0 Instructions

Homework is due Tuesday, March 18, 2024 at 23:59pm Central Time. Please refer to `https://courses.grainger.illinois.edu/cs446/sp2024/homework/hw/index.html` for course policy on homeworks and submission instructions.

1 Neural networks for simple functions

1. $$w_0 = w_1 = [1, -1]^\top$$

2. $$w_0 = [1, -1]^\top,\ w_1 = [m, -m]^\top,\ b = [\frac{b}{m}, -\frac{b}{m}]^\top$$

3. $$w_0 = 0,\ w_1 = 2b$$

4. $$w_0 = [1, 1, 1]^\top,\ b_0 = [2, 0, -2]^\top,\ w_1 = [3, -6, 3]^\top$$

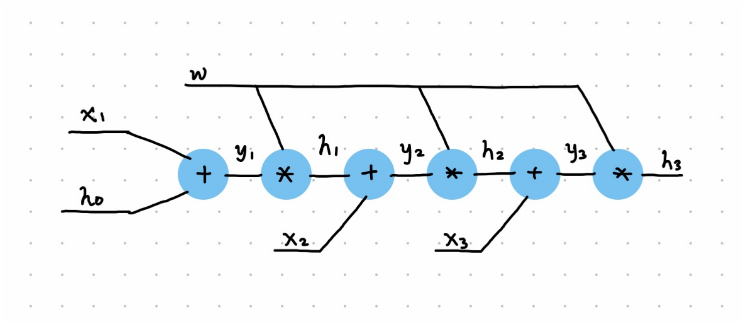
5. No. $x^2 = |x|^2$, while any transformation in $f(x)$ is either making linear transforms (w), or resulting in 0 (σ), in which $0 \leq |x|$. Any combination of them is impossible to produce a quadratic transform.

6. No. For $w_0x + b_0 > 0$, $|f(x) - x^2| = |w_1^\top w_0 x - x^2 + w_1^\top b_0|$. For $w_0 x + b_0 \leq 0$, $|f(x) - x^2| = |0 - x^2|$. Both of them are quadratic functions on x that will explode as x grows larger.

7. Yes. As x is limited in this case, we can select appropriate parameters to approximate x^2 with gradient descent:
$$\frac{\partial f}{\partial w_1} = \sigma(w_0 x + b_0)$$
$$\frac{\partial f}{\partial w_0} = w_1 \cdot \mathbb{1}\{x > 0\} \cdot x;\ \frac{\partial f}{\partial b_0} = w_1 \cdot \mathbb{1}\{x > 0\}$$

2 Backpropagation through time (BPTT)

1. As below:



2.
$$h_0 = 0,\ h_1 = w(x_1 + h_0) = wx_1,\ h_2 = w(x_2 + h_1) = wx_2 + w^2x_1,$$
$$h_3 = w(x_3 + h_2) = wx_3 + w^2x_2 + w^3x_1$$
$$y_1 = x_1 + h_0 = x_1,\ y_2 = x_2 + h_1 = x_2 + wx_1,\ y_3 = x_3 + h_2 = x_3 + wx_2 + w^2x_1$$

3.
$$\frac{\partial h_3}{\partial w} = x_3 + (h_2 + w\frac{\partial h_2}{\partial w}) = x_3 + wx_2 + w^2x_1 + w[x_2 + (h_1 + w\frac{\partial h_1}{\partial w})]$$
$$= x_3 + 2wx_2 + 2w^2x_1 + w^2(\frac{\partial h_1}{\partial w}) = x_3 + 2wx_2 + 3w^2x_1$$

4.
$$\frac{\partial f}{\partial h_1} = \frac{\partial h_T}{\partial h_1} = \prod_{t=1}^{T-1} \frac{\partial h_{t+1}}{\partial h_t} = \prod_{t=1}^{T-1} w\sigma'(x_t + h_{t-1}) = w^{T-1} \prod_{t=1}^{T-1} \sigma'(x_t + h_{t-1})$$

5. From the previous question we can see that if w is initialized too large or gradient components are mostly larger than 1 during the propagation, the gradient will tend to explode. In other cases that w is initialized small or gradient components are mostly less than 1, the gradient will tend to diminish to 0.

3 Transformers

1. Theoretically α_i can't be infinitely large since attention weights are supposed to be designed between 0 and 1. Practically the numerator in α_i would be less than denominator, and both of them can't reach 0 due to exponential. In this way, α_i can neither be neither infinitely large nor be 0.

2. q and k_i are pointing to similar directions, which differs from the rest k. $c = \sum \alpha_i v_i$ for all i satisfying such condition. This means that the token corresponding to q here is attached with greater attention to tokens corresponding to such i.

3. $q = m(k_1 + k_2)$, in which m is a really large number. In this case, we $\exp(k_i^\top q) = 1$ for $i \neq 1, 2$ and $\exp(k_i^\top q) = \exp(m)$ for $i \in \{1, 2\}$. If $\exp(m) \gg 1$, we have $c \approx \frac{1}{2}(v_1 + v_2)$.

