

Notes of Topology

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7/21/2024

前言

0. 本文档已共享在 github 仓库:<https://github.com/BacktoBedend/topology>

1. 如有发现或者看图片或排版不顺眼的地方, 请联系:

`zsg2218@foxmail.com`

(提示: 事实上作者及其合作者的拓扑成绩不太美妙, 制作这篇讲义练习 L^AT_EX 的成分居多, 可能无法给诸位学弟学妹提供答疑. 在此我们表示歉意!)

(请在主题中注明“拓扑 Latex 讲义挑刺”)

(尤其是后者, 这种情况很常见! 因为抄这篇讲义的时候及其合作者都是 L^AT_EX 新手, 难免会有将就和疏漏, 图画的不好看的肯定很多 (绝大多数都是手写版本讲义直接截图). 写完以后我们也没有太多精力把整个讲义排查一遍, 所以请读者们多多挑刺!)

在此, 我们对您的支持表示衷心感谢!

2. 本讲义整 (chao) 理 (xie) 自苏阳老师的拓扑基础课程手写讲义扫描件, 感谢他的精彩授课与辛勤付出! 苏阳老师人很好, 欢迎大家选修他的课!

3. 参考文献 (这里形式上就随便了一点, 毕竟不是学术文献)(并没有对着正文一个一个找, 所以必然有疏漏, 如有疏漏请联系上面的邮箱):

- 1) Munkres, Topology
- 2) Jänich, Topology
- 3) Armstrong, Basic Topology
- 4) Singer and Thorpe, Lecture Notes on Elementary Topology and Geometry
- 5) 尤承业, 基础拓扑学讲义
- 6) Allen Hatcher, Algebraic Topology

4. 讲义整 (chao) 理 (xie) 开始日期:2024 年 7 月 21 日,

初稿完稿日期:2024 年 8 月 2 日,

5. 作者尝试用 edge 自带的 pdf 浏览器翻阅此 pdf 发现非常卡顿.

在此推荐一款 pdf 阅读器:Sumatra PDF, 在作者的电脑上阅览此 pdf 较为流畅 (夹带私货.jpg)

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Chapter 1

Topological spaces and continuous maps

1.1 topological spaces

Definition 1.1. A topology (拓扑) on a set X is a collection \mathcal{T} of subsets of X having the following properties:

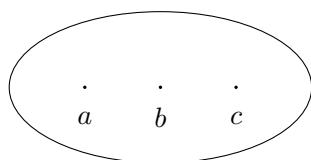
- (i) \emptyset and X are in \mathcal{T} .
- (ii) The union of the elements of any subcollection of \mathcal{T} is in \mathcal{T} .
- (iii) The intersection of the element any finite subcollection of \mathcal{T} is in \mathcal{T} .

A set X together with a topology \mathcal{T} is called a topological space (拓扑空间), denoted by (X, \mathcal{T}) or X . Elements in \mathcal{T} are called open sets (开集) of X , usually denoted by $U \subset X$.

Examples:

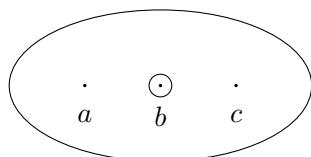
(1) $X = \{a, b, c\}$

(i)



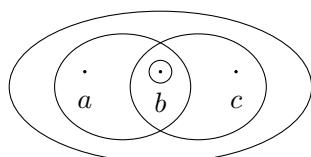
$$\mathcal{T} = \{\emptyset, X\}$$

(ii)



$$\mathcal{T} = \{\emptyset, X, \{b\}\}$$

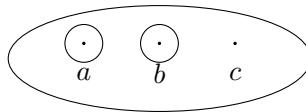
(iii)



$$\mathcal{T} = \{\emptyset, X, \{a, b\}, \{b, c\}, \{b\}\}$$

(iv) $\mathcal{T} = \text{all subsets of } X$.

non-example:



(2) X a set

- (i) $\mathcal{T} = \{\emptyset, X\}$: the trivial (平凡) topology.
- (ii) $\mathcal{T} = \text{all subsets of } X$: the discrete (离散) topology.

(3) X a set, $\mathcal{T}_f = \{U \subset X | X - U \text{ is finite or } \emptyset\} \cup \{\emptyset\}$.

Then \mathcal{T}_f is a topology on X , called the finite complement topology (余有限拓扑, 有限补拓扑).

Check:

(i) $X \in \mathcal{T}_f, \emptyset \in \mathcal{T}_f$.

(ii) $U_\alpha \in \mathcal{T}_f$ a family of open sets.

$$X - \bigcup_{\alpha} U_\alpha = \bigcap_{\alpha} (X - U_\alpha) \text{ is finite.}$$

$$\therefore \bigcup_{\alpha} \in \mathcal{T}_f.$$

(iii) $U_1, \dots, U_n \in \mathcal{T}_f$.

$$X - \bigcap_{i=1}^n U_i = \bigcup_{i=1}^n (X - U_i) \text{ is finite.}$$

$$\therefore \bigcap_{i=1}^n U_i \in \mathcal{T}_f.$$

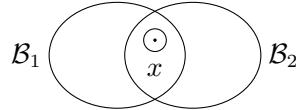
(4) X a set. $\mathcal{T}_c = \{U \subset X | X - U \text{ is countable or finite or } \emptyset\} \cup \{\emptyset\}$, then \mathcal{T}_c is a topology on X .

Definition 1.2. Let $\mathcal{T}, \mathcal{T}'$ be two topologies on a set X . If $\mathcal{T}' \subset \mathcal{T}$, we say that \mathcal{T}' is finer (细致) or larger than \mathcal{T} . \mathcal{T} is coarser (粗糙) (or smaller) than \mathcal{T}' . \mathcal{T}' is comparable with \mathcal{T} if either $\mathcal{T}' \supset \mathcal{T}$ or $\mathcal{T} \subset \mathcal{T}'$.

1.2 basis for a topology

Definition 1.3. Let X be a set. A basis for a topology (拓扑基) on X is a collection \mathcal{B} of subsets of X (called basis elements (基中的元素)) s.t.

- (1) for $\forall x \in X$, $\exists \mathcal{B} \in \mathcal{B}$ s.t. $x \in \mathcal{B}$.
- (2) if $x \in \mathcal{B}_1 \cap \mathcal{B}_2$, $\mathcal{B}_1, \mathcal{B}_2 \in \mathcal{B}$, then $\exists \mathcal{B}_3 \in \mathcal{B}$ s.t. $x \in \mathcal{B}_3 \subset \mathcal{B}_1 \cap \mathcal{B}_2$.



Lemma 1.4. If \mathcal{B} satisfies these two conditions, let \mathcal{T} be the collection of all unions of elements of \mathcal{B} , i.e.

$$\mathcal{T} = \left\{ U \subset X \mid U = \bigcup_{\alpha} \mathcal{B}_{\alpha}, \mathcal{B}_{\alpha} \in \mathcal{B} \right\}.$$

Then \mathcal{T} is a topology on X , called the topology generated by \mathcal{B} (由 \mathcal{B} 生成的拓扑).

Proof. (1) $\emptyset \in \mathcal{T}$, $X = \bigcup_{\mathcal{B} \in \mathcal{B}} \mathcal{B}$.

(2) if $U_{\alpha} \in \mathcal{T}$, then $\bigcup_{\alpha} U_{\alpha} \in \mathcal{T}$ by definition.

(3) If $U_1 = \bigcup_{\alpha} U_{\alpha}$, $U_2 = \bigcup_{\beta} \mathcal{B}'_{\beta}$, then

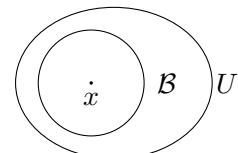
$$\begin{aligned} U_1 \cap U_2 &= \left(\bigcup_{\alpha} \mathcal{B}_{\alpha} \right) \cap \left(\bigcup_{\beta} \mathcal{B}'_{\beta} \right) \\ &= \bigcup_{\alpha, \beta} \underbrace{(\mathcal{B}_{\alpha} \cap \mathcal{B}'_{\beta})}_{\parallel} \quad \text{by (2)} \\ &\quad \bigcup_{\gamma} \mathcal{B}''_{\gamma} \end{aligned}$$

$$\therefore U_1 \cap U_2 \in \mathcal{T}.$$

□

An equivalent description of \mathcal{T} :

$$U \in \mathcal{T} \Leftrightarrow \forall x \in U, \exists \mathcal{B} \in \mathcal{B}, \text{s.t. } x \in \mathcal{B} \subset U.$$



Examples:

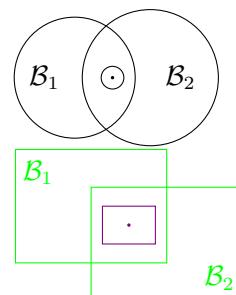
(1) $\mathcal{B} = \{(a, b) \subset \mathbb{R} \mid a, b \in \mathbb{R}\}$

The topology generated by \mathcal{B} is the standard topology on the real line \mathbb{R} .

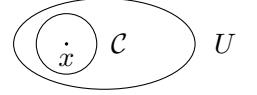
(2)

$\mathcal{B} = \{\text{all open discs in } \mathbb{R}^2\}$ generates
the standard topology on \mathbb{R}^2

$\mathcal{B}' = \{\text{all open rectangular regions in } \mathbb{R}^2\}$
generates the standard topology on \mathbb{R}^2



Lemma 1.5. (X, \mathcal{T}) a topological space, \mathcal{C} a collection of open sets, s.t. for any open set U , and each $x \in U$, $\exists \mathcal{C} \in \mathcal{C}$, s.t. $x \in \mathcal{C} \subset U$. Then \mathcal{C} is a basis for the topology \mathcal{T} of X .



Proof. (i) \mathcal{C} is a basis:

- (1) $\forall x \in X$, since X is open, $\exists \mathcal{C} \in \mathcal{C}$, s.t. $x \in \mathcal{C} \subset X$ $\therefore \bigcup_{\mathcal{C} \in \mathcal{C}} \mathcal{C} = X$.
- (2) Given $\mathcal{C}_1, \mathcal{C}_2 \in \mathcal{C}, \forall x \in \mathcal{C}_1 \cap \mathcal{C}_2$, since $\mathcal{C}_1 \cap \mathcal{C}_2$ is open, $\exists \mathcal{C} \in \mathcal{C}$ s.t. $x \in \mathcal{C} \subset \mathcal{C}_1 \cap \mathcal{C}_2$. $\therefore \mathcal{C}$ is a basis.

- (ii) Let \mathcal{T}' be the topology generated by \mathcal{C} , we need to show $\mathcal{T}' = \mathcal{T}$.

- (1) For any open set $U \in \mathcal{T}, \forall x \in U, \exists \mathcal{C} \in \mathcal{C}$, s.t. $x \in \mathcal{C} \subset U \therefore U$ is a union of open sets \mathcal{C} in \mathcal{C} , i.e. $U \in \mathcal{T}'$.
- (2) Any open set $U \in \mathcal{T}'$ is a union of open sets \mathcal{C} in $\mathcal{C} \therefore U \in \mathcal{T}$.

□

Lemma 1.6. Let \mathcal{B} and \mathcal{B}' be bases for the topologies \mathcal{T} and \mathcal{T}' , respectively, on X . Then the followings are equivalent.

- (1) \mathcal{T}' is finer than \mathcal{T} , i.e. $\mathcal{T}' \supset \mathcal{T}$
- (2) For any $\mathcal{B} \in \mathcal{B}, x \in \mathcal{B}, \exists \mathcal{B}' \in \mathcal{B}'$ s.t. $x \in \mathcal{B}' \subset \mathcal{B}$.

Proof.

- (2) \Rightarrow (1): $\forall U \in \mathcal{T}, \forall x \in U, \exists \mathcal{B} \in \mathcal{B}$ s.t. $x \in \mathcal{B} \subset U$.
 $\therefore \exists \mathcal{B}' \in \mathcal{B}'$ s.t. $x \in \mathcal{B}' \subset U$. $\therefore U$ is a union of elements in \mathcal{B}' , i.e. $U \in \mathcal{T}'$.
- (1) \Rightarrow (2): Given $\mathcal{B} \in \mathcal{B}, x \in \mathcal{B}$, now $\mathcal{B} \in \mathcal{T} \subset \mathcal{T}'$. $\therefore \mathcal{B} = \bigcup_{\alpha} \mathcal{B}'_{\alpha}$.
 $\therefore x \in \mathcal{B}'_{\alpha} \subset \mathcal{B}$ for some $\mathcal{B}'_{\alpha} \in \mathcal{B}'$.

□

Examples:

- (1) $\mathcal{B} = \{\text{open discs in } \mathbb{R}^2\}$, $\mathcal{B}' = \{\text{open rectangular regions in } \mathbb{R}^2\}$,

Then \mathcal{B} and \mathcal{B}' generate the same topology.



- (2) $\mathcal{B} = \{(a, b) \subset \mathbb{R} | a, b \in \mathbb{R}\}$ generates \mathcal{T} , the standard topology on \mathbb{R} .

$\mathcal{B}' = \{[a, b) \subset \mathbb{R} | a, b \in \mathbb{R}\}$ generates \mathcal{T}' , the lower limit topology (下极限拓扑) on \mathbb{R} , denoted by \mathbb{R}_l .

$\mathcal{B}'' = \{(a, b) \subset \mathbb{R} | a, b \in \mathbb{R}\} \cup \{(a, b) - K : (a, b \in \mathbb{R})\}$ where $K = \left\{ \frac{1}{n} \mid n \in \mathbb{Z}_+ \right\}$ generates \mathcal{T}'' , the K -topology on \mathbb{R} , denoted by \mathbb{R}_K .

Lemma 1.7. \mathbb{R}_l and \mathbb{R}_K are strictly finer than \mathbb{R} .

\mathbb{R}_l and \mathbb{R}_K are not comparable with one another.

Lemma 1.8. Let \mathcal{S} be a collection of subsets of X s.t. the union of elements in \mathcal{S} equals X . Let \mathcal{B} be the collection of all finite intersections of elements of \mathcal{S} , i.e.

$$\mathcal{B} = \left\{ \bigcup_{i=1}^n \mathcal{S}_i \mid \mathcal{S}_i \in \mathcal{S}, n \in \mathbb{N}_+ \right\}$$

Then \mathcal{B} is a basis, \mathcal{S} is called a subbasis (子基, wikipedia 上翻译为“准基”) of the topology \mathcal{T} generated by \mathcal{B} .

Proof. (1) $\bigcup_{\mathcal{S} \in \mathcal{S}} \mathcal{S} = X \Rightarrow \bigcup_{\mathcal{B} \in \mathcal{B}} \mathcal{B} = X$.

(2) For any $\mathcal{B}_1, \mathcal{B}_2 \in \mathcal{B}$, $x \in \mathcal{B}_1 \cap \mathcal{B}_2$, assume $\mathcal{B}_1 = \bigcap_{i=1}^n \mathcal{S}_i, \mathcal{B}_2 = \bigcap_{j=1}^m \mathcal{S}'_j$, then $\mathcal{B}_1 \cap \mathcal{B}_2 = \bigcap_{i=1}^n \mathcal{S}_i \cap \bigcap_{j=1}^m \mathcal{S}'_j \in \mathcal{B}$. \square

Remark: Topology \mathcal{T} :

in terms of a basis \mathcal{B} : open set $U = \bigcup_{\alpha} \mathcal{B}_{\alpha}$.

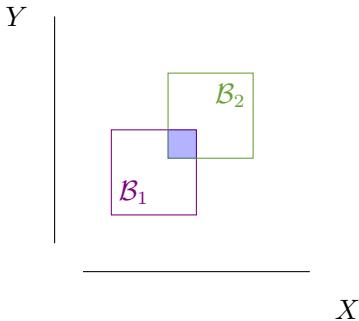
in terms of a subbasis \mathcal{S} : open set $U = \bigcup_{\alpha} \left(\bigcap_{i=1}^{n_{\alpha}} \mathcal{S}_i^{\alpha} \right)$.

1.3 the product topology on $X \times Y$

Definition 1.9. X, Y topological spaces. The product topology (乘积拓扑) on $X \times Y$ is the topology generated by the basis $\mathcal{B} = \{U \times V | U \subset X, V \subset Y \text{ open}\}$.

Check: \mathcal{B} is a basis.

- (1) $\emptyset \in \mathcal{B}, X \times Y \in \mathcal{B}$.
- (2) $\forall \mathcal{B}_1 = U_1 \times V_1, \mathcal{B}_2 = U_2 \times V_2 \in \mathcal{B}, \mathcal{B}_1 \cap \mathcal{B}_2 = (U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2) \in \mathcal{B}$.

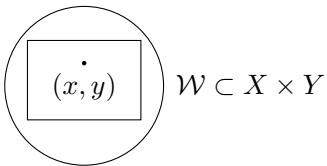


Theorem 1.10. Let $\mathcal{B}_X, \mathcal{B}_Y$ be a basis for the topology on X and Y respectively. Then the collection

$$\mathcal{D} = \{\mathcal{B}_1 \times \mathcal{B}_2 : \mathcal{B}_1 \in \mathcal{B}_X, \mathcal{B}_2 \in \mathcal{B}_Y\}.$$

is a basis for the product topology of $X \times Y$.

Proof. Recall: \mathcal{D} is a basis iff for any open set $\mathcal{W} \subset X \times Y, (x, y) \in \mathcal{W}, \exists \mathcal{B}_1, \mathcal{B}_2$ s.t. $(x, y) \in \mathcal{B}_1 \times \mathcal{B}_2 \subset \mathcal{W}$.
 $\because \mathcal{W}$ is open, $\therefore \exists U \subset X, V \subset Y$ open,



s.t. $(x, y) \in U \times V \subset \mathcal{W}, \exists \mathcal{B}_1 \in \mathcal{B}_X, \mathcal{B}_2 \in \mathcal{B}_Y$, s.t. $x \in \mathcal{B}_1 \subset U, y \in \mathcal{B}_2 \subset V, \therefore (x, y) \in \mathcal{B}_1 \times \mathcal{B}_2 \subset U \times V \subset \mathcal{W}$. \square

Example: The standard topology on $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ is the product topology of \mathbb{R} (with the standard topology) with itself. A basis is $\mathcal{B} = \{\text{open rectangular regions}\}$, another basis is $\mathcal{B}' = \{\text{open discs}\}$.

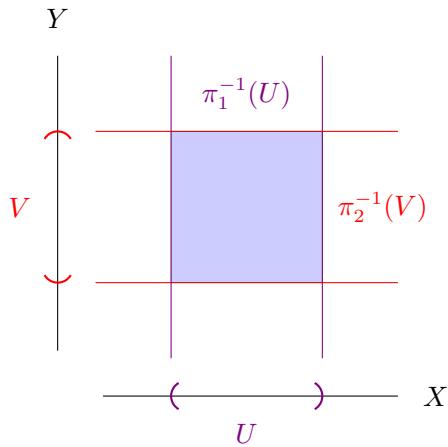
$$\begin{aligned} \text{Let } \pi_1 : X \times Y \rightarrow X, \quad \pi_2 : X \times Y \rightarrow Y \\ (x, y) \mapsto x \qquad \qquad (x, y) \mapsto y \end{aligned}$$

be the projections (投影) of $X \times Y$ onto its first and second factors, respectively. Let $U \subset X, V \subset Y$ be open sets, then $\pi_1^{-1}(U) = U \times Y, \pi_2^{-1}(V) = X \times V$ are open sets in $X \times Y$. $\pi_1^{-1}(U) \cap \pi_2^{-1}(V) = U \times V$.

Theorem 1.11. The collection

$$\mathcal{S} = \{\pi_1^{-1}(U) | U \text{ is open in } X\} \cup \{\pi_2^{-1}(V) | V \text{ is open in } Y\}$$

is a subbasis for the product topology on $X \times Y$.



Proof. Let \mathcal{T} be the product topology on $X \times Y$,

\mathcal{T}' be the topology generated by \mathcal{S} .

We need to show $\mathcal{T} = \mathcal{T}'$.

(i) every element of \mathcal{S} belongs to \mathcal{T} , i.e. open sets in \mathcal{T}' belong to \mathcal{T} , i.e. $\mathcal{T}' \subset \mathcal{T}$.

(ii) $\underbrace{U \times V}_{\text{a basis element for } \mathcal{T}} = \pi_1^{-1}(U) \cap \pi_2^{-1}(V) \in \mathcal{T}', \therefore \mathcal{T} \subset \mathcal{T}'$. □

1.4 the subspace topology

Definition 1.12. Let X be a topological space with topology \mathcal{T} . If Y is a subset of X , the collection

$$\mathcal{T}_Y = \{Y \cap U \mid U \in \mathcal{T}\}$$

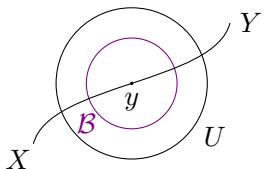
is a topology on Y , called the subspace topology (子空间拓扑). (Y, \mathcal{T}_Y) is called a subspace of (X, \mathcal{T}) .

Check: \mathcal{T}_Y is a topology on Y :

- (1) $\emptyset, Y \in \mathcal{T}_Y$.
- (2) $\bigcup_{\alpha} (Y \cap U_{\alpha}) = Y \cap \left(\bigcup_{\alpha} U_{\alpha} \right) \in \mathcal{T}_Y$.
- (3) $(Y \cap U_1) \cap (Y \cap U_2) = Y \cap (U_1 \cap U_2) \in \mathcal{T}_Y$.

Lemma 1.13. If \mathcal{B} is a basis for the topology of X , then the collection $\mathcal{B}_Y = \{\mathcal{B} \cap Y \mid \mathcal{B} \in \mathcal{B}\}$ is a basis for the subspace topology \mathcal{T}_Y .

Proof. For any $y \in Y \cap U \in \mathcal{T}_Y, \exists \mathcal{B} \in \mathcal{B}$ s.t. $y \in \mathcal{B} \subset U \subset Y$. $\therefore y \in \mathcal{B} \cap Y \subset U \cap Y$. \square



Remark: An open set in Y is not necessarily open in X . e.g. $Y = [0, 1] \subset \mathbb{R}$ for $a < 0, b \in (0, 1)$.

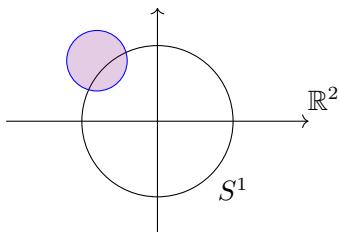
$$\begin{array}{c} (\quad [\quad) \quad] \quad \mathbb{R} \\ a \quad 0 \quad b \quad 1 \\ (a, b) \cap Y = [0, b) \text{ not open in } \mathbb{R}. \end{array}$$

Lemma 1.14. Let $Y \subset X$ be a subspace. If U is open in Y , and Y is open in X , then U is open in X .

Proof. $U = V \cap Y$ for some $V \subset X$ open, and Y is open in X , $\therefore U = V \cap Y$ is open in X . \square

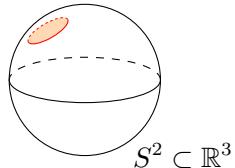
Examples:

- (1) $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\} \subset \mathbb{R}^2$ is the unit circle (1-sphere), with the subspace topology of \mathbb{R}^2 (with the standard topology).



A basis of the topology is the collection of open arcs.

- (2) the n -sphere $S^n (\subset \mathbb{R}^{n+1}) = \left\{ (x_1, x_2, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} x_i^2 = 1 \right\}$.



(3) the 2-torus

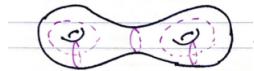


$$T^2 \subset \mathbb{R}^3$$

Question : Can you define the n -torus?

$$\text{Tips: } nT^2 = \underbrace{T^2 \# T^2 \# \cdots \# T^2}_{nT^{2'} s}$$

The closed orientable surface of genus 2



Question : Can you define the surface of genus g ?

Theorem 1.15. Let $A \subset X, B \subset Y$ be subspaces. Then the product topology on $A \times B$ is the same as the topology $A \times B$ inherits as a subspace of $X \times Y$.

Proof. Let $U \subset X, V \subset Y$ be open sets, then

$$\begin{array}{ccc}
 (U \times V) \cap (A \times B) & = & (U \cap A) \times (V \cap B) \\
 \uparrow & & \uparrow \quad \uparrow \\
 \underbrace{\text{a basis element for } X \times Y}_{\substack{\text{a basis element for} \\ \text{the subspace topology}}} & & \underbrace{\text{open sets in } A, B}_{\substack{\text{a basis element for} \\ \text{the product topology}}}
 \end{array}$$

□

Theorem 1.16. *X, Y topological spaces, $y \in Y$. Identify $X \times \{y\}$ with X . Then the subspace topology on $X \times \{y\} \subset X \times Y$ is the same as the topology on X .*

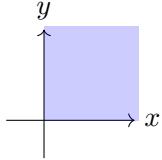
1.5 closed sets and limit points

1.5.1 closed sets

Definition 1.17. A subset A of a topological space X is said to be closed (闭) if $X - A$ is open.

Examples:

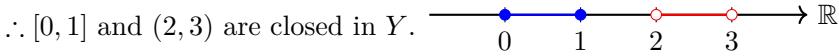
- (1) $[a, b] \subset \mathbb{R}$ is closed, since $\mathbb{R} - [a, b] = (-\infty, a) \cup (b, +\infty)$ is open.
- (2) $\{(x, y) \in \mathbb{R}^2 | x \geq 0 \text{ and } y \geq 0\}$ is closed, since the complement $= (-\infty, 0) \times \mathbb{R} \cup \mathbb{R} \times (-\infty, 0)$.



- (3) X with the finite complement topology, closed sets=finite sets.

- (4) X with the discrete topology, every set is closed.

- (5) $Y = [0, 1] \cup (2, 3) \subset \mathbb{R}$, $[0, 1] \subset Y$ is open, $(2, 3) \subset Y$ is open.



Theorem 1.18 (equivalent definition of topology). Let X be a topological space. Then the following conditions hold.

- (1) \emptyset and X are closed.
- (2) Arbitrary intersections of closed sets are closed.
- (3) Finite unions of closed sets are closed.

Proof. (1) $\emptyset = X - X$, $X = X - \emptyset$, X, \emptyset open $\Rightarrow \emptyset, X$ closed.

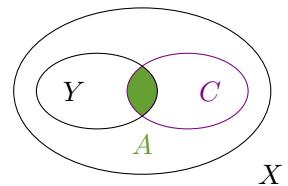
(2) $\{A_\alpha\}_{\alpha \in J}$ closed sets, then $X - \bigcap_{\alpha \in J} A_\alpha = \bigcup_{\alpha \in J} (X - A_\alpha)$ open.

(3) A_1, A_2, \dots, A_n closed sets, then $X - \bigcup_{i=1}^n A_i = \bigcap_{i=1}^n (X - A_i)$ open.

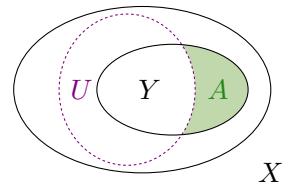
□

Theorem 1.19. $Y \subset X$ a subspace. Then $A \subset Y$ is closed if and only if $A = C \cap Y$ for some closed set $C \subset X$.

Proof. (i) Assume $A = C \cap Y$, $C \subset X$ closed. Then $Y - A = (X - C) \cap Y$, open in Y , since $X - C$ is open in X . $\therefore A$ is closed in Y .



- (ii) Assume A is closed in Y , then $Y - A$ is open, i.e. $Y - A = U \cap Y$ for some $U \subset X$ open. $\therefore A = (X - U) \cap Y$, and $X - U$ is closed in X .



□

Theorem 1.20. $Y \subset X$ a subspace, $A \subset Y$ closed, $Y \subset X$ closed. Then A is closed in X .

The proof is left for exercise.

1.5.2 closure and interior

X a topological space, $A \subset X$ a subset.

Definition 1.21. (i) The interior (内部) of A is the union of all open sets contained in A .

$$\text{int } A = \bigcup_{\substack{U \subset A \\ U \text{ open in } X}} U$$

This is the largest open set contained in A .

(ii) The closure (闭包) of A is the intersection of all closed sets containing A .

$$\overline{A} = \bigcap_{\substack{C \supset A \\ C \text{ closed in } X}} C$$

This is the smallest closed set containing A .

Remark:

$$\begin{array}{lll} \text{int } A & \subseteq A \subseteq & \overline{A} \\ \text{``$=$'' iff} & & \text{``$=$'' iff} \\ A \text{ is open} & & A \text{ is closed} \end{array}$$

Theorem 1.22. $A \subset Y \subset X$, $\overline{A} =$ the closure of A in X . Then the closure of A in Y equals $\overline{A} \cap Y$.

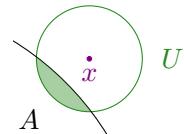
Proof. Let B be the closure of A in Y , we need to show $B = \overline{A} \cap Y$.

“ $B \subset \overline{A} \cap Y$ ”: \overline{A} is closed in X , $\therefore \overline{A} \cap Y$ is closed in Y , and $A \subset \overline{A} \cap Y$, $\therefore B \subset \overline{A} \cap Y$.

“ $B \supset \overline{A} \cap Y$ ”: B is closed in Y , $\therefore B = C \cap Y$ for some $C \subset X$, C closed, and $A \subset C$. $\therefore \overline{A} \subset C$, $\therefore \overline{A} \cap Y \subset C \cap Y = B$. \square

Theorem 1.23. Let $A \subset X$ be a subspace.

(a) Then $x \in \overline{A}$ iff every open set U containing x intersects A .



(b) Let \mathcal{B} be a basis for the topology on X , then $x \in \overline{A}$ iff every basis element containing x intersects A .

Proof. (a)

(i) If $x \in \overline{A}$, then $x \in U = X - \overline{A} \subset X - A$. U is open and $U \cap A = \emptyset$, i.e. \exists an open set U containing x and $U \cap A = \emptyset$.

(ii) If there exists an open set U s.t. $x \in U$ and $U \cap A = \emptyset$, then $C' = X - U$ is closed, $A \subset C'$ and $x \notin C'$.

$$\therefore x \notin \overline{A} = \bigcap_{\substack{C \supset A \\ C \text{ closed}}} C$$

(b) is a consequence of (a). \square

Definition 1.24. An open set U containing x is a neighborhood (邻域) of x .

Theorem 1.25. $x \in \overline{A}$ iff every neighborhood of X intersects A .

Examples:

$$(1) A = (0, 1] \subset \mathbb{R}, \overline{A} = [0, 1] \xrightarrow{\begin{array}{c} (\text{purple}) \\ 0 \quad 1 \end{array}} \mathbb{R}$$

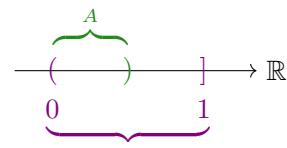
$$(2) B = \left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\}, \overline{B} = \{0\} \cup B.$$

(3) $\mathbb{Q} = \{\text{rational numbers}\} \subset \mathbb{R}$, then $\overline{\mathbb{Q}} = \mathbb{R}$.

$$(4) Y = (0, 1) \subset \mathbb{R}, A = \left(0, \frac{1}{2}\right).$$

The closure of A in \mathbb{R} is $\left[0, \frac{1}{2}\right]$.

The closure of A in Y is $\left[0, \frac{1}{2}\right] \cap Y = \left(0, \frac{1}{2}\right]$.



1.5.3 limit points

$A \subset X$ a subspace.

Definition 1.26. A point $x \in X$ is a limit point (极限点) (or cluster point, point of accumulation, 聚点) of A if every neighborhood of x intersects A in some point other than x .

In other words, x is a limit point of A iff $x \in \overline{A - \{x\}}$.

Examples:

- (1) $A = (0, 1] \subset \mathbb{R}$, limit points of $A = [0, 1]$.
- (2) $B = \left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\}$, then 0 is the only limit point of B .
- (3) $\mathbb{Q} \subset \mathbb{R}$ every limit point of \mathbb{R} is a limit point of \mathbb{Q} .

Theorem 1.27. Let $A \subset X$ be a subspace, let A' be the set of all limit points of A . Then $\overline{A} = A \cup A'$.

Proof. “ $A \cup A' \subset \overline{A}$ ”: $\forall x \in A', U$ a neighborhood of x . We have $U \cap A \neq \emptyset$. $\therefore x \in \overline{A}$. $\therefore A \cup A' \subset \overline{A}$.

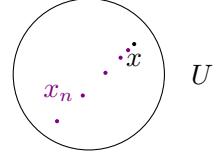
“ $A \cup A' \supset \overline{A}$ ”: For $\forall x \in \overline{A}$, if $x \in A$, then $x \in A \cup A'$; if $x \in \overline{A} - A$, then for any neighborhood U of x , $U \cap A \neq \emptyset$, and $x \notin U \cap A$. $\therefore x \in A'$. \square

Corollary 1.28. $A \subset X$ a subspace. A is closed iff A contains all its limit points.

Proof. A is closed $\Leftrightarrow A = \overline{A} = A \cup A'$. $\therefore A$ is closed $\Leftrightarrow A' \subset A$. \square

1.5.4 Hausdorff space

Definition 1.29. A sequence of points x_1, x_2, \dots in X converges (收敛) to $x \in X$ if for each neighborhood U of x , $\exists N$ s.t. for all $n \geq N, x_n \in U$.



Examples:

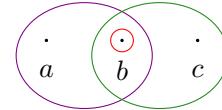
(i) $X = \mathbb{R}$

- (1) one-point set $\{x_0\}$ is closed.
- (2) A sequence cannot converge to more than one point.

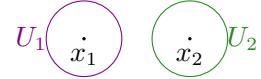
(ii) $X = \{a, b, c\}$

- (1) $\{b\}$ is not closed.

- (2) $\{x_n = b\}_{n \geq 0}$ converges to a, b and c !



Definition 1.30. A topological space X is called a Hausdorff space (豪斯多夫空间) if for each pair x_1, x_2 of distinct points of X , there exist neighborhoods U_1 of x_1 , U_2 of x_2 , s.t. $U_1 \cap U_2 = \emptyset$.



Theorem 1.31. Every finite set in a Hausdorff space X is closed.

Proof. It suffices to show that for every $x_0 \in X$, $\{x_0\}$ is closed. For $\forall x \neq x_0, \exists$ neighborhoods U of x , V of x_0 , s.t. $U \cap V = \emptyset \therefore x \in U \subset X - \{x_0\} \therefore X - \{x_0\}$ is open, $\{x_0\}$ is closed. \square

Theorem 1.32. If X is a Hausdorff space, then a sequence of points of X converges to at most one point of X .

Proof. Let $\{x_n\}_{n \geq 0}$ be a sequence converging to x . If $y \neq x$, then \exists neighborhoods U of x , V of y , s.t. $U \cap V = \emptyset$. $\exists N$, s.t. for all $n \geq N, x_n \in U \therefore \{x_n | n \geq N\} \cap V = \emptyset \therefore \{x_n\}$ does not converge to y . \square

Theorem 1.33. (a) The product of two hausdorff spaces is a Hausdorff space.

(b) A subspace of a Hausdorff space is a Hausdorff space.

The proof is left for exercise.

1.6 continuous maps(functions)

1.6.1 continuous maps

Definition 1.34. Let X, Y be topological spaces. A continuous map (连续映射) is a map $f : X \rightarrow Y$ s.t. for each open set $V \subset Y, f^{-1}(V)$ is an open set of X .

Remark:

- (1) If \mathcal{B} is a basis for the topology on Y , then any open set $V = \bigcup_{\alpha} \mathcal{B}_{\alpha}$ and $f^{-1}(V) = \bigcup_{\alpha} f^{-1}(\mathcal{B}_{\alpha})$. $\therefore f$ is continuous if $f^{-1}(\mathcal{B}_{\alpha})$ is open for every basis element $\mathcal{B} \in \mathcal{B}$.
- (2) If \mathcal{S} is a subbasis for the topology on Y , then any basis element $\mathcal{B} = \mathcal{S}_1 \cap \dots \cap \mathcal{S}_n$, and $f^{-1}(\mathcal{B}) = f^{-1}(\mathcal{S}_1) \cap \dots \cap f^{-1}(\mathcal{S}_n)$. $\therefore f$ is continuous if $f^{-1}(\mathcal{S})$ is open for every element \mathcal{S} in the subbasis \mathcal{S} .

Examples:

- (1) $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous $\Leftrightarrow f$ is continuous under the “ $\varepsilon - \delta$ ” definition.

$$\text{“}\Rightarrow\text{”} \quad \begin{array}{ccc} \overset{a}{\underset{x_0}{\overset{b}{\longrightarrow}}} & \xrightarrow{f} & \overset{f(x_0) - \varepsilon}{\underset{f(x_0)}{\overset{f(x_0) + \varepsilon}{\longrightarrow}}} \\ f^{-1}(V) & & \underbrace{\qquad\qquad\qquad}_{V=f(x_0-\varepsilon), f(x_0+\varepsilon)} \end{array}$$

$x_0 \in f^{-1}(V)$ open.

$\therefore \exists a, b$ s.t. $x_0 \in (a, b) \subset f^{-1}(V)$. Choose δ small enough s.t. $(x_0 - \delta, x_0 + \delta) \subset (a, b)$. Then $f((x_0 - \delta, x_0 + \delta)) \subset (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$. \square

Proof of “ \Leftarrow ” part is left for exercise.

- (2) (i) Continuous map $f : \mathbb{R} \rightarrow \mathbb{R}^n$ is called a continuous curve.
- (ii) Continuous map $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is called a continuous function.
- (iii) Continuous map $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is called a continuous vector field.
- (3) \mathbb{R}_l = the set of real numbers with lower limit topology, one of whose basis is $\mathcal{B} = \{[a, b) | a, b \in \mathbb{R}\}$.

Then:

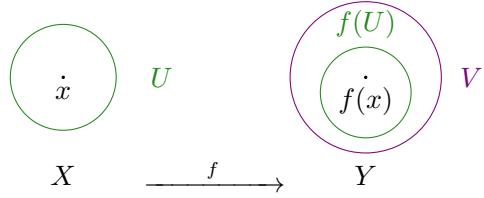
$f : \mathbb{R} \rightarrow \mathbb{R}_l, x \mapsto x$ is not continuous, since $f^{-1}([a, b)) = [a, b) \subset \mathbb{R}$ and the latter is not open.

$g : \mathbb{R}_l \rightarrow \mathbb{R}, x \mapsto x$ is continuous, since $g^{-1}((a, b)) = (a, b) = \bigcup_{n=1}^{\infty} \left[a + \frac{1}{n}, b\right)$ and the latter is open.

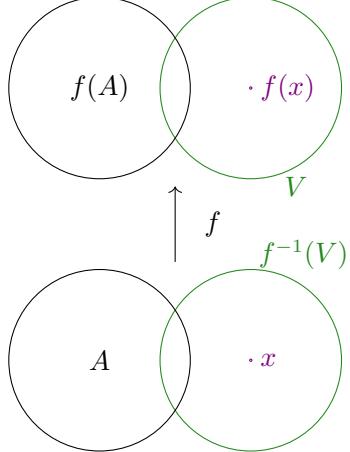
Theorem 1.35. X, Y topological spaces, $f : X \rightarrow Y$ a map. Then the followings are equivalent.

- (1) f is continuous.
- (2) For every subset $A \subset X$, $f(\overline{A}) \subset \overline{f(A)}$.
- (3) For every closed set $\mathcal{B} \subset Y$, $f^{-1}(\mathcal{B})$ is closed in X .
- (4) For each $x \in X$, and each neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subset V$.

If the condition (4) holds for the point $x \in X$, we say that f is continuous at x .



Proof. We show that (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) and (1) \Rightarrow (4) \Rightarrow (1).



Assume that f is continuous, $A \subset X, X \in \overline{A}$, we need to show $f(x) \in \overline{f(A)}$.

Let V be a neighborhood of $f(x)$, we need to show $f(A) \cap V \neq \emptyset$.

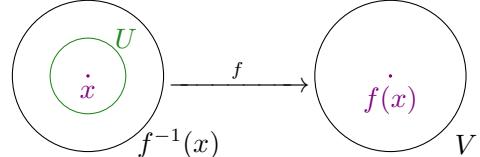
It suffices to show that $f^{-1}(V) \cap A \neq \emptyset$. But $f^{-1}(V)$ is a neighborhood of x , $x \in \overline{A} \Rightarrow f^{-1}(V) \cap A \neq \emptyset$.

(2) \Rightarrow (3): Let $\mathcal{B} \subset Y$ be a closed set, $A = f^{-1}(\mathcal{B})$. We need to show that $A = \overline{A}$. $f(A) = f(f^{-1}(\mathcal{B})) \subset \mathcal{B}$. If $x \in \overline{A}, f(x) \in f(\overline{A}) \subset \overline{f(A)} \subset \overline{\mathcal{B}} = \mathcal{B} \therefore x \in f^{-1}(\mathcal{B}) = A$, i.e. $\overline{A} = A$.

(3) \Rightarrow (1): Let $V \subset Y$ be an open set, $\mathcal{B} = Y - V$. Then $f^{-1}(\mathcal{B}) = X - f^{-1}(V)$ is closed. $\therefore f^{-1}(V)$ is open.

(1) \Rightarrow (4): Let $x \in X, V$ is a neighborhood of $f(x)$. Then $U = f^{-1}(V)$ is a nbhd of x , and $f(U) \subset V$.

(4) \Rightarrow (1): Let $V \subset Y$ be an open set. We need to show for every $x \in f^{-1}(V), \exists$ a neighborhood U of x and $U \subset f^{-1}(V)$.



Now V is a neighborhood of $f(x)$, by the hypothesis, \exists a neighborhood U of x s.t. $f(U) \subset V$, i.e. $U \subset f^{-1}(V)$. \square

Remark: Let $C(X, Y) \subset \text{Map}(X, Y)$ be the set of all continuous maps from X to Y . We will define a topology on $C(X, Y)$ later.

1.6.2 homeomorphisms

Definition 1.36. A bijection $f : X \rightarrow Y$ between topological spaces is called a homeomorphism (同胚) if both f and f^{-1} are continuous.

Notation: $f : X \xrightarrow{\cong} Y, X \cong Y$.

Remark: $f^{-1} : Y \rightarrow X$ is continuous means: $\forall U \subset X$ open, i.e. f maps open sets to open sets. Therefore a homeomorphism $f : X \rightarrow Y$ gives rise to a bijection between the topology \mathcal{T}_X on X and the topology \mathcal{T}_Y on Y :

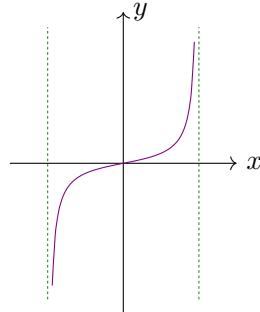
$$\mathcal{T}_x \xrightleftharpoons[f]{f^{-1}} \mathcal{T}_y$$

A property of topological spaces which is invariant under homeomorphisms is called a topological property (拓扑性质).

Examples:

$$(1) \quad \mathbb{R} \xrightleftharpoons[g]{f} \mathbb{R}, \text{ in which } f : x \mapsto 3x + 1, g : y \mapsto \frac{y - 1}{3}. \quad f \text{ is a homeomorphism.}$$

$$(2) \quad (-1, 1) \xrightleftharpoons[g]{f} \mathbb{R}, \text{ in which } f : x \mapsto \frac{x}{1 - x^2}, g : y \mapsto \frac{2y}{1 + \sqrt{1 + 4y^2}}. \quad f \text{ is a homeomorphism.}$$



$$(3) \quad \mathbb{R}_l \xrightleftharpoons[g]{f} \mathbb{R} \text{ in which } f : x \mapsto x \text{ is continuous, } g : y \mapsto y \text{ is not continuous. In fact, } \mathbb{R}_l \text{ and } \mathbb{R} \text{ are not homeomorphic since the topology } \mathbb{R}_l \text{ is finer than } \mathbb{R}. \quad$$

(4)

$$[0, 1] \subset \mathbb{R} \xrightarrow{f} S^1 \subset \mathbb{R}^2$$

$$t \mapsto (\cos 2\pi t, \sin 2\pi t)$$

f is continuous and bijective, but f^{-1} is not continuous: $f\left([0, \frac{1}{4}]\right) \subset S^1$ is not open. In fact, $[0, 1]$ is not homeomorphic to S^1 .

Let $f : X \rightarrow Y$ be an injective continuous map, $Z = f(X) \subset Y$, then $f' : X \rightarrow Z$ is bijective.

Definition 1.37. If $f' : X \rightarrow Z = f(X)$ is a homeomorphism, then the map $f : X \rightarrow Y$ is called a topological embedding (拓扑嵌入) of X in Y .

1.6.3 constructing continuous maps

Theorem 1.38. Let X, Y, Z be topological spaces

- (a) (constant map) the constant map $f : X \rightarrow Y, x \mapsto y_0$ is continuous for $\forall y_0 \in Y$.
- (b) (inclusion) If $A \subset X$ is a subspace, then the inclusion map $i_A : A \rightarrow X$ is continuous.
- (c) (composition) If $f : X \rightarrow Y, g : Y \rightarrow Z$ are continuous, then $g \circ f : X \rightarrow Z$ is continuous.
- (d) (restricting the domain) If $f : X \rightarrow Y$ is continuous, $A \subset X$ a subspace, then the restricted map $f|_A : A \rightarrow Y$ is continuous.
- (e) (restricting or expanding the range) If $f : X \rightarrow Y$ is continuous, $Z \subset Y$ a subspace s.t. $f(X) \subset Z$, then $g : X \rightarrow Z$ (obtained by restricting the range of f) is continuous; If $Y \subset Z$ is a subspace, then $h : X \rightarrow Z$ (obtained by expanding the range of f) is continuous.
- (f) (local formulation of continuity) If $X = \bigcup_{\alpha} U_{\alpha}$ is a union of open sets $\{U_{\alpha}\}$. Then $f : X \rightarrow Y$ is continuous iff $f|_{U_{\alpha}} : U_{\alpha} \rightarrow Y$ is continuous for all α .

Proof. (e) $\mathcal{B} \subset Z$ open, then $\mathcal{B} = Z \cap U$ for some $U \subset Y$ open, $g^{-1}(\mathcal{B}) = f^{-1}(U)$ open in X . (f) $V \subset Y$ open, $f^{-1}(V) \cap U_{\alpha} = \left(f|_{U_{\alpha}}\right)^{-1}(V)$, $f^{-1}(V) = \bigcup_{\alpha} (f^{-1}(V) \cap U_{\alpha})$. \square

Theorem 1.39 (the pasting lemma). Let $X = A \cup B$, where A, B are closed. Let $f : A \rightarrow Y, g : B \rightarrow Y$ be continuous. If $f(x) = g(x)$ for all $x \in A \cap B$, then f and g combine to give a continuous map

$$h : X \rightarrow Y, h(x) = \begin{cases} f(x) & x \in A \\ g(x) & x \in B \end{cases}$$

Proof. Let $C \subset Y$ be a closed set, then $h^{-1}(C) = f^{-1}(C) \cap g^{-1}(C)$ is a closed set, for both of the sets on the right side are closed. \square

Examples:

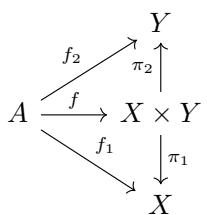
$$(1) h : \mathbb{R} \rightarrow \mathbb{R}, h(x) = \begin{cases} x & x \leq 0 \\ \frac{x}{2} & x \geq 0 \end{cases} \text{ is continuous.}$$

$$(2) f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = \begin{cases} x - 2 & x < 0 \\ x + 2 & x \geq 0 \end{cases} \text{ is not continuous.}$$

Theorem 1.40 (Maps into products). Let $f : A \rightarrow X \times Y, f(a) = (f_1(a), f_2(a))$ be a map. Then f is continuous iff the coordinate maps $f_1 : A \rightarrow X, f_2 : A \rightarrow Y$ are continuous.

