

1. (a) Since A is lower triangular, the eigenvalues are 2 and 1. To find the eigenspace corresponding to 2:

$$2I - A = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \quad \text{yielding} \quad \left\{ \begin{bmatrix} t \\ t \end{bmatrix} \mid t \in \mathbb{R} \right\}.$$

To find the eigenspace corresponding to 1:

$$I - A = \begin{bmatrix} -1 & 0 \\ -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{yielding} \quad \left\{ \begin{bmatrix} 0 \\ t \end{bmatrix} \mid t \in \mathbb{R} \right\}.$$

- (b) Observe that

$$\det(\lambda I - B) = \begin{vmatrix} \lambda - 1 & -1 \\ 2 & \lambda - 4 \end{vmatrix} = (\lambda - 1)(\lambda - 4) + 2 = \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3),$$

yielding eigenvalues 2 and 3. To find the eigenspace corresponding to 2:

$$2I - B = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \quad \text{yielding} \quad \left\{ \begin{bmatrix} t \\ t \end{bmatrix} \mid t \in \mathbb{R} \right\}.$$

To find the eigenspace corresponding to 3:

$$3I - B = \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix}, \quad \text{yielding} \quad \left\{ \begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} \mid t \in \mathbb{R} \right\}.$$

- (c) Observe that

$$\det(\lambda I - C) = \begin{vmatrix} \lambda - 1 & -1 \\ 1 & \lambda - 3 \end{vmatrix} = (\lambda - 1)(\lambda - 3) + 1 = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2,$$

yielding eigenvalue 2 only. To find the eigenspace corresponding to 2:

$$2I - C = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \quad \text{yielding} \quad \left\{ \begin{bmatrix} t \\ t \end{bmatrix} \mid t \in \mathbb{R} \right\}.$$

2. (a) Since A is upper triangular, the eigenvalues are 1, 2 and 3. To find the eigenspace corresponding to 1:

$$I - A = \begin{bmatrix} 0 & -1 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{yielding} \quad \left\{ \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} \mid t \in \mathbb{R} \right\}.$$

To find the eigenspace corresponding to 2:

$$2I - A = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{yielding} \quad \left\{ \begin{bmatrix} t \\ t \\ 0 \end{bmatrix} \mid t \in \mathbb{R} \right\}.$$

To find the eigenspace corresponding to 3:

$$3I - A = \begin{bmatrix} 2 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{yielding} \quad \left\{ \begin{bmatrix} 0 \\ -t \\ t \end{bmatrix} \mid t \in \mathbb{R} \right\}.$$

(b) Observe that

$$\begin{aligned} \det(\lambda I - B) &= \begin{vmatrix} \lambda & -1 & 0 \\ 2 & \lambda - 3 & 0 \\ 0 & 0 & \lambda - 2 \end{vmatrix} = (\lambda - 2) \begin{vmatrix} \lambda & -1 \\ 2 & \lambda - 3 \end{vmatrix} = (\lambda - 2)(\lambda(\lambda - 3) + 2) \\ &= (\lambda - 2)(\lambda^2 - 3\lambda + 2) = (\lambda - 2)^2(\lambda - 1), \end{aligned}$$

yielding eigenvalues 2 and 1. To find the eigenspace corresponding to 2:

$$2I - B = \begin{bmatrix} 2 & -1 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{yielding} \quad \left\{ \begin{bmatrix} \frac{t}{2} \\ t \\ s \end{bmatrix} \mid s, t \in \mathbb{R} \right\}.$$

To find the eigenspace corresponding to 1:

$$I - B = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{yielding} \quad \left\{ \begin{bmatrix} t \\ t \\ 0 \end{bmatrix} \mid t \in \mathbb{R} \right\}.$$

