

Lecture 3: Analytic Geometry

CSE4130: 기초머신러닝

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Roadmap

- (1) Norms
- (2) Inner Products
- (3) Lengths and Distances
- (4) Angles and Orthogonality
- (5) Orthonormal Basis
- (6) Orthogonal Complement
- (7) Inner Product of Functions
- (8) Orthogonal Projections
- (9) Rotations

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Norm

- A notion of the length of vectors
- **Definition.** A norm on a vector space V is a function $\|\cdot\| : V \mapsto \mathbb{R}$, such that for all $\lambda \in \mathbb{R}$ the following hold:
 - **Absolutely homogeneous:** $\|\lambda \mathbf{x}\| = |\lambda| \|\mathbf{x}\|$
 - **Triangle inequality:** $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$
 - **Positive definite:** $\|\mathbf{x}\| \geq 0$ and $\|\mathbf{x}\| \iff \mathbf{x} = \mathbf{0}$

Example for $V \in \mathbb{R}^n$

- **Manhattan Norm** (also called ℓ_1 norm) For $\mathbf{x} = [x_1, \dots, x_n] \in \mathbb{R}^n$,

$$\|\mathbf{x}\|_1 := \sum_{i=1}^n |x_i|$$

- **Euclidean Norm** (also called ℓ_2 norm) For $\mathbf{x} \in \mathbb{R}^n$,

$$\|\mathbf{x}\|_2 := \sqrt{\sum_{i=1}^n x_i^2} = \sqrt{\mathbf{x}^T \mathbf{x}}$$

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Motivation

- Need to talk about the length of a vector and the angle or distance between two vectors, where vectors are defined in abstract vector spaces
- To this end, we define the notion of **inner product** in an abstract manner.
- Dot product: A kind of inner product in vector space \mathbb{R}^n . $\mathbf{x}^T \mathbf{y} = \sum_{i=1}^n x_i y_i$
- **Question.** How can we generalize this and do a similar thing in some other vector spaces?

Formal Definition (1)

- A *bilinear mapping* Ω is a mapping with two arguments, and it is linear in each argument.
- **Definition.** Let V be a vector space and $\Omega : V \times V \rightarrow \mathbb{R}$ be a bilinear mapping that takes two vectors and maps them onto a real number. Then
 - Ω is called *symmetric* if $\Omega(\mathbf{x}, \mathbf{y}) = \Omega(\mathbf{y}, \mathbf{x})$ for all $\mathbf{x}, \mathbf{y} \in V$, i.e., the order of the arguments does not matter.
 - Ω is called *positive definite* if $\forall \mathbf{x} \in V \setminus \{\mathbf{0}\} : \Omega(\mathbf{x}, \mathbf{x}) > 0, \quad \Omega(\mathbf{0}, \mathbf{0}) = 0$.

Formal Definition (2)

- An inner product is a mapping $\langle \cdot, \cdot \rangle : V \times V \mapsto \mathbb{R}$ that satisfies the following conditions for all vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and all scalars $\lambda \in \mathbb{R}$:
 1. $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$
 2. $\langle \lambda \mathbf{v}, \mathbf{w} \rangle = \lambda \langle \mathbf{v}, \mathbf{w} \rangle$
 3. $\langle \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{w}, \mathbf{v} \rangle$
 4. $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$ and equal iff $\mathbf{v} = \mathbf{0}$
- The pair $(V, \langle \cdot, \cdot \rangle)$ is called an inner product space.

Example

- **Example.** $V = \mathbb{R}^n$ and the dot product $\langle \mathbf{x}, \mathbf{y} \rangle := \mathbf{x}^T \mathbf{y}$
- **Example.** $V = \mathbb{R}^2$ and $\langle \mathbf{x}, \mathbf{y} \rangle := x_1 y_1 - (x_1 y_2 + x_2 y_1) + 2x_2 y_2$
- **Example.** $V = \{\text{continuous functions in } \mathbb{R} \text{ over } [a, b]\}$, $\langle u, v \rangle := \int_a^b u(x)v(x)dx$

Symmetric, Positive Definite Matrix

- A square matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ that satisfies the following is called **symmetric, positive definite** (or just positive definite):

$$\forall \mathbf{x} \in V \setminus \{\mathbf{0}\} : \mathbf{x}^T \mathbf{A} \mathbf{x} > 0.$$

If only \geq in the above holds, then \mathbf{A} is called **symmetric, positive semidefinite**.

