MATH1023/MATH1062 Calculas

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October 7, 2024

1 Week1

1.1 Differential Equation

1. **Differential Equation(DE)**: A differential equation (DE) is a mathmatical equation that relates some function with its derivatives

2. **Order**: The order of a differential equation equals to a highest derivative occuring in it.

   - $\frac{dy}{dx} = -ky$ has order 1
   - $\frac{dy}{dx} = y^{18} + \frac{d^5y}{dx^2}y + x^2$ has order 5

3. **Standard Form**: The standard form of a first-order differential equation is

$$\frac{dy}{dx} = f(x, y)$$

4. **General Solution**: A general solution is a solution incoprating all constants of integration.

5. **Initial Condition**: An initial condition is a pair (x_0, y_0) such that $y(x_0) = y_0$

2 Week2

2.1 Direction Field

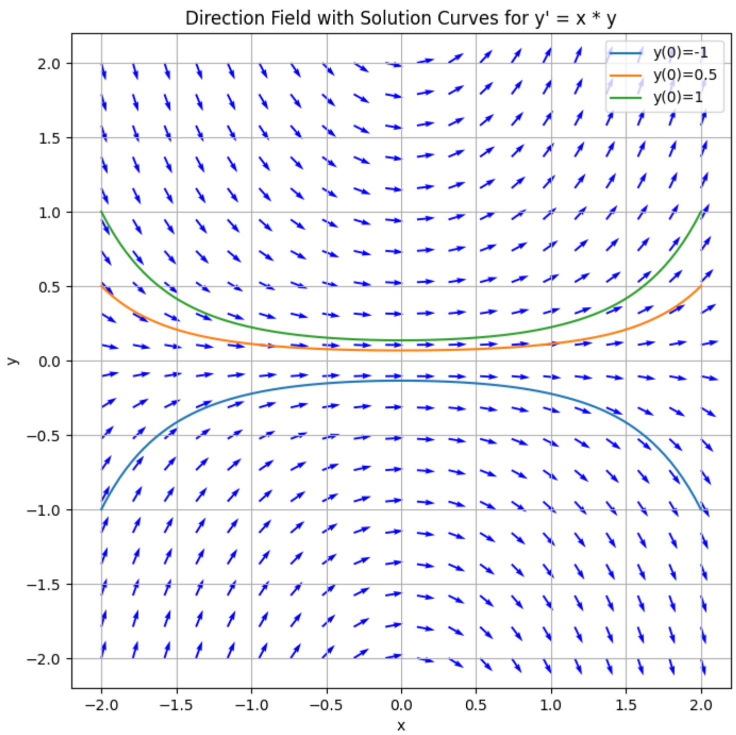
1. **Definition**: A direction field of a DE

$$y' = f(x, y)$$

   consists of a grid of short line segments with slope $f(a, b)$ drawn at points (a, b).So the line segment at (a, b) is tangent to any solution passing through (a, b)

2. **Example**:Draw some solution curves on the given direction field for the DE:

$$y' = xy$$



2.2 Separable equations

1. **Definition**: A first-order DE $y' = f(x, y)$ is called separable if there are functions $g(x)$ and $h(y)$ such that $f(x, y) = g(x)h(y)$, so a separable DE can be written

$$y' = g(x)h(y)$$

2. **Goal**: We want to find a method for solving separable DEs

3. **Method**: We can solve a separable DE:

$$\frac{dy}{dx} = g(x)h(y)$$

by separating variables.

Dividing both sides by $h(y)$ gives

$$\frac{1}{h(y)}\frac{dy}{dx} = g(x)$$

Intergrating both sides gives:

$$\int \frac{1}{h(y)} = \int g(x)dx$$

If we can find antiderivatives $H(y)$ for $\frac{1}{h(y)}$ and $G(x)$ for $g(x)$, then we have

$$H(y) = G(x) + C$$

3 Week3

3.1 Modelling Population Growth

1. **Constant Growth**: This occurs when the population x increases at a constant rate. The DE is

$$\frac{dx}{dt} = k$$

   where k is constant

2. **Exponential Growth**: The exponential growth model assumes the growth rate is proportional to the size of the population.