Proof. Let $\pi_1 : X \times Y \rightarrow X, \pi_2 : X \times Y \rightarrow Y$ be the projections, then π_1, π_2 are continuous. $f_i = \pi_i \circ f$.

f continuous $\Rightarrow f_i$ is continuous $i = 1, 2$



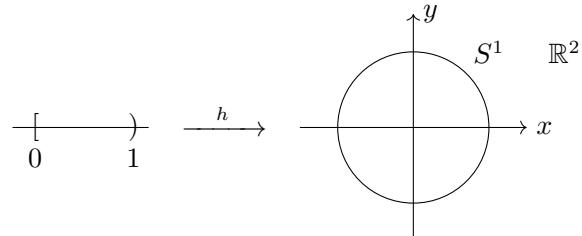
Conversely, assume f_1, f_2 are continuous. A basis for the topology on $X \times Y$ os $\{U \times V | U \subset X, V \subset Y \text{ open}\}$. It suffices to show that $f^{-1}(U \times V)$ is open but $f^{-1}(U \times V) = f_1^{-1}(U) \cap f_2^{-1}(V)$. \square

Example: A curve $\gamma : \mathbb{R} \rightarrow \mathbb{R}^2, x \mapsto (\gamma_1(x), \gamma_2(x))$ is continuous iff γ_1 and γ_2 are continuous functions.

e.g.

$$[0, 1) \xrightarrow{h} S^1$$

$$t \mapsto (\cos 2\pi t, \sin 2\pi t)$$



Caveat: A map $f : A \times B \rightarrow X$ which is continuous “in each variable separately” is **not** necessarily continuous.

1.6.4 categories and functors

Definition 1.41. A category (范畴) \mathcal{C} consists of

(i) a collection of objects (对象)

(ii) a set $\mathcal{C}(A, B)$ of morphism (态射) between any two objects A, B in \mathcal{C} satisfying

(1) a composition law $\begin{array}{ccc} \mathcal{C}(B, C) & \times & \mathcal{C}(A, B) \\ g & & f \\ \downarrow & & \downarrow \\ g \circ f & \mapsto & g \circ f \end{array}$ which is associative: $h \circ (g \circ f) = (h \circ g) \circ f$.

(2) \exists an identity morphism $\text{id}_A \in \mathcal{C}(A, A)$ for any A in \mathcal{C} , s.t. $\text{id} \circ f = f, f \circ \text{id} = f$.

Examples:

(1) The category of sets: $\mathcal{S}\text{ets}$

objects=sets, morphisms=maps between sets.

(2) The category of vector spaces over \mathbb{F} : $\text{Vect}_{\mathbb{F}}$.

(3) The category of groups: $\mathcal{G}\text{rps}$.

(4) The category of abelian groups: $\mathcal{A}\text{b}$.

(5) The category of (left) R -modules: $R\text{Mod}$.

(6) The category of topological spaces: $\mathcal{T}\text{op}$

objects=topological spaces

morphism=continuous maps

(7) If a category \mathcal{C} consists of only one object \cdot , then $\mathcal{C}(\cdot, \cdot)$ is a monoid (半群).

A group G can be viewed as a category with one object and all morphisms are invertible.

Definition 1.42. A functor (函子) $F : \mathcal{C} \rightarrow \mathcal{D}$ between two categories \mathcal{B}, \mathcal{D} assigns an object $F(A)$ of \mathcal{D} to each object A of \mathcal{C} , and a morphism $F(f) : F(A) \rightarrow F(B)$ of \mathcal{D} to each morphism $f : A \rightarrow B$ if \mathcal{C} , s.t.

$$F(\text{id}_A) = \text{id}_{F(A)}, F(g \circ f) = F(g) \circ F(f)$$

$$\mathcal{C} \xrightarrow{F} \mathcal{D}$$

$$\begin{array}{ccc} A & \longmapsto & F(A) \\ \downarrow f & & \downarrow F(f) \\ B & \longmapsto & F(B) \end{array}$$

In algebraic topology one studies functors

$$\mathcal{T}\text{op} \rightarrow \text{Vect}, \mathcal{G}\text{rps}, \mathcal{A}\text{b}$$

1.7 the quotient topology

X a topological space, a disjoint union is $X = \coprod_{\alpha \in A} X_\alpha$. Then there is a surjective map $p : X \rightarrow A, X \mapsto a$ if $x \in X_a$.

Question: How to define a topology on A s.t. p is continuous?

Answer: $U \subset A$ open iff $p^{-1}(U)$ is open in X .

Check: This is a topology on A

$$(1) \emptyset, A \in \mathcal{T}_A$$

$$(2) p^{-1} \left(\bigcup_{\alpha} U_{\alpha} \right) = \bigcup_{\alpha} p^{-1}(U_{\alpha}) \text{ open.}$$

$$(3) p^{-1}(U_1 \cap U_2) = p^{-1}(U_1) \cap p^{-1}(U_2) \text{ open.}$$

This is the largest topology on A s.t. p is continuous.

Definition 1.43. X, Y topological spaces, $p : X \rightarrow Y$ a surjective map is said to be a quotient map (商映射) provided that a subset $U \subset Y$ is open iff $p^{-1}(U)$ is open in X .

Remark: Equivalently, a surjective map $p : X \rightarrow Y$ is a quotient map if $\mathcal{B} \subset Y$ is closed iff $p^{-1}(\mathcal{B})$ is closed. (Since $p^{-1}(Y - \mathcal{B}) = X - p^{-1}(\mathcal{B})$).

Examples: A map $f : X \rightarrow Y$ an open map / closed map if the image of each open / closed set is open / closed. If $p : X \rightarrow Y$ is a surjective continuous map, and open or closed, then p is a quotient map.

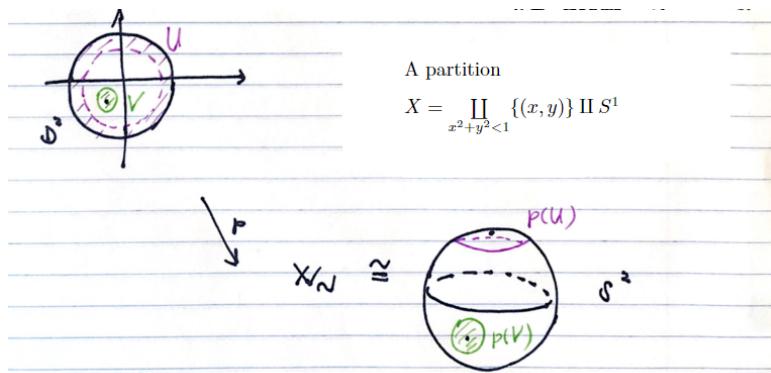
Definition 1.44. X a topological space, A a set, $p : X \rightarrow A$ a surjective map. Then there exists exactly one topology on A relative to which p is a quotient map; it's called the quotient topology (商拓扑) induced by p .

$$\text{a partition of } X, X = \coprod_{a \in A} X_a \Leftrightarrow \text{an equivalent relation } \sim \text{ on } X, \text{ with } A = X / \sim.$$

Definition 1.45. X / \sim with the quotient topology is called a quotient space (商空间) of X , or an identification space (等化空间).

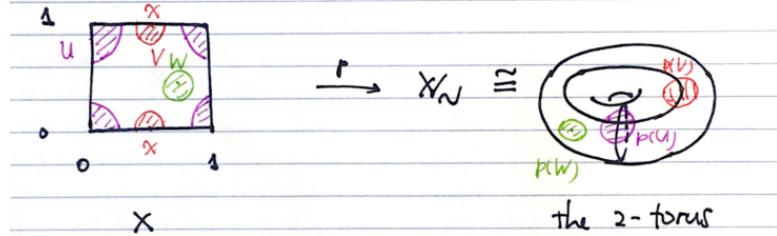
Examples:

$$(1) X = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \leq 1\} = D^2 \subset \mathbb{R}^2 \text{ the closed unit circle.}$$

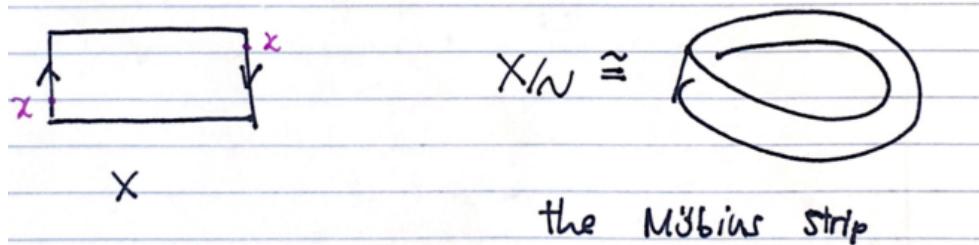


$$(2) X = [0, 1] \times [0, 1] \text{ a partition.}$$

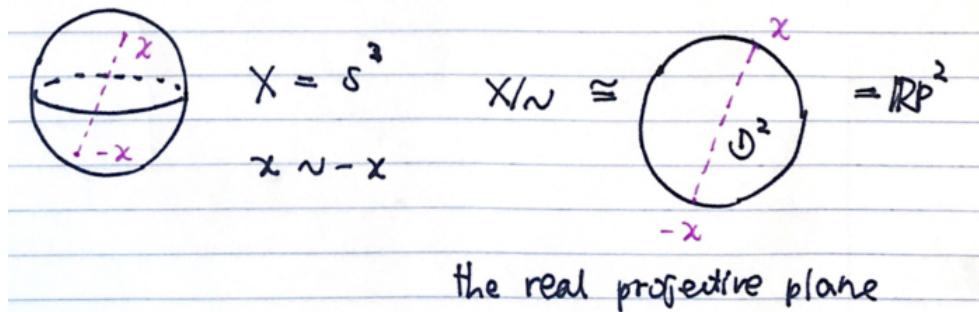
$$X = \coprod_{\substack{0 < x < 1 \\ 0 < y < 1}} \{(x, y)\} \amalg \coprod_{0 < x < 1} \{(x, 0), (x, 1)\} \amalg \coprod_{0 < y < 1} \{(0, y), (1, y)\} \amalg \{(0, 0), (0, 1), (1, 0), (1, 1)\}$$



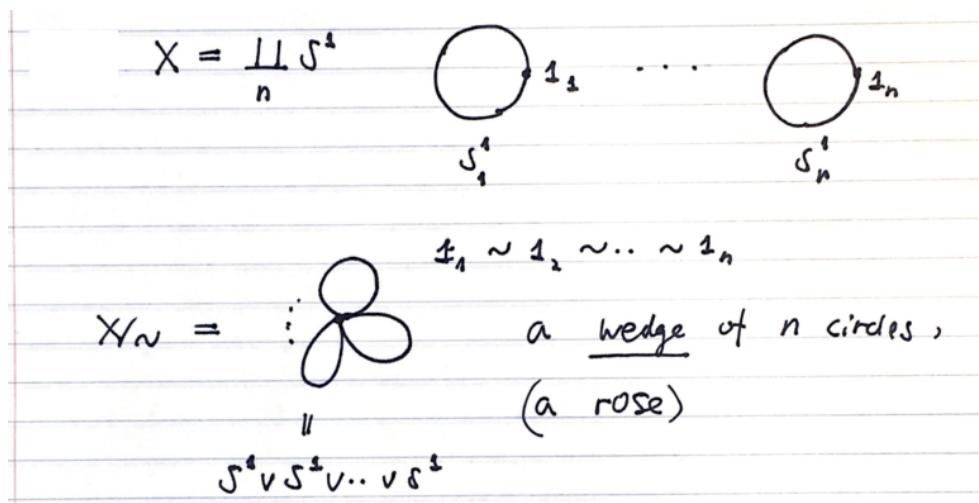
(3)



(4)



(5)



More generally we may construct $X \vee Y$.

Theorem 1.46. Let $p : X \rightarrow Y$ be a quotient map, $g : X \rightarrow Z$ be a map, that is constant on each $p^{-1}(y), \forall y \in Y$. Then g induces a map $f : Y \rightarrow Z$ s.t. $f \circ p = g$.

Then

- (1) f is continuous iff g is continuous.
- (2) f is quotient map iff g is a quotient map.

$$\begin{array}{ccc} x \in & X & \xrightarrow{p} Y \\ & \searrow g & \downarrow f \\ & g(x) \in & Z \end{array} \quad \exists y \downarrow \quad \exists f(y)$$

Proof. $\forall y \in Y$, for $x \in p^{-1}(y)$ defines $f(y) = g(y)$. This gives a well-defined map $f : Y \rightarrow Z$ s.t. $g = f \circ p$.

(a) f continuous $\Rightarrow g$ continuous.

Now assume g is continuous. For any $V \subset Z$ open, $g^{-1}(V) \subset X$ is open. But $g^{-1}(V) = p^{-1}(f^{-1}(V))$, and p is the quotient map.. $\therefore f^{-1}(V) \subset Y$ is open.

(b) f is a quotient map $\Rightarrow g$ is a quotient map. (Left for exercise)

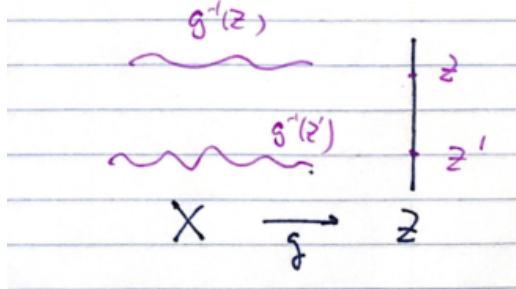
Now assume g is a quotient map. Then f is surjective.

$$V \subset Z \text{ open } \Leftrightarrow g^{-1}(V) \subset X \text{ open } (g^{-1}(V) = p^{-1}(f^{-1}(V))) \Leftrightarrow f^{-1}(V) \subset Y \text{ open} .$$

□

Corollary 1.47. Let $g : X \rightarrow Z$ be a surjective continuous map. Let X^* be the following collection of subsets of X : $X^* = \{g^{-1}(Z) | z \in Z\}$. Give X^* a quotient topology.

$$\coprod_{X^*} g^{-1}(z) = X \xrightarrow{g} Z \xleftarrow{f} X^*$$



(a) The map g induces a bijective continuous map $f : X^* \rightarrow Z$, which is a homeomorphism iff g is a quotient map.

(b) If Z is Hausdorff, so is X^* .

Proof. (a) f is bijective, and continuous by the theorem.

(i) f is a homeomorphism $\Rightarrow f$ is a quotient map

$$\Rightarrow g \text{ is a quotient map.}$$

(ii) g is a quotient map $\Rightarrow f$ is a quotient map, since f is bijective $\Rightarrow f$ is a homeomorphism.

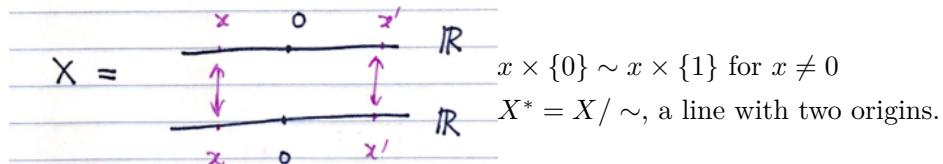
(b) For $x, y \in X^*, x \neq y \Rightarrow f(x) \neq f(y)$ in Z .

$\therefore \exists$ neighborhoods $U, V, f(x) \in U, f(y) \in V, U \cap V = \emptyset$. Then $f^{-1}(U)$ and $f^{-1}(V)$ are neighborhoods of x and y respectively, and $f^{-1}(U) \cap f^{-1}(V) = \emptyset$. □

Remark:

- (1) Composites of quotient maps are quotient maps.
- (2) The restriction of a quotient map $p : X \rightarrow Y$ to a subspace $A \subset X, p : A \rightarrow p(A)$, needs not to be a quotient map.
- (3) Products of quotient maps are not necessarily quotient maps.
- (4) The quotient space of a Hausdorff space is not necessarily Hausdorff.

e.g.



1.8 The product topology

Definition 1.48. Let $\{X_\alpha\}_{\alpha \in J}$ be an indexed family of sets. The cartesian product (笛卡尔积) $\coprod_{\alpha \in J} X_\alpha = \left\{ x : J \rightarrow \bigcup_{\alpha \in J} X_\alpha \mid x(\alpha) \in X_\alpha \right\}$ is the set of all J-tuples $(X_\alpha)_{\alpha \in J}$.

Theorem 1.49 (Axiom of choice). Given a collection of sets \mathcal{A} , there is a map $f : \mathcal{A} \rightarrow \bigcap_{A \in \mathcal{A}} A$ s.t. $f(A) \in A, \forall A \in \mathcal{A}$.

Definition 1.50. Let $\{X_\alpha\}_{\alpha \in J}$ be an induced family of topological spaces. The topology on $\prod_\alpha X_\alpha$ generates by a basis

$$\mathcal{B} = \left\{ \prod_{\alpha \in J} U_\alpha \mid U_\alpha \subset X_\alpha \text{ open} \right\}$$

is called the box topology (箱拓扑). The topology on $\prod_\alpha X_\alpha$ generated by the basis

$$\mathcal{B}' = \left\{ \prod_{\alpha \in J} U_\alpha \mid U_\alpha \subset X_\alpha \text{ open for all but finitely many } \alpha's \right\}$$

is called the product topology. In this topology $\prod_\alpha X_\alpha$ is called a product space.

Remark: Let $\pi_\beta : \prod_{\alpha \in J} X_\alpha \rightarrow X_\beta$ be the projection to the β -th coordinate. Then the product topology has a subbasis

$$\mathcal{S} = \{\pi_\beta^{-1}(U_\beta) \mid U_\beta \subset X_\beta \text{ open for some } \beta\}$$

Since $\pi_{\beta_1}^{-1}(U_{\beta_1}) \cap \dots \cap \pi_{\beta_n}^{-1}(U_{\beta_n}) = \prod_{\alpha \neq \beta_1, \dots, \beta_n} -\alpha U_\alpha$ where $U_\alpha = X_\alpha$ for $\alpha \neq \beta_1, \dots, \beta_n$.

For finite products, “box topology”=“product topology”.

In general, the box topology is finer than the product topology.

Theorem 1.51. Let $f : A \rightarrow \prod_\alpha X_\alpha, f(a) = (f_\alpha(a)) - \alpha \in J$ be a map, where $f_\alpha : A \rightarrow X_\alpha$. Let $\prod_\alpha X_\alpha$ have the product topology. Then f is continuous iff f_α is continuous for all α .

Proof. (a) Note that the projection $\pi_\beta : \prod_\alpha X_\alpha \rightarrow X_\beta$ is continuous, since for any $U_\beta \subset X_\beta$ open, $\pi_\beta^{-1}(U_\beta)$ is open in $\prod_\alpha X_\alpha$. $\therefore f_\beta = \pi_\beta \circ f$ is continuous for $\forall \beta$.

(b) Assume f_α is continuous for all α . We need to show f is continuous. It suffices to show for any element $\pi_\beta^{-1}(U_\beta)$ in the subbasis \mathcal{S} , $f^{-1}(\pi_\beta^{-1}(U_\beta))$ is open. But $f^{-1}(\pi_\beta^{-1}(U_\beta)) = (\pi_\beta \circ f)^{-1}(U_\beta) = f_\beta^{-1}(U_\beta)$ open in A . □

Example: $\mathbb{R}^\omega = \prod_{i=1}^{\infty} \mathbb{R}$, $f : \mathbb{R} \rightarrow \mathbb{R}^\omega, t \mapsto (t, t, \dots)$ (the diagonal)

Then f is continuous if \mathbb{R}^ω is given the product topology, but not continuous if \mathbb{R}^ω is given the box topology.

Let $B = (-1, 1) \times \left(-\frac{1}{2}, \frac{1}{2}\right) \times \left(-\frac{1}{3}, \frac{1}{3}\right) \times \dots \times \left(-\frac{1}{n}, \frac{1}{n}\right) \times \dots \subset \mathbb{R}^\omega$ be an open set in the box topology, then $f^{-1}(B) = \bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$.

Remark: (1) a family of topological spaces $Y_\lambda, \lambda \in \Lambda$,

a family of maps $f_\lambda : X \rightarrow Y_\lambda$.

Q : What is the smallest topology on X s.t. f_λ is continuous for all $\lambda \in \Lambda$?

A : $\mathcal{S} = \{f_\lambda^{-1}(U_\lambda) \mid \lambda \in \Lambda, U_\lambda \subset Y_\lambda \text{ open}\}$ is a subbasis, generates the desired topology (“weak topology”).

e.g.

(i) subspace topology $A \subset X$

(ii) product topology $X = \prod_\lambda X_\lambda$

(2) a family of topological spaces $X_\lambda, \lambda \in \Lambda$.

a family of maps $f_\lambda : X \rightarrow Y_\lambda$.

Q : What is the largest topology on X s.t. f_λ is continuous for all $\lambda \in \Lambda$?

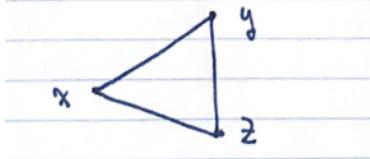
A : $\mathcal{T} = \{U \mid f_\lambda^{-1}(U) \subset X_\lambda \text{ open for all } \lambda \in \Lambda\}$ (“the strong topology”).

e.g. quotient map $f : X \rightarrow Y$.

1.9 the metric topology

Definition 1.52. A metric (度量) on a set X is a function $d : X \times X \rightarrow \mathbb{R}$ having the following properties:

- (1) $d(x, y) \geq 0$ for all $x, y \in X$; $d(x, y) = 0$ iff $x = y$.
- (2) $d(x, y) = d(y, x)$ (symmetric)
- (3) $d(x, y) + d(y, z) \geq d(x, z) \quad \forall x, y, z \in X$ (triangle inequality)

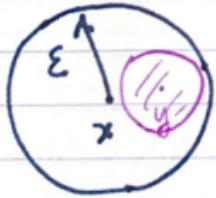


A set X together with a metric d is called a metric space (度量空间); $d(x, y)$ is called the distance between x and y ; $B_d(x, \varepsilon) = \{y \in X | d(x, y) < \varepsilon\}$ is called the ε -ball centered at x .

Definition 1.53. (X, d) a metric space, $\mathcal{B} = \{B_d(x, \varepsilon) | x \in X, \varepsilon > 0\}$ is a basis for a topology on X , called the metric topology induced by d .

Check: Claim: for any $y \in B(x, \varepsilon)$, $\exists \delta > 0$, s.t. $B(y, \delta) \subset B(x, \varepsilon)$.

Proof.



Take $\delta = \varepsilon - d(x, y) > 0$, then for any $z \in B(y, \delta), d(x, z) \leq d(x, y) + d(y, z) < \varepsilon \therefore B(y, \delta) \subset B(x, \varepsilon)$.

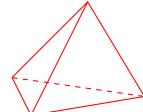
□

Given any balls B_1, B_2 , and $y \in B_1 \cap B_2, \exists \delta_1, \delta_2$ s.t. $B(y_i, \delta_i) \subset B_i (i = 1, 2)$. Let $\delta = \min \{\delta_1, \delta_2\}$, then $B(y, \delta) \subset B_1 \cap B_2$.

Lemma 1.54. A set U is open in the metric topology induced by d iff for each $y \in U$, $\exists \delta > 0$, s.t. $B_d(y, \delta) \subset U$.

Examples:

- (1) X a set, $d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$ is a metric on X , which induces the discrete topology.



- (2) The standard metric on $\mathbb{R} : d(x, y) = |x - y|$. The induced topology is the standard topology on \mathbb{R} .

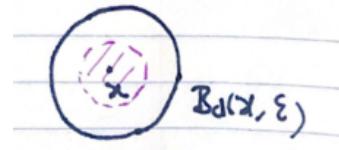
(3) The Euclidean metric on \mathbb{R}^n :

$$d(x, y) = \|x - y\| = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$$

The quarare metric on \mathbb{R}^n :

$$\rho(x, y) = \max\{|x_1 - y_1|, \dots, |x_n - y_n|\}$$

Lemma 1.55. Let d and d' be two metrics on X , with induced topologies \mathcal{T} and \mathcal{T}' respectively. Then \mathcal{T}' is the finer than \mathcal{T} iff for $\forall x \in X, \varepsilon > 0, \exists \delta > 0, \text{s.t. } B_{d'}(x, \delta) \subset B_d(x, \delta)$.



Proof. Suppose \mathcal{T}' is finer than \mathcal{T} . Let $B_d(x, \varepsilon)$ be a basis element for \mathcal{T} , then $\exists \delta > 0$, s.t. $B_{d'}(x, \delta) \subset B_d(x, \varepsilon)$.

Conversely, suppose the condition holds. For any open set $U \in \mathcal{T}, \forall x \in U, \exists \varepsilon > 0$, s.t. $B_d(x, \varepsilon) \subset U$. Then $\exists \delta > 0$, s.t. $B_{d'}(x, \delta) \subset B_d(x, \varepsilon) \therefore U$ is open in \mathcal{T}' . \square

Theorem 1.56. The topologies on \mathbb{R}^n induced by the euclidean metric d and the square metric ρ are the same as the product topology on \mathbb{R}^n .

Proof. (1)

$$\underbrace{\rho(x, y)}_{B_d(x, \varepsilon) \subset B_\rho(x, \varepsilon)} \leq d(x, y) \leq \underbrace{\sqrt{n}\rho(x, y)}_{B_\rho\left(x, \frac{\varepsilon}{\sqrt{n}}\right) \subset B_d(x, \varepsilon)}$$

$\therefore d$ and ρ induces the same topology.

(2)

- (i) Let $\mathcal{B} = (a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_n, b_n)$ be a basis element for the product topology, then for $\forall x = (x_1, \dots, x_n) \in \mathcal{B}, \exists \varepsilon_i > 0$ s.t. $(x_i - \varepsilon_i, x_i + \varepsilon_i) \subset (a_i, b_i)$. Let $\varepsilon = \min\{\varepsilon_1, \dots, \varepsilon_n\}$, then $B_\rho(x, \varepsilon) \subset \mathcal{B}$. \therefore the ρ -topology is finer than the product topology.
- (ii) Conversely, $B_\rho(x, \varepsilon) = (x_1 - \varepsilon, x_1 + \varepsilon) \times \cdots \times (x_n - \varepsilon, x_n + \varepsilon)$ is a basis element for the product topology. \therefore the product topology is finer than the ρ -topology.

\square

Definition 1.57. A topological space X is metrizable if there exists a metric d on the set X Which induces the topology of X .

Question: Is $\mathbb{R}^\omega = \prod_{n=1}^{\infty} \mathbb{R}$ metrizable?

Generalization of the d and ρ metrics:

$$(i) d(x, y) = \left(\sum_{i=1}^{\infty} (x_i - y_i)^2 \right)^{\frac{1}{2}}, \text{convergence problem} \rightarrow l^2\text{-topology.}$$

(ii) $\rho(x, y) = \sup\{|x_n - y_n| \mid n \in \mathbb{N}\}$. The topology is not well-defined, so we modify the metric $\bar{\rho}(x, y) = \sup\{\bar{d}(x_n, y_n) \mid x \in \mathbb{N}\}$ where $\bar{d}(x, y) = \min\{|x - y|, 1\} \rightarrow$ the uniform metric.

Theorem 1.58. For $x, y \in \mathbb{R}^\omega$, define $D(x, y) = \sup_n \left\{ \frac{\bar{d}(x_n, y_n)}{n} \right\}$. Then D is a metric that induces the product topology on \mathbb{R}^ω .

Proof. (i) D is a metric:

$$\frac{\bar{d}(x_n, z_n)}{n} \leq \frac{\bar{d}(x_n, y_n)}{n} + \frac{\bar{d}(y_n, z_n)}{n} \leq D(x, y) + D(y, z)$$

$$\therefore D(x, z) \leq D(x, y) + D(y, z).$$

(ii) $B_D(x, \varepsilon)$ is open in the product topology: Choose N large enough s.t. $\frac{1}{N} < \varepsilon$, let $V = (x_1 - \varepsilon, x_1 + \varepsilon) \times \cdots \times (x_N - \varepsilon, x_N + \varepsilon) \times \mathbb{R} \times \mathbb{R} \times \cdots$ be a neighborhood of x in the product topology.

Claim: $V \subset B_D(x, \varepsilon)$.

Proof. $y \in V, D(x, y) = \sup_n \left\{ \frac{\bar{d}(x_n, y_n)}{n} \right\}$ where $\begin{cases} \frac{\bar{d}(x_n, y_n)}{n} \leq \frac{1}{N} & \text{for } n \geq N \\ \bar{d}(x_n, y_n) < \varepsilon & \text{for } n < N \end{cases}$

$$\therefore D(x, y) < \varepsilon, y \in B_D(x, \varepsilon). \quad \square$$

(iii) Let $U = \prod_{i=1}^{\infty} U_i$ be a basis element for the product topology, where $U_i = \mathbb{R}$ for $i \neq \alpha_1, \dots, \alpha_n$. Given $x \in U$, choose $\varepsilon_i \leq 1$ s.t. $(x_i - \varepsilon_i, x_i + \varepsilon_i) \subset U_i$ for $i = \alpha_1, \dots, \alpha_n$. Let $\varepsilon = \min \left\{ \frac{\varepsilon_i}{i} \mid i = \alpha_1, \dots, \alpha_n \right\}$.

Claim: $B_D(x, \varepsilon) \subset U$.

Proof. $\forall y \in B_D(x, \varepsilon), \frac{\bar{d}(x_i, y_i)}{i} < \varepsilon \leq \frac{\varepsilon_i}{i}$ for $i = \alpha_1, \dots, \alpha_n \therefore \bar{d}(x_i, y_i) < \varepsilon_i \leq 1 \therefore d(x_i, y_i) = \bar{d}(x_i, y_i) < \varepsilon_i \therefore y_i \in U_i, y \in U$. \square

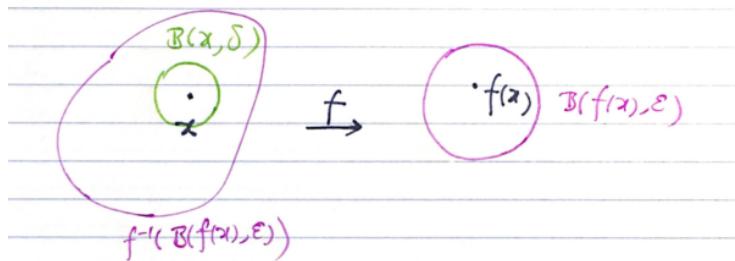
\square

Remark:

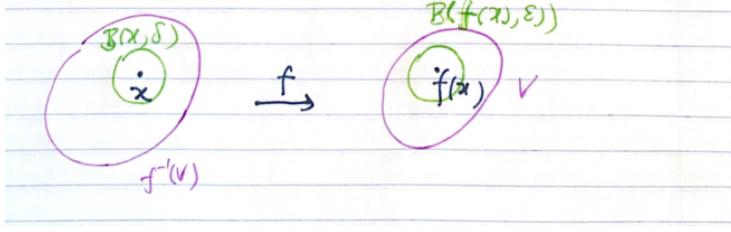
- (1) $A \subset X$ a subspace of a metric space, then A, d is a metric space.
- (2) A metric space (X, d) is a Hausdorff space.
- (3) A countable product $\prod_{i=1}^{\infty} X_i$ of metric spaces is metrizable.

Theorem 1.59. $(X, d_X), (Y, d_Y)$ metric spaces. A map $f : X \rightarrow Y$ is continuous iff for $\forall x \in X, \varepsilon > 0, \exists \delta > 0$, s.t. $d_X(x, y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon$.

Proof. (i) Assume f is continuous. Given $x \in X, \varepsilon > 0, f^{-1}(B(f(x), \varepsilon))$ is open. $\therefore \exists \delta > 0$, s.t. $B(x, \delta) \subset f^{-1}(B(f(x), \varepsilon))$.



- (ii) Assume the ε, δ condition holds. Let $V \subset Y$ be open. For $\forall x \in f^{-1}(V), \exists \varepsilon > 0$, s.t. $B(f(x), \varepsilon) \subset V$.
 $\therefore \exists \delta > 0$, s.t. $B(x, \delta) \subset f^{-1}(B(f(x), \varepsilon)) \subset f^{-1}(V)$. $\therefore f^{-1}(V)$ is open.



□

Lemma 1.60 (The sequence lemma). *X a topological space, $A \subset X$. If there is a sequence of points of A converging to x , then $x \in \overline{A}$; the converse holds if X is metrizable.*

Proof. (i) Suppose $x_n \rightarrow x$, where $x_n \rightarrow A$. Then any neighborhood U of x , $U \cap A \neq \emptyset$. $\therefore x \in \overline{A}$.

- (ii) Suppose (X, d) is a metric space. For $x \in \overline{A}$, $B\left(x, \frac{1}{n}\right) \cap A \neq \emptyset$ for all $n \in \mathbb{N}$. Choose $x_n \in B\left(x, \frac{1}{n}\right) \cap A$. Then for any neighborhood U of x , $B\left(x, \frac{1}{n}\right) \subset U$ for n large. $\therefore x_n \in U$ for x large. i.e. $x_n \rightarrow x$.



□

Theorem 1.61. *Let $f : X \rightarrow Y$. If f is continuous, then for every converging sequence $x_n \rightarrow x$ in X , $f(x_n)$ converges to $f(x)$; the converse holds if X is metrizable.*

Proof. (i) Assume f is continuous. Given $x_n \rightarrow x$, let V be a neighborhood of $f(x)$, then $f^{-1}(V)$ is a neighborhood of x . $\therefore \exists N$ s.t. $x_n \in f^{-1}(V)$ for $n \geq N$. Therefore, $f(x_n) \in V$ for $n \geq N$, i.e. $f(x_n) \rightarrow f(x)$.

- (ii) Assume the convergent condition holds. Let $A \subset X$ be a subset, we show that $f(\overline{A}) \subset \overline{f(A)}$. For $x \in \overline{A}$, \exists a sequence $x_n \rightarrow x$, $x_n \in A$. Then $f(x_n) \rightarrow f(x)$, and $f(x_n) \in f(A)$. $\therefore f(x) \in \overline{f(A)}$. $\therefore f(\overline{A}) \subset \overline{f(A)}$.

□

Theorem 1.62. *If $f, g : X \rightarrow \mathbb{R}$ are continuous functions, then $f \neq g$, $f \cdot g$ and f/g (If $g(x) \neq 0$ for all x) are continuous.*

Proof. $X \xrightarrow{f \times g} \mathbb{R} \times \mathbb{R} \xrightarrow{\cdot} \mathbb{R}$. □

Definition 1.63. *Let $f_n : X \rightarrow Y$ be a sequence of maps from a set X to a metric space (Y, d) . The sequence (f_n) converges uniformly (一致收敛) to the map $f : X \rightarrow Y$ if given $\varepsilon > 0$, \exists an integer N s.t. $d(f_n(x), f(x)) < \varepsilon$ for all $n > N$ and all $x \in X$.*

Theorem 1.64 (Uniform limit theorem). *Let $f_n : X \rightarrow Y$ be a sequence of continuous maps from a topological space to a metric space Y . If (f_n) converges uniformly to f , then f is continuous.*

Proof. Let $V \subset Y$ open, $x_0 \in f^{-1}(V)$, $y_0 = f(x_0)$, $B(y_0, \varepsilon) \subset V$. Choose N s.t. for all $n \geq N$, $x \in X$, $d(f_n(x), f(x)) < \frac{\varepsilon}{3}$. f_N is continuous $\Rightarrow f_N^{-1}\left(B\left(f_N(x_0), \frac{\varepsilon}{3}\right)\right) = U$ is a neighborhood of x_0 .

Claim: $f(U) \subset B(y_0, \varepsilon) \subset V$.

Proof. $\forall x \in U$,

$$\begin{cases} d(f(x), f_N(x)) < \frac{\varepsilon}{3} \\ d(f_N(x), f_N(x_0)) < \frac{\varepsilon}{3} \\ d(f_N(x_0), f(x_0)) < \frac{\varepsilon}{3} \end{cases}$$

$$\therefore d(f(x), f(x_0)) < \varepsilon.$$

□

□

Chapter 2

Topological properties

2.1 connected spaces and path connected spaces

Definition 2.1. Let X be a topological space. A separation of X is a pair U, V of disjoint nonempty open subsets of X whose union is X . The space X is said to be connected (连通) if there does not exist a separation of X .

The following is an alternative definition.

Definition 2.2. A space X is connected if and only if the only subsets of X that are both open and closed are \emptyset and X .

Examples:

- (1) $X = \{a, b\}$ with the trivial topology is connected.
- (2) $Y = [-1, 0] \cup (0, 1]$ is not connected.
- (3) $\mathbb{Q} \subset \mathbb{R}$ with the subspace topology is not connected. Choose $a \in \mathbb{R} - \mathbb{Q}$, then
 $((-\infty, a) \cap \mathbb{Q}) \cup ((a, +\infty) \cap \mathbb{Q}) = \mathbb{Q}$.
- (4) There is a counter-example.



Lemma 2.3. Let $X = C \cup D$ be a separation, $Y \subset X$ a connected subspace, then $Y \subset C$ or $Y \subset D$.

Proof. $C, D \subset X$ are open, so $C \cap Y, D \cap Y$ open in Y . Y is connected $\Rightarrow C \cap Y$ or $D \cap Y = \emptyset$. \square

Theorem 2.4. The union of a collection of connected subspaces of X that have a point in common is connected.

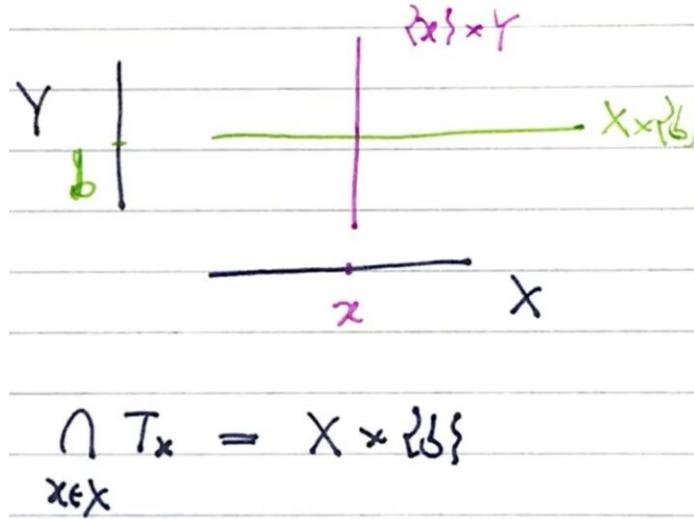
Proof. Let $\{A_\alpha\}$ be a collection of connected subspaces of X , $p \in \bigcap A_\alpha$, $Y = \bigcup A_\alpha$. we prove that Y is connected. Suppose $Y = C \cup D$ is a separation, $p \in C$ then $A_\alpha \subset C, \forall \alpha$, so $Y = C$. \square

Theorem 2.5. *The image of a connected space under a continuous map is connected.*

Proof. Let $f : X \rightarrow Y$ be a continuous map, $Z = f(x) \subset Y$. Let $Z = A \cup B$ be a separation, then $X = f^{-1}(A) \cup f^{-1}(B)$ is a separation, a contradiction. \square

Theorem 2.6. *A finite cartesian product of connected spaces is connected.*

Proof. Let X, Y be connected spaces. we shall show that $X \times Y$ is connected. Let $T_x = X \times \{b\} \cup \{x\} \times Y$, then T_x is connected $\forall x \in X$. Now $X \times Y = \bigcup_{x \in X} T_x$ and $\bigcap_{x \in X} T_x = X \times \{b\}$. So $X \times Y$ is connected.



\square

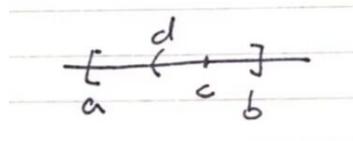
Remark:

- (i) \mathbb{R}^ω in the box topology is not connected.
 - (ii) An arbitrary product of connected spaces is connected in the product topology, e.g. \mathbb{R}^ω .
-

Theorem 2.7. *The real line \mathbb{R} is connected, and so are intervals and rays in \mathbb{R} .*

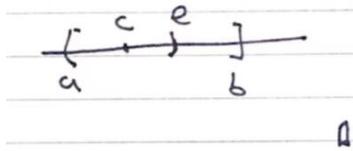
Proof. Assume $\mathbb{R} = A \cup B$ is a separation, $a \in A, b \in B, a < b, A_0 = A \cap [a, b], B_0 = B \cap [a, b], c = \sup A_0 \in [a, b]$

- (i) if $c \in B_0$, then $c > a$ (since A is open). B_0 is open in $[a, b] \Rightarrow \exists d < c$, s.t. $(d, c] \subset B_0$, then $\sup A_0 \leq d < c$.



- (ii) if $c \in A_0$, then $c < b$ (since B is open). A_0 is open in $[a, b] \Rightarrow \exists e > c$, s.t. $[c, e) \subset A_0$. So $\sup A_0 \geq e > c$.

\square



Theorem 2.8 (Intermediate value theorem). *If $f : X \rightarrow \mathbb{R}$ is a continuous function, X is a connected space. Let $a, b \in X$, $f(a) < f(b)$. Then for any $r \in (f(a), f(b))$, $\exists c \in X$, s.t. $f(c) = r$.*

Proof. Assume $r \notin f(x)$, let $A = f(x) \cap (-\infty, r)$. $B = f(x) \cap (r, +\infty)$, then $f(x) = A \cup B$ is a separation, a contradiction since $f(X)$ is connected. \square

Definition 2.9. Define an equivalence relation \sim on a topological space X : $x \sim y$ if \exists a connected subspace of X containing both x and y . The equivalence classes are called connected components (连通分支) of X .

Theorem 2.10. The components of X are connected disjoint subspaces of X whose union is X , such that each nonempty connected subspace of X intersects only one of them.

i.e. $X = \coprod_{\alpha} X_{\alpha}$, X_{α} connected components, $\forall A \subset X, \exists! \alpha$, s.t. $A \subset X_{\alpha}$.

Proof. (i) $X_{\alpha} \cap X_{\beta} = \emptyset, X = \bigcup_{\alpha} X_{\alpha}$ by definition.

(ii) $A \subset X$ connected, assume \exists components C_1, C_2 , s.t. $x_1 \in A \cap C_1, x_2 \in A \cap C_2$, then $x_1 \sim x_2$, so $C_1 = C_2$. So $\exists!$ one component C , s.t. $A \subset C$

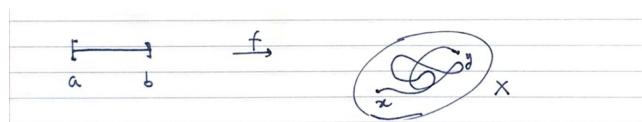
(iii) we show each component C is connected: choose $x_0 \in C$, $\forall x \in C$, $\exists A_x$ connected. s.t. $x_0, x \in A_x$ (since $x_0 \sim x$), and $A_x = C$. $C = \bigcup_{x \in C} A_x$ and $x_0 \in \bigcap_{x \in C} A_x$. So C is connected. \square

Remark: $X = \coprod_{\alpha} X_{\alpha}$, X_{α} connected components. Then $\overline{X}_{\alpha} = X_{\alpha}$ (in general, $A \subset X$ connected $\rightarrow \overline{A}$ connected).

So each component is closed. if there are only finitely many components, then each component is also open.

e.g. $\mathbb{Q} \subset \mathbb{R}$, each point is a component, closed but not open.

Definition 2.11. A path (道路) in a topological space X from x to y is a continuous map $f : [a, b] \rightarrow X$ s.t. $f(a) = x, f(b) = y$.



A space X is said to be path connected (道路连通) if every pair of points in X can be joined by a path in X .

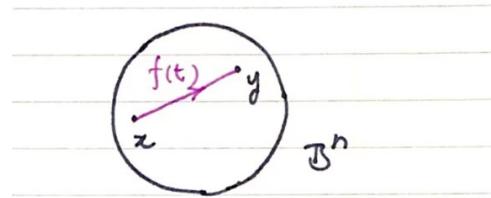
Remark: path-connected \rightarrow connected, since $[a, b]$ is connected.

Examples:

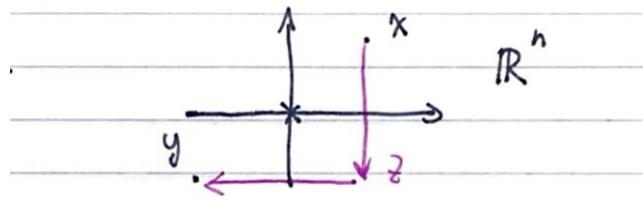
- (1) The unit ball $B^n = \left\{ x \in R^n \mid (x_1^2 + x_2^2 + \dots + x_n^2)^{1/2} \leq 1 \right\}$ is path-connected.

$$f(t) = (1-t)x + ty, t \in [0, 1].$$

$$\|f(t)\| \leq (1-t)\|x\| + t\|y\| \leq 1$$



- (2) the punctured Euclidean space $R^n - \{0\}$ is path-connected for $n > 1$.



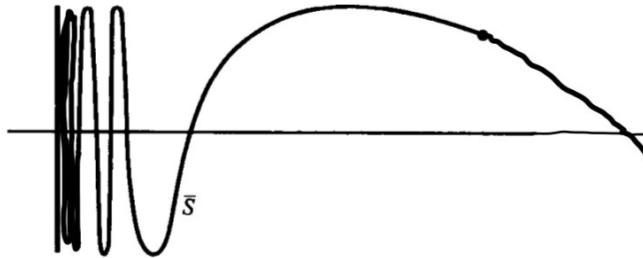
- (3) the unit sphere $S^{n-1} = \{x \in R^n \mid \|x\| = 1\} \subset R^n$ is path-connected for $n > 1$.

$$g : R^n - \{0\} \longrightarrow S^{n-1}, x \mapsto \frac{x}{\|x\|}$$

Remark: $GL_n(\mathbb{R}) \subset M_n(\mathbb{R}) = R^{n^2}$. Is $GL_n(\mathbb{R})$ path-connected?

Examples:

- (1) The topologists' sine curve



$$S = \left\{ \left(x, \sin \frac{1}{x} \right) \mid 0 < x \leq 1 \right\} \subset R^2.$$

\bar{S} = the closure of S in R^2 is called the topologist's sine curve. $\bar{S} = S \cup \{0\} \times [-1, 1]$. S is connected $\rightarrow \bar{S}$ is connected.

Claim. \bar{S} is not path-connected.

Proof. Let $f : [a, c] \rightarrow \overline{S}$ be a path s.t. $f(a) = 0, f(c) \in S, f^{-1}(\{0\} \times [-1, 1]) \subset [a, c]$ is closed. So \exists a largest element b .

Then $f : [a, c] \rightarrow \overline{S}$ with $f(b) \in \{0\} \times [-1, 1], f(t) \in S (t > b)$. Assume $b = 0, c = 1$, then $f : [0, 1] \rightarrow \overline{S}$ with $f(0) \in \{0\} \times [-1, 1], f(t) \in S$, for $t > 0$. Let $f(t) = (x(t), y(t))$, then $x(0) = 0, x(t) > 0$ for $t > 0$. $y(t) = \frac{1}{\sin x(t)}$ for $t > 0$. We find a sequence $t_n, t_n \rightarrow 0$ with $y(t_n) = (-1)^n$, as follows: take $0 < x_n < x(\frac{1}{n})$ s.t. $\sin \frac{1}{x_n} = (-1)^n$. $x : [0, 1] \rightarrow \mathbb{R}$ a continuous function with $x(0) = 0$.

Intermediate value theorem $\rightarrow \exists t_n \in (0, \frac{1}{n}), \text{s.t. } x(t_n) = x_n$. So $y(t_n) = \frac{1}{\sin x_n} = (-1)^n$.

□

Definition 2.12. Define an equivalence relation \sim on X : $x \sim y$ if \exists a path in X from x to y . The equivalence classes are called the path components of X . (道路连通分支).

