Exercise sheet 2: Hidden Markov models II

Exercise 1 - The occasionally cheating casino

In a casino they use a fair die most of the time, but occasionally they switch to a loaded die. The loaded die has a probability 0.5 to show number six and probability 0.1 for the numbers one to five. Assume that the casino switches from a fair to a loaded die with probability 0.05 before each roll, and that the probability of switching back is 0.1. The probability to start a game with the fair die is 0.9.

Question 1A Find the probability $P(\mathcal{O}|M)$ for $\mathcal{O} = 1662$ and the given HMM using the forward algorithm.

Hint 1 : Formulae

$$\begin{aligned}\alpha_1(i) =& \pi_i \times b_{i,o_1}\\ \alpha_{t+1}(j) =& \sum_{i\in\{F,L\}} \alpha_t(i) \times a_{i,j} \times b_{j,o_{t+1}}\end{aligned}$$

Hint 2 : Calculation Method

$$\begin{aligned}\alpha_1(F) =& \pi_F \times b_{F,1} = 0.9 \times \frac{1}{6} = 0.15\\ \alpha_1(L) =& \pi_F \times b_{L,1} = 0.1 \times 0.1 = 0.01\end{aligned}$$

$$\begin{aligned}\alpha_2(F) =& \alpha_1(F) \times a_{F,F} \times b_{F,6} + \alpha_1(L) \times a_{L,F} \times b_{F,6} = 0.15 \times 0.95 \times \frac{1}{6} + 0.01 \times 0.1 \times \frac{1}{6} = 0.0239167\\ \alpha_2(L) =& \alpha_1(F) \times a_{F,L} \times b_{L,6} + \alpha_1(L) \times a_{L,L} \times b_{L,6} = 0.15 \times 0.05 \times 0.5 + 0.01 \times 0.9 \times 0.5 = 0.00825\\ \alpha_3(F) =& 0.023917 \times 0.95 \times \frac{1}{6} + 0.00825 \times 0.1 \times \frac{1}{6} = 0.00392\\ \alpha_3(L) =& 0.023917 \times 0.05 \times 0.5 + 0.00825 \times 0.9 \times 0.5 = 0.00431\\ \alpha_4(F) =& 0.00392 \times 0.95 \times \frac{1}{6} + 0.00431 \times 0.1 \times \frac{1}{6} = 0.000693\\ \alpha_4(L) =& 0.00392 \times 0.05 \times 0.1 + 0.00431 \times 0.9 \times 0.1 = 0.000407\end{aligned}$$

Solution

$$P(\mathcal{O} = 1662) = \alpha_4(F) + \alpha_4(L) = 0.000693 + 0.000407 = 0.0011$$

Question 1B Given the result of Question 1A, do you expect a higher probability for the observations $\mathcal{O} = 1666$ and $\mathcal{O} = 1262$?

Hint 1 It has something to do with the emission probabilities of the different states.

Solution As state L has a high probability to emit a six, observations with more sixes are more likely.

$$P(\mathcal{O} = 1666) > P(\mathcal{O} = 1662) > P(\mathcal{O} = 1262)$$

Question 1C Find the most probable path through the HMM that produces the sequence $\mathcal{O} = 1662$.

Hint 1 : Formulae

$$\begin{aligned}
\delta_1(i) =& \pi_i \times b_{i,o_1} \\
\delta_{t+1}(j) =& max_{i \in \{F,L\}} \delta_t(i) \times a_{i,j} \times b_{j,o_{t+1}} \\
q_t^* =& argmax_{1 \leq i \leq n} \{\delta_t(i) a_{i,q_{t+1}^*}\}
\end{aligned}$$

Hint 2 : Intermediate calculations

$$\begin{aligned}
\delta_1(F) =& \pi_F \times b_{F,1} = 0.9 \times \frac{1}{6} = 0.15 \\
\delta_1(L) =& \pi_F \times b_{L,1} = 0.1 \times 0.1 = 0.01
\end{aligned}$$

$$\begin{aligned}
\delta_2(F) =& max(\delta_1(F) \times a_{F,F} \times b_{F,6}, \delta_1(L) \times a_{L,F} \times b_{F,6}) = max(0.02375, 0.00016) = 0.02375 \\
\delta_2(L) =& max(\delta_1(F) \times a_{F,L} \times b_{L,6}, \delta_1(L) \times a_{L,L} \times b_{L,6}) = max(0.00375, 0.0045) = 0.0045 \\
\delta_3(F) =& max(0.00376, 0.000075) = 0.00376 \\
\delta_3(L) =& max(0.00059375, 0.002025) = 0.002025 \\
\delta_4(F) =& max(0.0005953, 0.00003375) = 0.0005953 \\
\delta_4(L) =& max(0.0000188, 0.00018225) = 0.00018225
\end{aligned}$$

$$P(\mathcal{P}^*, \mathcal{O}) = max(\delta_4(F), \delta_4(L)) = 5.95 \times 10^{-4}$$

Solution

$$\begin{aligned}
q_4^* =& argmax_{i \in \{F,L\}}(\delta_4(i)) = F \\
q_3^* =& argmax_{i \in \{F,L\}}(\delta_3(i) \times a_{i,q_4^*}) = argmax(F : 0.00376 \times 0.95, L : 0.002025 \times 0.1) = F \\
q_2^* =& argmax_{i \in \{F,L\}}(\delta_2(i) \times a_{i,q_3^*}) = argmax(F : 0.02375 \times 0.95, L : 0.0045 \times 0.1) = F \\
q_1^* =& argmax_{i \in \{F,L\}}(\delta_1(i) \times a_{i,q_2^*}) = argmax(F : 0.15 \times 0.95, L : 0.01 \times 0.1) = F \\
\Rightarrow& Q^* = FFFF
\end{aligned}$$

The best path is therefore to stay in state F.