   The general form of a DE modelling exponential growth is

$$\frac{dx}{dt} = kx$$

   where k is constant

3. **Logistic Growth**: Exponential growth is not a realistic growth model for all values of t. A small animal population with unlimited resources of food and space may show exponential growth initially

   As the population gets larger there will be food shortages, overcrowding, and other factors that slow down the growth rate.

   The growth rate k should decrease as the population x increases.

   Since k is no longer constant, we write $k = g(x)$, so the DE becomes

$$\frac{dx}{dt} = g(x)x$$

A small population can growth exponentially, se we want $g(x) \approx k$ when $x \approx 0$. But as x increases $g(x)$ should decrease.

The simplest formula with this behaviour is

$$g(x) = k - ax$$

So the DE becomes

$$\tfrac{dx}{dt} = (k - ax)x$$

We introduce a new constant $b = \frac{k}{a}$ so

$$(k - ax)x = ax(\tfrac{k}{a} - x) = ax(b - x)$$

Let $\frac{b}{a} = b$, the logistic DE is then given by

$$\tfrac{dx}{dt} = ax(b - x)$$

4 week4

4.1 First-order linear DEs

1. **First-order linear differential equation**: A first-order linear differential equation is a DE of the form:

   $$\tfrac{dy}{dx} + p(x)y = q(x)$$

   $\frac{dy}{dx}$ and y occur only linearly

2. **How to solve first-order linear DEs ?**: The idea is multiplying the DE by a function $r(x)$ give:

   $$r(x)\tfrac{dy}{dx} + r(x)p(x) = r(x)q(x)$$

   If we can find $r(x)$ such that:

   $$r(x)\tfrac{dy}{dx} + r(x)p(x) = \tfrac{d}{dx}(r(x)y(x))$$

   then the DE becomes:

   $$\tfrac{d}{dx}(r(x)y(x)) = r(x)q(x)$$

   Integrating with respect to x gives:

   $$\int \tfrac{d}{dx}(r(x)y(x))dx = \int (r(x)q(x))dx$$
   $$\rightarrow$$
   $$r(x)y(x) = \int r(x)q(x)dx + C$$

so the general solution is

$$y = \frac{1}{r(x)}[\int r(x)q(x)dx + C]$$

3. **Integrating factor**: The function

$$r(x) = e^{\int p(x)dx}$$

is an intergrating factor for the first-order linear DE

$$\frac{dy}{dx} + p(x)y = q(x)$$

4. **General Solution** the general solution of the DE is

$$y = \frac{1}{r(x)}[\int r(x)q(x)dx + C]$$

5 week5

5.1 Higher order differential equations

Higher order DEs involve higher order derivatives.For example, the DE:

$$\frac{d^2y}{dx^2} + f(x, y)\frac{dy}{dx} = g(x, y)$$

is a second-order differential equation.

1. Solving higher-order DEs is harder.
2. The general solution of a second-order DE has 2 degrees of freedom, so needs two initial conditions.
3. The general solution of an nth-order DE has n degrees of freedom, so needs n initial conditions.

5.2 Second-order linear DEs with constant coefficients

1. **Definition** A second-order linear differential equation is a DE that can be expressed in the form:

$$\frac{d^2y}{dx} + f_1(x)\frac{dy}{dx} + f_0(x)y = g(x)$$

The DE is linear in y and its derivatives.

2. **homogeneous/inhomogeneous**
   - The DE is homogeneous if $g(x) = 0$
   - The DE is inhomogeneous if $g(x) \neq 0$

