An Hamiltonian that does swap between $|1,0\rangle$ & $|0,1\rangle$ states is the Jaynes Cummings Hamiltonian

considering 2 two-level systems

$$H_{int} = \sigma_1^+ \sigma_2^- + \sigma_1^- \sigma_2^+$$

$$= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{array}{c} \begin{matrix} |00\rangle & |01\rangle & |10\rangle & |11\rangle \end{matrix} \\ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \begin{array}{c} \\ \langle 00| \\ \langle 01| \\ \langle 10| \\ \langle 11| \end{array}$$

$|10\rangle\langle 01|$

$|01\rangle\langle 10|$

For our case the total Hamiltonian is

(Treating the Qubit & resonator as TLS)

$$H_{total} = \hbar \omega_q \frac{\sigma_q^z}{2} + \hbar \omega_r \frac{\sigma_r^z}{2} + \hbar g \left(\sigma_q^+ \sigma_r^- + \sigma_q^- \sigma_r^+ \right)$$

$$+ \hbar \Omega_q(t) \cdot \sigma_q^x + \hbar \Omega_r(t) \sigma_r^x$$

The control Hamiltonian looks like

$$\hbar \Omega_q(t) \cdot \sigma_q^x \otimes \mathbb{I}_r + \hbar \Omega_r(t) \cdot \mathbb{I}_q \otimes \sigma_r^x$$

$$= \hbar \Omega_q(t) \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} + \hbar \Omega_r(t) \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$= \hbar \begin{array}{c} \begin{array}{cccc} |00\rangle & |01\rangle & |10\rangle & |11\rangle \end{array} \\ \begin{pmatrix} 0 & \Omega_r(t) & \Omega_q(t) & 0 \\ \Omega_r(t) & 0 & 0 & \Omega_q(t) \\ \Omega_q(t) & 0 & 0 & \Omega_r(t) \\ 0 & \Omega_q(t) & \Omega_r(t) & 0 \end{pmatrix} \end{array} \begin{array}{c} \\ \langle 00| \\ \langle 01| \\ \langle 10| \\ \langle 11| \end{array}$$

Checking if it makes sense.

$|01\rangle\langle 00|$ → resonator excitation
$|00\rangle\langle 01|$ → resonator de-excitation

} Both by $\Omega_r(t)$

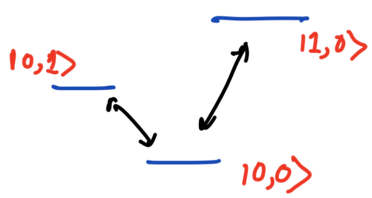
Similarly for qubit excitation & de-excitation

$|10\rangle\langle 00|$ & $|00\rangle\langle 10|$

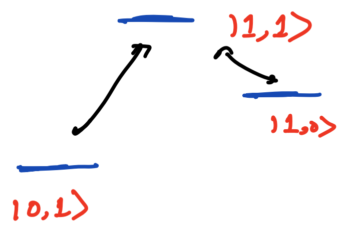
Now same for $|11\rangle\langle 01|$ → Qubit excitation

$|01\rangle\langle 11|$ → Qubit deexcitation

So the only way to go from $|10\rangle$ to $|01\rangle$ is by going via $|11\rangle$ or $|00\rangle$



or



(Assuming $\omega_q > \omega_r$)

Vee-type

Λ-type

Now increasing the number of levels of the Qubit
(To verify if $|2,0\rangle \rightarrow |0,1\rangle$ transition is allowed)

$$H_{drive} = \hbar\,\Omega_q(t)\left(a + a^{+}\right) + \hbar\,\Omega_r(t)\,\sigma_r^{x}$$

$$= \hbar\,\Omega_q(t)\left[\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}\right] \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$+ \hbar\,\Omega_r(t)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \hbar\Omega_q(t)\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$+ \hbar\Omega_r(t)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \hbar\Omega_q(t)\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & \sqrt{2} & 0 \\ 0 & 1 & 0 & 0 & 0 & \sqrt{2} \\ 0 & 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 & 0 \end{pmatrix}$$

$$+ \hbar\Omega_r(t)\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

$$= \hbar \begin{array}{c} \begin{array}{cccccc} |00\rangle & |01\rangle & |10\rangle & |11\rangle & |20\rangle & |21\rangle \end{array} \\ \begin{pmatrix} 0 & 0 & \Omega_q(t) & \Omega_r(t) & 0 & 0 \\ 0 & 0 & 0 & \Omega_q(t) & \Omega_r(t) & 0 \\ \Omega_q(t) & 0 & 0 & 0 & \sqrt{2}\Omega_q(t) & \Omega_r(t) \\ \Omega_r(t) & \Omega_q(t) & 0 & 0 & 0 & \sqrt{2}\Omega_q(t) \end{pmatrix} \end{array} \begin{array}{c} \\ \langle 00| \\ \langle 01| \\ \langle 10| \\ \langle 11| \end{array}$$

$$
\begin{array}{c}
\begin{pmatrix}
0 & \Omega_r(t) & \sqrt{2}\,\Omega_q(t) & 0 & 0 & 0 \\
0 & 0 & \Omega_r(t) & \sqrt{2}\,\Omega_q(t) & 0 & 0
\end{pmatrix}
\end{array}
\begin{array}{l}
\langle 20| \\
\langle 21|
\end{array}
$$

This is allowed by the resonator drive