The University of Sydney

MATH2022 Linear and Abstract Algebra

Semester 1 **Week 11 Longer Solutions**

1. (a) $-2\mathbf{w} = (-4, -4, 8)$.

(b) $\| -2\mathbf{w}\| = |-2|\|\mathbf{w}\| = 2\|\mathbf{w}\| = 2\sqrt{4+4+16} = 2\sqrt{24} = 4\sqrt{6}$.

(c) $\left\| \dfrac{-2}{\|\mathbf{w}\|}\mathbf{w} \right\| = \dfrac{2}{\|\mathbf{w}\|}\|\mathbf{w}\| = 2$.

(d) $\mathbf{u} + \mathbf{v} = (2, -2, 2)$.

(e) $\|\mathbf{u}+\mathbf{v}\| = \sqrt{4+4+4} = \sqrt{12} = 2\sqrt{3}$.

(f) $\|\mathbf{u}\| + \|\mathbf{v}\| = \sqrt{1+9+4} + \sqrt{1+1+0} = \sqrt{14} + \sqrt{2}$.

Observe that

$$\|\mathbf{u}\| + \|\mathbf{v}\| = \sqrt{14} + \sqrt{2} = \sqrt{2}(\sqrt{7}+1) > \sqrt{2}\sqrt{7} > \sqrt{2}\sqrt{6} = 2\sqrt{3} = \|\mathbf{u}+\mathbf{v}\|$$

so that the triangle inequality is indeed holding.

2. We have $\mathbf{u}\cdot\mathbf{v} = 2 - 1 + 2 = 3$ and

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{3}{\sqrt{4+1+1}\sqrt{1+1+4}} = \frac{3}{6} = \frac{1}{2},$$

so that $\theta = \dfrac{\pi}{3}$.

3. (a) That the tetrahedron is regular is immediate by observing that all the lengths between vertices are the same:

$$\begin{aligned} \|\overrightarrow{P_0P_1}\| &= \sqrt{1+1+0} = \sqrt{2}, & \|\overrightarrow{P_0P_2}\| &= \sqrt{1+0+1} = \sqrt{2}, \\ \|\overrightarrow{P_0P_3}\| &= \sqrt{0+1+1} = \sqrt{2}, & \|\overrightarrow{P_1P_2}\| &= \sqrt{0+1+1} = \sqrt{2}, \\ \|\overrightarrow{P_1P_3}\| &= \sqrt{1+0+1} = \sqrt{2}, & \|\overrightarrow{P_2P_3}\| &= \sqrt{1+1+0} = \sqrt{2}. \end{aligned}$$

(b) The centre of the tetrahedron is the point $Q(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, and it suffices to find the angle θ between $\overrightarrow{QP_0}$ and $\overrightarrow{QP_1}$. But

$$\cos\theta = \frac{\overrightarrow{QP_0}\cdot\overrightarrow{QP_1}}{\|\overrightarrow{QP_0}\|\|\overrightarrow{QP_1}\|} = \frac{(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})\cdot(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})}{\|(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})\|\|(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})\|} = \frac{-\frac{1}{4}}{\frac{3}{4}} = -\frac{1}{3},$$

so that $\theta = \cos^{-1}(-\frac{1}{3}) \approx 109.5°$.

4. Let θ be the angle inscribed in a semicircle. Then θ is the angle between

$$\mathbf{u}+\mathbf{v} \quad \text{and} \quad \mathbf{u}-\mathbf{v}$$

for some directed line segments $\mathbf{u}$ and $\mathbf{v}$ that join the centre of the circle to its circumference, but such that $\mathbf{u} \neq \mathbf{v}$ and $\mathbf{u} \neq -\mathbf{v}$. But

