Basis and Dimension

- Recall: A set X spans a vector space V

$$\langle X \rangle = V$$

: Linear independence of a set of vectors X gave "efficient" spanning set of $\langle X \rangle$.

- Today: Combine these two ideas

⇝ Basis of a vector space

Key notions: describe arbitrary vectors

: recover approach via matrices

: define "dimension" = degrees of freedom.

• Definition: Let V be a vector space over F.

A set of vectors $\mathcal{B} \subset V$ is called a basis of V if

① $\mathcal{B}$ spans V: $\langle \mathcal{B} \rangle = V$

② $\mathcal{B}$ is linearly independent.

- That is,

① For any $\underset{\sim}{v} \in V$, we can write

$$\underset{\sim}{v} = \lambda_1 \underset{\sim}{b}_1 + \lambda_2 \underset{\sim}{b}_2 + \cdots + \lambda_K \underset{\sim}{b}_K$$

for some $\lambda_i \in F$, $\underset{\sim}{b}_i \in \mathcal{B}$.

② There are no "redundancies": No $\underset{\sim}{b}_K \in \mathcal{B}$ is a linear combination of the others.

• Remark: This definition allows $|\mathcal{B}| = \infty$!

- But, we will typically have $|\mathcal{B}| < \infty$ for our examples and applications.

Example: Standard bases of F^n

- F^2 : $\mathcal{B} = \{(1,0),\ (0,1)\}$

- F^3 : $\mathcal{B} = \{(1,0,0),\ (0,1,0),\ (0,0,1)\}$

- F^n : $\{\underset{\sim}{e}_1, \underset{\sim}{e}_2, \dots, \underset{\sim}{e}_n\}$, $\underset{\sim}{e}_i = (0, \dots, 0, \overset{i^{th} \text{ place}}{\overset{\downarrow}{1}}, 0, \dots, 0)$

- Check: span?

  Yes! $(\lambda_1, \dots, \lambda_n) \in F^n$

  $\Rightarrow \lambda_1 \underset{\sim}{e}_1 + \lambda_2 \underset{\sim}{e}_2 + \dots + \lambda_n \underset{\sim}{e}_n \in \langle \underset{\sim}{e}_1, \dots, \underset{\sim}{e}_n \rangle$

  : LI?

  Yes! $\lambda_1 \underset{\sim}{e}_1 + \dots + \lambda_n \underset{\sim}{e}_n = \underset{\sim}{0}$

  $\Rightarrow (\lambda_1, \dots, \lambda_n) = (0, \dots, 0)$

  $\Rightarrow \lambda_i = 0 \ \forall i.$

- Example: $\mathbb{P}_K = \{a_0 + a_1 x + \dots + a_K x^K \mid a_i \in F\}$

has a basis

$$\mathcal{B} = \{1, x, x^2, \dots, x^K\}$$ ← "Monomial" basis.

- Span: clear
- LI: proven last lecture
  - Thought experiment: Can't produce x^K by just sums of x^n, $n<K$.
- Remark: $\mathbb{P}_K$ has other bases, which are often used because of nice properties. More later...

Coordinate vectors

- Theorem: Let $\mathcal{B} = \{\underset{\sim}{b}_1, \dots, \underset{\sim}{b}_n\}$ be a basis of a vector space V. Then for any $\underset{\sim}{v} \in V$, there exist unique $\lambda_1, \lambda_2, \dots, \lambda_n \in F$ such that

$$\underset{\sim}{v} = \lambda_1 \underset{\sim}{b}_1 + \lambda_2 \underset{\sim}{b}_2 + \dots + \lambda_n \underset{\sim}{b}_n.$$

Proof

Since $\langle \mathcal{B} \rangle = V$, the existence of some $\lambda_1, \dots, \lambda_n$ is immediate.