If $g(x)=0, f_0(x)=a, f_1(x)=b$ for $a,b\in\mathbb{R}$, then we have a homogeneous second-order linear differential equation with constant coefficient:

$$\frac{d^2y}{dx^2}+a\frac{dy}{dx}+by=0$$

3. Solve the above DE :

$$\frac{d^2y}{dx^2}+a\frac{dy}{dx}+by=0$$

- **Observation 1**: y is a linear combination of its first two derivatives, so we try:

  $$y(x)=e^{mx}$$

  We have

  $$\frac{dy}{dx}=me^{mx}, \frac{d^2y}{dx^2}=m^2e^{mx}$$

- **Observation 2**: Find m such that $y=Ce^{mx}$ satisfies the DE

  $$\frac{d^2y}{dx^2}+a\frac{dy}{dx}+by=0$$

  substituting y and its derivatives we get:

  $$\begin{gathered} Cm^2e^{mx}+aCme^{mx}+bCe^{mx}=0 \\ \Rightarrow Ce^{mx}(m^2+am+b)=0 \\ \Rightarrow m=\frac{-a\pm\sqrt{a^2-ab}}{2} \end{gathered}$$

  So we have 2 solutions

  $$m_1=\frac{-a+\sqrt{a^2-ab}}{2}, m_2=\frac{-a-\sqrt{a^2-ab}}{2}$$

- **Observation 3:** Show that if $m=m_1,m_2$ are solutions of m^2+am+b,then $y=C_1e^{m_1x}+C_2e^{m_2x}$, satisfies the DE

  $$\frac{d^2y}{dx^2}+a\frac{dy}{dx}+by=0$$

  we have

  $$\begin{gathered} y=C_1e^{m_1x}+C_2e^{m_2x} \\ \Rightarrow \frac{dy}{dx}=m_1C_1e^{m_1x}+m_2C_2e^{m_2x} \\ \Rightarrow \frac{d^2y}{dx^2}=m_1^2C_1e^{m_1x}+m_2^2C_2e^{m_2x} \end{gathered}$$

  substituting into the DE we get

  $$\begin{gathered} 2 \\ \Rightarrow \frac{dy}{dx}=m_1C_1e^{m_1x}+m_2C_2e^{m_2x} \\ \Rightarrow \frac{d^2y}{dx^2}=m_1^2C_1e^{m_1x}+m_2^2C_2e^{m_2x} \end{gathered}$$

substituting into the DE we get

$$\begin{aligned}\frac{d^2y}{dx^2}+a\frac{dy}{dx}+by &= m_1^2C_1e^{m_1x}+m_2^2C_2e^{m_2x}+a\left(m_1C_1e^{m_1x}+m_2C_2e^{m_2x}\right)+b\left(C_1e^{m_1x}+C_2e^{m_2x}\right)\\ &= C_1e^{m_1x}\left(m_1^2+am_1+b\right)+C_2e^{m_2x}\left(m_2^2+am_2+b\right)\\ &=0\end{aligned}$$

- **formal solution:** We now have a good candidate for a general solution of the DE:

$$y = C_1e^{m_1x}+C_2e^{m_2x}$$

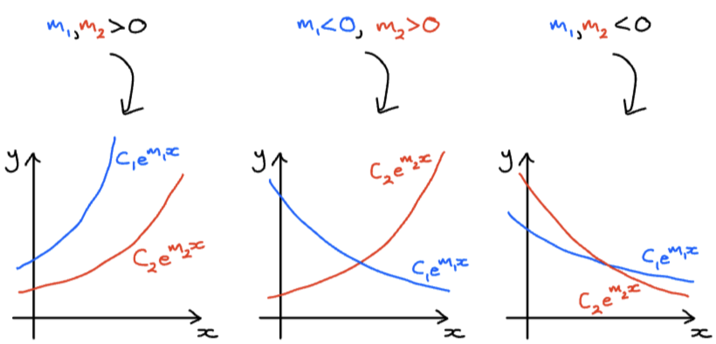
Where $m_1=\frac{-a+\sqrt{a^2-4b}}{2}$,$m_2=\frac{-a-\sqrt{a^2-4b}}{2}$ are solutions of $m^2+am+b=0$. We have 3 cases to consider:

