Theorems: Let A be a stochastic matrix.

1. 1 is an eigenvalue of A, and so A has at least one steady-state probability vector, $\underline{v}$, which is fixed by A. $A\underline{v} = 1\underline{v}$

2. All eigenvalues of A are less than or equal to 1 in magnitude.

3. If A is regular (some power of A has all positive entries), then:

   A. $\underline{v}$ is unique There is only 1 steady state vector. up to scalar multiple

   $A(\lambda \underline{v}) = \lambda A \underline{v} = \lambda \underline{v}$

   B. $\lim_{n\to\infty} A^n = [\underline{v}\ \underline{v} \cdots \underline{v}]$

   C. For any probability vector, $\underline{x}$, $\lim_{n\to\infty} A^n \underline{x} = \underline{v}$.

   D. 1 is the dominant eigenvalue.

Better A: There is only one probability steady state vector.

Proof: We only prove Theorem 1.

First note that any square matrix has the same eigenvalues as its transpose.

$$\chi_A(\lambda) = |\lambda I - A| = |(\lambda I - A)^T| = |\lambda I - A^T| = \chi_{A^T}(\lambda).$$

Let $A = [a_{ij}]$ be a $n\times n$ stochastic matrix.

Then $\underbrace{[1\ 1 \cdots 1]}_{1\times n} \underbrace{A}_{n\times n} = \left[\sum_{i=1}^n a_{i1}\ \sum_{i=1}^n a_{i2} \cdots \sum_{i=1}^n a_{in}\right]_{1\times n}$

$= [1\ 1 \cdots 1]$

$$[1 \cdots 1]A = [1 \cdots 1]$$

Take the transpose of both sides:

$$([1\ 1 \cdots 1]A)^T = [1\ 1 \cdots 1]^T$$

$$A^T[1\ 1 \cdots 1]^T = [1\ 1 \cdots 1]^T$$

$$A^T\begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix} = 1\begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix}$$

So, $\lambda=1$ is an eigenvalue of A^T with corresponding eigenvector $\begin{bmatrix}1\\\vdots\\1\end{bmatrix}$. Not an eigenvector of A.

Hence $\lambda=1$ is an eigenvalue of A. $\square$

Example: Check this using $E=\begin{bmatrix}0.8 & 0.3 & 0.2\\0.1 & 0.2 & 0.6\\0.1 & 0.5 & 0.2\end{bmatrix}$.

$$[1\ 1 \cdots 1]\begin{bmatrix}0.8 & 0.3 & 0.2\\0.1 & 0.2 & 0.6\\0.1 & 0.5 & 0.2\end{bmatrix} = [1\ 1 \cdots 1]$$

So $$E^T\begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix} = \begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix}$$

So 1 is an eigenvalue of E^T.

Hence 1 is an eigenvalue of E.

Example: Find the unique steady-state probability vector for $E = \begin{bmatrix} 0.8 & 0.3 & 0.2 \\ 0.1 & 0.2 & 0.6 \\ 0.1 & 0.5 & 0.2 \end{bmatrix}$.

Since E is stochastic, $\lambda = 1$ is an eigenvalue.
Let's find the associated eigenspace.

$$I - E = \begin{bmatrix} 0.2 & -0.3 & -0.2 \\ -0.1 & 0.8 & -0.6 \\ -0.1 & -0.5 & 0.8 \end{bmatrix} \sim \begin{bmatrix} 2 & -3 & -2 \\ 1 & -8 & 6 \\ 1 & 5 & -8 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -8 & 6 \\ 2 & -3 & -2 \\ 1 & 5 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & -8 & 6 \\ 0 & 13 & -14 \\ 0 & 13 & -14 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -8 & 6 \\ 0 & 1 & \frac{-14}{13} \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{array}{c} \begin{bmatrix} 1 & 0 & \frac{-34}{13} \\ 0 & 1 & \frac{-14}{13} \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccc} x & y & z \end{array} \end{array}$$

$$x - \frac{34}{13} z = 0$$

$$y - \frac{14}{13} z = 0$$

Let $z = t \in \mathbb{R}$

So $y = {}^{14}/_{13}\, t$ and $x = \frac{34}{13} t$.

So, the eigenspace is $\left\{ t \begin{bmatrix} 34/13 \\ 14/13 \\ 1 \end{bmatrix} \middle|\ t \in \mathbb{R} \right\}$

Hence $E\begin{bmatrix} t\,{}^{34}/_{13} \\ t\,{}^{14}/_{13} \\ t \end{bmatrix} = \begin{bmatrix} t\,{}^{34}/_{13} \\ t\,{}^{14}/_{13} \\ t \end{bmatrix}$ for any $t \in \mathbb{R}$.

However, we need a probability vector.

Choose t so that

$$t\left(\frac{34}{13} + \frac{14}{13} + 1\right) = 1$$

$$t\left(\frac{61}{13}\right) = 1$$

$$t = {}^{13}/_{61}.$$

Now, the unique probability steady state eigenvector is

$$\underline{v} = \begin{bmatrix} \frac{34}{13} \times \frac{13}{61} \\ {}^{14}/_{13} \times \frac{13}{61} \\ {}^{13}/_{61} \end{bmatrix} = \begin{bmatrix} {}^{34}/_{61} \\ {}^{14}/_{61} \\ {}^{13}/_{61} \end{bmatrix}$$

Check: $E\underline{v} = \begin{bmatrix} & \end{bmatrix}\begin{bmatrix} \\ \end{bmatrix}$

exercise.

