THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Calculus Tutorial 2 (Week 3)

MATH1062/MATH1023: Mathematics 1B (Calculus) Semester 2, 2024

Questions marked with * are harder questions.

Material covered

(1) Separable differential equations

Summary of essential material

Here is a table of standard integrals.

- $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \qquad (n \neq -1)$
- $\displaystyle\int \frac{dx}{x} = \ln|x| + C$
- $\displaystyle\int e^x\,dx = e^x + C$
- $\displaystyle\int \sin(x)\,dx = -\cos(x) + C$
- $\displaystyle\int \cos(x)\,dx = \sin(x) + C$
- $\displaystyle\int \sec^2(x)\,dx = \tan(x) + C$
- $\displaystyle\int \operatorname{cosec}^2(x)\,dx = -\cot(x) + C$
- $\displaystyle\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C$
- $\displaystyle\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

We consider $\ln(x) \equiv \log_e(x)$, where $e = 2.71828\ldots$ and make use of hte exponential and log laws.

- $e^{a+b} = e^a e^b$
- $e^{a-b} = e^a e^{-b} = \dfrac{e^a}{e^b}$
- $\ln(ab) = \ln a + \ln b$
- $\ln a^b = b \ln a$
- $\ln e^x = x \ln e = x$
- $e^{\ln x} = x$

Questions to complete during the tutorial

1. Consider the differential equation $\dfrac{dy}{dx} = 2xy$.

(a) Calculate the general solution of the equation.

(b) Calculate and sketch the particular solution that passes through $(x, y) = (0, 2)$.

2. Which of the following differential equations are separable? For those that are not, justify your answer. Write those that are in separated form and solve them.

(a) $\dfrac{dy}{dx} = 5$

(b) $\dfrac{dy}{dx} = xy + 5$

(c) $\dfrac{dy}{dx} = x + y$

(d) $\dfrac{dy}{dx} = \dfrac{x+1}{2xy}$

(e) $\dfrac{dy}{dx} = x\sqrt{4 - y^2}$

(f) $\dfrac{dy}{dx} = \dfrac{x + \cos y}{x^3\sqrt{x^2 - 16}}$

(g) $\dfrac{dy}{dx} = \dfrac{e^y}{(a-x)} + 3e^y$ (h) $y\dfrac{dy}{dx} = (x-y^2)\sin y$ (j) $\dfrac{dy}{dx} = \dfrac{(9+y^4)\sin(x)}{y}$

(i) $\dfrac{dy}{dx} = \dfrac{\sin 2x}{e^y}$

***3.** The velocity v of objects falling in viscous fluids varies in t according to

$$\frac{dv}{dt} = g - kv. \tag{1}$$

where $g \simeq 9.81 m/s^2$ is the gravitational aceleration and k is the friction constant.

(a) Calculate the particular solution with initial condition $v(0) = 0$.

(b) What is the terminal velocity of the falling object and when is it obtained? When does the falling object attain 99% of its terminal velocity?

(c) Calculate the terminal velocities of a glass marble with diameter $1cm$ sinking in: (i) honey and (ii) olive oil. In viscous fluids $k = 18\dfrac{\mu\rho_s}{q^2(\rho_0 - \rho_s)}$, where ρ_s and μ are, respectively, the density and kinematic viscosity of the surrounding medium and ρ_0 and q are, respectively, the density and diameter of the falling object. The densities of honey, olive oil and glass are roughly $\rho_{honey} = 1450kg/m^3$, $\rho_{oil} = 910kg/m^3$ and $\rho_{glass} = 3000kg/m^3$ respectively. The kinematic viscosities of honey and oil are $\mu_{honey} = 73.6 \times 10^{-6} m^2/s$ and $\mu_{oil} = 43.2 \times 10^{-6} m^2/s$.

(d) What are the terminal velocities for a marble with diameter $10cm$?

***4.** (a) Describe all solutions to the differential equation $\dfrac{dy}{dx} = e^{-x}y^2 - e^{-x} + y^2 - 1$.

Hint: Use partial fractions.

(b) Determine particular solutions satisfying initial conditions $y(e) = 1$ and $y(0) = \dfrac{1+e}{1-e}$.

Short answers to selected exercises

1. (a) $y = Ae^{x^2}$ (b) $y = 2e^{x^2}$

2. (a) $y(x) = 5x + c$

(b) Not separable

(i) $y(x) = \ln\left(-\frac{1}{2}\cos(2x) + C\right)$

(j) $y = \pm\sqrt{3\tan(-6\cos(x) + C)}$

3. (c) Roughly $0.79m/s$ and $2.89m/s$.