(c) Observe that

$$\begin{aligned} \det(\lambda I - C) &= \begin{vmatrix} \lambda + 7 & 2 & -6 \\ 2 & \lambda - 1 & -2 \\ 10 & 2 & \lambda - 9 \end{vmatrix} = \begin{vmatrix} \lambda + 7 & 2 & \lambda + 1 \\ 2 & \lambda - 1 & 0 \\ 10 & 2 & \lambda + 1 \end{vmatrix} \\ &= \begin{vmatrix} \lambda - 3 & 0 & 0 \\ 2 & \lambda - 1 & 0 \\ 10 & 2 & \lambda + 1 \end{vmatrix} = (\lambda - 3)(\lambda - 1)(\lambda + 1), \end{aligned}$$

yielding eigenvalues 3, 1 and -1 . To find the eigenspace corresponding to 3:

$$3I - C = \begin{bmatrix} 10 & 2 & -6 \\ 2 & 2 & -2 \\ 10 & 2 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & -8 & 4 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix},$$

yielding $\left\{ \begin{bmatrix} \frac{t}{2} \\ \frac{t}{2} \\ t \end{bmatrix} \mid t \in \mathbb{R} \right\}$. To find the eigenspace corresponding to 1:

$$I - C = \begin{bmatrix} 8 & 2 & -6 \\ 2 & 0 & -2 \\ 10 & 2 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

yielding $\left\{ \begin{bmatrix} t \\ -t \\ t \end{bmatrix} \mid t \in \mathbb{R} \right\}$. To find the eigenspace corresponding to -1 :

$$-I - C = \begin{bmatrix} 6 & 2 & -6 \\ 2 & -2 & -2 \\ 10 & 2 & -10 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & 8 & 0 \\ 0 & 12 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

yielding $\left\{ \begin{bmatrix} t \\ 0 \\ t \end{bmatrix} \mid t \in \mathbb{R} \right\}$.

3. We have $A\mathbf{v} = \lambda\mathbf{v}$ for some nonzero vector \mathbf{v} .

(a) If $A^2 = 0$, the zero matrix, then, denoting the zero column vector by $\mathbf{0}$, we have

$$\mathbf{0} = A^2\mathbf{v} = A(A\mathbf{v}) = A(\lambda\mathbf{v}) = \lambda(A\mathbf{v}) = \lambda^2\mathbf{v},$$

so that $\lambda^2 = 0$, since \mathbf{v} is nonzero, so that $\lambda = 0$.

(b) Suppose that $A^2 = A$. Then

$$\lambda\mathbf{v} = A\mathbf{v} = A^2\mathbf{v} = A(A\mathbf{v}) = A(\lambda\mathbf{v}) = \lambda(A\mathbf{v}) = \lambda^2\mathbf{v}.$$

Hence $\mathbf{0} = \lambda^2\mathbf{v} - \lambda\mathbf{v} = (\lambda^2 - \lambda)\mathbf{v} = \lambda(\lambda - 1)\mathbf{v}$, so that $\lambda(\lambda - 1) = 0$, since \mathbf{v} is nonzero. Thus $\lambda = 0$ or $\lambda = 1$.

(c) Suppose that $A^2 = I$. Then

$$\mathbf{v} = I\mathbf{v} = A^2\mathbf{v} = A(A\mathbf{v}) = A(\lambda\mathbf{v}) = \lambda(A\mathbf{v}) = \lambda^2\mathbf{v}.$$

Hence $\mathbf{0} = \lambda^2\mathbf{v} - \mathbf{v} = (\lambda^2 - 1)\mathbf{v} = (\lambda - 1)(\lambda + 1)\mathbf{v}$, so that $(\lambda - 1)(\lambda + 1) = 0$, since \mathbf{v} is nonzero. Thus $\lambda = 1$ or $\lambda = -1$.

4. (a) We have $\alpha^6 = \beta^2 = 1$ and $\beta\alpha = \alpha^{-1}\beta = \alpha^5\beta$, from which it follows quickly, by moving all α 's to the left and β 's to the right and collapsing, that

$$G = \{1, \alpha, \alpha^2, \alpha^3, \alpha^4, \alpha^5, \beta, \alpha\beta, \alpha^2\beta, \alpha^3\beta, \alpha^4\beta, \alpha^5\beta\}.$$

If the order in which the generators are written is reversed then powers of α stay the same (including the identity element 1 vacuously), and β stays the same. However, rewriting $\alpha^i\beta$ backwards, for $1 \leq i \leq 5$ produces

$$\beta\alpha^i = \alpha^{-i}\beta = \alpha^{6-i}\beta$$

so that the list of reflections $\alpha\beta, \alpha^2\beta, \alpha^3\beta, \alpha^4\beta, \alpha^5\beta$ is duplicated but in reverse order.

