# MATH1023/MATH1062 Calculas

Usyd Mingyuan Ba

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## 1 Week1

### 1.1 Differential Equation

1. **Differential Equation(DE)**: A differential equation (DE) is a mathmatical equation that relates some function with its derivatives

2. **Order**: The order of a differential equation equals to a highest derivative occuring in it.

   - $\frac{dy}{dx} = -ky$ has order 1
   - $\frac{dy}{dx} = y^{18} + \frac{d^5y}{dx^2}y + x^2$ has order 5

3. **Standard Form**: The standard form of a first-order differential equation is

$$\frac{dy}{dx} = f(x, y)$$

4. **General Solution**: A general solution is a solution incoprating all constants of integration.

5. **Initial Condition**: An initial condition is a pair $(x_0, y_0)$ such that $y(x_0) = y_0$

## 2 Week2

### 2.1 Direction Field

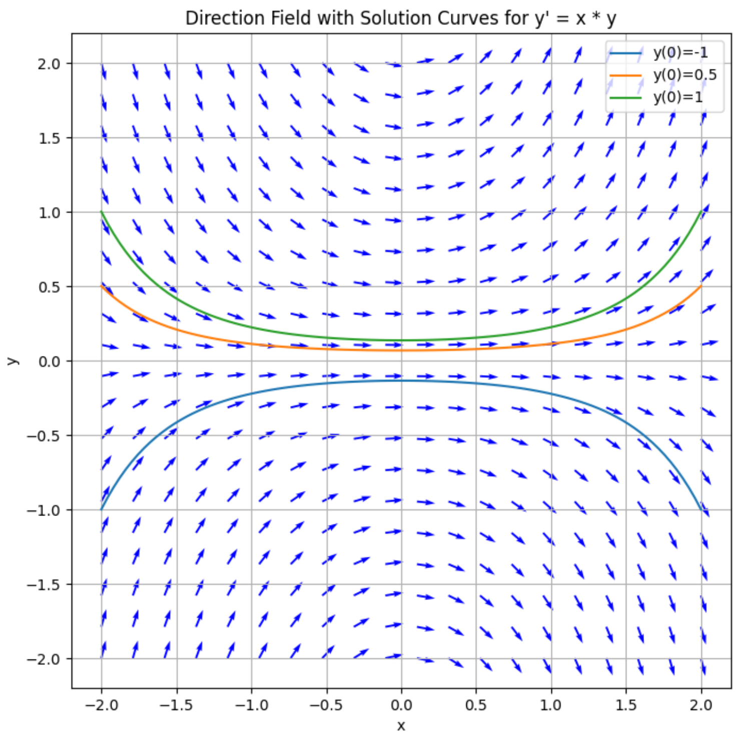
1. **Definition**: A direction field of a DE

$$y' = f(x, y)$$

   consists of a grid of short line segments with slope $f(a, b)$ drawn at points $(a, b)$.So the line segment at $(a, b)$ is tangent to any solution passing through $(a, b)$

2. **Example**:Draw some solution curves on the given direction field for the DE:

$$y' = xy$$




## 2.2 Separable equations

1. **Definition**: A first-order DE $y' = f(x, y)$ is called separable if there are functions $g(x)$ and $h(y)$ such that $f(x, y) = g(x)h(y)$, so a separable DE can be written

$$y' = g(x)h(y)$$

2. **Goal**: We want to find a method for solving separable DEs

3. **Method**: We can solve a separable DE:

$$\frac{dy}{dx} = g(x)h(y)$$

by separating variables.

Dividing both sides by $h(y)$ gives

$$\frac{1}{h(y)}\frac{dy}{dx} = g(x)$$

Intergrating both sides gives:

$$\int \frac{1}{h(y)} = \int g(x)dx$$

If we can find antiderivatives $H(y)$ for $\frac{1}{h(y)}$ and $G(x)$ for $g(x)$, then we have

$$H(y) = G(x) + C$$

# 3 Week3

## 3.1 Modelling Population Growth

1. **Constant Growth**: This occurs when the population $x$ increases at a constant rate. The DE is

   $$\frac{dx}{dt} = k$$

   where k is constant

2. **Exponential Growth**: The exponential growth model assumes the growth rate is proportional to the size of the population.

   The general form of a DE modelling exponential growth is

   $$\frac{dx}{dt} = kx$$

   where k is constant

3. **Logistic Growth**: Exponential growth is not a realistic growth model for all values of $t$. A small animal population with unlimited resources of food and space may show exponential growth initially

   As the population gets larger there will be food shortages, overcrowding, and other factors that slow down the growth rate.

   The growth rate k should decrease as the population x increases.

   Since k is no longer constant, we write $k = g(x)$, so the DE becomes

   $$\frac{dx}{dt} = g(x)x$$

A small population can growth exponentially, se we want $g(x) \approx k$ when $x \approx 0$. But as x increases $g(x)$ should decrease.

The simplest formula with this behaviour is

$$g(x) = k - ax$$

So the DE becomes

$$\tfrac{dx}{dt} = (k - ax)x$$

We introduce a new constant $b = \frac{k}{a}$ so

$$(k - ax)x = ax(\tfrac{k}{a} - x) = ax(b - x)$$

Let $\frac{b}{a} = b$, the logistic DE is then given by

$$\tfrac{dx}{dt} = ax(b - x)$$

# 4 week4

## 4.1 First-order linear DEs

1. **First-order linear differential equation**: A first-order linear differential equation is a DE of the form:

$$\tfrac{dy}{dx} + p(x)y = q(x)$$

$\frac{dy}{dx}$ **and** $y$ **occur only linearly**

2. **How to solve first-order linear DEs ?**: The idea is multiplying the DE by a function $r(x)$ give:

$$r(x)\tfrac{dy}{dx} + r(x)p(x) = r(x)q(x)$$

If we can find $r(x)$ such that:

$$r(x)\tfrac{dy}{dx} + r(x)p(x) = \tfrac{d}{dx}(r(x)y(x))$$

then the DE becomes:

$$\tfrac{d}{dx}(r(x)y(x)) = r(x)q(x)$$

Integrating with respect to x gives:

$$\int \tfrac{d}{dx}(r(x)y(x))dx = \int(r(x)q(x))dx$$
$$\rightarrow$$
$$r(x)y(x) = \int r(x)q(x)dx + C$$

so the general solution is

$$y = \frac{1}{r(x)}[\int r(x)q(x)dx + C]$$

3. **Integrating factor**: The function

$$r(x) = e^{\int p(x)dx}$$

is an intergrating factor for the first-order linear DE

$$\frac{dy}{dx} + p(x)y = q(x)$$

4. **General Solution** the general solution of the DE is

$$y = \frac{1}{r(x)}[\int r(x)q(x)dx + C]$$

# 5 week5

## 5.1 Higher order differential equations

Higher order DEs involve higher order derivatives.For example, the DE:

$$\frac{d^2y}{dx^2} + f(x,y)\frac{dy}{dx} = g(x,y)$$

is a second-order differential equation.

