

Exercise sheet 1: Hidden Markov models

Exercise 1 - Two kinds of dice



A casino uses two kinds of dice: 98% of dice are fair and 2% are loaded. The loaded die has a probability of 0.5 to show number six and 0.1 for the numbers one to five.

Question 1A When we pick up a die from a table at random, what is the probability of rolling a six?

Hint 1 : Formulae

$$L = \text{Loaded} \quad F = \text{Fair} \quad O = \text{Observation} \quad P(O) = P(F) \times P(O|F) + P(L) \times P(O|L)$$

Hint 2 : Calculation Method

$$P(6) = 0.98 \times \frac{1}{6} + 0.02 \times \frac{1}{2}$$

Solution

$$P(6) = 0.173\bar{3}$$

Question 1B We pick up a die from a table at random and roll [6 6 6]. What is the probability, that the die is loaded.

Hint 1 : Formulae

$$P(L|O) = \frac{P(L, O)}{P(O)} \quad P(L, O) = P(O|L) \times P(L)$$

Hint 2 : Calculation Method

$$\begin{aligned} P(L|O) &= \frac{P(O|L) \times P(L)}{P(O|L) \times P(L) + P(O|F) \times P(F)} \\ &= \frac{\left(\frac{1}{2}\right)^3 \times 0.02}{\left(\frac{1}{2}\right)^3 \times 0.02 + \left(\frac{1}{6}\right)^3 \times 0.98} \end{aligned}$$

Solution

$$P(L|O) = 35.53\%$$

Question 1C How many sixes in a row would we need to roll to be at least 90% sure that the die is loaded?

Hint 1 : Formulae

$$P(L|O) = \frac{P(O|L) \times P(L)}{P(O|L) \times P(L) + P(O|F) \times P(F)}$$

Hint 2 : Calculation Method

$$P(L|O) = \frac{\frac{2}{100} \times (\frac{1}{2})^n}{\frac{2}{100} \times (\frac{1}{2})^n + \frac{98}{100} \times (\frac{1}{6})^n} \geq 0.9 \quad | \text{ split } (\frac{1}{6})^n \quad (1)$$

$$\iff \frac{\frac{2}{100} \times (\frac{1}{2})^n}{\frac{2}{100} \times (\frac{1}{2})^n + \frac{98}{100} \times (\frac{1}{2})^n \times (\frac{1}{3})^n} \geq \frac{9}{10} \quad | \text{ factorize} \quad (2)$$

$$\iff \frac{\frac{2}{100} \times (\frac{1}{2})^n}{\frac{2}{100} \times (\frac{1}{2})^n \times (1 + 49 \times (\frac{1}{3})^n)} \geq \frac{9}{10} \quad | \text{ simplify, given } n > 0 \quad (3)$$

$$\iff \frac{1}{1 + 49 \times (\frac{1}{3})^n} \geq \frac{9}{10} \quad | \text{ cross-multiply, given } n > 0 \quad (4)$$

$$\iff \frac{9}{10} (1 + 49 \times (\frac{1}{3})^n) \leq 1 \quad | \text{ rewrite} \quad (5)$$

$$\iff (\frac{1}{3})^n \leq \frac{1}{441} \quad | \ln() \quad (6)$$

$$\iff n \times \ln(\frac{1}{3}) \leq \ln(\frac{1}{441}) \quad | \times \frac{1}{\ln(\frac{1}{3})} \quad (7)$$

$$\iff n \geq \frac{\ln(\frac{1}{441})}{\ln(\frac{1}{3})} \quad (8)$$

$$\iff n \geq 5.542487... \quad (9)$$

Solution

$n = 6$, as only Integers make sense here (just trying would also work)