(b) Using the facts that $\alpha^6 = \beta^2 = 1$ and $\beta\alpha^i\beta = \alpha^{-i}$ for all i , we have

$$\beta\alpha^5\beta^5\alpha^{-3}\beta^{-3}\alpha^8\beta = \beta\alpha^5\beta\alpha^3\beta\alpha^2\beta = \beta\alpha^5\alpha^{-3}\alpha^2\beta = \beta\alpha^4\beta = \alpha^{-4} = \alpha^2.$$

(c) Put

$$A = \begin{bmatrix} \cos \pi/3 & -\sin \pi/3 \\ \sin \pi/3 & \cos \pi/3 \end{bmatrix} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix},$$

the rotation matrix that rotates 60 degrees anticlockwise, and

$$B = \begin{bmatrix} \cos 0 & \sin 0 \\ \sin 0 & -\cos 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

the reflection matrix that reflects through the horizontal axis (but in fact any reflection matrix will work). Then $A^6 = B^2 = I$ and

$$BAB = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix} = A^{-1},$$

so that, as in (a), it follows that

$$H = \{I, A, A^2, A^3, A^4, A^5, B, AB, A^2B, A^3B, A^4B, A^5B\}.$$

Because all of these powers of A are distinct (corresponding to multiples of one sixth rotation), it follows that all elements of H are distinct. The mapping $\alpha \mapsto A$ and $\beta \mapsto B$ induces a bijection from G to H that respects the group multiplication since, on the one hand, for $0 \leq i, j \leq 5$ and $0 \leq k, \ell \leq 1$,

$$(\alpha^i \beta^k)(\alpha^j \beta^\ell) = \begin{cases} \alpha^{i+j} \beta^\ell & \text{if } k = 0, \\ \alpha^{i-j} \beta^{1+\ell} & \text{if } k = 1, \end{cases}$$

whilst, on the other hand,

$$(A^i B^k)(A^j B^\ell) = \begin{cases} A^{i+j} B^\ell & \text{if } k = 0, \\ A^{i-j} B^{1+\ell} & \text{if } k = 1, \end{cases}$$

where addition or subtraction of exponents is taken mod 6 and mod 2.

5. We have $\alpha^6 = \beta^2 = 1$, since α is a single 6-cycle and β is a product of disjoint transpositions (2-cycles). Further $\beta^{-1} = \beta$ and

$$\beta^{-1} \alpha \beta = \beta \alpha \beta = (1 \ 6)(2 \ 5)(3 \ 4)(1 \ 2 \ 3 \ 4 \ 5 \ 6)(1 \ 6)(2 \ 5)(3 \ 4) = (1 \ 6 \ 5 \ 4 \ 3 \ 2) = \alpha^{-1}.$$

Thus

$$\beta \alpha = \beta \alpha \beta \beta = \alpha^{-1} \beta = \alpha^5 \beta.$$

Collecting all the α 's to the left and β 's to the right, and simplifying, it follows that

$$\begin{aligned} \langle \alpha, \beta \rangle &= \{1, \alpha, \alpha^2, \alpha^3, \alpha^4, \alpha^5, \beta, \alpha\beta, \alpha^2\beta, \alpha^3\beta, \alpha^4\beta, \alpha^5\beta\} \\ &= \{1, (1 \ 2 \ 3 \ 4 \ 5 \ 6), (1 \ 3 \ 5)(2 \ 4 \ 6), (1 \ 4)(2 \ 5)(3 \ 6), (1 \ 5 \ 3)(2 \ 6 \ 4), (1 \ 6 \ 5 \ 4 \ 3 \ 2), \\ &\quad (1 \ 6)(2 \ 5)(3 \ 4), (1 \ 5)(2 \ 4), (1 \ 4)(2 \ 3)(5 \ 6), (1 \ 3)(4 \ 6), (1 \ 2)(3 \ 6)(4 \ 5), (2 \ 6)(3 \ 5)\}. \end{aligned}$$