1. Solving higher-order DEs is harder.
2. The general solution of a second-order DE has 2 degrees of freedom, so needs two initial conditions.
3. The general solution of an nth-order DE has n degrees of freedom, so needs n initial conditions.

## 5.2 Second-order linear DEs with constant coefficients

1. **Definition** A second-order linear differential equation is a DE that can be expressed in the form:

$$\frac{d^2y}{dx} + f_1(x)\frac{dy}{dx} + f_0(x)y = g(x)$$

The DE is linear in $y$ and its derivatives.

2. **homogeneous/inhomogeneous**
   - The DE is homogeneous if $g(x) = 0$
   - The DE is inhomogeneous if $g(x) \neq 0$

If $g(x) = 0, f_0(x) = a, f_1(x) = b$ for $a, b \in \mathbb{R}$, then we have a homogeneous second-order linear differential equation with constant coefficient:

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0$$

3. Solve the above DE :

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0$$

- **Observation 1**: y is a linear combination of its first two derivatives, so we try:

  $$y(x) = e^{mx}$$

  We have

  $$\frac{dy}{dx} = me^{mx}, \frac{d^2y}{dx^2} = m^2e^{mx}$$

- **Observation 2**: Find m such that $y = Ce^{mx}$ satisfies the DE

  $$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0$$

  substituting y and its derivatives we get:

  $$\begin{gathered} Cm^2e^{mx} + aCme^{mx} + bCe^{mx} = 0 \\ \Rightarrow Ce^{mx}(m^2 + am + b) = 0 \\ \Rightarrow m = \frac{-a \pm \sqrt{a^2 - ab}}{2} \end{gathered}$$

  So we have 2 solutions

  $$m_1 = \frac{-a + \sqrt{a^2 - ab}}{2}, m_2 = \frac{-a - \sqrt{a^2 - ab}}{2}$$

- **Observation 3:** Show that if $m = m_1, m_2$ are solutions of $m^2 + am + b$,then $y = C_1e^{m_1x} + C_2e^{m_2x}$, satisfies the DE

  $$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0$$

  we have

  $$\begin{gathered} y = C_1e^{m_1x} + C_2e^{m_2x} \\ \Rightarrow \frac{dy}{dx} = m_1C_1e^{m_1x} + m_2C_2e^{m_2x} \\ \Rightarrow \frac{d^2y}{dx^2} = m_1^2C_1e^{m_1x} + m_2^2C_2e^{m_2x} \end{gathered}$$

  substituting into the DE we get

  $$\begin{gathered} 2 \\ \Rightarrow \frac{dy}{dx} = m_1C_1e^{m_1x} + m_2C_2e^{m_2x} \\ \Rightarrow \frac{d^2y}{dx^2} = m_1^2C_1e^{m_1x} + m_2^2C_2e^{m_2x} \end{gathered}$$

substituting into the DE we get

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = m_1^2 C_1 e^{m_1 x} + m_2^2 C_2 e^{m_2 x} + a\left(m_1 C_1 e^{m_1 x} + m_2 C_2 e^{m_2 x}\right) + b\left(C_1 e^{m_1 x} + C_2 e^{m_2 x}\right)$$
$$= C_1 e^{m_1 x}\left(m_1^2 + am_1 + b\right) + C_2 e^{m_2 x}\left(m_2^2 + am_2 + b\right)$$
$$= 0$$

- **formal solution:** We now have a good candidate for a general solution of the DE:

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$$

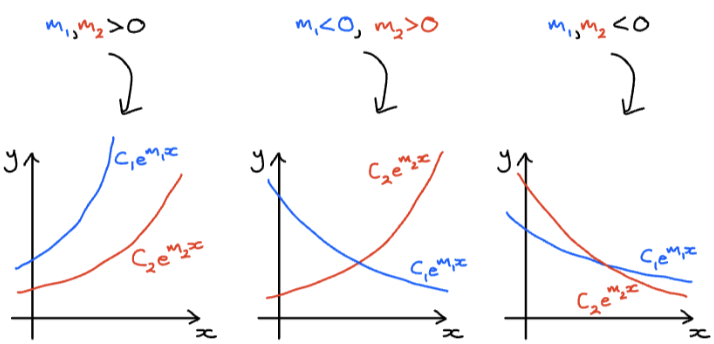
Where $m_1 = \frac{-a+\sqrt{a^2-4b}}{2}, m_2 = \frac{-a-\sqrt{a^2-4b}}{2}$ are solutions of $m^2+am+b=0$. We have 3 cases to consider:

  - **Case 1**: For $a^2 > 4b$ we have 2 distinct real solutions

$$m_1 \neq m_2, m_1, m_2 \in \mathbb{R}$$

The general solution is

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$$




  - **Case 2**: For $a^2 < 4b$ we have 2 distinct complex solutions:

$$m_1, m_2 = \frac{-a \pm \sqrt{a^2-4b}}{2} = \frac{-a \pm 2ik}{2} = -\frac{a}{2} \pm ik$$
where $k = \frac{1}{2}\sqrt{4b-a^2} > 0$

Using Euler's formula:

$$e^{ikx} = cos(kx) + isin(kx)$$

We have

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$$

$$= C_1 e^{\left(-\frac{a}{2}+ik\right)x} + C_2 e^{\left(-\frac{a}{2}-ik\right)x}$$

