

Exercise sheet 1: Hidden Markov models

Exercise 1 - Two kinds of dice



A casino uses two kinds of dice: 98% of dice are fair and 2% are loaded. The loaded die has a probability of 0.5 to show number six and 0.1 for the numbers one to five.

1a)

When we pick up a die from a table at random, what is the probability of rolling a six?

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Hint 1 : Formulae

$$L = \text{Loaded} \quad F = \text{Fair} \quad \mathcal{O} = \text{Observation} \quad P(\mathcal{O}) = P(F) \times P(\mathcal{O}|F) + P(L) \times P(\mathcal{O}|L)$$

Hint 2 : Calculation Method

$$P(6) = 0.98 \times \frac{1}{6} + 0.02 \times \frac{1}{2}$$

Solution

$$P(6) = 0.173\bar{3}$$

1b)

Hide We pick up a die from a table at random and roll [6 6 6]. What is the probability, that the die is loaded.

Hint 1 : Formulae

$$P(L|\mathcal{O}) = \frac{P(L, \mathcal{O})}{P(\mathcal{O})} P(L, \mathcal{O}) = P(\mathcal{O}|L) \times P(L)$$

Hint 2 : Calculation Method

$$\begin{aligned}
P(L|\mathcal{O}) &= \frac{P(\mathcal{O}|L) \times P(L)}{P(\mathcal{O}|L) \times P(L) + P(\mathcal{O}|F) \times P(F)} \\
&= \frac{\left(\frac{1}{2}\right)^3 \times 0.02}{\left(\frac{1}{2}\right)^3 \times 0.02 + \left(\frac{1}{6}\right)^3 \times 0.98}
\end{aligned}$$

Solution

$$P(L|\mathcal{O}) = 35.53\%$$

1c)

How many sixes in a row would we need to roll to be at least 90% sure that the die is loaded?

Hide**Hint 1 : Formulae**

$$P(L|\mathcal{O}) = \frac{P(\mathcal{O}|L) \times P(L)}{P(\mathcal{O}|L) \times P(L) + P(\mathcal{O}|F) \times P(F)}$$

Hint 2 : Calculation Method

$$P(L|\mathcal{O}) = \frac{\frac{2}{100} \times \left(\frac{1}{2}\right)^n}{\frac{2}{100} \times \left(\frac{1}{2}\right)^n + \frac{98}{100} \times \left(\frac{1}{6}\right)^n} \geq 0.9 \quad | \text{ split } \left(\frac{1}{6}\right)^n \quad (1)$$

$$\iff \frac{\frac{2}{100} \times \left(\frac{1}{2}\right)^n}{\frac{2}{100} \times \left(\frac{1}{2}\right)^n + \frac{98}{100} \times \left(\frac{1}{2}\right)^n \times \left(\frac{1}{3}\right)^n} \geq \frac{9}{10} \quad | \text{ factorize } \quad (2)$$

$$\iff \frac{\frac{2}{100} \times \left(\frac{1}{2}\right)^n}{\frac{2}{100} \times \left(\frac{1}{2}\right)^n \times \left(1 + 49 \times \left(\frac{1}{3}\right)^n\right)} \geq \frac{9}{10} \quad | \text{ simplify, given } n > 0 \quad (3)$$

$$\iff \frac{1}{1 + 49 \times \left(\frac{1}{3}\right)^n} \geq \frac{9}{10} \quad | \text{ cross-multiply, given } n > 0 \quad (4)$$

$$\iff \frac{9}{10} (1 + 49 \times \left(\frac{1}{3}\right)^n) \leq 1 \quad | \text{ rewrite } \quad (5)$$

$$\iff \left(\frac{1}{3}\right)^n \leq \frac{1}{441} \quad | \ln() \quad (6)$$

$$\iff n \times \ln\left(\frac{1}{3}\right) \leq \ln\left(\frac{1}{441}\right) \quad | \times \frac{1}{\ln\left(\frac{1}{3}\right)} \quad (7)$$

$$\iff n \geq \frac{\ln\left(\frac{1}{441}\right)}{\ln\left(\frac{1}{3}\right)} \quad (8)$$

$$\iff n \geq 5.542487... \quad (9)$$

Solution

$n = 6$, as only Integers make sense here (just trying would also work)