Theorem 2.13. The path components of X are path-connected disjoint subspaces of X whose union is X , such that each nonempty path-connected subspaces of X intersects only one of them.

Definition 2.14. A space X is said to be locally connected (局部连通) at x , if for every neighborhood μ of x , \exists a connected neighborhood V of x , contained in μ . If X is locally connected at each $x \in X$, it is said to be locally connected.

Examples:

- (1) the topologists' sine curve is connected but not locally connected.
- (2) $\mathbb{Q} \subset \mathbb{R}$ is not locally connected.

2.2 compactness

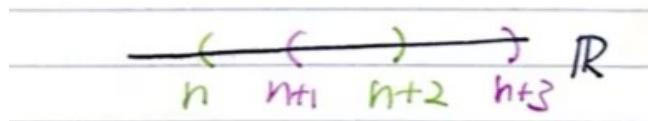
Definition 2.15. A covering (覆盖) of a space X is a collection \mathcal{A} of subspaces whose union is equal to X . It is called an open covering (开覆盖) if elements in \mathcal{A} are open subsets.

Definition 2.16. A space X is called compact (紧致) if every open covering of X contains a finite subcollection that also covers X .

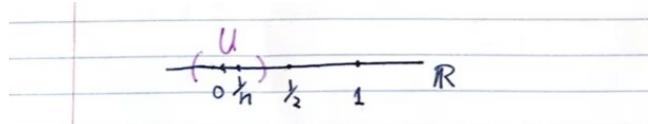
Examples:

- (1) $X = \mathbb{R}$, $\mathcal{A} = \{(n, n+1) \mid n \in \mathbb{Z}\}$. \nexists finite sub-covering.

So \mathbb{R} is not compact.



- (2) $X = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\} \subset \mathbb{R}$ is compact.



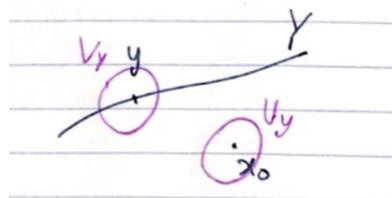
Theorem 2.17. Every closed subspace of a compact space is compact.

Proof. Let $Y \subset X$ be a closed subspace, X is compact. Let $\{A_\alpha\}_{\alpha \in J}$ be an open covering of Y , then \exists open sets $U_\alpha \subset X$, s.t. $A_\alpha = U_\alpha \cap Y$. $\{U_\alpha \mid \alpha \in J\} \cup \{X - Y\}$ is an open covering of X , therefore $\exists U_{\alpha_1}, U_{\alpha_2}, \dots, U_{\alpha_n}$, s.t. $Y \subset \bigcup_{i=1}^n U_{\alpha_i}$. So $\bigcup_{i=1}^n A_{\alpha_i} = Y$. \square

Theorem 2.18. Every compact subspace of a Hausdorff space is closed.

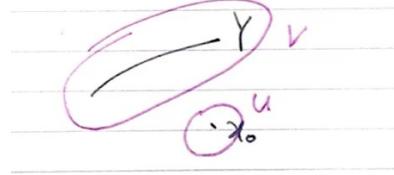
Proof. Let $Y \subset X$ be a compact subspace, X is Hausdorff. We shall show that $X - Y$ is open.

For $\forall x_0 \in X - Y$, $y \in Y$, \exists neighborhoods U_y of x_0 , and V_y of y , s.t. $U_y \cap V_y = \emptyset$. Now $\{V_y \cap Y \mid y \in Y\}$ is an open covering of Y . So \exists finitely many y_1, y_2, \dots, y_n , s.t. $(V_{y_1} \cap Y) \cup \dots \cup (V_{y_n} \cap Y) = Y$. Let $V_{y_1} \cup \dots \cup V_{y_n} = V$, then $Y \subset V$. let $U = U_{y_1} \cup \dots \cup U_{y_n}$ be an open neighborhood of x_0 , then $U \cap V = \emptyset$. So $U \cap Y = \emptyset$.



\square

Lemma 2.19. If Y is a compact subspace of a Hausdorff space X , and $x_0 \notin Y$. Then \exists disjoint open sets U and V , of X , containing x_0 and Y , respectively.



Example : $(a, b] \subset \mathbb{R}$ and $(a, b) \subset \mathbb{R}$ are not compact, since \mathbb{R} is Hausdorff.

Remark: Given a $(\mathbb{R}, \text{finite complement topology})$. Then

- (i) not Hausdorff.
- (ii) every subset is compact.
- (iii) proper closed subsets = finite sets.

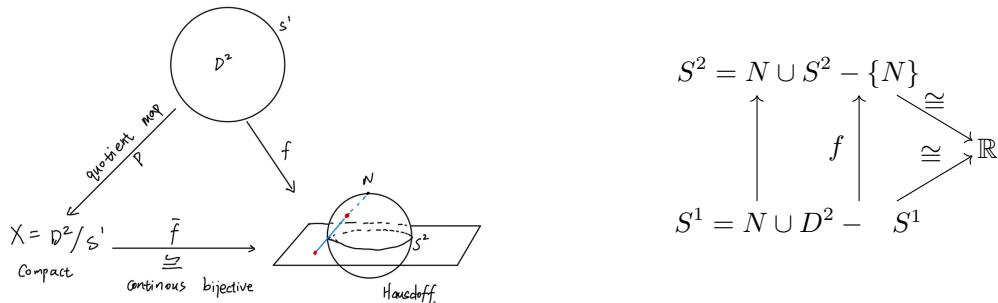
Theorem 2.20. The image of a compact space under a continuous map is compact.

Proof. Let $f : X \rightarrow Y$ be a continuous map, X is compact. Let \mathcal{A} be an open covering of $f(X)$, then $\{f^{-1}(A) | A \in \mathcal{A}\}$ is an open covering of X . So \exists finitely many A_1, \dots, A_n s.t. $f^{-1}(A_1) \cup \dots \cup f^{-1}(A_n) = X$. So $A_1 \cup \dots \cup A_n = f(X)$. \square

Theorem 2.21. Let $f : X \rightarrow Y$ be a bijective continuous map, X is compact, Y is Hausdorff, then f is a homeomorphism.

Proof. We need to show that inverse map $f^{-1} : Y \rightarrow X$ is continuous. Let $A \subset X$ be closed subset, we want to show $(f^{-1})^{-1}(A) = f(A) \subset Y$ is closed. X is compact and $A \subset X$ is closed $\Rightarrow A$ is compact. $\Rightarrow f(A)$ is compact. Y is Hausdorff $\Rightarrow f(A)$ is closed. \square

example:



Theorem 2.22. The product of finitely many compact spaces is compact.

Proof.

Step 1: Let X, Y be compact, $N \subset X \times Y$ be an open neighborhood of the “slice” $x_0 \times Y$. We prove that \exists a neighborhood W of $x_0 \in X$, s.t. $W \times Y \subset N$. $W \times Y$ is called a tube about $x_0 \times Y$. Basis elements of product topology have the form $U \times V$, $U \subset X$, $V \subset Y$ open. $N \subset X \times Y$ open $\Rightarrow N$ is covered by $U \times V$ with $U \times V \subset N$. Since $x_0 \times Y$ is compact, \exists finitely many such basis elements $U_1 \times V_1, \dots, U_n \times V_n$ s.t. $x_0 \times Y \subset \bigcup_{i=1}^n (U_i \times V_i)$. and $x_0 \times Y \cap (U_i \times V_i) \neq \emptyset$. Define $W = U_1 \cap \dots \cap U_n$.

Claim : $W \times Y \subset \bigcup_{i=1}^n (U_i \times V_i)$

Proof. $\forall (x, y) \in W \times Y, \exists i$ s.t. $(x_0, y) \in U_i \times V_i$

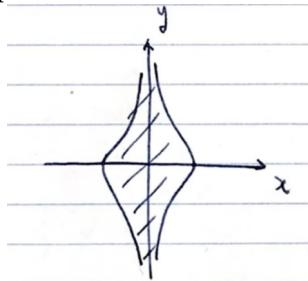
$x \in W \subset U_i$, So $(x, y) \in U_i \times V_i$.

So $W \times Y \subset \bigcup_{i=1}^n (U_i \times V_i) \subset N$. \square

Step 2: Let \mathcal{A} be an open covering of $X \times Y$. For $\forall x \in X$, Since $X \times Y$ is compact, \exists finitely many A_1, \dots, A_m in \mathcal{A} s.t. $Y \subset A_1 \cup \dots \cup A_m = N$. By step 1, \exists a tube $W_x \times Y$ s.t. $x \times Y \subset W_x \times Y \subset N$.

$\{W_x | x \in X\}$ is an open covering of X , $\therefore \exists$ finitely many $W_1 \dots W_k$, s.t. $\bigcup_{i=1}^k W_i = X$. $\therefore (W_1 \times Y) \cup \dots \cup (W_k \times Y) = X \times Y$, Each $W_i \times Y$ is covered by finitely many elements in \mathcal{A} . \square

Example:



$$Y = \text{the } y\text{-axis}, N = \left\{ (x, y) \mid |x| < \frac{1}{y^2 + 1} \right\}$$

Then \nexists a tube for $0 \times Y$.

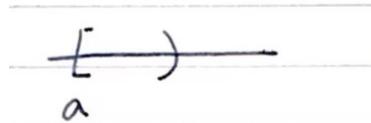
Theorem 2.23 (Tychonoff). *An arbitrary product of compact spaces is compact in the product topology.*

Theorem 2.24. *A closed interval $[a, b] \subset \mathbb{R}$ is compact.*

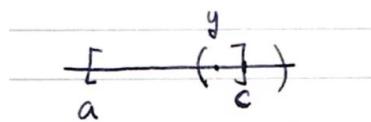
Proof. Let \mathcal{A} be an open covering of $[a, b]$,

$C = \{y \in [a, b] | [a, b] \text{ can be covered by finitely many elements in } \mathcal{A}\}$. Let $c = \sup C$, then we must have

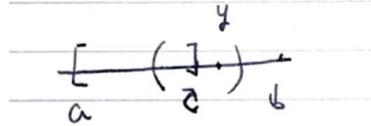
(i) $c > a$, since



(ii) $c \in C$, since



(iii) $c = b$, since



□

Theorem 2.25. A subspace $A \in \mathbb{R}^n$ is compact if and only if it is closed and is bounded in the euclidean metric d or square metric ρ .

Proof. $\rho(x, y) \leq d(x, y) \leq \sqrt{n}\rho(x, y)$ ∴ it suffices to consider ρ .

(i) Assume A is compact. Then A is closed. Consider $\{B_\rho(0, m) | m \in \mathbb{N}\}$, $\bigcup_{m=1}^{\infty} B_\rho(0, m) = \mathbb{R}^n$. A is compact.
 $\therefore \exists$ an M , s.t. $A \subset B_\rho(0, M)$, i.e., A is bounded.

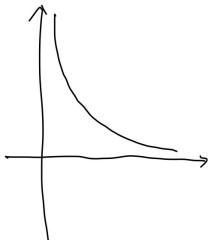
(ii) Assume A is closed and bounded, i.e. $\exists N$, s.t. $\rho(x, y) \leq N \forall x, y \in A$. Choose $x_0 \in A$, with $\rho(x_0, 0) = b$. Then $\rho(x_0, 0) \leq N + b$, $\forall x \in A$. $\therefore A \subset [-(N + b), N + b]^n$ closed and $[-(N + b), N + b]^n$ is compact. $\therefore A$ is compact.

□

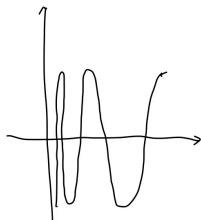
Examples:

(1) the unit sphere $S^{n-1} \subset \mathbb{R}^n$, unit ball $B^n \subset \mathbb{R}^n$ are compact.

(2) $A = \{(x, \frac{1}{x}) | 0 < x \leq 1\} \subset \mathbb{R}^2$ closed but not bounded, hence not compact.



(3) $S = \left\{ \left(x, \sin \frac{1}{x} \right) \middle| 0 < x \leq 1 \right\}$ bounded but not closed hence not compact.

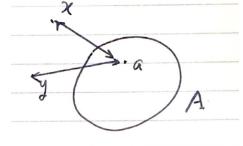


Theorem 2.26 (Extreme value theorem). $f : X \rightarrow \mathbb{R}$ a continuous function, X is compact. Then $\exists c, d \in X$, s.t. $f(c) \leq f(x) \leq f(d)$ for $\forall x \in X$.

Proof. $A = f(x) \subset \mathbb{R}$ is compact, hence is bounded and closed. bounded $\Rightarrow \inf A = m > -\infty$, $\sup A = M < +\infty$. closed $\Rightarrow m \in A, M \in A$. \square

Definition 2.27. (X, d) is metric space, $A \subset X$ a non-empty subspace.

Then for each $x \in X$, the distance from x to A (从 x 到 A 的距离) is $d(x, A) = \inf \{d(x, a) | a \in A\}$. This is a continuous function in X :



Check: $\forall x, y \in X$,

$$d(x, A) \leq d(x, a) \leq d(x, y) + d(y, a) \therefore d(x, A) \leq d(x, y) + d(y, A), \text{ i.e. } |d(x, A) - d(y, A)| \leq d(x, y).$$

$A \subset X$ a bounded subspace, the diameter (直径) of A is

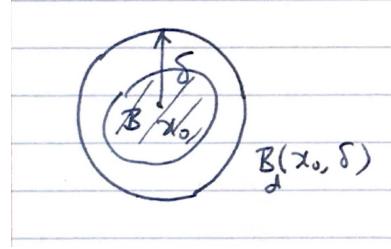
$$\text{diam}(A) = \sup \{d(a_1, a_2) | a_1, a_2 \in A\}.$$

Lemma 2.28 (the Lebesgue number lemma). Let \mathcal{A} be an open covering of a metric space (X, d) . if X is compact, then \exists a $\delta > 0$, s.t. for each subspace of diameter $< \delta$, there exists an elements of \mathcal{A} containing it. The number δ is called a Lebesgue number of \mathcal{A} .

Proof. Assume $X \notin \mathcal{A}$. Choose a finite collection $\{A_1, \dots, A_n\}$ of \mathcal{A} that covers X . Let $C_i = X - A_i$, define $f : X \rightarrow \mathbb{R}$, $f(x) = \frac{1}{n} \sum_{i=1}^n d(x, C_i)$ a continuous function.

$\forall x \in X, \exists A_i$ s.t. $x \in A_i$, and A_i is open. $\therefore d(x, C_i) > 0 \therefore f(x) > 0$. Let $\delta = \min \{f(x) | x \in X\} > 0$.

We show that δ is a Lebesgue number of \mathcal{A} . Let $B \in X$ with $\text{diam}(B) < \delta$, choose $x_0 \in B$, then $B \subset B_d(x_0, \delta)$. Now $\delta \leq f(x_0) = \frac{1}{n} \sum_{i=1}^n d(x_0, C_i) \leq \max_{i=1 \dots n} d(x_0, C_i) = d(x_0, C_m)$. Then $B_d(x_0, \delta) \subset X - C_m = A_m$.



\square

Definition 2.29. A topological space X is said to be sequentially compact (列紧) if every sequence of points of X has a convergent subsequence.

Theorem 2.30. If X is a metrizable space, then the following are equivalent.

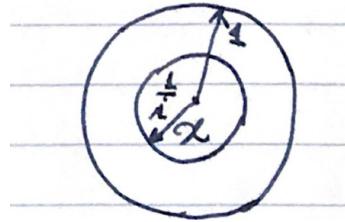
(1) X is compact.

(2) X is sequentially compact.

Proof. (1) \Rightarrow (2):

Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in X , $A = \{x_n | n \in \mathbb{N}\}$. If A is finite, then \exists infinitely many $n \in \mathbb{N}$, s.t. $x_n = x$, giving a convergent subsequence. Now assume A is infinite.

- (i) A has a limit point: if not, then $A = \bar{A}$ is closed (recall that $\bar{A} = A' \cup A$). For $\forall a \in A$, \exists a neighborhood U_a s.t. $U_a \cap A = a$. Now $\{U_a | a \in A\} \cup \{X - A\}$ is an open covering of X , and X is compact, $\therefore \exists$ a finite subcovering, therefore A is finite.
- (ii) Let x be a limit point of A , we will find a subsequence (x_{n_i}) converging to x ; Choose $x_{n_1 \in B(x, 1)}$, inductively, since $B(x, \frac{1}{i}) \cap A$ is infinite, we may choose $n_i > n_{i-1}$ s.t. $x_{n_i} \in B(x, \frac{1}{i})$. The subsequence (x_{n_i}) converges to x .



□

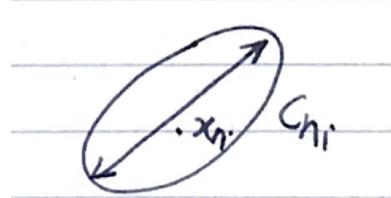
(2) \Rightarrow (1):

Step 1 : the Lebesgue number lemma holds for X , i.e. for an open covering \mathcal{A} of X , \exists a $\delta > 0$, s.t. for any $B \subset X$ with $\text{diam } B < \delta$, $\exists A \in \mathcal{A}$ s.t. $B \subset A$.

Proof. Assume there is no such δ . Then for $\forall n \in \mathbb{N}$, $\exists C_n \subset X$, s.t. $\text{diam } C_n < \frac{1}{n}$, but C_n is not contained in any $A \in \mathcal{A}$. Choose $x_n \in C_n$ for each n , then by assumption, \exists a subsequence (x_{n_i}) of (x_n) , converging to a . $\exists A \in \mathcal{A}$, s.t. $a \in A$. $\exists \varepsilon > 0$, s.t. $B(a, \varepsilon) \subset A$.

- if i is large enough s.t. $\frac{1}{n_i} < \frac{\varepsilon}{2}$, then $C_{n_i} \subset B(x_{n_i}, \frac{\varepsilon}{2})$
- i is large enough then $d(x_{n_i}, a) < \frac{\varepsilon}{2}$.
- \Rightarrow for i large enough $C_{n_i} \subset B(x_{n_i}, \frac{\varepsilon}{2}) \subset B(a, \varepsilon) \subset A$.

□



Step 2 : given $\varepsilon > 0$, \exists a finite covering of X by open $\varepsilon - balls$.

Proof. Assume \exists an $\varepsilon > 0$, s.t. X cannot be covered by finitely many open $\varepsilon - balls$. Start from $x_1 \in X$, $\exists x_2 \notin B(x_1, \varepsilon)$, $x_3 \notin B(x_1, \varepsilon) \cup B(x_2, \varepsilon)$, and so on. We get a sequence (x_n) s.t. $d(x_{n+1}, x_i) \geq \varepsilon$ for $i = 1, \dots, n$. Therefore (x_n) has no convergent subsequence.

□

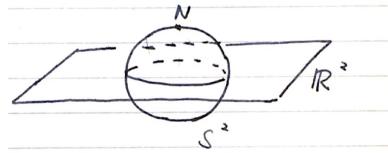
Step 3 : sequentially compact \Rightarrow compact.

Proof. Let \mathcal{A} be an open covering of X , δ be a Lebesgue number of \mathcal{A} , $\varepsilon = \frac{\delta}{3}$. There is a finite covering of X by open ε -balls B_1, \dots, B_n . Now $\text{diam } B_i \leq \frac{2\delta}{3} < \delta$. $\therefore \exists A_i \in \mathcal{A}$ s.t. $B_i \subset A_i$. $\therefore \{A_1, \dots, A_n\}$ is a finite covering of X . \square

Definition 2.31. If Y is a compact Hausdorff space and X is a proper subspace of Y whose closure equals Y , then Y is said to be a compactification (緊化) of X .

Examples:

- (1) $X = (a, b)$, $Y = [a, b]$
- (2) $X = \mathbb{R}$, $Y = S^1$
- (3) $X = \mathring{D}^2$ the open 2-ball, $Y = D^2$ the closed 2-ball
- (4) $X = \mathbb{R}^2$, $Y = S^2$



Definition 2.32. A space X is said to be locally compact at x , if there is some compact subspace C of X that contains a neighborhood of x . If X is locally compact at each x , X is said to be locally compact.

Examples:

- (1) \mathbb{R} is locally compact, $x \in (a, b) \subset [a, b]$
- (2) \mathbb{R}^n is locally compact.
- (3) \mathbb{R}^ω is not locally compact.
- (4) $\mathbb{Q} \subset \mathbb{R}$ is not locally compact.

Theorem 2.33. Let X be a space. Then X is locally compact Hausdorff if and only if there exists a space Y satisfying the following conditions.

- (1) X is a subspace of Y .
- (2) $Y - X$ consists of a single point ∞ .
- (3) Y is a compact Hausdorff space.

if Y and Y' are two spaces satisfying these conditions, then there is a homeomorphism of Y with Y' that equals the identity map on X .

Remark: if X is compact, then $Y = X \cup \{\infty\}$.

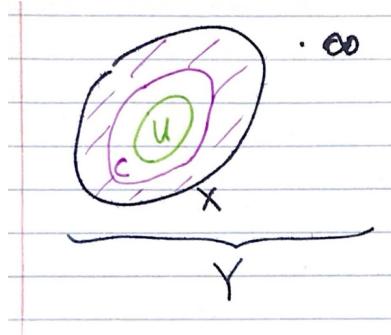
if X is not compact, then ∞ is a limit point of X and $\bar{X} = Y$. In this case Y is called the one-point compactification of X . (一点緊化)

e.g. $X = \mathbb{R}^n$, one-point compactification = S^n .

Proof. (1) uniqueness :

$$\begin{array}{ccc} Y & \xrightarrow{1:1} & Y', \quad \infty \mapsto \infty' \\ \cup & & \cup \\ X & \xrightarrow{\text{id}} & X \end{array}$$

- (2) (i) construction of Y , $Y = X \cup \{\infty\}$ with topology $\mathcal{T} = \{U|U \subset X \text{ open}\} \cup \{Y - C|C \subset X \text{ compact}\}$
 Check : intersection:



- $U_1 \cap U_2$ open
- $(Y - C_1) \cap (Y - C_2) = Y - (C_1 \cup C_2)$
- $U \cap (Y - C) = U \cap (X - C)$

C closed, $\therefore X - C$ open.

union:

- $\bigcup_{\alpha} U_{\alpha} = U$ open
- $\bigcup_{\beta} (Y - C_{\beta}) = Y - \bigcap_{\beta} C_{\beta} = Y - C$.
- $(\bigcup_{\alpha} U_{\alpha}) \cup \bigcup_{\beta} (Y - C_{\beta}) = U \cup (Y - C) = Y - (C - U)$, $C - U$ is compact.

(ii) X is a subspace of Y .

(iii) Y is compact: let \mathcal{A} be an open covering of Y . Then $\exists A_1 \in \mathcal{A}$, $A_1 = Y - C$. $A' = \{A \in \mathcal{A} | A \neq A_1\}$ is an open covering of C . $\therefore \exists$ a finite subcovering.

(iv) Y is Hausdorff: to separate $x \in X$ and $y = \infty$, we need the local compactness of X .

(3) the converse is true.

□

Definition 2.34. A subset A of a space X is said to be dense (稠密) in X if $\overline{A} = X$.

2.3 countability

Definition 2.35. If a space X has countable basis for its topology, then X is said to satisfy the Second countability axiom (第二可数性公理) or to be second-countable (第二可数的).

Examples:

$$(1) \mathbb{R} : \mathcal{B} = \{(a, b) | a, b \in \mathbb{Q}\}$$

$$\mathbb{R}^n : \mathcal{B} = \{B(x, r) | x \in \mathbb{Q}^n, r \in \mathbb{Q}\}$$

$$\text{or } \mathcal{B}' = \left\{ \prod_{i=1}^n [a_i, b_i] \mid a_i, b_i \in \mathbb{Q} \right\}.$$

$$(2) \mathbb{R}_\omega : \mathcal{B} = \left\{ \prod_{n=1}^{\infty} U_n \mid U_n = (a_n, b_n), a_n, b_n \in \mathbb{Q} \right\} \text{ for finitely many values of } n; U_n = \mathbb{R} \text{ for all other } n.$$

Theorem 2.36. A subset of a second-countable space is second-countable; a countable product of second countable spaces is second-countable.

Proof. (i) Let \mathcal{B} be a countable basis for X , $A \subset X$ a subspace,

then $\{A \cap B | B \in \mathcal{B}\}$ is a countable basis for A .

(ii) Let \mathcal{B}_n be a countable basis for X_n , $n \in \mathbb{N}$.

Then

$$\left\{ \prod_{n=1}^{\infty} U_n \mid U_n \in \mathcal{B}_n \text{ for finitely many values of } n, \text{ and } U_n = X_n \text{ for all other } n \right\}.$$

□

Theorem 2.37. Suppose that X is second-countable. Then

(i) Every open covering of X contains a countable subcollection covering X .

(ii) There exists a countable subset of X that is dense in X .

Proof. Let $\mathcal{B} = \{\mathcal{B}_n\}$ be a countable basis of X .

(i) Let \mathcal{A} be an open covering. For each $n \in \mathbb{N}$, if \exists some $A \in \mathcal{A}$ s.t. $B_n \subset A$, then we choose such an element and denote it by A_n . $\mathcal{A}' = \{\text{all such } A_n\}$ is a countable subcollection of \mathcal{A} .

Claim : \mathcal{A}' is a covering of X .

Proof. $\forall x \in X$, $\exists A \in \mathcal{A}$ s.t. $x \in A$. $\therefore \exists$ a basis element B_n s.t. $x \in B_n \subset A$ $\therefore A_n$ is defined for this n and $x \in B_n \subset A_n$.

□

(ii) For each $n \in \mathbb{N}$, choose $x_n \in B_n$, let $D = \{x_n | n \in \mathbb{N}\}$. Then for $\forall x \in X$, any neighborhood U of X , $\exists B_n \subset U$ $x_n \in B_n \subset U$ $\therefore D \cap U = \emptyset \therefore x \in \overline{D}$.

□

Remark:

- compact $\iff \exists$ finite sub-covering
- second-countable $\implies \exists$ countable sub-covering (Lindelöf)
- paracompact $\iff \exists$ locally finite refinement.

2.4 separation axioms

Definition 2.38. A space X is said to be Hausdorff if for each $x, y \in X$, $x \neq y$, there exist disjoint open sets containing x and y , respectively. (T_2 axiom)

Remark: X Hausdorff \Rightarrow one-point sets are closed in X .

Definition 2.39. Suppose that one-point sets are closed in X .

- (1) X is said to be regular (正则) if for each point x , and closed set B , $x \notin B$, there exist disjoint open sets containing x and B , respectively. (T_3 axiom)
- (2) X is said to be normal (正规) if for each pair of disjoint closed sets A, B , there exist disjoint open sets containing A and B , respectively. (T_4 axiom)

Remark: normal \Rightarrow regular \Rightarrow Hausdorff.

Lemma 2.40. Let X be a topological space, assume one-point sets in X are closed.

- (1) X is regular \iff for each $x \in X$ and a neighborhood U of x . \exists a neighborhood V of x , s.t. $\overline{V} \subset U$.



- (2) X is normal \iff for any closed set $A \subset X$ and a neighborhood U of A , \exists a neighborhood V of A , s.t. $\overline{V} \subset U$.

Proof. (1) “ \Rightarrow ”: Given x, U , let $B = X - U$, then \exists open sets V, W , s.t. $x \in V$, $B \subset W$, $V \cap W = \emptyset$. For $\forall y \in B$, W is a neighborhood of y and $W \cap V = \emptyset \therefore y \notin \overline{V} \therefore \overline{V} \cap B = \emptyset$ i.e. $\overline{V} \subset U$.

(2) “ \Leftarrow ”: Given x and B , let $U = X - B$, a neighborhood of x . Then \exists a neighborhood V of x s.t. $\overline{V} \subset U$. $X - \overline{V}$ is a neighborhood of B and $V \cap (X - \overline{V}) = \emptyset$, \square

Theorem 2.41. (1) A subspace of a Hausdorff space is Hausdorff; a product of Hausdorff spaces is Hausdorff.

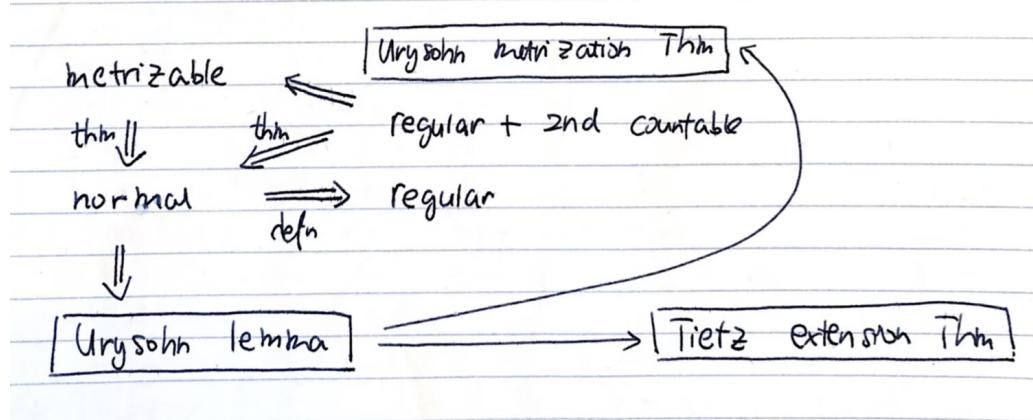
- (2) A subspace of a regular space is regular; a product of regular spaces is regular.

Proof. (2).

- (i) $Y \subset X$ a subspace, then one-point sets are closed in Y . Let $x \in Y$, $B \subset Y$ closed, $x \notin B$. Let \overline{B} = the closure of B in X , then $\overline{B} \cap Y = B$. Now $x \notin \overline{B}$, X is regular $\therefore \exists$ disjoint open sets U, V , $x \in U$, $\overline{B} \subset Y$. Then $U \cap Y$, $V \cap Y$ are disjoint neighborhoods of x and B in Y .
- (ii) Let $\{X_\alpha\}$ be a family of regular spaces. $X = \prod_\alpha X_\alpha$. Then X is Hausdorff, hence one-point sets are closed. Let $x = (x_\alpha) \in X$, U be a neighborhood of x in X . Choose a basis element $\prod_\alpha U_\alpha \subset U$, then for $\forall \alpha$, \exists a neighborhood V_α of x_α in X_α . Then $V = \prod_\alpha V_\alpha$ is a neighborhood of x and $\overline{V} = \prod_\alpha \overline{V}_\alpha$ (by the following theorem) $\subset U$

□

Theorem 2.42. Let $\{X_\alpha\}$ be a family of spaces, $A_\alpha \subset X_\alpha$ be subspaces. Then $\prod_\alpha \overline{A}_\alpha = \overline{\prod_\alpha A_\alpha}$ (in the product or the box product).

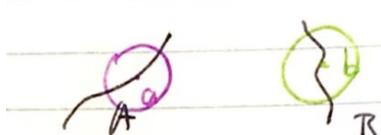


Theorem 2.43. Every metrizable space is normal.

Proof. metrizable \implies one-point sets are closed. Let X be a metric space with metric d , $A, B \subset X$ be disjoint closed sets.

$$\forall a \in A, \exists \varepsilon_a, \text{s.t. } B(a, \varepsilon_a) \cap B = \emptyset$$

$$\forall b \in B, \exists \varepsilon_b, \text{s.t. } B(b, \varepsilon_b) \cap A = \emptyset.$$



Define $U = \bigcup_{a \in A} B\left(a, \frac{\varepsilon_a}{2}\right)$, $V = \bigcup_{b \in B} B\left(b, \frac{\varepsilon_b}{2}\right)$ open neighborhoods of A and B , respectively.

Claim : $U \cap V = \emptyset$. proof: $\forall z \in U \cap V, \exists a \in A, b \in B, \text{s.t. } z \in B\left(a, \frac{\varepsilon_a}{2}\right) \cap B\left(b, \frac{\varepsilon_b}{2}\right)$
 $\therefore d(a, b) \leq d(z, a) + d(z, b) < \frac{1}{2}(\varepsilon_a + \varepsilon_b) \leq \max\{\varepsilon_a, \varepsilon_b\}$.

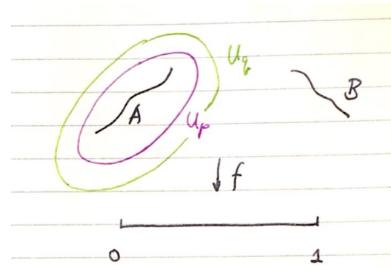
□

Theorem 2.44. Every compact Hausdorff space is normal.

Theorem 2.45 (Urysohn lemma). Let X be a normal space, $A, B \subset X$ be disjoint closed subsets. Then there exists a continuous function $f : X \rightarrow [0, 1]$ s.t. $f(A) = \{0\}$, $f(B) = \{1\}$.

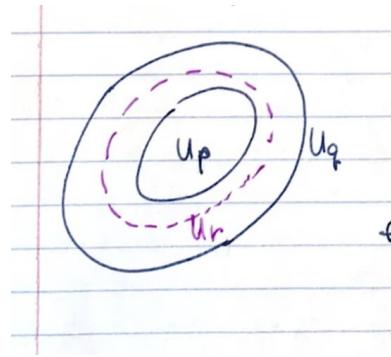
Proof. Step 1:

Let $P = \mathbb{Q} \cap [0, 1]$, we shall define for each $p \in P$ an open set $U_p \subset X$, s.t. $p < q \implies \overline{U_p} \subset U_q$. (*)



Let $U_1 = X - B$, choose U_0 an open neighborhood of A s.t. $\overline{U_0} \subset U_1$ (by the normality of X).

Induction: let p_n be the first n elements of P , suppose that U_p is defined for all $p \in P$ and $p < q \implies U_p \subset U_q$. (e.g. $P_1 = \{0\}$, $P_2 = \{0, 1\}$, $P_3 = \dots$). Let r be the $(n+1)-st$ element in P , $P_{n+1} = \{p\} \cup P_n$, p be the immediate predecessor of r ; q be the immediate successor of r .
normality of $X \implies \exists$ an open set $U_r \subset X$, s.t. $\overline{U_p} \subset U_r$, $\overline{U_r} \subset U_q$.



Check the condition (*): $\forall s \in P_n$

either $s \leq p < r \implies \overline{U_s} \subset U_p \subset U_r$ or $s \geq q > r \implies \overline{U_r} \subset U_q \subset U_s$.

Step 2:

for $\forall p \in \mathbb{Q}$, define $U_P = \emptyset$ if $p < 0$, $U_p = X$ if $p > 1$. Then we still have $p < q \implies \overline{U_p} \subset U_q$.

Step 3:

Given $x \in X$, let $\mathbb{Q}(x) = \{p \in \mathbb{Q} | x \in U_p\}$, then $\mathbb{Q} \cap (1, +\infty) \subset \mathbb{Q}(x) \subset \mathbb{Q} \cap [0, +\infty)$. Define $f(x) = \inf \mathbb{Q}(x) = \inf \{p \in \mathbb{Q} | x \in U_p\}$.

Step 4: f is the desired function.

(a) $x \in A \implies x \in U_p$ for all $p \geq 0 \implies f(x) = 0$

$x \in B \implies x \notin U_p$ for all $p \leq 1 \implies f(x) = 1$

(b) Fact

(1) $x \in \overline{U_r} \implies f(x) \leq r$

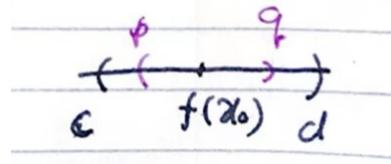
(2) $x \notin U_r \implies f(x) \geq r$

Proof. (1) $x \in \overline{U_r} \implies x \in U_s$ for all $s > r \therefore f(x) = \inf \mathbb{Q}(x) \leq r$

(2) $x \notin U_r \implies x \notin U_s$ for all $s < r \therefore f(x) = \inf \mathbb{Q}(x) \geq r$.

□

- (c) f is continuous: Given $x_0 \in X$, and an open interval (c, d) s.t. $f(x_0) \in (c, d)$, We need to find a neighborhood U of x_0 , s.t. $f(U \subset (c, d))$. Choose $p, q \in \mathbb{Q}$, s.t. $c < p < f(x_0) < q < d$.



Let $U = U_q - \overline{U_p}$

$$\left. \begin{array}{l} \text{(i)} \quad f(x_0) < q \xrightarrow{(2)} x_0 \in U_q \\ \quad \quad \quad f(x_0) > p \xrightarrow{(1)} x_0 \notin \overline{U_p} \end{array} \right\} \Rightarrow x_0 \in U$$

$$\begin{aligned} \text{(ii)} \quad & \forall x \in U, x \in U_q \subset \overline{U_q} \Rightarrow f(x) \leq q < d \\ & x \notin \overline{U_p} \Rightarrow x \notin U_r \Rightarrow f(x) \geq p > c. \end{aligned}$$

□

Theorem 2.46 (Urysohn metrization theorem). *Every regular space with a countable basis is metrizable,*

e.g. $\mathbb{R}^\omega = \prod_{n=1}^{\infty} \mathbb{R}$ is metrizable.

Proof. (sketch)

step 1 :

\exists continuous functions

$\{f_n : X \rightarrow [0, 1] | n \in \mathbb{N}\}$ s.t. $\forall x_0 \in X$ and a neighborhood U of x_0 , $\exists n \in \mathbb{N}$, s.t. $f_n(x_0) > 0$, $f_n|_{X-U} = 0$. construction: let $\{B_n\}$ be a countable basis, for all $n, m \in \mathbb{N}$ s.t. $\overline{B_n} \subset B_m$, by the Urysohn lemma, $\exists g_{n,m} : X \rightarrow [0, 1]$ s.t. $g_{n,m}(\overline{B_n}) = \{1\}$, $g_{n,m}(X - B_m) = \{0\}$.

step 2:

define $F : X \rightarrow \mathbb{R}^\omega$, $x \mapsto (f_1(x), f_2(x) \dots)$

(i) F is continuous

(ii) F is injective

(iii) $F : X \rightarrow Z \subset \mathbb{R}^\omega$ is an open map. $\therefore F : X \rightarrow \mathbb{R}^\omega$ is an embedding, \mathbb{R}^ω is metrizable $\Rightarrow X$ is metrizable.

□

Theorem 2.47 (Tietz extension theorem). *Let X be a normal space, $A \subset X$ be a closed subspace.*

- (a) *Any continuous function of A into $[a, b]$ may be extended to a continuous function of X into $[a, b]$.*
- (b) *Any continuous function of A into \mathbb{R} may be extended to a continuous function of X into \mathbb{R} .*

$$A \xrightarrow{f} [a, b], \mathbb{R}$$

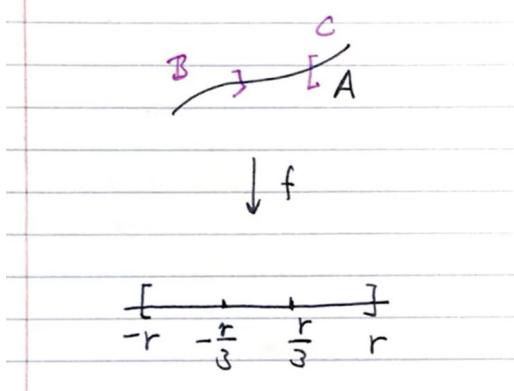
\cap

$\exists \bar{f}?$

X

Proof. (sketch)

step 1 : Given a continuous function $f : A \rightarrow [-r, r]$, we may construct a continuous function $g : X \rightarrow \left[-\frac{1}{3}r, \frac{1}{3}r\right]$ s.t. $|g(a) - f(a)| \leq \frac{2}{3}r \forall a \in A$.



$$I_1 = \left[-r, -\frac{1}{3}r\right]$$

$$I_2 = \left[-\frac{1}{3}r, \frac{1}{3}r\right]$$

$$I_3 = \left[\frac{1}{3}r, r\right]$$

$$B = f^{-1}(I_1), C = f^{-1}(I_3)$$

Urysohn lemma $\Rightarrow \exists$ a continuous function $g : X \rightarrow \left[-\frac{1}{3}r, \frac{1}{3}r\right]$ s.t. $g(B) = -\frac{1}{3}r, g(C) = \frac{1}{3}r$.

step 2: Given $f : A \rightarrow [-1, 1]$, we extend it to X :

$$(i) r = 1, \exists g_1 : X \rightarrow \left[-\frac{1}{3}, \frac{1}{3}\right] \text{ s.t. } |f(a) - g_1(a)| \leq \frac{2}{3} \forall a \in A$$

$$(ii) r = \frac{2}{3}, \exists g_2 : X \rightarrow \left[-\frac{1}{3} \cdot \frac{2}{3}, \frac{1}{3} \cdot \frac{2}{3}\right] \text{ s.t. } |f(a) - g_1(a) - g_2(a)| \leq \left(\frac{2}{3}\right)^2 \forall a \in A$$

Inductively, $\exists g_{n+1}$, s.t.

$|g_{n+1}(x)| \leq \frac{1}{3} \cdot \left(\frac{2}{3}\right)^2, |f(a) - g_1(a) \dots g_{n+1}(a)| \leq \left(\frac{2}{3}\right)^{n+1} \forall a \in A$. Let $S_k(x) = \sum_{n=1}^k g_n(x)$, continuous, converges uniformly to a continuous function $g : X \rightarrow [-1, 1]$, and $g|_A = f$.

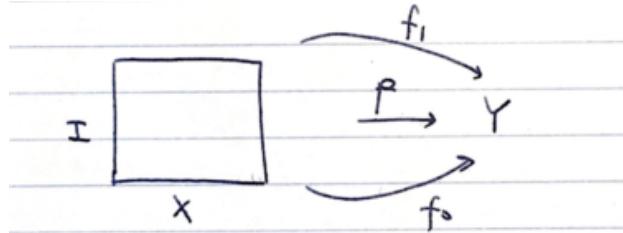
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Chapter 3

The Fundamental Group

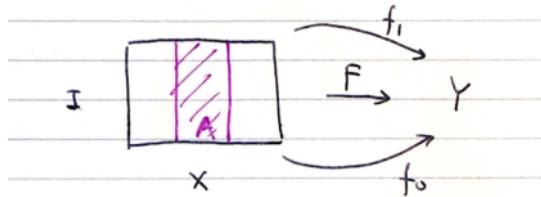
3.1 homotopy of paths

Definition 3.1. Let $f_0, f_1 : X \rightarrow Y$ be continuous maps. A homotopy (同伦) between f_0 and f_1 is a continuous map $F : X \times [0, 1] \rightarrow Y$ s.t. $F(x, 0) = f_0(x)$, $F(x, 1) = f_1(x)$. f_0 and f_1 are said to be homotopic, denoted by $f_0 \simeq f_1$. If $f \simeq C$, a constant map, we say that f is null-homotopic (零伦的).



Let $f : X \rightarrow Y$ be a continuous map. If there exists a continuous map $g : Y \rightarrow X$, s.t. $g \circ f \cong \text{id}_X$ and $f \circ g \cong \text{id}_Y$, then f is called a homotopy equivalence (同伦等价), g is a homotopy inverse of f , X and Y are homotopy equivalent.

Definition 3.2. Let $A \subset X$ be a subspace, a homotopy relative to A is a continuous map $F : X \times I \rightarrow Y$, s.t. $F(a, t) = F(a, 0)$ for $\forall a \in A, t \in I$. Then $f_0 = F(\cdot, 0)$ and $f_1 = F(\cdot, 1)$ are called homotopic relative to A, denoted by $f_0 \simeq_{\text{rel } A} f_1$.

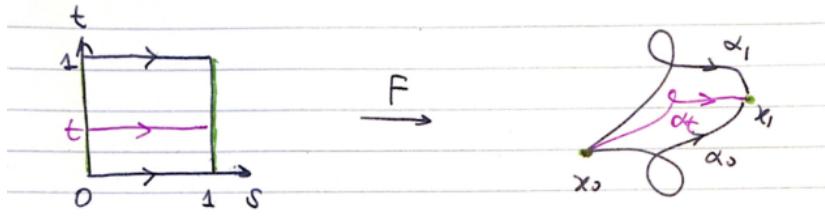


$\text{Map}(X, Y)/\simeq =$ the set of path components of $\text{Map}(X, Y)$

Remark: $= \pi_0 \text{Map}(X, Y) = [X, Y] = ?$

the homotopy set

Definition 3.3. A path in X is a continuous map $\alpha : I \rightarrow X$. Two paths α_0 and α_1 are path homotopic, denoted by $\alpha_0 \simeq_p \alpha_1$, if $\alpha_0(0) = \alpha_1(0) = x_0$, $\alpha_0(1) = \alpha_1(1) = x_1$, and α_0 and α_1 are homotopic rel $\{0, 1\}$, i.e. \exists a continuous map (a path homotopy) $F : X \times I \rightarrow X$ s.t. $F(s, 0) = \alpha_0(s)$, $F(s, 1) = \alpha_1(s)$, $F(0, t) = x_0$, $F(1, t) = x_1$ for $\forall s \in I, t \in I$.



Lemma 3.4. The relation \simeq and \simeq_p are equivalence relations.

Proof. (i) $f \simeq f$ by $F(x, t) = f(x), \forall t \in I, x \in X$.

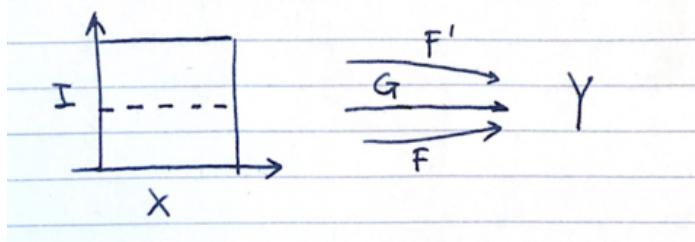
(ii) If $F : f_0 \simeq f_1$ is a homotopy between f_0 and f_1 . Then $G(x, t) = F(x, 1-t)$ is a homotopy between f_1 and f_0 .

(iii) If $F : f_0 \simeq f_1, F' : f_1 \simeq f_2$ are homotopies, then $G : X \times [0, 1] \rightarrow Y$, defined as

$$G(x, t) = \begin{cases} F(x, 2t) & t \in \left[0, \frac{1}{2}\right] \\ F'(x, 2t-1) & t \in \left[\frac{1}{2}, 1\right] \end{cases}$$

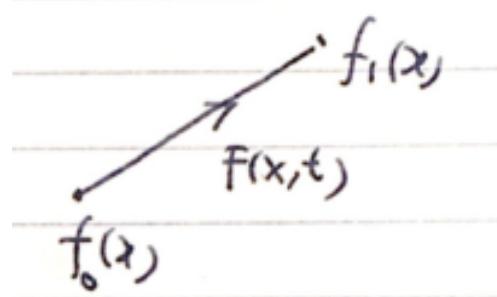
is a homotopy between f_0 and f_2 . (Continuity by the pasting lemma)

□

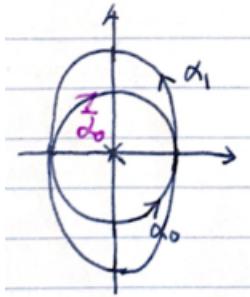


Examples:

- $f_0, f_1 : X \rightarrow \mathbb{R}^2$ are always homotopic by the straight-line homotopy $F(x, t) =$
- (1) $(1-t)f_0(x) + tf_1(x)$. The same holds for path-homotopy, the same holds if \mathbb{R}^2 is related by a convex subspace $A \subset \mathbb{R}^n$.



- (2) $X = \mathbb{R}^2 - \{0\}$ The punctured plane, $\alpha_0, \alpha_1 : S^1 \rightarrow X, \alpha_0(s) = (\cos \pi s, \sin \pi s)$, $\alpha_1(s) = (\cos \pi s, 2 \sin \pi s)$ are homotopic by a straight-line homotopy.
- But α_0 is not homotopic to $\bar{\alpha}_0(s) = (\cos \pi s, -\sin \pi s)$ in X .