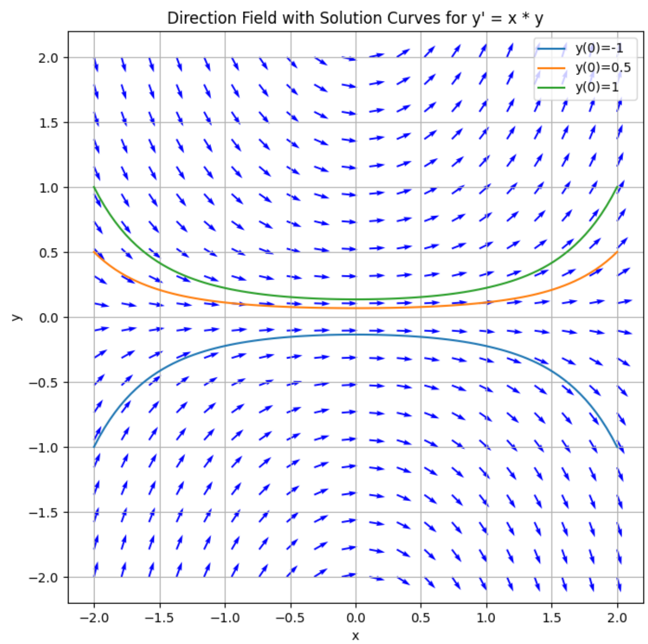
$$= e^{-\frac{a}{2}x}\left(C_1 e^{ikx} + C_2 e^{-ikx}\right)$$

$$= e^{-\frac{a}{2}x}\left(C_1\left(\cos(kx) + i\sin(kx)\right) + C_2\left(\cos(kx) - i\sin(kx)\right)\right)$$

$$= e^{-\frac{a}{2}x}\left(\left(C_1 + C_2\right)\cos(kx) + i\left(C_1 - C_2\right)\sin(kx)\right)$$

$$= e^{-\frac{a}{2}x}\left(D_1\cos(kx) + D_2\sin(kx)\right)$$

So the general solution is: $y = e^{-\frac{a}{2}x}\left(D_1\cos(kx) + D_2\sin(kx)\right)$




- **Case 3**: For $a^2 = 4b$ we have 1 real solution:

  $$m_1 = m_2 = \frac{-a\pm\sqrt{a^2-4b}}{2} = -\frac{a}{2}$$

  Our solution becomes

  $$y = C_1e^{-\frac{a}{2}x} + C_2e^{-\frac{a}{2}x}$$

  $$= \left(C_1 + C_2\right)e^{-\frac{a}{2}x}$$

  $$= De^{-\frac{a}{2}x}$$

  Here, $D$ is a constant ($D = C_1 + C_2$), which means we only have 1 degree of freedom, so this is not a general solution.

We look for a general solution of the form

$$y = f(x)e^{-\frac{a}{2}x}$$

Substituting $y$ and its derivatives into the differential equation (DE)

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0$$

gives

$$e^{-\frac{a}{2}x}\left(f''(x) + \frac{1}{4}(4b - a^2)f(x)\right) = 0 \quad \text{(exercise)}$$

Since $e^{-\frac{a}{2}x} \neq 0$,

$$f''(x) = \frac{1}{4}(a^2 - 4b)f(x) = 0$$

which implies

$$f'(x) = C_2$$

$$f(x) = C_2x + C_1$$

Hence, the general solution is

$$y = (C_1 + C_2x)e^{-\frac{a}{2}x}$$

# 6 week6

## Simple harmonic motion

- Periodic bhaviour without damping is modelled by the DE

$$\frac{d^2x}{dt^2} + bx = 0, b > 0$$
$$\text{or}$$
$$\ddot{x} + w_0^2 x = 0$$

- We can express the solution as

$$x = Acos(w_0 t + \phi)$$

  - $A$ = amplitude
  - $w_0$ = frequency
  - $\phi$ = phase
  - $T = \frac{2\pi}{w_0}$ = period

## Damped harmonic oscillator

- Periodic behaviour with damping is modelled by the DE:

$$\frac{d^2x}{dt^2} + a\frac{dx}{dt} + bx = 0$$

  with $a = 2\gamma$,$b = \omega_0^2$, or

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0$$

- The characteristic equation is

$$m^2 + am + b = 0$$

  which has solution

$$m = \frac{-a \pm \sqrt{a^2 - 4b}}{2} = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$$

## Inhomogeneous second-order linear DEs with constant coefficients

- An inhomogeneous scond-order linear differential equation with constant coefficients is a DE that can be expressed in the form:

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = g(x)$$

- **theorem**: Let $y_p(x)$ be a particular solution of an inhomogeneous linear DE and let $y_h(x)$ be the general solution of the corresponding homogeneous DE. Then the general solution of the inhomogeneous DE is the

$$y(x) = y_h(x) + y_p(x)$$

- systems of first-order linear DEs with constant coefficients: A system of two first-order DEs with constant coefficients has the form:

$$\frac{dx}{dt} = ax + by \texttt{ (*)},$$
$$\frac{dy}{dt} = cx + fy \texttt{ (**)}$$

to solve this system, we follow the following steps:

1. Differentiate `(*)`

$$\frac{d^2x}{dt^2} = a\frac{dx}{dt} + b\frac{dy}{dt} \texttt{ (I)}$$

2. Substitude the right hand side of of `(**)` into `(I)`

$$\frac{d^2x}{dt^2} = a\frac{dx}{dt} + b(cx + fy) \texttt{ (II)}$$

3. Rearrange `(*)` to make y the subject

$$y = \frac{1}{b}(\frac{dx}{dt} - ax) \texttt{ (III)}$$

4. Substitude the right hand side of `(III)` into `(II)`

$$\frac{d^2x}{dt^2} = a\frac{dx}{dt} + b(cx + \frac{f}{b}(\frac{dx}{dt} - ax)) \rightarrow \frac{d^2x}{dt^2} = (a+f)\frac{dx}{dt} + (bx - af)x$$

5. Solve the DE

$$\frac{d^2x}{dt^2} - (a+f)\frac{dx}{dt} - (bc - af)x = 0 \text{ for x.}$$

6. Substitute x into `(**)` and solve the first-order linear DE for y

$$\frac{dy}{dt} = cx + fy \rightarrow \frac{dy}{dt} + p(t)y = q(t)$$

# 7 Week7

## 2-dimensional plane

- The **2-dimensional plane**, often called the $(x, y)$-plane, can be represented by the set

$$\mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}$$

- The **graph** of a function

$$f : D \rightarrow \mathbb{R}, \quad y = f(x), \quad D \subseteq \mathbb{R}$$

is given by the set

$$\{(x, y) \in \mathbb{R}^2 \mid y = f(x), x \in D\}$$

- Curves in the plane can also be given by parametric equations:

$$x = f(t), \quad y = g(t)$$

where $t$ is a parameter.

