If λ is an eigenvalue, the eigenspace of M corresponding to λ is:

$$\{\underline{v} \mid M\underline{v} = \lambda \underline{v}\}$$
$$= \{\underline{v} \mid (\lambda I - M)\,\underline{v} = \underline{0}\}$$
$$= \{\text{eigenvectors}\} \,\dot{\cup}\, \{\underline{0}\}$$

Example: Find the eigenspaces associated to 2, and 3, where $M = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}$.

$\lambda = 2$ $\quad 2I - M = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -2 & +1 \\ -2 & 1 \end{bmatrix}$

$\left[\begin{array}{cc|c} -2 & 1 & 0 \\ -2 & 1 & 0 \end{array}\right]$ Let's simplify: $\begin{bmatrix} -2 & 1 \\ -2 & 1 \end{bmatrix} \sim \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix}$

y is free

Let $\underline{v} = \begin{bmatrix} x \\ y \end{bmatrix}$

Then $-2x + y = 0$

Let $y = t \in \mathbb{R}$

Then $-2x + t = 0$

$+2x = t$

$x = \frac{1}{2}t$

So $\underline{v} = \begin{bmatrix} \frac{1}{2}t \\ t \end{bmatrix} = t\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$ $\left(\text{or } s\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$

where $t \in \mathbb{R}$.

$t \neq 0$.

$\lambda = 3$

$\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

If $\underline{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ then $x - y = 0$

Let $y = r$, $r \in \mathbb{R}$.

So $\underline{v} = r \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $r \neq 0$.

What about finding Eigenvalues?

Theorem: A scalar λ is an eigenvalue of M if and only if $\det(\lambda I - M) = 0$.

Proof:

$\Rightarrow$ Let λ be an eigenvalue of M.

So there exists $\underline{v} \neq \underline{0}$ such that

$M\underline{v} = \lambda \underline{v}$. That is,

$(\lambda I - M)\underline{v} = \underline{0}$.

Assume that $(\lambda I - M)^{-1}$ exists.

Then $(\lambda I - M)^{-1}(\lambda I - M)\underline{v} = (\lambda I - M)^{-1}\underline{0}$

So $\underline{v} = \underline{0}$. Contradiction!

So $(\lambda I - M)^{-1}$ does NOT exist.

Hence $\det(\lambda I - M) = 0$.

$\Leftarrow$

Now suppose that $\det(\lambda I - M) = 0$,
So $(\lambda I - M)$ is NOT invertible.
So the RREF of $(\lambda I - M)$ has at least one row of zeros.
Therefore one of the columns of the coefficient matrix doesn't have a leading entry. Hence, we have a free variable. Ie, there is at least one non-zero solution to the homogeneous system of equations. So it is an eigenvector.

Definition: The determinant $\det(\lambda I - M)$

is a polynomial in λ with leading term λ^n, and is called the characteristic polynomial of M. We write

$$\chi(\lambda) = \chi_M(\lambda) = \det(\lambda I - M).$$

Note: Some people call $\det(M - \lambda I)$

the characteristic polynomial, which is the same if n is even, and the negative if n is odd.

Example: Find the Eigenvalues of $M = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}$.

$$\det(\lambda I - M) = \left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} \right| = \begin{vmatrix} \lambda-4 & 1 \\ -2 & \lambda-1 \end{vmatrix}$$

$$= (\lambda-4)(\lambda-1)+2$$
$$= \lambda^2 - 5\lambda + 6$$
$$= (\lambda-2)(\lambda-3)$$

So $\lambda = 2, 3$.

Note: Not all real matrices have real Eigenvalues (example later). However, all complex matrices have complex Eigenvalues. Why?

The Fundamental Theorem of Algebra says that every non-constant polynomial has a root in $\mathbb{C}$.

Section 7.2 The Cayley-Hamilton Theorem

Arthur Cayley (1821–1895) discovered abstract group axioms

William Hamilton (1805–1865) discovered arithmetic of quaternions ← extends complex numbers into higher dimensions

The Cayley-Hamilton Theorem is a remarkable connection between determinants and matrix arithmetic.

In a theoretical sense, it can help you simplify high powers of a matrix, find the inverse of a matrix and is helpful in solving ordinary differential equations. It is used in many practical applications such as computer programming, coding and engineering - any situation where Eigenvalues are required. For example, in structural analysis and signal processing.

avoiding dangerous resonance

filtering/noise reduction

Consider a square nxn matrix $A = [a_{ij}]_{\substack{i=1,\dots,n \\ j=1,\dots,n}}$ with entries from a field, F. Then the characteristic polynomial of A is:

$$\chi_A(\lambda) = \det(\lambda I - A) = \begin{vmatrix} \lambda - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \lambda - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & \lambda - a_{nn} \end{vmatrix}$$

$$= b_0 + b_1\lambda + b_2\lambda^2 + \cdots + b_{n-1}\lambda^{n-1} + \lambda^n$$

for some $b_0, b_1, \dots, b_n \in F$.

Theorem: $\chi(A) = 0$ ← zero matrix

That is, $b_0 I + b_1 A + b_2 A^2 + \ldots + b_{n-1}A^{n-1} + \cancel{b_n} A^n = 0$ ← zero matrix

or "Every square matrix is a root of it's own characteristic polynomial."

careful! $\chi(A) \neq \det(AI - A)$ ✗ Find the polynomial first. THEN sub in A.

Example 1: Confirm the Cayley-Hamilton theorem using the matrix $M = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}$.

Then use it to find M^{-1}.

Recall that $\chi_M(\lambda) = \lambda^2 - 5\lambda + 6$

Check: $M^2 - 5M + 6I$

$$= \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} - 5\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & -5 \\ 10 & -1 \end{bmatrix} - \begin{bmatrix} 20 & -5 \\ 10 & 5 \end{bmatrix} + \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Since $M^2 - 5M + 6I = 0$

$$M^2 - 5M = -6I$$

$$M(M - 5I) = -6I$$

$$M \underbrace{\left[-\tfrac{1}{6}(M - 5I)\right]} = I$$

So $M^{-1} = -\frac{1}{6}(M - 5I)$

$= -\frac{1}{6}\left(\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}\right)$

$= -\frac{1}{6}\begin{bmatrix} -1 & -1 \\ 2 & -4 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$

Example 2: Confirm the Cayley-Hamilton theorem using the reflection matrix

$$M = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}.$$

$\chi_M(\lambda) = \left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \right|$

$= \begin{vmatrix} \lambda - \cos\theta & -\sin\theta \\ -\sin\theta & \lambda + \cos\theta \end{vmatrix}$

$= (\lambda - \cos\theta)(\lambda + \cos\theta) - \sin^2\theta$

$= \lambda^2 - \cos^2\theta - \sin^2\theta$

$\boxed{\chi(\lambda) = \lambda^2 - 1} = (\lambda - 1)(\lambda + 1)$

Two eigenvalues: $1, -1$

Check: $\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} - I$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - I = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$