Definition 3.5. If α_0 is a path in X from x_0 to x_1 , α_1 is a path from x_1 to x_2 , we define the product

$$\alpha_0 * \alpha_1 \text{ to be the path } \alpha_2(s) = \begin{cases} \alpha_0(2s) & s \in \left[0, \frac{1}{2}\right] \\ \alpha_1(2s - 1) & s \in \left[\frac{1}{2}, 1\right] \end{cases} \text{ from } x_0 \text{ to } x_2.$$

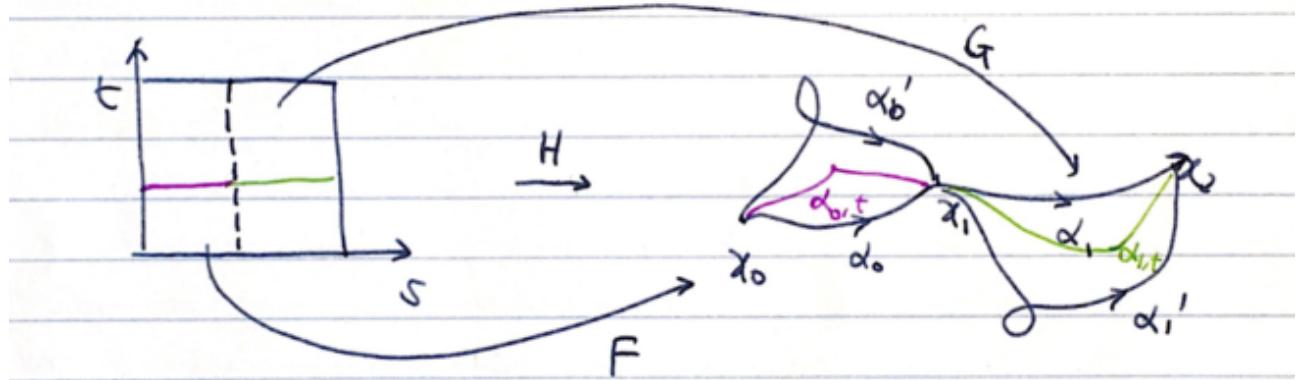
The product operation induces a well-defined operation on path-homotopy classes:

$$[\alpha_0] * [\alpha_1] := [\alpha_0 * \alpha_1].$$

Check: Given path-homotopies $F : \alpha_0 \simeq_F \alpha'_0$, $G : \alpha_1 \simeq_P \alpha'_1$. Define

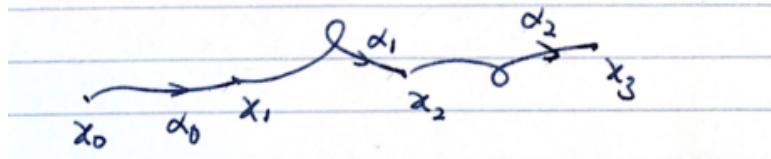
$$H(s, t) = \begin{cases} F(2s, t) & s \in \left[0, \frac{1}{2}\right] \\ G(2s - 1, t) & s \in \left[\frac{1}{2}, 1\right] \end{cases}$$

a homotopy between $\alpha_0 * \alpha_1$ and $\alpha'_0 * \alpha'_1$.



Theorem 3.6. The operation $*$ has the following properties:

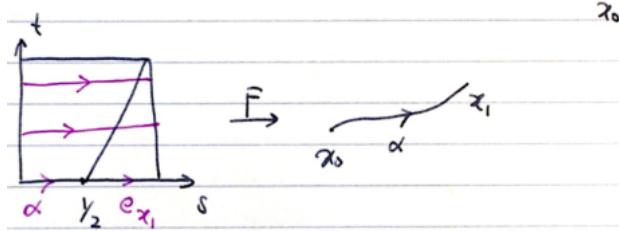
- (1) (associativity) If $[\alpha_0] * ([\alpha_1] * [\alpha_2])$ is defined, then so is $([\alpha_0] * [\alpha_1]) * [\alpha_2]$, and they are equal:



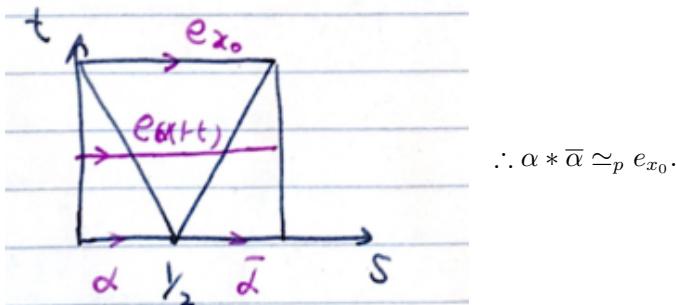
- (2) (right and left identities) If α is a path from x_0 to x_1 , then $[\alpha] * [e_{x_1}] = [\alpha]$, $[e_{x_0}] * [\alpha] = [\alpha]$ where e_x denotes the constant path at $x \in X$.

- (3) (inverse) If α is a path from x_0 to x_1 , let $\bar{\alpha}$ be the path $\bar{\alpha}(s) = \alpha(1-s)$. It is called the inverse of α . Then $[\alpha] * [\bar{\alpha}] = [e_{x_0}]$, $[\bar{\alpha}] * [\alpha] = [e_{x_1}]$.

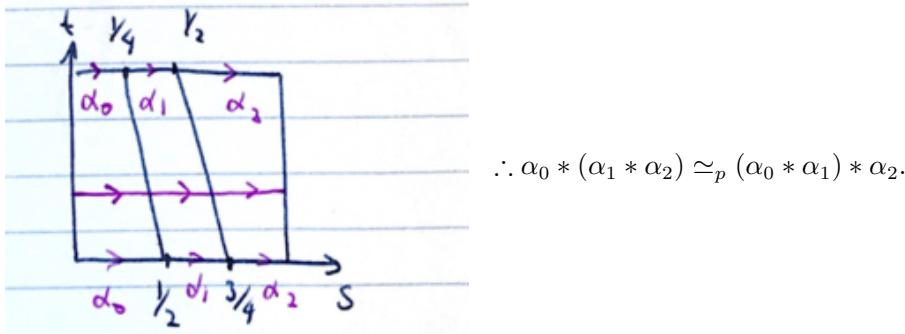
Proof. (2)



$$(3) \quad F(s, t) = \begin{cases} \alpha\left(\frac{2s}{1+t}\right) & s \in \left[0, \frac{1+t}{2}\right] \\ x_1 & s \in \left[\frac{1+t}{2}, 1\right] \end{cases} \therefore \alpha * e_{x_1} \simeq_p \alpha.$$



(1)



□

Definition 3.7. A groupoid is a category in which every morphism is an isomorphism.

Examples:

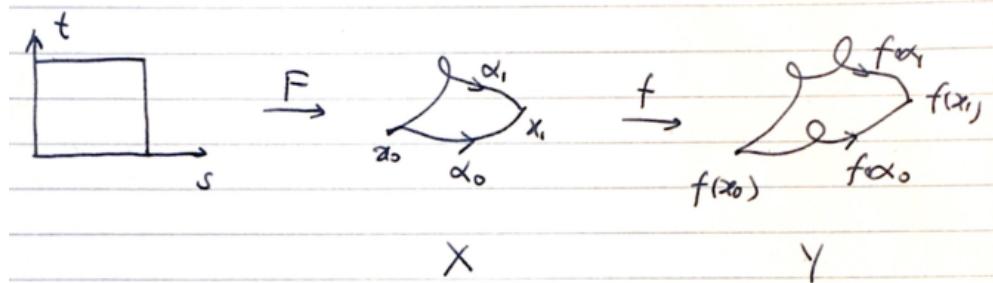
- (1) A group is a groupoid with one object.
- (2) For any space X , the fundamental groupoid $\prod(X)$ is a category whose objects are the points of X , and whose morphisms are path homotopy classes of paths.

Naturality/functoriality:

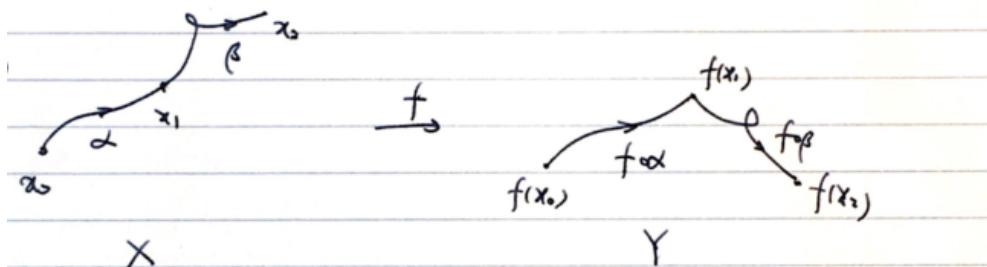
Lemma 3.8. Let $f : X \rightarrow Y$ be a continuous map.

- (1) If F is a path homotopy between the paths α_0 and α_1 in X , then $f \circ F$ is a path homotopy between the paths $f \circ \alpha_0$ and $f \circ \alpha_1$.
- (2) Let α and β be paths in X , with $\alpha(1) = \beta(0)$, then $f \circ (\alpha * \beta) = (f \circ \alpha) * (f \circ \beta)$.

Proof. (1)



(2)



□

$$\Pi_1 : \mathcal{T}_{\text{op}} \rightarrow \text{Groupoids}$$

Therefore $X \mapsto \pi_1(X)$ is a functor.

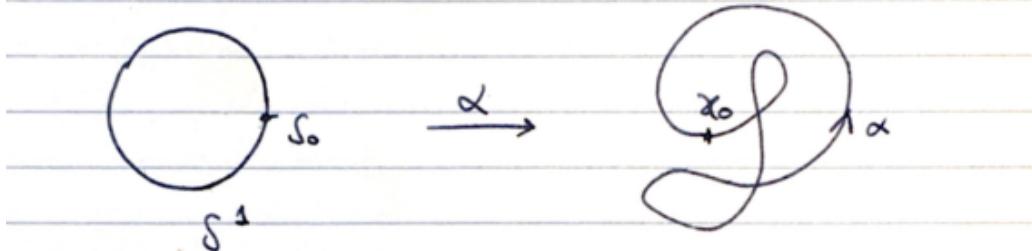
$$(X \xrightarrow{f} Y) \mapsto f_* \pi_1(X)$$

3.2 the fundamental group

Definition 3.9. Let X be a space, $x_0 \in X$ be a point. A path α in X with $\alpha(0) = \alpha(1) = x_0$ is called a loop (环路) based at x_0 . The set of path homotopy classes of loops at x_0 with the operation $*$, is called the fundamental group (基本群) (the first homotopy group, 一阶同伦群) of X , relative to the base point x_0 , denoted by $\pi_1(X, x_0)$.

Alternative definition:

$$\pi_1(X, x_0) = \text{Map}((S^1, s_0), (X, x_0)) / \text{based homotopy}$$



$$\pi_0(X, x_0) = \text{Map}((S^0, s_0), (X, x_0)) / \text{based homotopy a set.}$$

$$\pi_0(X, x_0) = 0 \Leftrightarrow X \text{ is path-connected.}$$

$$\pi_n(X, x_0) = \text{Map}((S^n, s_0), (X, x_0)) / \text{based homotopy an abelian group for } n \geq 2.$$

Example:

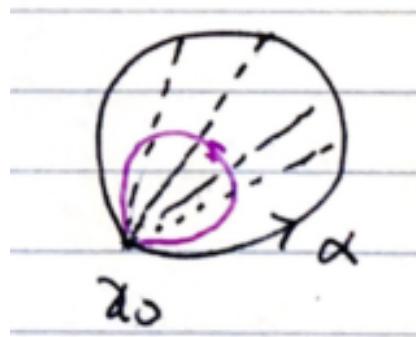
$$\pi_1(\mathbb{R}^n, x_0) = \{e\}$$

Straight-line homotopy

$$F(s, t) = (1 - t)\alpha(s) + tx_0$$

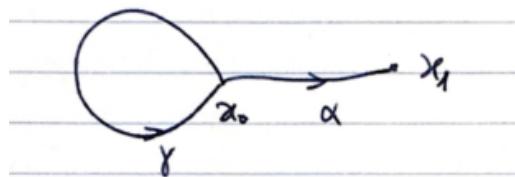
For any $A \subset \mathbb{R}^n$ convex,

$$\pi_1(A, x_0) = \{e\}$$



Definition 3.10. Let α be a path in X from x_0 to x_1 . We define a map

$$\begin{aligned} \Phi_\alpha : \pi_1(X, x_0) &\rightarrow \pi_1(X, x_1) \\ [\gamma] &\mapsto [\bar{\alpha}] * [\gamma] * [\alpha] \end{aligned}$$



Theorem 3.11. The map Φ_α is a group isomorphism.

Proof. (i) Φ_α is a homomorphism:

$$\begin{aligned} &\Phi_\alpha([\gamma_1]) * \Phi_\alpha([\gamma_0]) \\ &= ([\bar{\alpha}] * [\gamma_1] * [\alpha]) * ([\bar{\alpha}] * [\gamma_2] * [\alpha]) \\ &= [\bar{\alpha}] * [\gamma_1] * [\gamma_2] * [\alpha] \\ &= \Phi_\alpha([\gamma_1] * [\gamma_2]) \end{aligned}$$

(ii) Φ_α has an inverse homomorphism $\Phi_{\bar{\alpha}}$:

$$\Phi_\alpha \circ \Phi_{\bar{\alpha}}([\gamma]) = [\bar{\alpha}] * [\alpha] * [\gamma] * [\bar{\alpha}] * [\alpha] = [\gamma].$$

Similar for $\Phi_{\bar{\alpha}} \circ \Phi_\alpha$.

□

Corollary 3.12. *If X is path connected, then for any $x_0, x_1 \in X$, $\pi_1(X, x_0)$ is isomorphic to $\pi_1(X, x_1)$.*

Definition 3.13. *A space X is said to be simply connected (单连通) if it is path-connected and $\pi_1(X, x_0) = 0$.*

Definition 3.14. *Let $f : X \rightarrow Y$ be a continuous map, $f(x_0) = y_0$. Define $f_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0)$ by $f_*([\gamma]) = [f \circ \gamma]$. f_* is called the homomorphism induced by f .*

Check:

$$\begin{array}{ccccccc} \gamma & \simeq_p & \gamma' & \Rightarrow f \circ & \simeq_p & f \circ \gamma' \\ \text{(i) well-defined:} & & & F & & f \circ F \\ & & & & & & \end{array}$$

$$\text{(ii) homomorphism: } f \circ (\gamma_1 * \gamma_2) = (f \circ \gamma_1) * (f \circ \gamma_2)$$

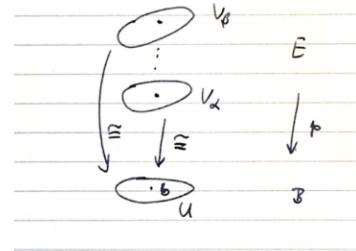
Theorem 3.15. *If $f : (X, x_0) \rightarrow (Y, y_0)$, $g : (Y, y_0) \rightarrow (Z, z_0)$ are continuous maps, then $(g \circ f)_* = g_* \circ f_*$; if $\text{id} : (X, x_0) \rightarrow (X, x_0)$ is the identity map, then id_* is the identity homomorphism.*

Proof. $(g \circ f)_*[\gamma] = [g \circ f \circ \gamma] = [g \circ (f \circ \gamma)] = g_*([f \circ \gamma]) = g_*(f_*[\gamma]).$ □

Corollary 3.16. *If $h : X \rightarrow Y$ is a homeomorphism, $h(x_0) = y_0$, then $h_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0)$ is an isomorphism.*

3.3 covering spaces

Definition 3.17. Let $p : E \rightarrow B$ be a continuous surjective map. If for every point $b \in B$, there exists a neighborhood U s.t. $p^{-1}(U)$ is a union of disjoint open sets V_α in E , $p^{-1}(U) = \coprod_\alpha V_\alpha$ and for each α , $p|_{V_\alpha} : V_\alpha \rightarrow U$ is a homeomorphism. Then p is called a covering map (覆/复叠/迭映射), and E is said to be a covering space (覆叠空间) of B .



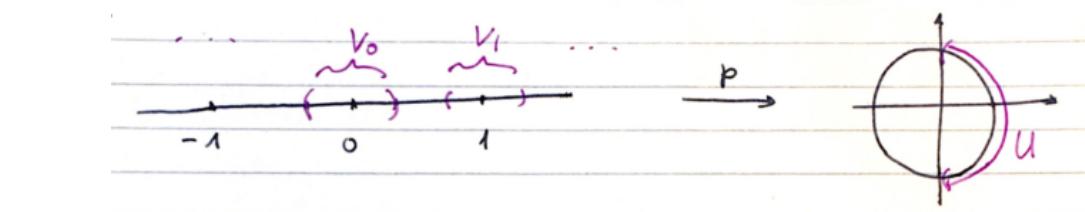
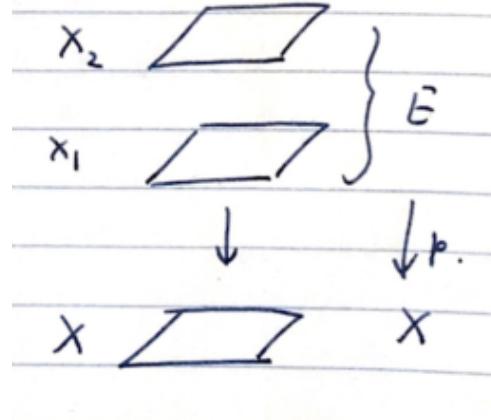
Remark:

(1) $p^{-1}(b)$ has discrete topology.

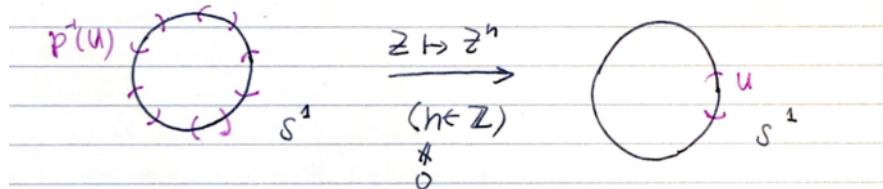
(2) p is an open map.

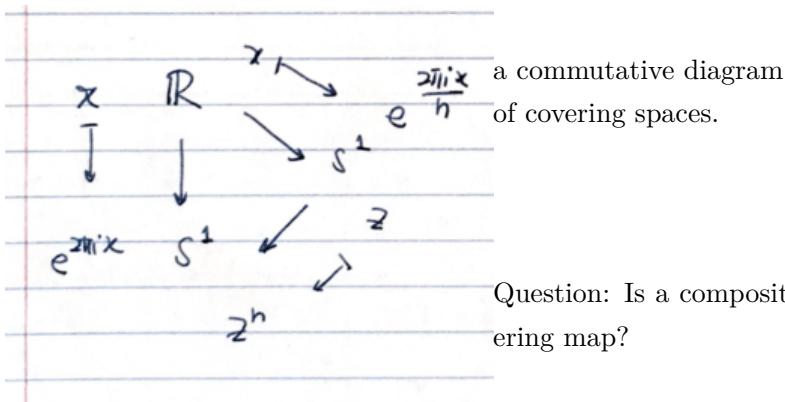
Examples:

$$(1) \quad \begin{array}{ccc} E = X \times \{1, \dots, n\} & \xrightarrow{r} X & \text{a trivial covering map} \\ (x, i) & \mapsto x & \end{array}$$



(3) $p : S^1 \rightarrow S^1, z \mapsto z^n$ is a covering map.

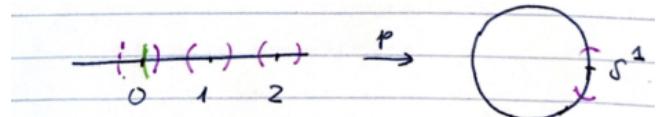




Question: Is a composite of covering maps a covering map?

Remark: A covering map $p : E \rightarrow B$ is a local homomorphism, (i.e. $\forall e \in E, \exists$ a neighborhood V of e s.t. $p|_V$ is a homeomorphism). But the converse is not true:

$$p : \mathbb{R}_+ \longrightarrow S^1 (\mathbb{R}_+ = \mathbb{R}_{>0})$$



is a local homeomorphism, but not a covering map.

Theorem 3.18. If $p : E \rightarrow B, p' : E' \rightarrow B'$ are covering maps, then $p \times p' : E \times E' \rightarrow B \times B'$ is a covering map.

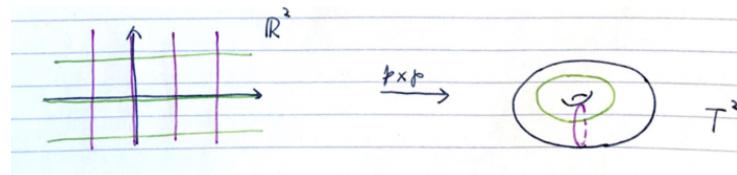
Proof.

$$\begin{aligned} (p \times p')^{-1}(U \times U') &= p^{-1}(U) \times (p')^{-1}(U') \\ &= \coprod_{\alpha} V_{\alpha} \times \coprod_{\beta} V'_{\beta} \\ &= \coprod_{\alpha, \beta} (V_{\alpha} \times V'_{\beta}) \end{aligned}$$

So $p \times p' : V_{\alpha} \times V'_{\beta} \rightarrow U \times U'$ is a homeomorphism. \square

Examples:

(1) $p \times p : \mathbb{R}^2 = \mathbb{R} \times \mathbb{R} \rightarrow S^1 \times S^1 = T^2$ is a covering map.



(2) $B_0 = S^1 \times \{\ast\} \cup \{\ast\} \times S^1 \subset T^2$

the figure-eight space  $= S^1 \vee S^1$

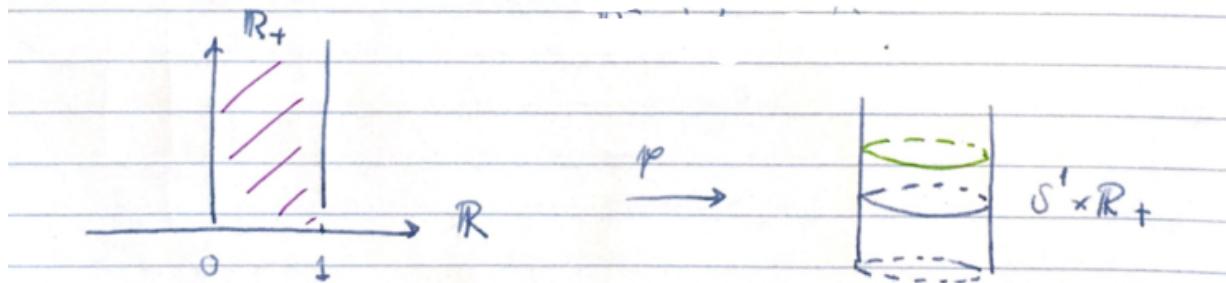
$$E_0 = p^{-1}(B_0) = (\mathbb{R} \times \mathbb{Z}) \cup (\mathbb{Z} \times \mathbb{R})$$

the “infinite grid” $E_0 \xrightarrow{p} B_0$ is a covering map.

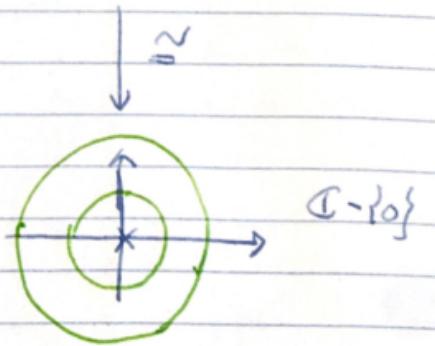
Question: Can you find other covering spaces of $S^1 \vee S^1$?

(3)

$\times \text{id} : \mathbb{R} \times \mathbb{R}_+ \rightarrow S^1 \times \mathbb{R}_+, (x, t) \mapsto (e^{2\pi i x}, t)$ is a covering map, in which $S^1 \times \mathbb{R}_+ \cong \mathbb{R}^2 - \{0\}$ by $\varphi : (z, t) \mapsto (t, z)$.



the Riemann
surface of $f(z) = \log z$



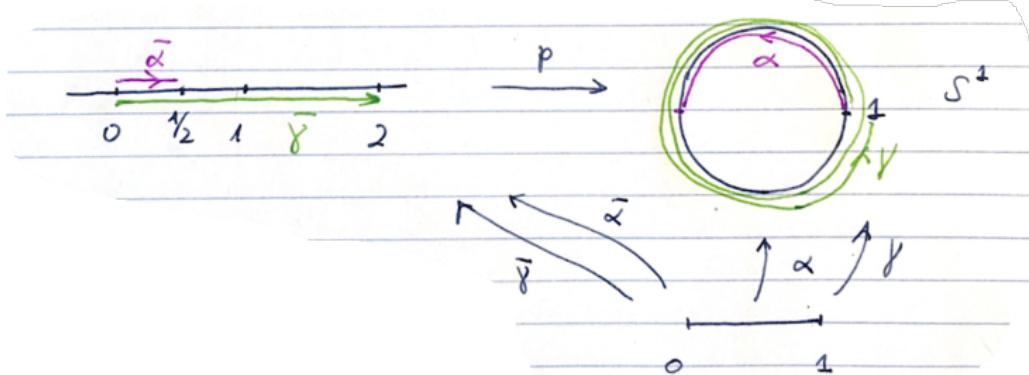
3.4 the fundamental group of the circle

Definition 3.19. Let $p : E \rightarrow B$ be a map, $f : X \rightarrow B$ be a continuous map. A lifting (提升) of f is a map $\bar{f} : X \rightarrow E$ s.t. $p \circ \bar{f} = f$.

$$\begin{array}{ccc} & E & \\ \bar{f} & \nearrow & \downarrow p \\ X & \xrightarrow{f} & B \end{array}$$

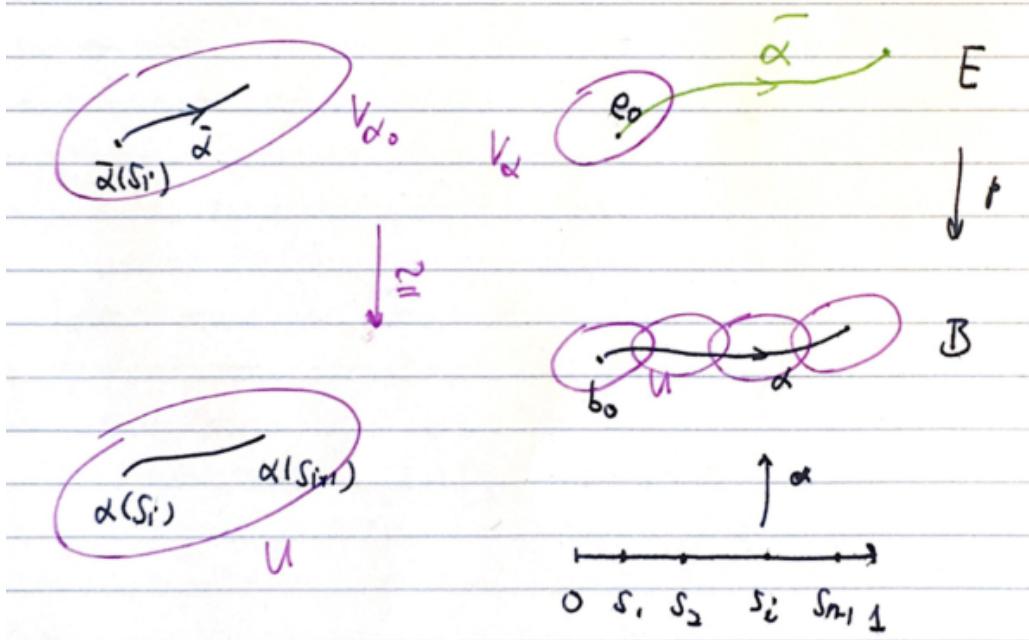
Example: $p : \mathbb{R} \rightarrow S^1$ the covering map.

The path $\alpha : [0, 1] \rightarrow S^1, \alpha(s) = (\cos \pi s, \sin \pi s)$, lifts to the path $\bar{\alpha} : [0, 1] \rightarrow \mathbb{R}, \bar{\alpha}(s) = \frac{s}{2}$.



The loop $\gamma : [0, 1] \rightarrow S^1, \gamma(s) = (\cos 4\pi s, \sin 4\pi s)$ lifts to the path $\bar{\gamma} : [0, 1] \rightarrow \mathbb{R}, \bar{\gamma}(s) = 2s, \bar{\gamma}(0) = 0, \bar{\gamma}(1) = 2$.

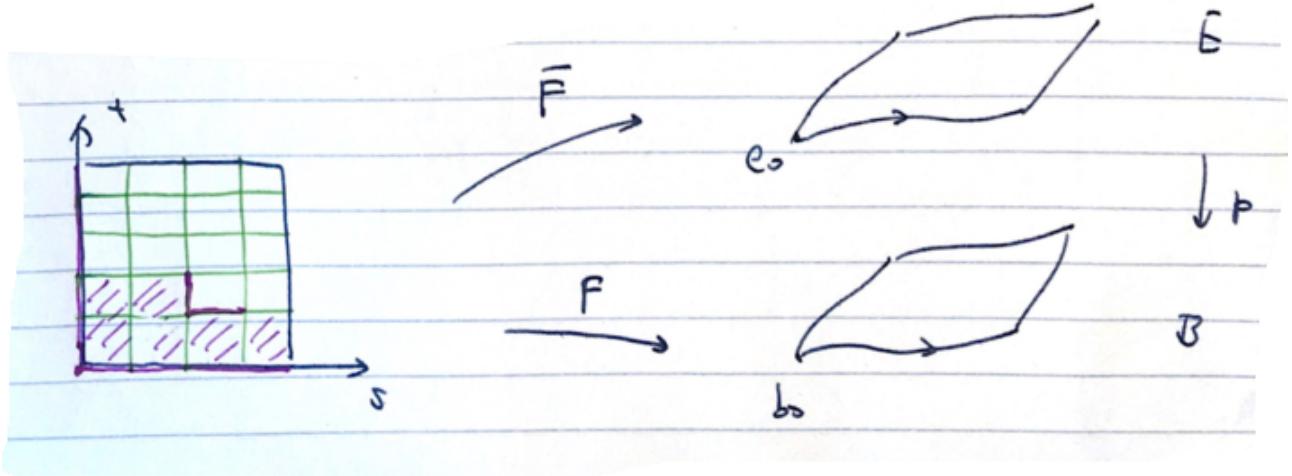
Lemma 3.20. Let $p : E \rightarrow B$ be a covering map, $p(e_0) = b_0$. Then any path $\alpha : [0, 1] \rightarrow B$ beginning at b_0 has a unique lifting to a path $\bar{\alpha} : [0, 1] \rightarrow E$ beginning at e_0 .



Proof. (1) Existence: cover B by evenly-covered open sets U , find a subdivision $O = s_0 < s_1 < \dots < s_n = 1$ of $[0, 1]$ s.t. $[s_i, s_{i+1}]$ lies in such an U (Lebesgue number lemma). Define the lifting $\bar{\alpha}$ inductively:

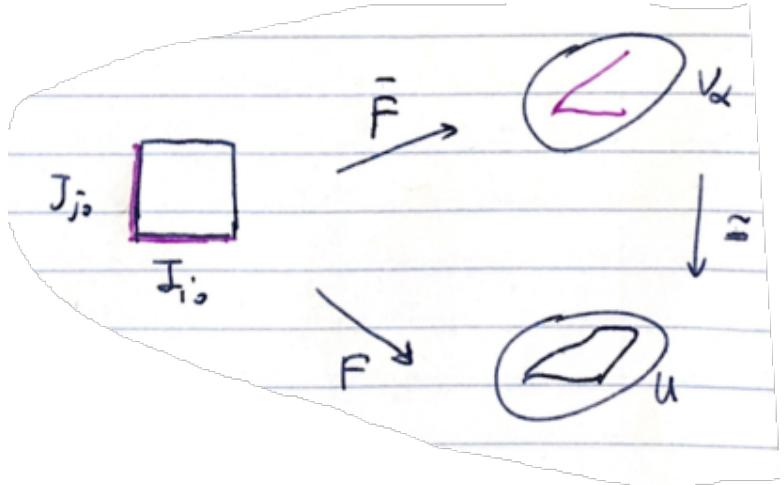
- (i) $\bar{\alpha}(0) = e_0$
- (ii) Suppose $\bar{\alpha}(s)$ is defined for $0 \leq s \leq s_i$, define $\bar{\alpha}$ on $[s_i, s_{i+1}]$ as follows: $f([s_i, s_{i+1}]) \subset U, p^{-1}(U) = \coprod_{\alpha} V_{\alpha}$. Assume $\bar{\alpha}(s_i) \in V_{\alpha_0}, p|_{V_{\alpha_0}} : V_{\alpha_0} \xrightarrow{\cong} U$ a homeomorphism. \therefore define $\bar{\alpha}(s) = (p|_{V_{\alpha}})^{-1}(\alpha(s))$ for $s \in [s_i, s_{i+1}]$
- (2) uniqueness: Assume $\bar{\alpha}'$ is another lifting of α with $\bar{\alpha}'(0) = e_0 = \bar{\alpha}(0)$. Assume $\bar{\alpha}'(s) = \bar{\alpha}(s)$ on $[0, s_i]$. $\bar{\alpha}'$ is a lifting of $\alpha \Rightarrow \bar{\alpha}'([s_i, s_{i+1}]) \subset \coprod_{\alpha} V_{\alpha}$. $\bar{\alpha}'([s_i, s_{i+1}])$ is connected $\Rightarrow \bar{\alpha}'([s_i, s_{i+1}]) \subset V_{\alpha_0}$ since $\bar{\alpha}'(s_i) = \bar{\alpha}(s_i) \in V_{\alpha_0}$. $p|_{V_{\alpha_0}} : V_{\alpha_0} \rightarrow U$ is a homeomorphism $\Rightarrow \bar{\alpha}' = \bar{\alpha}$ on $[s_i, s_{i+1}]$. \square

Lemma 3.21. Let $p : E \rightarrow B$ be a covering map, $p(e_0) = b_0$. Let $F : I \times I$ be a continuous map, $F(0,0) = b_0$. Then there is a unique lifting of F to a continuous map $\bar{F} : I \times I \rightarrow E$ s.t. $\bar{F}(0,0) = e_0$. If F is a path homotopy, then so is \bar{F} .



Proof. (1) Define $\bar{F}(0,0) = e_0$, extend to $\bar{F} : 0 \times I \cup I \times 0 \rightarrow E$ by the preceding lemma. Choose subdivisions $0 = s_0 < s_1 < \dots < s_m = 1, 0 = t_0 < t_1 < \dots < t_n = 1$, with $I_i = [s_{i-1}, s_i], J_j = [t_{j-1}, t_j]$, s.t. $F(I_i) \times J_j \subset U$ with $p^{-1}(U) = \coprod_{\alpha} (V_{\alpha}, p|_{V_{\alpha}} : V_{\alpha} \xrightarrow{\cong} U)$ a homeomorphism.

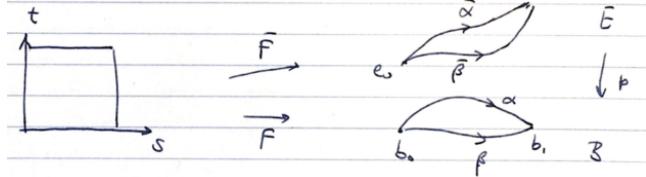
Extend \bar{F} inductively to $I_1 \times J_1, I_2 \times J_1, \dots, I_1 \times J_2, I_2 \times J_2, \dots$. Assume \bar{F} is defined on $A, I_{i_0} \times J_{j_0}$ is the next rectangle. Since $\bar{F}(A \cap (I_{i_0} \times J_{j_0}))$ is connected it is connected in some V_{α} , $p|_{V_{\alpha}} : V_{\alpha} \rightarrow U$ is a homeomorphism. \therefore define $\bar{F}(x) = p^{-1}(F(x))$ for $\forall x \in I_{i_0} \times J_{j_0}$. And $\bar{F} : X \times I \rightarrow E$ is unique since in each step the extension is unique.



(2) If F is a path homotopy, i.e. $F(0 \times I) = b_0$, then $\bar{F}(0 \times I) \subset= e_0$. Similiar for $\bar{F}(1 \times I)$. \square

Theorem 3.22. Let $p : E \rightarrow B$ be a covering map, $p(e_0) = b_0$. If α and β are two paths in B from b_0 to b_1 , let $\bar{\alpha}$ and $\bar{\beta}$ be their liftings to paths in E , $\bar{\alpha}(0) = \bar{\beta}(0) = e_0$. If α and β are path homotopic, then $\bar{\alpha}(1) = \bar{\beta}(1)$, and $\bar{\alpha}$ and $\bar{\beta}$ are path homotopic.

Proof. Let $F : X \times I \rightarrow B$ be the path homotopy between α and β .



Let $\bar{F} : I \times I \rightarrow E$ be the lifting of F with $\bar{F}(0, 0) = e_0$. By the preceding lemma, $\bar{F}(0 \times I) = e_0, \bar{F}(1 \times I) = e_1, \bar{F}(0, s) = \bar{\alpha}(s), \bar{F}(1, s) = \bar{\beta}(s)$. \square

Slogan: Paths have liftings; path-homotopic paths have path-homotopic liftings.

Definition 3.23. Let $p : E \rightarrow B$ be a covering map, fix $b_0 \in B$ and $e_0 \in p^{-1}(b_0)$. Given an element $[\gamma] \in \pi_1(B, b_0)$, let $\bar{\gamma}$ be the lifting of γ in E with $\bar{\gamma}(0) = e_0$. Let $\phi([\gamma]) = \bar{\gamma}(1)$. Then ϕ is a well-defined map.

$$\begin{aligned}\Phi : \pi_1(B, b_0) &\rightarrow p^{-1}(b_0) \\ [\gamma] &\mapsto \bar{\gamma}(1)\end{aligned}$$

(Since $\gamma \simeq_p \gamma' \Rightarrow \bar{\gamma}(1) = \bar{\gamma}'(1)$) called the lifting correspondence, depending on the choice of e_0 .

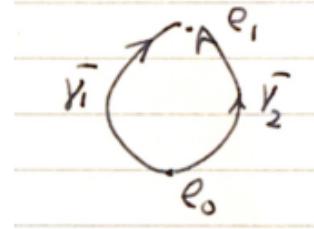
Theorem 3.24. Let $p : E \rightarrow B$ be a covering map, $p(e_0) = b_0$. If E is path connected, then the lifting correspondence

$$\phi : \pi_1(B, b_0) \rightarrow p^{-1}(b_0)$$

is surjective. If E is simply connected, it is bijective.

Proof. (1) If E is path connected, given $e_1 \in p^{-1}(b_0)$, let $\bar{\gamma}$ be a path from e_0 to e_1 , then $\gamma = p \circ \bar{\gamma}$ is a loop at b_0 with $\phi([\gamma]) = \bar{\gamma}(1) = e_1$.

(2) Assume E is simply connected, $[\gamma_1], [\gamma_2] \in \pi_1(B, b_0)$ s.t. $\phi([\gamma_1]) = \phi([\gamma_2])$, $\bar{\gamma}_1, \bar{\gamma}_2$ be the liftings of γ_1 and γ_2 , $\bar{\gamma}_1(0) = \bar{\gamma}_2(0) = e_0$. E is simply connected $\Rightarrow \bar{\gamma}_1 \simeq_p \bar{\gamma}_2 (\bar{\gamma}_1 \simeq_p \bar{\gamma}_2 \Leftrightarrow \bar{\gamma}_1 * (-\bar{\gamma}_2) \simeq_p e_{e_0} \Rightarrow \gamma_1 \simeq_p \gamma_2)$



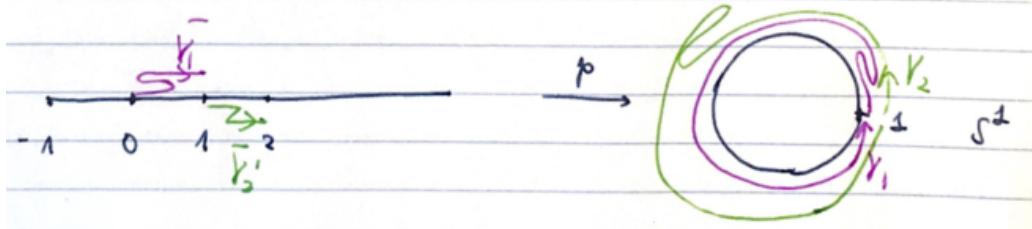
\square

Theorem 3.25. The fundamental group of S^1 is isomorphic to the additive group of integers, $\pi_1(S^1) \cong \mathbb{Z}$.

Proof. Let $p : \mathbb{R} \rightarrow S^1, x \mapsto e^{i2\pi x}$ be the covering map. $e_0 = 0 \in \mathbb{R}, b_0 = 1 \in S^1 \subset \mathbb{C}$. Then $p^{-1}(b_0) = \mathbb{Z}$, \mathbb{R} is 1-connected $\Rightarrow \phi : \pi_1(S^1, b_0) \rightarrow \mathbb{Z}$ is bijective. \square

Claim: ϕ is a homomorphism.

Proof.



given $[\gamma_1], [\gamma_2] \in \pi_1(S^1, b_0)$, let $\bar{\gamma}_1, \bar{\gamma}_2$ be the liftings in \mathbb{R} with $\bar{\gamma}_1(0) = \bar{\gamma}_2(0) = 0$. Let $\phi([\gamma_1]) = \bar{\gamma}_1(1) = n, \phi([\gamma_2]) = \bar{\gamma}_2(1) = m$. Let $\bar{\gamma}'_2$ be the path $\bar{\gamma}'_2(s) = n + \bar{\gamma}_2(s)$, then $\bar{\gamma}'_2$ is a lifting of γ_2 , with $\bar{\gamma}'_2(0) = n = \bar{\gamma}_1(1)$.

$\therefore \bar{\gamma}_1 * \bar{\gamma}'_2$ is a lifting of $\gamma_1 * \gamma_2$, begins at 0, and $\bar{\gamma}'_2(1) = n + m$. Therefore $\Phi([\gamma_1] * [\gamma_2]) = (\bar{\gamma}_1 * \bar{\gamma}'_2)(1) = \bar{\gamma}'_2(1) = n + m$. \square

Theorem 3.26. Let $p : E \rightarrow B$ be a covering map, $p(e_0) = b_0$.

- (1) The homomorphism $\varphi_* : \pi_1(E, e_0) \rightarrow \pi_1(B, b_0)$ is a monomorphism.
- (2) Let $H = p_0(\pi_1(E, e_0))$. The lifting correspondence induces an injective map: $\Phi : H \setminus \pi_1(B, b_0) \rightarrow p^{-1}(b_0)$ of the set of the right cosets of H into $p^{-1}(b_0)$. Φ is bijective if E is path connected.
- (3) If γ is a loop in B based at b_0 , then $[\gamma] \in H$ if and only if γ lifts to a loop based at e_0 .

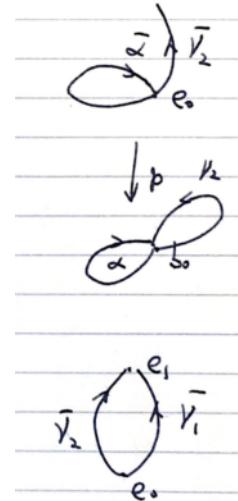
Proof. (1) Let $\bar{\gamma}$ be a loop in E at e_0 s.t. $p_*[\bar{\gamma}]$ is the identity element in $\pi_1(B, b_0)$. Let F be a path homotopy between $p \circ \bar{\gamma}$ and $e_{b_0} \cdot \bar{F}$ be the lifting of F with $\bar{F}(0, 0) = e_0$. Then $\bar{F}(0, s) = \bar{\gamma}(s) \cdot \bar{F}(1, s) = e_{e_0}$, \bar{F} is a path homotopy between $\bar{\gamma}$ and e_{e_0} .

(2) Let $[\gamma_1], [\gamma_2] \in \pi_1(B, b_0)$. Claim: $\phi([\gamma_1]) = \phi([\gamma_2]) \Leftrightarrow [\gamma_1] \in H * [\gamma_2]$.

Proof: Let $\bar{\gamma}_1, \bar{\gamma}_2$ be liftings of γ_1 and γ_2 at e_0 .

(i) if $[\gamma_1] \in H * [\gamma_2]$, then $[\gamma_1] = [\alpha * \gamma_2]$ for some $[\alpha] \in H$. There exists a loop $\bar{\alpha}$ at e_0 s.t. $\alpha = p \circ \bar{\alpha}$. Now $\bar{\alpha} * \bar{\gamma}_2$ is a lifting of $\alpha * \gamma_2$ at e_0 . $\therefore \bar{\gamma}_1$ and $\bar{\alpha} * \bar{\gamma}_2$ must have the same ending point, i.e. $\bar{\gamma}_1(1) = \bar{\alpha} * \bar{\gamma}_2(1)$.

(ii) If $\bar{\gamma}_1(1) = \bar{\gamma}_2(1)$, then $\bar{\alpha} = \bar{\gamma}_1 * (-\bar{\gamma}_2)$ is a loop at e_0 and $[\bar{\gamma}_1] = [\bar{\alpha} * \bar{\gamma}_2]$. Let $F : \bar{\gamma}_1 \simeq_p \bar{\alpha} * \bar{\gamma}_2$ be a path homotopy, then $F = p \circ \bar{F}$ is a path homotopy between γ_1 and $\alpha * \gamma_2$, where $\alpha = p \circ \bar{\alpha}$ is a loop at b_0 , i.e. $[\gamma_1] = [\alpha] * [\gamma_2] \in H * [\gamma_2]$.



E path connected $\Rightarrow \phi$ is surjective $\Rightarrow \Phi$ is bijective.

(3) Since $\phi([\gamma_1]) = \phi([\gamma]) \Leftrightarrow [\gamma_1] * H * [\gamma_2]$, take $\gamma_1 = \gamma, \gamma_2 = e_{b_0}$, then $[\gamma] \in H \Leftrightarrow \phi([\gamma]) = \phi([e_{b_0}]) = e_0$. i.e. $[\gamma] \in H \Leftrightarrow \bar{\gamma}$ is a loop at e_0 . \square

3.5 applications

Definition 3.27. Let $A \subset X$ be a subspace. A retraction (收缩映射) of X onto A is a continuous map $\gamma : X \rightarrow A$ s.t. $\gamma|_A = \text{id}_A$. If such a map γ exists, we say that A is a retract (收缩核) of X .

Lemma 3.28. If A is a retract of X , $j : A \rightarrow X$ denote the inclusion map, then the induced homomorphism $j_* : \pi_1(A, a) \rightarrow \pi_1(X, a)$ is injective.

Proof. Let $r : X \rightarrow A$ be a retraction, then $\text{id}_A = r \circ j : A \xrightarrow{j} X \xrightarrow{r} A$. $\therefore \text{id} = r_* \circ j_* : \pi_1(A, a) \rightarrow \pi_1(X, a) \rightarrow \pi_1(A, a)$. $\therefore j_*$ is injective. \square

Theorem 3.29. S^1 is not a retract of D^2 .

Proof. $j_* : \pi_1(S^1, b_0) \rightarrow \pi_1(D^2, b_0) = 0$ is not injective, in which $\pi_1(S^1, b_0) \cong \mathbb{Z}$. \square

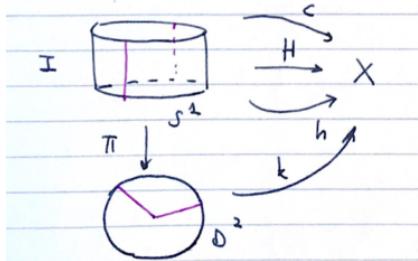
Lemma 3.30. Let $h : S^1 \rightarrow X$ be a continuous map. Then the followings are equivalent.

