The University of Sydney

MATH2022 Linear and Abstract Algebra

Semester 1 **Week 8 Longer Solutions**

1. Certainly $\mathbb{C}$ is an abelian group under addition. To become a real vector space we just take scalar multiplication to be multiplication of a complex number by a real number. Since $\mathbb{C}$ is a field, and $\mathbb{R}$ is a subset of $\mathbb{C}$, we immediately have

$$(\forall \lambda, \mu \in \mathbb{R})(\forall \mathbf{v}, \mathbf{w} \in \mathbb{C}) \quad (\lambda + \mu)\mathbf{v} \ = \ \lambda\mathbf{v} + \mu\mathbf{v} \quad \text{and} \quad \lambda(\mathbf{v} + \mathbf{w}) \ = \ \lambda\mathbf{v} + \lambda\mathbf{w}$$

and

$$(\forall \mathbf{v} \in \mathbb{C}) \quad 1\mathbf{v} \ = \ \mathbf{v} \ .$$

This verifies that $\mathbb{C}$ is a vector space over $\mathbb{R}$. We may identify a complex number $z = a + bi$, where $a, b \in \mathbb{R}$ with the geometric vector $\mathbf{v}$ that is the position vector of the point (a, b) in the xy-plane. Mapping z to $\mathbf{v}$ clearly yields a bijection that respects vector addition and scalar multiplication, so yields a vector space isomorphism between $\mathbb{C}$ and the vector space of geometric vectors in the xy-plane. Further

$$\mathbb{C} \ = \ \{a + bi \mid a, b \in \mathbb{R}\} \ = \ \{a1 + bi \mid a, b \in \mathbb{R}\} \ = \ \langle 1, i \rangle \ ,$$

so that $\{1, i\}$ spans $\mathbb{C}$. (This corresponds to the fact that the usual unit vectors $\mathbf{i}$ and $\mathbf{j}$ span the vector space of geometric vectors in the plane.)

2. Geometrically, S_1 describes a line in the plane of slope -1 passing through the origin. We claim it is a subspace of $\mathbb{R}^2$. Certainly S_1 is nonempty, since $(0, 0) \in S_1$. Suppose that $\mathbf{v}, \mathbf{w} \in S_1$ and $\lambda, \mu \in \mathbb{R}$. Then $\mathbf{v} = (x_1, y_1)$ and $\mathbf{w} = (x_2, y_2)$ for some $x_1, x_2, y_1, y_2 \in \mathbb{R}$, and, by definition of membership of S_1,

$$x_1 + y_1 \ = \ 0 \ = \ x_2 + y_2 \ .$$

But, we have

$$\lambda\mathbf{v} + \mu\mathbf{w} \ = \ \lambda(x_1, y_1) + \mu(x_2, y_2) \ = \ (\lambda x_1 + \mu x_2, \lambda y_1 + \mu y_2) \ = \ (x', y') \ ,$$

where $x' = \lambda x_1 + \mu x_2$, $y' = \lambda y_1 + \mu y_2$, and, further,

$$\begin{aligned} x' + y' \ &= \ \lambda x_1 + \mu x_2 + \lambda y_1 + \mu y_2 \\ &= \ \lambda(x_1 + y_1) + \mu(x_2 + y_2) \ = \ \lambda(0) + \mu(0) \ = \ 0 \ . \end{aligned}$$

This shows that $\lambda\mathbf{v} + \mu\mathbf{w} \in S_1$, so that S_1 is closed under taking linear combinations, completing the verification that it is a subspace of $\mathbb{R}^2$.

Geometrically, S_2 is a line of slope -1 with y-intercept 1. It does not pass through the origin, so does not contain the zero vector $(0, 0)$, so cannot be a subspace of $\mathbb{R}^2$.

Geometrically, S_3 is a "half-plane" including all points on the line S_1 and all points in the plane situated vertically above points on S_1, where we think of "vertical" as meaning in the positive y-direction. This is not a subspace of $\mathbb{R}^2$, however, as it is not closed under scalar multiplication. To see this, for example, observe that $(1, 0) \in S_3$, since $1 + 0 = 1 \geq 0$, but $-(1, 0) = (-1, 0) \notin S_3$, since $-1 + 0 = -1 < 0$.