We want $\gamma \in \text{Sym}(6)$ such that $\gamma^{-1} \alpha \gamma = \alpha^{-1}$, so we look exhaustively for permutations that rewrite the cycle $(1 \ 2 \ 3 \ 4 \ 5 \ 6)$ as all possible cycles that represent α^{-1} , namely,

$$(1 \ 6 \ 5 \ 4 \ 3 \ 2), (6 \ 5 \ 4 \ 3 \ 2 \ 1), (5 \ 4 \ 3 \ 2 \ 1 \ 6), (4 \ 3 \ 2 \ 1 \ 6 \ 5), (3 \ 2 \ 1 \ 6 \ 5 \ 4), (2 \ 1 \ 6 \ 5 \ 4 \ 3),$$

producing the following possibilities, in the same order:

$$\gamma = (2\ 6)(3\ 5) = \alpha^5\beta, \quad \gamma = (1\ 6)(2\ 5)(3\ 4) = \beta, \quad \gamma = (1\ 5)(2\ 4) = \alpha\beta,$$

$$\gamma = (1\ 4)(2\ 3)(5\ 6) = \alpha^2\beta, \quad \gamma = (1\ 3)(4\ 6) = \alpha^3\beta, \quad \gamma = (1\ 2)(3\ 6)(4\ 5) = \alpha^4\beta.$$

These possibilities exhaust precisely all of the reflections of the form $\alpha^i\beta$ for $0 \leq i \leq 5$.

6. (a) We have that

$$M^2 = \begin{bmatrix} 2 & 2 & 4 \\ 2 & 5 & 8 \\ -1 & -2 & -3 \end{bmatrix} \begin{bmatrix} 2 & 2 & 4 \\ 2 & 5 & 8 \\ -1 & -2 & -3 \end{bmatrix} = \begin{bmatrix} 4 & 6 & 12 \\ 6 & 13 & 24 \\ -3 & -6 & -11 \end{bmatrix},$$

and

$$3M - 2I = \begin{bmatrix} 6 & 6 & 12 \\ 6 & 15 & 24 \\ -3 & -6 & -9 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 4 & 6 & 12 \\ 6 & 13 & 24 \\ -3 & -6 & -11 \end{bmatrix},$$

which verifies that $M^2 = 3M - 2I$. Further

$$\begin{aligned} \chi(\lambda) &= \det(\lambda I - M) = \begin{vmatrix} \lambda - 2 & -2 & -4 \\ -2 & \lambda - 5 & -8 \\ 1 & 2 & \lambda + 3 \end{vmatrix} = \begin{vmatrix} \lambda - 2 & -2 & -4 \\ -2 & \lambda - 5 & -8 \\ \lambda - 1 & 0 & \lambda - 1 \end{vmatrix} \\ &= \begin{vmatrix} \lambda + 2 & -2 & -4 \\ 6 & \lambda - 5 & -8 \\ 0 & 0 & \lambda - 1 \end{vmatrix} = (\lambda - 1) \begin{vmatrix} \lambda + 2 & -2 \\ 6 & \lambda - 5 \end{vmatrix} \\ &= (\lambda - 1)((\lambda + 2)(\lambda - 5) + 12) = (\lambda - 1)(\lambda^2 - 3\lambda + 2) = (\lambda - 1)^2(\lambda - 2). \end{aligned}$$

(b) By the Cayley-Hamilton Theorem,

$$\chi(M) = (M - I)^2(M - 2I) = 0.$$

This is consistent with the first part of (a), since $M^2 = 3M - 2I$ implies that

$$(M - I)(M - 2I) = M^2 - 3M + 2I = 0,$$

which in turn implies $\chi(M) = 0$, since $(\lambda - 1)(\lambda - 2)$ divides $\chi(\lambda)$.