- (1) h is nullhomotopic, i.e. $h \simeq C$.
- (2) h extends to a continuous map $k : D^2 \rightarrow X$, i.e.

$$\begin{array}{ccc} S^1 & \xrightarrow{h} & X \\ i \downarrow & \nearrow k & \\ D^2 & & \end{array}$$

- (3) $h_* : \pi_1(S^1, b_0) \rightarrow \pi_1(X, h(b_0))$ is the trivial homomorphism.

Proof. (1) \Rightarrow (2) Let $H : S^1 \times I \rightarrow X$ be a homotopy between h and a constant map c .



Let $\pi : S^1 \times I \rightarrow D^2, (x, t) \mapsto (1-t)x$. Then π is continuous, closed and surjective, so it is a quotient map. $H|_{S^1 \times \{1\}} = C$. $\therefore H$ induces a continuous map $k : D^2 \rightarrow X, k|_{S^1} = h$.

(2) \Rightarrow (3)

$$\begin{array}{ccc} S^1 & \xrightarrow{h} & X \\ i \downarrow & \nearrow k & \\ D^2 & & \end{array} \quad \therefore h_* = k_* \circ j_*.$$

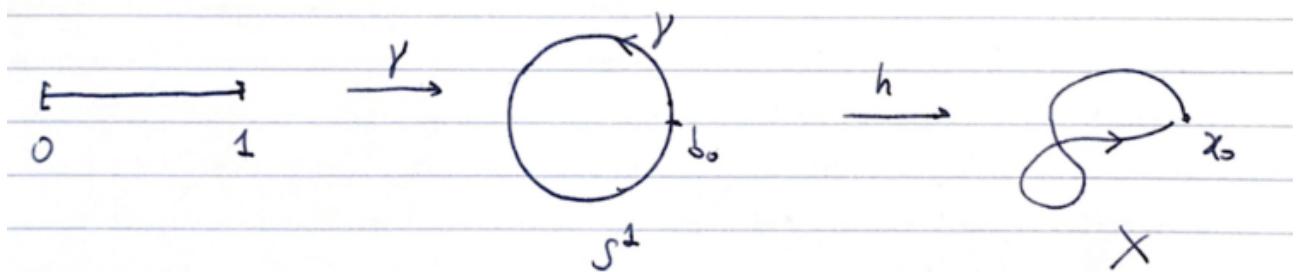
$\pi_1(S^1, b_0) \rightarrow \pi_1(D^2, b_0) \rightarrow \pi_1(X, h(b_0))$

\parallel

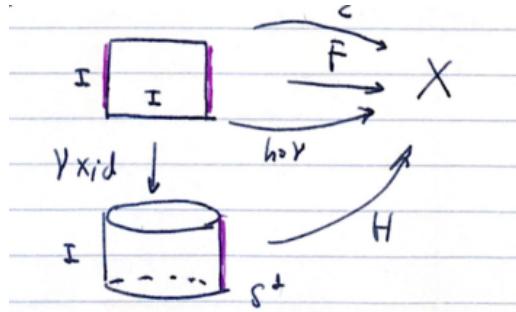
0

$\therefore h_*$ is the trivial homomorphism.

(3) \Rightarrow (1): A generator of $\pi_1(S^1, b_0) \cong \mathbb{Z}$ is represented by the loop $\gamma : [0, 1] \rightarrow S^1, x \mapsto e^{i2\pi x}$.



Let $h(b_1) = x_0$, if $h_* : \pi_1(S^1, b_0) \rightarrow \pi_1(X, x_0)$ is trivial, then $h \circ \gamma \simeq_p e_{x_0}$. Let $F : I \times I \rightarrow X$ be a path homotopy.



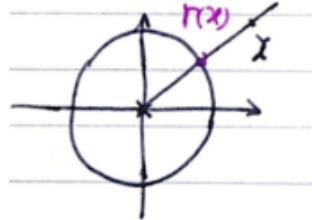
The map $\gamma \times \text{id} : I \times I \times S^1 \times I, (x, t) \mapsto (e^{i2\pi x}, t)$ is a quotient map (since continuous, closed and surjective). $F(0 \times I) = x_0 = F(1 \times I)$. $\therefore F$ induces a continuous map $H : S^1 \times I \rightarrow X$, which is a homotopy between h and c .

□

Corollary 3.31. *The inclusion map $j : S^1 \rightarrow \mathbb{R}^2 - 0$ is not nullhomotopic. the identity map $\text{id} : S^1 \rightarrow S^1$ is not nullhomotopic.*

Proof.

\exists a retraction $r : \mathbb{R}^2 - 0 \rightarrow S^1, x \mapsto \frac{x}{|x|}$
 $\therefore j_* : \pi_1(S^1, b_0) \rightarrow \pi_1(\mathbb{R}^2 - 0, b_0)$ is injective not trivial.
 Similar for $\text{id} : S^1 \rightarrow S^1$.

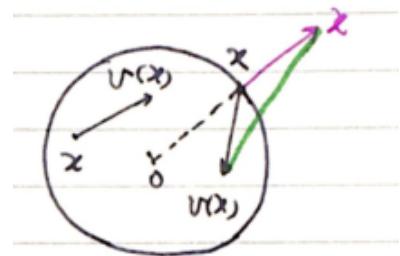


□

Theorem 3.32. *Given a nonvanishing continuous vector field $v : D^2 \rightarrow \mathbb{R}^2 - 0$ on D^2 , there exists a point $x_1 \in S^1$ s.t. $v(x_1)$ points directly inward, and a point $x_2 \in S^1$ s.t. $v_*(x_2)$ points directly outward.*

Proof. Assume $v(x)$ does not point directly inward for $\forall x \in S^1$. Consider $h = v|_{S^1} : S^1 \rightarrow \mathbb{R}^2 - 0$.

Claim: h is homotopic to the inclusion map $j : S^1 \hookrightarrow \mathbb{R}^2 - 0$, hence not nullhomotopic.



proof.

We have a straight-line homotopy $F : S^1 \times I \rightarrow \mathbb{R}^2 - 0, F(x, t) = (1-t)v(x) + tx$.

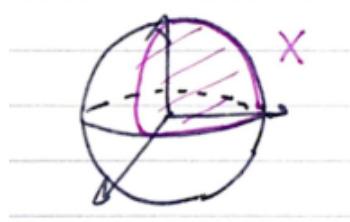
On the other hand, h extends to $v : D^2 \rightarrow \mathbb{R}^2 - 0$, hence is nullhomotopic, a contradiction.

For the directly put-pointing point x_2 , consider $v'(x) = -v(x)$.

□

Theorem 3.33 (Brouwer fixed-point theorem). *If $f : D^n \rightarrow D^n$ is a continuous map, then there exists a point $x \in D^n$ s.t. $f(x) = x$.*

Proof. Let $X = S^{n-1} \cap \mathbb{R}_{\geq 0}^n$, then X is homeomorphic to D^{n-1} . $f : X \rightarrow X, x \mapsto Ax/|Ax|$ has a fixed point $x_0 \in X$, i.e. $Ax_0 = |Ax_0| \cdot x_0$.

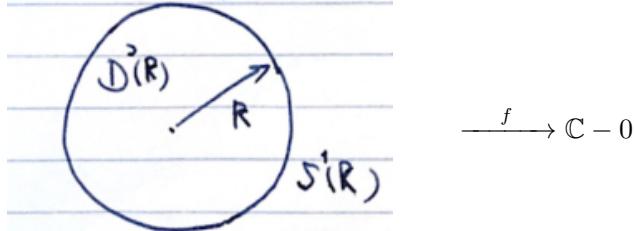


□

Theorem 3.34 (The fundamental theorem of algebra). *A polynomial $f(z) = z^n + a_{n-1}z^{n-1} + \dots + a_0 \in \mathbb{C}[z](n > 0)$ has a root in \mathbb{C} .*

Proof. Assume $f(z) \neq 0$ for $\forall z \in \mathbb{C}$.

Let $D^2(R)$ be the closed disc of radius R . $\partial D^2(R) = S^1(R)$ the circle of radius R .



Let $\varphi = f|_{S^1(R)} : S^1(R) \rightarrow \mathbb{C} - 0$,

Claim: For R large enough, φ is homotopic to $g : S^1(R) \rightarrow \mathbb{C} - 0, g(z) = z^n$.

Proof. We have a straight-line homotopy between φ and g :

$$F : S^1(R) \times I \rightarrow \mathbb{C}, F(z, t) = (1-t)\varphi(z) + tz^n = z^n + (1-t)(a_{n-1}z^{n-1} + \dots + a_0)$$

For R large enough, $|F(z, t)| \geq |z|^n - (|a_{n-1}|) > 0$.

$$\therefore F : S^1(R) \times I \rightarrow \mathbb{C} - 0.$$

φ extends to $f|_{D^2(R)} : D^2(R) \rightarrow \mathbb{C} - 0$, hence is nullhomotopic $\Rightarrow g$ is nullhomotopic for R large enough.

Claim: g is not nullhomotopic for all $R > 0$.

Proof.

$$\begin{aligned} h : S^1(R) &\xrightarrow{g} \mathbb{C} - 0 \xrightarrow{r} S^1 \\ z &\mapsto z^n, \quad z \mapsto z/|z| \end{aligned}$$

$S^1(R)$ 	$\xrightarrow{h} S^1$ $\xrightarrow{R \cdot e^{i\pi x}} \xrightarrow{e^{i\pi l x}}$ $\circ \quad 1$	$h_* : \pi_1(S^1(R)) \rightarrow S^1$ $\xrightarrow{\cong} \mathbb{Z}$ $1 \mapsto n$
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$\therefore h$ is not homotopic $\Rightarrow g$ is not nullhomotopic. □

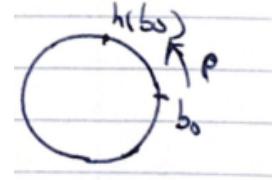
Theorem 3.35 (Borsuk-Ulam theorem). *Given a continuous map $f : S^n \rightarrow \mathbb{R}^n (n \geq 0)$, there is a point $x \in S^n$ s.t. $f(x) = f(-x)$.*

Definition 3.36. *The antipode of a point $x \in S^n$ is the point $-x$. A map $h : S^n \rightarrow S^n$ is antipode-preserving (保对径点的) if $h(-x) = -h(x)$ for all $x \in S^n$.*

Theorem 3.37. *If $h : S^1 \rightarrow S^1$ is a continuous antipode-preserving map, then h is not nullhomotopic.*

Proof. Let $b_0 = 1 \in S^1$, $\rho : S^1 \rightarrow S^1$ be a rotation s.t. $\rho(h(b_0)) = b_0$. Then $\rho \circ h$ is also antipode-preserving. $\rho \simeq \text{id}_{S^1} \Rightarrow h \simeq C$ iff $\rho \circ h \simeq C$.

\therefore It suffices to prove the theorem for those h with $h(b_0) = b_0$.



Consider the covering map $q : S^1 \rightarrow S^1$, $q(z) = z^2$.

$$q_* : \pi_1(S^1, b_0) \rightarrow \pi_1(S^1, b_0)$$

$$\begin{array}{ccc} \mathbb{R} & & \mathbb{R} \\ \mathbb{Z} & \xrightarrow{\cdot 2} & \mathbb{Z} \end{array}$$

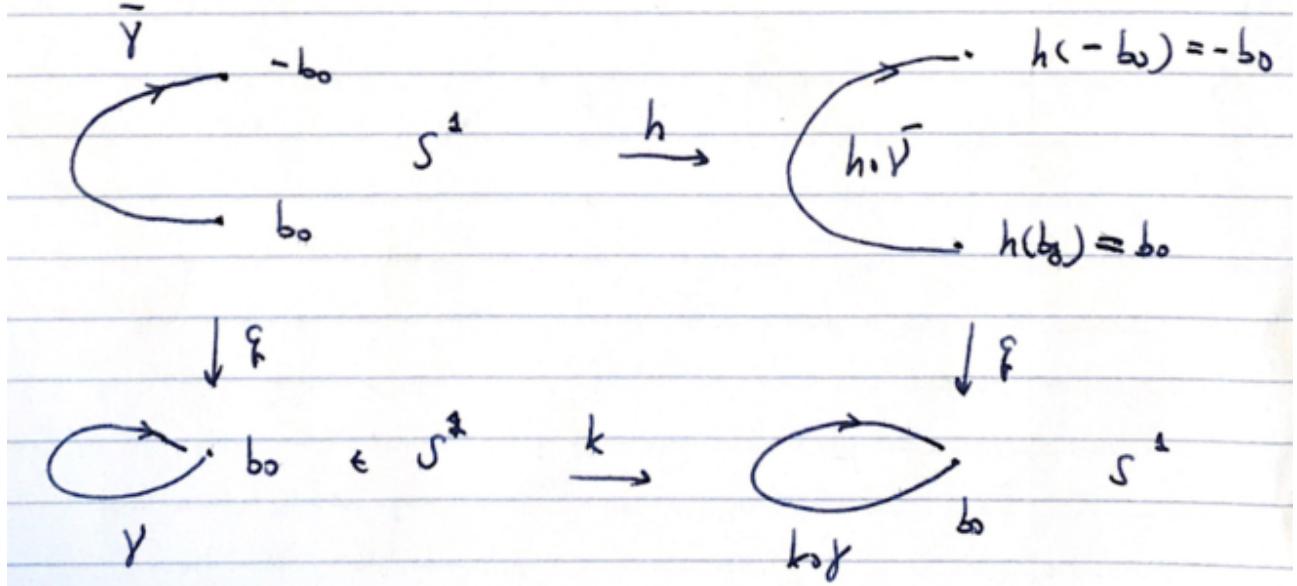
is a multiplication by 2.

Since $q(h(-z)) = q(-h(z)) = q(h(z))$, h induces a map $k : S^1 \rightarrow S^1$ s.t. $k \circ q = q \circ h$, k is continuous since q is a quotient map.

$$\begin{array}{ccc} S^1 & \xrightarrow{h} & S^1 \\ \downarrow q & & \downarrow q \\ S^1 & \xrightarrow{k} & S^1 \end{array}$$

Claim: $k_* : \pi_1(S^1, b_0) \rightarrow \pi_1(S^1, b_0)$ is nontrivial.

Proof.



Let $\bar{\gamma}$ be a path in S^1 from b_0 to $-b_0$, then $\gamma = q \circ \bar{\gamma}$ is a loop at b_0 representing a nontrivial element $[\gamma] \in \pi_1(S^1, b_0)$. Now $h \circ \bar{\gamma}$ is a lifting of $k \circ \gamma$ with $[k \circ \gamma] = k_*[\gamma] \in \pi_1(S^1, b_0)$. Since $h \circ \bar{\gamma}$ is a path from b_0 to $-b_0$, $k_*[\gamma] \neq e$. Therefore $k_* : \pi_1(S^1, b_0) \rightarrow \pi_1(S^1, b_0)$ is nontrivial, hence injective.

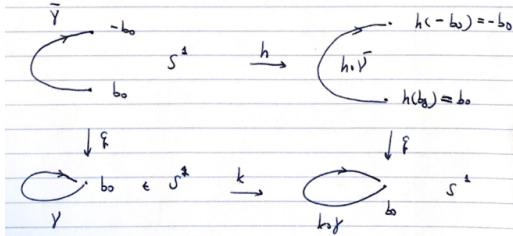
$k \circ q = q \circ h \Rightarrow k_* \circ q_* = q_* \circ h_* : \pi_1(S^1, b_0) \rightarrow \pi_1(S^1, b_0)$ h_* is injective, hence h is not nullhomotopic.

$$\begin{array}{ccc} \pi_1(S^1, b_0) & \xrightarrow{h_*} & \pi_1(S^1, b_0) \\ \downarrow q_* = \cdot 2 & & \downarrow q_* = \cdot 2 \\ \pi_1(S^1, b_0) & \xrightarrow{k_*} & \pi_1(S^1, b_0) \end{array}$$

□

Theorem 3.38. *There is no continuous antipode-preserving map $g : S^2 \rightarrow S^1$.*

Proof. Suppose $g : S^2 \rightarrow S^1$ is such a map.



Take S^1 to be the equator of S^2 ,
 $h = g|_{S^1} : S^1 \rightarrow S^1$.

Then h is continuous and antipode-preserving, $\therefore h$ is not nullhomotopic.

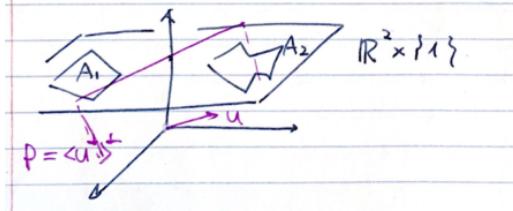
But g is an extension of h to $D_+^2 \cong D^2$, hence h is nullhomotopic, a contradiction. \square

Proof of the Borsuk-Ulam theorem for S^2

Let $f : S^2 \rightarrow \mathbb{R}^2$ be a continuous map s.t. $f(x) \neq f(-x)$ for all $x \in S^2$. Then the map $g : S^2 \rightarrow S^1, x \mapsto \frac{f(x) - f(-x)}{|f(x) - f(-x)|}$ is continuous and $g(-x) = -g(x), \forall x \in S^2$. \square

Theorem 3.39 (the bisection theorem). *Given two bounded polygonal regions A_1 and A_2 in \mathbb{R}^2 , there exists a line L in \mathbb{R}^2 that bisects each of them.*

Proof. Let $\mathbb{R}^2 = \mathbb{R}^2 \times \{1\} \subset \mathbb{R}^3$.



For $u \in S^2$, let $\rho = \langle u \rangle^\perp = \{x \in \mathbb{R}^3 | \langle x, u \rangle = 0\}$ a plane

$v = \{x \in \mathbb{R}^3 | \langle x, u \geq 0 \rangle\}$

$f_i(u) = \text{Area}(A_i \cap V)$ ($i = 1, 2$) a continuous function in U . Then $f_i(U) + f_i(-U) = \text{Area}(A_i)$.

$F : S^2 \rightarrow \mathbb{R}^2, F(u) = (f_1(u), f_2(u))$.

Borsuk-Ulam theorem $\Rightarrow \exists u \in S^2$ s.t. $F(u) = F(-u)$.

$\therefore f_i(u) = f_i(-u) = \frac{1}{2} \text{Area}(A_i)$. The corresponding line $L = P \cap (\mathbb{R}^2 \times \{1\})$ bisects A_1 and A_2 . \square

Theorem 3.40 (invariance of domain, 区域不变性). *U $\subset \mathbb{R}^2$ an open set, $f : U \rightarrow \mathbb{R}^2$ continuous and injective. Then $f(U) \subset \mathbb{R}^2$ is open and the inverse function $f^{-1} : f(U) \rightarrow U$ is open and the inverse function $f^{-1} : f(U) \rightarrow U$ is continuous, i.e. f is a homeomorphism.*

A simple closed curve (简单闭曲线) C is a space homeomorphic to S^1 .

Theorem 3.41 (the Jordan curve theorem). *Let $C \subset S^2$ be a simple closed curve. Then C separates S^2 into two components W_1 and W_2 , i.e. $S^2 - C = W_1 \sqcup W_2$. Each of W_i has C as its boundary, i.e. $\overline{W}_i - W_i = C$.*

Theorem 3.42 (Schöenflies theorem). *Let $C \subset S^2$ be a simple closed curve with $S^2 - C = W_1 \cup W_2$. Then \overline{W}_i is homeomorphic to the closed disc D^2 .*

Counter example in dimension 3: the Alexander horned sphere.

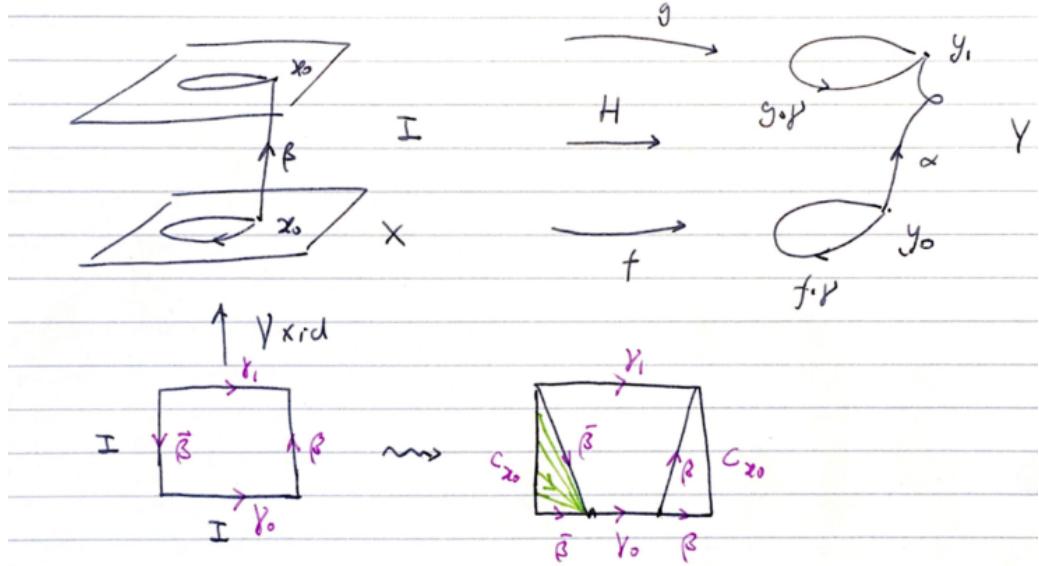
3.6 homotopy type

Theorem 3.43. Let $f, g : X \rightarrow Y$ be continuous maps, $f(x_0) = y_0, g(x_0) = y_1$. Let $H : X \times I \rightarrow Y$ be a homotopy between f and g , $\alpha(t) = H(x_0, t)$ be the path in Y from y_0 to y_1 . Then there is a commutative diagram

$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow{f_*} & \pi_1(Y, y_0) \\ & \searrow g_* & \downarrow \cong_{\Phi_\alpha} \\ & & \pi_1(Y, y_1) \end{array}$$

Especially, if $f(x_0) = y_0 = g(x_0)$, H is a homotopy relative to x_0 , then $f_* = g_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0)$.

Proof. Let $\gamma : I \rightarrow X$ be a loop at x_0 , we need to show $f_*[\gamma] = \Phi_\alpha(f_*[\gamma])$, i.e. $g \circ \gamma \simeq_p \bar{\alpha} * (f \circ \gamma) * \alpha$. Consider $\gamma \times \text{id} : I \times I \rightarrow X \times I$, let $\gamma_0(t) = (\gamma(t), 0), \gamma_1(t) = (\gamma(t), 1), \beta(t) = (x_0, t)$. Then $\gamma \times \text{id}$ is a path homotopy between γ_1 and $\bar{\beta} * \gamma_0 \beta$, $H \circ (\gamma \times \text{id})$ is a path homotopy between $g \circ \gamma$ and $\bar{\alpha} * (f \circ \gamma) * \alpha$.



□

Corollary 3.44. If $f : X \rightarrow Y$ is nullhomotopic, then f_* is the trivial homomorphism.

Theorem 3.45. If $f : X \rightarrow Y$ is a homotopy equivalence, then $f_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, f(x_0))$ is an isomorphism.

Proof. Let $g : Y \rightarrow X$ be a homotopy inverse of f , $g \circ f \simeq \text{id}_X \implies$

$$\begin{array}{ccccc} \pi_1(X, x_0) & \xrightarrow{f_*} & \pi_1(Y, f(x_0)) & \xrightarrow{g_*} & \pi_1(X, gf(x_0)) \\ & & & & \downarrow \cong_{\Phi_\alpha} \\ & & & & \pi_1(X, x_0) \end{array}$$

id

$$\therefore (\Phi_{\alpha \circ g_*}) \cdot f_* = \text{id} : \pi_1(X, x_0) \rightarrow \pi_1(X, x_0).$$

$$f \circ g \simeq \text{id}_Y \implies$$

$$\begin{array}{ccccc} \pi_1(Y, f(x_0)) & \xrightarrow{g_*} & \pi_1(X, gf(x_0)) & \xrightarrow{f_*} & \pi_1(Y, fgf(x_0)) \\ & & & & \downarrow \Phi_{beta} \cong \\ & & & & \pi_1(Y, f(x_0)) \end{array}$$

id

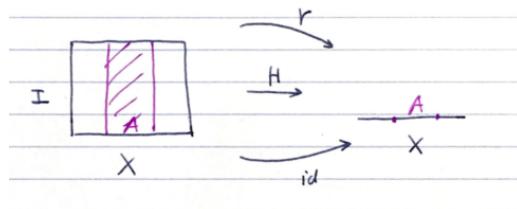
∴

$$\begin{array}{ccc}
 \pi_1(X, x_0) & \xrightarrow{f_*} & \pi_1(Y, f(x_0)) \\
 \cong \uparrow \Phi_\alpha & & \cong \uparrow \Phi_{f \circ \alpha} \\
 \pi_1(X, gf(x_0)) & \xrightarrow{f_*} & \pi_1(Y, fgf(x_0)) \\
 \uparrow g_* & \swarrow \Phi_\beta & \\
 \pi_1(Y, f(x_0)) & &
 \end{array}$$

$$\therefore f_* \circ (\Phi_\alpha \circ g_* \circ \Phi_\beta \circ (\Phi_{f \circ \alpha})^{-1}) = \text{id} : \pi_1(Y, f(x_0)) \rightarrow \pi_1(Y, f(x_0)). \quad \square$$

Slogan: homotopy equivalent spaces have isomorphic fundamental groups.

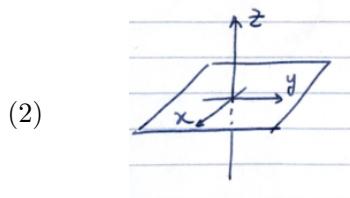
Definition 3.46. Let $A \subset X$ be a subspace. A deformation retraction (形变收缩) is a homotopy $H : X \times I \rightarrow X$ s.t. $H(x, 0) = x, H(x, 1) \in A$ for all $x \in X$, and $H(a, t) = a$ for all $a \in A, t \in I$. $r : X \rightarrow A, r(x) = H(x, 1)$ is a retraction. A is a deformation retract (形变收缩核) of X .



Examples:

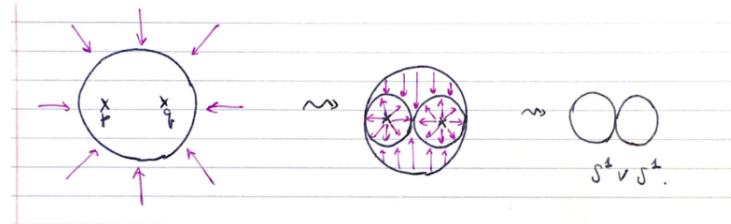


$$\begin{aligned}
 S^n &\subset \mathbb{R}^{n+1} - 0 \\
 r : \mathbb{R}^{n+1} - 0 &\longrightarrow S^n, x \mapsto x/|x| \\
 H(x, t) &= (1-t)x + t \cdot \frac{x}{|x|}
 \end{aligned}$$



$$\begin{aligned}
 X &= \mathbb{R}^3 - \text{the } z\text{-axis has } \mathbb{R}^2 - 0 \text{ as a deformation retract.} \\
 H(x, y, z, t) &= (x, y, (1-t)z) \\
 \therefore \mathbb{R}^3 - \text{the } z\text{-axis} &\simeq \mathbb{R}^2 - 0 \simeq S^1 \\
 \pi_1(X) &\simeq \mathbb{Z}.
 \end{aligned}$$

(3) the double punctured plane $X = \mathbb{R}^2 - \{p, q\}$



3.7 the Seifert-van Kampen theorem

Theorem 3.47. Let $p_\alpha : \prod_\alpha X_\alpha \rightarrow X_\alpha$ be the projection. Then the induced homomorphism $\pi_{p_{\alpha*}} : \pi_1\left(\prod_\alpha X_\alpha\right) \rightarrow \prod_\alpha \pi_1(X_\alpha)$ is an isomorphism.

Proof.

$$\pi_{p_{\alpha*}} : \pi_1\left(\prod_\alpha X_\alpha\right) \rightarrow \prod_\alpha \pi_1(X_\alpha)$$

$$\text{surjective: } (\gamma_\alpha(t)) \mapsto (\gamma_\alpha(t))$$

$$\text{injective: } (H_\alpha(t)) \mapsto (H_\alpha(t))$$

□

Example: $T^n = \underbrace{S^1 \times \cdots \times S^1}_{n \text{ fold}}$, the n -torus, $\pi_1(T^n) \cong \bigoplus_n \mathbb{Z} = \mathbb{Z}^n$.

Definition 3.48. Let $\{G_\alpha\}$ be a family of groups. A word with letters in $\{G_\alpha\}$ is $g_1 g_2 \cdots g_m$ with $m > 0$, $g_i \in G_{\alpha_i}, g_i \neq e$.

It is reduced if $\alpha_1 \neq \alpha_{i+1}$. The free product of $\{G_\alpha\}$ is the group $G = *_\alpha G_\alpha$, where

- $*_\alpha G_\alpha =$ the set of reduced words with letters in $\{G_\alpha\}$ together with the empty word.

- Group operation=juxtaposition and reduction:

$$(g_1 \cdots g_m) \cdot (h_1 \cdots h_n) = g_1 \cdots g_m h_1 \cdots h_n \rightsquigarrow \text{reduced word.}$$

- inverse element: $(g_1 \cdots g_m)^{-1} = g_m^{-1} \cdots g_1^{-1}$.

- associativity law:

$$\underbrace{g_1 \cdots g_m h_1 \cdots h_n}_{\text{reduction}} \cdots \underbrace{h_i \cdots h_j \cdots h_n k_1 \cdots k_l}_{\text{reduction}}$$

(c.f. [Hatcher, p.41])

Examples:

- (1) a free group (自由群) is a free product of any number of copies of \mathbb{Z} .

$$F_n = \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{n \text{ fold}} \text{ the free group of rank } n.$$

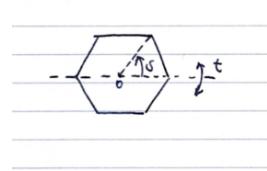
$$\text{e.g. } F_2 = \{a^{m_1} b^{n_1} a^{m_2} b^{n_2} \cdots a^{m_k} b^{n_k} \mid m_i, n_i \in \mathbb{Z}, k \in \mathbb{N}\}.$$

- (2) $\mathbb{Z}/2 * \mathbb{Z}/2 = \{e, a, b, ab, ba, aba, bab, \dots\}$.

\exists a split short exact sequence:

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z}/2 * \mathbb{Z}/2 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & 1 \\ & & \parallel & & \xrightarrow{\quad a \quad \leftrightarrow \quad 1 \quad} & & & & \\ & & \{(ab)^n | \mathbb{Z}\} & & w \longmapsto & & \text{Length of } w \bmod 2 & & \end{array}$$

$\therefore \mathbb{Z}/2 * \mathbb{Z}/2 \cong \mathbb{Z} \rtimes \mathbb{Z}/2$, s semi-direct product.



the finite dihedral group
a regular n -gon
 $D_{2n} = \langle s, t | s^n, t^2, ts = st \rangle$
the dihedral group (二面体群)

$$1 \longrightarrow \mathbb{Z}/n \longrightarrow D_{2n} \longrightarrow \mathbb{Z}/2 \longrightarrow 1$$

There is an embedding $G_\alpha \xrightarrow[\alpha]{i_\alpha} *_\alpha G_\alpha$, $g \mapsto g$.

If there is a collection of homomorphisms $\{\varphi_\alpha : G_\alpha \rightarrow H\}$, then there is a unique extension $\varphi : *_\alpha G_\alpha \rightarrow H$ s.t. $\varphi_\alpha = i_\alpha \circ \varphi : G_\alpha \rightarrow H$ given by $\varphi(g_1 \cdots g_m) = \varphi_{\alpha_1}(g_1) \cdots \varphi_{\alpha_m}(g_m)$.

Remark: in the category of abelian groups, $\bigoplus_\alpha G_\alpha$ satisfies the property.

Especially, the family of homomorphisms $\{i_k : G_k \rightarrow G_1 \times \cdots \times G_n\}$ extends to a surjective homomorphism $\bigoplus_{i=1}^n G_i \rightarrow G_1 \times \cdots \times G_n$.

Consider $F_n = *_n \mathbb{Z} \twoheadrightarrow \mathbb{Z}^n = \bigoplus_n \mathbb{Z}$. Any homomorphism $\varphi : F_n \rightarrow A$ to an abelian group A factors through $\bar{\varphi} : \mathbb{Z}^n \rightarrow A$,

$$\begin{array}{ccc} F_n & \twoheadrightarrow & \mathbb{Z}^n \\ \varphi \downarrow & \swarrow \bar{\varphi} & \text{, since } \\ A & & \end{array} \quad \begin{array}{ccccc} \mathbb{Z} & \xrightarrow{i_k} & F_n & \xrightarrow{\varphi} & A \\ \downarrow & \searrow & \downarrow & & \swarrow \bar{\varphi} \\ \mathbb{Z}^n & & & & \end{array}$$

$\therefore \mathbb{Z}^n$ is the abelianization of F_n , i.e. $F_n/[F_n, F_n]$. $\mathbb{Z}^n \not\cong \mathbb{Z}^m$ if $m \neq n \Rightarrow F_n \not\cong F_m$ if $m \neq n$.

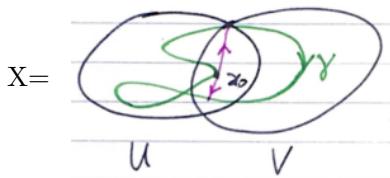
Remark: A subgroup of \mathbb{Z}^n is a free abelian group of rank $\leq n$. A subgroup of F_n is a free group, rank may be $> n$. e.g. $F_n < F_2, F_\infty < F_2$.

Let $X = U \cup V$, where U and V are open in X ; assume that U, V and $U \cap V$ are path connected, $x_0 \in U \cap V$.

$$\pi_1(U \cap V) \xrightarrow{i_1} \pi_1(U)$$

Then we have a commutative diagram

$$\begin{array}{ccc} \pi_1(U \cap V) & \xrightarrow{i_1} & \pi_1(U) \\ \downarrow i_2 & & \downarrow j_1 \\ \pi_1(V) & \xrightarrow{j_2} & \pi_1(X) \end{array}$$



Theorem 3.49 (Seifert-van Kampen). *Under the above assumption, the homomorphism $\Phi : \pi_1(U) * \pi_1(V) \rightarrow \pi_1(X)$ is surjective, the kernel of Φ is the normal subgroup generated by all elements of the form $i_1(\alpha)i_2(\alpha)^{-1}, \alpha \in \pi_1(U \cap V)$.*

$$\therefore \pi_1(X) \cong \pi_1(U) * \pi_1(V) / \langle i_1(\alpha)i_2(\alpha)^{-1} \mid \alpha \in \pi_1(U \cap V) \rangle_N.$$

Corollary 3.50. *Under the same hypothesis,*

(1) *if $U \cap V$ is simply connected, then*

$$\pi_1(U) * \pi_1(V) \rightarrow \pi_1(X)$$

is an isomorphism.

(2) *if V is simply connected, then*

$$\pi_1(V) / \langle \text{Im } i_* : \pi_1(U \cap V) \rightarrow \pi_1(U) \rangle_N$$

Examples:

(1)

$$\pi_1(S^n) = 0, (n \geq 2)$$

$$S^n = U \cup V$$

$$U \cong \overset{\circ}{D} \cong V$$



$$U \cap V \cong S^{n-1} \times (-1, 1) \cong S^{n-1}, \therefore \pi_1(S^n) = \pi_1(U) * \pi_1(V)/N = 0$$

(2) (i)

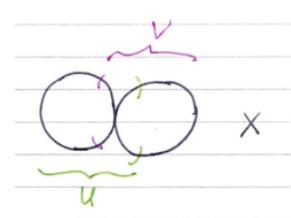
$$X = S^1 \vee S^1 \text{ "figure eight"}$$

$$X = U \cap V, U \simeq S^1 \simeq V,$$

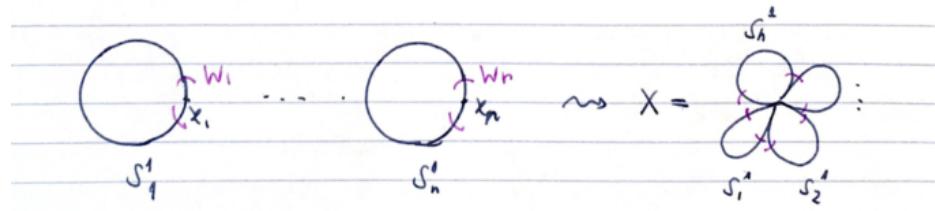
$$U \cap V \simeq \text{pt.}$$

$$\therefore \pi_1(S^1 \vee S^1) \cong \pi_1(S^1) * \pi_1(S^1) \cong \mathbb{Z} * \mathbb{Z} = F_2.$$

(Compare with $\pi_1(S^1 \times S^1 \cong \mathbb{Z}^2)$)



- (ii) in general, let $(S^1_1, x_1), \dots, (S^1_n, x_n)$ be n circles. $X = \bigvee_n S^1 = S^1_1 \cup \dots \cup S^1_n / x_1 \sim x_2 \sim \dots \sim x_n$
the quotient space is a wedge of n circles.



$U \subset X$ is open $\Leftrightarrow U \cap S^1_i$ is open in S^1_i for all i ,

Let $W_i \subset S^1_i$ be an open arc containing x_i ,

$$U_i = S^1_i \vee \bigvee_{j \neq i} U_j, \text{ then } X = \bigcup_{i=1}^n U_i, U_i \simeq S^1, U_i \cap U_j = W_i \cap W_j \simeq \text{pt.} \text{ Inductively, } \pi_1(X) \cong \underbrace{\mathbb{Z} * \dots * \mathbb{Z}}_{n \text{ fold}} = F_n.$$

- (iii) $X = \bigvee S^1$ a countably infinite wedge of circles. $\pi_1(X) = \pi_1\left(\lim_{n \rightarrow \infty} \bigvee_n S^1\right) = \lim_{n \rightarrow \infty} \pi_1\left(\bigvee_n S^1\right) = \lim_{n \rightarrow \infty} F_n = F_\infty.$

"taking π_1 commutes with direct limits"

- (iv) C_n = the circle of radius $\frac{1}{n}$, center $\left(\frac{1}{n}, 0\right)$. $Y = \bigcup_{n=1}^{\infty} C_n \subset \mathbb{R}^2$ with the subspace topology.

(the hawaii's earring)

\exists a retraction $r_n : Y \rightarrow C_n$ collapsing all $C_i (i \neq n)$ to 0. $\rho_n = (r_n)_* : \pi_1(Y) \rightarrow \pi_1(C_n) \rightsquigarrow \rho = \prod \rho_n : \pi_1(Y) \rightarrow \prod_{n=1}^{\infty} \pi_1(C_n) = \prod_{\infty} \mathbb{Z}$ a homomorphism.

ρ is surjective: $\forall (k_1, k_2, \dots, k_n, \dots) \in \prod_{\infty} \mathbb{Z}$,

$$\gamma : [0, 1] \rightarrow Y, \gamma \left|_{\left[1 - \frac{1}{n}, 1 - \frac{1}{n+1}\right]} : \left[1 - \frac{1}{n}, 1 - \frac{1}{n+1}\right] \rightarrow C_n\right.$$

represents $k_n \in \pi_1(C_n)$. γ is continuous.



$\prod_{\infty} \mathbb{Z}$ is uncountable $\Rightarrow \pi_1(\gamma)$ is uncountable!

whereas $\pi_1 \left(\bigvee_{\infty} S^1 \right) = F_{\infty}$ is countable.

Consider the retraction $r : Y \rightarrow C_1 \cup \dots \cup C_n$ collapsing all $C_i (i > n)$ to 0. Then $r_* : \pi_1(Y) \rightarrow \pi_1(C_1 \cup \dots \cup C_n) = F_n$ is surjective. $\Rightarrow \pi_1(Y)$ is nonabelian.

Generators and relations:

G a group, $S \subset G$ is a set of generators if $\forall g \in G, g = S_1^{n_1} \cdots S_k^{n_k}, s_i \in S, n_i \in \mathbb{Z}$. Let $\{a_{\alpha}\}_{\alpha \in J}$ be a set of generators of G , $F \cong *_{\infty} \mathbb{Z}$ be the free group generated by $\{a_{\alpha}\}_{\alpha \in J}$, then there is a surjective homomorphism $h : F \rightarrow G, h(a_{\alpha}) = a_{\alpha}$. Let $N = \ker h$, called the relation subgroup, each element of N is called a relation. If $\{r_{\beta}\}$ is a set of N s.t. they normally generate N , i.e. N is the smallest normal subgroup of F that contains $\{r_{\beta}\}$, we call the family $\{r_{\beta}\}$ a complete set of relations for G .

Definition 3.51. A presentation (表现) of a group G consists of a set of generators $\{a_{\alpha}\}$, along with a complete set of relations $\{r_{\beta}\}$. Notation: $G = \langle a_{\alpha} | r_{\beta} \rangle$. If $\{a_{\alpha}\}$ is finite, G is finitely generated (有限生成); if both $\{a_{\alpha}\}$ and $\{r_{\beta}\}$ are finite, G is finitely presented (有限表现), $\{a_{\alpha}\}, \{r_{\beta}\}$ form a finite presentation of G .

Examples:

$$(1) F_n = \langle a, \dots, a_n \rangle.$$

$$\mathbb{Z}/n = \langle a | a^n \rangle, \mathbb{Z}^2 = \langle a, b | [a, b] \rangle$$

$$D_{2n} = \langle s, t | s^n, t^2, sts \rangle$$

$$\mathbb{Z}/2 * \mathbb{Z}/2 = \langle a, b | a^2, b^2 \rangle$$

$$= D_{\infty} = \mathbb{Z} \rtimes \mathbb{Z}/2 = \langle s, t | t^2, sts \rangle (s = ab, t = a)$$

e.x. find a representation for $S_3, \text{SL}_2(\mathbb{Z})$.

$$(2) \mathbb{Q}$$
 is not finitely generated.

$$(3) \text{ If } G_1 = \langle a_{\alpha} | r_{\beta} \rangle, G_2 = \langle b_{\alpha'} | s_{\beta'} \rangle, \text{ then } G_1 * G_2 = \langle a_{\alpha}, b_{\alpha'} | r_{\beta}, s_{\beta'} \rangle.$$

Remark: A group has various presentations. Given two finite presentations $G_1 = \langle a_{\alpha} | r_{\beta} \rangle$ and $G_2 = \langle b_{\alpha'} | s_{\beta'} \rangle$, to determine if G_1 is isomorphic to G_2 is an unsolvable problem!

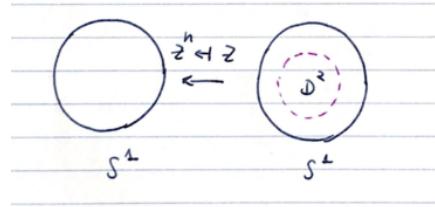
Examples:

$$(1) f : S^1 \rightarrow S^1, z \mapsto z^n (n \geq 1)$$

$X_n = D \sqcup S^1 / (z \sim f(z) \quad z \in S^1) = S^1 \cup_f D^2$ the quotient space (compact Hausdorff)

$$X_1 \cong D^2$$

X_n the n -fold dunce cap



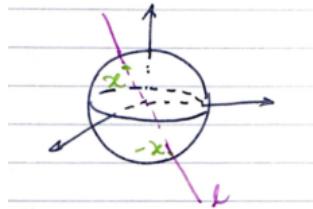
$$X = U \cup V$$

$$\begin{array}{ccc} \parallel & \parallel \\ \overset{\circ}{D}^2 & (D^2 - 0) \cup S^1 / \sim \xrightarrow{\simeq} S^1 \end{array}$$

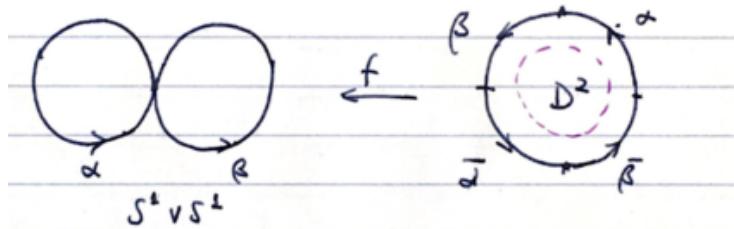
$$U \cap V = \overset{\circ}{D}^2 - 0 \simeq S^1 \left(\frac{1}{2} \right).$$

$$\begin{aligned} \pi_1(X) &= \pi_1(V) / (\text{Im } (\pi_1(U \cap V) \rightarrow \pi_1(V)))_N \\ &\cong \pi_1(V) / (\text{Im } f_*)_N \\ &= \mathbb{Z}/n\mathbb{Z} = \langle a | a^n \rangle \end{aligned}$$

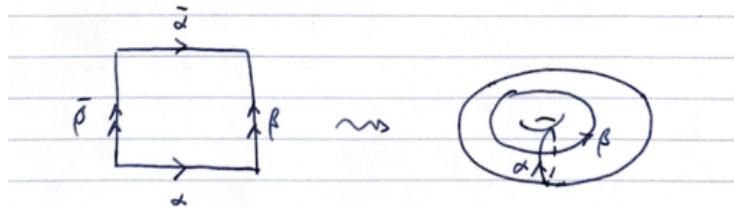
$$\begin{aligned} X_2 = \mathbb{R}P^2 &= \text{the space of straight lines in } \mathbb{R}^3 \text{ through } O \\ &= \text{the real projective plane (实射影平面)} \\ &= \mathbb{R}^3 - 0 / (x \sim \lambda x | \lambda \neq 0) \\ &= S^2 /_{x \sim -x} = D^2 /_{(x \sim -x | x \in S^1)} \end{aligned}$$



$$(2) f : S^1 \rightarrow S^1 \vee S^1$$



$$((S^1 \vee S^1) \cup D^2) /_{(x \sim f(x) | x \in S^1)} = (S^1 \vee S^1) \cup_f D^2 = T^2$$



$$T^2 = U \cup V$$

$$\begin{array}{ccc} \parallel & \parallel \\ \overset{\circ}{D}^2 & (S^1 \vee S^1) \cup_f (D^2 - 0) & \xrightarrow{\simeq} S^1 \vee S^1 \end{array}$$

$$U \cap V = \overset{\circ}{D}^2 - O \simeq S^1 \left(\frac{1}{2} \right)$$

$$\therefore \pi_1(T^2) = \pi_1(V) / \langle \text{Im } f_* \rangle_N = \mathbb{Z} * \mathbb{Z} / \langle [a, b] \rangle_N \cong \mathbb{Z}^2$$

$$(\text{Claim:}) \langle [a, b] \rangle_N = [F_1, F_2]$$

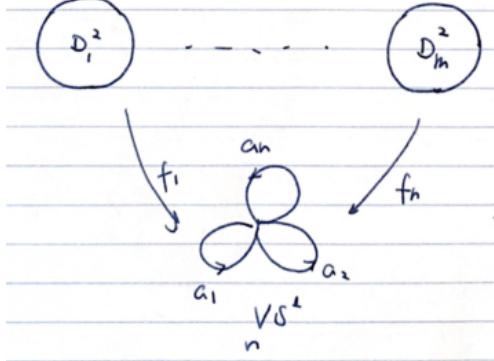
$$\text{Proof. (i) } \langle [a, b] \rangle_N \subset [F_1, F_2]$$

(ii) $F_2/\langle [a, b] \rangle_N$ is generated by aN, bN where $N = \langle [a, b] \rangle_N$. But $aN \cdot bN = bN \cdot aN, \therefore F_2/\langle [a, b] \rangle_N$ is abelian, $\therefore [F_1, F_2] \subset \langle [a, b] \rangle_N$)

(3) in general, given a finitely presented group

$$G = \langle a_1, \dots, a_n | r_1, \dots, r_m \rangle$$

$$X = \left(\bigvee_n S^1 \cup \bigcup_{i=1}^m D_i^2 \right) / \sim \text{ a 2-dim complex (复形)}$$



$f_i : S_i^1 \rightarrow \bigvee_n S^1$ representing $r_i \in \pi_1 \left(\bigvee_n S^1 \right)$, then

$$\pi_1(X) \cong F_n / \langle r_1, \dots, r_m \rangle_N$$

$$= \langle a_1, \dots, a_n | r_1, \dots, r_m \rangle$$

$$= G$$

\therefore the classification of finite 2-dim complexes is unsolvable.

e.g. \exists a finite 2-dim complex X with $\pi_1(X) \cong D_{2n}, S_n, \text{SL}_2(\mathbb{Z})$. What is your favourite f.p. group?

