MATH2022 Assignment 1

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1 Q1

In group G, for each element $\alpha \neq e$, there is a unique inverse α^{-1} such that $\alpha\alpha^{-1} = e$ (*Inverse Element,according to the definition of Group*). Since G has an even number of elements, the total number of non-identity elements is odd. This means that if we try to partition them into pairs like (α, α^{-1}), there must be at least one element that cannot be paired with a distinct element. The unpaired element must therefore be its own inverse, implying $\alpha = \alpha^{-1}$. For this α, it follows that $\alpha^2 = \alpha\alpha = e$.

2 Q2

(a)

$$\begin{aligned}\det(A) &= -2\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} + 4\begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} \\ &= -2 + 4 \cdot (-2) \\ &= -10 \\ &= 0(mod\ 5)\end{aligned}$$

Therefore, Matrix A is not invertible over Z_5.

(b)

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 2 \\ 1 & 1 & 2 & 3 \\ 0 & 2 & 4 & 4 \end{array}\right] \xrightarrow{R_2=R_1-R_2} \left[\begin{array}{ccc|c} 1 & 3 & 1 & 2 \\ 0 & 2 & 4 & 4 \\ 0 & 2 & 4 & 4 \end{array}\right]$$

$$\xrightarrow{R_3=R_2-R_3} \left[\begin{array}{ccc|c} 1 & 3 & 1 & 2 \\ 0 & 2 & 4 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow[]{\substack{R_1=R_1+R_2 \\ R_2=\frac{1}{2}R_2}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

we have: $\begin{cases} x = 1 \\ y + 2z = 2 \end{cases}$, let $z = t, t \in Z_5$,the solution is :

$$\therefore \begin{cases} x = 1 \\ y = 2 - 2t \mod 5 \quad , \quad t \in \mathbb{Z}_5 \\ z = t \end{cases}$$

Since z can take any value in $Z_5 = 0, 1, 2, 3, 4$,and for each z,there is exactly 1 solution for x and y.Thus,there are 5 solutions for this system.

3 Q3

(a)

True. To perform the conjugation $\beta^{-1}\alpha\beta$, we simply replace each α_i in every cycle with $\alpha_i\beta$:

$$\beta^{-1}\alpha\beta = (\beta^{-1}\alpha_1\beta)(\beta^{-1}\alpha_2\beta)(\beta^{-1}\alpha_3\beta)\ldots(\beta^{-1}\alpha_k\beta)$$

where k represents the number of cycles in α, and k is odd. It doesn't change the number of elements in a permutation.The conjugation is also odd.

(b)

False.If $C = \mathbf{0}$,the statement "If $AC = BC$, then $A = B$" still holds true when $A \neq B$.

(c)

True. We have $\det(A) = \det(A^T)$. If $A^T = -A$, then $\det(A) = \det(-A)$, and since $\det(-A) = (-1)^n \det(A)$ for an $n \times n$ matrix, where $n = 3$ in this case, it follows that $\det(A) = -\det(A)$ which indicates that $\det(A) = 0$.

4 Q4

Assume that the eigenvalue of eigenvector $\mathbf{v}$ is λ,so that by the description above, we have:

$$A\mathbf{v} = \lambda\mathbf{v}$$

To prove that $B\mathbf{v}$ is also an eigenvector of A with the same eigenvalue λ as $\mathbf{v}$, we need to show that $A(B\mathbf{v}) = \lambda(B\mathbf{v})$.

Proof

$$\begin{gathered} A(B\mathbf{v}) \\ = (AB)\mathbf{v} \\ = B(A\mathbf{v}) \\ = B\lambda\mathbf{v} \\ = \lambda(B\mathbf{v}) \end{gathered}$$

We have $A(B\mathbf{v}) = \lambda(B\mathbf{v})$. Thus, the product $B\mathbf{v}$ is also an eigenvector of A with eigenvalue λ.