### 3-dimensional space

- **3-dimensional space** can be represented by the set

$$\mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$$

- **Right-handed system**: The $x, y, z$ axes are a right-handed system. The positive $x, y, z$ directions are determined by the right-hand rule:
  1. Point the fingers of your right hand in the positive $x$-direction.
  2. Curl your fingers in the positive $y$-direction.
  3. Your thumb points in the positive $z$-direction.

### Curves in $\mathbb{R}^3$

- Curves in $\mathbb{R}^3$ can be represented using parametric equations:

$$x = f(t), \quad y = g(t), \quad z = h(t)$$

- There is no way of turning these parametric equations of a curve in space intoa single Cartesion equation

### Surfaces in $\mathbb{R}^3$

- A **surface** in $\mathbb{R}^3$ is given by a single equation involving $x, y, x$
- The general form of a plane is $ax + by + cz = d$
- The general form of a **sphere** with radius $r$ and centre $(a, b, c)$ is $(x - a)^2(y - b)^2 + (z - c)^2 = r^2$
- The general form of a **paraboloid** is given by

$$z = c \pm ((x - a)^2 + (y - b)^2)$$

## 8 Week8

### Function of one variable

- **Definition**: Recall that a function of one real variable

$$f : D \to \mathbb{R}, D \subseteq \mathbb{R}$$

is a rule that assigns to each number $x \in D$ a number $f(x) \in \mathbb{R}$

- The **domain** of $f$ is the set D of allowed inpus.
- The **natural domain** of $f$ is the largest subset of R of allowed inputs.

## Function of 2 variables

- **Definition:** A function of 2 real variables:

$$f : D \to \mathbb{R}, D \subseteq \mathbb{R}^2$$

  is a rule that assigns to each pair $(x, y) \in D$ a number $f(x, y) \in \mathbb{R}$

- The **domain** of f is the set D of allowed inpus
- The **natrual domian of** $f$ is the largest subset of $\mathbb{R}^2$ of allowed inputs

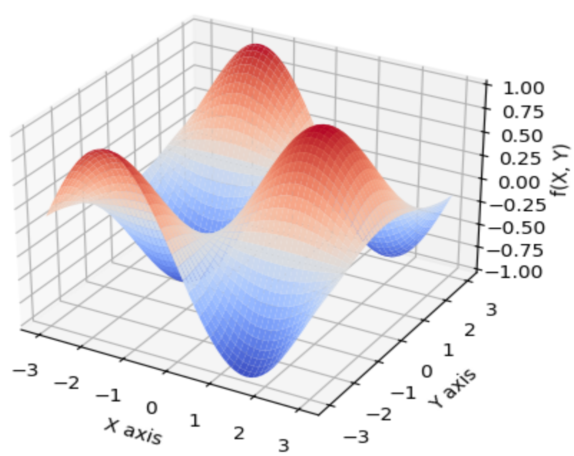
## Graphs of functions

- The graph of a function of 2 variables:

$$f : D \to \mathbb{R}$$

  is the set of points

$$\{(x, y, f(x, y)) \in \mathbb{R}^3 | (x, y) \in D\}$$




- We can not get a full sphere as a function. It fails the vertical line test.

## Level Curves

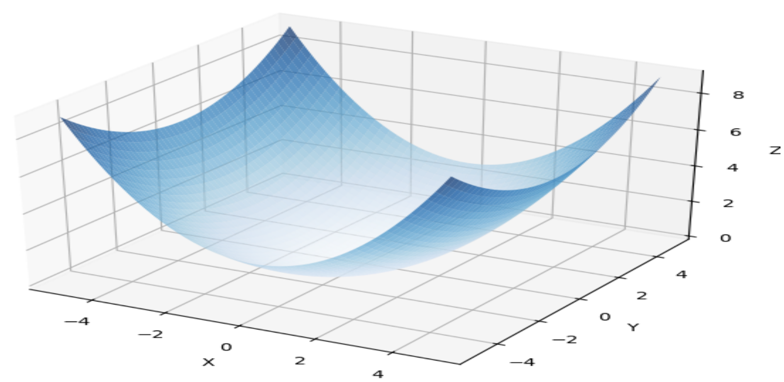
- **Definition:** A level cureve of a function $f(x, y)$ is a curve in $\mathbb{R}^2$ defined by

$$f(x, y) = c$$

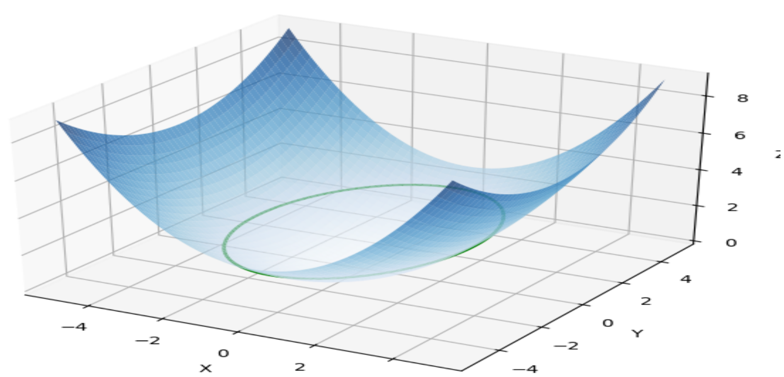
for a constant $c \in \mathbb{R}$

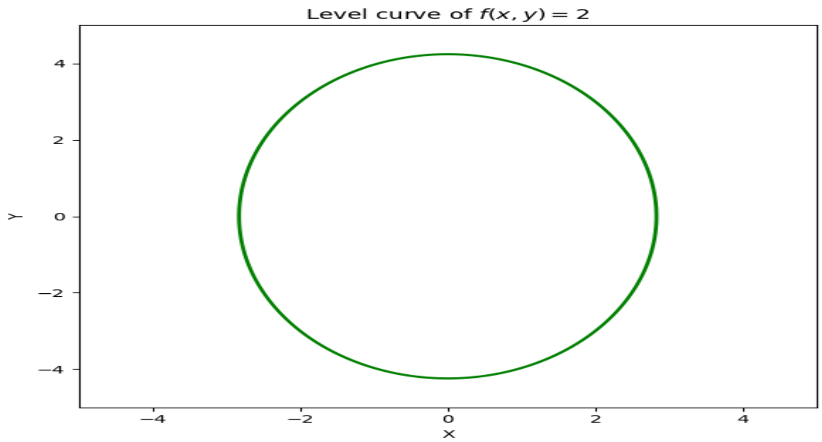
- The level curves $f(x, y) = c$ are the intersections of the surface $z = f(x, y)$ with the planes $z = c$