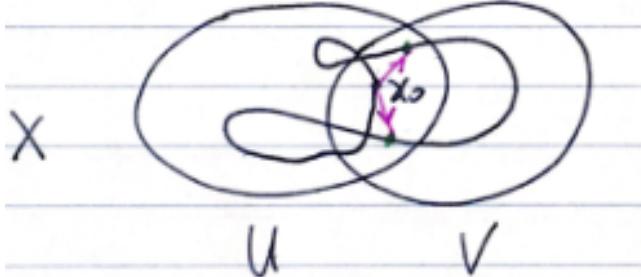
(4) more general, given $a \in \pi_1(X, x_0)$, represented by $\gamma : (S^1, b_0) \rightarrow (X, x_0)$. Let $X \cup_\gamma D^2 = X \cup D^2 /_{(x \sim \gamma(x) | x \in S^1)}$. Then $\pi_1(X \cup_\gamma D^2) = \pi_1(X) / \langle a \rangle_N$, i.e. a is “killed” in $X \cup_\gamma D^2$.

Proof of the Seifert-van Kanpen Theorem

(sketch)

(c.f.[基础拓扑学 尤承业])

(1) $\pi_1(U) * \pi_1(V) \xrightarrow{\Phi} \pi_1(X)$ is surjective:



Let $\gamma : [0, 1] \rightarrow X$ be a loop at x_0 .

\exists a subdivision $0 = a_0 < a_1 < \dots < a_n = 1$, s.t. $\gamma([a_1, a_{i+1}]) \subset U$ or V , $\gamma(a_i) \in U \cap V$. (Lebesgue number lemma).

Choose a path α_i from x_0 to $\gamma(a_i)$, $i = 1, \dots, n - 1$.

Let $\gamma_i = \gamma|_{[a_{i-1}, a_i]}$, $\beta_i = \alpha_{i-1} * \gamma_i * \bar{\alpha}_i$, a loop at x_0 in U or V . Then

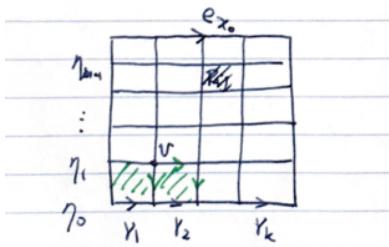
$$\begin{aligned}\gamma &\simeq_p (\alpha_0 * \gamma_1 * \bar{\alpha}_1) * (\alpha_1 * \gamma_2 * \bar{\alpha}_2) * \dots * (\alpha_{n-1} * \gamma_n * \bar{\alpha}_n) \\ &= \beta_1 * \dots * \beta_n\end{aligned}$$

$$\therefore [\gamma] = \Phi(\beta_1 \beta_2 \dots \beta_n).$$

(2) let $\langle i_1(a)i_2(a)^{-1} | a \in \pi_1(U \cap V) \rangle_N = N$, clearly $N \subset \ker \Phi$. Need to show $\ker \Phi \subset N$, it suffices to show that $\pi = \pi_1(U) * \pi_1(V)/N \xrightarrow{\Phi} \pi_1(X)$ is injective.

Let γ be a loop in U or V , $[\gamma]$ be the element in $\pi_1(U)$ or $\pi_1(V)$, $\langle \gamma \rangle$ be the element in π . If γ is a loop in $U \cap V$, $\langle \gamma \rangle$ is a well-defined element in π . For $w \in \pi$, $w = [\gamma_1] \dots [\gamma_k]N = \langle \gamma_1 \rangle \dots \langle \gamma_k \rangle$.

If $\Phi(w) = e \in \pi_1(X)$, then there is a path homotopy $H : X \times I \rightarrow X$.

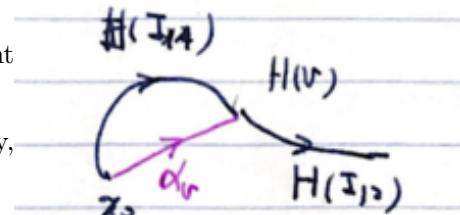


Subdivide $I \times I$ into rectangles R_{ij} s.t. $H(R_{ij}) \subset U$ or V . For each vertex v of R_{ij} , choose a path α_v from x_0 to $H(v)$ in U or V .

Consider the loop η_1 in X by inserting $\bar{\alpha}_v \alpha_v$, we have an element

$$\langle \eta_1^1 \rangle \dots \langle \eta_1^k \rangle \in \pi,$$

and $\langle \eta_1^1 \rangle \dots \langle \eta_1^k \rangle = \langle \gamma_1 \rangle \dots \langle \gamma_k \rangle$ using $H|_{R_{11}}$. Inductively, $\langle \gamma_1 \rangle \dots \langle \gamma_k \rangle = \langle e_{x_0} \rangle = e \in \pi$.



3.8 additional topics

3.8.1 knots and links

(c.f.[Armstrong, Basic Topology])

Definition 3.52. A knot (扭结) is a subspace of \mathbb{R}^3 or S^3 which is homeomorphic to S^1 .



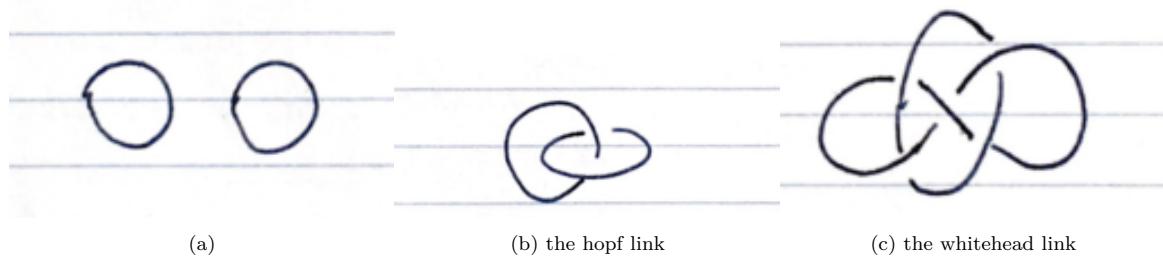
(a) the trivial knot

(b) the trefoil knot(三叶结)

(c) the figure eight knot (8 字结)

A link (链环) L is a disjoint union of knots.

$$L = K_1 \cup K_2 \cup \dots \cup K_n (g \geq 2).$$



(a)

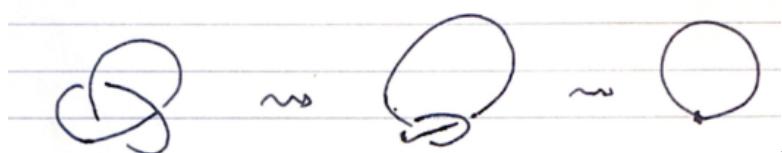
(b) the hopf link

(c) the whitehead link

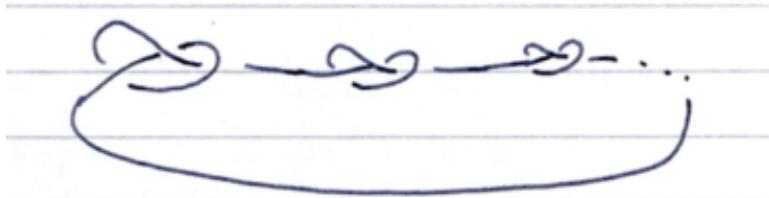
the Borromean ring

Two knots K_1 and K_2 are equivalent if \exists a homeomorphism $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ s.t. $f(K_1) = K_2$, and f is isotopic (同痕) to id. (i.e. \exists a homotopy $H : \mathbb{R}^3 \times I \rightarrow \mathbb{R}^3$ between f and id, and $H(-, t) : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is a homeomorphism $\forall t \in I$).

(To avoid:



A knot is tame if it is equivalent to a polygonal knot.



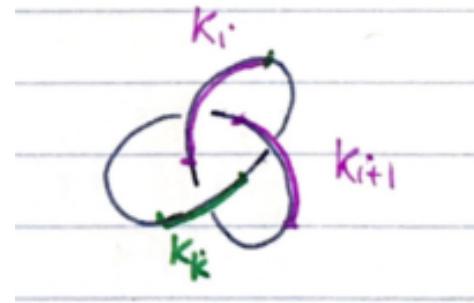
A wild knot:

$\pi_1(\mathbb{R}^3 - K)$ is called the knot group. It is an invariant of the knot K .

$$\mathbb{R}^3 - K = (\mathbb{R}_{\geq 0}^3 - K) \cup (\mathbb{R}_{\leq 0}^3 - K)$$

\Downarrow
 $*$

$$\pi_1(\mathbb{R}_{\geq 0}^3 - K) \cong \langle x_1 \cdots x_n \rangle$$



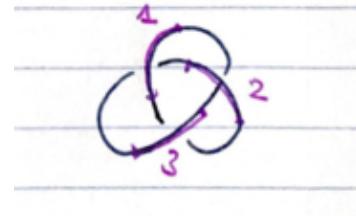
Each underpass between K_i and K_{i+1} , separated by K_k gives a relation $x_i x_k = x_k x_{i+1}$.

$$\therefore \pi_1(\mathbb{R}^3 - K) \cong \langle x_1 \cdots x_n \mid x_i x_k x_{i+1}^{-1} x_k^{-1} i = 1, \dots, n-1 \rangle \text{ (the Wirtinger presentation)}$$

e.g. K = the trefoil knot

generators: x_1, x_2, x_3 ,

relations: $x_1 x_3 = x_3 x_2, x_2 x_1 = x_1 x_3$



$$x_3 = x_1^{-1} x_2 x_1 \rightsquigarrow \pi_1(\mathbb{R}^3 - K) \cong \langle x_1, x_2 \mid x_2 x_1 = x_1^{-1} x_2 x_1 x_2 \rangle$$

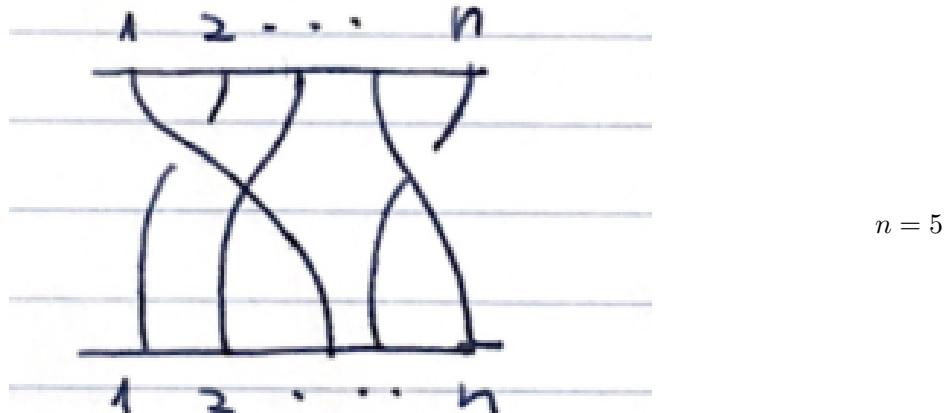
$$\begin{array}{ccc} & \cong \langle x_1, x_2 \mid x_1 x_2 x_1 = x_2 x_1 x_2 \rangle \\ \downarrow & & \\ x_1 & & x_2 \\ \downarrow & & \downarrow \\ S_3 & (1\ 2) & (2\ 3) \end{array}$$

$$\therefore \pi_1(\mathbb{R}^3 - K) \not\cong \pi_1(\mathbb{R}^3 - \text{circle}) \cong \mathbb{Z}.$$

\therefore the trefoil knot is not trivial.

3.8.2 braids

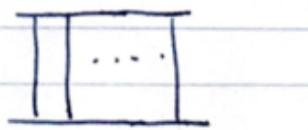
A braid (辫) of n strings is an object as follows:



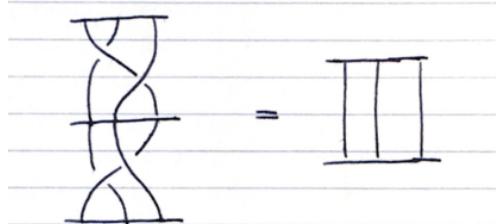
The set of (equivalences classes of) n -braids B_n has a group structure by concatenation:



- neutral element = the trivial braid



- inverse element = the reversed braid

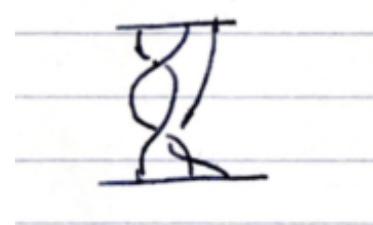


B_n is called the (Artin) braid group. (辫群)

e.g. $B_2 \cong \mathbb{Z}$

$$1 \rightarrow PB_n \rightarrow B_n \rightarrow S_n \rightarrow 1$$

\parallel
the pure braid group



a presentation of B_n :

$$\sigma_1 = \begin{array}{c} 1 \ 2 \ \cdots \ n \\ \diagup \quad \diagdown \\ \text{---} \end{array}, \quad \sigma_2 = \begin{array}{c} 1 \ 2 \ 3 \ \cdots \ n \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ \text{---} \end{array}, \dots,$$

$$\sigma_{n-1} = \begin{array}{c} 1 \ \cdots \ n-1 \ n \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ \text{---} \end{array}, \text{ Then } B_n \text{ is generated by } \sigma_1, \dots, \sigma_{n-1}.$$

$$\text{e.g. } \begin{array}{c} 1 \ 2 \ 3 \\ \diagup \quad \diagdown \quad \diagup \\ \text{---} \end{array} = \begin{array}{c} \sigma_1 \\ \diagup \quad \diagdown \\ \text{---} \end{array} \sigma_2^{-1} = \sigma_1 \cdot \sigma_2^{-1}.$$

relations: (1) $\sigma_i \sigma_j = \sigma_j \sigma_i$ for $|i - j| \geq 2$

$$\begin{array}{c} i \ i+1 \ i+2 \ i+3 \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ \text{---} \end{array}$$

(2) $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ for all i .

$$\begin{array}{c} i \ i+1 \ i+2 \\ \sigma_i \\ \sigma_{i+1} \\ \sigma_i \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} = \begin{array}{c} i \ i+1 \ i+2 \\ \sigma_{i+1} \\ \sigma_i \\ \sigma_{i+1} \\ \text{---} \end{array}$$

$\therefore B_n = \langle \sigma_1 \cdots \sigma_{n-1} | [\sigma_i, \sigma_j] | i - j \geq 2, \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \forall i \rangle$.

Definition 3.53. The unordered configuration space (构型空间) of n points in \mathbb{R}^2 is $U\text{conf}_n(\mathbb{R}^2) = \{(x_1, \dots, x_n) \in (\mathbb{R}^2)^n \mid x_\varepsilon \neq x_j, \forall \varepsilon \neq j\} / S_n$.

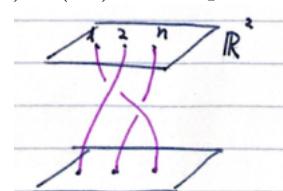
A loop $\gamma : I \rightarrow U\text{conf}_n(\mathbb{R}^2)$ at the base point $(1, 2, \dots, n) \in (\mathbb{R}^2)^n$ corresponds to a braid.

$$\therefore \pi_1(U\text{conf}_n(\mathbb{R}^2)) \cong B_n$$

$$\pi_1(\text{conf}_n(\mathbb{R}^2)) \cong PB_n$$

||

the ordered configuration space.



3.8.3 pushout

(1) given sets and maps, let $\rho = X \sqcup Y / (f(z) \sim g(z)) | z \in Z$

$$\begin{array}{ccc}
 Z & \xrightarrow{f} & X \\
 g \downarrow & & \downarrow i_1 \\
 Y & \xrightarrow{i_2} & P
 \end{array}
 \quad
 \begin{array}{c}
 j_1 \\
 \searrow \\
 \curvearrowleft u \\
 \nearrow j_2 \\
 Q
 \end{array}$$

(P, i_1, i_2) satisfies the universal property:
 for any $j_1 : X \rightarrow Q, j_2 : Y \rightarrow Q$,
 s.t. $j_1 \circ f = j_2 \circ g$, $\exists!$ a map
 $u : P \rightarrow Q$, s.t. $j_1 = i_1 \circ u, j_2 = i_2 \circ u$.

P, i_1, i_2 is unique up to isomorphism, called the pushout of the diagram

$$\left(\begin{array}{ccc} Z & \xrightarrow{f} & X \\ \downarrow g & & \\ Y & & \end{array} \right).$$

(2) topological spaces and continuous maps

$$\begin{array}{ccc}
 Z & \xrightarrow{f} & X \\
 g \downarrow & & \downarrow i_1 \\
 Y & \xrightarrow{i_2} & P
 \end{array}
 \quad
 \begin{array}{c}
 j_1 \\
 \searrow \\
 \curvearrowleft \quad u \quad \curvearrowright \\
 j_2 \qquad \qquad \qquad Q
 \end{array}$$

$\rho = X \sqcup Y /_{(f(z) \sim g(z) | z \in Z)}$
 quotient space
 satisfies the same universal property in $\mathcal{T}\text{op}$.

(3) groups and homomorphisms

$$\begin{array}{ccc}
 H & \xrightarrow{f_1} & G_1 \\
 f_2 \downarrow & & \downarrow i_1 \\
 G_2 & \xrightarrow{i_2} & P
 \end{array}
 \quad
 \begin{array}{c}
 j_1 \\
 \searrow \\
 u \\
 \dashrightarrow \\
 \nearrow \\
 j_2
 \end{array}
 \quad
 Q$$

$\rho = G_1 * G_2 / \langle f_1(h)f_2(h)^{-1} | h \in H \rangle_N$ satisfies the same universal property in $\mathcal{G}p$.

If $H < G_1, H < G_2, P$ is denoted by $G_1 *_H G_2$, called the amalgamated product of G_1 and G_2 over H .

e.g.

$$\mathrm{SL}_2(\mathbb{Z}) = \mathbb{Z}/4 *_{\mathbb{Z}/2} \mathbb{Z}/6$$

$$\mathcal{T}\text{op}_* \xrightarrow{\pi_1} \mathcal{G}\text{p}$$

Chapter 4

Surfaces

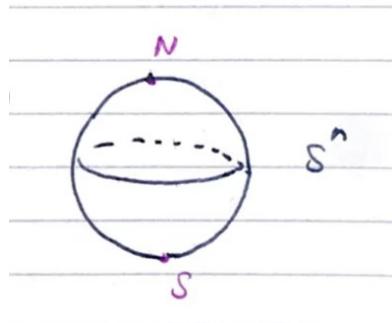
4.1 introduction

Definition 4.1. An n -dimensional topological manifold (拓扑流形) M^N is a second countable Hausdorff space which is locally homeomorphic to the euclidean space space \mathbb{R}^n , i.e. for $\forall x \in M$, \exists an open neighborhood U of x , and a homeomorphism $h : U \xrightarrow{\cong} \mathbb{R}^n$.

Examples.

$$\mathbb{R}^n, S^n = (S^n - \{N\}) \cup (S^n - \{S\})$$

$$T^n = S^1 \times \cdots \times S^1$$



Question. What is a classification of an n -dimensional (compact) manifolds?

Theorem 4.2. A compact connected 1-dimensional manifold is homeomorphic to S^1 .

4.2 the classification of closed Surfaces

Examples of compact Surfaces (closed surfaces, 闭曲面).

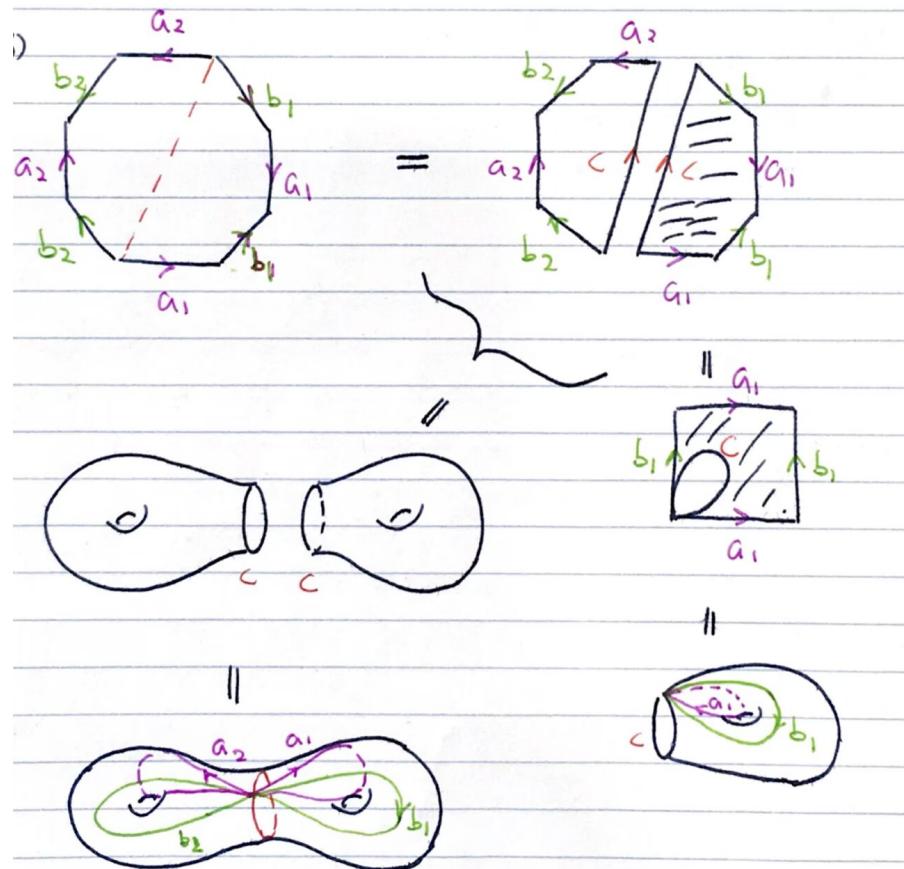
(1) S^2 the 2-sphere



$P^2 = \mathbb{R}P^2 =$

the projective plane

(3)



the 2-torus $2T^2 = T^2 \# T^2 = (T^2 - D^2) \cup_{\partial} (T^2 - D^2)$ the conected sum (连通和) of T^2 and T^2 .

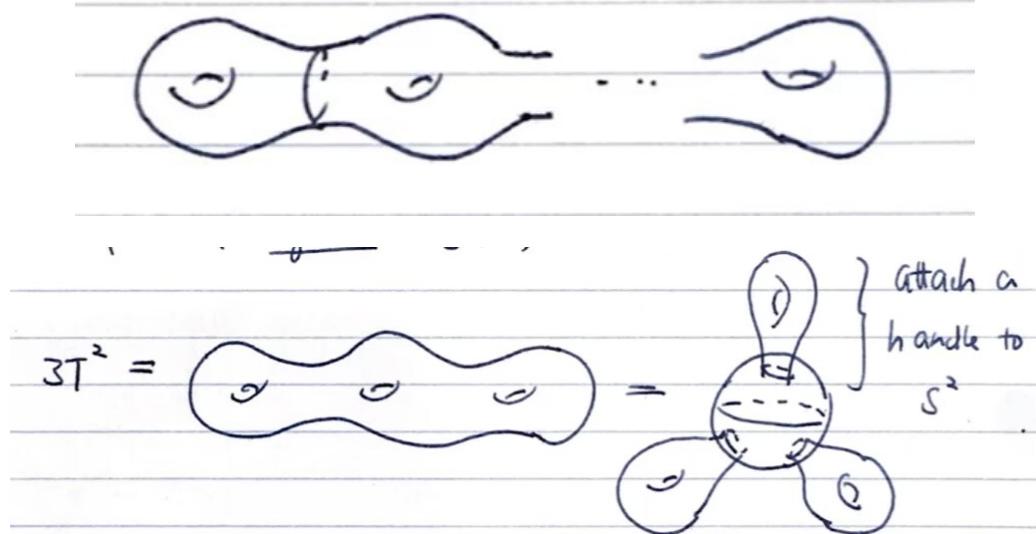
In general, take a regular $4n$ -gon, label the edges

$a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \dots a_n b_n a_n^{-1} b_n^{-1}$, glue the corresponding edges, we get a closed surface

$nT^2 = T^2 \# T^2 \# \dots \# T^2$ (n -fold) called the n -torus, or the orientable (可定向) closed surface of genus (亏格) n .

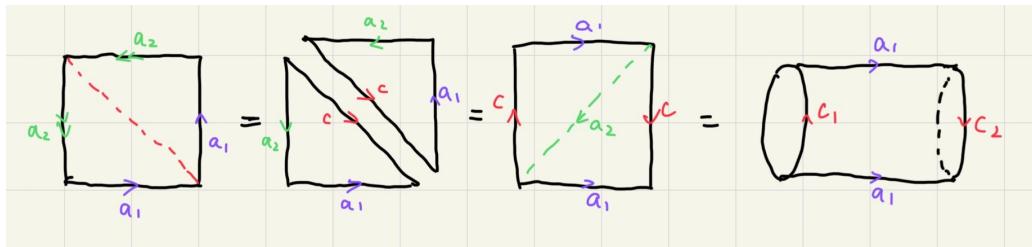
Remark:

(4)



$$\begin{array}{c}
 \text{Diagram showing the decomposition of a surface into } \mathbb{R}P^2 - D^2 \text{ and } \mathbb{R}P^2 - \overset{\circ}{D}{}^2. \\
 \text{The first part shows a square with edges labeled } a_1, a_2, a_1, a_2 \text{ being glued to form a surface with a boundary component } c. \\
 \text{The second part shows the resulting surface being glued to another surface with boundary } c, \text{ resulting in } 2P^2 = P^2 \# P^2, \text{ which is the connected sum of two projective planes.}
 \end{array}$$

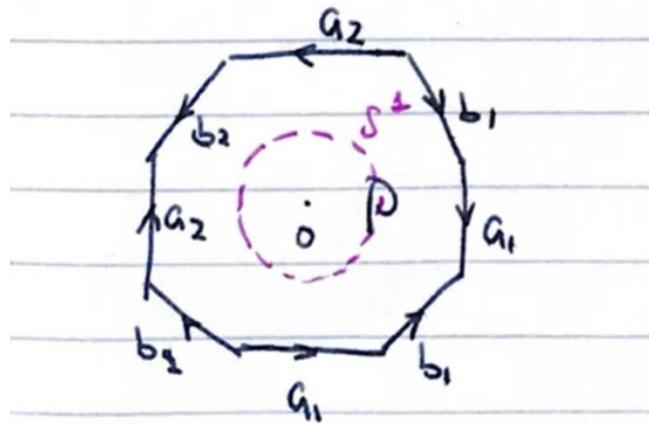
another point of view:



In general, take a regular $2m$ -gon, label the edges $a_1a_1a_2a_2 \dots a_ma_m$, glue the corresponding edges, we get a closed surface $mP^2 = P^2 \# \dots \# P^2$ (m fold), called the non-orientable closed surface of genus m .

Theorem 4.3 (The classification of closed Surfaces). *A closed surface is homeomorphic to exactly one of the surfaces nT^2 ($n \geq 0$) or mP^2 ($m \geq 1$).*

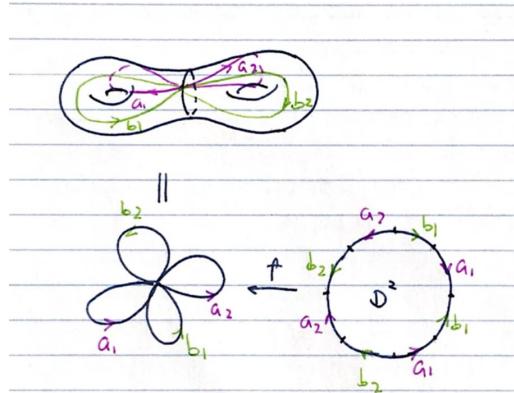
Proof. (1) the “models” nT^2 and mP^2 are not homeomorphic to each other.



$$\begin{aligned} nT^2 &= U \cup V \\ &\parallel \quad \parallel \\ \overset{\circ}{P} &\quad (P - 0)/ \sim^{p, \simeq} \bigvee_{2n} S^1 \end{aligned}$$

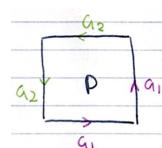
$$U \cap V = \overset{\circ}{P} - 0 \simeq S^1$$

$$\begin{aligned} \therefore \pi_1(nT^2) &\cong \pi_1(V)/_{\langle Im\pi_1(U \cap V) \rightarrow \pi_1(V) \rangle_N} \\ &= \left\langle a_1, b_1, \dots, a_n, b_n \mid a_1 b_1 a_1^{-1} b_1^{-1} \dots a_n b_n a_n^{-1} b_n^{-1} = \prod_{i=1}^n [a_i, b_i] \right\rangle \end{aligned}$$



$$\pi_1(nT^2)_{ab} (\cong H_1(nT^2)) = \left\langle a_1, b_1 \dots a_n, b_n \mid \prod_{i=1}^n [a_i, b_i] \right\rangle_{ab}$$

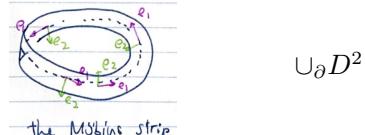
similarly



$$\pi_1(mP^2) \cong \langle a_1, \dots, a_m \mid a_1^2 a_2^2 \dots a_m^2 \rangle$$

$$H_1(mP^2) = \pi_1(mP^2)_{ab} \cong \mathbb{Z}^m /_{2(e_1 + \dots + e_m)} \cong \mathbb{Z}/2 \oplus \mathbb{Z}^{m-1}$$

□

 Remark: $P^2 =$

 Any mP^2 contains a Möbius strip, hence is not orientable.

Let v_1, \dots, v_k be points in \mathbb{R}^n . They are in general position if $v_1 - v_0, \dots, v_k - v_0$ are linearly independent. A k -simplex (k -常形) with vertices v_0, \dots, v_k is $\sigma = \left\{ x = \sum_{i=0}^k \lambda_i v_i \in \mathbb{R}^n \mid \sum_{i=0}^k \lambda_i = 1, \lambda_i \geq 0 \right\} \subset \mathbb{R}^n$

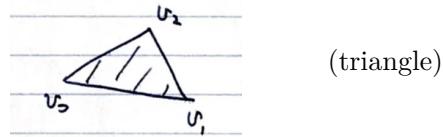
e.g. 0-simplex :



1-simplex :



2-simplex :



3-simplex :

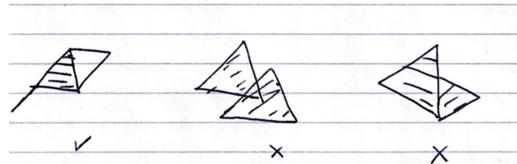


If σ, τ are simplices, and the vertices of τ form a subset of the vertices of σ , then we say that τ is a face of σ .

Definition 4.4. A finite collection K of simplices in some Euclidean space \mathbb{R}^n is called a (finite) simplicial complex (单纯复形) If

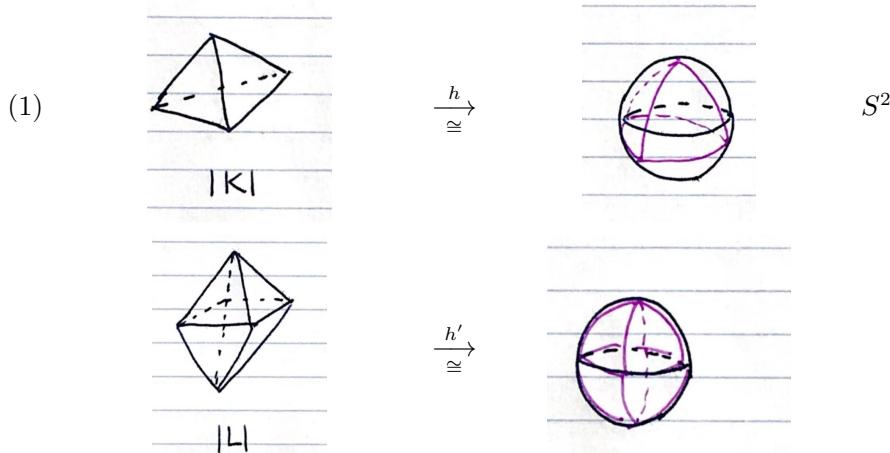
- (i) for $\forall \sigma \in K$, all the faces of σ are elements in K
- (ii) for $\forall \sigma, \tau \in K$, if $\sigma \cap \tau \neq \emptyset$, then σ and τ intersect in a common face.

e.g.

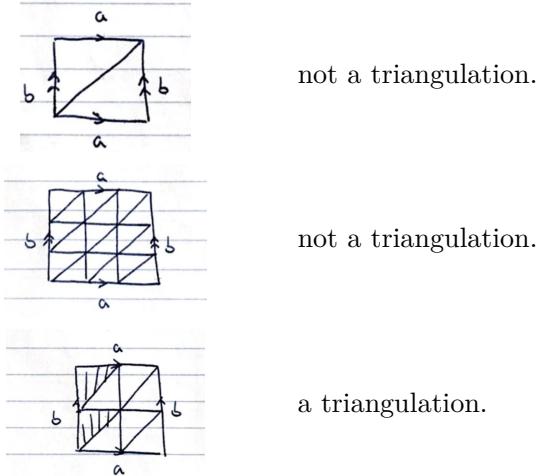


The geometric realization of a simplicial complex K is $|K| = \bigcup_{\sigma \in K} \sigma \subset \mathbb{R}^n$, a subspace of \mathbb{R}^n , a polyhedron (多面体).

Definition 4.5. A triangulation (单纯剖分/三角剖分) of a topological space X consists of a simplicial complex K and a homeomorphism $h : |K| \xrightarrow{\cong} X$. Such a space X is called triangulable (可三角剖分的).

Examples


(2) T^2 :



Theorem 4.6. A compact Topological manifold of $\dim \leq 3$ is triangulable. In each dimension $n \geq 4$, there exist non-triangulable topological manifolds.

diam 1 : S^1 is triangulable



diam 2 : closed surfaces are triangulable. (Radó 1925)

diam 3 : compact 3-manifolds are triangulable. (Möise 1950)

diam 4 : Freedman (1980)

diam ≥ 5 : Manolescu (2016)

Definition 4.7. Let K be a finite simplicial complex, denote the number of i -simplices of K by m_i , the Euler characteristic (欧拉示性数) of K is $\chi(K) = \sum_{i=1}^n (-1)^i m_i \in \mathbb{Z}$.

Definition and theorem 4.8.

Let X be a triangulable space, the Euler characteristic of X is $\chi(X) = \chi(K)$, where K is a triangulation of X . $\chi(X)$ is independent of the choice of triangulations, it is a topological invariant.

Examples

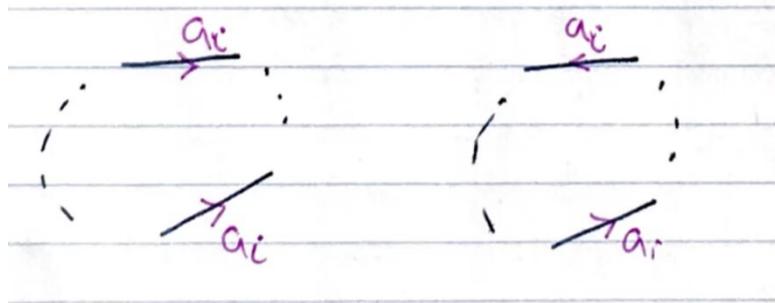
(i) $\chi(S^2) = \chi(\text{triangle}) = 4 - 6 + 4 = 2$

(ii) $\chi(T^2) = \chi(\text{square}) = 0$

proof of the classification theorem of surfaces

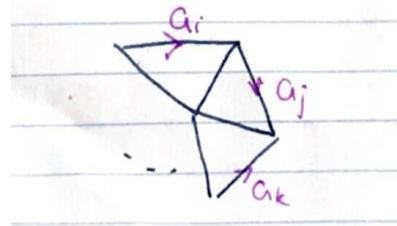
(2) every closed surface is homeomorphic to nT^2 or mP^2 .

- (i) Let P be a $2n$ -gon in \mathbb{R}^2 , E be the set of edges of P , a labelling of the edges of P is a 2-to-1 map $E \xrightarrow{l} \{a_1, \dots, a_n\}$. Given an orientation of each edge of P . Identifying the edges of P according to the labelling and orientables, we get a quotient space P/\sim .



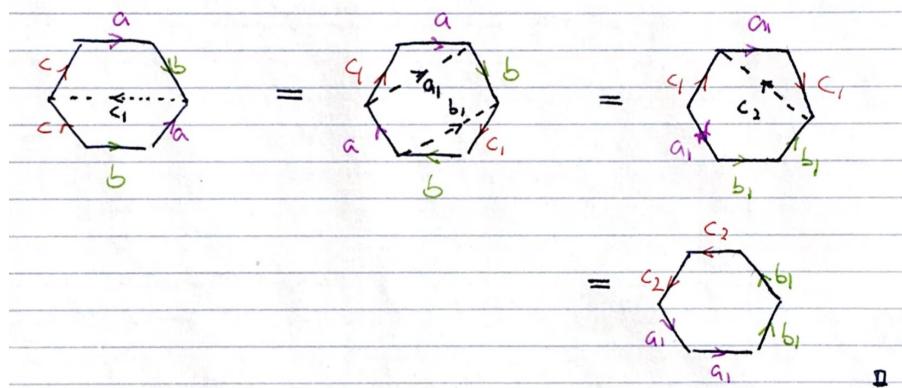
if $S \cong P/\sim$, we say this is a polygonal presentation of S . As a corollary of the triangulability of closed surfaces;

Lemma 4.9. Every closed surface has a polygonal presentation.



- (ii) transform a polygonal presentation to the standard presentation of nT^2 or mP^2 : $a_1b_1a_1^{-1}b_1^{-1}\dots$ or $a_1a_1a_2a_2\dots$

Example. $P^2 \# T^2 \cong 3P^2 \cong P^2 \# \text{Klein bottle.}$

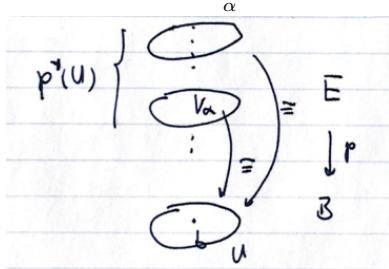


Chapter 5

Classification of covering spaces

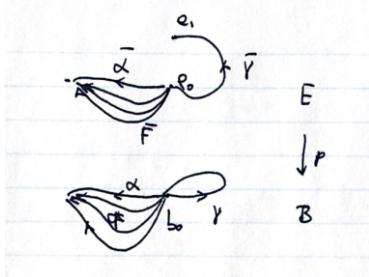
5.1 revision

Recall a surjective continuous map $p : E \rightarrow B$ is a covering map, if for any $b \in B$, \exists a neighborhood U of b , s.t. $p^{-1}(U) = \coprod_{\alpha} V_{\alpha}$, $V_{\alpha} \subset E$ open, and $p|_{V_{\alpha}} : V_{\alpha} \rightarrow U$ is a homomorphism.



Lifting properties (Chapter 3 §4) Let $p(e_0) = b_0$.

- (1) A path $\alpha : [0, 1] \rightarrow B$ with $\alpha_0 = b_0$ has a unique lifting $\bar{\alpha} : [0, 1] \rightarrow E$ with $\bar{\alpha}(0) = e_0$.
- (2) A (path) homotopy $F : I \times I \rightarrow B$ can be lifted to a (path) homotopy $\bar{F} : I \times I \rightarrow E$.



- (3) lifting correspondence: $\pi_1(B, b_0) \rightarrow p^{-1}(b_0)$, $[\gamma] \mapsto \bar{\gamma}(1)$

- if E is path connected.
- if E is simply connected.

Let $H_0 = p_*(\pi_1(E, e_0)) < \pi_1(B, b_0)$ a subgroup

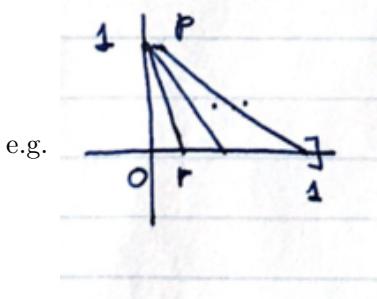
Question: Does H_0 determine (E, p) ?

5.2 equivalence of covering spaces

Definition 5.1. Let $p : E \rightarrow B, p' : E' \rightarrow B$ be covering maps. They are said to be equivalent if there exists a homeomorphism $h : E \rightarrow E'$ such that $p = p' \circ h$. The homeomorphism h is called an equivalence of covering maps/spaces.

$$\begin{array}{ccc} E & \xrightarrow{\quad h \quad \cong} & E' \\ & \searrow p & \swarrow p' \\ & B & \end{array}$$

Definition 5.2. A space X is said to be locally path connected at x if for every neighborhood U of x , there exists a path-connected neighborhood V of x contained in U . If X is locally path connected at each $x \in X$, it is said to be locally path connected.



$T =$ the union of all the segments joining $p = (0, 1)$ to $(0, r)$ for all $r \in \mathbb{Q} \cap [0, 1]$.

Then T is path connected, but not locally path connected except at p .

Theorem 5.3. Let $p : E \rightarrow B, p' : E' \rightarrow B$ be covering maps, $p(e_0) = b_0 = p'(e'_0), H_0 = p_*(\pi_1(E, e_0)), H'_0 = p'_*(\pi_1(E^1, e'_0))$. Then there is an equivalence $h : E \rightarrow E'$ s.t. $h(E_0) = e'_0$ iff $H_0 = H'_0$. If h exists, it is unique. (Assume all spaces are path connected and locally path connected.)

Proof. “ \Rightarrow ”: $H_0 = p_*(\pi_1(E, e_0)) = p'_*(h_*(\pi_1(E', e'_0)))$.

h a homeomorphism Ra $h_* : \pi_1(E, e_0) \rightarrow \pi_1(E', e'_0)$ an isomorphism.

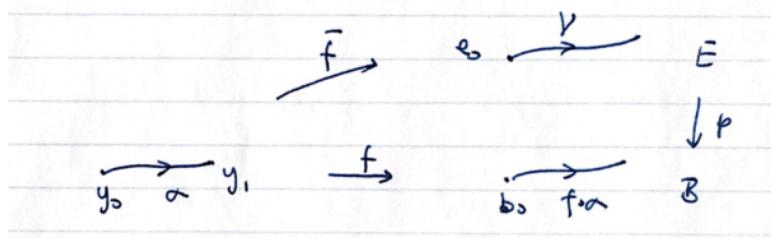
(to be continued) □

Lemma 5.4 (the general lifting lemma). Let $p : E \rightarrow B$ be a covering map, $p(e_0) = b_0$; $f : Y \rightarrow B$ be a continuous map, $f(y_0) = b_0$. Assume all spaces are path connected and locally path connected. Then the map f can be lifted to a map $\bar{f} : Y \rightarrow E$ s.t. $\bar{f}(y_0) = e_0$ iff $f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(E, e_0)) = H_0$. If such a lifting exists, it is unique.

$$\begin{array}{ccc} E & \ni e_0 & \\ \bar{f} \nearrow & & \downarrow p \\ y_0 \in Y & \xrightarrow{f} & B \ni b_0 \end{array}$$

Proof. (1) “ \Rightarrow ”: $f_*(\pi_1(Y, y_0)) = p_*(\bar{f}_*(\pi_1(Y, y_0))) \subset p_*(\pi_1(E, e_0))$.

(2) uniqueness: given $y_1 \in Y$, choose a path α in Y from y_0 to y_1 , then the path $f \circ \alpha$ in B has a unique lifting γ in E beginning at e_0 .



If there exists a lifting \bar{f} of f , then $\bar{f} \circ \alpha$ is also a lifting of $f \circ \alpha$ in E beginning at e_0 .

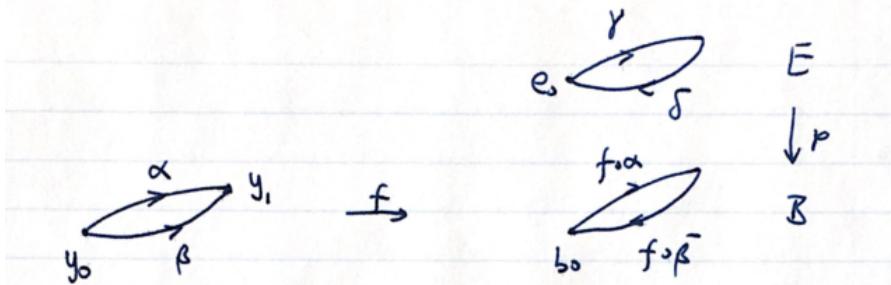
$$\therefore \gamma(1) = \bar{f}(\alpha(1)) = \bar{f}(y_1).$$

(3) " \Leftarrow " We define the lifting \bar{f} by the same construction, i.e. define $\bar{f}(y_1) = \gamma(1)$. We need to show

(i) \bar{f} is well-defined (independent of the choice of α).

(ii) \bar{f} is continuous.

(i) Let β be another path from y_0 to y_1 .



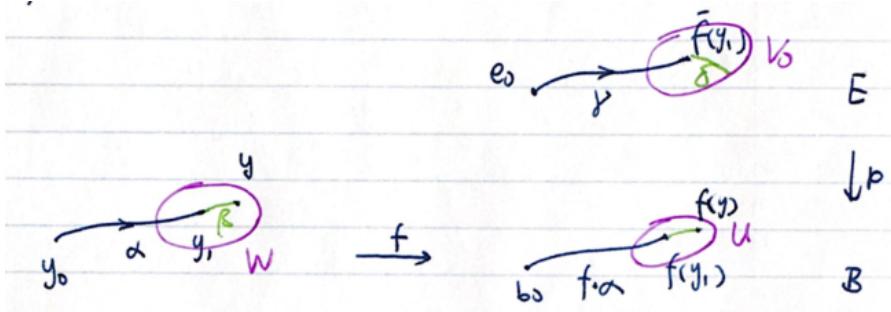
Let S be the lifting of $f \circ \bar{\beta}$ beginning at $\gamma(1)$, then $\gamma * \delta$ is a lifting of the loop $f \circ (\alpha * \bar{\beta})$. By the hypothesis $f * (\pi_1(Y, y_0)) \subset p_*(\pi_1(E, e_0))$, $[f \circ (\alpha * \bar{\beta})] \in \text{Imp}_*$. $\therefore \gamma * \delta$ is a loop in E . $\therefore \bar{\delta}$ is the lifting of $f \circ \beta$ beginning at e_0 , $\gamma(1) = \bar{\delta}(1)$, $\therefore \bar{f}$ is well-defined.

(ii) For any $y_1 \in Y$, any neighborhood N of $\bar{f}(y_1)$, we find a neighborhood W of y_1 s.t. $\bar{f}(W) \subset N$ as follows:

Choose a path-connected neighborhood U of $f(y_1)$, s.t. $p^{-1}(U) = \coprod_{\alpha} V_{\alpha}$, $\bar{f}(y_1) \in V_0$ and $V_0 \subset N$, $p_0 = p|_{V_0} : V_0 \xrightarrow{\cong} U$ a homeomorphism. f is continuous and Y is locally path connected $\Rightarrow \exists$ a path-connected neighborhood W of y_1 , s.t. $f(W) \subset U$.