- $\mathbf{A}_1 = \begin{pmatrix} 9 & 6 \\ 6 & 5 \end{pmatrix}$ is positive definite.
- $\mathbf{A}_2 = \begin{pmatrix} 9 & 6 \\ 6 & 3 \end{pmatrix}$ is not positive definite.

Inner Product and Positive Definite Matrix (1)

- Consider an n -dimensional vector space V with an inner product $\langle \cdot, \cdot \rangle$ and an ordered basis $B = (\mathbf{b}_1, \dots, \mathbf{b}_n)$ of V .
- Any $\mathbf{x}, \mathbf{y} \in V$ can be represented as: $\mathbf{x} = \sum_{i=1}^n \psi_i \mathbf{b}_i$ and $\mathbf{y} = \sum_{j=1}^n \lambda_j \mathbf{b}_j$ for some ψ_i and λ_j , $i, j = 1, \dots, n$.

$$\langle \mathbf{x}, \mathbf{y} \rangle = \left\langle \sum_{i=1}^n \psi_i \mathbf{b}_i, \sum_{j=1}^n \lambda_j \mathbf{b}_j \right\rangle = \sum_{i=1}^n \sum_{j=1}^n \psi_i \langle \mathbf{b}_i, \mathbf{b}_j \rangle \lambda_j = \hat{\mathbf{x}}^T \mathbf{A} \hat{\mathbf{y}},$$

where $\mathbf{A}_{ij} = \langle \mathbf{b}_i, \mathbf{b}_j \rangle$ and $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are the coordinates w.r.t. B .

Inner Product and Positive Definite Matrix (2)

- Then, if $\forall \mathbf{x} \in V \setminus \{\mathbf{0}\} : \mathbf{x}^T \mathbf{A} \mathbf{x} > 0$ (i.e., \mathbf{A} is symmetric, positive definite), $\hat{\mathbf{x}}^T \mathbf{A} \hat{\mathbf{y}}$ legitimately defines an inner product (w.r.t. B)
- Properties
 - The kernel of \mathbf{A} is only $\{\mathbf{0}\}$, because $\mathbf{x}^T \mathbf{A} \mathbf{x} > 0$ for all $\mathbf{x} \neq \mathbf{0} \implies \mathbf{A} \mathbf{x} \neq \mathbf{0}$ if $\mathbf{x} \neq \mathbf{0}$.
 - The diagonal elements a_{ii} of \mathbf{A} are all positive, because $a_{ii} = \mathbf{e}_i^T \mathbf{A} \mathbf{e}_i > 0$.

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Length

- Inner product naturally induces a norm by defining:

$$\|x\| := \sqrt{\langle x, x \rangle}$$

- Not every norm is induced by an inner product
- **Cachy-Schwarz inequality.** For the induced norm by the inner product,

$$|\langle x, y \rangle| \leq \|x\| \|y\|$$

Example 3.5 (Lengths of Vectors)

Distance

- Now, we can introduce a notion of distance using a norm as:

Distance. $d(\mathbf{x}, \mathbf{y}) := \|\mathbf{x} - \mathbf{y}\| = \sqrt{\langle \mathbf{x} - \mathbf{y}, \mathbf{x} - \mathbf{y} \rangle}$

- If the dot product is used as an inner product in \mathbb{R}^n , it is **Euclidian distance**.
- **Note.** The distance between two vectors does **NOT** necessarily require the notion of an inner product. Norm is just sufficient.
- Generally, if the following is satisfied, it is a suitable notion of distance, called **metric**.
 - **Positive definite.** $d(\mathbf{x}, \mathbf{y}) \geq 0$ for all \mathbf{x}, \mathbf{y} and $d(\mathbf{x}, \mathbf{y}) = 0 \iff \mathbf{x} = \mathbf{y}$
 - **Symmetric.** $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x})$
 - **Triangle inequality.** $d(\mathbf{x}, \mathbf{z}) \leq d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z})$

Angle, Orthogonal, and Orthonormal

- Using C-S inequality,

$$-1 \leq \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\|\mathbf{x}\| \|\mathbf{y}\|} \leq 1$$

- Then, there exists a unique $\omega \in [0, \pi]$ with

$$\cos \omega = \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\|\mathbf{x}\| \|\mathbf{y}\|}$$

- We define ω as the **angle** between \mathbf{x} and \mathbf{y} .
- **Definition.** If $\langle \mathbf{x}, \mathbf{y} \rangle = 0$, in other words their angle is $\pi/2$, we say that they are **orthogonal**, denoted by $\mathbf{x} \perp \mathbf{y}$. Additionally, if $\|\mathbf{x}\| = \|\mathbf{y}\| = 1$, they are **orthonormal**.