Graph of $f(x, y) = \frac{x^2}{4} + \frac{y^2}{9}$




Graph of $f(x, y) = c$ where $c = 2$






## Partial derivatives

- **Definition:** for a sufficiently smooth function of 2 variables

$$f : D \to \mathbb{R}, D \subseteq \mathbb{R}^2$$

The partial derivative of $f$ with respect to $x$ at $(x, y) = (a, b)$ is:

$$f_x(a, b) = \frac{\partial f}{\partial x}\bigg|_{(x,y)=(a,b)} = \lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h}$$

and the partial derivative of $f$ with respect to $y$ at $(x, y) = (a, b)$ is

$$f_x(a, b) = \frac{\partial f}{\partial y}\bigg|_{(x,y)=(a,b)} = \lim_{h\to 0} \frac{f(a,b+h)-f(a,b)}{h}$$

- **Terminology**: If $f_x(a, b) = \frac{\partial f}{\partial x}\Big|_{(a,b)}$ exists for all $(a, b) \in D$, then we say that $f$ is diffrentiable with respect to $x$ on D and we write

$$f_x(x, y) = \frac{\partial f}{\partial x}(x, y)$$

for the derivative function of f w.r.t. x.

- Similarly, If $f_y(a, b) = \frac{\partial f}{\partial y}\Big|_{(a,b)}$ exists for all $(a, b) \in D$, then we say that $f$ is diffrentiable with respect to $y$ on D and we write

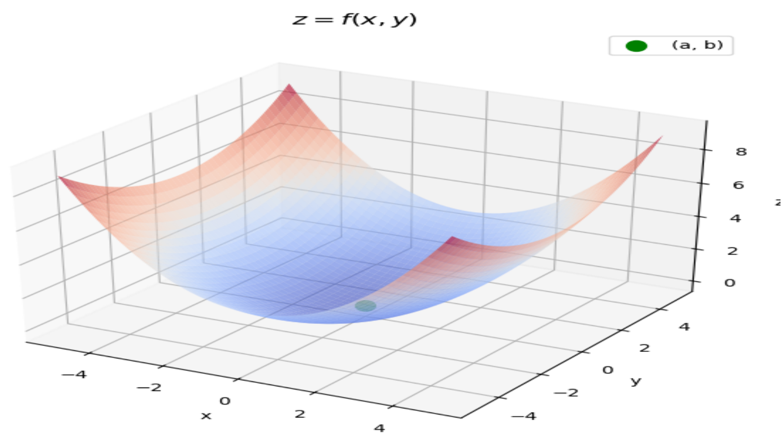
$$f_y(x,y) = \frac{\partial f}{\partial y}(x,y)$$

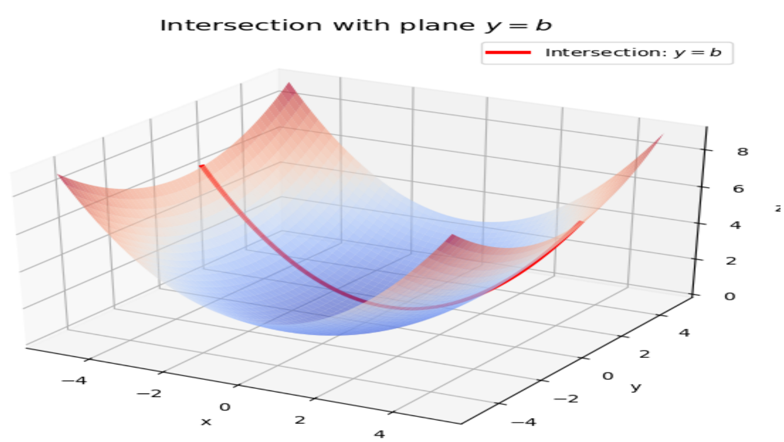
for the derivative function of f w.r.t. y.

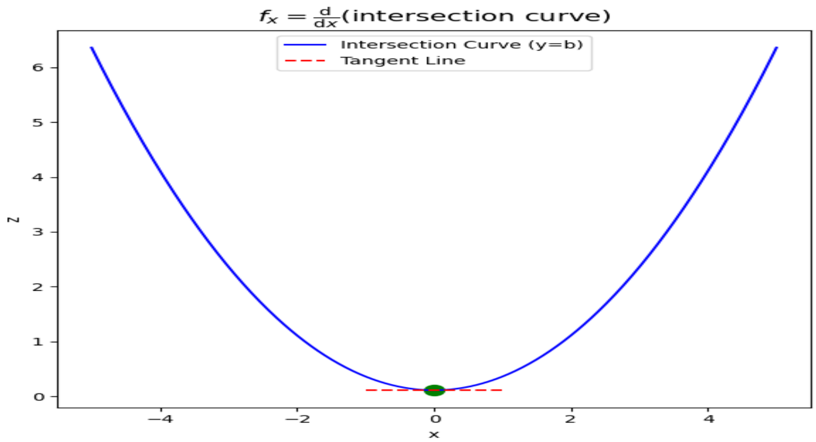
- What do partial derivatives measure? For a sufficiently smooth function:

$$f : D \to \mathbb{R}, D \subset \mathbb{R}$$

the partial derivatives $f_x = \frac{\partial f}{\partial x}$ measure the rate of change of $f$ on the $x$ direction.






- Here we have the intersection of the surface $z = f(x, y)$ and the plane $y = b$ is the function of one variable given by

$$g(x) = f(x, b)$$

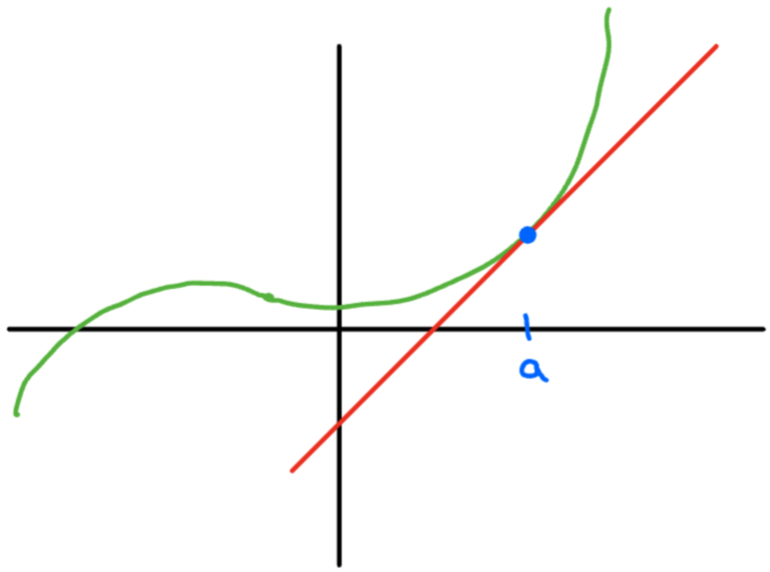
The gradient of the tangent to this curve at $x = a$ is given by

$$\begin{aligned} g'(a) &= \lim_{h\to 0} \frac{g(a+h)-g(a)}{h} \\ &= \lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h} \\ &= f_x(a, b) \end{aligned}$$

