Example 3: Confirm the Cayley-Hamilton theorem for the rotation matrix

$$M = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}.$$

$$\chi_M(\lambda) = \left| \begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix} - \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix} \right|$$

$$= \begin{vmatrix}\lambda - \cos\theta & \sin\theta\\ -\sin\theta & \lambda - \cos\theta\end{vmatrix}$$

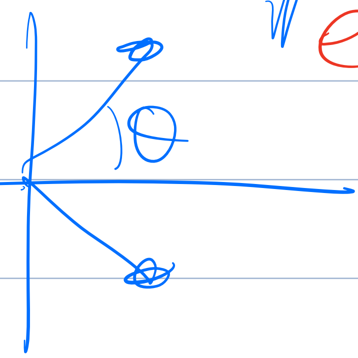
$$= (\lambda - \cos\theta)^2 + \sin^2\theta$$

$$= \lambda^2 - 2\cos\theta\,\lambda + \cos\theta^2 + \sin^2\theta$$

$$= \lambda^2 - 2\cos\theta\,\lambda + 1$$

$$= (\lambda - \underbrace{(\cos\theta + i\sin\theta)}_{e^{i\theta}})(\lambda - \underbrace{(\cos\theta - i\sin\theta)}_{e^{-i\theta}})$$

$$= (\lambda - e^{i\theta})(\lambda - e^{-i\theta})$$



2 eigenvalues.

Now check $\chi(M)$.

$$\chi(M) = M^2 - 2(\cos\theta)M + I$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} - 2\cos\theta\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta - \sin^2\theta & -2\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix} - 2\cos\theta\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix} - \begin{bmatrix} 2\cos^2\theta & -\sin 2\theta \\ \sin 2\theta & 2\cos^2\theta \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos 2\theta - 2\cos^2\theta + 1 & 0 \\ 0 & \cos 2\theta - 2\cos^2\theta + 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\lambda I - M = \begin{bmatrix} \lambda+1 & 0 \\ 0 & \lambda+1 \end{bmatrix}$$

Consider the case where $\theta = \pi$.

Then $M = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$ and there is only a single eigenvalue of -1, since

$$\chi(\lambda) = (\lambda + 1)^2.$$

A similar thing happens for $\theta = 0$.

Exercise: Find the inverse matrices of the reflection and rotation matrices.

Example 4: Find the eigenvalues of the following matrix over $\mathbb{Z}_5$, and find it's inverse.

$$M = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\chi(\lambda) = \det(\lambda I - M) = \begin{vmatrix} \lambda-1 & -1 & 0 \\ -1 & \lambda & -1 \\ 0 & -1 & \lambda-1 \end{vmatrix}$$

$C_3 \to C_3 + (\lambda-1)C_2$

$$= \begin{vmatrix} \lambda-1^{+} & -1^{-} & 1-\lambda^{+} \\ -1^{-} & \lambda & \lambda^2-\lambda-1 \\ 0^{+} & -1^{-} & 0^{+} \end{vmatrix}$$

$0 + (\lambda-1)(-1) \leftarrow$

$-1 + (\lambda-1)\lambda$

$(\lambda-1) + (\lambda-1)(-1)$

expand along bottom row

$$= \begin{vmatrix} \lambda-1 & 1-\lambda \\ -1 & \lambda^2-\lambda-1 \end{vmatrix}$$

$$= (\lambda-1)(\lambda^2-\lambda-1) + (1-\lambda)$$

$$= (\lambda-1)(\lambda^2-\lambda-1) - (\lambda-1)$$

$$= (\lambda-1)\left[\lambda^2-\lambda-1 - 1\right]$$

$$= (\lambda-1)(\lambda^2-\lambda-2)$$

$$= (\lambda-1)(\lambda-2)(\lambda+1)$$

So the eigenvalues are $1, -1, 2$.

or $1, 4, 2$

Now to find M^{-1}:

from*

$$\chi(\lambda) = \lambda^3 - 2\lambda^2 - \lambda + 2$$

By the theorem

$$M^3 - 2M^2 - M + 2I = 0$$

$$M^3 - 2M^2 - M = -2I$$

$$M(M^2 - 2M - I) = -2I$$

$$M\left[-\frac{1}{2}(M^2 - 2M - I)\right] = I$$

$$\text{So } M^{-1} = -\frac{1}{2}(M^2 - 2M - I)$$

Check:

$$M^{-1} = \begin{bmatrix} 3 & 3 & 2 \\ 3 & 2 & 3 \\ 2 & 3 & 3 \end{bmatrix}$$

Section 7.3 Diagonalisation over a Field

The Linear Algebra Principle: Move in straight lines whenever you can.

The Conjugation Principle: To do something difficult, change your position/viewpoint so that you can do something easily, then return.

Diagonalisation combines these two concepts. We use eigenvalues and eigenvectors to “straighten-out” a matrix. Then we can use the conjugation principle. $M = PDP^{-1}$

Let M be an nxn matrix with eigenvalues $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$.