Geometrically, S_4 is the perimeter of the circle of radius 1 centred at the origin. It is not a subspace of $\mathbb{R}^2$ since it does not contain the origin.

3. Geometrically, S_1 describes a plane in space that passes through the origin, and we claim it is a subspace of $\mathbb{R}^3$. Certainly S_1 is nonempty, since $(0,0,0) \in S_1$. Suppose that $\mathbf{v}, \mathbf{w} \in S_1$ and $\lambda, \mu \in \mathbb{R}$. Then $\mathbf{v} = (x_1, y_1, z_1)$ and $\mathbf{w} = (x_2, y_2, z_2)$ for some $x_1, x_2, y_1, y_2, z_1, z_2 \in \mathbb{R}$, and, by definition of membership of S_1,

$$x_1 + y_1 + z_1 \;=\; 0 \;=\; x_2 + y_2 + z_2 \;.$$

But, we have

$$\lambda\mathbf{v}+\mu\mathbf{w} \;=\; \lambda(x_1,y_1,z_1)+\mu(x_2,y_2,z_2) \;=\; (\lambda x_1+\mu x_2, \lambda y_1+\mu y_2, \lambda z_1+\mu z_2) \;=\; (x', y', z') \;,$$

where $x' = \lambda x_1 + \mu x_2$, $y' = \lambda y_1 + \mu y_2$, $z' = \lambda z_1 + \mu z_2$, and, further,

$$\begin{aligned} x' + y' + z' &= \lambda x_1 + \mu x_2 + \lambda y_1 + \mu y_2 + \lambda z_1 + \mu z_2 \\ &= \lambda(x_1 + y_1 + z_1) + \mu(x_2 + y_2 + z_2) \;=\; \lambda(0) + \mu(0) \;=\; 0 \;. \end{aligned}$$

This shows that $\lambda\mathbf{v} + \mu\mathbf{w} \in S_1$, so that S_1 is closed under taking linear combinations, completing the verification that it is a subspace of $\mathbb{R}^3$.

Geometrically, S_2 is a plane is space that does not pass through the origin. Indeed $(0,0,0)$ does not satisfy the criterion for membership of S_2, since $2(0) + 3(0) + 4(0) = 0 \neq 1$. Since S_2 does not contain the zero vector, it cannot be a subspace of $\mathbb{R}^3$.

Geometrically, S_3 is a "half-space" including all points on the plane S_1. We obtain all points of S_3 by moving, starting from points on S_1, away from S_1 parallel to the z-axis, but in the direction of the negative z-axis. This is not a subspace of $\mathbb{R}^3$, however, as it is not closed under scalar multiplication. To see this, for example, observe that $(0,0,-1) \in S_3$, since $2(0) + 3(0) + 4(-1) = -4 \leq 0$, but $-(0,0,-1) = (0,0,1) \notin S_3$, since $2(0) + 3(0) + 4(1) = 4 > 0$.

Geometrically, S_4 is the surface and interior of the sphere of radius 1 centred at the origin. It is not a subspace of $\mathbb{R}^3$ since it is not closed under addition. For example $(1,0,0), (0,1,0) \in S_4$ but $(1,0,0) + (0,1,0) = (1,1,0) \notin S_4$ since $1^2 + 1^2 + 0^2 = 2 > 1$.

4. We have

$$A \;=\; \begin{bmatrix} 1 & 1 & 5 \\ 2 & 3 & 13 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 5 \\ 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix} ,$$

$$B \;=\; \begin{bmatrix} 1 & -1 & -2 \\ 3 & -2 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \end{bmatrix} ,$$

which has a different reduced row echelon form to A, so that A and B are not row equivalent. Hence the row spaces of A and B are different. However

$$C \;=\; \begin{bmatrix} 1 & -1 & -1 \\ 4 & -3 & -1 \\ 3 & -1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 3 \\ 0 & 2 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} ,$$

which has the same nonzero rows as the reduced row echelon form of A. Hence A and C have identical row spaces.

