dt} = kH + \dfrac{b}{At}$

(b) $\dfrac{dH}{dt} = \dfrac{kH}{At}$

(c) $\dfrac{dH}{dt} = kAtH^2$

(d) $\dfrac{dH}{dt} = \dfrac{k}{A}\left(H + \dfrac{1}{t}\right)$

**3.** A population of rabbits $R$ is observed to double in size every 5 years. If $t$ is measured in years, identify the differential equation modeling this behaviour?

(a) $\dfrac{dR}{dt} = \ln(5)\dfrac{R\,t}{2}$

(b) $\dfrac{dR}{dt} = \dfrac{R\,t}{2}$

(c) $\dfrac{dR}{dt} = \ln(2)\dfrac{R}{5}$

(d) $\dfrac{dR}{dt} = 2^{t/5}$

**4.** Let $C$ be an arbitrary constant. Identify the general solution of $\dfrac{dy}{dx} = \dfrac{y^2}{x^2}$.

(a) $y = \dfrac{C}{x-1}$

(b) $y = -\dfrac{C}{x}$

(c) $y = \dfrac{1}{x^{-1} - C}$

(d) $y = \dfrac{1}{x + C}$

**5.** A tank holds 1000 liters of water. A solution with a salt concentration of 0.02 kg per liter is added at a rate of 10 liters per minute. The solution is kept mixed and is drained from the tank at 10 liters per minute. Let $m(t)$ be the amount (in kg) of salt in the tank after $t$ minutes. Which differential equation models this mixing problem?

(a) $\dfrac{dm}{dt} = \dfrac{m}{0.02} - 10$ (b) $\dfrac{dm}{dt} = 0.02 - \dfrac{m}{10}$ (c) $\dfrac{dm}{dt} = 4 - \dfrac{m}{1000}$

(d) $\dfrac{dm}{dt} = 0.2 - \dfrac{m}{100}$

**6.** Which differential equation models exponential growth?

(a) $\dfrac{dx}{dt} = kx$ (b) $\dfrac{dx}{dt} = kx - at$ (c) $\dfrac{dx}{dt} = t^2 - at$

(d) $\dfrac{dx}{dt} = kt(a-x)$

**7.** The general solution of a first-order differential equation is given by $y(x) = \dfrac{1}{x}(\sin(x) + C)$. Find the particular solution for which $y = 1$ when $x = \pi$.

(a) $y(x) = \dfrac{\pi}{x}(2\sin(x) + 1)$ (b) $y(x) = \dfrac{1}{x}(\sin(x) + \pi)$

(c) $y(x) = \dfrac{2}{x}(\sin(\pi x) - \dfrac{1}{2})$ (d) $y(x) = \dfrac{\pi}{x}(\sin(x) + 1)$

**8.** Simple harmonic motion is described by

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

where $\omega$ is a constant. Which statement in the list below is true?

(a) Simple harmonic motion is described by a linear first-order differential equation.

(b) The frequency of the oscillation in the above equation is $\omega$.

(c) A particular solution of the above equation tends to infinity as $t$ tends to infinity.

(d) The general solution depends on one arbitrary constant.

**9.** For $f(x,y) = \ln(x\sqrt{y})$, calculate $f_y(0.5, 4)$.

(a) 0 (b) $\dfrac{1}{8}$ (c) $\dfrac{1}{4}$ (d) 1

**10.** For $f(x,y) = 3y + x^2y^2 - 2x$, calculate $f_y(1,2)$.

(a) 8 (b) 7 (c) 6 (d) 5

**11.** The equation of the tangent plane to the surface $z = \sin(xy) + x\cos(y)$ at the point where $x = 2$ and $y = 0$ is:

(a) $z = x - 2$ (b) $z = 2y$ (c) $z = x + 2y$
(d) $z = x + 2y - 4$

**12.** Find $\dfrac{\partial z}{\partial y}$ at $(1, 0, 1)$, given that $\ln(x + y + z) = x + 2y + 3z$.

(a) $\dfrac{4}{5}$ (b) $\dfrac{3}{5}$ (c) $-\dfrac{3}{5}$ (d) $-\dfrac{4}{5}$

**13.** The gradient of the the function $f(x, y) = xe^{-y}$ at the point $(1, 0)$ is:

(a) $\mathbf{i} - \mathbf{j}$ (b) $\mathbf{i}$ (c) $2\mathbf{i}$ (d) $\mathbf{j}$

**14.** Given that $\nabla g(2, 1) = 3\mathbf{i} - \mathbf{j}$, what is the directional derivative of $g$ at $(2, 1)$, in the direction of $\frac{1}{5}(4\mathbf{i} - 3\mathbf{j})$ ?

(a) 1 (b) 2 (c) 3 (d) 4

**15.** A function $f(x, y)$ has a critical point at $(a, b)$. If $f_{yy}(a, b) = 0$, and $f_{xy}(a, b) \neq 0$ and all second order partial derivatives are continuous at $(a, b)$ what can be said about the critical point using the second derivative test?

(a) The critical point is a local maximum.

(b) The critical point is a local minimum.

(c) The critical point is a saddle point.

(d) Nothing, as we do not have sufficient information to apply the second derivative test.

(e) Nothing, as the second derivative test is inconclusive.