These are the solutions to the chracteristic polynomial $\chi(\lambda) = \det(\lambda I - M)$

with corresponding eigenvectors $\underline{v}_1, \underline{v}_2, \underline{v}_3, ..., \underline{v}_n$ (respectively).

$$\text{So } M\underline{v}_i = \lambda_i \underline{v}_i \quad \text{for } i = 1, .., n.$$

Let P be the nxn matrix whose columns are the eigenvectors. So

$$P = [\underline{v}_1 \ \underline{v}_2 \ \ldots \ \underline{v}_n].$$

Then $MP = M[\underline{v}_1 \ \underline{v}_2 \ \ldots \ \underline{v}_n]$

$$= [M\underline{v}_1 \ M\underline{v}_2 \ \ldots \ M\underline{v}_n]$$

convince yourself!

$$= [\lambda_1\underline{v}_1 \ \lambda_2\underline{v}_2 \ \ldots \ \lambda_n\underline{v}_n]$$

$$= [\underline{v}_1 \ \underline{v}_2 \ \ldots \ \underline{v}_n] \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ & & \ddots \\ 0 & & & \lambda_n \end{bmatrix}$$

check!

$= PD$ where $D = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$

That is, $MP = PD.$

If P is invertible then we can rearrange to $M = PDP^{-1}.$

We say that M has been diagonalised. Also, M is the conjugate of D by the inverse of P.

Rearranging again, we see that D is the conjugate of M by P: $D = P^{-1}MP$

Note the most important word on this page: IF.

The matrix P is not always invertible. However, we have a useful sufficient condition to guarantee invertibility.

Theorem: If the eigenvalues of an nxn matrix are all different, then the matrix, P, whose columns are the eigenvectors, is invertible. It follows that M is diagonalisable.

I will prove the theorem in the case of only 2 distinct eigenvalues. This can be extended to the general case.

Lemma: If λ_1 and λ_2 are distinct eigenvalues associated to eigenvectors $\underline{v}_1$ and $\underline{v}_2$ then $\underline{v}_1$ and $\underline{v}_2$ are linearly independent.

Proof: Suppose λ_1, λ_2 are distinct eigenvalues associated to eigenvectors $\underline{v}_1$ and $\underline{v}_2$.

Let $\alpha \underline{v}_1 + \beta \underline{v}_2 = \underline{0}$ *, $\alpha, \beta \in F$.

① Multiply $*$ by M: $\quad M\alpha \underline{v}_1 + M\beta \underline{v}_2 = M\underline{0}$

$$\alpha M\underline{v}_1 + \beta M \underline{v}_2 = \underline{0}$$

$$\alpha \lambda_1 \underline{v}_1 + \beta \lambda_2 \underline{v}_2 = \underline{0}$$

② Multiply $*$ by λ_1: $\quad \alpha \lambda_1 \underline{v}_1 + \beta \lambda_1 \underline{v}_2 = \underline{0}$

Now subtract ② $-$ ①

$$\beta \lambda_1 \underline{v}_2 - \beta \lambda_2 \underline{v}_2 = \underline{0}$$

$$\beta \underbrace{(\lambda_1 - \lambda_2)}_{\neq 0} \underbrace{\underline{v}_2}_{\neq \underline{0}} = \underline{0}$$

So $\beta = 0$.

Sub $\beta = 0$ into $*$: $\quad \alpha \underline{v}_1 = \underline{0}$

Therefore $\alpha = 0$.

$\square$

Example: Let $x \in \mathbb{R}, |x| < 1$, and put

$$M = \begin{bmatrix} 1 & 1 \\ 0 & x \end{bmatrix}.$$

Find a closed formula for M^n and hence prove that

$$1 + x + x^2 + \ldots + x^{n-1} = \frac{x^n - 1}{x - 1}.$$

① Find the eigenvalues of M.

$$|\lambda I - M| = \left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & x \end{bmatrix} \right| = \begin{vmatrix} \lambda - 1 & -1 \\ 0 & \lambda - x \end{vmatrix}$$

$$= (\lambda - 1)(\lambda - x)$$

So $\lambda = 1, x$

② Find the eigenvectors.

$\boxed{\lambda = 1}$

$R_2 \to R_2 + (1-x) R_1$ then $R_1 \to -R_1$

$$\lambda I - M = \begin{bmatrix} 0 & -1 \\ 0 & 1-x \end{bmatrix} \sim \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Let $\underline{v_1} = \begin{bmatrix} a \\ b \end{bmatrix}$, then a is free. So let $a = t \in \mathbb{R}$.

Also, $b = 0$. So $\underline{v_1} = t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $t \neq 0$.

$\boxed{\lambda = x}$ $\quad xI - M = \begin{bmatrix} x-1 & -1 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -\frac{1}{x-1} \\ 0 & 0 \end{bmatrix}$

Let $\underline{v}_x = \begin{bmatrix} a \\ b \end{bmatrix}$. $\quad R_1 \to \frac{1}{x-1} R_1$

This is ok because $|x| < 1 \Rightarrow x \neq 1$

Then b is free so

let $b = s \in \mathbb{R}$.

~~We know~~ that $a - b/_{x-1} = 0$

So $a = \frac{1}{x-1} s$.

Hence $\underline{v}_x = s \begin{bmatrix} \frac{1}{x-1} \\ 1 \end{bmatrix} \quad s \in \mathbb{R}, s \neq 0.$