  - **Case 1**: For $a^2>4b$ we have 2 distinct real solutions

$$m_1\neq m_2, m_1,m_2\in\mathbb{R}$$

The general solution is

$$y = C_1e^{m_1x}+C_2e^{m_2x}$$



  - **Case 2**: For $a^2<4b$ we have 2 distinct complex solutions:

$$m_1,m_2=\frac{-a\pm\sqrt{a^2-4b}}{2}=\frac{-a\pm 2ik}{2}=-\frac{a}{2}\pm ik$$
$$\text{where } k=\frac{1}{2}\sqrt{4b-a^2}>0$$

Using Euler's formula:

$$e^{ikx}=cos(kx)+isin(kx)$$

We have

$$y = C_1e^{m_1x}+C_2e^{m_2x}$$

$$=C_1e^{\left(-\frac{a}{2}+ik\right)x}+C_2e^{\left(-\frac{a}{2}-ik\right)x}$$

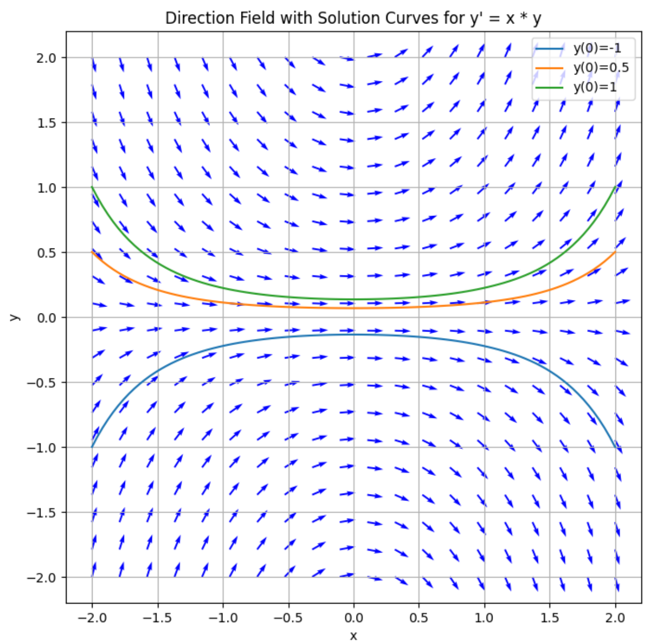
$$=e^{-\frac{a}{2}x}\left(C_1e^{ikx}+C_2e^{-ikx}\right)$$

$$= e^{-\frac{a}{2}x}\left(C_1\left(\cos(kx) + i\sin(kx)\right) + C_2\left(\cos(kx) - i\sin(kx)\right)\right)$$

$$= e^{-\frac{a}{2}x}\left(\left(C_1 + C_2\right)\cos(kx) + i\left(C_1 - C_2\right)\sin(kx)\right)$$

$$= e^{-\frac{a}{2}x}\left(D_1\cos(kx) + D_2\sin(kx)\right)$$

So the general solution is: $y = e^{-\frac{a}{2}x}\left(D_1\cos(kx) + D_2\sin(kx)\right)$



– **Case 3**: For $a^2 = 4b$ we have 1 real solution:

$$m_1 = m_2 = \frac{-a \pm \sqrt{a^2 - 4b}}{2} = -\frac{a}{2}$$

Our solution becomes

$$y = C_1 e^{-\frac{a}{2}x} + C_2 e^{-\frac{a}{2}x}$$

$$= \left(C_1 + C_2\right) e^{-\frac{a}{2}x}$$

$$= De^{-\frac{a}{2}x}$$

Here, D is a constant ($D = C_1 + C_2$), which means we only have 1 degree of freedom, so this is not a general solution.

We look for a general solution of the form

$$y = f(x)e^{-\frac{a}{2}x}$$

Substituting y and its derivatives into the differential equation (DE)

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0$$

gives

$$e^{-\frac{a}{2}x}\left(f''(x) + \frac{1}{4}(4b - a^2)f(x)\right) = 0 \quad \text{(exercise)}$$