$$\|\mathbf{u}\| = \|\mathbf{v}\| = r$$

is the radius of the circle. Then

$$\begin{aligned}\cos\theta &= \frac{(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}-\mathbf{v})}{\|\mathbf{u}+\mathbf{v}\|\|\mathbf{u}-\mathbf{v}\|} = \frac{\mathbf{u}\cdot\mathbf{u}-\mathbf{v}\cdot\mathbf{v}}{\|\mathbf{u}+\mathbf{v}\|\|\mathbf{u}-\mathbf{v}\|} = \frac{\|\mathbf{u}\|^2-\|\mathbf{v}\|^2}{\|\mathbf{u}+\mathbf{v}\|\|\mathbf{u}-\mathbf{v}\|} \\ &= \frac{r^2-r^2}{\|\mathbf{u}+\mathbf{v}\|\|\mathbf{u}-\mathbf{v}\|} = \frac{0}{\|\mathbf{u}+\mathbf{v}\|\|\mathbf{u}-\mathbf{v}\|} = 0\,,\end{aligned}$$

so that $\theta = \dfrac{\pi}{2}$.

5. We have $\|\mathbf{u}\| = \sqrt{4+0+1+9} = \sqrt{14}$, $\|\mathbf{v}\| = \sqrt{25+16+49+1} = \sqrt{91}$,

$$\|\mathbf{u}+\mathbf{v}\| = \|(7,4,6,2)\| = \sqrt{49+16+36+4} = \sqrt{105}\,,$$

and $\mathbf{u}\cdot\mathbf{v} = 10+0-7-3 = 0$, so that $\theta = \dfrac{\pi}{2}$. Observe that

$$\begin{aligned}\|\mathbf{u}+\mathbf{v}\| &= \sqrt{105} = \sqrt{14+91} < \sqrt{14+19+2\sqrt{(14)(19)}} \\ &= \sqrt{(\sqrt{14}+\sqrt{91})^2} = \sqrt{14}+\sqrt{91} = \|\mathbf{u}\|+\|\mathbf{v}\|\,,\end{aligned}$$

consistent with the triangle inequality. However,

$$\|\mathbf{u}+\mathbf{v}\|^2 = 105 = 14+91 = \|\mathbf{u}\|^2+\|\mathbf{v}\|^2\,.$$

This is consistent with what one might expect from an analogue of the Theorem of Pythagoras, since one might expect $\mathbf{u}$ and $\mathbf{v}$ to form a "right-angled triangle".

6. (a) We have $\mathbf{v} = \mathbf{i}+2\mathbf{j}$.

(b) Observe that $\|\mathbf{v}\| = \sqrt{1+4} = \sqrt{5}$, so that $\widehat{\mathbf{v}} = \frac{1}{\sqrt{5}}(\mathbf{i}+2\mathbf{j})$.

(c) The position vector of the point $(-1,5)$ is the vector $\mathbf{u} = -\mathbf{i}+5\mathbf{j}$.

(d) The projection of $\mathbf{u}$ in the direction of $\mathbf{v}$ is

$$\mathrm{proj}_{\mathbf{v}}\mathbf{u} = (\mathbf{u}\cdot\widehat{\mathbf{v}})\widehat{\mathbf{v}} = \frac{1}{5}(-1+10)(\mathbf{i}+2\mathbf{j}) = \frac{9}{5}\mathbf{i}+\frac{18}{5}\mathbf{j}\,.$$

(e) The nearest point on the line is $(\frac{9}{5},\frac{18}{5})$, and the distance is

$$\begin{aligned}\|\mathbf{u}-\mathrm{proj}_{\mathbf{v}}\mathbf{u}\| &= \|-\mathbf{i}+5\mathbf{j}-(\tfrac{9}{5}\mathbf{i}+\tfrac{18}{5}\mathbf{j})\| = \|-\tfrac{14}{5}\mathbf{i}+\tfrac{7}{5}\mathbf{j}\| \\ &= \frac{1}{5}\sqrt{196+49} = \frac{1}{5}\sqrt{245} = \frac{7\sqrt{5}}{5}\,.\end{aligned}$$