Example

- Orthogonality is defined by a given inner product. Thus, different inner products may lead to different results about orthogonality.
- **Example.** Consider two vectors $\mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$
- Using the dot product as the inner product, they are orthogonal.
- However, using $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{y}$, they are not orthogonal.

$$\cos \omega = \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\|\mathbf{x}\| \|\mathbf{y}\|} = -\frac{1}{3} \implies \omega \approx 1.91 \text{ rad} \approx 109.5^\circ$$

Orthogonal Matrix

- **Definition.** A square matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ is an **orthogonal matrix**, iff its columns (or rows) are **orthonormal** so that

$$\mathbf{A}\mathbf{A}^T = \mathbf{I} = \mathbf{A}^T\mathbf{A}, \text{ implying } \mathbf{A}^{-1} = \mathbf{A}^T.$$

- We can use $\mathbf{A}^{-1} = \mathbf{A}^T$ for the definition of orthogonal matrices.
- Fact 1. \mathbf{A}, \mathbf{B} : orthogonal $\implies \mathbf{AB}$: orthogonal
- Fact 2. \mathbf{A} : orthogonal $\implies \det(\mathbf{A}) = \pm 1$
- The linear mapping Φ by orthogonal matrices preserve **length** and **angle** (for the dot product)

$$\|\Phi(\mathbf{A})\|^2 = \|\mathbf{Ax}\|^2 = (\mathbf{Ax})^T(\mathbf{Ax}) = \mathbf{x}^T\mathbf{A}^T\mathbf{Ax} = \mathbf{x}^T\mathbf{x} = \|\mathbf{x}\|^2$$

$$\cos \omega = \frac{(\mathbf{Ax})^T(\mathbf{Ay})}{\|\mathbf{Ax}\| \|\mathbf{Ay}\|} = \frac{\mathbf{x}^T\mathbf{A}^T\mathbf{Ay}}{\sqrt{\mathbf{x}^T\mathbf{A}^T\mathbf{Ax} \mathbf{y}^T\mathbf{A}^T\mathbf{Ay}}} = \frac{\mathbf{x}^T\mathbf{y}}{\|\mathbf{x}\| \|\mathbf{y}\|}$$

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Orthonormal Basis

- Basis that is orthonormal, i.e., they are all orthogonal to each other and their lengths are 1.
- Standard basis in \mathbb{R}^n , $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$, is orthonormal.
- **Question.** How to obtain an orthonormal basis?

1. Use Gaussian elimination to find a basis for a vector space spanned by a set of vectors.
 - Given a set $\{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ of unorthogonal and unnormalized basis vectors. Apply Gaussian elimination to the augmented matrix $(\mathbf{B}\mathbf{B}^T | \mathbf{B})$
2. Constructive way: Gram-Schmidt process (we will cover this later)

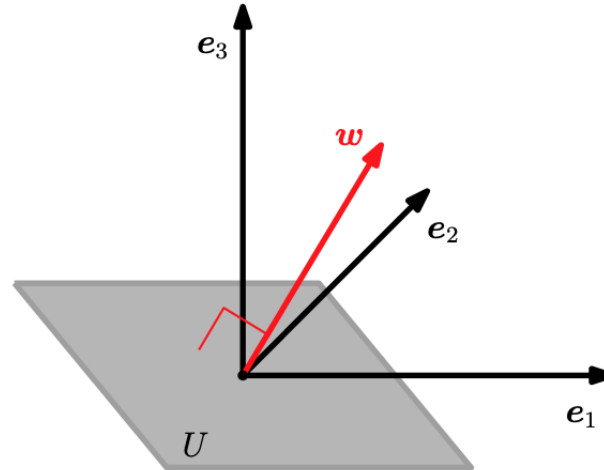
Orthogonal Complement (1)