Claim: $\bar{f}(W) \subset V_0$

Proof.



$\forall y \in W$, choose a path β in W from y_1 to y , $f \circ \beta$ is a path in U from $f(y_1)$ to $f(y)$, and $\delta = p_0^{-1} \circ f \circ \beta$ is a path in V_0 .

$\therefore \gamma * \delta$ is a lifting of $f \circ (\alpha * \beta)$, $\bar{f}(y) \in V_0$. □

Proof of theorem 5.3 (continued)

Proof. \Leftarrow : Assume $H_0 = H'_0$.

$$\begin{array}{ccc} & E' & \\ h \swarrow & \downarrow p' & \\ E & \xrightarrow{p} & B \\ & k \searrow & \\ & E & \end{array} \quad \begin{array}{l} H_0 \subset H'_0 \Rightarrow p \text{ has a lifting } h \text{ s.t. } h(e_0) = e'_0. \\ H'_0 \subset H_0 \Rightarrow p' \text{ has a lifting } k \text{ s.t. } k(e'_0) = e_0 \end{array}$$

$$\begin{array}{ccc} & E & \\ k \circ h \swarrow & \uparrow p & \\ E & \xrightarrow{p} & B \\ & \text{id} \nearrow & \\ & B & \end{array} \quad \begin{array}{l} k \circ h : E \rightarrow E \text{ is a lifting of } p, \text{ by the uniqueness of liftings,} \\ k \circ h = \text{id}_E. \text{ Similarly } h \circ k = \text{id}_{E'} \end{array}$$

□

Lemma 5.5. Let $p : E \rightarrow B$ be a covering map, $e_0, e_1 \in p^{-1}(b_0)$, $H_i = p_*(\pi_1(E, e_i))$, $i = 0, 1$.

- (i) If γ is a path from e_0 to e_1 , $\alpha = p \circ \gamma$ is the loop in B , then $[\alpha] * H_1 * [\alpha]^{-1} = H_0$, i.e. H_0 and H_1 are conjugate by $[\alpha]$.
- (ii) Conversely, given e_0 and a subgroup H of $\pi_1(B, b_0)$ conjugate to H_0 , there exists a point $e_1 \in p^{-1}(b_0)$ s.t. $H_1 = H$.

Proof. (i)

The handwritten diagram shows two loops γ and $\bar{\gamma}$ in E starting at e_0 and e_1 respectively, with a map h between them. Below, a commutative diagram relates fundamental groups:

$$\begin{array}{ccc} \pi_1(E, e_1) & \xrightarrow{\cong} & \pi_1(E, e_0) \\ \downarrow p_* & \lrcorner & \downarrow p_* \\ \pi_1(B, b_0) & \xrightarrow{\cong} & \pi_1(B, b_0) \end{array}$$

On the right, conjugacy is shown as $[\gamma * h * \bar{\gamma}]$ and $[\alpha] * [h] * [\alpha]^{-1}$. The bottom part shows $\alpha = p \circ \gamma$ and the conclusion $\therefore [\alpha] * H_1 * [\alpha]^{-1} = H_0$.

- (ii) Assume $H_0 = [\alpha] * H * [\alpha]^{-1}$ for some $[\alpha] \in \pi_1(B, b_0)$. Let γ be the lifting of α in E beginning at e_0 , $\gamma(1) = e_1$. Then by (i), $H_0 = [\bar{\alpha}] * H_1 * [\bar{\alpha}]^{-1}$, i.e. $H = H_1$.

□

Theorem 5.6. Let $p : E \rightarrow B, p' : E' \rightarrow B$ be covering maps, $p(e_0) = b_0 = p'(e'_0)$. Assume all spaces are path connected and locally path connected. Then the covering maps p and p' are equivalent if and only if $H_0 = p_*(\pi_1(E_0, e_0))$ and $H'_0 = p'_*(\pi_1(E', e'_0))$ are conjugate in $\pi_1(B, b_0)$.

Therefore we have an injective map

$$\left\{ \begin{array}{l} \text{equivalence classes of} \\ \text{covering spaces of } B \end{array} \right\} \xrightarrow{\text{injective}} \left\{ \begin{array}{l} \text{conjugacy classes of} \\ \text{subgroups of } \pi_1(B, b_0) \end{array} \right\}$$

Proof. “ \Rightarrow ”: If $h : E \rightarrow E'$ is an equivalence. Let $e'_1 = h(e_0)$, $H'_1 = p_*(\pi_1(E', e'_1))$. Then $H_0 = H'_1$ and H_0 and H'_1 are conjugate.

“ \Leftarrow ”: If H_0 and H'_1 are conjugate, then $\exists e'_1 \in (p')^{-1}(b_0)$ s.t. $H_0 = H'_1 = p'_*(\pi_1(E', e'_1))$. Then \exists an equivalence $h : E \rightarrow E'$ s.t. $h(e_0) = e'_1$. \square

Example: $B = S^1$, $\pi_1(S^1) \cong \mathbb{Z}$. (Conjugacy classes) of subgroups of \mathbb{Z} are $n\mathbb{Z}, n \in \mathbb{N} \cup \{0\}$. They are realized by the covering maps: $S^1 \rightarrow S^1, z \mapsto z^n (n \geq 1)$ and $\mathbb{R} \rightarrow S^1, x \mapsto e^{i2\pi x}$. \therefore These are all the equivalences classes of covering maps of S^1 . Note that the trivial covering map $S^1 \times \{1, \dots, n\} \rightarrow S^1$ is not considered.

Remark: If B is locally path connected, then $B = \coprod_{\alpha} B_{\alpha}$ is open, in which B_{α} 's are path components of B . \therefore we may only consider the case B locally path connected and path connected. In this case, if $p : E \rightarrow B$ is a covering map, $E = \coprod_{\alpha} E_{\alpha}$, E_{α} 's are path components of E , then $p : E_{\alpha} \rightarrow B$ is a covering map, E_{α} is path connected and locally path connected.

Definition 5.7. Let $p : E \rightarrow B$ be a covering map. If E is simply-connected, then E is called a universal covering space of B (无有覆盖/泛覆盖). Any two universal coverings of B are equivalent since $\pi_1(E) = 0 = \pi_1(E')$.

\therefore we speak of “the” universal covering of B .

$$\begin{array}{ccc} E' & \xrightarrow[\cong]{h} & E \\ & \searrow p' & \swarrow p \\ & B & \end{array}$$

Theorem 5.8. Let $p : E \rightarrow B$ be the universal covering space. Given any covering map $r : Y \rightarrow B$, there is a covering map $q : E \rightarrow Y$ s.t. $r \circ q = p$.

$$\begin{array}{ccc} E & & \\ \downarrow \beta & \nearrow q & \\ Y & & \\ \downarrow r & \nearrow q & \\ B & & \end{array}$$

Slogan: The universal covering space is a covering space of all covering spaces.

Proof.

$$\begin{array}{ccc} & Y \ni y_0 & \\ & \downarrow r & \\ e_0 \in E & \xrightarrow[p]{q} & B \ni b_0 \\ & \nearrow q & \end{array} \quad \begin{array}{l} p_*(\pi_1(E, e_0)) = 0 \subset r_*(\pi_1(Y, y_0)) \\ \therefore \exists \text{ a lifting } q \text{ of } p. \\ \text{by the following lemma, } q : E \rightarrow Y \text{ is a covering map.} \end{array} \quad \square$$

Lemma 5.9. In the following commutative diagram, p, q, r are continuous maps, X, Y, Z are path connected and locally path connected.

$$\begin{array}{ccc} X & & \\ \downarrow p & \nearrow q & \\ Y & & \\ \downarrow r & \nearrow q & \\ Z & & \end{array}$$

Then

- (i) If p and r are covering maps, so is q .

(ii) If p and q are covering maps, so is r .

(compare with exercise 15.5)(We cannot find the original exercise, so we failed to provide more information about it. But as compensation, we put exercise1 of §80 in [Munkres] below.)

1. Let $q : X \rightarrow Y$ and $r : Y \rightarrow Z$ be maps; let $p = r \circ q$.

(a) Let q and r be covering maps. Show that if Z has a universal covering space, then p is a covering map.

*(b) Give an example where q and r are covering maps but p is not.

Key word for the proof: lifting of paths.

Examples: $\mathbb{R} \rightarrow S^1, \mathbb{R}^2 \rightarrow T^2, S^2 \rightarrow P^2$.

More examples from group actions.

5.3 covering transformations

Definition 5.10. An equivalence of a covering map $p : E \rightarrow B$ with itself is called a covering transformation (覆盖变换).

The set of all covering transformations forms a group, called the group of covering transformations (覆盖变换群), denoted by $C(E, p, B)$.

$$\begin{array}{ccc} E & \xrightarrow{\stackrel{h}{\cong}} & E \\ & \searrow p & \downarrow \\ & B & \end{array}$$

Recall: the conjugacy class of $H_0 = p_*(\pi_1(E, e_0))$ determines the equivalence class of (E, p) .

Question: Does H_0 determine the group $C(E, p, B)$?

Definition 5.11. Let H be a subgroup of G , the normalizer (正规化子) of H in G is $N(H) = \{g \in G | gHg^{-1} = H\}$. $N(H)$ is a subgroup of G , H is a normal subgroup of $N(H)$, and $N(H)$ is the largest such group.

Given a covering transformation $h : E \rightarrow E$, by the uniqueness of lifting, h is determined by $h(e_0) \in p^{-1}(b_0)$.

\therefore define an injective map

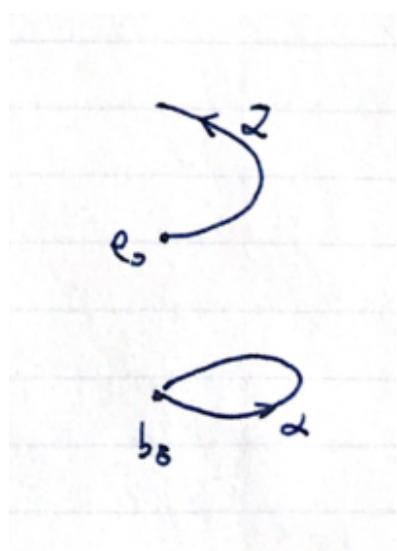
$$\begin{aligned} C(E, p, B) &\xrightarrow{\Phi} p^{-1}(b_0) = F \\ h &\mapsto h(e_0) \end{aligned}$$

$$\begin{array}{ccc} E \ni e_0 & & \\ \nearrow h \cong & & \downarrow p \\ e_0 \in E & \xrightarrow[p]{} & B \ni b_0 \end{array}$$

Recall we also have the bijective lifting correspondence:

$$\begin{aligned} \pi_1(B, b_0)/_{H_0} &\xrightarrow[1:1]{\Phi} p^{-1}(b_0) \\ [\alpha] &\mapsto \bar{\alpha}(1) \end{aligned}$$

$$\begin{array}{ccc} C(E, p, B) & \xrightarrow[\text{inj}]{\Psi} & p^{-1}(b_0) & \bar{\alpha}(1) \\ h \mapsto h(e_0) & \cong \uparrow \Phi & & \uparrow \\ \pi_1(B, b_0)/_{H_0} & \ni [\alpha] & & \end{array}$$



Lemma 5.12. $\text{Im } \Psi = \Phi(N(H_0)/_{H_1})$.

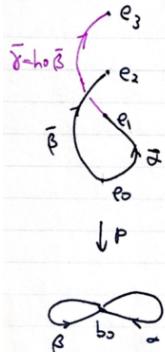
Proof. Given $e_1 = \bar{\alpha}(1) \in p^{-1}(b_0)$, \exists a covering transformation $h : E \rightarrow E$ s.t. $h(e_0) = e_1$ iff $p_*(\pi_1(E, e_0)) = H_0 = H_1 = p_*(\pi_1(E, e_1))$. But we know $H_0 = [\alpha]*H_1*[alpha]^{-1}$. $\therefore H_0 = H_1$ iff $[\alpha] \in N(H_0)$.

$$\begin{array}{ccc} E \ni e_0 & & \\ h \swarrow \cong \quad \downarrow p & & \\ e_0 \in E \xrightarrow[p]{} B \ni b_0 & & \end{array}$$

□

Theorem 5.13. *The bijection $\Phi^{-1} \cdot \Psi : C(E, p, B) \rightarrow N(H_0)/H_0$ is an isomorphism of groups.*

Proof. We need to show $\Phi^{-1} \cdot \Psi$ is a homomorphism. Let $h, k : E \rightarrow E$ be covering transformations, $h(e_0) = e_1, k(e_0) = e_2$, then $\Psi(h) = e_1, \Psi(k) = e_2$. Choose path $\bar{\alpha}$ from e_0 to e_1 , $\bar{\beta}$ from e_0 to e_2 , $\alpha = p \circ \bar{\alpha}, \beta = p \circ \bar{\beta}$, then $\Phi^{-1} \circ \Psi(h) = [\alpha] \cdot H_0, \Phi^{-1} \circ \Psi(k) = [\beta] H_0$. Let $h(k(e_0)) = e_3$, then we need to show $\Phi([\alpha * \beta] \cdot H_0) = e_3$.



Let $\bar{\gamma} = h \circ \bar{\beta}$ be a path from e_1 to e_3 .

Then $p \circ \bar{\gamma} = \beta$.

$\therefore \bar{\alpha} * \bar{\gamma}$ is a lifting of

$\alpha * \beta$ beginning at e_0 .

$\therefore \Phi([\alpha * \beta] \cdot H_0) = \bar{\gamma}(1) = e_3$

□

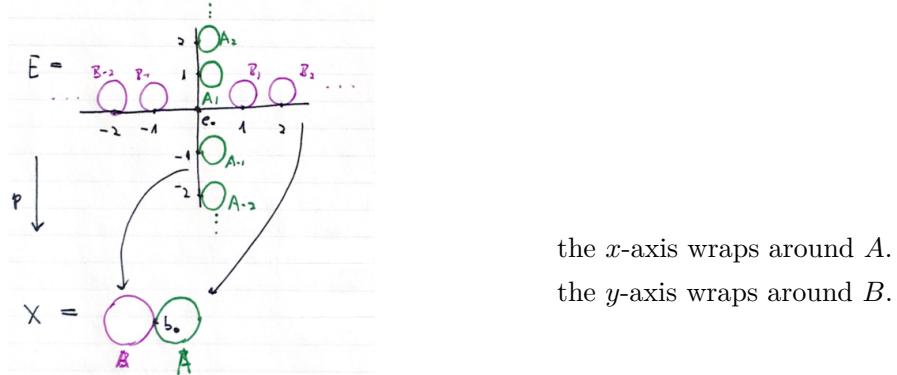
Corollary 5.14. *The group H_0 is a normal subgroup of $\pi_1(B, b_0)$ if and only if for every pair of points e_1, e_2 of $p^{-1}(b_0)$, there is a covering transformation $h : E \rightarrow E$ with $h(e_1) = e_2$. In this case there is an isomorphism $\Phi^{-1} \circ \Phi : C(E, p, B) \xrightarrow{\cong} \pi_1(B, b_0)/H_0$. Such a covering map is called a regular covering (正规覆盖).*

Corollary 5.15. *Let $p : E \rightarrow B$ be the universal covering, then $C(E, p, B) \cong \pi_1(B, b_0)$.*

Examples:

(1) If $\pi_1(B)$ is abelian, then any covering map is regular e.g. $B = S^1, T^n$, or a topological group G .

(2)



The loop A lifts to

$$\begin{cases} \text{a loop in } E \text{ at base points } (0, n) & (n \neq 0) \\ \text{a line segment in } E \text{ at base points } (n, 0) & \forall n \in \mathbb{Z}. \end{cases}$$

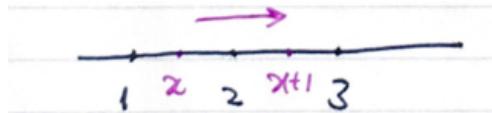
Similar for the loop B .

\therefore a covering transformation h must fix e_0 , $\therefore h = \text{id}$.

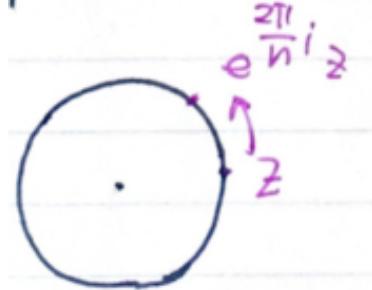
Definition 5.16. Let X be an topological space, $\text{Homeo}(X)$ be the group of self-homeomorphisms of X , G be a group. An action (作用) of G on X is a homomorphism $G \rightarrow \text{Homeo}(X)$, i.e. we assign to each $g \in G$ a homeomorphism $g : X \xrightarrow{\cong} X$ s.t. $g(h(x)) = (gh)(x) \forall x \in X, g, h \in G$. (Notation: $G \curvearrowright X$) The action is effective if $G \rightarrow \text{Homeo}(X)$ is injective. It suffices to consider effective actions.

Examples:

- (1) $\mathbb{Z} \curvearrowright \mathbb{R} : 1 \cdot x = x + 1$ translation



- (2) $\mathbb{Z}/n \curvearrowright S^1 : 1 \cdot z = e^{\frac{2\pi}{n}i} \cdot z$ rotation



- (3) $\mathbb{Z} \curvearrowright S^1 : 1 \cdot z = e^{2\pi\alpha i} \cdot z$ rotation

- if $\alpha = m/n$, then this action factors through \mathbb{Z}/n :

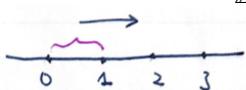
$$\begin{array}{ccc} \mathbb{Z} & \longrightarrow & \mathbb{Z}/n \\ & \searrow & \downarrow \\ & & \text{Homeo}(S^1) \end{array}$$

- if α is irrational, then this action is effective.

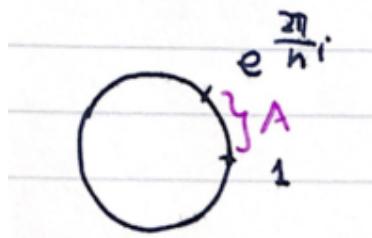
Definition 5.17. The orbit (轨道) of $x \in X$ is $\text{Orb}(x) = \{g(x) | g \in G\}$. The orbit space X/G is the quotient space obtained from X by the equivalence relation $x \sim g(x), \forall g \in G, x \in X$.

Examples:

- (1) $\mathbb{Z} \curvearrowright \mathbb{R}$ translation: $\mathbb{R}/\mathbb{Z} = [0, 1]/_{0 \sim 1} \cong S^1$.

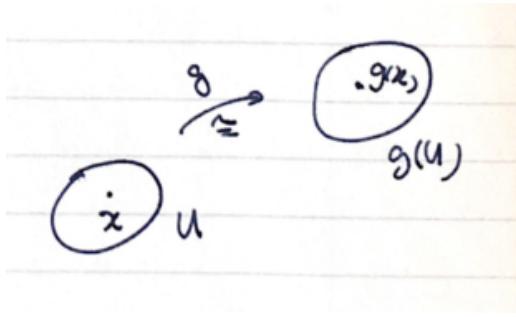


- (2) $\mathbb{Z}/n \curvearrowright S^1$ rotation: $S^1/\mathbb{Z}/n = A/_{1 \sim e^{\frac{2\pi}{n}i}} \cong S^1$



- (3) each orbit if the irrational rotation action $\mathbb{Z} \curvearrowright S^1$ is dense in S^1 .

Definition 5.18. An action $G \curvearrowright X$ is called free (自由) if $g(x) \neq x, \forall x \in X, g \neq e$. In other words, the map $G \rightarrow \text{Orb}(x), g \mapsto g(x)$ is bijective for all $x \in X$. An action $G \curvearrowright X$ is called properly discontinuous (真不连续) if for every $x \in X$, \exists a neighborhood U of x , s.t. $g(U) \cap U = \emptyset, \forall g \neq e$. (This implies $g_1(U) \cap g_2(U) = \emptyset, \forall g_1, g_2 \in G$).

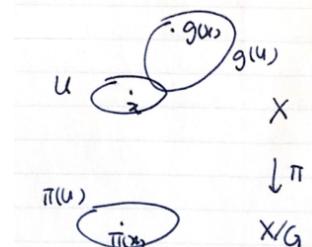


Remark:

- (i) properly discontinuous \Rightarrow free
- (ii) the irrational rotation action $\mathbb{Z} \curvearrowright S^1$ is free but not properly discontinuous.
- (iii) if G is finite, then free \Rightarrow properly discontinuous. (Assume X is Hausdorff)

Theorem 5.19. Let X be path connected and locally path connected, $G \curvearrowright X$ be an action. The quotient map $X \xrightarrow{\pi} X/G$ is a covering map if and only if the action is properly discontinuous. In this case, the covering map π is regular and G is its group of covering transformations.

Proof. (i) π is an open map: $X \xrightarrow{\pi} X/G$. Let $U \subset X/G$ is open. $\pi_1(U)$ open $\Leftrightarrow \pi^{-1}(\pi(U))$ open, and $\pi^{-1}(\pi(U)) = \bigcup_{g \in G} g(U)$ open.



(ii) Suppose the action is properly discontinuous, we show that π is a covering map: $\forall x \in X$, let U be a neighborhood of x s.t. $g(U) \cap U = \emptyset, \forall g \neq e$. Then $\pi(U)$ is an open neighborhood of $\pi(x)$, and $\pi^{-1}(\pi(U)) = \coprod_{g \in G} g(U)$ is a disjoint union of open sets $G(U)$. $\pi : U \rightarrow \pi(U)$ is continuous, open and bijective, hence a homeomorphism. (If $\exists y, gy \in U$, then $gy \in U \cap g(U)$, lightening sign)

$$\begin{array}{ccc} U & \xrightarrow[g]{\cong} & g(U) \\ & \searrow \pi & \downarrow \pi \\ & & \pi(U) \end{array}$$

(iii) Suppose $\pi : X \rightarrow X/G$ is a covering map, we show that the action is properly discontinuous:

$$\begin{array}{ccc} X & & \ni x \\ \pi \downarrow & & \downarrow \\ X/G & & \ni \pi(x) \subset V \end{array}$$

Given $x \in X$, let V be a neighborhood of $\pi(x)$ s.t.
 $\pi^{-1}(V) = \coprod_{\alpha} U_{\alpha}, \pi : U_{\alpha} \xrightarrow{\cong} V$.

$\circlearrowleft g(x) \quad g(U_\alpha)$

Assume $x \in U_\alpha$,
then $g(U_\alpha) \cap U_\alpha = \emptyset$
for $\forall g \neq e$.
 \therefore the action is
properly discontinuous.

$\circlearrowleft x \quad U_\alpha$

$\downarrow \pi$
 $\circlearrowleft \cdot \quad V$

(iv) Assume $\pi : X \rightarrow X/G$ is a covering map, we show $G = C(X, \pi, X/G)$.

$$(1) \quad \begin{array}{ccc} X & \xrightarrow{\stackrel{g}{\cong}} & X \\ & \searrow \pi & \downarrow \pi \\ & X/G & \end{array}$$

$\therefore g \in G$ is a covering transformation.

(2) given a covering transformation $\varphi : X \xrightarrow{\cong} X$. If $\varphi(x_1) = x_2$ then x_1 and x_2 are in the same orbit.
 $\therefore \exists g \in G$ s.t. $g(x_1) = x_2$. By the uniqueness of lifting, $\varphi = g \therefore G = C(X, \pi, X/G)$.

(3) Since for any two points x_1, x_2 in the same orbit, $\exists g \in G = C(X, \pi, X/G)$ s.t. $g(x_1) = x_2$, the covering is regular. (p.186 Corollary).

“regular covering=the group of covering transformations acts transitively”

□

Theorem 5.20. If $p : X \rightarrow B$ is a regular covering map and $G = C(X, p, B)$ is its group of covering transformations. Then there is a homeomorphism $k : X/G \rightarrow B$ s.t.

$$\begin{array}{ccc} X & & \\ \pi \downarrow & \searrow p & \\ X/G & \xrightarrow{\stackrel{k}{\cong}} & B \end{array}$$

regular covering

Slogan: \parallel

properly discontinuous action

Proof. (1) π is a quotient map, p is constant on each orbit $\Rightarrow \exists$ a continuous map $k : X/G \rightarrow B$.

(2) p is a quotient map (continuous, surjective and open), π is constant on each $p^{-1}(b) \Rightarrow \exists$ a continuous map $B \rightarrow X/G$, which is the inverse of k .

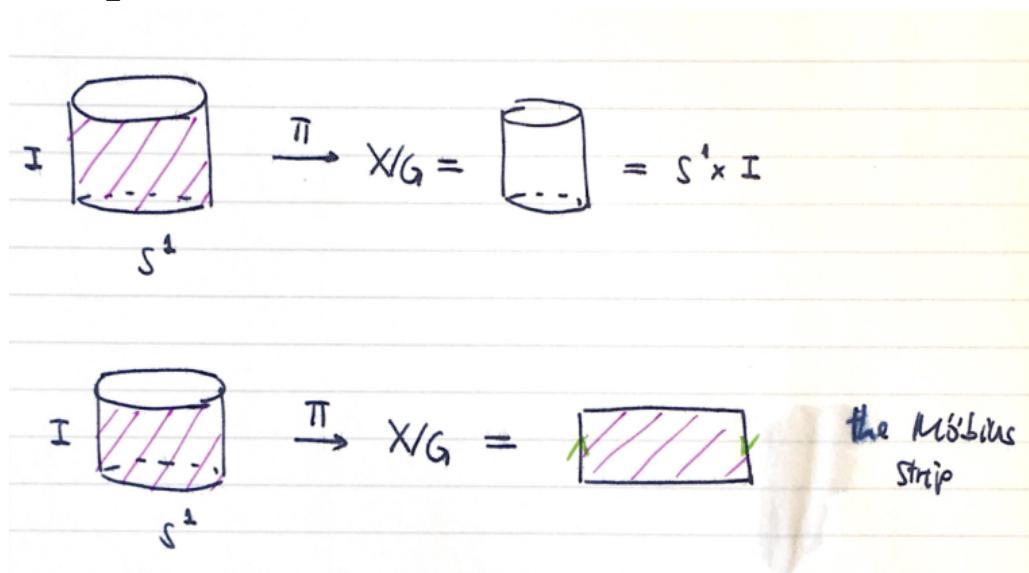
□

Remark: $G \curvearrowright X$ properly discontinuous, there is a short exact sequence $1 \rightarrow \pi_1(X) \rightarrow \pi_1(X/G) \rightarrow G \rightarrow 1$.

Example: $X = S^1 \times [0, 1]$ two $G = \mathbb{Z}/2$ actions on X :

$$X \xrightarrow[\cong]{h} X, (x, t) \rightarrow (-x, t)$$

$$X \xrightarrow[\cong]{k} X, (x, t) \rightarrow (-x, 1 - t)$$



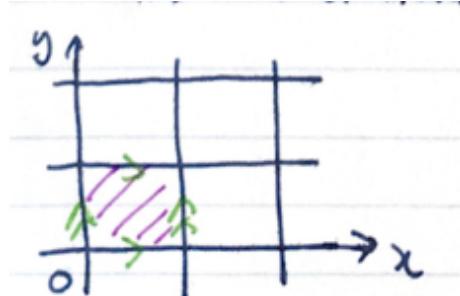
Example: X path connected, locally path connected, simply connected, $G \curvearrowright X$ properly discontinuous, Then $X \xrightarrow{\pi} X/G$ is the universal covering.

(1)

$$\begin{aligned} \mathbb{Z} \curvearrowright \mathbb{R} \text{ translations} &\rightsquigarrow \mathbb{R} \xrightarrow{\pi} S^1 \\ \mathbb{Z}^2 \curvearrowright \mathbb{R}^2 \text{ translations} &\rightsquigarrow \mathbb{R}^2 \xrightarrow{\pi} T^2 \end{aligned}$$

In general,

$$\mathbb{Z}^n \curvearrowright \mathbb{R}^n \text{ translations} \rightsquigarrow \mathbb{R}^n \xrightarrow{\pi} T^n$$



(2)

\mathbb{Z}^2 generates a, b , $\mathbb{Z} \curvearrowright \mathbb{R}^2$ as follows:

$$a : (x, y) \mapsto (x + 1, y),$$

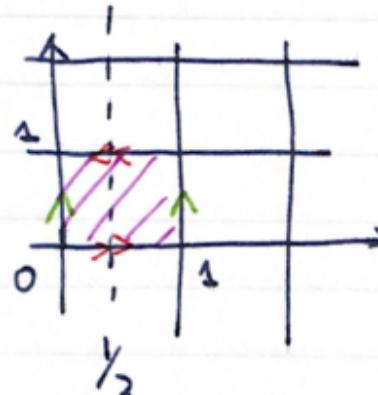
$$Sb : (x, y) \mapsto (1 - x, y + 1)$$

$$b \circ a : (x, y) \mapsto (x + 1, y) \mapsto (-x, y + 1)$$

$$a \circ b : (x, y) \mapsto (1 - x, y + 1) \mapsto (-x, y + 1)$$

$$\therefore a \circ b = b \circ a$$

$$\mathbb{R}^2 / \mathbb{Z}^2 = \text{the Klein bottle}$$



$$(3) \mathbb{Z}/2 \curvearrowright S^2, x \mapsto -x, S^2 / \mathbb{Z}/2 = P^2$$

in general $\mathbb{Z}/2 \curvearrowright S^n, x \mapsto -x, S^n / \mathbb{Z}/2 = \mathbb{RP}^n$, the n -dim projective space.

5.4 existence of covering spaces

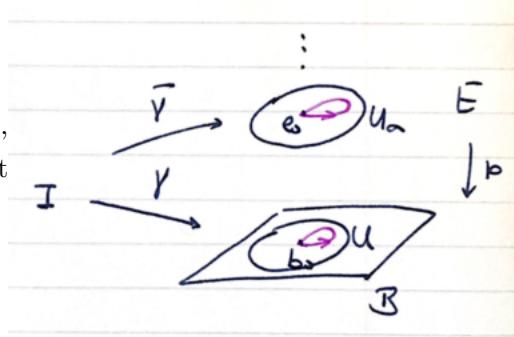
Recall: $\begin{cases} \text{equivalence classes of} \\ \text{covering spaces of } B \end{cases} \xrightarrow{\text{inj}} \begin{cases} \text{conjugacy classes of} \\ \text{subgroups of } \pi_1(B, b_0) \end{cases}$,
 $(E \xrightarrow{p} B) \mapsto p_*(\pi_1(E, e_0)) = H$.

Question: when is the map surjective?

Lemma 5.21. Let $p : E \rightarrow B$ be the universal covering, $p(e_0) = b_0$. Then there is a neighborhood U of b_0 s.t. the inclusion $i : U \rightarrow B$ induces the trivial homomorphism

$$i_* : \pi_1(U, b_0) \rightarrow \pi_1(B, b_0)$$

Proof. Let U be a neighborhood of b_0 that is evenly covered by p , i.e. $\pi^{-1}(U) = \coprod_{\alpha} U_{\alpha}, p : U_{\alpha} \xrightarrow{\cong} U$. Let $e_0 \in U_{\alpha}$,



$$\pi_1(E, e_0) \xrightarrow{p_*} \pi_1(B, b_0)$$

$$[\bar{\gamma}] \quad \mapsto \quad [\gamma]$$

$$\pi_1(E, e_0) = 0 \Rightarrow [\gamma] = 0.$$

∴

$$\pi_1(U, b_0) \longrightarrow \pi_1(B, b_0)$$

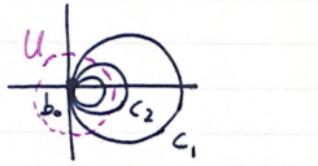
$$[\gamma] \quad \mapsto \quad [\gamma]$$

is trivial. □

Example:

$$X = \bigcup_{n=1}^{\infty} C_n \text{ the hawaiian earring}$$

$$C_n = \text{the circle of radius } \frac{1}{n}, \text{ centered at } \left(\frac{1}{n}, 0\right).$$



Claim: For any neighborhood U of b_0 , $i_* : \pi_1(U, b_0) \rightarrow \pi_1(X, b_0)$ is nontrivial.

Proof. Choose n large s.t. $C_n \subset U$, consider the retraction $r : X \rightarrow C_n$, $r|_{C_n} = \text{id}$, $r(C_m) = b_0$ ($m \neq n$).

$$\begin{array}{ccccccc}
 C_n & \xrightarrow{j} & U & \xrightarrow{i} & X & \xrightarrow{r} & C_n \quad \text{induce} \\
 & \searrow \text{id} & & & & \nearrow & \\
 \pi_1(C_n, b_0) & \xrightarrow{j_*} & \pi_1(U, b_0) & \xrightarrow{i_*} & \pi_1(X, b_0) & \xrightarrow{r_*} & \pi_1(C_n, b_0) \\
 & \searrow \text{id} & & & & \nearrow & \\
 & \parallel \mathbb{Z} & & & & & \parallel \mathbb{Z}
 \end{array}$$

$\therefore i_* : \pi_1(U, b_0) \rightarrow \pi_1(X, b_0)$ is nontrivial.

$\therefore X$ has no universal covering. \square

Definition 5.22. A space B is said to be semilocally simply connected if for each $b \in B$, there is a neighborhood U of b , s.t. the homomorphism $i_* : \pi_1(U, b) \rightarrow \pi_1(B, b)$ induced by the inclusion is trivial.
e.g. B is a manifold or B is simply connected.

Theorem 5.23. Let B be path connected, locally path and semilocally simply connected. Let $b_0 \in B$, given a subgroup H of $\pi_1(B, b_0)$, there exists a covering map $\varphi : E \rightarrow B$ (E is path connected) and a point $e_0 \in p^{-1}(b_0)$ s.t. $p_*\pi_1(E, e_0) = H$.

Under the assumption on B

$$\left\{ \begin{array}{l} \text{equivalence classes of} \\ \text{covering spaces of } B \end{array} \right\} \xrightarrow{1:1} \left\{ \begin{array}{l} \text{conjugacy classes of} \\ \text{subgroups of } \pi_1(B, b_0) \end{array} \right\}$$

Corollary 5.24. The space B has a universal covering space if and only if B is path connected, locally path connected and semilocally path connected.

idea of the proof :

(1) recall if $E \xrightarrow{p} B$ is a covering map, E path connected. $\forall e \in E$, let $\bar{\alpha}$ be a path from e_0 to e , then $\alpha = p \circ \bar{\alpha}$ is a path in B from b_0 to $b = \alpha(1)$, and $\bar{\alpha}$ is a lifting of α . Take another path β from b_0 to b , let $\bar{\beta}$ be the lifting of β beginning at e_0 , then

$$\bar{\alpha}(1) = \bar{\beta}(1) \Leftrightarrow [\alpha * (-\beta)] \in H = p_*\pi_1(E, e_0)$$

(2) construction of E : \mathcal{P} = the set of all paths in B beginning at b_0 . Define an equivalence relation on \mathcal{P} : $\alpha \sim \beta \Leftrightarrow \alpha(1) = \beta(1)$ and $[\alpha * (-\beta)] \in H$.

Let $E = \mathcal{P}/\sim$ be the set of equivalence classes denote by the equivalence class of a path a path α by $\alpha^\#$, define $p : E \rightarrow B$, $\alpha^\# \mapsto \alpha(1)$.

We need to show E is a path connected topological space, p is a covering map.

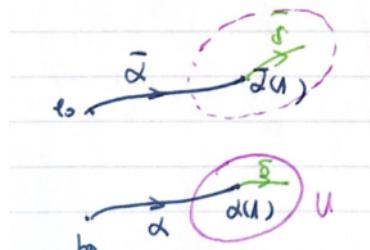
(3) topology on E : there are two ways to topologize E .

(i) give \mathcal{P} the compact-open topology and $E = \mathcal{P}/\sim$ the quotient topology.

(ii) given $\alpha \in \mathcal{P}$, let U be any path connected neighborhood of $\alpha(1)$.

Define $B(U, \alpha) = \{(\alpha * \delta)^\# \mid \delta \text{ is a path in } U \text{ beginning at } \alpha(1)\}$.

Then $B(U, \alpha)$ is a basis of a topology on E .



(4) $p : E \rightarrow B$ is continuous, open and a covering map.

(5) lifting of path:

$e_0 = (\text{the constant path at } b_0)^\# \in E$,

then $p(e_0) = b_0$.

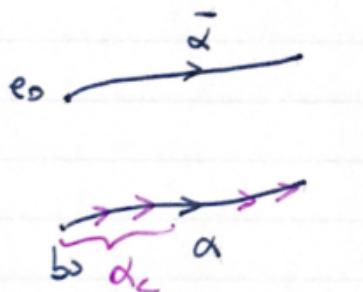
For any $e = \alpha^\# \in E, C \in [0, 1]$,

let $\alpha_C : I \rightarrow B, \alpha_C(t) = \alpha(tc)$.

Then $\bar{\alpha} : I \rightarrow E, c \mapsto (\alpha_C)^\#$ is a path in E from E_0

to e ,

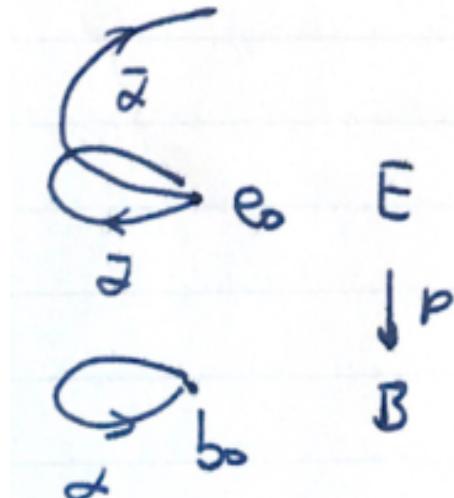
a lifting of α , $\bar{\alpha}(1) = \alpha^\# = e$.



Need to verify the continuity of $\bar{\alpha}$.

$\therefore E$ is path connected.

(6) compute $p_*\pi_1(E, e_0)$.



$$\begin{array}{c} [\alpha] \in p_*\pi_1(E, e_0) \\ \Updownarrow \\ \bar{\alpha} \text{ is a loop in } E \\ \Updownarrow \\ \alpha^\# = \bar{\alpha}(1) = e_0 \iff [\alpha] \in H \end{array}$$

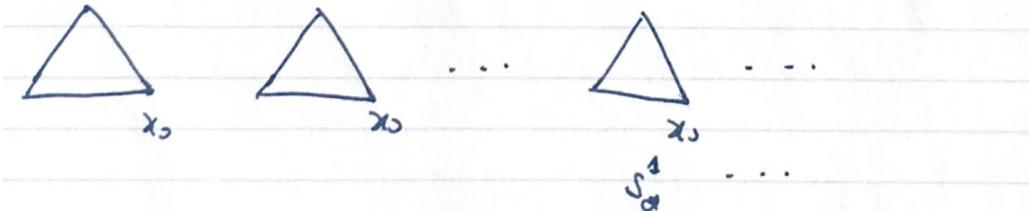
$\therefore p_*\pi_1(E, e_0) = H$.

5.5 application to free groups

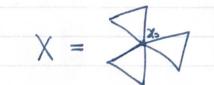
Theorem 5.25. If H is a subgroup of a free group, then H is free.

sketch of proof .

Let F be a free group with generators $\{\alpha | \alpha \in J\}$, $F = *_J \mathbb{Z}$. Let X be a wedge of circles $S_\alpha^1 (\alpha \in J)$ $X = \bigvee_{\alpha \in J} S_\alpha^1$, $x_0 \in X$ be the common point, then $\pi_1(X, x_0) \cong *_J \mathbb{Z} = F$.



X is a 1-dimensional simplicial complex.

e.g.  X is path connected, locally path connected and semilocally simply connected.

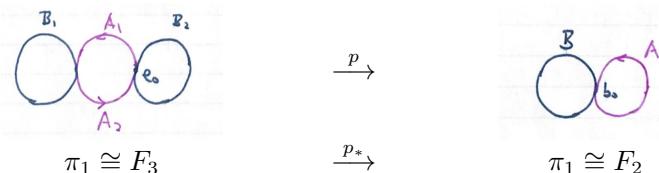
\therefore There is a covering space $p : E \rightarrow X$ s.t. $p_*(\pi_1(E, e_0)) = H$. $\therefore \pi_1(E, e_0) \cong H$.

Fact .

- (1) A covering space of a 1-dimensional simplicial complex is a 1-dimensional simplicial complex.
- (2) the fundamental group of a 1-dimensional simplicial complex K is a free group. (one can find a maximal tree T in K , $K \cong K/T \cong \vee S^1$.)

Examples

(1)

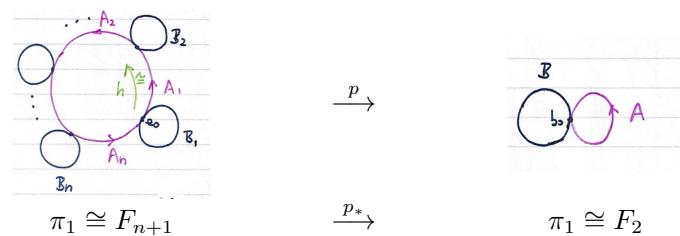


$$[F_2 : p_*(F_3)] = |F_2 / p_*(F_3)| = p^{-1}(b_0) = 2$$

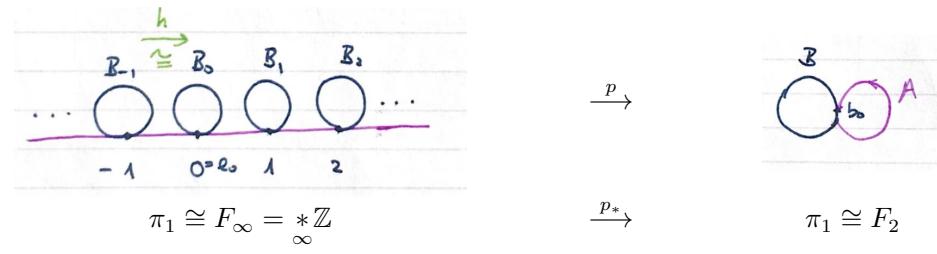
$\therefore F_3 \triangleleft F_2$ a normal subgroup of index 2.

Question , if $F_2 = \langle a, b \rangle$, find generators of $p_* F_3$

(2) in general



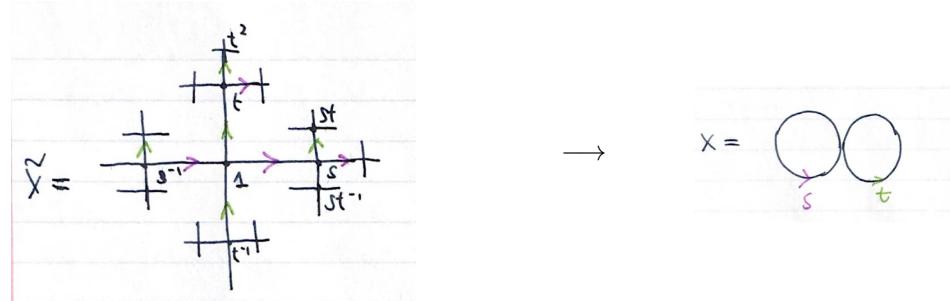
$$F_{n+1} \triangleleft F_2 \quad \text{index} = n. \quad F_2/F_{n+1} = \mathbb{Z}/n$$



$$\therefore F_\infty \triangleleft F_2 \quad F_2/F_\infty \cong \mathbb{Z}$$

(3) the universal covering space of $X = S^1 \vee S^1$.

\tilde{X} is a 1-dimensional(infinite) simplicial complex, vertices $V = \pi_1(X) \cong F_2 = \langle s, t \rangle$. edges $E = \{(g, gt), (g, gs) | g \in \pi_1(X)\}$



an infinite four-valent tree (antenna space) contractible. ([Ref](#) . [Brown, Cohomology of Groups]. [Office Hours with a Geometric Group Theorist] [包志强])

Additional topics

(1) the universal covering of closed surfaces.

three geometries

(i) the Euclidean geometry \mathbb{E}^2 : $\text{Isom}(\mathbb{E}^2) = \mathbb{R}^2 \rtimes O(2)$

curvature = 0 (flat)



$$L = 2\pi r$$



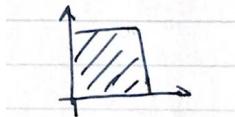
$$\alpha + \beta + \gamma = \pi.$$

- $\mathbb{Z}^2 \hookrightarrow \mathbb{R}^2$ by translations, $\mathbb{R}^2/\mathbb{Z}^2 = T^2$

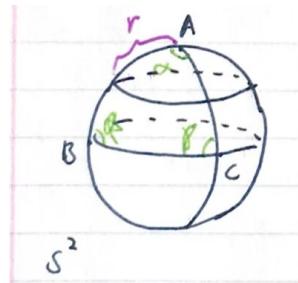
- $\mathbb{Z}^2 \curvearrowright \mathbb{R}^2$ by translations and reflections $\mathbb{R}^2/\mathbb{Z}^2 =$ the klein bottle K

$\therefore T^2, K$ inhent a geometry of curvature 0,

$$\chi(T^2) = 0 = \chi(K).$$



(ii) the spherical geometry S^2 :



$$Isom(S^2) = O(3)$$

curvature = 1 (positively curved)

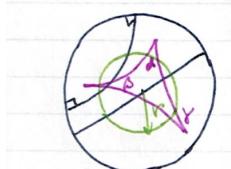
$$L = 2\pi r, \pi < \alpha + \beta + \gamma < 3\pi$$

$$\mathbb{Z}^2 \curvearrowright S^2 \text{ by isometry, } S^2/\mathbb{Z}^2 = \rho^2$$

$$\chi(S^2) = 2, \quad \chi(\rho^2) = 1$$

(iii) the hyperbolrc geometry \mathbb{H}^2 the poincaré disk model :

$$\mathbb{H}^2 = \{z \in \mathbb{C} \mid |z| < 1\}$$



$$ds^2 = 4 \cdot \frac{dx^2 + dy^2}{(1 - r^2)^2}$$

$$d(z_1, z_2) = \cosh^{-1} \left(1 + \frac{2|z_1 - z_2|^2}{(1 - |z_1|^2)(1 - |z_2|^2)} \right)$$

curvature = -1 (negativaly curved)

$$L = 2\pi \sinh r, \quad \alpha + \beta + \gamma < \pi.$$

the poincaré upper half-palne nodel:

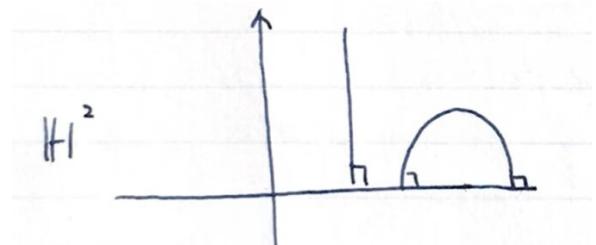
$$Isom^+(\mathbb{H}^2) = PSL_2(\mathbb{R}) = SL_2(\mathbb{R}) / (\pm I)$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}), \quad A \cdot z = \frac{az + b}{cz + d} \text{ (Möbius transformation)} \text{ For any closed surfaces}$$

$$S = nT^2 \ (n \geq 2), mP^2 \ (m \geq 3). \exists G < Isom(\mathbb{H}^2),$$

$$G \curvearrowright \mathbb{H}^2 \text{ properly discontinuously. s.t. } \mathbb{H}^2/G \cong S.$$

\therefore the universal covering space of S is $\mathbb{H}^2 \cong \mathbb{R}^2$. and S inhents a geometry of curvature-1.