5. Writing $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ as rows of a matrix A and row reducing we get

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & 1 & -2 \\ 3 & 6 & 3 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 3 & -8 \\ 0 & 0 & 6 & -16 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 1/3 \\ 0 & 0 & 1 & -8/3 \\ 0 & 0 & 0 & 0 \end{bmatrix} .$$

Writing $\mathbf{w}_1$ and $\mathbf{w}_2$ as rows of a matrix B and row reducing we get

$$B = \begin{bmatrix} 1 & 2 & -4 & 11 \\ 2 & 4 & -5 & 14 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -4 & 11 \\ 0 & 0 & 3 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 1/3 \\ 0 & 0 & 1 & -8/3 \end{bmatrix} ,$$

which is in row echelon form with identical nonzero rows as the row echelon form above for A. Hence the rows of A and the rows of B span identical row spaces. This verifies that $S_1 = S_2$.

6. (a) The zero vector is unique, for if $\mathbf{0}$ and $\mathbf{0}'$ are both zeros in a vector space then they both act as additive identity elements, so that

$$\mathbf{0} = \mathbf{0} + \mathbf{0}' = \mathbf{0}' .$$

(b) Suppose that $\mathbf{u}$ is a vector and both $\mathbf{v}$ and $\mathbf{w}$ act as negatives of $\mathbf{u}$, that is,

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u} = \mathbf{0} = \mathbf{u} + \mathbf{w} = \mathbf{w} + \mathbf{u} .$$

Then, in particular,

$$\mathbf{v} = \mathbf{v} + \mathbf{0} = \mathbf{v} + (\mathbf{u} + \mathbf{w}) = (\mathbf{v} + \mathbf{u}) + \mathbf{w} = \mathbf{0} + \mathbf{w} = \mathbf{w} ,$$

which verifies that the negative of $\mathbf{u}$ is unique.

(c) Let $\mathbf{v} \in V$ and put $\mathbf{w} = 0\mathbf{v}$. We show that $\mathbf{w} = \mathbf{0}$. Observe first that

$$\mathbf{w} + \mathbf{w} = 0\mathbf{v} + 0\mathbf{v} = (0 + 0)\mathbf{v} = 0\mathbf{v} = \mathbf{w} .$$

Hence

$$\mathbf{0} = -\mathbf{w} + \mathbf{w} = -\mathbf{w} + (\mathbf{w} + \mathbf{w}) = (-\mathbf{w} + \mathbf{w}) + \mathbf{w} = \mathbf{0} + \mathbf{w} = \mathbf{w} ,$$

that is, $\mathbf{w} = \mathbf{0}$, as required.

(d) Let $\lambda \in F$ and put $\mathbf{w} = \lambda\mathbf{0}$. We show that $\mathbf{w} = \mathbf{0}$. Observe first that

$$\mathbf{w} + \mathbf{w} = \lambda\mathbf{0} + \lambda\mathbf{0} = \lambda(\mathbf{0} + \mathbf{0}) = \lambda\mathbf{0} = \mathbf{w} .$$

Hence, as in part (c),

$$\mathbf{0} = -\mathbf{w} + \mathbf{w} = -\mathbf{w} + (\mathbf{w} + \mathbf{w}) = (-\mathbf{w} + \mathbf{w}) + \mathbf{w} = \mathbf{0} + \mathbf{w} = \mathbf{w} ,$$

that is, $\mathbf{w} = \mathbf{0}$, as required.

(e) Observe, for $\mathbf{v} \in V$, that

$$\mathbf{v} + (-1)\mathbf{v} = 1\mathbf{v} + (-1)\mathbf{v} = (1 + (-1))\mathbf{v} = (1 - 1)\mathbf{v} = 0\mathbf{v} = \mathbf{0} ,$$

by part (c), so that $(-1)\mathbf{v}$ acts as a negative of $\mathbf{v}$, so must be $-\mathbf{v}$, since the negative of $\mathbf{v}$ is unique, by part (b).

(f) Let $\mathbf{v} \in V$ and $\lambda \in F$, and suppose that $\lambda\mathbf{v} = \mathbf{0}$. Suppose that $\lambda \neq 0$. Then λ^{-1} exists in F so that, by part (d),

$$\mathbf{0} \;=\; \lambda^{-1}\mathbf{0} \;=\; \lambda^{-1}(\lambda\mathbf{v}) \;=\; (\lambda^{-1}\lambda)\mathbf{v} \;=\; 1\mathbf{v} \;=\; \mathbf{v}\,,$$

that is, $\mathbf{v} = \mathbf{0}$. This proves that either $\lambda = 0$ or $\mathbf{v} = \mathbf{0}$, as required.