7. (a) Clearly neither $\mathbf{b}_1$ nor $\mathbf{b}_2$ is a scalar multiple of the other and both satisfy the equation defining W. Hence they form a basis for W (since W is two-dimensional). It suffices then to check that $\mathbf{b}_1$ and $\mathbf{b}_2$ are orthogonal and have length 1. The first follows because

$$\mathbf{b}_1\cdot\mathbf{b}_2 = -\frac{1}{\sqrt{6}}+0+\frac{1}{\sqrt{6}} = 0\,,$$

and the second because

$$\mathbf{b}_1\cdot\mathbf{b}_1 = \frac{1}{2}+0+\frac{1}{2} = 1 \qquad\text{and}\qquad \mathbf{b}_2\cdot\mathbf{b}_2 = \frac{1}{3}+\frac{1}{3}+\frac{1}{3} = 1\,.$$

(b) We have

$$\begin{aligned}\mathrm{proj}_W\mathbf{v} &= (\mathbf{v}\cdot\mathbf{b}_1)\mathbf{b}_1 + (\mathbf{v}\cdot\mathbf{b}_2)\mathbf{b}_2 \;=\; -\frac{1}{\sqrt{2}}\mathbf{b}_1 - \frac{7}{\sqrt{3}}\mathbf{b}_2 \\ &= (-\tfrac{1}{2}, 0, -\tfrac{1}{2}) + (\tfrac{7}{3}, -\tfrac{7}{3}, -\tfrac{7}{3}) \;=\; (\tfrac{11}{6}, -\tfrac{7}{3}, -\tfrac{17}{6})\,.\end{aligned}$$

(c) Hence the closest point on W to $(4, 2, -5)$ is $(\frac{11}{6}, -\frac{7}{3}, -\frac{17}{6})$ and the distance is

$$\begin{aligned}\|\mathbf{u} - \mathrm{proj}_{\mathbf{v}}\mathbf{u}\| &= \|(4,2,-5) - (\tfrac{11}{6}, -\tfrac{7}{3}, -\tfrac{17}{6})\| \;=\; \|(\tfrac{13}{6}, \tfrac{13}{3}, -\tfrac{13}{6})\| \\ &= \frac{13}{6}\|(1,2,-1)\| \;=\; \frac{13\sqrt{6}}{6}\,.\end{aligned}$$

8. (a) Observe that

$$\begin{aligned}(\lambda\mathbf{u} - \mathbf{v})\cdot(\lambda\mathbf{u} - \mathbf{v}) &= \lambda^2\mathbf{u}\cdot\mathbf{u} - 2\lambda\mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v} \\ &= \frac{(\mathbf{u}\cdot\mathbf{v})^2}{\|\mathbf{u}\|^4}\mathbf{u}\cdot\mathbf{u} - 2\frac{(\mathbf{u}\cdot\mathbf{v})^2}{\|\mathbf{u}\|^2} + \mathbf{v}\cdot\mathbf{v} \\ &= \frac{(\mathbf{u}\cdot\mathbf{v})^2}{\|\mathbf{u}\|^2} - 2\frac{(\mathbf{u}\cdot\mathbf{v})^2}{\|\mathbf{u}\|^2} + \mathbf{v}\cdot\mathbf{v} \\ &= \|\mathbf{v}\|^2 - \frac{(\mathbf{u}\cdot\mathbf{v})^2}{\|\mathbf{u}\|^2}\,.\end{aligned}$$

(b) If $\mathbf{v}$ is a scalar multiple of $\mathbf{u}$, say $\mathbf{v} = \mu\mathbf{u}$ for some scalar μ, then

$$|\mathbf{u}\cdot\mathbf{v}| \;=\; |\mathbf{u}\cdot\mu\mathbf{u}| \;=\; |\mu(\mathbf{u}\cdot\mathbf{u})| \;=\; |\mu|\|\mathbf{u}\|^2 \;=\; \|\mathbf{u}\|\big(|\mu|\|\mathbf{u}\|\big) \;=\; \|\mathbf{u}\|\|\mu\mathbf{u}\| \;=\; \|\mathbf{u}\|\|\mathbf{v}\|\,.$$