- Consider D -dimensional vector space V and M -dimensional subspace $W \subset V$. The **orthogonal complement** U^\perp is a $(D - M)$ -dimensional subspace of V and contains all vectors in V that are orthogonal to every vector in U .
- $U \cap U^\perp = \mathbf{0}$
- Any vector $\mathbf{x} \in V$ can be uniquely decomposed into:

$$\mathbf{x} = \sum_{m=1}^M \lambda_m \mathbf{b}_m + \sum_{j=1}^{D-M} \psi_j \mathbf{b}_j^\perp, \quad \lambda_m, \psi_j \in \mathbb{R},$$

where $(\mathbf{b}_1, \dots, \mathbf{b}_M)$ and $(\mathbf{b}_1^\perp, \dots, \mathbf{b}_{D-M}^\perp)$ are the **bases** of U and U^\perp , respectively.

Orthogonal Complement (2)



- The vector \mathbf{w} with $\|\mathbf{w}\| = 1$, which is orthogonal to U , is the basis of U^\perp .
- Such \mathbf{w} is called **normal vector** to U .
- For a linear mapping represented by a matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$, the solution space of $\mathbf{A}\mathbf{x} = 0$ is $\text{row}(\mathbf{A})^\perp$, where $\text{row}(\mathbf{A})$ is the row space of \mathbf{A} (i.e., span of row vectors). In other words, $\text{row}(\mathbf{A})^\perp = \ker(\mathbf{A})$

Inner Product of Functions

- **Remind:** $V = \{\text{continuous functions in } \mathbb{R} \text{ over } [a, b]\}$, the following is a proper inner product.

$$\langle u, v \rangle := \int_a^b u(x)v(x)dx$$

- **Example.** Choose $u(x) = \sin(x)$ and $v(x) = \cos(x)$, where we select $a = -\pi$ and $b = \pi$. Then, since $f(x) = u(x)v(x)$ is odd (i.e., $f(-x) = -f(x)$),

$$\int_{-\pi}^{\pi} u(x)v(x)dx = 0.$$

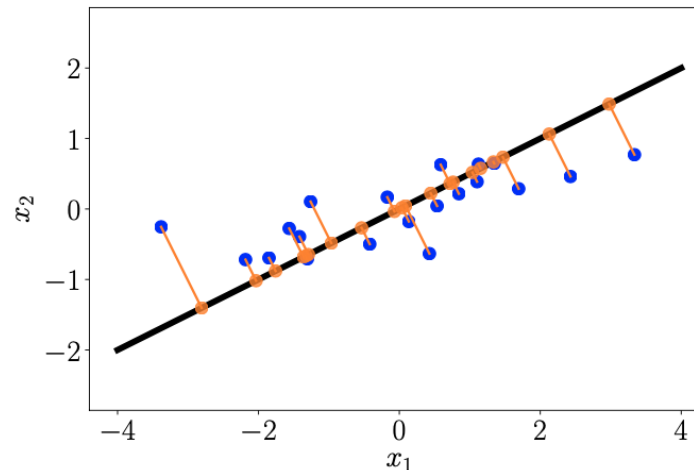
- Thus, u and v are orthogonal.
- Similarly, $\{1, \cos(x), \cos(2x), \cos(3x), \dots, \}$ is orthogonal over $[-\pi, \pi]$.

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Projection: Motivation

- Big data: high dimensional
- However, most information is contained in a few dimensions
- **Projection**: A process of reducing the dimensions (hopefully) without loss of much information¹
- **Example**. Projection of 2D dataset onto 1D subspace



¹In **Chap.10**, we will formally study this with the topic of PCA (Principal Component Analysis).

Definition of Projection

- **Definition.** Let V be a vector space and $U \in V$ a subspace of V . A linear mapping $\pi : V \rightarrow U$ is called a *projection* if $\pi^2 = \pi \circ \pi = \pi$
- Since linear mappings can be expressed by transformation matrices, the preceding definition applies equally to a special kind of transformation matrices, the *projection matrices* \mathbf{P}_π , which exhibit the property that $\mathbf{P}_\pi^2 = \mathbf{P}_\pi$.

