THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Calculus Tutorial 1 (Week 2)

MATH1062/MATH1023: Mathematics 1B (Calculus) Semester 2, 2024

Questions marked with * are harder questions.

Material covered

(1) Models and differential equations

Summary of essential material

Here are some useful properties of certain *trigonometric functions* that may be useful:

$$\tan^2(x) + 1 = \sec^2(x)$$

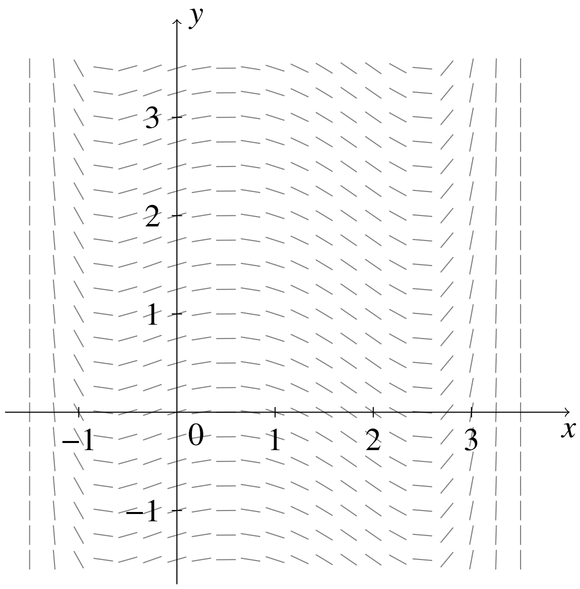
and

$$\int \sec^2(x)\, dx = \tan(x) + C.$$

Questions to complete during the tutorial

1. Find the particular solution of $\dfrac{dy}{dx} = \dfrac{1}{1+x^2}$, where $y(1) = \pi/4$.

2. The differential equation $\dfrac{dy}{dx} = f(x)$ has a direction field given by the diagram below.



(a) On the direction field draw the graphs of two solutions of $dy/dx = f(x)$, where one solution $y = g(x)$ passes through the point $(0, 1)$ and the other solution $y = h(x)$ satisfies the equation $h(1) = 0$.

(b) Do the graphs of $y = g(x)$ and $y = h(x)$ intersect? If not, why not?

3. Evaluate $\displaystyle\int \frac{x^2}{(a^2-x^2)^{3/2}}\,dx$ by making the substitution $x = a\sin u$, where a is some nonzero constant.

***4.** Which of the following differential equations are separable? Write those that are in separated form and solve them.

(a) $\displaystyle y\frac{dy}{dx} = (x-y^2)\sin y$

(b) $\displaystyle \frac{dy}{dx} = \frac{x+1}{2xy}$

(c) $\displaystyle \frac{dy}{dx} = \frac{x+\cos y}{x^3\sqrt{x^2-16}}$

(d) $\displaystyle \frac{dy}{dx} = \frac{a^2e^y}{(a^2-x^2)^{3/2}} - \frac{e^y}{(a^2-x^2)^{1/2}}$

***5.** An animal population has a net growth rate per unit population which varies with the seasons, being positive in summer and negative in winter. Let $x(t)$ be the size of the population at time t, which is measured in years. The following differential equation is suggested as a model for this situation:

$$\frac{dx}{dt} = (k\cos 2\pi t)x \quad (k \text{ a positive constant}).$$

(a) What is the period of $\cos 2\pi t$?

(b) What time of year do you think $t = 0$ represents ?

(c) Can you explain why x has been multiplied by $(k\cos 2\pi t)$ in this model?

(d) Solve the equation to find $x(t)$, given that $x = x_0$ at $t = 0$.

(e) Does $x(t)$ have a limiting value as $t \to \infty$?

(f) What are the maximum and minimum values of x and when do they occur?

Short answers to selected exercises

1. $y = \tan^{-1} x$

3. $\displaystyle \frac{x}{\sqrt{a^2-x^2}} - \sin^{-1}\frac{x}{a} + C$

4. (b) $y^2 = x + \ln x + C$

(d) $\displaystyle -e^{-y} = \frac{x}{\sqrt{a^2-x^2}} - \sin^{-1}\frac{x}{a} + C$

5. (d) $x = x_0 e^{(k\sin 2\pi t)/2\pi}$