Conversely, suppose that $\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|$. Since $\mathbf{u} \neq \mathbf{0}$, we may consider the scalar $\lambda = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|^2}$. By part (a),

$$\begin{aligned}\|\lambda\mathbf{u} - \mathbf{v}\|^2 &= (\lambda\mathbf{u} - \mathbf{v})\cdot(\lambda\mathbf{u} - \mathbf{v}) \;=\; \|\mathbf{v}\|^2 - \frac{(\mathbf{u}\cdot\mathbf{v})^2}{\|\mathbf{u}\|^2} \\ &= \|\mathbf{v}\|^2 - \frac{\|\mathbf{u}\|^2\|\mathbf{v}\|^2}{\|\mathbf{u}\|^2} \;=\; \|\mathbf{v}\|^2 - \|\mathbf{v}\|^2 \;=\; 0\,,\end{aligned}$$

so that $\lambda\mathbf{u} - \mathbf{v} = \mathbf{0}$. Hence $\mathbf{v} = \lambda\mathbf{u}$, so that $\mathbf{v}$ is a scalar multiple of $\mathbf{u}$.

9. Let $\mathbf{u} = (u_1, \ldots, u_n), \mathbf{v} = (v_1, \ldots, v_n), \mathbf{w} = (w_1, \ldots, w_n) \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$. Then

$$\begin{aligned}(\mathbf{u} + \mathbf{v})\cdot\mathbf{w} &= (u_1 + v_1, \ldots, u_n + v_n)\cdot(w_1, \ldots, w_n) \\ &= (u_1 + v_1)w_1 + \ldots + (u_n + v_n)w_n \\ &= u_1w_1 + v_1w_1 + \ldots + u_nw_n + v_nw_n \\ &= u_1w_1 + \ldots + u_nw_n + v_1w_1 + \ldots + v_nw_n \\ &= \mathbf{u}\cdot\mathbf{w} + \mathbf{v}\cdot\mathbf{w}\,,\end{aligned}$$

and

$$\begin{aligned}\lambda(\mathbf{u}\cdot\mathbf{v}) &= \lambda(u_1v_1 + \ldots + u_nv_n) \\ &= (\lambda u_1)v_1 + \ldots + (\lambda u_n)v_n \;=\; u_1(\lambda v_1) + \ldots + u_n(\lambda v_n) \\ &= (\lambda u_1, \ldots, \lambda u_n)\cdot(v_1, \ldots, v_n) \;=\; (u_1, \ldots, u_n)\cdot(\lambda v_1, \ldots, \lambda v_n) \\ &= (\lambda\mathbf{u})\cdot\mathbf{v} \;=\; \mathbf{u}\cdot(\lambda\mathbf{v})\,.\end{aligned}$$

10. We have

$$\begin{aligned}\|\mathbf{u}+\mathbf{v}\|^2+\|\mathbf{u}-\mathbf{v}\|^2 &= (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v})+(\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v})\\ &= \mathbf{u}\cdot\mathbf{u}+2\mathbf{u}\cdot\mathbf{v}+\mathbf{v}\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{u}-2\mathbf{u}\cdot\mathbf{v}+\mathbf{v}\cdot\mathbf{v}\\ &= 2\mathbf{u}\cdot\mathbf{u}+2\mathbf{v}\cdot\mathbf{v} \;=\; 2\|\mathbf{u}\|^2+2\|\mathbf{v}\|^2\ .\end{aligned}$$

Interpreted in $\mathbb{R}^2$, this result says that the sum of the squares of the diagonals of a parallelogram add up to the sum of the squares of the lengths of the sides.