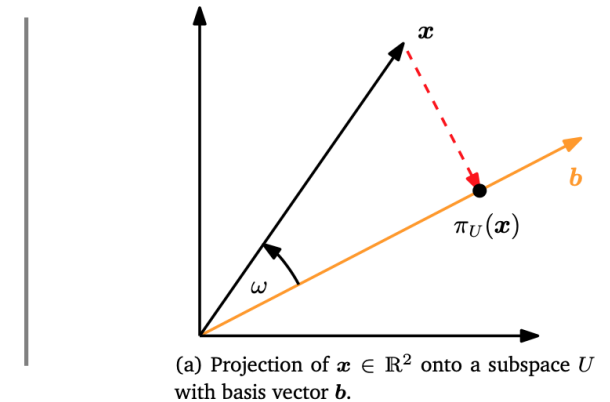
Projection onto Lines (1D Subspaces)

- Consider a 1D subspace $U \subset \mathbb{R}^n$ spanned by the basis \mathbf{b} .
- For $\mathbf{x} \in \mathbb{R}^n$, what is its projection $\pi_U(\mathbf{x})$ onto U (assume the dot product)?

$$\langle \mathbf{x} - \pi_U(\mathbf{x}), \mathbf{b} \rangle = 0 \xleftrightarrow{\pi_U(\mathbf{x}) = \lambda \mathbf{b}} \langle \mathbf{x} - \lambda \mathbf{b}, \mathbf{b} \rangle = 0$$
$$\implies \lambda = \frac{\langle \mathbf{b}, \mathbf{x} \rangle}{\|\mathbf{b}\|^2} = \frac{\mathbf{b}^\top \mathbf{x}}{\|\mathbf{b}\|^2}, \text{ and } \pi_U(\mathbf{x}) = \lambda \mathbf{b} = \frac{\mathbf{b}^\top \mathbf{x}}{\|\mathbf{b}\|^2} \mathbf{b}$$

- Projection matrix $\mathbf{P}_\pi \in \mathbb{R}^{n \times n}$ in $\pi_U(\mathbf{x}) = \mathbf{P}_\pi \mathbf{x}$

$$\pi_U(\mathbf{x}) = \lambda \mathbf{b} = \mathbf{b} \lambda = \frac{\mathbf{b} \mathbf{b}^\top}{\|\mathbf{b}\|^2} \mathbf{x}, \quad \mathbf{P}_\pi = \frac{\mathbf{b} \mathbf{b}^\top}{\|\mathbf{b}\|^2}$$

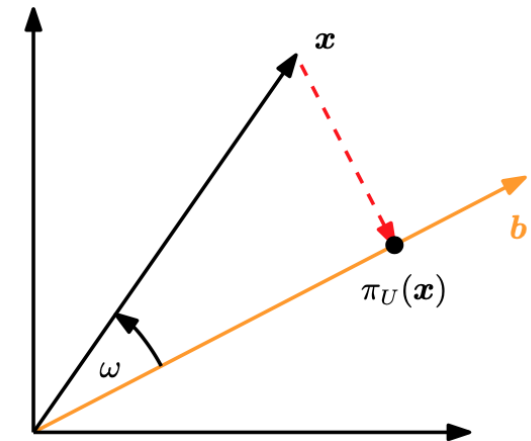


Inner Product and Projection

- We project \mathbf{x} onto \mathbf{b} , and let $\pi_{\mathbf{b}}(\mathbf{x})$ be the projected vector.
- **Question.** Understanding the inner product $\langle \mathbf{x}, \mathbf{b} \rangle$ from the projection perspective?

$$\langle \mathbf{x}, \mathbf{b} \rangle = \|\pi_{\mathbf{b}}(\mathbf{x})\| \times \|\mathbf{b}\|$$

- In other words, the inner product of \mathbf{x} and \mathbf{b} is the product of (length of the projection of \mathbf{x} onto \mathbf{b}) \times (length of \mathbf{b})



(a) Projection of $\mathbf{x} \in \mathbb{R}^2$ onto a subspace U with basis vector \mathbf{b} .

