The University of Sydney

MATH2022 Linear and Abstract Algebra

Semester 1 — **First Quiz Practice Exercises** — 2024

The first quiz is on March 21 on canvas. It will be an online quiz of twelve multiple choice questions for which you will have 40 minutes. These questions will be similar to the ones below.

1. Which one of the following forms a field under addition and multiplication?

(a) $\mathbb{N}$ (b) $\mathbb{Z}_2$ (c) $\mathbb{Z}_4$ (d) $\mathbb{Z}_6$ (e) $\mathbb{Z}_8$

Answer: (b)

2. Which one of the following is not a field under addition and multiplication?

(a) $\mathbb{Q}$ (b) $\mathbb{R}$ (c) $\mathbb{C}$ (d) $\mathbb{Z}$ (e) $\mathbb{Z}_{13}$

Answer: (d)

3. If today is Thursday, what day of the week will it be after 2018^{2018} days have elapsed?

(a) Friday (b) Saturday (c) Sunday
(d) Monday (e) Tuesday

Answer: (d)

4. Which one of the following statements is true?

(a) $\frac{2}{3} = 5$ in $\mathbb{Z}_{11}$. (b) $\frac{3}{4} = 4$ in $\mathbb{Z}_{13}$. (c) $\frac{3}{4} = 2$ in $\mathbb{Z}_7$.
(d) $\frac{2}{3} = 6$ in $\mathbb{Z}_{13}$. (e) $\frac{3}{4} = 7$ in $\mathbb{Z}_{11}$.

Answer: (b)

5. Consider the following matrix

$$M = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 4 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

with entries from $\mathbb{Z}_7$. Working over $\mathbb{Z}_7$, which of the following is true?

(a) $\det M = 0$ (b) $\det M = 4$ (c) $\det M = 5$
(d) $\det M = 2$ (e) $\det M = 6$

Answer: (e)

6. Consider the following system of equations over $\mathbb{Z}_5$:

$$\begin{array}{rcrcrcrcl} x & + & 2y & & & + & w & = & 1 \\ 2x & + & y & + & z & & & = & 2 \\ x & + & y & + & 2z & + & 2w & = & 1 \end{array}$$

Working over $\mathbb{Z}_5$, how many distinct solutions are there for (x, y, z, w)?

(a) infinitely many (b) no solutions (c) exactly one

(d) exactly five (e) exactly twenty-five

Answer: (e)

7. Find the unique solution to the following matrix equation

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

working over $\mathbb{Z}_2$.

(a) $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ (b) $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ (c) $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

(d) $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ (e) $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

Answer: (e)

8. Find the value of λ such that the system

$$\begin{array}{rcrcrcl} x & & & + & z & = & 1 \\ x & + & y & + & \lambda z & = & 2 \\ 2x & - & \lambda y & - & 4z & = & 6 \end{array}$$

is inconsistent over $\mathbb{R}$, but has more than one solution over $\mathbb{Z}_7$.

(a) $\lambda = 0$ (b) $\lambda = 1$ (c) $\lambda = 2$

(d) $\lambda = 3$ (e) $\lambda = 4$

Answer: (d)

9. Consider the matrix

$$M = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}$$

with entries from $\mathbb{Z}_2$. Which of the following is row equivalent to M and in reduced row echelon form?

(a) $\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

Answer: (c)

10. Consider the following matrices over $\mathbb{R}$, where θ is a real number:

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \qquad T_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

Which one of the following statements is true?

(a) $R_{\pi/3}^3 = I = T_{\pi/2}^2$ (b) $R_{2\pi/3}^3 = I = T_{2\pi/3}^3$ (c) $R_{\pi/4}^4 = I = T_{\pi/4}^4$

(d) $R_{\pi/2}T_{2\pi/3}R_{\pi/2} = T_{4\pi/3}$ (e) $T_{\pi/2}R_{2\pi/3}T_{\pi/2} = R_{4\pi/3}$

Answer: (e)

11. Consider the real matrix

$$M = \begin{bmatrix} 3 & 9 \\ 3 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 \\ 3 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and elementary matrices

$$E_1 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}, \quad E_4 = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}.$$

Use the chain of equivalences above to find a correct expression for M as a product of these elementary matrices.

(a) $M = E_2E_4E_1E_3$ (b) $M = E_2E_4E_3E_1$ (c) $M = E_4E_3E_1E_2$

(d) $M = E_2E_1E_3E_4$ (e) $M = E_3E_1E_4E_2$

Answer: (b)

12. Suppose that A, B and P are real square matrices such that P is invertible and $\lambda \in \mathbb{R}$ such that

$$P(A - \lambda I)P^{-1} \;=\; B\,,$$

where I denotes the identity matrix. Which of the of the following is a correct expression for A?

(a) $A = P^{-1}BP + \lambda I$ (b) $A = PBP^{-1} + \lambda I$ (c) $A = P^{-1}BP + \lambda P$
(d) $A = P^{-1}B + \lambda P^{-1}$ (e) $A = P^{-1}BP + \lambda P^{-1}$

Answer: (a)

13. Suppose that a, b, c, x are elements of a group G such that

$$axcba^{-1} \;=\; b\,.$$

Which one of the following is a correct expression for x?

