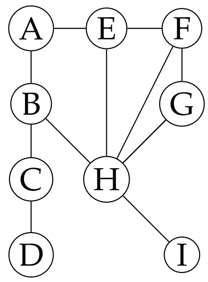
Warm-up

Problem 1. Consider the following undirected graph.



a) Starting from A, give the layers the breadth-first search algorithm finds.

b) Starting from A, give the order in which the depth-first search algorithm visits the vertices.

Problem solving

Problem 2. An undirected graph $G = (V, E)$ is said to be bipartite if its vertex set V can be partitioned into two sets A and B such that $E \subseteq A \times B$. Design an $O(n+m)$ algorithm to test if a given input graph is bipartite using the following guide:

a) Suppose we run BFS from some vertex $s \in V$ and obtain layers $L_1, \ldots, L_k$. Let (u, v) be some edge in E. Show that if $u \in L_i$ and $v \in L_j$ then $|i - j| \leq 1$.

b) Suppose we run BFS on G. Show that if there is an edge (u, v) such that u and v belong to the same layer then the graph is not bipartite.

c) Suppose G is connected and we run BFS. Show that if there are no intra-layer edges then the graph is bipartite.

d) Put together all the above to design an $O(n+m)$ time algorithm for testing bipartiness.

Problem 3. Give an $O(n)$ time algorithm to detect whether a given undirected graph contains a cycle. If the answer is yes, the algorithm should produce a cycle. (Assume adjacency list representation.)

Problem 4. Let $G = (V, E)$ be an n vertex graph. Let s and t be two vertices. Argue that if $\text{dist}(s, t) > n/2$ then there there exists a vertex $u \neq s, t$ such that every path from s to t goes through u.

Problem 5. In a directed graph, a *get-stuck* vertex has in-degree $n-1$ and out-degree 0. Assume the adjacency matrix representation is used. Design an $O(n)$ time algorithm to test if a given graph has a get-stuck vertex. Yes, this problem can be solved without looking at the entire input matrix.

Problem 6. Let G be an undirected graph with vertices numbered $1 \ldots n$. For a vertex i define $\text{small}(i) = \min\{j : j \text{ is reachable from } i\}$, that is, the smallest vertex reachable from i. Design an $O(n+m)$ time algorithm that computes small(i) for *every* vertex in the graph.

Problem 7. In a connected undirected graph $G = (V, E)$, a vertex $u \in V$ is said to be a cut vertex if its removal disconnects G; namely, $G[V - u]$ is not connected.

The aim of this problem is to adapt the algorithm for cut edges from the lecture, to handle cut vertices.

a) Derive a criterion for identifying cut vertices that is based on the `down-and-up`$[\cdot]$ values defined in the lecture.

b) Use this criterion to develop an $O(n+m)$ time algorithm for identifying all cut vertices.

Problem 8. Let T be a rooted tree. For each vertex $u \in T$ we use T_u to denote the subtree of T made up by u and all its descendants. Assume each vertex $u \in T$ has a value $A[u]$ associated with it. Let $B[u] = \min\{A[v] : v \in T_u\}$. Design an $O(n)$ time algorithm that given A, computes B.

Problem 9. Let G be a connected undirected graph. Design a linear time algorithm for finding all cut edges by using the following guide:

a) Derive a criterion for identifying cut edges that is based on the `down-and-up`$[\cdot]$ values defined in the lecture.

b) Use this criterion to develop an $O(n+m)$ time algorithm for identifying all cut edges.

Advanced problem solving

Problem 10. Let G be a directed graph with vertices numbered $1 \ldots n$. For a vertex i define `small`$(i) = \min\{j : j \text{ is reachable from } i\}$, that is, the smallest vertex reachable from i. Design an $O(n+m)$ time algorithm that computes small(i) for *every* vertex in the graph.