Example

- $\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

$$\mathbf{P}_\pi = \frac{\mathbf{b}\mathbf{b}^\top}{\|\mathbf{b}\|^2} = \frac{1}{9} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} (1 \ 2 \ 2) = \frac{1}{9} \begin{pmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{pmatrix}$$

For $\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$,

$$\pi_U(\mathbf{x}) = \mathbf{P}_\pi \mathbf{x} = \frac{1}{9} \begin{pmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{9} \begin{pmatrix} 5 \\ 10 \\ 10 \end{pmatrix} \in \text{span}\left[\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}\right]$$

Projection onto General Subspaces

- $\mathbb{R}^n \rightarrow 1\text{-Dim}$

- A basis vector \mathbf{b} in 1D subspace

$$\pi_U(\mathbf{x}) = \frac{\mathbf{b}\mathbf{b}^\top \mathbf{x}}{\mathbf{b}^\top \mathbf{b}}, \quad \lambda = \frac{\mathbf{b}^\top \mathbf{x}}{\mathbf{b}^\top \mathbf{b}}$$

$$\mathbf{P}_\pi = \frac{\mathbf{b}\mathbf{b}^\top}{\mathbf{b}^\top \mathbf{b}}$$

- $\mathbb{R}^n \rightarrow m\text{-Dim}, (m < n)$

- A basis matrix

$$\mathbf{B} = (\mathbf{b}_1, \dots, \mathbf{b}_m) \in \mathbb{R}^{n \times m}$$

$$\pi_U(\mathbf{x}) = \mathbf{B}(\mathbf{B}^\top \mathbf{B})^{-1} \mathbf{B}^\top \mathbf{x}, \quad \boldsymbol{\lambda} = (\mathbf{B}^\top \mathbf{B})^{-1} \mathbf{B}^\top \mathbf{x}$$

$$\mathbf{P}_\pi = \mathbf{B}(\mathbf{B}^\top \mathbf{B})^{-1} \mathbf{B}^\top$$

- $\lambda \in \mathbb{R}^1$ and $\boldsymbol{\lambda} \in \mathbb{R}^m$ are the coordinates in the projected spaces, respectively.
- $(\mathbf{B}^\top \mathbf{B})^{-1} \mathbf{B}^\top$ is called **pseudo-inverse**.
- How to derive is analogous to the case of 1-D lines (see pp. 71).

Example: Projection onto 2D Subspace

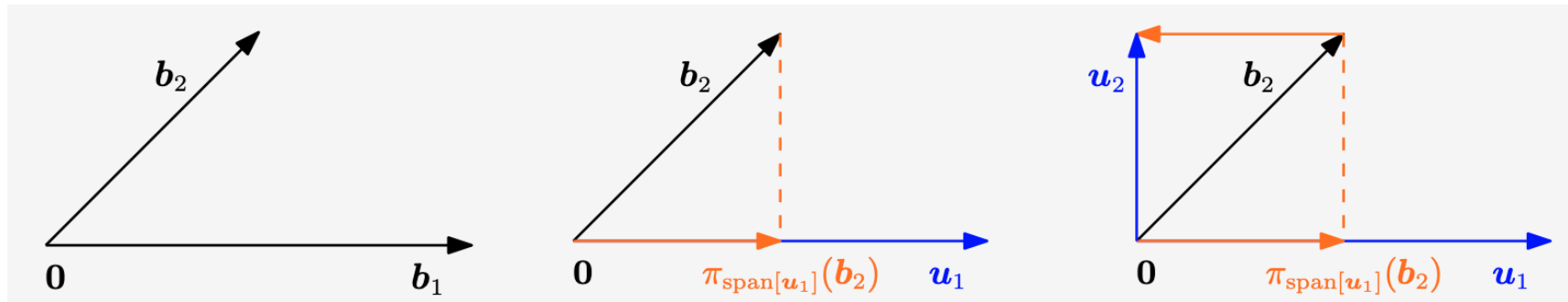
- $U = \text{span}\left[\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}\right] \subset \mathbb{R}^3$ and $\mathbf{x} = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}$. Check that $\{(1 \ 1 \ 1)^\top, (0 \ 1 \ 2)^\top\}$ is a basis.
- Let $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}$. Then, $\mathbf{B}^\top \mathbf{B} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 3 & 5 \end{pmatrix}$
- Can see that $\mathbf{P}_\pi = \mathbf{B}(\mathbf{B}^\top \mathbf{B})^{-1} \mathbf{B}^\top = \frac{1}{6} \begin{pmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{pmatrix}$, and
$$\pi_U(\mathbf{x}) = \frac{1}{6} \begin{pmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{pmatrix} \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix}$$

Gram-Schmidt Orthogonalization Method (G-S method)