(c) The formula holds trivially for $k = 1$, and also for $k = 2$ by part (a), which starts an induction. Suppose $k > 2$ and, as inductive hypothesis, that

$$M^{k-1} = (2^{k-1} - 1)M + (2 - 2^{k-1})I.$$

Then

$$\begin{aligned} M^k &= MM^{k-1} = M((2^{k-1} - 1)M + (2 - 2^{k-1})I) = (2^{k-1} - 1)M^2 + (2 - 2^{k-1})M \\ &= (2^{k-1} - 1)(3M - 2I) + (2 - 2^{k-1})M = (3(2^{k-1} - 1) + 2 - 2^{k-1})M + (2 - 2^k)I \\ &= (2^k - 1)M + (2 - 2^k)I, \end{aligned}$$

which verifies the inductive step, completing the proof for all positive k . The formula also holds trivially for $k = 0$. One can prove the formula for negative k also by induction. A direct verification is to calculate as follows, for positive k :

$$\begin{aligned}
M^k & \left((2^{-k} - 1)M + (2 - 2^{-k})I \right) \\
&= \left((2^k - 1)M + (2 - 2^k)I \right) \left((2^{-k} - 1)M + (2 - 2^{-k})I \right) \\
&= \left((2^k - 1)(2^{-k} - 1) \right) M^2 + \left((2^k - 1)(2 - 2^{-k}) + (2 - 2^k)(2^{-k} - 1) \right) M \\
&\quad + (2 - 2^k)(2 - 2^{-k})I \\
&= (2 - 2^k - 2^{-k})M^2 + (2^{k+1} - 3 + 2^{-k} + 2^{-k+1} - 3 + 2^k)M \\
&\quad + (5 - 2^{k+1} - 2^{-k+1})I \\
&= (2 - 2^k - 2^{-k})(3M - 2I) + (3(2^k) - 6 + 3(2^{-k}))M + (5 - 2^{k+1} - 2^{-k+1})I \\
&= \left(6 - 3(2^k) - 3(2^{-k}) + 3(2^k) - 6 + 3(2^{-k}) \right) M \\
&\quad + \left(-4 + 2^{k+1} + 2^{-k+1} + 5 - 2^{k+1} - 2^{-k+1} \right) I \\
&= I,
\end{aligned}$$

which verifies that

$$M^{-k} = (M^k)^{-1} = (2^{-k} - 1)M + (2 - 2^{-k})I,$$

so that the formula holds for all integers k .

(d) The formula gives

$$\begin{aligned}
M^5 &= (2^5 - 1)M + (2 - 2^5)I = 31 \begin{bmatrix} 2 & 2 & 4 \\ 2 & 5 & 8 \\ -1 & -2 & -3 \end{bmatrix} - \begin{bmatrix} 30 & 0 & 0 \\ 0 & 30 & 0 \\ 0 & 0 & 30 \end{bmatrix} \\
&= \begin{bmatrix} 32 & 62 & 124 \\ 62 & 125 & 248 \\ -31 & -62 & -123 \end{bmatrix},
\end{aligned}$$

$$\begin{aligned}
M^{-1} &= (2^{-1} - 1)M + (2 - 2^{-1})I = -\frac{1}{2} \begin{bmatrix} 2 & 2 & 4 \\ 2 & 5 & 8 \\ -1 & -2 & -3 \end{bmatrix} + \begin{bmatrix} \frac{3}{2} & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & \frac{3}{2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2} & -1 & -2 \\ -1 & -1 & -4 \\ \frac{1}{2} & 1 & 3 \end{bmatrix},
\end{aligned}$$

and

$$\begin{aligned}
M^{-5} &= (2^{-5} - 1)M + (2 - 2^{-5})I = -\frac{31}{32} \begin{bmatrix} 2 & 2 & 4 \\ 2 & 5 & 8 \\ -1 & -2 & -3 \end{bmatrix} + \begin{bmatrix} \frac{63}{32} & 0 & 0 \\ 0 & \frac{63}{32} & 0 \\ 0 & 0 & \frac{63}{32} \end{bmatrix} \\
&= \frac{1}{32} \begin{bmatrix} 1 & -62 & -124 \\ -62 & -92 & -248 \\ 31 & 62 & 156 \end{bmatrix}.
\end{aligned}$$

7. (a) We have

$$A^2 = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} = \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix},$$

after simplifying, which corresponds to a rotation of the plane 2θ radians.

