The University of Sydney
School of Mathematics and Statistics

Solutions to Calculus Exercises 1 (Week 1)

MATH1062/MATH1023: Mathematics 1B (Calculus) — Semester 2, 2024

Material covered

(1) Models and differential equations

Assumed Knowledge

Integration techniques. Trigonometric functions, exponential function and logarithms. (Taylor series and binomial series.)

Objectives

1. Given a verbal description of a simple model, to be able to express it as a mathematical equation.
2. Recognise ordinary differential equations.
3. Given an ordinary first order differential equation, to be able to transform it into standard form.
4. Be familiar with general properties of (first order) differential equations.

Exercises

Questions marked with * are harder questions.

1. Find the general solution by antidifferentiation and sketch the solution curves of:

(a) $\dfrac{dy}{dx} = \cos 2x$

Solution: $y = \displaystyle\int \cos 2x\, dx = \frac{1}{2}\sin 2x + C$. The set of solution curves is a set of sine curves, each with amplitude $\dfrac{1}{2}$ and period π, displaced vertically one above the other.

(b) $\dfrac{dy}{dx} = \cosh x$

Solution: $y = \displaystyle\int \cosh x\, dx = \sinh x + C$. The set of solution curves is obtained by displacing $\sinh x$ vertically.

2. Find the general solution of the following differential equations, and in each case give also the particular solution satisfying the initial condition $y(0) = 2$.

(a) $\dfrac{dy}{dx} = 20xe^{5x^2}$

Solution: Note that $20xe^{5x^2} = 2df/dx e^{f(x)}$ for $f(x) = 5x^2$. General solution is thus $y = 2e^{5x^2} + C$. The condition $y(0) = 2$ then requires $2 = 2e^{5\times 0^2} + C = 2 + C$, so $C = 0$, and the particular solution is $y = 2e^{5x^2}$.

(b) $\dfrac{dy}{dx} = 6x^2 + \cos x$

Solution: General solution is $y = 2x^3 + \sin x + C$. The condition $y(0) = 2$ then requires $2 = C$, so the particular solution is $y = 2x^3 + \sin x + 2$.

(c) $\dfrac{dy}{dx} = x \cos x$

Solution: Use integration by parts to get the general solution

$$y = x \sin x - \int \sin x \, dx = x \sin x + \cos x + C.$$

The initial condition $y(0) = 2$ then requires $2 = 1 + C$, so $C = 1$, and the particular solution is $y = x \sin x + \cos x + 1$.

(d) $\dfrac{dy}{dx} = x^2 e^x$

Solution: Using integration by parts (twice) we get the general solution

$$\begin{aligned} y &= x^2 e^x - \int 2xe^x \, dx \\ &= x^2 e^x - \left(2xe^x - \int 2e^x \, dx\right) \\ &= x^2 e^x - 2xe^x + 2e^x + C. \end{aligned}$$

The initial condition $y(0) = 2$ then requires $2 = 2 + C$, so $C = 0$, and the particular solution is $y = x^2 e^x - 2xe^x + 2e^x$.

3. Put the following first order differential equations into standard form.

(a) $x^2 \dfrac{dy}{dx} + xy = 1$

Solution: The standard form of a first order ordinary differential equation is $\dfrac{dy}{dx} = f(x, y)$. It follows that we have for the solution: $\dfrac{dy}{dx} = \dfrac{1 - xy}{x^2}$

(b) $\dfrac{1}{\sin(t)}\left(t\dfrac{dy}{dt} - y\right) = t^2$

Solution: $\dfrac{dy}{dt} = \dfrac{y + t^2 \sin(t)}{t}$

***4.** Newton's law of gravitation states that the acceleration of an object at a distance r from the centre of an object of mass M is given by

$$\frac{d^2r}{dt^2} = -\frac{GM}{r^2},$$

where G is the universal gravitational constant.

(a) Use the identity

$$\frac{d^2r}{dt^2} = \frac{d}{dr}\left(\frac{1}{2}v^2\right),$$

combined with integration with respect to r. Determine the resulting constant of integration using the condition $u = v(R)$ and show that

$$v^2 - u^2 = \frac{2GM}{r} - \frac{2GM}{R}.$$

Solution: Substituting,

$$\frac{d}{dr}\left(\frac{1}{2}v^2\right) = -\frac{GM}{r^2},$$

and performing the antidifferentiation,

$$\frac{1}{2}v^2 = -\int \frac{GM}{r^2}\,dr = \frac{GM}{r} + C.$$

We must choose C so that $v = u$ when $r = R$; so substituting, we want

$$\frac{1}{2}u^2 = \frac{GM}{R} + C.$$

Replacing C by this expression gives (after rearrangement)

$$v^2 - u^2 = \frac{2GM}{r} - \frac{2GM}{R}.$$

(b) Now write $r = R + s$ where s is the height of the object above the surface of the Earth, radius R and mass M. Use the binomial series to expand the factor $(1 + s/R)^{-1}$ to show that, close to the surface of the Earth,

$$v^2 \approx u^2 - 2gs,$$

for some constant g. Find the expression for g.

Reminder: The binomial series is $(1 + x)^{-1} = 1 - x + x^2 - x^3 + \cdots$, which converges for $|x| < 1$.

Solution: Writing $r = R + s$,

$$v^2 - u^2 = \frac{2GM}{(R+s)} - \frac{2GM}{R} = \frac{2GM}{R}\left(1 + \frac{s}{R}\right)^{-1} - \frac{2GM}{R}.$$

Using the binomial theorem on $(1 + s/R)^{-1}$, we obtain

$$v^2 = u^2 + \frac{2GM}{R}\left(1 - \frac{s}{R} + \frac{s^2}{R^2} + \ldots\right) - \frac{2GM}{R} = u^2 - \frac{2GMs}{R^2} + \ldots$$

Retaining only the leading term,

$$v^2 \approx u^2 - 2\left(\frac{GM}{R^2}\right)s,$$

so $g = GM/R^2$.