- Constructively transform any basis $(\mathbf{b}_1, \dots, \mathbf{b}_n)$ of n -dimensional vector space V into an orthogonal/orthonormal basis $(\mathbf{u}_1, \dots, \mathbf{u}_n)$ of V
- Iteratively construct as follows

$$\mathbf{u}_1 := \mathbf{b}_1$$

$$\mathbf{u}_k := \mathbf{b}_k - \pi_{\text{span}[\mathbf{u}_1, \dots, \mathbf{u}_{k-1}]}(\mathbf{b}_k), \quad k = 2, \dots, n \quad (*)$$



Example: G-S method

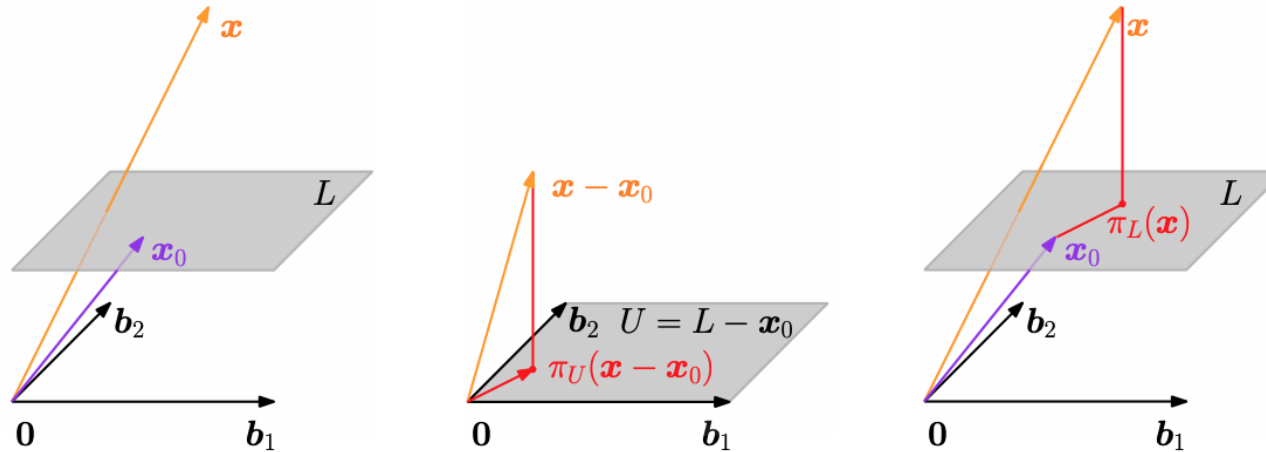
- A basis $(\mathbf{b}_1, \mathbf{b}_2) \in \mathbb{R}^2$, $\mathbf{b}_1 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and $\mathbf{b}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

- $\mathbf{u}_1 = \mathbf{b}_1 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and

$$\mathbf{u}_2 = \mathbf{b}_2 - \pi_{\text{span}[\mathbf{u}_1]}(\mathbf{b}_2) = \frac{\mathbf{u}_1 \mathbf{u}_2^\top}{\|\mathbf{u}_1\|} \mathbf{b}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- \mathbf{u}_1 and \mathbf{u}_2 are orthogonal. If we want them to be orthonormal, then just normaliation would do the job.

Projection onto Affine Subspaces



- Affine space: $L = \mathbf{x}_0 + U$
- Affine subspaces are not vector spaces
- Idea: (i) move \mathbf{x} to a point in U , (ii) do the projection, (iii) move back to L

$$\pi_L(\mathbf{x}) = \mathbf{x}_0 + \pi_U(\mathbf{x} - \mathbf{x}_0)$$

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Rotation

- Length and angle preservation: two properties of linear mappings with **orthogonal matrices**. Let's look at some of their special cases.
- A linear mapping that rotates the given coordinate system by an angle θ .
- Basis change
- $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}$ and $\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix}$
- Rotation matrix $\mathbf{R}(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$
- Properties
 - Preserves distance: $\|\mathbf{x} - \mathbf{y}\| = \|\mathbf{R}_\theta(\mathbf{x}) - \mathbf{R}_\theta(\mathbf{y})\|$
 - Preserves angle

Questions?

References

- [1] This lecture slide is mainly based upon <https://yung-web.github.io/home/courses/mathml.html> (made by Prof. Yung Yi, KAIST EE)
- [2] Marc Peter Deisenroth, A. Aldo Faisal, and Cheng Soon Ong. Mathematics for machine learning. Cambridge University Press, 2020.