It turns out that the stochastic matrices are closed under matrix multiplication. To prove this, we first need the following Lemma.

Lemma: Let A be a stochastic matrix. If $\underline{v}$ is a probability vector then so is $A\underline{v}$.

Proof: Let $A = [a_{ij}]_{\substack{i=1,\dots,n \\ j=1,\dots,n}}$ be a stochastic matrix and let $\underline{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ be a probability vector.

Then, for each $1 \le j \le n$,

column sum of jth column of A

$$\sum_{i=1}^{n} a_{ij} = \sum_{i=1}^{n} v_i = 1.$$

$$A\underline{v} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$$

Let w_i be the i-th entry of $A\underline{v}$.

We will show that each w_i is non-negative.

$$w_i = [a_{i1}\ a_{i2} \cdots a_{in}] \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$= \sum_{j=1}^{n} a_{ij} v_j$$

Each a_{ij} and v_j is non-negative by definition. Hence, so is w_i.

We will show that $\sum_{i=1}^{n} w_i = 1$.

$$\sum_{i=1}^{n} w_i = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} v_j \right)$$

$$= \sum_{j=1}^{n} \left(\sum_{i=1}^{n} a_{ij} v_j \right)$$

$$= \sum_{j=1}^{n} \left(\sum_{i=1}^{n} a_{ij} \right) v_j$$

$$= \sum_{j=1}^{n} 1 \, v_j = \sum_{j=1}^{n} v_j$$

$$= 1.$$

$\square$

We can now prove the following:

Proposition: The set of stochastic nxn matrices is closed under matrix multiplication.

Proof: For the purposes of this proof, denote the set of stochastic nxn matrices by S_n.

Let $A, B \in S_n$. Then $B = [\underline{v_1} \ \underline{v_2} \ \cdots \ \underline{v_n}]$ for some probability vectors, $\underline{v_i}$.

Then $AB = A[\underline{v_1} \ \underline{v_2} \ \cdots \ \underline{v_n}]$

$$= [A\underline{v_1} \ A\underline{v_2} \ \cdots \ A\underline{v_n}]$$

Since each $A\underline{v_i}$ is a probability vector, we are done.

$\square$

Section 8 Intro to Vector Spaces and Linear Transformations

Section 8.1 Cartesian Products

Vector spaces are ubiquitous in mathematics.They generalise and abstract properties of the Cartesian xy-plane. Linear transformations are functions which “preserve” the vector space structure.

Firstly, we need to define some terminology and notation.

Definition: The Cartesian product of sets $A_1, A_2, \ldots, A_n$ is the set

$$A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\}$$

If $A_1, A_2, \ldots, A_n$ have binary operations then the Cartesian product inherits these operations coordinate-wise.

Cartesian product operation

eg. $(a_1, a_2, a_3) \oplus (b_1, b_2, b_3) =$

$$(a_1 +_1 b_1,\ a_2 +_2 b_2,\ a_3 +_3 b_3)$$

operation in A_1 operation in A_2 operation in A_3

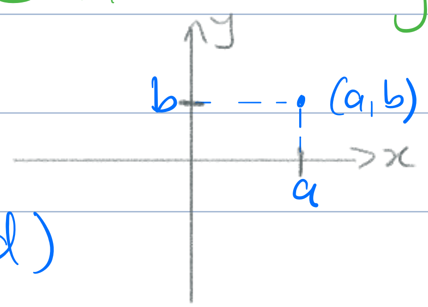
Example: The usual Cartesian plane, or xy-plane.

$$\mathbb{R} \times \mathbb{R} = \{(x, y) \mid x, y \in \mathbb{R}\}$$

(devised by Descartes in the 17th century)

Also denoted $\mathbb{R}^2$ or $\mathbb{R} \oplus \mathbb{R}$.

$(a, b) + (c, d)$
$= (a + c, b + d)$



Example: $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{(a,b)\,|a,b \in \mathbb{Z}_2\}$

$= \{(0,0),(0,1),(1,0),(1,1)\}$

and, for example

$(1,1)+(1,1)=(1+1,1+1)=(0,0)$

$(0,1)+(1,0)=(0+1,1+0)=(1,1)$

Example: $\mathbb{Z}_2 \times \mathbb{Z}_3 = \{(a,b)\,|a \in \mathbb{Z}_2, b \in \mathbb{Z}_3\}$

$= \{(0,0),(0,1),(0,2),(1,0),(1,1),(1,2)\}$

and, for example

$(1,1)+(1,1)=(1+1,1+1)=(0,2)$

$(1,1)+(0,2)=(1+0,1+2)=(1,0)$

$(1,1)+(1,0)=(1+1,1+0)=(0,1)$

If $A_1 = A_2 = \cdots = A_n$ then we call $A_1 \times A_2 \times \cdots \times A_n$ a Cartesian power. Most commonly,

$$A = A_2 = \ldots = A_n = F$$

where F is a field, and we write

$$F^n = F \times \ldots \times F$$
$$= \{(a_1, a_2, .., a_n)\,|a_1, a_2, \ldots, a_n \in F\},$$

which has coordinate-wise addition and multiplication inherited from F:

$$(a_1, \ldots, a_n) + (b_1, \ldots, b_n) = (a_1 + b_1, \ldots, a_n + b_n) \quad *$$
$$(a_1, \ldots, a_n)\,(b_1, \ldots, b_n) = (a_1 b_1, \ldots, a_n b_n)$$

It also has scalar multiplication (for $\lambda \in F$): $\lambda\,(a_1, \ldots, a_n) = (\lambda a_1, \ldots, \lambda a_n) \quad **$