Exercise 2 - Profile HMMs

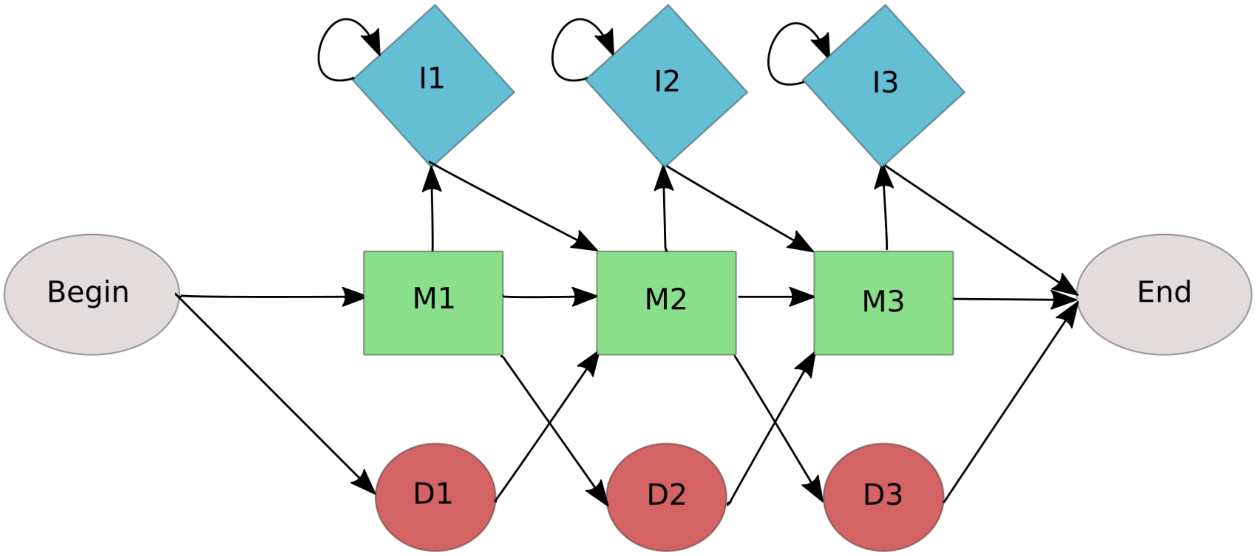
Profile HMMs define a position specific scoring scheme which can be used to search databases for homologous sequences.

The following multiple alignment of DNA sequences is given:

```
AC---A
A----A
AG---T
TTGGGT
**   *
```

Question 2A Draw the graphical representation of the profile HMM for the given multiple alignment.

Solution



Question 2B Find the state sequences that correspond to each row in the alignment.

Solution

```
AC---A M1 M2 M3
A----A M1 D2 M3
AG---T M1 M2 M3
TTGGGT M1 M2 I2 I2 I2 M3
```

Question 2C Compute the following emission probabilities with maximum likelihood estimation: $b_{M_1,A}$, $b_{M_1,G}$, $b_{M_1,C}$, $b_{M_1,T}$.

Formulae

$b_{i,k} = Pr[r_t = k | q_t = i]$, with t a point in time $(t \in T) q_1 ... q_t ... q_T$ random variables for the states$r_1 ... r_t ... r_T$ random variables

Solution $b_{M_1,A} = \frac{3}{4}$

$b_{M_1,G} = 0$

$b_{M_1,C} = 0$

$b_{M_1,T} = \frac{1}{4}$

Question 2D Compute the following transition probabilities with maximum likelihood estimation: a_{M_2,M_3}, a_{M_2,I_2}, a_{M_2,D_3}

Formulae

$$a_{i,j} = Pr[q_t = j | q_{t-1} = i]$$

Solution $a_{M_2,M_3} = \frac{2}{2+1+0} = \frac{2}{3}$

$a_{M_2,I_2} = \frac{1}{2+1+0} = \frac{1}{3}$

$a_{M_2,D_3} = \frac{0}{2+1+0} = 0$

Question 2E Repeat the calculations from c) and d) using a pseudo-count of 1.

Solution $b_{M_1,A} = \frac{3+1}{4+4} = \frac{1}{2}$

$b_{M_1,G} = \frac{0+1}{4+4} = \frac{1}{8}$

$b_{M_1,C} = \frac{0+1}{4+4} = \frac{1}{8}$

$b_{M_1,T} = \frac{1+1}{4+4} = \frac{1}{4}$

$a_{M_2,M_3} = \frac{2+1}{2+1+0+3} = \frac{1}{2}$

$a_{M_2,I_2} = \frac{1+1}{2+1+0+3} = \frac{1}{3}$

$a_{M_2,D_3} = \frac{0+1}{2+1+0+3} = \frac{1}{6}$