**End of Multiple Choice Section**
**Make sure that your answers are entered on the Multiple Choice Answer Sheet**

## Extended Answer Section

*There are* **four** *questions in this section, each with a number of parts.*
*Write your answers in the space provided below each part. There is extra space at the end of the paper.*

**There are *three* questions in this sample exam. In the exam, you will have approximately 15 minutes per question.**

**1.** (a) The number $x$ in a population satisfies the logistic equation

$$\frac{dx}{dt} = 2x(10 - x),$$

where $t$ is the time in years. If the population is initially 2, find the time it takes to increase to 4. Leave your answer in exact form.

(b) The velocity $v$ of a particle follows the first-order differential equation

$$\frac{dv}{dt} = v + e^t,$$

where $t$ is time.

($i$) If at $t = 0$ the velocity is $v = -3$ determine whether the particle is experiencing a positive or negative acceleration (change in velocity $v(t)$) at $t = 0$? Justify your answer.

($ii$) Compute how the velocity $v$ depends on time $t$ by finding the general solution of the equation.

(*iii*) Find the particular solution of the equation with initial condition $v(0) = -3$.

(*iv*) In an experiment designed to track the particle described in part (*iii*), two detectors are placed at a distance 10 of the particle's initial position (one in the positive direction and one in the negative direction). Determine the detectors, if any, at which the particle will be tracked. Detectors track particles that cross their position but do not affect their movement.

**2.** (a) A tank contains 1000 litres of water with 20 kg of dissolved salt. Pure water flows into the tank at a rate of 20 L/min, the solution drains at the same rate. You can assume the solution in the tank is well mixed.

($i$) Write down the differential equation for the mass $m$ of salt in the tank after $t$ minutes.

($ii$) Solve the equation and find the amount of salt after 10 minutes.

(*iii*) Now suppose the pure water inflow is cut to 10 L/min, but the tank is still drained at the rate of 20 L/min. Set up the differential equation for the mass $m$ of salt at time $t$.

(*iv*) Solve the equation from the part (*iii*) with the initial condition $m(0) = 20$ kg and find the concentration of salt in the water at the time when the tank is drained completely.

(b) Let $F(x, y) = \ln(2x^2 - y)$.

($i$) Calculate the first order partial derivatives of $F(x, y)$ at the point $(1, 0)$.

($ii$) Find the equation of the tangent plane to the surface $z = F(x, y)$ at the point $P = (1, 0, \ln 2)$.

**3.** (a) Let $g(x, y) = ye^{-x}$.

($i$) Calculate the gradient vector $\nabla g(x, y)$. Hence find $\nabla g(0, 2)$.

($ii$) Find the directional derivative of $g(x, y)$ at the point $(0, 2)$ in the direction $\dfrac{\mathbf{i}}{\sqrt{2}} + \dfrac{\mathbf{j}}{\sqrt{2}}$.

(b) $(i)$ Write down the differential of $h(x, y) = x^{1/3}y$. Find the expression for this differential when $x = 8$ and $y = 3$.

$(ii)$ Use part $(i)$ to find an approximation for $2.7 \times \sqrt[3]{8.2}$.

(c) Let $f(x,y) = \dfrac{1}{x\,y}$. Find the point $(x, y)$ such that $0 \le x \le 2$, $0 \le y \le 2$ and the distance from the origin $(0, 0, 0)$ to the point $(x, y, f(x, y))$ on the surface $z = f(x, y)$ is minimal.

**Note**: The distance between two points $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ in $\mathbb{R}^3$ is given by $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$.

## Table of Standard Integrals

**1.** $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$

**2.** $\displaystyle\int \frac{dx}{x} = \ln|x| + C$

**3.** $\displaystyle\int e^x\,dx = e^x + C$

**4.** $\displaystyle\int \sin x\,dx = -\cos x + C$

**5.** $\displaystyle\int \cos x\,dx = \sin x + C$

**6.** $\displaystyle\int \tan x\,dx = -\ln\big|\cos x\big| + C$

**7.** $\displaystyle\int \cot x\,dx = \ln\big|\sin x\big| + C$

**8.** $\displaystyle\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

**9.** $\displaystyle\int \sec^2 x\,dx = \tan x + C$

**10.** $\displaystyle\int \operatorname{cosec}^2 x\,dx = -\cot x + C$

**11.** $\displaystyle\int \sec x\,dx = \ln\big|\sec x + \tan x\big| + C$

**12.** $\displaystyle\int \operatorname{cosec} x\,dx = \ln\big|\operatorname{cosec} x - \cot x\big| + C$

**13.** $\displaystyle\int \sinh x\,dx = \cosh x + C$

**14.** $\displaystyle\int \cosh x\,dx = \sinh x + C$

**15.** $\displaystyle\int \tanh x\,dx = \ln\cosh x + C$

**16.** $\displaystyle\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C \quad (|x| < a)$

**17.** $\displaystyle\int \frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C = \ln\left(x + \sqrt{x^2+a^2}\right) + C'$

**18.** $\displaystyle\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + C = \ln\left(x + \sqrt{x^2-a^2}\right) + C' \quad (x > a)$

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**Linearity:** $\displaystyle\int \big(\lambda f(x) + \mu g(x)\big)\,dx = \lambda\int f(x)\,dx + \mu\int g(x)\,dx$

**Integration by substitution:** $\displaystyle\int f(u(x))\frac{du}{dx}\,dx = \int f(u)\,du$

**Integration by parts:** $\displaystyle\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$

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**End of Extended Answer Section**