(b) We have

$$B^2 = \begin{bmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{bmatrix} \begin{bmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I,$$

after simplifying, which corresponds to a rotation of the plane 0 radians (the identity mapping).

(c) From general facts about rotation matrices (see Q4(a)(b) of Week 2 Exercises),

$$A^{-3} = (A^{-1})^3 = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}^3 = \begin{bmatrix} \cos 3\theta & \sin 3\theta \\ -\sin 3\theta & \cos 3\theta \end{bmatrix},$$

which corresponds to a rotation of the plane -3θ radians.

(d) We have

$$AB = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{bmatrix} = \begin{bmatrix} \cos(\theta + 2\phi) & \sin(\theta + 2\phi) \\ \sin(\theta + 2\phi) & -\cos(\theta + 2\phi) \end{bmatrix},$$

after simplifying, which corresponds to a reflection of the plane through the line through the origin making an angle of $\frac{\theta+2\phi}{2}$ with the positive x -axis.

(e) As a special case of the previous calculation, taking $\phi = \theta$, we have that

$$AC = \begin{bmatrix} \cos 3\theta & \sin 3\theta \\ \sin 3\theta & -\cos 3\theta \end{bmatrix},$$

which corresponds to a reflection of the plane through the line through the origin making an angle of $\frac{3\theta}{2}$ with the positive x -axis.

(f) We have

$$BA = \begin{bmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} = \begin{bmatrix} \cos(2\phi - \theta) & \sin(2\phi - \theta) \\ \sin(2\phi - \theta) & -\cos(2\phi - \theta) \end{bmatrix},$$

after simplifying, which corresponds to a reflection of the plane through the line through the origin making an angle of $\frac{2\phi-\theta}{2}$ with the positive x -axis.

(g) We have

$$BC = \begin{bmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{bmatrix} \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} = \begin{bmatrix} \cos 2(\phi - \theta) & -\sin 2(\phi - \theta) \\ \sin 2(\phi - \theta) & \cos 2(\phi - \theta) \end{bmatrix},$$

after simplifying, which corresponds to a rotation of the plane $2(\phi - \theta)$ radians.

(h) We have, by (d) and (f),

$$\begin{aligned}
ABA &= \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \\
&= \begin{bmatrix} \cos(\theta + 2\phi) & \sin(\theta + 2\phi) \\ \sin(\theta + 2\phi) & -\cos(\theta + 2\phi) \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \\
&= \begin{bmatrix} \cos(\theta + 2\phi - \theta) & \sin(\theta + 2\phi - \theta) \\ \sin(\theta + 2\phi - \theta) & -\cos(\theta + 2\phi - \theta) \end{bmatrix} = \begin{bmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{bmatrix} = B.
\end{aligned}$$

Alternatively, by Q4(a)(d) of Week 2 Exercises,

$$ABA = ABAI = ABABB = A(BAB)B = AA^{-1}B = IB = B.$$

(i) By Q4(a)(d) of Week 2 Exercises,

$$BA^2B = (A^2)^{-1} = A^{-2},$$

which corresponds to a rotation of the plane -2θ radians.

(j) We have, by (f) and (g),

$$\begin{aligned}
BAC &= \begin{bmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \\
&= \begin{bmatrix} \cos(2\phi - \theta) & \sin(2\phi - \theta) \\ \sin(2\phi - \theta) & -\cos(2\phi - \theta) \end{bmatrix} \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \\
&= \begin{bmatrix} \cos(2\phi - 3\theta) & -\sin(2\phi - 3\theta) \\ \sin(2\phi - 3\theta) & \cos(2\phi - 3\theta) \end{bmatrix},
\end{aligned}$$

which corresponds to a rotation of the plane $2\phi - 3\theta$ radians.

