The University of Sydney
School of Mathematics and Statistics

Calculus Tutorial 3 (Week 4)

MATH1062/MATH1023: Mathematics 1B (Calculus) — Semester 2, 2024

Questions marked with * are harder questions.

Material covered

(1) Applications of separable equations

Summary of essential material

Recall that we can use the theory of separable differential equations to model growth and pollution problems.

Questions to complete during the tutorial

1. The rate of change of N radioactive atoms is proportional to N.

(a) Write a differential equation for the number of radioactive atoms N.

(b) Solve this differential equation for N in terms of N_0, where N_0 is the value of N at time $t = 0$.

(c) Calculate the content of radioactive carbon C^{14} (as a percentage of the living value) of a fossilised tree that is claimed to be 3000 years old. Note that the half-life of C^{14} is 5730 years.

2. An animal population has a net growth rate per unit population which varies with the seasons, being positive in summer and negative in winter. Let $x(t)$ be the size of the population at time t, which is measured in years. The following differential equation is suggested as a model for this situation:

$$\frac{dx}{dt} = (k\cos 2\pi t)x \quad (k \text{ a positive constant}).$$

(a) What is the period of $\cos(2\pi t)$?

(b) What time of year do you think $t = 0$ represents ?

(c) Solve the equation to find $x(t)$, given that $x = x_0$ at $t = 0$.

(d) Does $x(t)$ have a limiting value as $t \to \infty$?

(e) What are the maximum and minimum values of x and when do they occur?

***3.** A trout farmer believes that, if she doesn't remove any fish from a certain tank, the population of fish is modelled by the differential equation

$$\frac{dP}{dt} = 2P - 0.01P^2,$$

where P is the number of fish after t years.

(a) Suppose the farmer initially stocks the tank with 50 fish.
   (i) What is the maximum number of fish the tank can support long-term and according to this model?
   (ii) Find P for which the population is increasing most rapidly.
   (iii) Find an explicit formula for P in terms of t.
   (iv) How long will it be before there are 199 fish in the tank?
   (v) Sketch a graph of $y = P(t)$.

(b) Now suppose that the farmer initially stocks the tank with 250 fish. Find the solution $P(t)$, and sketch its graph.

(c) If the trout farmer were to remove 75 fish from the tank each year, then the differential equation modelling population size would be

$$\frac{dP}{dt} = 2P - 0.01P^2 - 75.$$

   (i) Find a solution to this differential equation, supposing that the farmer initially stocks the tank with 60 fish.
   (ii) According to this model, what is the size of the fish population in the long-term?
   (iii) What should the farmer expect to happen if she initially stocks the tank with 40 fish?

4. When people smoke, carbon monoxide is released into the air. In a room of volume $50\,\text{m}^3$, smokers introduce air containing $0.05\,\text{mg/m}^3$ of carbon monoxide at the rate of $0.002\,\text{m}^3/\text{min}$. Assume that the smoky air mixes immediately with the rest of the air, and that the mixture is pumped through an air purifier at a rate of $0.002\,\text{m}^3/\text{min}$. The purifier removes all the carbon monoxide from the air passing through it.

(a) Write a differential equation for $m(t)$, the mass of carbon monoxide in the room at time t, where t is measured in minutes.
Hint: $\dfrac{dm}{dt}$ = mass rate in − mass rate out, see lecture notes.

(b) Solve the differential equation, assuming that there was no carbon monoxide in the room initially.

(c) What happens to the value of $m(t)$ in the long run?

(d) It is dangerous for people to be in the room if the mass of carbon monoxide per unit volume reaches $0.001\,\text{mg/m}^3$. How long does it take for this to happen?

Short answers to selected exercises

1. (c) 69.6%

2. (d) $x = x_0 e^{(k\sin(2\pi t))/2\pi}$

3. (a)
   (i) 200
   (ii) 100
   (iii) $P(t) = \dfrac{200}{1 + 3e^{-2t}}$
   (iv) 3.19 years

(b) $P = 200/(1 - 0.2e^{-2t})$

(c)

(i) $P = \dfrac{150(1 + 3e^{-t})}{1 + 9e^{-t}}$ (ii) 150 (iii) 0