7. Suppose first that S is closed under addition and scalar multiplication, and let $\mathbf{v}, \mathbf{w} \in S$ and $\lambda, \mu \in F$. Then $\lambda\mathbf{v} \in S$ and $\mu\mathbf{w} \in S$, since S is closed under scalar multiplication, so that

$$\lambda\mathbf{v} + \mu\mathbf{w} \in S\,,$$

now because S is closed under addition. This completes the verification that S is closed under taking linear combinations. Conversely, suppose S is closed under taking linear combinations, and let $\mathbf{v}, \mathbf{w} \in S$ and $\lambda \in F$. Then

$$\mathbf{v} + \mathbf{w} \;=\; 1\mathbf{v} + 1\mathbf{w} \;\in\; S \qquad \text{and} \qquad \lambda\mathbf{v} \;=\; \lambda\mathbf{v} + 0\mathbf{w} \;\in\; S\,,$$

in each case because S is closed under taking linear combinations. But this verifies that S is closed under addition and scalar multiplication.

8. Since S and T are subspaces of V then they are both nonempty and closed under addition and scalar multiplication. In particular, there is some $\mathbf{v} \in S$, so that

$$\mathbf{0} \;=\; 0\mathbf{v} \in S\,,$$

since S is closed under scalar multiplication. Similarly $\mathbf{0} \in T$, so that $\mathbf{0} \in S \cap T$. Thus $S \cap T$ is nonempty. If $\mathbf{v}, \mathbf{w} \in S \cap T$ and $\lambda \in F$ then certainly $\mathbf{v}, \mathbf{w} \in S$, so that $\mathbf{v} + \mathbf{w}$ and $\lambda\mathbf{v} \in S$, by closure properties of S, and similarly for T, so that

$$\mathbf{v} + \mathbf{w} \;\in\; S \cap T \quad \text{and} \quad \lambda\mathbf{v} \;\in S \cap T\,.$$

Thus $S \cap T$ is closed under addition and scalar multiplication, so is a subspace of V.

9. Let S be a subspace of a vector space V. Certainly S is nonempty and, because of its closure properties, the addition and scalar multiplication of V restricts to addition and scalar multiplication in S. As in the solution to the previous exercise, S contains the zero vector. Also, if $\mathbf{v} \in S$ then $-\mathbf{v} = (-1)\mathbf{v} \in S$, since S is closed under scalar multiplication. Thus addition is associative and commutative (inherited from V) and S contains an additive identity element (the zero from V) and has additive inverses (because of closure under taking negatives of vectors). Thus S is an abelian group with respect to addition. Compatibility with scalar multiplication is inherited from V, so S is a vector space over the same underlying field.

10. The zero vector in F^X is the so-called *zero function* $\mathbf{0} : X \to F$ that maps each $x \in X$ to $0 \in F$. This is because, by definition of addition of functions, if $f \in F^X$ then

$$(\mathbf{0} + f)(x) \;=\; \mathbf{0}(x) + f(x) \;=\; 0 + f(x) \;=\; f(x)\,,$$

so that $\mathbf{0} + f = f$. If $f \in F^X$ then the negative of f denoted by $-f$ is given by the rule

$$(-f)(x) \;=\; -f(x)$$

for all $x \in X$. This is because

$$(f + (-f))(x) \;=\; f(x) + (-f)(x) \;=\; f(x) + (-f(x)) \;=\; f(x) - f(x) \;=\; 0\,,$$

for all $x \in X$, so that $f + (-f) = \mathbf{0}$, the zero function.

11. Certainly, we have

$$M\mathbf{0}^\top = \mathbf{0}$$

(interpreting the first $\mathbf{0}$ as an element of F^n and the second $\mathbf{0}$ as a column vector with m 0's), so that $\mathbf{0} \in S$. Hence S is nonempty. Suppose that $\mathbf{v}, \mathbf{w} \in S$ and $\lambda, \mu \in F$. Then $M\mathbf{v}^\top = \mathbf{0}$ and $M\mathbf{w}^\top = \mathbf{0}$, so that

$$M(\lambda\mathbf{v} + \mu\mathbf{w})^\top = \lambda M\mathbf{v}^\top + \mu M\mathbf{w}^\top = \lambda\mathbf{0} + \mu\mathbf{0} = \mathbf{0} .$$

Thus $\lambda\mathbf{v} + \mu\mathbf{w} \in S$, so that S is closed under taking linear combinations, so that S is a subspace of F^n.