- How do we calculate partial derivatives?
  - To calculate $f_x = \frac{\partial f}{\partial x}$
    1. Imagine $y$ is a constant
    2. Differentiate as a function of one variable $x$.
  - To calculate $f_y = \frac{\partial f}{\partial y}$
    1. Imagine $x$ is a constant
    2. Differentiate as a function of one variable $y$.

## Tangent lines and plane

- **tangent lines**: Given a differentiable function of 1 variable, we can consider the tangent line at $x = a$.




The equation of the tangent line is:

$$y = f(a) + f'(a)(x - a)$$

The tangent is the best linear approximation to $f$ near $x = a$

- **Tangent plane:** Consider the tangent plane at a point $(a, b)$ of a "nice" function of two variables

$$f : D \to \mathbb{R}, D \subseteq \mathbb{R}^2$$

The tangent plane is the best linear approximation to $f$ near $(x, y) = (a, b)$. It should have the same first-order partial derivatives.

The equation of the tangent plane to $f$ at $(a, b)$ is

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

# 9 Week 9

## Approximating values of functions using tangents

- **Differential:** The differential of a differentiable function $y = f(x)$ is

$$dy = f'(x)dx$$

In Leibniz notation $dy = \frac{dy}{dx}dx$

- **Differential:** The differential of a differentiable function $z = f(x, y)$ is

$$dz = f_x(x, y)dx + f_y(x, y)dy$$

- **Approximation:** If $(x, y)$ is near $(a, b)$ then we have

$$\begin{gathered} f(x, y) \approx f(a, b) + dz \\ = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \\ = \text{z value of equation of tangent plane} \end{gathered}$$

## The total derivative

- **Definition:** If $z = f(x, y), x = g(t), y = h(t)$ are differentiable functions, then the total derivative of z with respect to $t$ at $t = a$ is:

$$\frac{dz}{dt} = \lim_{k \to 0} \frac{f(g(a + k), h(a + k)) - f(g(a), h(a))}{k}$$

- To calculate the total derivative, we user the total derivative rule:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

- **Chain Rule**: If $z = f(x, y), x = g(s, t), y = h(s, t)$ are differentiable functions, we have:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

## Implicit Differentiation

- **Implicit function theorem:(IFT):** Let $C \subseteq \mathbb{R}^2$ be a curve defined by $f(x, y) = k$ for some differentiable function $f : D \to \mathbb{R}, D \subseteq \mathbb{R}^2$ and $k \in \mathbb{R}$. If $(a, b) \in D, f(a, b) = k$ and $f_y(a.b) \neq 0$ then C can be described around $(a, b)$ by a function

$$y = g(x)$$

- **Application of the IFT:** If we can apply the IFT then we can find $\frac{dy}{dx}$ using the following method:

    1. Start with $f(x, y) = k$
    2. Use the IFT to express $y$ locally as a function of $x$ and substitute into the formula for the curve

$$f(x, g(x)) = k$$

3. Use the chain rule to differentiate with respect to x

$$f_x \frac{dx}{dx} + f_y \frac{dg}{dx} = 0$$

4. Solve for $\frac{dg}{dx}$

$$\frac{dg}{dx} = -\frac{fx}{fy}$$

- **A formula for $\frac{dy}{dx}$:**

$$\left.\frac{dy}{dx}\right|_{x=a} = -\frac{f_x(a,b)}{f_y(a,b)}$$

# Week 10+11

## Directional derivatives

- **Definition:** Let $\underset{\sim}{u}$ be a nonzero vector with $\underset{\sim}{\hat{u}} = u_1 \underset{\sim}{i} + u_2 \underset{\sim}{j}$, and let $f(x,y)$ be a differentiable function. The directional derivative of $f$ at $(a,b)$ in the direction of $\underset{\sim}{u}$ is:

$$\left(D_{\underset{\sim}{\hat{u}}} f\right)(a,b) = \lim_{h \to 0} \frac{f(a + u_1 h, b + u_2 h) - f(a,b)}{h}$$

- Remarks
  - $(D_{\underset{\sim}{i}})(a,b) = f_x(a,b)$
  - $(D_{\underset{\sim}{j}})(a,b) = f_y(a,b)$

- If $f(x,y)$ is differentiable, and $\underset{\sim}{\hat{u}} = u_1 \underset{\sim}{i} + u_2 \underset{\sim}{j}$ is a unit vector, then

$$D_{\underset{\sim}{\hat{u}}} f(a,b) = f_x(a,b)u_1 + f_y(a,b)u_2$$

- How to compute $D_{\underset{\sim}{u}} f$ (Method 1)
  1. Find the unit vector in the direction
  2. Find the first-order partial derivatives at the point
  3. Apply the above formula

## Gradient Vector

- **Definition:** Suppose we haves a differentiable function :

$$f : D \to R, D \subseteq R^2$$

The gradient of f is the vecotr valued function :

$$\nabla f : D \to R^2$$

defined by

$$\nabla f(x, y) = f_x(x, y)\underset{\sim}{i} + f_y(x, y)\underset{\sim}{j}$$

- How to compute $D_{\underset{\sim}{u}} f$ (Method 2):

$$D_{\underset{\sim}{u}} f(a, b) = \nabla f(a, b)\underset{\sim}{\hat{u}}$$

1. Find the unit vector $\underset{\sim}{\hat{u}}$
2. Find the gradient $\nabla f$ at the point
3. Evaluate the dot product $\nabla f(a, b)\underset{\sim}{\hat{u}}$