8. We have

$$\chi(\lambda) = \det(\lambda I - M) = \begin{vmatrix} \lambda - a & -b \\ -c & \lambda - d \end{vmatrix} = (\lambda - a)(\lambda - d) - bc = \lambda^2 - (a + d)\lambda + ad - bc,$$

and

$$\begin{aligned}
\chi(M) &= M^2 - (a + d)M + (ad - bc)I \\
&= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} - (a + d) \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix} \\
&= \begin{bmatrix} a^2 + bc & ab + bd \\ ca + dc & cb + d^2 \end{bmatrix} - \begin{bmatrix} a^2 + ad & ab + db \\ ac + dc & ad + d^2 \end{bmatrix} + \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix} \\
&= \begin{bmatrix} a^2 + bc - a^2 - ad + ad - bc & ab + bd - ab - db \\ ca + dc - ac - dc & cb + d^2 - ad - d^2 + ad - bc \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},
\end{aligned}$$

the zero matrix, verifying of the Cayley-Hamilton Theorem for 2×2 matrices.

9. (a) We have that $M\mathbf{v} = \lambda\mathbf{v}$ for some nonzero column vector \mathbf{v} . We claim that $M^k\mathbf{v} = \lambda^k\mathbf{v}$ for all positive integers k , and verify this by induction. This is

clearly true if $k = 1$. Suppose that $k \geq 1$ and assume as inductive hypothesis that $M^k \mathbf{v} = \lambda^k \mathbf{v}$. Then

$$M^{k+1} \mathbf{v} = (MM^k) \mathbf{v} = M(M^k \mathbf{v}) = M(\lambda^k \mathbf{v}) = \lambda^k (M \mathbf{v}) = \lambda^k (\lambda \mathbf{v}) = \lambda^{k+1} \mathbf{v} ,$$

which establishes the inductive step, and completes the proof of our claim. Thus, for all positive integers k , we have that λ^k is an eigenvalue of M^k (and further that \mathbf{v} is always a corresponding eigenvector).

- (b) Again, we have that $M \mathbf{v} = \lambda \mathbf{v}$ for some nonzero column vector \mathbf{v} . If $\lambda = 0$ then $M \mathbf{v} = 0 \mathbf{v} = \mathbf{0}$, so that

$$\mathbf{v} = I \mathbf{v} = (M^{-1} M) \mathbf{v} = M^{-1} (M \mathbf{v}) = M^{-1} \mathbf{0} = \mathbf{0} ,$$

which contradicts that \mathbf{v} is a nonzero column vector. Hence λ is nonzero. Further,

$$\mathbf{v} = \frac{1}{\lambda} (\lambda \mathbf{v}) = \lambda^{-1} M \mathbf{v} ,$$

so that

$$M^{-1} \mathbf{v} = M^{-1} (\lambda^{-1} M \mathbf{v}) = \lambda^{-1} (M^{-1} M) \mathbf{v} = \lambda^{-1} I \mathbf{v} = \lambda^{-1} \mathbf{v} .$$

This proves that λ^{-1} is an eigenvalue of M^{-1} (and further that \mathbf{v} is a corresponding eigenvector).

10. Let $M = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$ where $\theta \in \mathbb{R}$ be a rotation matrix. Working over \mathbb{C} , we have

$$\begin{aligned} \chi(\lambda) &= \det(\lambda I - M) = \begin{vmatrix} \lambda - \cos \theta & \sin \theta \\ -\sin \theta & \lambda - \cos \theta \end{vmatrix} = (\lambda - \cos \theta)^2 + \sin^2 \theta \\ &= \lambda^2 - 2\lambda \cos \theta + \cos^2 \theta + \sin^2 \theta = \lambda^2 - 2\lambda \cos \theta + 1 , \end{aligned}$$

with roots

$$\lambda = \frac{2 \cos \theta \pm \sqrt{4 \cos^2 \theta - 4}}{2} = \cos \theta \pm i \sin \theta = e^{\pm i \theta}$$

where $i = \sqrt{-1}$. If θ is an even multiple of π then $M = I$ and $\lambda = 1$ is the unique eigenvalue. If θ is an odd multiple of π then $M = -I$ and $\lambda = -1$ is the unique eigenvalue. In both these cases, the eigenspace comprises all column vectors with entries from \mathbb{C} . Suppose then that θ is not a multiple of π , so that $\sin \theta \neq 0$. Finding the eigenspace corresponding to $\lambda = e^{i \theta} = \cos \theta + i \sin \theta$:

