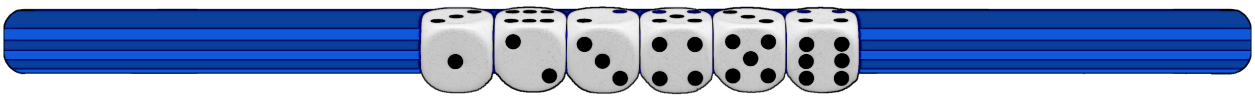
Exercise sheet 1: Hidden Markov models

Exercise 1 - Two kinds of dice



A casino uses two kinds of dice: 98% of dice are fair and 2% are loaded. The loaded die has a probability of 0.5 to show number six and 0.1 for the number one to five.

Question A When we pick up a die from a table at random, what is the probability of rolling a six?

Hint 1 : Formulae

$$L = \text{Loaded} \quad F = \text{Fair} \quad O = \text{Observation} P(O) = P(F) \times P(O|F) + P(L) \times P(O|L)$$

Hint 2 : Calculation Method

$$P(6) = 0.98 \times \frac{1}{6} + 0.02 \times \frac{1}{2}$$

Solution

$$P(6) = 0.173\bar{3}$$

Question B We pick up a die from a table at random and roll [6 6 6]. What is the probability, that the die is loaded.

Hint 1 : Formulae

$$P(L|O) = \frac{P(L,O)}{P(O)} P(L,O) = P(O|L) \times P(L)$$

Hint 2 : Calculation Method

$$\begin{aligned} P(L|O) &= \frac{P(O|L) \times P(L)}{P(O|L) \times P(L) + P(O|F) \times P(F)} \\ &= \frac{(\frac{1}{2})^3 \times 0.02}{(\frac{1}{2})^3 \times 0.02 + (\frac{1}{6})^3 \times 0.98} \end{aligned}$$

Solution

$$P(L|O) = 35.53\%$$

Question C How many sixes in a row would we need to roll to be at least 90% sure that the die is loaded?

Hint 1 : Formulae

$$P(L|O) = \frac{P(O|L) \times P(L)}{P(O|L) \times P(L) + P(O|F) \times P(F)}$$

Hint 2 : Calculation Method

$$\begin{aligned}
P(L|O) = \frac{\frac{2}{100} \times (\frac{1}{2})^n}{\frac{2}{100} \times (\frac{1}{2})^n + \frac{98}{100} \times (\frac{1}{6})^n} &\geq 0.9 && |\text{ split } (\frac{1}{6})^n && (1)\\
\iff \frac{\frac{2}{100} \times (\frac{1}{2})^n}{\frac{2}{100} \times (\frac{1}{2})^n + \frac{98}{100} \times (\frac{1}{2})^n \times (\frac{1}{3})^n} &\geq \frac{9}{10} && |\text{ factorize} && (2)\\
\iff \frac{\frac{2}{100} \times (\frac{1}{2})^n}{\frac{2}{100} \times (\frac{1}{2})^n \times (1 + 49 \times (\frac{1}{3})^n)} &\geq \frac{9}{10} && |\text{ simplify, given } n > 0 && (3)\\
\iff \frac{1}{1 + 49 \times (\frac{1}{3})^n} &\geq \frac{9}{10} && |\text{ cross-multiply, given } n > 0 && (4)\\
\iff \frac{9}{10}(1 + 49 \times (\frac{1}{3})^n) &\leq 1 && |\text{ rewrite} && (5)\\
\iff (\frac{1}{3})^n &\leq \frac{1}{441} && |\ \ln() && (6)\\
\iff n \times ln(\frac{1}{3}) &\leq ln(\frac{1}{441}) && | \times \frac{1}{ln(\frac{1}{3})} && (7)\\
\iff n &\geq \frac{ln(\frac{1}{441})}{ln(\frac{1}{3})} && && (8)\\
\iff n &\geq 5.542487... && && (9)
\end{aligned}$$

Solution

$n = 6$ as only Integers make sense here (just trying would also work)

Exercise 2 - The occasionally cheating casino

In a casino they use a fair die most of the time, but occasionally they switch to a loaded die. The loaded die has a probability 0.5 to show number six and probability 0.1 for the numbers one to five. Assume that the casino switches from a fair to a loaded die with probability 0.05 before each roll, and that the probability of switching back is 0.1. The probability to start a game with the fair die is 0.9.

Question C Draw a graphical representation of the described Hidden Markov model.

Solution a