Exercise sheet 2: Hidden Markov models II

Exercise 1 - The occasionally cheating casino

In a casino they use a fair die most of the time, but occasionally they switch to a loaded die. The loaded die has a probability 0.5 to show number six and probability 0.1 for the numbers one to five. Assume that the casino switches from a fair to a loaded die with probability 0.05 before each roll, and that the probability of switching back is 0.1. The probability to start a game with the fair die is 0.9.

Question 1A Find the probability $P(\mathcal{O}|M)$ for $\mathcal{O} = 1662$ and the given HMM using the forward algorithm.

Hint 1 : Formulae

$$\begin{aligned}\alpha_1(i) =& \pi_i \times b_{i,o_1} \\ \alpha_{t+1}(j) =& \sum_{i\in\{F,L\}} \alpha_t(i) \times a_{i,j} \times b_{j,o_{t+1}}\end{aligned}$$

Hint 2 : Calculation Method

$$\begin{aligned}\alpha_1(F) =& \pi_F \times b_{F,1} = 0.9 \times \frac{1}{6} = 0.15 \\ \alpha_1(L) =& \pi_F \times b_{L,1} = 0.1 \times 0.1 = 0.01\end{aligned}$$

$$\begin{aligned}\alpha_2(F) =& \alpha_1(F) \times a_{F,F} \times b_{F,6} + \alpha_1(L) \times a_{L,F} \times b_{F,6} = 0.15 \times 0.95 \times \frac{1}{6} + 0.01 \times 0.1 \times \frac{1}{6} = 0.0239167 \\ \alpha_2(L) =& \alpha_1(F) \times a_{F,L} \times b_{L,6} + \alpha_1(L) \times a_{L,L} \times b_{L,6} = 0.15 \times 0.05 \times 0.5 + 0.01 \times 0.9 \times 0.5 = 0.00825 \\ \alpha_3(F) =& 0.023917 \times 0.95 \times \frac{1}{6} + 0.00825 \times 0.1 \times \frac{1}{6} = 0.00392 \\ \alpha_3(L) =& 0.023917 \times 0.05 \times 0.5 + 0.00825 \times 0.9 \times 0.5 = 0.00431 \\ \alpha_4(F) =& 0.00392 \times 0.95 \times \frac{1}{6} + 0.00431 \times 0.1 \times \frac{1}{6} = 0.000693 \\ \alpha_4(L) =& 0.00392 \times 0.05 \times 0.1 + 0.00431 \times 0.9 \times 0.1 = 0.000407\end{aligned}$$

Solution

$$P(\mathcal{O} = 1662) = \alpha_4(F) + \alpha_4(L) = 0.000693 + 0.000407 = 0.0011$$

Question 1B Given the result of Question 1A, do you expect a higher probability for the observations $\mathcal{O} = 1666$ and $\mathcal{O} = 1262$?

Hint 1 It has something to do with the emission probabilities of the different states.

Solution As state L has a high probability to emit a six, observations with more sixes are more likely.

$$P(\mathcal{O} = 1666) > P(\mathcal{O} = 1662) > P(\mathcal{O} = 1262)$$

Question 1C Find the most probable path through the HMM that produces the sequence $\mathcal{O} = 1662$.

Hint 1 : Formulae

$$\begin{aligned}
\delta_1(i) =& \pi_i \times b_{i,o_1} \\
\delta_{t+1}(j) =& max_{i\in\{F,L\}}\delta_t(i) \times a_{i,j} \times b_{j,o_{t+1}} \\
q_t^* =& argmax_{1\leq i\leq n}\{\delta_t(i)a_{i,q^*_{t+1}}\}
\end{aligned}$$

Hint 2 : Intermediate calculations

$$\begin{aligned}
\delta_1(F) =& \pi_F \times b_{F,1} = 0.9 \times \frac{1}{6} = 0.15 \\
\delta_1(L) =& \pi_F \times b_{L,1} = 0.1 \times 0.1 = 0.01 \\
\\
\delta_2(F) =& max(\delta_1(F) \times a_{F,F} \times b_{F,6}, \delta_1(L) \times a_{L,F} \times b_{F,6}) = max(0.02375, 0.00016) = 0.02375 \\
\delta_2(L) =& max(\delta_1(F) \times a_{F,L} \times b_{L,6}, \delta_1(L) \times a_{L,L} \times b_{L,6}) = max(0.00375, 0.0045) = 0.0045 \\
\delta_3(F) =& max(0.00376, 0.000075) = 0.00376 \\
\delta_3(L) =& max(0.00059375, 0.002025) = 0.002025 \\
\delta_4(F) =& max(0.0005953, 0.00003375) = 0.0005953 \\
\delta_4(L) =& max(0.0000188, 0.00018225) = 0.00018225
\end{aligned}$$

$$P(\mathcal{P}^*, \mathcal{O}) = max(\delta_4(F), \delta_4(L)) = 5.95 \times 10^{-4}$$

Solution

$$\begin{aligned}
q_4^* =& argmax_{i\in\{F,L\}}(\delta_4(i)) = F \\
q_3^* =& argmax_{i\in\{F,L\}}(\delta_3(i) \times a_{i,q_4^*}) = argmax(F : 0.00376 \times 0.95, L : 0.002025 \times 0.1) = F \\
q_2^* =& argmax_{i\in\{F,L\}}(\delta_2(i) \times a_{i,q_3^*}) = argmax(F : 0.02375 \times 0.95, L : 0.0045 \times 0.1) = F \\
q_1^* =& argmax_{i\in\{F,L\}}(\delta_1(i) \times a_{i,q_2^*}) = argmax(F : 0.15 \times 0.95, L : 0.01 \times 0.1) = F \\
& \Rightarrow Q^* = FFFF
\end{aligned}$$

The best path is therefore to stay in state F.