- Property: If the angle between $\nabla f$ and $\underset{\sim}{u}$ is $\theta$, then we have:

$$D_{\underset{\sim}{\hat{u}}} f(a, b) = \nabla f(a, b)\underset{\sim}{\hat{u}} = ||\nabla f(a, b)||\,||\underset{\sim}{\hat{u}}|| \cos\theta = ||\nabla f(a, b)|| \cos\theta \leq ||\nabla f(a, b)||$$

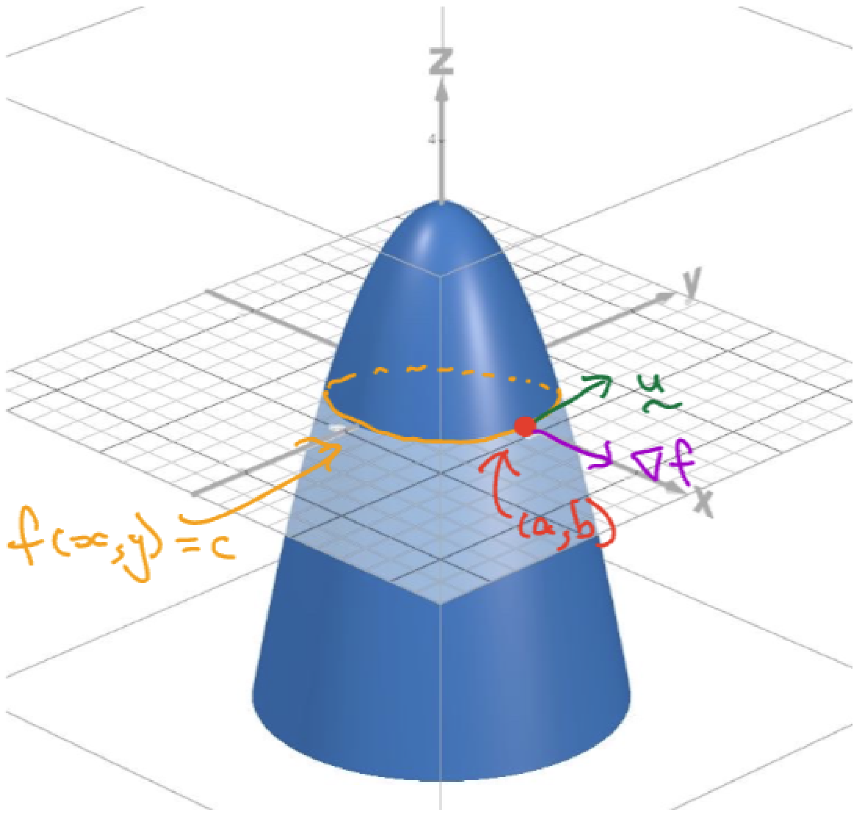
That is the maximum value of $D_{\underset{\sim}{\hat{u}}} f(a, b)$ is $||\nabla f(a, b)||$. This is attained when $\cos\theta = 1 \leftrightarrow \theta = 0 \leftrightarrow \nabla f$ and $\underset{\sim}{u}$ have the same direction.

## Directional derivatives and level curves

- **Topic:** $D_{\underset{\sim}{\hat{u}}} f(a, b) = 0$, that is $\underset{\sim}{u}$ is orthogonal to $\nabla f(a, b)$. That is

$$D_{\underset{\sim}{\hat{u}}} f(a, b) = \nabla f(a, b)\underset{\sim}{\hat{u}} = 0$$

- **Application Of Level Curve:** Consider a differentiable function $f(x, y)$. Let $(a, b)$ be a point such that $f(a, b) = c$, and let $vectu$ be the tangent vector to the level curve $f(x, y) = c$ at $(a, b)$.




$f(x,y)$ is a constant on the level curve on the $f(x,y) = c$, so the directional derivative of f at $(a,b)$ in the direction of the tangent vector $\underset{\sim}{u}$ must be zero. So we have

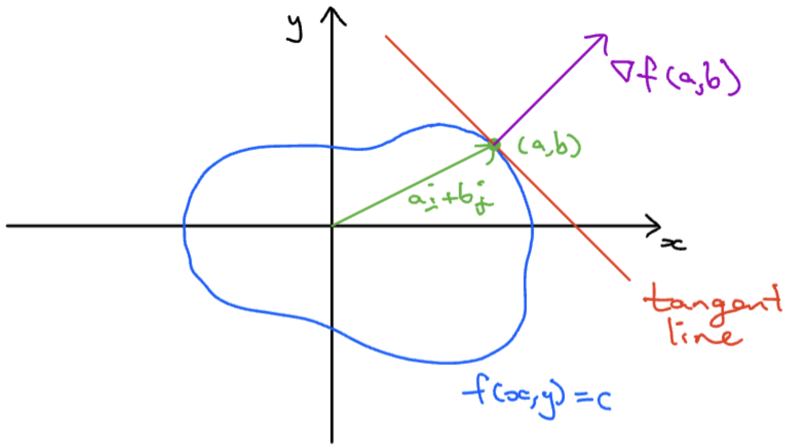
$$D_{\underset{\sim}{\hat{u}}} f(a,b) = \nabla f(a,b) \underset{\sim}{\hat{u}} = 0$$

Hence, $\nabla f(a,b)$ is orthogonal to the level curve $f(x,y) = f(a,b)$

- **Equation of a tangent to a level curve:** Given a differentiable function

$$f : D \to R, D \subseteq R^2$$

we can find an equation of the tangent line to a level curve $f(x,y) = c$ at $(a,b)$




- The tangent line has normal form:

$$\nabla f(a,b)((x-a)\underset{\sim}{i}+(y-b)\underset{\sim}{j})=0$$

- general form:

$$f_x(a,b)(x-a)+f_y(a,b)(y-b)=0$$

- **property**

Recall that:

$$D_{\underset{\sim}{u}}f=\nabla f\underset{\sim}{\hat{u}}=||\nabla f||\cos\theta$$

Thus, we have

$$-||\nabla f||\leq D_{\underset{\sim}{u}}f\leq ||\nabla f||$$

- The maximum value $D_{\underset{\sim}{u}}f=||\nabla f||$ occurs when

$$\cos\theta=1\leftrightarrow\theta=0\leftrightarrow\underset{\sim}{u}=k\nabla f$$

$\underset{\sim}{u}=k\nabla f\rightarrow$ same direction

- The minimum value $D_{\underset{\sim}{u}}f=-||\nabla f||$ occurs when

$$\cos\theta=-1\leftrightarrow\theta=\pi\leftrightarrow\underset{\sim}{u}=-k\nabla f$$