Exercise 2 - The occasionally cheating casino



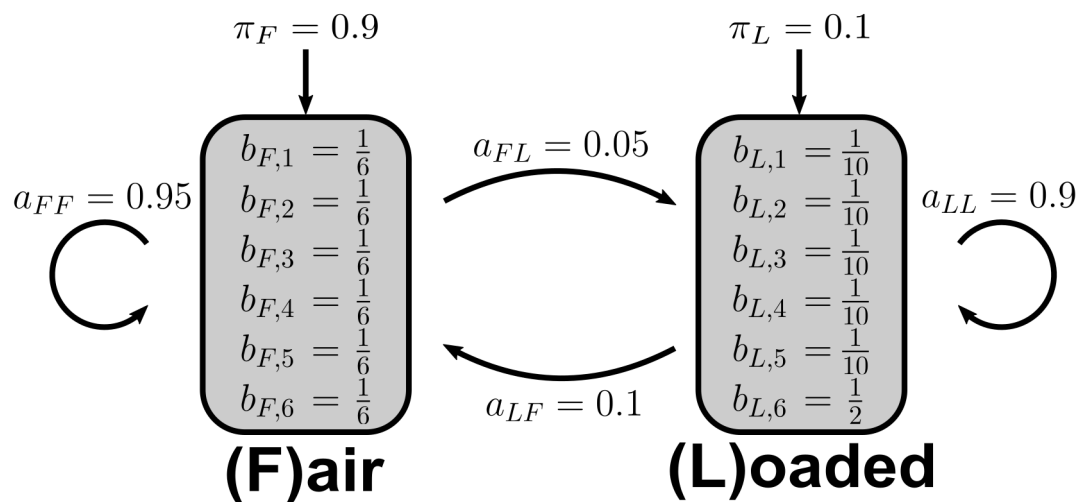
In a casino they use a fair die most of the time, but occasionally they switch to a loaded die. The loaded die has a probability 0.5 to show number six and probability 0.1 for the numbers one to five. Assume that the casino switches from a fair to a loaded die with probability 0.05 before each roll, and that the probability of switching back is 0.1. The probability to start a game with the fair die is 0.9.

2a)

Draw a graphical representation of the described Hidden Markov model.

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Solution



2b)

Given an observed sequence of outcomes $\mathcal{O} = 3661634$ and two possible state sequences $s_1 = LLLFFFF$ and $s_2 = FFFFFFFF$ (where F = Fair and L = Loaded), what are the joint probabilities $P(\mathcal{O}, s_1)$ and $P(\mathcal{O}, s_2)$ in the HMM described above?

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Hint 1 : Formulae

$$P(\mathcal{O}, s_x) = P(\mathcal{O}|s_x) \times P(s_x)$$

Hint 2 : Calculation Method

$$P(s_1) = \pi_L \times a_{LL}^2 \times a_{LF} \times a_{FF}^3 = 0.1 \times 0.9^2 \times 0.1 \times (0.95)^3 = 0.0069$$

$$P(s_2) = \pi_F \times a_{FF}^6 = 0.9 \times 0.95^6 = 0.6616$$

$$P(\mathcal{O}|s_1) = b_{L,3} \times b_{L,6}^2 \times b_{F,1} \times b_{F,6} \times b_{F,3} \times b_{F,4} = 0.1 \times 0.5^2 \times \left(\frac{1}{6}\right)^4 = 1.9 \times 10^{-5}$$

$$P(\mathcal{O}|s_2) = b_{F,3} \times b_{F,6}^2 \times b_{F,1} \times b_{F,6} \times b_{F,3} \times b_{F,4} = \left(\frac{1}{6}\right)^7 = 3.57 \times 10^{-6}$$

$$P(\mathcal{O}, s_1) = P(\mathcal{O}|s_1) \times P(s_1) = 1.9 \times 10^{-5} \times 0.0069$$

$$P(\mathcal{O}, s_2) = P(\mathcal{O}|s_2) \times P(s_2) = 3.57 \times 10^{-6} \times 0.6616$$

Solution

$$P(\mathcal{O}, s_1) = 1.34 \times 10^{-7}$$

$$P(\mathcal{O}, s_2) = 2.36 \times 10^{-6}$$

2c)

Give an observation $\mathcal{O} = 1662$, how many possible state sequences exist in the described HMM?

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Hint 1 The actual observation does not matter in this case because all emission probabilities are > 0 . There are 2^4 possible state sequences.

Solution There are 16 possible state sequences.

Exercise 3 - Programming assignment

For the programming tasks, please follow the instructions given in GitHub Classroom under the following links.

1) Intro and warm-up python programming tasks:

https://classroom.github.com/a/i_FyA5SR

2) Hidden Markov Models tasks:

<https://classroom.github.com/a/E-FEwES4>