11. For $\mathbf{u},\mathbf{v}\in\mathbb{R}^n$, the Cauchy-Schwarz inequality gives

$$\begin{aligned}\|\mathbf{u}+\mathbf{v}\|^2 &= (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) \;=\; \mathbf{u}\cdot\mathbf{u}+2\mathbf{u}\cdot\mathbf{v}+\mathbf{v}\cdot\mathbf{v}\\ &= \|\mathbf{u}\|^2+2\mathbf{u}\cdot\mathbf{v}+\|\mathbf{v}\|^2 \;\le\; \|\mathbf{u}\|^2+2|\mathbf{u}\cdot\mathbf{v}|+\|\mathbf{v}\|^2\\ &\le \|\mathbf{u}\|^2+2\|\mathbf{u}\|\|\mathbf{v}\|+\|\mathbf{v}\|^2 \;=\; \big(\|\mathbf{u}\|+\|\mathbf{v}\|\big)^2\end{aligned}$$

so that, taking square roots of nonnegative numbers,

$$\|\mathbf{u}+\mathbf{v}\| \;\le\; \|\mathbf{u}\|+\|\mathbf{v}\|\ ,$$

which is the triangle inequality.

12. Using the triangle inequality, we have, for all $\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbb{R}^n$,

$$d(\mathbf{u},\mathbf{w}) \;=\; \|\mathbf{u}-\mathbf{w}\| \;=\; \|(\mathbf{u}-\mathbf{v})+(\mathbf{v}-\mathbf{w})\| \;\le\; \|\mathbf{u}-\mathbf{v}\|+\|\mathbf{v}-\mathbf{w}\| \;=\; d(\mathbf{u},\mathbf{v})+d(\mathbf{v},\mathbf{w})\ .$$

13. Since $\mathbf{v}$ and $\mathbf{w}$ are orthogonal, we have $\langle\mathbf{v},\mathbf{w}\rangle=0$. Hence, by properties of the inner product,

$$\begin{aligned}\|\mathbf{v}+\mathbf{w}\|^2 &= \langle\mathbf{v}+\mathbf{w},\mathbf{v}+\mathbf{w}\rangle \;=\; \langle\mathbf{v},\mathbf{v}\rangle+\langle\mathbf{v},\mathbf{w}\rangle+\langle\mathbf{w},\mathbf{v}\rangle+\langle\mathbf{w},\mathbf{w}\rangle\\ &= \langle\mathbf{v},\mathbf{v}\rangle+0+0+\langle\mathbf{w},\mathbf{w}\rangle \;=\; \|\mathbf{v}\|^2+\|\mathbf{w}\|^2\ .\end{aligned}$$

14. If $\mathbf{v}$ is orthogonal to everything in W then, in particular, $\mathbf{v}$ is orthogonal to each of $\mathbf{v}_1,\ldots,\mathbf{v}_n$, which are clearly elements of W. Suppose, conversely, that $\mathbf{v}$ is orthogonal to each of $\mathbf{v}_1,\ldots,\mathbf{v}_n$. Let $\mathbf{w}$ be any element of W, so

$$\mathbf{w} \;=\; \lambda_1\mathbf{v}_1+\ldots+\lambda_n\mathbf{v}_n$$

for some $\lambda_1,\ldots,\lambda_n\in\mathbb{R}$. Then, by properties of the inner product,

$$\langle\mathbf{v},\mathbf{w}\rangle \;=\; \lambda_1\langle\mathbf{v},\mathbf{v}_1\rangle+\ldots+\lambda_n\langle\mathbf{v},\mathbf{v}_n\rangle \;=\; \lambda_1 0+\ldots+\lambda_n 0 \;=\; 0\ ,$$

so that $\mathbf{v}$ is orthogonal to w.