Since $e^{-\frac{a}{2}x} \neq 0$,

$$f''(x) = \frac{1}{4}(a^2 - 4b)f(x) = 0$$

which implies

$$f'(x) = C_2$$

$$f(x) = C_2x + C_1$$

Hence, the general solution is

$$y = (C_1 + C_2x)e^{-\frac{a}{2}x}$$

6 week6

Simple harmonic motion

- Periodic bhaviour without damping is modelled by the DE

$$\frac{d^2x}{dt^2} + bx = 0, b > 0$$
$$\text{or}$$
$$\ddot{x} + w_0^2 x = 0$$

- We can express the solution as

$$x = Acos(w_0 t + \phi)$$

  - A = amplitude
  - w_0 = frequency
  - ϕ = phase
  - $T = \frac{2\pi}{w_0}$ = period

Damped harmonic oscillator

- Periodic behaviour with damping is modelled by the DE:

$$\frac{d^2x}{dt^2} + a\frac{dx}{dt} + bx = 0$$

  with $a = 2\gamma$,$b = \omega_0^2$, or

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0$$

- The characteristic equation is

$$m^2 + am + b = 0$$

  which has solution

$$m = \frac{-a \pm \sqrt{a^2 - 4b}}{2} = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$$

Inhomogeneous second-order linear DEs with constant coefficients

- An inhomogeneous scond-order linear differential equation with constant coefficients is a DE that can be expressed in the form:

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = g(x)$$

- **theorem**: Let $y_p(x)$ be a particular solution of an inhomogeneous linear DE and let $y_h(x)$ be the general solution of the corresponding homogeneous DE. Then the general solution of the inhomogeneous DE is the

$$y(x) = y_h(x) + y_p(x)$$

- systems of first-order linear DEs with constant coefficients: A system of two first-order DEs with constant coefficients has the form:

$$\frac{dx}{dt} = ax + by \texttt{ (*)},$$
$$\frac{dy}{dt} = cx + fy \texttt{ (**)}$$

to solve this system, we follow the following steps:

1. Differentiate `(*)`
$$\frac{d^2x}{dt^2} = a\frac{dx}{dt} + b\frac{dy}{dt} \texttt{ (I)}$$
2. Substitude the right hand side of of `(**)` into `(I)`
$$\frac{d^2x}{dt^2} = a\frac{dx}{dt} + b(cx + fy) \texttt{ (II)}$$
3. Rearrange `(*)` to make y the subject
$$y = \frac{1}{b}(\frac{dx}{dt} - ax) \texttt{ (III)}$$
4. Substitude the right hand side of `(III)` into `(II)`
$$\frac{d^2x}{dt^2} = a\frac{dx}{dt} + b(cx + \frac{f}{b}(\frac{dx}{dt} - ax)) \to \frac{d^2x}{dt^2} = (a+f)\frac{dx}{dt} + (bx - af)x$$
5. Solve the DE
$$\frac{d^2x}{dt^2} - (a+f)\frac{dx}{dt} - (bc - af)x = 0 \text{ for x.}$$
6. Substitute x into `(**)` and solve the first-order linear DE for y
$$\frac{dy}{dt} = cx + fy \to \frac{dy}{dt} + p(t)y = q(t)$$

7 Week7

2-dimensional plane

- The **2-dimensional plane**, often called the (x, y)-plane, can be represented by the set
$$\mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}$$

- The **graph** of a function

$$f : D \to \mathbb{R}, \quad y = f(x), \quad D \subseteq \mathbb{R}$$

is given by the set

$$\{(x, y) \in \mathbb{R}^2 \mid y = f(x), x \in D\}$$

- Curves in the plane can also be given by parametric equations:

$$x = f(t), \quad y = g(t)$$

where t is a parameter.

3-dimensional space

- **3-dimensional space** can be represented by the set

$$\mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$$

- **Right-handed system**: The x, y, z axes are a right-handed system. The positive x, y, z directions are determined by the right-hand rule:
  1. Point the fingers of your right hand in the positive x-direction.
  2. Curl your fingers in the positive y-direction.
  3. Your thumb points in the positive z-direction.