4. $$\|q - k_i\|_2^2 = \|q\|_2^2 + \|k_i\|_2^2 + 2q^\top k_i = \|q\|_2^2 + 2q^\top k_i$$
$$\exp(\frac{-\|q - k_i\|_2^2}{2\sigma}) = \exp(\frac{\|q\|_2^2}{2}) \cdot \exp(q^\top k_i)$$
$$\implies \beta_i = \frac{\exp(\frac{\|q\|_2^2}{2}) \cdot \exp(q^\top k_i)}{\exp(\frac{\|q\|_2^2}{2}) \cdot \sum_j \exp(q^\top k_i)} = \alpha_i$$

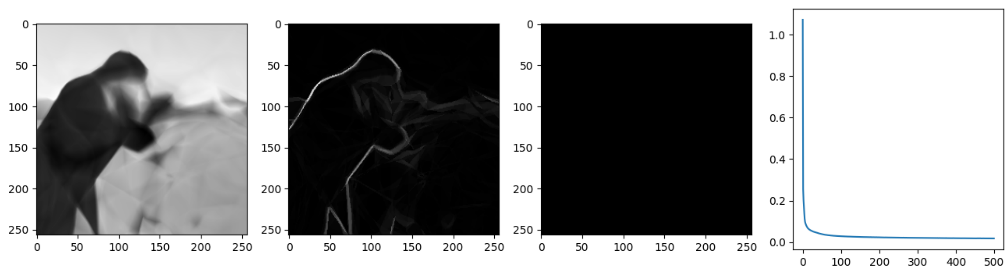
5. Knowing the SVD of P, we write P as:
$$P = U\Sigma V^\top$$
Given $\text{rank}(P) = k$, we have:
$$P = \sum_{i=1}^{k} \sigma_i u_i v_i^\top$$
with singular values σ_i and singular vectors u_i, v_i. There are n of u_i, v_i in dimension d, so computation of single $u_i v_i^\top$ costs $O(nd)$. Therefore, computation of P above can cost only $O(nkd)$.

4 Resnet

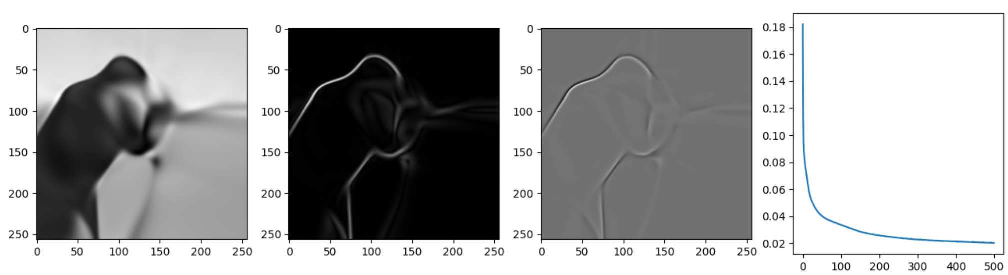
2. BatchNorm2d layer will center the output of Conv2d layer around 0, which eliminates the bias.

5 Coding: Image overfitting

5. ReLU:
   (a) epochs = 500
   (b) learning rate = 10^{-4}
   (c) batch size = 800



   Tanh:
   (a) epochs = 500
   (b) learning rate = 10^{-4}
   (c) batch size = 800

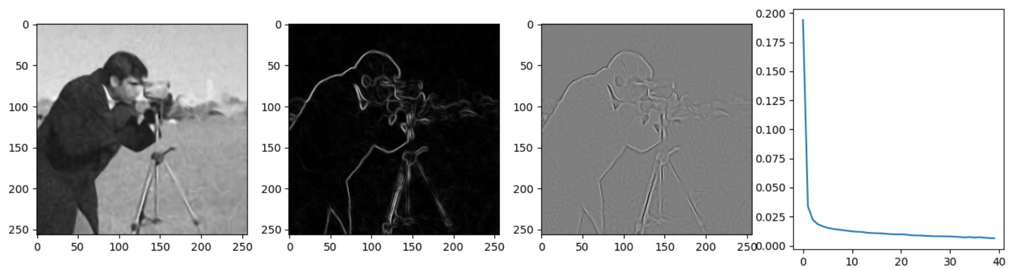


Sin:

(a) epochs = 40

(b) learning rate = 10^{-4}

(c) batch size = 800



6.

$$\nabla^2 \text{ReLU}(\mathbf{x}) = \nabla^2 \max\{0, \mathbf{x}\} = \frac{\partial^2 \max\{0, \mathbf{x}\}}{\partial x_1^2} + \frac{\partial^2 \max\{0, \mathbf{x}\}}{\partial x_2^2}$$
$$= \frac{\partial \mathbb{1}\{x > 0\}}{\partial x_1} + \frac{\partial \mathbb{1}\{x > 0\}}{\partial x_2} = 0$$

7. Decrease the learning rate appropriately; learning rate scheduling.

8. Initial loss is larger, the model converges slower, and the steady loss is higher. The experiment seems failed. Sin activation is picked for the experiment below.

(a) epochs = 100

(b) learning rate = 10^{-4}

(c) batch size = 800