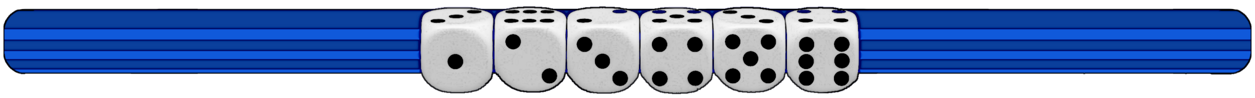
Exercise sheet 1: Hidden Markov models

Exercise 1 - Two kinds of dice



A casino uses two kinds of dice: 98% of dice are fair and 2% are loaded. The loaded die has a probability of 0.5 to show number six and 0.1 for the numbers one to five.

Question 1A When we pick up a die from a table at random, what is the probability of rolling a six?

Hint 1 : Formulae

$$L = \text{Loaded} \quad F = \text{Fair} \quad \mathcal{O} = \text{Observation} P(\mathcal{O}) = P(F) \times P(\mathcal{O}|F) + P(L) \times P(\mathcal{O}|L)$$

Hint 2 : Calculation Method

$$P(6) = 0.98 \times \frac{1}{6} + 0.02 \times \frac{1}{2}$$

Solution

$$P(6) = 0.173\bar{3}$$

Question 1B We pick up a die from a table at random and roll [6 6 6]. What is the probability, that the die is loaded.

Hint 1 : Formulae

$$P(L|\mathcal{O}) = \frac{P(L, \mathcal{O})}{P(\mathcal{O})} P(L, \mathcal{O}) = P(\mathcal{O}|L) \times P(L)$$

Hint 2 : Calculation Method

$$\begin{aligned} P(L|\mathcal{O}) &= \frac{P(\mathcal{O}|L) \times P(L)}{P(\mathcal{O}|L) \times P(L) + P(\mathcal{O}|F) \times P(F)} \\ &= \frac{(\frac{1}{2})^3 \times 0.02}{(\frac{1}{2})^3 \times 0.02 + (\frac{1}{6})^3 \times 0.98} \end{aligned}$$

Solution

$$P(L|\mathcal{O}) = 35.53\%$$

Question 1C How many sixes in a row would we need to roll to be at least 90% sure that the die is loaded?

Hint 1 : Formulae

$$P(L|\mathcal{O}) = \frac{P(\mathcal{O}|L) \times P(L)}{P(\mathcal{O}|L) \times P(L) + P(\mathcal{O}|F) \times P(F)}$$

Hint 2 : Calculation Method

$$\begin{aligned}
P(L|\mathcal{O}) = \frac{\frac{2}{100} \times (\frac{1}{2})^n}{\frac{2}{100} \times (\frac{1}{2})^n + \frac{98}{100} \times (\frac{1}{6})^n} &\geq 0.9 && \mid \text{split } (\frac{1}{6})^n && (1)\\
\iff \frac{\frac{2}{100} \times (\frac{1}{2})^n}{\frac{2}{100} \times (\frac{1}{2})^n + \frac{98}{100} \times (\frac{1}{2})^n \times (\frac{1}{3})^n} &\geq \frac{9}{10} && \mid \text{factorize} && (2)\\
\iff \frac{\frac{2}{100} \times (\frac{1}{2})^n}{\frac{2}{100} \times (\frac{1}{2})^n \times (1 + 49 \times (\frac{1}{3})^n)} &\geq \frac{9}{10} && \mid \text{simplify, given } n > 0 && (3)\\
\iff \frac{1}{1 + 49 \times (\frac{1}{3})^n} &\geq \frac{9}{10} && \mid \text{cross-multiply, given } n > 0 && (4)\\
\iff \frac{9}{10}(1 + 49 \times (\frac{1}{3})^n) &\leq 1 && \mid \text{rewrite} && (5)\\
\iff (\frac{1}{3})^n &\leq \frac{1}{441} && \mid \ln() && (6)\\
\iff n \times ln(\frac{1}{3}) &\leq ln(\frac{1}{441}) && \mid \times \frac{1}{ln(\frac{1}{3})} && (7)\\
\iff n &\geq \frac{ln(\frac{1}{441})}{ln(\frac{1}{3})} && && (8)\\
\iff n &\geq 5.542487... && && (9)
\end{aligned}$$