For uniqueness, we use linear independence of $\mathcal{B}$:

Suppose $\mu_1, \mu_2, \dots, \mu_n \in F$ also satisfy

$$\underset{\sim}{v} = \mu_1 \underset{\sim}{b}_1 + \mu_2 \underset{\sim}{b}_2 + \dots + \mu_n \underset{\sim}{b}_n .$$

$$\Rightarrow \lambda_1 \underset{\sim}{b}_1 + \lambda_2 \underset{\sim}{b}_2 + \dots + \lambda_n \underset{\sim}{b}_n = \mu_1 \underset{\sim}{b}_1 + \mu_2 \underset{\sim}{b}_2 + \dots + \mu_n \underset{\sim}{b}_n .$$

$$\Rightarrow (\lambda_1 - \mu_1) \underset{\sim}{b}_1 + (\lambda_2 - \mu_2) \underset{\sim}{b}_2 + \dots + (\lambda_n - \mu_n) \underset{\sim}{b}_n = \underset{\sim}{0} .$$

By $\mathcal{B}$ LI, $(\lambda_1 - \mu_1) = (\lambda_2 - \mu_2) = \dots = (\lambda_n - \mu_n) = 0$.

$$\Rightarrow \lambda_i = \mu_i \quad \forall i$$

∎

- Definition: Let $\mathcal{B} = \{\underset{\sim}{b}_1, \dots \underset{\sim}{b}_n\}$ be a basis of a vector space V.
  Let $\underset{\sim}{v} \in V$ and $\lambda_1, \dots, \lambda_n$ the unique scalars s.t.

$$\underset{\sim}{v} = \lambda_1 \underset{\sim}{b}_1 + \dots + \lambda_n \underset{\sim}{b}_n.$$

The coordinate vector or coordinates of $\underset{\sim}{v}$ with respect to $\mathcal{B}$ is given by

$$[\underset{\sim}{v}]_{\mathcal{B}} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix}$$

- Tells us how far to move along our new independent directions.

- Ex: F^n, $\mathcal{B} = \{\underset{\sim}{e}_1, \underset{\sim}{e}_2, \dots, \underset{\sim}{e}_n\}$

$$\Rightarrow [(\lambda_1, \dots, \lambda_n)]_{\mathcal{B}} = [\lambda_1 \underset{\sim}{e}_1 + \dots + \lambda_n \underset{\sim}{e}_n]_{\mathcal{B}}$$

$$= \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix}$$

- Key computational tool: Finding coordinate vectors is often done by solving system of linear equations.

- Example: Set $\mathcal{B} = \{(1,1), (2,-1)\} \subset \mathbb{R}^2$.
  Show $\mathcal{B}$ is a basis, and find $[(3,-7)]_{\mathcal{B}}$.

  - $\mathcal{B}$ is LI, since $|\mathcal{B}| = 2$ and $(1,1) \neq \lambda(2,-1)$ for any scalar λ.

  - For $\mathcal{B}$ to span $\mathbb{R}^2$, consider $(x,y) \in \mathbb{R}^2$. Then we need $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha(1,1) + \beta(2,-1) = (x,y)$$

$$\Rightarrow (\alpha + 2\beta, \alpha - \beta) = (x,y)$$

$$\Rightarrow \begin{cases} \alpha + 2\beta = x \\ \alpha - \beta = y \end{cases}$$

$$\Rightarrow \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

- Now, solve for α and β:

$$\left(\begin{array}{cc|c} 1 & 2 & x \\ 1 & -1 & y \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 2 & x \\ 0 & -3 & y-x \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 2 & x \\ 0 & 1 & \frac{x-y}{3} \end{array}\right)$$

$$\sim \left(\begin{array}{cc|c} 1 & 0 & \frac{x+2y}{3} \\ 0 & 1 & \frac{x-y}{3} \end{array}\right)$$

$$\Rightarrow \quad \alpha = \frac{x+2y}{3}, \quad \beta = \frac{x-y}{3}.$$

- Since solutions exist for all (x,y), get $\mathcal{B}$ spans $\mathbb{R}^2$.