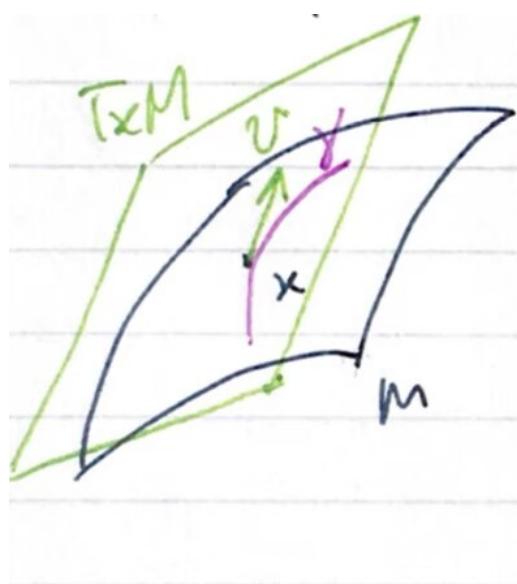


$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$

$$\{z \in \mathbb{C} | Re Z > 0\}$$

$$\chi(nT^2) = 2 - 2n, \quad \chi(mP^2) = 2 - m.$$

In general, if (M^n, g) is a closed Riemannian manifold of curvature ≤ 0 , then the exponential map.



exp:

$$\begin{aligned} T_x M &\longrightarrow M \\ v &\longmapsto \gamma_v(1) \end{aligned}$$

is a covering map (cartan's Thm)

$$\therefore \tilde{M} \cong T_x M \cong \mathbb{R}^n.$$

(2)

$$\begin{aligned} \mathrm{GL}_n^+(\mathbb{R}) &= \{A \in \mathrm{M}_n(\mathbb{R}) \mid \det A \neq 0\} \quad \text{a Lie group} \\ &\cup \\ \mathrm{SO}(n) &= \{A \in \mathrm{M}_n(\mathbb{R}) \mid A \cdot A^t = I, \det A = 1\} \\ &\quad \text{a maximal compact subgroup of } \mathrm{GL}_n(\mathbb{R}) \end{aligned}$$

Then the space of left cosets

$\mathrm{GL}_n^+(\mathbb{R})/\mathrm{SO}(n)$ = the space of positive definite

quadratic forms on \mathbb{R}^n

$\cap |$ open, convex

$$\mathrm{symm}_n(\mathbb{R}) \cong \mathbb{R}^{\frac{n(n+1)}{2}}$$

$$\therefore \mathrm{GL}_n^+(\mathbb{R})/\mathrm{SO}(n) \cong \mathbb{R}^{\frac{n(n+1)}{2}}$$

Let $\Gamma < \mathrm{GL}_n^+(\mathbb{R})$ be a discrete group, the Γ acts on $\mathrm{GL}_n^+(\mathbb{R})/\mathrm{SO}(n)$ by left translation: $\gamma \cdot A\mathrm{SO}(n) = \gamma A \cdot \mathrm{SO}(n)$

Fact . if Γ is torsion free, then the action is properly discontinuous.

$$\therefore {}_r \backslash \mathrm{GL}_n^+(\mathbb{R})/\mathrm{SO}(n) \text{ has universal covering space } \mathbb{R}^{\frac{n(n+1)}{2}}.$$

e.g. $\mathrm{GL}_n^+(\mathbb{Z}) = \mathrm{SL}_n(\mathbb{Z}) \subset \mathrm{GL}_n^+(\mathbb{R})$ has torsion elements.

$$\begin{array}{ccccccc} 1 & \longrightarrow & \Gamma(\mathbb{N}) & \longrightarrow & \mathrm{SL}_n(\mathbb{Z}) & \rightarrow & \mathrm{SL}_n(\mathbb{Z}/N) \\ & & \parallel & & & & \\ & & \{A \in \mathrm{SL}_n(\mathbb{Z}) \mid A \equiv I \pmod{N}\} & & & & \end{array}$$

the principal congruent subgroup (同余子群) of level N .

(i) $[\mathrm{SL}_n(\mathbb{Z}) : \Gamma(N)] < \infty$ since $|\mathrm{SL}_n(\mathbb{Z}/N)| < \infty$

(ii) For $N > \geq 3$, $\Gamma(N)$ is torsion free.

Ref . [Brown, Cohomology of Groups p.38].

Chapter 6

the homology theory

The fundamental group functor

$$\begin{aligned}\mathcal{T}_{\text{op}}_* &\xrightarrow{\pi_1} \mathcal{G}\text{p}, (X, x_0) \mapsto \pi_1(X, x_0) \\ ((X, x_0) &\xrightarrow{f} (Y, y_0)) \mapsto f_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0).\end{aligned}$$

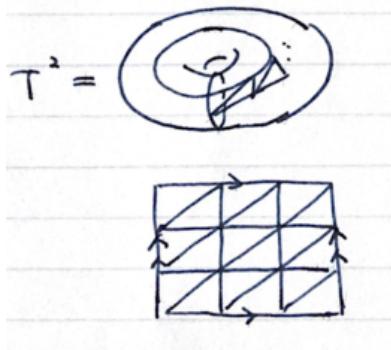
homotopy invariance, applications, computation: (Seifert-van Kampen Theorem).

The homology group functors $n \geq 0$

$$\mathcal{T}_{\text{op}} \xrightarrow{H_n} \mathcal{A}\text{b} \text{ (the category of abelian groups)}$$

$$\begin{aligned}X &\mapsto H_n(X) \\ (X &\xrightarrow{f} Y) \mapsto f_* : H_n(X) \rightarrow H_n(Y)\end{aligned}$$

e.g.



$$\begin{array}{ccc} & \mathbb{R}^2 & \\ \bar{f} \nearrow & & \downarrow \pi \\ S^2 & \xrightarrow{f} & T^2 \\ \therefore f \text{ is nullhomotopic.} & & \end{array}$$

6.1 simplicial homology

Definition 6.1. A chain complex (C_*, ∂) (链复形) of abelian groups consists of a sequence of abelian groups $\{C_n | n \in \mathbb{Z}\}$ (called chain groups (链群)) and homomorphisms $\{\partial_n : C_n \rightarrow C_{n-1} | n \in \mathbb{Z}\}$ (called boundary homomorphisms (边缘同态)) s.t.

$$\partial_{n-1} \cdot \partial_n = 0, \forall n \in \mathbb{Z}$$

$$\rightarrow \cdots C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} C_{n-2} \rightarrow \cdots$$

$Z_n = \ker \partial_n$ is called the group of closed chains (闭链群) or the group of n -cycles. $B_n = \text{Im} \partial_{n+1}$ (in which $B_n \subseteq Z_n$) is called the group of boundary chains (边缘链群).

$$\partial_n \circ \partial_{n+1} = 0 \Rightarrow B_n \subseteq Z_{n+1}$$

For two cycles $z_1, z_2 \in Z_n$, if $z_1 - z_2 \in B_n$, i.e. $\exists C \in C_{n+1}$, s.t. $z_1 - z_2 = \partial_{n+1}(C)$, then they are called homologous (同调的).

$H_n(C_*, \partial) = Z_n / B_n = \ker \partial_n / \text{Im} \partial_{n+1}$ is the n -dim homology group (同调群) of the chain complex (C_*, ∂) .

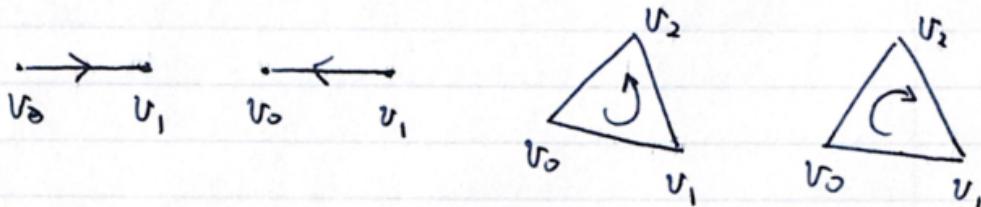
Let v_0, \dots, v_k be $(k+1)$ points in \mathbb{R}^n of general position, (i.e. $v_1 - v_0, \dots, v_k - v_0$ are linearly independent), the k -simplex with vertices v_0, \dots, v_k is

$$\sigma = \left\{ \sum_{i=0}^k \lambda_i v_i \in \mathbb{R}^n \middle| \sum_{i=0}^k \lambda_i = 1, \lambda_i \geq 0 \right\}.$$

An orientation (定向) of σ is an ordering of the vertices v_0, \dots, v_k . Two orderings are equivalent if they differ by an even permutation.

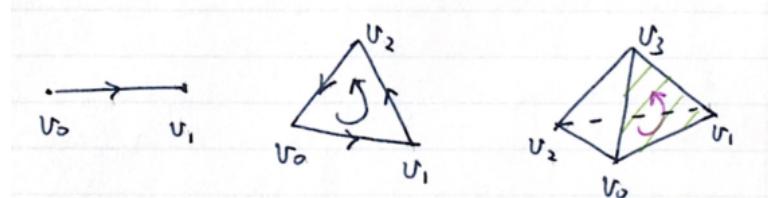
\therefore there are exactly two orientations on a k -simplex σ when $k \geq 1$.

e.g.



We denote such an oriented simplex by (v_0, \dots, v_k) , and the simplex with the opposite orientation by $-(v_0, \dots, v_k)$. An orientation on a k -simplex σ induces orientations on its $(k-1)$ -faces by $(-1)^i(v_0, \dots, \hat{v}_i, \dots, v_k)$

e.g.



The boundary of an oriented simplex (v_0, \dots, v_k) is

$$\partial(v_0, \dots, v_k) = \sum_{i=0}^k (-1)^i (v_0, \dots, \hat{v}_i, \dots, v_k).$$

$$e.g. \partial v_0 = 0, \quad \partial(v_0, v_1) = v_1 - v_0.$$

$$\partial(v_0, v_1, v_2) = (v_1, v_2) - (v_0, v_2) + (v_0, v_1).$$

Let K be a finite simplicial complex, (a collection of simplices s.t.

(i) if $\sigma \in K$, then all the faces of σ are in K .

(ii) any two simplices in K intersect properly: $\sigma \cap \tau$ is a common face of σ and τ .

)

Fix an orientation of each simplex in K , let

$$C_p(K) = \left\{ \sum n_\sigma \cdot \sigma \mid \sigma \in K \text{ an orientated } p\text{-simplex}, n_\sigma \in \mathbb{Z} \right\}$$

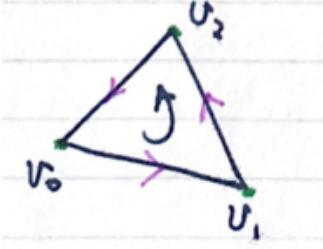
be the free abelian group generated by oriented p -simplices of K . Define the boundary homomorphism

$$\partial_p : C_p(K) \rightarrow C_{p-1}(K),$$

$$\sigma = (v_0, \dots, v_p) \mapsto \partial_p(\sigma) = \sum_{i=0}^p (-1)^i (v_0, \dots, \hat{v}_i, \dots, v_p)$$

Lemma 6.2. $\partial_p \cdot \partial_{p+1} : C_{p+1}(K) \rightarrow C_p(K) \rightarrow C_{p-1}(K)$ is the zero homomorphism.

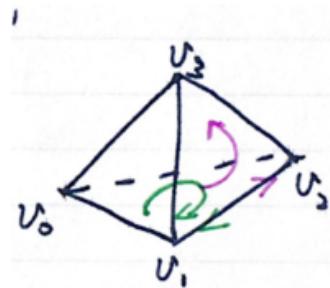
e.g.



$$\begin{aligned} & \partial_1 \cdot \partial_2(v_0, v_1, v_2) \\ &= \partial_1((v_1, v_2) - (v_0, v_2) + (v_0, v_1)) \\ &= v_2 - v_1 - (v_2 - v_0) + v_1 - v_0 = 0 \end{aligned}$$

Proof. Let $\sigma = (v_0, \dots, v_{p+1})$ be an oriented $(p+1)$ -simplex.

$$\begin{aligned} \partial^2(v_0, \dots, v_{p+1}) &= \partial \left(\sum_{i=0}^{p+1} (-1)^i (v_0, \dots, \hat{v}_i, \dots, v_{p+1}) \right) \\ &= \sum_{i=0}^{p+1} (-1)^i \left(\sum_{j=0}^{i-1} (-1)^j (v_0, \dots, \hat{v}_j, \dots, \hat{v}_i, \dots, v_{p+1}) \right. \\ &\quad \left. + \sum_{j=i+1}^{p+1} (-1)^{j-1} (v_0, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_{p+1}) \right) \\ &= 0 \end{aligned}$$



$\therefore (C_*(K), \partial) = \{C_p(K), \partial_p | p \geq 0\}$ is a chain complex of abelian groups.

$C_p(K) \supseteq \ker \partial_p = Z_p(K) \supseteq \text{Im} \partial_{p+1} = B_p(K)$ do not depend on the orientations on the simplices. The p -th homology group (同调群) of the simplices complex K is $H_p(K) = Z_p(K)/B_p(K)$.

If K is a finite simplicial complex, then $C_p(K)$ is a finitely generated free abelian group, so is $Z_p(K) = \ker \partial_p$. $\therefore H_p(K)$ is a finitely generated abelian group. \square

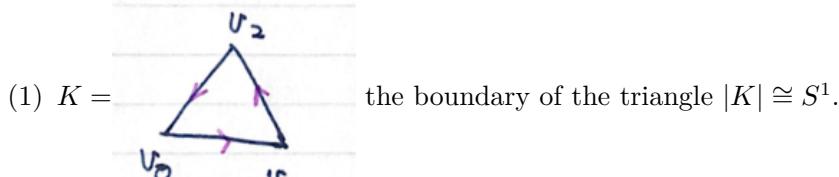
Theorem 6.3 (the structure of f.g. abelian groups). *Let A be a finitely generated abelian group, then $A \cong F \oplus T$, where $F \cong \mathbb{Z}^n$ is a free abelian group of rank r (the rank of A), T is the torsion subgroup of A , a finite abelian group.*

$$T = T_{p_i} \oplus \cdots \oplus T_{p_r}, p_1, \dots, p_r \in \mathbb{P}$$

$$T_{p_i} = \mathbb{Z}/p_i^{\alpha_i} \oplus \cdots \oplus \mathbb{Z}/p_i^{\alpha_{l_i}}$$

Definition 6.4. The rank of $H_p(K)$ is called the p -th Betti number of K , denoted by $\beta_p(K)$.

Examples:



$$\cdots \rightarrow C_2(K) \xrightarrow{\partial_2} C_1(K) \xrightarrow{\partial_1} C_0(K) \xrightarrow{\partial_0} 0 \\ = 0 \quad \cong \mathbb{Z}^3 \quad \cong \mathbb{Z}^3$$

basis of $C_1(K)$:

$$\begin{cases} e_1 = (v_0, v_1) \\ e_2 = (v_1, v_2) \\ e_3 = (v_2, v_0) \end{cases}$$

basis of $C_0(K)$: v_0, v_1, v_2 .

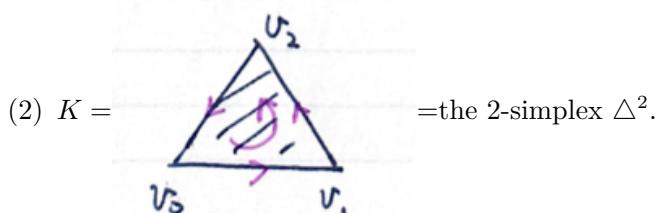
In which:

$$\begin{cases} e_1 = (v_0, v_1) \mapsto (v_1 - v_0) \\ e_2 = (v_1, v_2) \mapsto (v_2 - v_1) \\ e_3 = (v_2, v_0) \mapsto (v_0 - v_2) \end{cases}$$

$$H_0(K) = \frac{\ker \partial_0}{\text{Im } \partial_1} = \frac{C_0(K)}{\text{Im } \partial_1} \cong \mathbb{Z} \text{ generated by } [v_0] (= [v_1] = [v_2])$$

$$H_1(K) = \frac{\ker \partial_1}{\text{Im } \partial_2} = \ker \partial_1 = \mathbb{Z} \text{ generated by } [e_1 + e_2 + e_3].$$

$$H_p(K) = 0, p \geq 2.$$



$$\cdots \rightarrow C_3(K) \xrightarrow{\partial_3} C_2(K) \xrightarrow{\partial_2} C_1(K) \xrightarrow{\partial_1} C_0(K) \xrightarrow{\partial_0} 0 \\ \cong Z \quad \cong \mathbb{Z}^3$$

basis of $C_0(K) : e_1, e_2, e_3$.

basis of $C_2(K) : \sigma = (v_0, v_1, v_2) \mapsto e_1 + e_2 + e_3$

$$H_1(K) = \frac{\ker \partial_1}{\text{Im } \partial_2} = 0$$

$$H_2(K) = \frac{\ker \partial_2}{\text{Im } \partial_3} = \ker \partial_2 = 0.$$

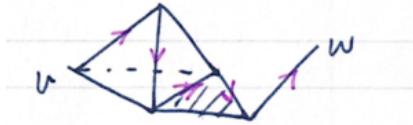
- (3) Two vertices v, w of a simplicial complex K are in the same (path) component of $|K|$. (Remark: $|K|$ is locally path connected.)

\Updownarrow

v is joined to w by an edge path, i.e. \exists vertices v_1, \dots, v_k s.t.

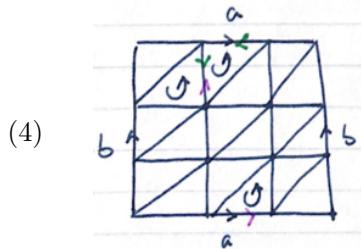
$(v, v_1), (v_1, v_2), \dots, (v_k, w)$ are edges in K .

\Updownarrow



v and w are homologous as 0-cycles $\partial((v, v_1) + (v_1, v_2) + \dots + (v_k, w)) = v - w$.

Theorem 6.5. $H_0(K) \cong \mathbb{Z}^r$, $r = \text{the number of path components of } |K|$.



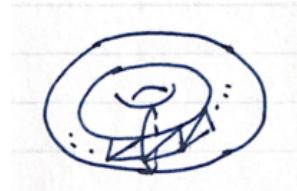
Let K be a triangulation of the torus T^2 , there is a compatible way to orient the 2-simplices $\{\sigma_1, \dots, \sigma_n\}$, s.t. the two induced orientations on an edge are opposite.

$$\dots \rightarrow C_3(K) \xrightarrow{\partial_3} C_2(K) \xrightarrow{\partial_2} C_1(K) \rightarrow \\ \parallel \\ 0$$

$$\sigma = \sigma_1 + \dots + \sigma_n \mapsto 0$$

and $\ker \partial_2 \cong \mathbb{Z}$ is generated by σ .

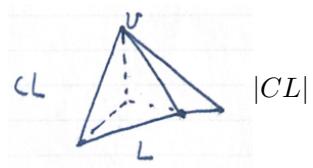
$\therefore H_2(K) = \ker \partial_2 \cong \mathbb{Z}, \sigma = [K]$ called the fundamental class.



The same holds for any closed orientable surface $S = nT^2 (n \geq 0)$, but not for non-orientable surfaces.

- (5) Let L be a finite simplicial complex, assume $L \subset \mathbb{R}^n$. let $v = (0, \dots, 0, 1) \in \mathbb{R}^{n+1}$. The cone on L is the simplicial complex

$$K = CL = \{v, (v, v_0, \dots, v_k), (v_0, \dots, v_k) | (v_0, \dots, v_k) \in L\}$$



$$|CL| = C|L| = |L| \times [0, 1] / |L| \times \{1\}$$

Define a homomorphism $d : C_p(K) \rightarrow C_{p+1}(K)$,

$$\begin{cases} \sigma = (v_0, \dots, v_p) \mapsto (v, v_0, \dots, v_p) & \sigma \in L \\ \sigma \mapsto 0 & \sigma \notin L. \end{cases}$$

Then $\partial d(\sigma) = \sigma - d(\partial(\sigma)) (p > 0)$

Check: if $\sigma \notin L$, then $\sigma = (v, v_0, \dots, v_{p-1})$.

$$\text{LHS} = 0, \partial\sigma = (v_0, \dots, v_{p-1}) + \sum_{i=0}^{p-1} (-1)^{i+1} (v, v_0, \dots, \hat{v}_i, \dots, v_{p-1}) \therefore d(\partial\sigma) = (v, v_0, \dots, v_{p-1}) = \sigma.$$

$$\therefore \text{RHS} = \sigma - d\partial(\sigma) = 0$$

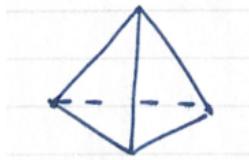
\therefore For any p -cycle z , we have $\partial d(z) = z - d\partial(z) = z$.

$\therefore z$ is null homologous. $\therefore H_p(K) = 0. \forall p > 0$.

And $h_0(K) \cong \mathbb{Z}$ since $|K| = C|L|$ is path connected.

- (6) Let Δ^{n+1} be the $(n+1)$ -simplex, $n > 0$. Δ^{n+1} is the cone on Δ^n . $\therefore H_p(\Delta^{n+1}) = 0, \forall p > 0$.

Let Σ^n be the simplicial complex consisting of all faces of Δ^{n+1} of dimension $\leq n$, (the “boundary” of Δ^{n+1}), then $|\Sigma^n| \cong S^n$.



And according to the diagram below,

$$H_p(\Sigma^n) \cong H_p(\Delta^{n+1}), 0 \leq p \leq n-1.$$

$$H_n(\Sigma^n) = Z_n(\Sigma^n) = \ker \partial_n = \ker \partial'_n \xrightarrow{(a)} \text{Im} \partial'_{n+1} \xrightarrow{[(b)]} \mathbb{Z}$$

$$(a) : H_n(\Delta^{n+1}) = \frac{\ker \partial'_n}{\text{Im} \partial'_{n+1}} = 0$$

$$(b) : 0 = H_{n+1}(\Delta^{n+1}) = \ker \partial'_{n+1}$$

$$\begin{array}{ccccccc} 0 & \longrightarrow & C_n(\Sigma^n) & \xrightarrow{\partial_n} & C_m(\Sigma^n) & \longrightarrow & \cdots \longrightarrow C_0(\Sigma^n) \longrightarrow 0 \\ & & \downarrow = & & \downarrow = & & \\ 0 & \longrightarrow & C_{n+1}(\Delta^{n+1})_{\cong \mathbb{Z}} & \xrightarrow{\partial'_{n+1}} & C_n(\Delta^{n+1}) & \xrightarrow{\partial'_n} & C_{n-1}(\Delta^{n+1}) \longrightarrow \cdots \longrightarrow C_0(\Delta^{n+1}) \longrightarrow 0 \end{array}$$

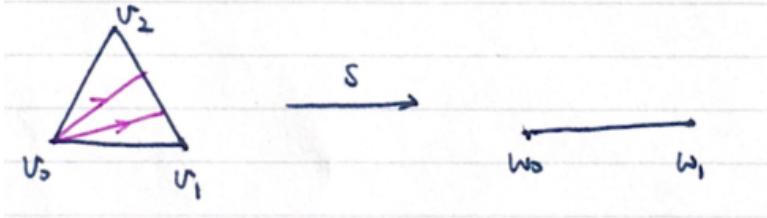
6.2 topological invariance

Definition 6.6. Let K, L be simplicial complexes, $|K|, |L|$ be their geometric realizations. A map $s : |K| \rightarrow |L|$ is called simplicial if it takes simplices of K linearly onto simplices of L .

Explanation:

$$\begin{array}{ll} \text{vertices} \rightarrow & \text{vertices} \\ \text{edges} \rightarrow & \text{vertices, or edges} \\ \text{triangles} \rightarrow & \text{vertices, edges or triangles} \\ \vdots & \end{array}$$

If $\sigma = (v_0, \dots, v_k)$ is a k -simplex of K , $x = \sum_{i=0}^k \lambda_i v_i \in \sigma$. then $s(x) = \sum_{i=0}^k \lambda_i s(v_i)$. We say s is non-degenerate on σ if $\dim s(\sigma) = \dim \sigma$.



e.g.

$$s(v_0) = w_0, s(v_1) = w_1, s(v_2) = w_2.$$

Remark: A simplicial map is continuous, since

- (i) continuous on each simplex;
- (ii) the glueing lemma.

Let $s : |K| \rightarrow |L|$ be a simplicial map, if induces a homomorphism between the chain groups

$$S_p : C_p(K) \rightarrow C_p(L), \sigma = (v_0, \dots, v_p) \mapsto \begin{cases} (s(v_0), \dots, s(v_p)) & \text{if } s \text{ is non-deg on } \sigma \\ 0 & \text{otherwise} \end{cases}$$

Lemma 6.7. The following diagram commutes.

$$\begin{array}{ccc} C_p(K) & \xrightarrow{\partial_p} & C_{p-1}(K) \\ \downarrow S_p & & \downarrow S_{p-1} \\ C_p(L) & \xrightarrow{\partial_p} & C_{p-1}(L) \end{array}$$

Proof. let $\sigma = (v_0, \dots, v_p)$, we need to show $\partial S_p(\sigma) = S_{p-1}(\partial\sigma)$

- (i) if s is non-degenerate on σ , then

$$\begin{aligned} \text{LHS} &= \partial(s(v_0), \dots, s(v_p)) \\ &= \sum_{i=0}^p (-1)^i (s(v_0), \dots, \widehat{s(v_i)}, \dots, s(v_p)) \\ \text{RHS} &= S_{p-1} \left(\sum_{i=0}^p (-1)^i (v_0, \dots, \widehat{v_i}, \dots, v_p) \right) \\ &= \sum_{i=0}^p (-1)^i (s(v_0), \dots, \widehat{s(v_i)}, \dots, s(v_p)) \end{aligned}$$

(ii) if s is degenerate on σ , suppose $s(v_j) = s(v_k), j < k$. Then LHS = 0,

$$\begin{aligned} \text{RHS} &= s \left(\sum_{i=0}^p (-1)^i (v_0, \dots, \hat{v}_i, \dots, v_p) \right) \\ &= (-1)^j s(v_0, \dots, \hat{v}_j, \dots, v_p) + (-1)^k s(v_0, \dots, \hat{v}_k, \dots, v_p) \\ &= \begin{cases} 0 & \text{if } s \text{ is generated on } (v_0, \dots, \hat{v}_j, \dots, v_p) \\ & \text{or } \dots \dots \dots (v_0, \dots, \hat{v}_i, \dots, v_p) \\ (-1)^j s(v_0, \dots, \hat{v}_j, \dots, v_p) & \\ +(-1)^k s(v_0, \dots, \hat{v}_k, \dots, v_p) & \text{otherwise} \end{cases} \end{aligned}$$

But $s(v_0, \dots, \hat{v}_j, \dots, v_p) = (-1)^{j-k-1} s(v_0, \dots, \hat{v}_k, \dots, v_p)$. \square

$\{S_p : C_p(K) \rightarrow C_p(L) | p \geq 0\}$ is called a chain map (链映射) between the chain complexes $(C_* K, \partial)$ and $(C_* (L), \partial)$.

$$\begin{array}{ccccccc} \cdots & \longrightarrow & C_{p+1}(K) & \xrightarrow{\partial} & C_p(K) & \xrightarrow{\partial} & C_{p-1}(K) \longrightarrow \cdots \\ & & \downarrow S_{p+1} & \curvearrowright & \downarrow S_p & \curvearrowright & \downarrow S_{p-1} \\ \cdots & \longrightarrow & C_{p+1}(L) & \xrightarrow{\partial} & C_p(L) & \xrightarrow{\partial} & C_{p-1}(L) \longrightarrow \cdots \end{array}$$

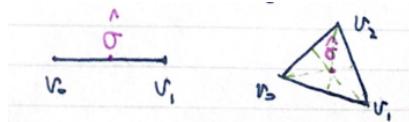
- (i) \forall cycle $z \in Z_p(K), \partial s(z) = s(\partial(z)) = 0, \therefore s(z) \in Z_p(L).$
- (ii) $\forall C \in C_{p+1}(K), \partial C \in B_p(K), s(\partial C) = \partial s(C) \in B_p(L), \therefore$

$$C_p(K) \supset Z_p(K) \supset B_p(K)$$

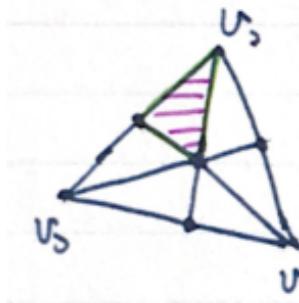
$$\begin{array}{ccc} & \downarrow S_p & \downarrow S_p & \downarrow S_p \\ C_p(L) & \supset & Z_p(L) & \supset & B_p(L) \end{array}$$

\therefore the chain map induces a homomorphism between the homology groups $S_{*p} : H_p(K) \rightarrow H_p(L), \forall p \geq 0$.

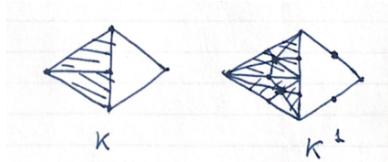
Let $\sigma = (v_0, \dots, v_k)$ be a k -simplex, its barycenter (重心) is $\hat{\sigma} = \frac{1}{k+1}(v_0 + \dots + v_k)$.



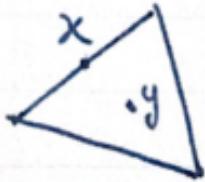
The barycentric subdivision (重心重分) of σ is a simplicial complex consisting of the simplices $(\hat{\sigma}_0, \dots, \hat{\sigma}_l)$, where $\sigma_0, \dots, \sigma_l$ are faces of σ , $\dim \sigma_0 < \dim \sigma_1 < \dots < \dim \sigma_l$.



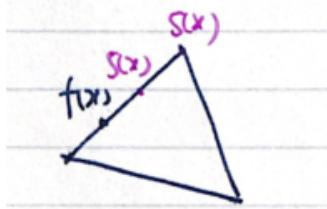
Let K be a simplicial complex, the barycentric subdivision of K is the union of the barycentric subdivisions of all its simplices, denoted by K^1 . Inductively we define the m -th barycentric subdivision of K by $K^m = (K^{m-1})^1$. Clearly $|K|^m = |K|$.



Definition 6.8. Let K be a simplicial complex, for any $x \in |K|$, there is a unique simplex σ of K s.t. x is the interior of σ . σ is called the carrier of x .



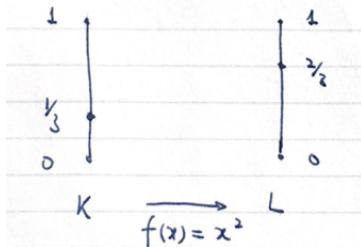
Definition 6.9. A simplicial map $s : |K| \rightarrow |L|$ is a simplicial approximation (单纯逼近) of a continuous map $f : |K| \rightarrow |L|$ if $s(x)$ lies in the carrier of $f(x)$ for each $x \in |K|$.



Simplicial approximation theorem

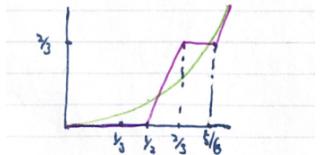
Let $f : |K| \rightarrow |L|$ be a continuous map, then $\exists m \in \mathbb{N}$ s.t. $f : |K^m| \rightarrow |L|$ has a simplicial approximation $s : |K^m| \rightarrow |L|$.

e.g.



There is no simplicial approximation of f .

\exists a simplicial approximation $s : |K^2| \rightarrow |L|$

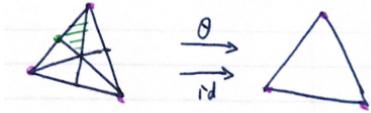


Let K^1 be the barycentric subdivision of K , we want to show

$H_p(K^1) \cong H_p(K)$ ($\forall p \geq 0$). Define a chain map $\{\eta_p : C_p(K) \rightarrow C_p(K^1) | p \geq 0\}$, $\eta_p : C_p(K) \rightarrow C_p(K^1), \sigma \mapsto \sum_i \sigma(i)$, the subdivision map.



There is a chain map $\{\theta_p : C_p(K^1) \rightarrow C_p(K) | p \geq 0\}$ induced by the simplicial approximation $\theta : |K^1| \rightarrow |K|$ of $\text{id} : |K^1| \rightarrow |K|$.



Theorem 6.10. $\eta_{p*} H_p(K^1) \rightarrow H_p(K)$ is an isomorphism, $(\eta_{p*})^{-1} = \theta_{p*}$.

\therefore barycentric subdivision does not change the homology groups of a simplicial complex.

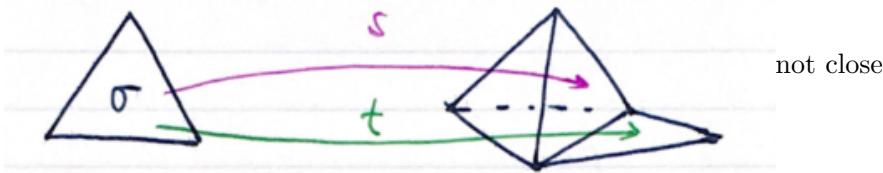
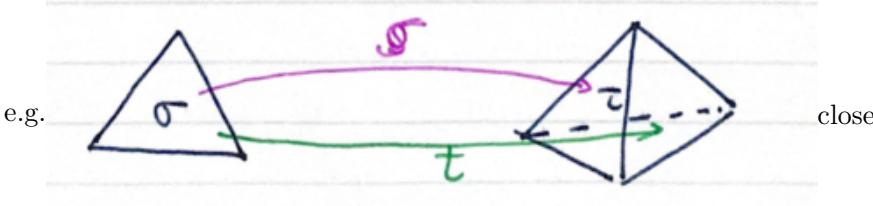
Now give a continuous map $f : |K| \rightarrow |L|$, let $s : |K^m| \rightarrow |L|$ be a simplicial approximation of f , we define $f_{*p} : H_p(K) \xrightarrow[\cong]{\eta_p^m} H_p(K^m) \xrightarrow{S_*} H_p(L)$ to be the homomorphism between homology groups induced by f . We need to show:

(i) f_* does not depend on the simplicial approximation

(ii) If f and g are homotopic, $f \simeq g : |K| \rightarrow |L|$, then $f_* = g_* : H_p(K) \rightarrow H_p(L), \forall p \geq 0$.

Facts:

(1) If $s, t : |K| \rightarrow |L|$ are “close” simplicial maps, (i.e. for each simplex $\sigma \in K, \exists$ a simplex $\tau \in L$, s.t. $s\sigma$ and $t\sigma$ are faces of τ).



Then $s_* = t_* : H_p(K) \rightarrow H_p(L)$ for all p .

(2) If $f, g : |K| \rightarrow |L|$ are homotopic maps, we may find a barycentric subdivision K^m and a sequence of simplicial maps $s_1, \dots, s_n : |K^m| \rightarrow |L|$, s.t. s_1 is a simplicial approximation of f , s_i and s_{i+1} are close for $1 \leq i \leq n-1$, and s_n is a simplicial approximation of g .

Theorem 6.11. (1) Any continuous map $f : |K| \rightarrow |L|$ induces a homomorphism $f_* : H_p(K) \rightarrow H_p(L) (\forall p \geq 0)$.

(2)

(i) $(\text{id})_* : H_p(K) \rightarrow H_p(K)$ is the identity homomorphism.

(ii) Given $f : |K| \rightarrow |L|, g : |L| \rightarrow |M|$, then $(g \circ f)_* = g_* \circ f_* : H_p(K) \rightarrow H_p(M), \forall p \geq 0$.

(3) If $f, g : |K| \rightarrow |L|$ are homotopic maps, then $f_* = g_* : H_p(K) \rightarrow H_p(L)$.

Corollary 6.12. If K and $|L|$ are homotopy equivalent, then $H_p(K) \cong H_p(L)$ for all p .

Proof. Let $f : |K| \rightarrow |L|$ be a homotopy equivalence, $g : |L| \rightarrow |K|$ be a homotopy inverse of F , then

$$H_p(K) \xrightarrow{f_*} H_p(L) \xrightarrow{g_*} H_p(K)$$

id

$$H_p(L) \xrightarrow{g_*} H_p(K) \xrightarrow{f_*} H_p(L)$$

id

$\therefore f_* : H_p(K) \rightarrow H_p(L)$ is an isomorphism. \square

Proof of the theorem :

(1) let $s : |K^{m+1}| \rightarrow |L|, t : |K^n| \rightarrow |L|$ be simplicial approximations of $f : |K| \rightarrow |L|, n \geq m$, $\theta : |K^n| \rightarrow |K^m|$ be a (iterated) simplicial approximation of id.

$$\begin{array}{ccc} |K^m| & \longrightarrow & |L| \\ \theta \uparrow & \nearrow t & \\ |K^n| & & \end{array}$$

Then both $s \circ \theta$ and t are simplicial approximations of f , since $s(x) \in$ the carrier of $f(x) \mapsto s(\theta(x)) \in$ the carrier of $f(x)$. $\therefore s \circ \theta$ and t are close ($\forall \theta \in K^n$, let $x \in \sigma, \tau \in L$ be the largest simplex of L containing the carrier of $f(x)$ as a face, then $s\theta(\sigma)$ and $t(\sigma)$ are both faces of τ).

$$\Rightarrow t_* = s_* \cdot \theta_* : H_p(K^n) \xrightarrow{\theta_*} H_p(K^m) \xrightarrow{s_*} H_p(L).$$

$$\begin{array}{ccccc} H_p(K) & \xrightarrow{\eta^m} & H_p(K^m) & \xrightarrow{s_*} & H_p(L) \\ \parallel & \nearrow \eta^{n-m} & \downarrow \theta & \nearrow t_* & \\ H_p(K) & \xrightarrow{\eta^n} & H_p(K^n) & & \end{array}$$

(2) (i) By definition.

(ii) Let $t : |L^n| \rightarrow |M|$ be a simplicial approximation of g , $s : |K^m| \rightarrow |L^n|$ be a simplicial approximation of $f : |K^m| \rightarrow |L^n|, \theta : |L^n| \rightarrow |L|$ be a (iterated) simplicial approximation of id, then $\theta \circ s$ is a simplicial approximation of $f : |K^m| \rightarrow |L|, t \circ s : |K^m| \rightarrow |M|$ is a simplicial approximation of $g \circ f : |K^m| \rightarrow |M|$.

$$\begin{array}{ccccc} H_p(K^m) & \xrightarrow{s_*} & H_p(L^n) & & \\ \uparrow \eta^m & & \theta_* \downarrow & \nearrow t_* & \\ (g \circ f)_* : H_p(K) & \xrightarrow{f_*} & H_p(L) & \xrightarrow{g_*} & H_p(M) \end{array}$$

i.e. $(g \circ f)_* = g_* \circ f_*$.

(3) By the above facts, we have simplicial maps

$s_1 \cdots s_n : |K^m| \rightarrow |L|$, inducing

$$\begin{array}{c} S_{n*} \\ \parallel \\ \vdots \\ \parallel \\ H_p(K) \xrightarrow{\eta^m} H_p(K^m) \xrightarrow{S_{1*}} H_p(L) \end{array}$$

$\therefore f_* = s_{1*} \circ \eta^m = s_{n*} \circ \eta^m = g_*$.

Applications:

(1)

Theorem 6.13 (invariance of dimension). \mathbb{R}^m is homeomorphic to \mathbb{R}^n if and only if $m = n$.

Proof. Let $h : \mathbb{R}^m \rightarrow \mathbb{R}^n$ be a homeomorphism, then $S^{m-1} \simeq \mathbb{R}^m - \{0\} \cong r^n - \{h(0)\} \simeq S^{n-1}$.

$$\text{But } H_p(S^k) \cong \begin{cases} \mathbb{Z} & p = 0, k \\ 0 & \text{otherwise} \end{cases}$$

□

(2) Brower fixed-point Theorem

A continuous map $f : D^n \rightarrow D^n$ has a fixed point.

Proof. If there is a fixed-point free map $f : D^n \rightarrow D^n$, then we may construct a retraction $r : D^n \rightarrow S^{n-1}$, $\text{id} : S^{n-1} \xrightarrow{i} D^n \xrightarrow{r} S^{n-1}$ induce

$$\text{id} : H_{n-1}(S^{n-1}) \xrightarrow{i_*} H_{n-1}(D^n) \xrightarrow{r_*} H_{n-1}(S^{n-1}) \cong \mathbb{Z}$$

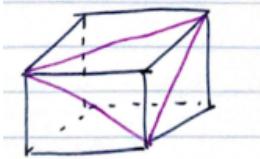
a contradiction. □

(3) Let K be a finite simplicial complex, the Euler characteristic

$$\chi(K) = \sum_{q=0}^n (-1)^q (\# \text{ of } q\text{-simplices of } K).$$

Euler-Poincaré Formula : $\chi(K) = \sum_{q=0}^n (-1)^q \beta_q$, where $\beta_q = \text{rank } H_q(K)$ is the q -th Betti number of K .

Corollary 6.14. Let $P \subset \mathbb{R}^3$ be a convex polytope, $V = \# \text{ of vertices}$, $E = \# \text{ of edges}$, $F = \# \text{ of faces}$, then $V_E + F = 2$.



Proof.

$$P \cong S^2, \chi(S^2) = 2.$$

$C_p(K; \mathbb{Q})$ = the \mathbb{Q} -vector space generated by oriented p -simplices of K .

$$\partial_p : C_p(K; \mathbb{Q}) \rightarrow C_{p-1}(K; \mathbb{Q}), \sigma = (v_0, \dots, v_p) \mapsto \sum_{i=0}^p (-1)^i (v_0, \dots, \hat{v}_i, \dots, v_p).$$

Then $\{(C_p(K; \mathbb{Q}), \partial_p) | p \geq 0\}$ is a chain complex of \mathbb{Q} -vector spaces, its homology group $H_p(K; \mathbb{Q}) = \ker \partial_p / \text{Im } \partial_{p+1}$ is called the p -th homology group of K with rational coefficients, a \mathbb{Q} -vector space.

$$\dim C_p(K; \mathbb{Q}) = \# \text{ of } p\text{-simplices of } K.$$

A standard fact in linear algebra \Rightarrow

$$\sum_{p=0}^n (-1)^p \dim_{\mathbb{Q}} C_p(K; \mathbb{Q}) = \sum_{p=0}^n (-1)^p \dim \mathbb{Q} H_p(K; \mathbb{Q})$$

□

Lemma 6.15. $\dim_{\mathbb{Q}} H_p(K; \mathbb{Q}) = \text{rank } H_p(K)$.

Proof. Let A be an abelian group, $A \otimes_{\mathbb{Z}} \mathbb{Q} = \{\sum a_i \otimes r_i | a_i \in A, r_i \in \mathbb{Q}\}$ with relations:

$$\begin{cases} na \otimes r = a \otimes nr, n \in \mathbb{Z} \\ a \otimes r + a' \otimes r = (a + a') \otimes r, a \otimes r + a \otimes r' = a \otimes (r + r') \end{cases}$$

□

Fact: $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$, a short exact sequence of abelian group, then $0 \rightarrow A \otimes \mathbb{Q} \xrightarrow{f \otimes \text{id}} B \otimes \mathbb{Q} \xrightarrow{g \otimes \text{id}} C \otimes \mathbb{Q} \rightarrow 0$ is a short exact sequence. ($- \otimes \mathbb{Q}$ is an exact functor)

Proof. Assume $f\left(\sum_i a_i \otimes r_i\right) = 0$, i.e. $\sum_i f(a_i) \otimes r_i = 0$.

Let $r_i = \frac{m_i}{n_i}, \prod_j \frac{n_j}{n_i} = l_i$, then $\sum_i f(a_i) \otimes \frac{m_i}{n_i} = 0 \Rightarrow \sum_i m_i l_i f(a_i) \otimes 1 = 0$.

$\therefore f\left(\sum_i m_i l_i a_i\right) = 0, \therefore \sum_i m_i l_i a_i = 0 \Leftrightarrow \sum_i a_i \otimes \frac{m_i}{n_i} = 0$
 $\therefore f \otimes \text{id}$ is injective.

Now from $0 \rightarrow Z_p(K) \rightarrow C_p(K) \xrightarrow{\partial_p} B_{p-1}(K) \rightarrow 0$, we have $0 \rightarrow Z_p(K) \otimes \mathbb{Q} \rightarrow C_p(K) \otimes \mathbb{Q} \xrightarrow{\partial_p \otimes \text{id}} B_{p-1}(K) \otimes \mathbb{Q} \rightarrow 0$.

$\therefore Z_p(K) \otimes \mathbb{Q} = \ker(\partial_p \otimes \text{id}), B_{p-1}(K) \otimes \mathbb{Q} = \text{Im}(\partial_p \otimes \text{id})$.

From $0 \rightarrow B_p(K) \rightarrow Z_p(K) \rightarrow H_p(K) \rightarrow 0$, we have

$$0 \rightarrow B_p(K) \otimes \mathbb{Q} \longrightarrow Z_p(K) \otimes \mathbb{Q} \rightarrow H_p(K) \otimes \mathbb{Q} \rightarrow 0 \\ = \text{Im}(\partial_{p+1} \otimes \text{id}) \qquad \qquad \qquad = \ker(\partial_p \otimes \text{id})$$

$\therefore H_p(K) \otimes \mathbb{Q} \cong \ker(\partial_p \otimes \text{id}) / (\partial_{p+1} \otimes \text{id}) = H_p(k; \mathbb{Q})$,
 at the same time RHS $\cong (\mathbb{Z}^\beta \oplus T) \otimes \mathbb{Q} \cong \mathbb{Q}^{\beta_p}$

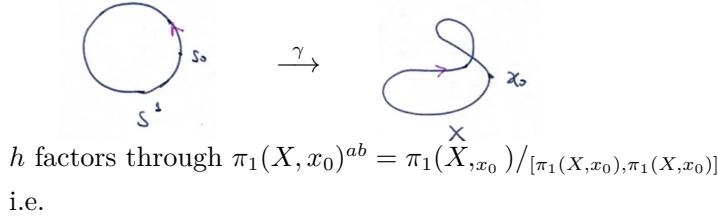
□

(4) the Hurewicz homomorphism

$$h : \pi_1(X, x_0) \longrightarrow H_1(X)$$

$$[\gamma : (S^1, s_0) \longrightarrow (X, x_0)] \longrightarrow \gamma_*[S^1]$$

where $[S^1] \in H_1(S^1) \cong \mathbb{Z}$ a generator



$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow{h} & H_1(X) \\ \downarrow & \nearrow \bar{h} & \\ \pi_{X, x_0}/[\pi_1(X, x_0), \pi_1(X, x_0)] & & \end{array}$$

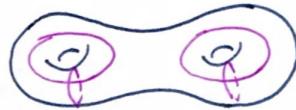
Theorem 6.16. $\bar{h} : \pi_1(X, x_0)/[\pi_1(X, x_0), \pi_1(X, x_0)] \longrightarrow H_1(X)$ is an isomorphism.

Example

$$S = gT^2 \text{ closed orientable surface of genus } g \\ \pi_1(S) \cong \left\langle a_1, b_1, \dots, a_g, b_g \mid \prod_{i=1}^g [a_i, b_i] = 1 \right\rangle$$

$H_1(S) \cong \pi_1(S)^{ab} = \mathbb{Z}^{2g}$ with basis $\{a_1, b_1, \dots, a_g, b_g\}$.

p	0	1	2
$H_p(S)$	\mathbb{Z}	\mathbb{Z}^{2g}	\mathbb{Z}



$$\chi(S) = 1 - 2g + 1 = 2 - 2g$$

$S = mP^2$ closed non-orientable surface of genus m.

$$\pi_1(S) \cong \left\langle a_1, \dots, a_m \mid \prod_{i=1}^m a_i^2 = 1 \right\rangle$$

$$H_1(S) \cong \pi_1(S)^{ab} = \mathbb{Z}^{m-1} \oplus \mathbb{Z}/_2$$

p	0	1	2
$H_p(S)$	\mathbb{Z}	$\mathbb{Z}^{m-1} \oplus \mathbb{Z}/_2$	\mathbb{Z}