12. Every subspace of V contains the zero vector, using the argument given above in a previous solution, so the zero vector lies in the intersection of all subspaces of V. Certainly $\{\mathbf{0}\}$ is a subspace of V since $\mathbf{0} + \mathbf{0} = \mathbf{0} = \lambda\mathbf{0}$ for all scalars λ. Further it contains $\emptyset$ (as do all sets). Thus

$$\{\mathbf{0}\} \subseteq \bigcap\{S \mid S \text{ is a subspace of } V \text{ containing } \emptyset\} \subseteq \{\mathbf{0}\} ,$$

so these sets are in fact equal. This verifies that $\{\mathbf{0}\}$ is the smallest subspace of V containing $\emptyset$, as required.

13. The zero square matrix is symmetric so is in S, so S is nonempty. Suppose that $A, B \in S$ and $\lambda, \mu \in F$. Then

$$(\lambda A + \mu B)^\top = \lambda A^\top + \mu B^\top = \lambda A + \mu B ,$$

so that S is closed under taking linear combinations. This verifies that S is a subspace of $\mathrm{Mat}_{n,n}$. If $n = 2$ then

$$S = \left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} \,\middle|\, a, b, c \in F \right\} = \left\{ a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \,\middle|\, a, b, c \in F \right\} .$$

Hence a spanning set for S in this case is

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} .$$

14. (a) The zero polynomial is in $\mathbb{P}_n$ for each n, so certainly also in $\mathbb{P}$, so both are nonempty. A linear combination of two polynomials is clearly a polynomial, and if both have degree at most n, then so does the linear combination. This shows that both $\mathbb{P}_n$ and $\mathbb{P}$ are closed under taking linear combinations. Hence $\mathbb{P}$ is a subspace of $\mathbb{R}^\mathbb{R}$ and $\mathbb{P}_n$ is a subspace of $\mathbb{P}$.

(b) By definition of a polynomial of degree at most n, it is a linear combination of powers of x from $1 = x^0$ up to and including x^n. Thus $\{1, x, \ldots, x^n\}$ spans $\mathbb{P}_n$.

(c) The mapping

$$a_0 + a_1x + \ldots + a_nx^n \mapsto (a_0, a_1, a_2, \ldots, a_n) ,$$

for $a_0, a_1, \ldots, a_n \in \mathbb{R}$, is a bijection from $\mathbb{P}_n$ to $\mathbb{R}^{n+1}$, which clearly preserves addition and scalar multiplication, so is a vector space isomorphism.

(d) Suppose that S is a finite subset of $\mathbb{P}$. Then there is a positive integer n such that all polynomials in S have degree less than n. Hence $\langle S\rangle \subseteq \mathbb{P}_n$, since $\mathbb{P}_n$ is closed under taking linear combinations. But $x^{n+1} \in \mathbb{P}\backslash\mathbb{P}_n$, so certainly $x^{n+1} \notin \langle S\rangle$, so that S cannot span $\mathbb{P}$.

15. Let S be the set of bounded functions from $\mathbb{R}^{\mathbb{R}}$. Certainly the zero function is bounded so is in S. Hence S is nonempty. Suppose that $f, g \in S$ and $\lambda, \mu \in \mathbb{R}$. There there exist $K, L \geq 0$ such that

$$|f(x)| \ < \ K \quad \text{and} \quad |g(x)| < L \ ,$$

for all $x \in \mathbb{R}$. But then, by definition of addition and scalar multiplication of functions, and the triangle inequality,

$$\begin{aligned} |(\lambda f + \mu g)(x)| \ &= \ |\lambda f(x) + \mu g(x)| \ \leq \ |\lambda f(x)| + |\mu g(x)| \ = \ |\lambda||f(x)| + |\mu||g(x)| \\ &\leq \ |\lambda|K + |\mu|L \ , \end{aligned}$$

for all $x \in \mathbb{R}$, which shows that $\lambda f + \mu g$ is bounded. Thus S is closed under taking linear combinations so is a subspace of $\mathbb{R}^{\mathbb{R}}$.