15. Let $f,g,h\in V$ and $\lambda\in\mathbb{R}$. Then

$$\langle f,g\rangle \;=\; \int_a^b f(x)g(x)\,dx \;=\; \int_a^b g(x)f(x)\,dx \;=\; \langle g,f\rangle\ ,$$

$$\begin{aligned}\langle f+g,h\rangle &= \int_a^b (f(x)+g(x))h(x)\,dx \;=\; \int_a^b f(x)h(x)+g(x)h(x)\,dx\\ &= \int_a^b f(x)h(x)\,dx+\int_a^b g(x)h(x)\,dx \;=\; \langle f,h\rangle+\langle g,h\rangle\ ,\end{aligned}$$

$$\langle \lambda f, g\rangle \;=\; \int_a^b \lambda f(x)g(x)\,dx \;=\; \lambda\int_a^b f(x)g(x)\,dx \;=\; \lambda\langle f,g\rangle\,,$$

and

$$\langle f, f\rangle \;=\; \int_a^b f(x)f(x)\,dx \;=\; \int_a^b \big(f(x)\big)^2\,dx \;\geq\; 0\,,$$

since, in this last case, the integrand is nonnegative throughout the interval. Certainly

$$\langle \mathbf{0}, \mathbf{0}\rangle \;=\; \int_a^b \mathbf{0}(x)\mathbf{0}(x)\,dx \;=\; \int_a^b 0\,dx \;=\; 0\,.$$

Suppose that $f \in V$ such that $\langle f, f\rangle = 0$, so that $\int_a^b \big(f(x)\big)^2\,dx = 0$. We show that $f = \mathbf{0}$. Suppose to the contrary that $f \neq \mathbf{0}$. Then $f(x) \neq 0$ for some $x \in [a, b]$. By continuity of f, there is some $\varepsilon > 0$ and some interval I of positive width $\delta > 0$ such that

$$|f(y)| \;\geq\; \epsilon \qquad (\forall y \in I)\,.$$

But then

$$\int_a^b \big(f(x)\big)^2\,dx \;\geq\; \epsilon^2\delta \;>\; 0\,,$$

which contradicts that $\int_a^b \big(f(x)\big)^2\,dx = 0$. Hence $f = \mathbf{0}$. This proves that $\langle f, f\rangle = 0$ if and only if $f = \mathbf{0}$, and completes the verification that V is an inner product space.

16. (a) We have

$$\langle f, f\rangle \;=\; \int_{-1}^1 1^2\,dx \;=\; 2\,, \quad \langle g, g\rangle \;=\; \int_{-1}^1 x^2\,dx \;=\; \frac{2}{3}, \quad \langle h, h\rangle \;=\; \int_{-1}^1 x^6\,dx \;=\; \frac{2}{7}\,,$$

so that $\|f\| = \sqrt{2}$, $\|g\| = \sqrt{\frac{2}{3}}$ and $\|h\| = \sqrt{\frac{2}{7}}$. Further

$$\langle f, g\rangle \;=\; \int_{-1}^1 x\,dx \;=\; 0\,, \quad \langle f, h\rangle \;=\; \int_{-1}^1 x^3\,dx \;=\; 0\,, \quad \langle g, h\rangle \;=\; \int_{-1}^1 x^4\,dx \;=\; \frac{2}{5}\,.$$

Observe also that

$$\|f - g\|^2 \;=\; \int_{-1}^1 (1-x)^2\,dx \;=\; \left[-\frac{(1-x)^3}{3}\right]_{-1}^1 \;=\; \frac{8}{3}\,,$$

so that the distance between f and g is $\sqrt{\frac{8}{3}} = 2\sqrt{\frac{2}{3}}$. The pairs f, g and f, h are orthogonal.

(b) We have $\langle p, q\rangle = 0$ if and only if $\int_{-1}^1 x^{m+n}\,dx = 0$, which occurs if and only if

$$\left[\frac{x^{m+n+1}}{m+n+1}\right]_{-1}^1 \;=\; 0\,,$$

which, in turn, occurs if and only if $1^{m+n+1} = (-1)^{m+n+1}$, that is, $m + n + 1$ is even, or equivalently, $m + n$ is odd. Thus p and q are orthogonal if and only if m and n have different parity.