Curves in $\mathbb{R}^3$

- Curves in $\mathbb{R}^3$ can be represented using parametric equations:

$$x = f(t), \quad y = g(t), \quad z = h(t)$$

- There is no way of turning these parametric equations of a curve in space intoa single Cartesion equation

Surfaces in $\mathbb{R}^3$

- A **surface** in $\mathbb{R}^3$ is given by a single equation involving x, y, x
- The general form of a plane is $ax + by + cz = d$
- The general form of a **sphere** with radius r and centre (a, b, c) is $(x - a)^2(y - b)^2 + (z - c)^2 = r^2$
- The general form of a **paraboloid** is given by

$$z = c \pm ((x - a)^2 + (y - b)^2)$$

8 Week8

Function of one variable

- **Definition**: Recall that a function of one real variable

$$f : D \to \mathbb{R}, D \subseteq \mathbb{R}$$

is a rule that assigns to each number $x \in D$ a number $f(x) \in \mathbb{R}$

- The **domain** of f is the set D of allowed inpus.
- The **natural domain** of f is the largest subset of R of allowed inputs.

Function of 2 variables

- **Definition:** A function of 2 real variables:

$$f : D \to \mathbb{R}, D \subseteq \mathbb{R}^2$$

is a rule that assigns to each pair $(x, y) \in D$ a number $f(x, y) \in \mathbb{R}$

- The **domain** of f is the set D of allowed inpus
- The **natrual domian of** f is the largest subset of $\mathbb{R}^2$ of allowed inputs

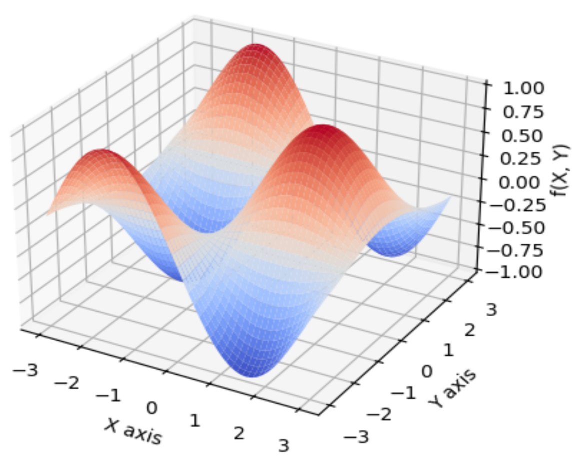
Graphs of functions

- The graph of a function of 2 variables:

$$f : D \to \mathbb{R}$$

is the set of points

$$\{(x, y, f(x, y)) \in \mathbb{R}^3 | (x, y) \in D\}$$



- We can not get a full sphere as a function. It fails the vertical line test.