(a) $x = a^{-1}ba(cb)^{-1}$ (b) $x = (ba)^{-1}c^{-1}ba$ (c) $x = a^{-1}b(cb)^{-1}a$
(d) $x = c^{-1}(ba^{-1})^{-1}a^{-1}b$ (e) $x = (ba^{-1})^{-1}a^{-1}bc^{-1}$

Answer: (a)

14. Consider the permutations

$$\alpha \;=\; (5\;2\;1\;4\;3)\,, \qquad \beta \;=\; (1\;3)(2\;4\;6)\,, \qquad \gamma \;=\; (1\;2\;4)(3\;5\;6)$$

of $\{1, 2, 3, 4, 5, 6\}$ expressed in cycle notation. Which one of the following is correct?

(a) α and γ are odd, and β is even. (b) α and γ are even, and β is odd.
(c) α and β are even, and γ is odd. (d) α and β are odd, and γ is even.
(e) β and γ are even, and α is odd.

Answer: (b)

15. Consider the group G of symmetries of a square, generated by a rotation α and a reflection β. Simplify the following expression in G:

$$\alpha\beta\alpha\beta^3\alpha^{-3}\beta^{-1}\alpha^{-1} \;=$$

(a) $\alpha\beta$ (b) $\alpha^2\beta$ (c) $\alpha^3\beta$
(d) α^3 (e) α^2

Answer: (b)

16. Consider the permutations

$$\alpha \;=\; (1\;2\;3\;4)(5\;6\;7)\,, \qquad \beta \;=\; (1\;3)(2\;4)\,, \qquad \gamma \;=\; (1\;2\;3)(4\;5)(6\;7)$$

of $\{1, 2, 3, 4, 5, 6, 7\}$ expressed in cycle notation. Simplify the permutation

$$\delta \;=\; \alpha\beta\gamma^{-1}\,,$$

composing from left to right:

(a) $\delta = (1\ 5\ 7\ 4\ 2\ 3)$ (e) $\delta = (1\ 5\ 7\ 4)$ (d) $\delta = (1\ 3\ 2\ 4\ 7\ 5)$

(c) $\delta = (1\ 4\ 7\ 5)$ (b) $\delta = (1\ 5\ 7\ 4)(3\ 2)$

Answer: (a)

17. Consider the permutations

$$\alpha \;=\; (1\ 3)(2\ 4\ 6\ 5) \qquad \text{and} \qquad \beta \;=\; (1\ 4\ 2\ 5)(6\ 3)$$

of $\{1, 2, 3, 4, 5, 6\}$ expressed in cycle notation. Which one of the following is a correct expression for the permutation

$$\gamma = \beta^{-1}\alpha\beta$$

where we compose from left to right?

(a) $\gamma = (4\ 6)(5\ 2\ 1\ 3)$ (b) $\gamma = (5\ 6)(4\ 3\ 1\ 2)$ (c) $\gamma = (4\ 6)(1\ 5\ 2\ 3)$

(d) $\gamma = (4\ 6)(5\ 3\ 1\ 2)$ (e) $\gamma = (5\ 6)(4\ 1\ 3\ 2)$

Answer: (c)

18. Consider the permutations

$$\alpha \;=\; (1\ 3\ 2)(4\ 6\ 5) \qquad \text{and} \qquad \gamma \;=\; (4\ 2\ 5)(6\ 1\ 3)$$

of $\{1, 2, 3, 4, 5, 6\}$ expressed in cycle notation. Which one of the following is a correct expression for a permutation β with the property

$$\gamma \;=\; \beta^{-1}\alpha\beta$$

where we compose from left to right?

(a) $\beta = (1\ 4\ 6)(2\ 3\ 5)$ (b) $\beta = (1\ 4\ 2)(3\ 6\ 5)$ (c) $\beta = (1\ 6\ 2\ 3)$

(d) $\beta = (1\ 3\ 6\ 4)(3\ 5)$ (e) $\beta = (1\ 3\ 2\ 6)$

Answer: (c)

19. Which one of the following configurations is possible to reach from the 8-puzzle

1	2	3
4	5	6
7	8	

by moving squares in and out of the space?

(a)

4	1	5
6	2	3
7	8	

(b)

2	3	4
8	7	5
1	6	

(c)

7	6	4
2	8	3
1	5	

(d)

4	8	2
5	7	1
3	6	

(e)

8	6	4
3	1	5
2	7	

Answer: (e)

20. Which one of the following configurations is impossible to reach from the 8-puzzle

1	2	3
4	5	6
7	8	

by moving squares in and out of the space?

(a)

2	4	
8	1	5
7	6	3

(b)

1	2	3
6		8
7	5	4

(c)

7	1	2
3		6
4	8	5

(d)

	1	4
3	2	7
6	5	8

(e)

3	1	4
	8	2
6	5	7

Answer: (a)