Exercise 2 - The occasionally cheating casino

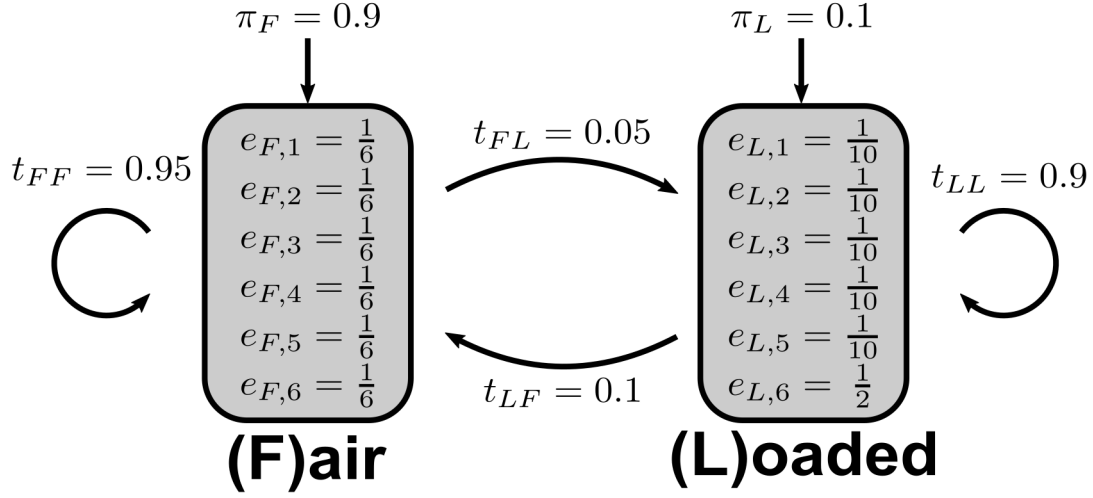


In a casino they use a fair die most of the time, but occasionally they switch to a loaded die. The loaded die has a probability 0.5 to show number six and probability 0.1 for the numbers one to five. Assume that the

casino switches from a fair to a loaded die with probability 0.05 before each roll, and that the probability of switching back is 0.1. The probability to start a game with the fair die is 0.9.

Question 2A Draw a graphical representation of the described Hidden Markov model.

Solution



Question 2B Given an observed sequence of outcomes $O = 3661634$ and two possible state sequences $s_1 = LLLFFFF$ and $s_2 = FFFFFFF$ (where F = Fair and L = Loaded), what are the joint probabilities $P(O, p_1)$ and $P(O, p_2)$ in the HMM described above?

Hint 1 : Formulae

$$P(O, p_x) = P(O|p_x) \times P(p_x)$$

Hint 2 : Calculation Method

$$P(s_1) = \pi_L \times t_{LL}^2 \times t_{LF} \times t_{FF}^3 = 0.1 \times 0.9^2 \times 0.1 \times (0.95)^3 = 0.0069$$

$$P(s_2) = \pi_F \times t_{FF}^6 = 0.9 \times 0.95^6 = 0.6616$$

$$P(O|s_1) = t_{L,3} \times t_{L,6}^2 \times t_{F,1} \times t_{F,6} \times t_{F,3} \times t_{F,4} = 0.1 \times 0.5^2 \times \left(\frac{1}{6}\right)^4 = 1.9 \times 10^{-5}$$

$$P(O|s_2) = t_{F,3} \times t_{F,6}^2 \times t_{F,1} \times t_{F,6} \times t_{F,3} \times t_{F,4} = \left(\frac{1}{6}\right)^7 = 3.57 \times 10^{-6}$$

$$P(O, s_1) = P(O|s_1) \times P(s_1) = 1.9 \times 10^{-5} \times 0.0069$$

$$P(O, s_2) = P(O|s_2) \times P(s_2) = 3.57 \times 10^{-6} \times 0.6616$$

Solution

$$P(O, s_1) = 1.34 \times 10^{-7}$$

$$P(O, s_2) = 2.36 \times 10^{-6}$$

Question 2C Give an observation $O = 1662$, how many possible state sequences exist in the described HMM?

Hint 1 : The actual observation does not matter in this case because all emission probabilities are > 0 . This there are 2^4 possible state sequences.

Solution There are 16 possible state sequences.

Exercise 3 - Programming assignment

The programming assignments are available via Github Classroom and contain automatic tests.

The programming assignment of this exercise sheet is available [here](#).