Solution

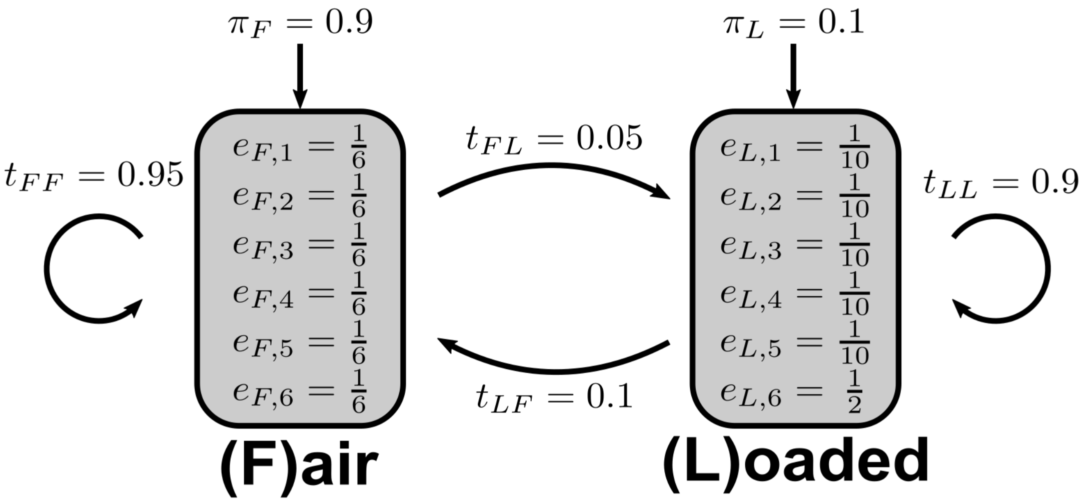
$n = 6$, as only Integers make sense here (just trying would also work)

Exercise 2 - The occasionally cheating casino

In a casino they use a fair die most of the time, but occasionally they switch to a loaded die. The loaded die has a probability 0.5 to show number six and probability 0.1 for the numbers one to five. Assume that the casino switches from a fair to a loaded die with probability 0.05 before each roll, and that the probability of switching back is 0.1. The probability to start a game with the fair die is 0.9.

Question 2A Draw a graphical representation of the described Hidden Markov model.

Solution



Question 2B Given an observed sequence of outcomes $\mathcal{O} = 3661634$ and two possible state sequences $s_1 = LLLFFFF$ and $s_2 = FFFFFFF$ (where F = Fair and L = Loaded), what are the joint probabilities $P(\mathcal{O}, p_1)$ and $P(\mathcal{O}, p_2)$ in the HMM described above?

Hint 1 : Formulae

$$P(\mathcal{O}, p_x) = P(\mathcal{O}|p_x) \times P(p_x)$$

Hint 2 : Calculation Method

$$\begin{aligned}
P(s_1) =& \pi_L \times t_{LL}^2 \times t_{LF} \times t_{FF}^3 = 0.1 \times 0.9^2 \times 0.1 \times (0.95)^3 = 0.0069 \\
P(s_2) =& \pi_F \times t_{FF}^6 = 0.9 \times 0.95^6 = 0.6616 \\
P(\mathcal{O}|s_1) =& t_{L,3} \times t_{L,6}^2 \times t_{F,1} \times t_{F,6} \times t_{F,3} \times t_{F,4} = 0.1 \times 0.5^2 \times (\frac{1}{6})^4 = 1.9 \times 10^{-5} \\
P(\mathcal{O}|s_2) =& t_{F,3} \times t_{F,6}^2 \times t_{F,1} \times t_{F,6} \times t_{F,3} \times t_{F,4} = (\frac{1}{6})^7 = 3.57 \times 10^{-6} \\
P(\mathcal{O}, s_1) =& P(\mathcal{O}|s_1) \times P(s_1) = 1.9 \times 10^{-5} \times 0.0069 \\
P(\mathcal{O}, s_2) =& P(\mathcal{O}|s_2) \times P(s_2) = 3.57 \times 10^{-6} \times 0.6616
\end{aligned}$$

Solution

$$\begin{aligned}
P(\mathcal{O}, s_1) = 1.34 \times 10^{-7} \\
P(\mathcal{O}, s_2) = 2.36 \times 10^{-6}
\end{aligned}$$

Question 2C Give an observation $\mathcal{O} = 1662$, how many possible state sequences exist in the described HMM?

Hint 1 The actual observation does not matter in this case because all emission probabilities are > 0. This there are 2^4 possible state sequences.

Solution There are 16 possible state sequences.

Exercise 3 - Programming assignment

Under Construction
