Exercise sheet 2: Hidden Markov models II

Exercise 1 - The occasionally cheating casino

In a casino they use a fair die most of the time, but occasionally they switch to a loaded die. The loaded die has a probability 0.5 to show number six and probability 0.1 for the numbers one to five. Assume that the casino switches from a fair to a loaded die with probability 0.05 before each roll, and that the probability of switching back is 0.1. The probability to start a game with the fair die is 0.9.

Question 1A Find the probability $P(\mathcal{O}|M)$ for $\mathcal{O} = 1662$ and the given HMM using the forward algorithm.

Hint 1 : Formulae

$$\alpha_1(i) = \pi_i \times b_{i,o_1}$$
$$\alpha_{t+1}(j) = \sum_{i \in \{F,L\}} \alpha_t(i) \times a_{i,j} \times b_{j,o_{t+1}}$$

Hint 2 : Calculation Method

$$\alpha_1(F) = \pi_F \times b_{F,1} = 0.9 \times \frac{1}{6} = 0.15$$
$$\alpha_1(L) = \pi_F \times b_{L,1} = 0.1 \times 0.1 = 0.01$$

$$\alpha_2(F) = \alpha_1(F) \times a_{F,F} \times b_{F,6} + \alpha_1(L) \times a_{L,F} \times b_{F,6} = 0.15 \times 0.95 \times \frac{1}{6} + 0.01 \times 0.1 \times \frac{1}{6} = 0.0239167$$
$$\alpha_2(L) = \alpha_1(F) \times a_{F,L} \times b_{L,6} + \alpha_1(L) \times a_{L,L} \times b_{L,6} = 0.15 \times 0.05 \times 0.5 + 0.01 \times 0.9 \times 0.5 = 0.00825$$
$$\alpha_3(F) = 0.023917 \times 0.95 \times \frac{1}{6} + 0.00825 \times 0.1 \times \frac{1}{6} = 0.00392$$
$$\alpha_3(L) = 0.023917 \times 0.05 \times 0.5 + 0.00825 \times 0.9 \times 0.5 = 0.00431$$
$$\alpha_4(F) = 0.00392 \times 0.95 \times \frac{1}{6} + 0.00431 \times 0.1 \times \frac{1}{6} = 0.000693$$
$$\alpha_4(L) = 0.00392 \times 0.05 \times 0.1 + 0.00431 \times 0.9 \times 0.1 = 0.000407$$

Solution

$$P(\mathcal{O} = 1662) = \alpha_4(F) + \alpha_4(L) = 0.000693 + 0.000407 = 0.0011$$

Question 1B Given the result of Question 1A, do you expect a higher probability for the observations $\mathcal{O} = 1666$ and $\mathcal{O} = 1262$?

Hint 1 It has something to do with the emission probabilities of the different states.

Solution As state L has a high probability to emit a six, observations with more sixes are more likely.

$$P(\mathcal{O} = 1666) > P(\mathcal{O} = 1662) > P(\mathcal{O} = 1262)$$

Question 1C Find the most probable path through the HMM that produces the sequence $\mathcal{O} = 1662$.

Hint 1 todo

Solution todo