- Now, $(x,y) = \frac{x+2y}{3}(1,1) + \frac{x-y}{3}(2,-1)$

$$\Rightarrow [(x,y)]_{\mathcal{B}} = \begin{bmatrix} \frac{x+2y}{3} \\ \frac{x-y}{3} \end{bmatrix}$$

$$\Rightarrow [(3,-7)]_{\mathcal{B}} = \begin{bmatrix} -11/3 \\ 10/3 \end{bmatrix}.$$

• Example : Let $W = \left\langle \begin{pmatrix}1\\2\\-1\end{pmatrix}, \begin{pmatrix}2\\0\\1\end{pmatrix} \right\rangle \subset \mathbb{R}^3$, with basis $\mathcal{B} = \left\{ \begin{pmatrix}1\\2\\-1\end{pmatrix}, \begin{pmatrix}2\\0\\1\end{pmatrix} \right\}$ ← clearly LI, and span by definition!

Find $\left[\begin{pmatrix}-4\\4\\-5\end{pmatrix} \right]_{\mathcal{B}}$.

- Again, need α, β such that

$$\alpha \begin{pmatrix}1\\2\\-1\end{pmatrix} + \beta \begin{pmatrix}2\\0\\1\end{pmatrix} = \begin{pmatrix}-4\\4\\-5\end{pmatrix}$$

$\rightsquigarrow$ Solve $\left(\begin{array}{cc|c} 1 & 2 & -4 \\ 2 & 0 & 4 \\ -1 & 1 & -5 \end{array}\right)$.

• So, we solve

$$\left(\begin{array}{cc|c} 1 & 2 & -4 \\ 2 & 0 & 4 \\ -1 & 1 & -5 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 1 & 2 & -4 \\ -1 & 1 & -5 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 2 & -6 \\ 0 & 1 & -3 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{array}\right)$$

$\Rightarrow \alpha = 2$, $\beta = -3$

$\Rightarrow \left[\begin{pmatrix}-4\\4\\-5\end{pmatrix} \right]_{\mathcal{B}} = \begin{bmatrix}2\\-3\end{bmatrix}$.

- Notice: even though $\begin{pmatrix} -4 \\ 4 \\ -3 \end{pmatrix}$ is in $\mathbb{R}^3$, it's coordinate vector wrt $\mathcal{B}$ has only two entries!

- Definition: The dimension of a vector space V is the size of any basis $\mathcal{B}$ of V.

  - That is, the dimension is the number of coordinates we need to describe our vectors.
  - In our previous example, $|\mathcal{B}| = 2$, so we say $\dim W = 2$

- Remark: Any basis?!
  - Theorem: Any two bases of a vector space $\mathcal{B}_1$, $\mathcal{B}_2$, satisfy
    $$|\mathcal{B}_1| = |\mathcal{B}_2|.$$
  - So, our definition makes sense.

- Example: $\dim(\mathbb{R}^2) = 2$
  $\dim(\mathbb{R}^3) = 3$
  $\dim(F^n) = n$

  $\Bigg\}$ Just use standard basis.

- Example: $\dim(\mathbb{P}_k) = k+1$, $|\mathcal{B} = \{1, x, .., x^k\}| = k+1$.

Rank of a matrix

- Let M be a matrix over a field F.
  - The row rank of M is the $\dim(\text{Row}(M))$
  - The column rank of M is the $\dim(\text{Col}(M))$.

- Theorem: row rank (M) = column rank (M).
  ↳ This one is kind of difficult.

- Definition: We say the rank of M is the row rank or column rank of M:

$$\text{rank}(M) = \text{rowrank}(M) = \text{colrank}(M)$$

- Procedure: Row reduce M (for row space) or M^T (for column space) and find nonzero rows.

  ⇝ Produces a basis for row/column space, # = dimension

- Example: Let $M = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$ over $\mathbb{Z}_2$.

  Find rank M, basis of row space, basis of column space.

  - Check: $M \sim \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

So, rank $M = 3$

row space has basis

$(1,0,1,1), (0,1,0,1), (0,0,1,0)$.

- $M^T = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

So, row space of M^T has basis

$(1,0,1,0), (0,1,0,1), (0,0,1,1)$

$\Rightarrow$ column space of M has basis

$\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$.

- Example: Let $M = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix}$ working over $\mathbb{R}$.

Find rank M by row reducing both M, M^T.

$M \sim \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$M^T \sim \begin{pmatrix} 1 & 5 & 9 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$