$$\begin{bmatrix} i \sin \theta & \sin \theta \\ -\sin \theta & i \sin \theta \end{bmatrix} \sim \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix} \sim \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix} ,$$

yielding $\left\{ \begin{bmatrix} iz \\ z \end{bmatrix} \mid z \in \mathbb{C} \right\}$. Finding the eigenspace corresponding to $\lambda = e^{-i \theta} = \cos \theta - i \sin \theta$:

$$\begin{bmatrix} -i \sin \theta & \sin \theta \\ -\sin \theta & -i \sin \theta \end{bmatrix} \sim \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \sim \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix} ,$$

yielding $\left\{ \begin{bmatrix} -iz \\ z \end{bmatrix} \mid z \in \mathbb{C} \right\}$.

11. The (i, ℓ) -entry of $A(\text{adj} A)$ is

$$\sum_{j=1}^n (-1)^{\ell+j} a_{ij} \det A_{\ell j} .$$

If $i = \ell$ then this is the expansion along the ℓ th row of A , yielding $\det A$. If $i \neq \ell$ then this is the expansion along the ℓ th row of the matrix obtained by replacing the ℓ th row of A by the i th row, yielding the determinant of a matrix with two identical rows, which is zero. This proves $A(\text{adj} A) = (\det A)I$, and then the rest follows quickly.

12. Given $M\mathbf{x} = \mathbf{c}$, we have

$$\mathbf{x} = M^{-1}\mathbf{c} = \frac{1}{\det M}(\text{adj} M)\mathbf{c} ,$$

so that, for each i ,

$$x_i = \frac{1}{\det M} \sum_{j=1}^n (-1)^{j+i} \det(M_{ji}) c_j = \frac{\det M_i}{\det M} ,$$

since $\det M_i = \sum_{j=1}^n (-1)^{j+i} c_j \det M_{ji}$, expanding down the i th column. In the given system,

$$M = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 5 & 6 \\ 3 & 1 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{c} = \begin{bmatrix} -4 \\ -3 \\ -1 \end{bmatrix} ,$$

so that

$$\det M = \begin{vmatrix} 2 & 3 & 4 \\ 5 & 5 & 6 \\ 3 & 1 & 2 \end{vmatrix} = 2 \begin{vmatrix} 5 & 6 \\ 1 & 2 \end{vmatrix} - 3 \begin{vmatrix} 5 & 6 \\ 3 & 2 \end{vmatrix} + 4 \begin{vmatrix} 5 & 5 \\ 3 & 1 \end{vmatrix} = -8 ,$$

$$\det M_1 = \begin{vmatrix} -4 & 3 & 4 \\ -3 & 5 & 6 \\ -1 & 1 & 2 \end{vmatrix} = -4 \begin{vmatrix} 5 & 6 \\ 1 & 2 \end{vmatrix} - 3 \begin{vmatrix} -3 & 6 \\ -1 & 2 \end{vmatrix} + 4 \begin{vmatrix} -3 & 5 \\ -1 & 1 \end{vmatrix} = -8 ,$$

$$\det M_2 = \begin{vmatrix} 2 & -4 & 4 \\ 5 & -3 & 6 \\ 3 & -1 & 2 \end{vmatrix} = 2 \begin{vmatrix} -3 & 6 \\ -1 & 2 \end{vmatrix} + 4 \begin{vmatrix} 5 & 6 \\ 3 & 2 \end{vmatrix} + 4 \begin{vmatrix} 5 & -3 \\ 3 & -1 \end{vmatrix} = -16 ,$$

$$\det M_3 = \begin{vmatrix} 2 & 3 & -4 \\ 5 & 5 & -3 \\ 3 & 1 & -1 \end{vmatrix} = 2 \begin{vmatrix} 5 & -3 \\ 1 & -1 \end{vmatrix} - 3 \begin{vmatrix} 5 & -3 \\ 3 & -1 \end{vmatrix} - 4 \begin{vmatrix} 5 & 5 \\ 3 & 1 \end{vmatrix} = 24 ,$$

yielding

$$x = \frac{-8}{-8} = 1 , \quad y = \frac{-16}{-8} = 2 , \quad z = \frac{24}{-8} = -3 .$$