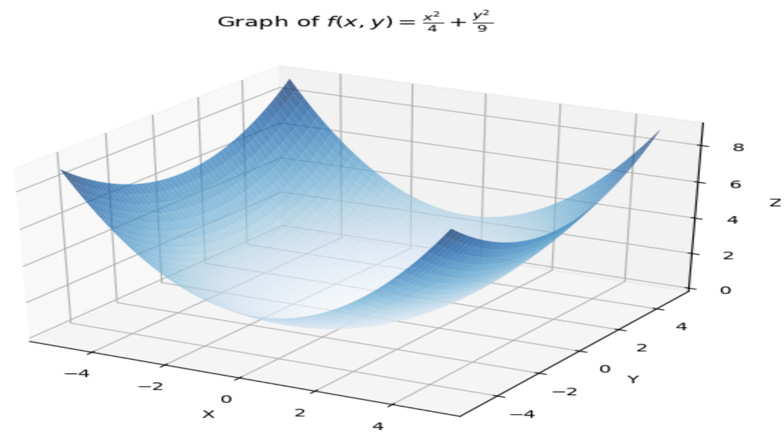
Level Curves

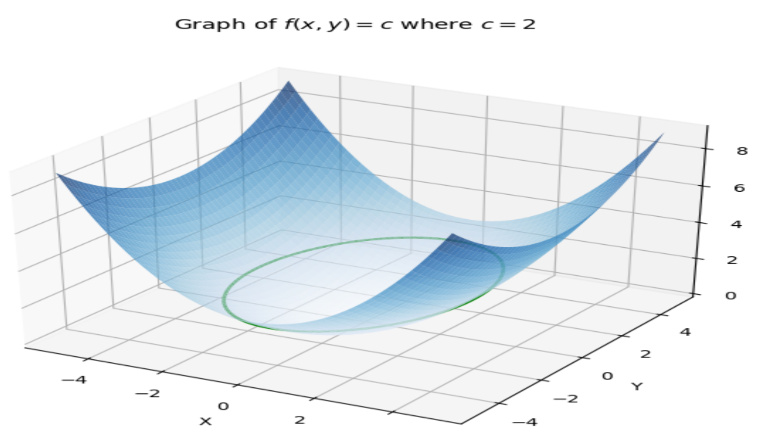
- **Definition:** A level cureve of a function $f(x, y)$ is a curve in $\mathbb{R}^2$ defined by

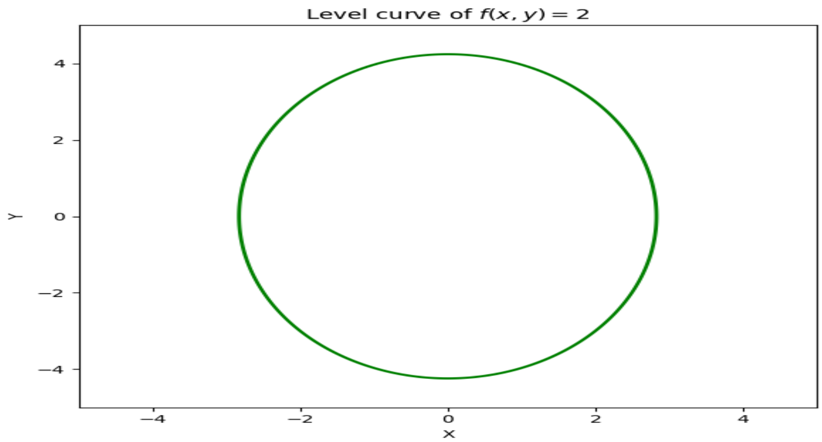
$$f(x, y) = c$$

for a constant $c \in \mathbb{R}$

- The level curves $f(x, y) = c$ are the intersections of the surface $z = f(x, y)$ with the planes $z = c$







Partial derivatives

- **Definition:** for a sufficiently smooth function of 2 variables

$$f : D \to \mathbb{R}, D \subseteq \mathbb{R}^2$$

The partial derivative of f with respect to x at $(x, y) = (a, b)$ is:

$$f_x(a,b) = \frac{\partial f}{\partial x}\bigg|_{(x,y)=(a,b)} = \lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h}$$

and the partial derivative of f with respect to y at $(x, y) = (a, b)$ is

$$f_x(a,b) = \frac{\partial f}{\partial y}\bigg|_{(x,y)=(a,b)} = \lim_{h\to 0} \frac{f(a,b+h)-f(a,b)}{h}$$

- **Terminology**: If $f_x(a,b) = \frac{\partial f}{\partial x}\bigg|_{(a,b)}$ exists for all $(a, b) \in D$, then we say that f is diffrentiable with respect to x on D and we write

$$f_x(x,y) = \frac{\partial f}{\partial x}(x,y)$$

for the derivative function of f w.r.t. x.

- Similarly, If $f_y(a,b) = \frac{\partial f}{\partial y}\bigg|_{(a,b)}$ exists for all $(a, b) \in D$, then we say that f is diffrentiable with respect to y on D and we write

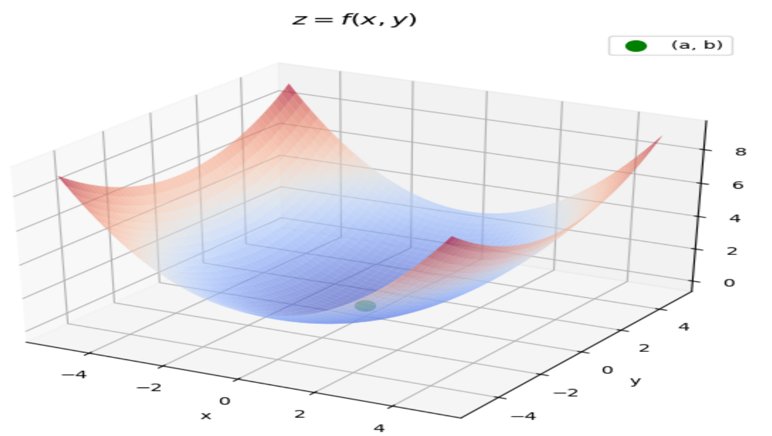
$$f_y(x,y) = \frac{\partial f}{\partial y}(x,y)$$