$\underset{\sim}{u}=-k\nabla f\rightarrow$ opposite direction

Now, suppose $\underset{\sim}{u} = u_i \underset{\sim}{i} + u_2 \underset{\sim}{j}$ is tangent to the level curve $f(x, y) = c$

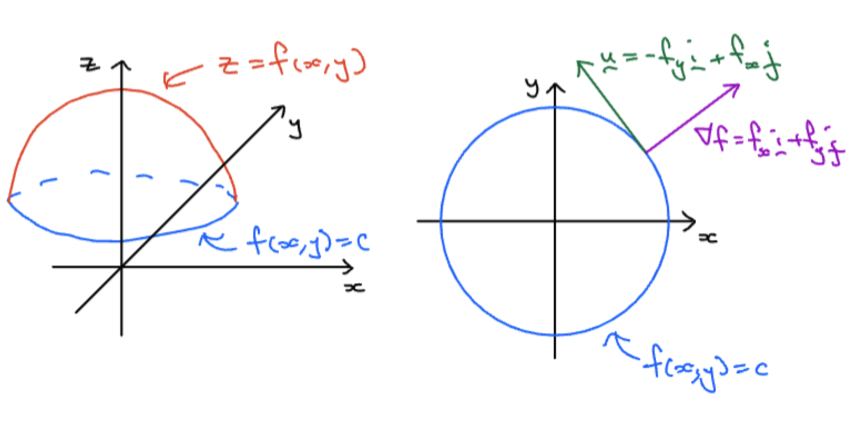
Since $f$ is constant on the level curve, the directional derivative in the direction of $\underset{\sim}{u}$ must be zero:

$$D_{\underset{\sim}{u}} f = \nabla f \underset{\sim}{u} = u_1 f_x + u_2 f_y = 0$$

So $\nabla f = f_x \underset{\sim}{i} + f_y \underset{\sim}{j}$ is orthogonal to $\underset{\sim}{u}$, and we can take $u_1 = -f_y$,$u_2 = f_x$, so

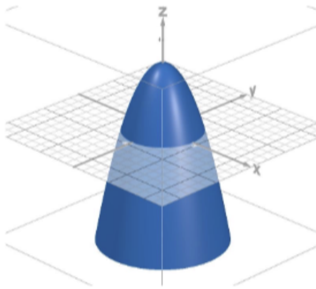
$$\underset{\sim}{u} = -f_y \underset{\sim}{i} + f_x \underset{\sim}{j}$$

is tangent to the level curve.




## Critical Points

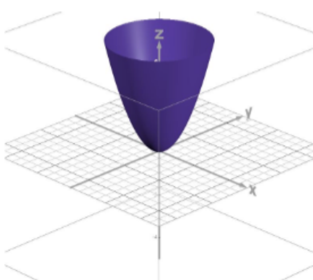
- A critical point of a differentiable function $f : D \to, D \subseteq R^2$ is a point $(a, b) \in D$ such that:
$$\nabla f(a, b) = \underset{\sim}{0}$$
- Types of Critical Points



A local maximum is a point $(a,b) \in D$ if

$$f(a,b) \geqslant f(x,y)$$

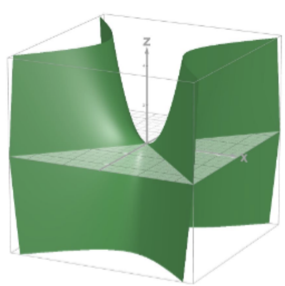
for all $(x,y)$ in an open disc around $(a,b)$



A local minimum is a point $(a,b) \in D$ if

$$f(a,b) \leqslant f(x,y)$$

for all $(x,y)$ in an open disc around $(a,b)$



A saddle point is a point $(a,b) \in D$ which is a critical point that is not a local maximum and not a local minimum.

- **discriminant**: The dicriminant of a differentiable function $f : S \to R, S \subseteq R^2$ with differentiable first-order partial derivatives is

$$D(x,y) = f_{xx}(x,y)f_{yy}(x,y) - f_{xy}(x,y)^2$$

- **Sceond derivative test**: Let $f : S \to R, S \subseteq R^2$ be a differentiable function, and $(a,b) \in S$ a critical point of f. Then:
  - $D(a,b) < 0 \implies (a,b)$ is a saddle point
  - $D(a,b) > 0, f_{xx}(a,b) < 0 \implies (a,b)$ is a local maximum
  - $D(a,b) > 0, f_{xx}(a,b) > 0 \implies (a,b)$ is a local minimum

### High Order partial derivavtives

- Suppose we haves a differentiable function:

$$f : D \to R, D \subseteq R^2$$

with first-order partial derivatives. We define:

$$f_{xx} = (f_x)_x = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

differentiate w.r.t. x twice

$$f_{xy} = (f_x)_y = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

differentiate w.r.t. x first, then y

$$f_{yx} = (f_y)_x = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

differentiate w.r.t. y first, then x

$$f_{yy} = (f_y)_y = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

differentiate w.r.t y twice

- **Theorem:** If $f$ has continuouts partial derivatives near $(a, b)$ then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

## week 12

### Global Extrema

- **Definition:** Lef $f : S \to R, s \subseteq R^2$ be a function of 2 variables.
  - A point $(a, b) \in S$ is a global maximum of $f$

    $$f(a, b) \geq f(x, y)$$

    for all $(x, y) \in S$
  - A point $(a, b) \in S$ is a global minimum of $f$

    $$f(a, b) \leq f(x, y)$$

    for all $(x, y) \in S$

- **When can global extrema occur**
  - A set $S \subseteq R^2$ is closed if it contains all points in its boundary
  - A set $S \subseteq R^2$ is bounded if it is contained in a sufficiently large disc

- **Fact:** If $f : S \to R, S \subseteq R^2$ is continuous, and S is closed and bounded, then f attains a global maximum and global minimum on S.

- **Method for finding global extrema:** If $f : S \to \mathbb{R}$ is differentiable (hence continuous) and $S \subset \mathbb{R}^2$ is closed and bounded, then we can find the global extrema on $S$ by:

  1. Finding all critical points of $f$ on $S$ by solving $\nabla f = 0$.
  2. Parameterizing the boundary of $S$ by $x(t), y(t)$, $t \in [a, b]$, and finding any critical points of $f$ restricted to the boundary of $S$ by solving $g'(t) = 0$ for $g(t) = f(x(t), y(t))$.
  3. Finding any endpoints of the boundary.
  4. Comparing function values at all points found above.