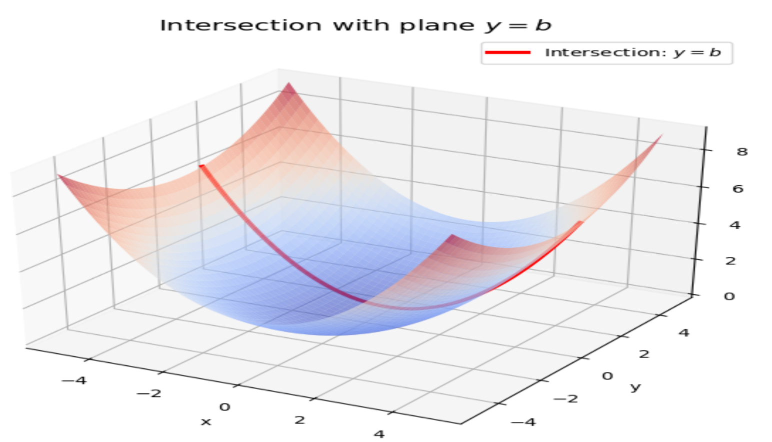
for the derivative function of f w.r.t. y.

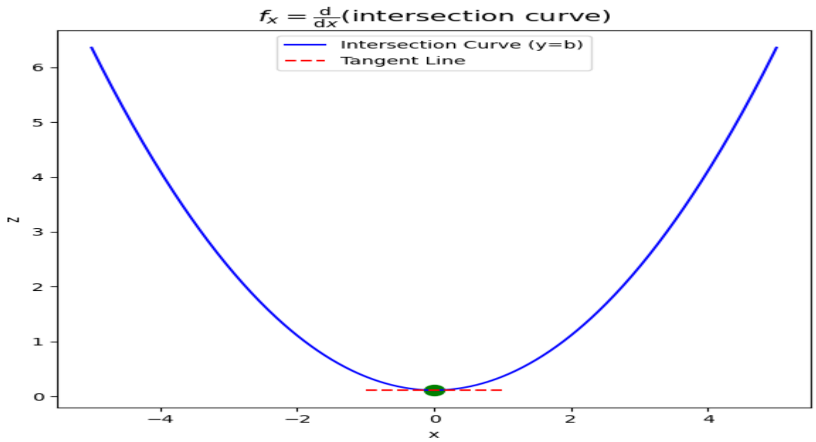
- What do partial derivatives measure? For a sufficiently smooth function:

$$f : D \to \mathbb{R}, D \subset \mathbb{R}$$

the partial derivatives $f_x = \frac{\partial f}{\partial x}$ measure the rate of change of f on the x direction.







- Here we have the intersection of the surface $z = f(x, y)$ and the plane $y = b$ is the function of one variable given by

$$g(x) = f(x, b)$$

The gradient of the tangent to this curve at $x = a$ is given by

$$\begin{aligned} g'(a) &= \lim_{h \to 0} \frac{g(a+h) - g(a)}{h} \\ &= \lim_{h \to 0} \frac{f(a+h, b) - f(a, b)}{h} \\ &= f_x(a, b) \end{aligned}$$

- How do we calculate partial derivatives?
  - To calculate $f_x = \frac{\partial f}{\partial x}$
    1. Imagine y is a constant
    2. Differentiate as a function of one variable x.
  - To calculate $f_y = \frac{\partial f}{\partial y}$
    1. Imagine x is a constant
    2. Differentiate as a function of one variable y.