Notes of Topology

lectured by Su Yang, compiled by Qiuqiuren and 查无此人

7/21/2024

前言

1. 如有发现或者看图片或排版不顺眼的地方, 请务必联系以下两位中的一位:

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(请在主题中注明 “拓扑 Latex 讲义挑刺”)

(尤其是后者, 这种情况很常见! 因为抄这篇讲义的时候两位作者都是 LaTeX 新手, 难免会有将就和疏漏, 图画的不好看的肯定很多 (绝大多数都是手写版本讲义直接截图). 写完以后我们也没有太多精力把整个讲义排查一遍, 所以请读者们多多挑刺!)

在此, 我们对您的支持表示衷心感谢!

2. 本讲义整 (chao) 理 (xie) 自苏阳老师的拓扑基础课程手写讲义扫描件, 感谢他的精彩授课与辛勤付出! 苏阳老师人很好, 欢迎大家选修他的课!

3. 参考文献 (这里形式上就随便了一点, 毕竟不是学术文献)(并没有对着正文一个一个找, 所以必然有疏漏, 如有疏漏请联系上面的邮箱):

1) Munkres, Topology

2) Jänich, Topology

3) Armstrong, Basic Topology

4) Singer and Thorpe, Lecture Notes on Elementary Topology and Geometry

5) 尤承业, 基础拓扑学讲义

6) Allen Hatcher, Algebraic Topology

4. 讲义整 (chao) 理 (xie) 开始日期:2024 年 7 月 21 日,

初稿完稿日期:2024 年 8 月 2 日,

5. 作者尝试用 edge 自带的 pdf 浏览器翻阅此 pdf 发现非常卡顿.

在此推荐一款 pdf 阅读器:Sumatra PDF, 在作者的电脑上阅览此 pdf 较为流畅 (夹带私货.jpg)

Contents

Chapter 1

Topological spaces and continuous maps

1.1 topological spaces

Definition 1.1. *A topology (拓扑) on a set X is a collection $\mathcal{T}$ of subsets of X having the following properties:*

(i) $\varnothing$ and X are in $\mathcal{T}$.

(ii) The union of the elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

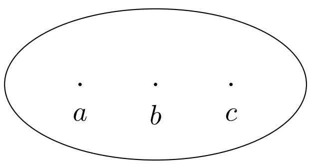
(iii) The intersection of the element any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

A set X together with a topology $\mathcal{T}$ is called a topological space (拓扑空间), denoted by $(X, \mathcal{T})$ or X. Elements in $\mathcal{T}$ are called open sets (开集) of X, usually denoted by $U \subset X$.

Examples:

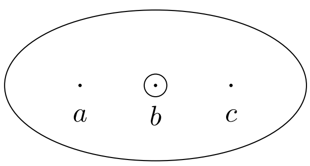
(1) $X = \{a, b, c\}$

(i)



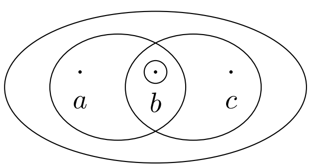
$\mathcal{T} = \{\varnothing, X\}$

(ii)



$\mathcal{T} = \{\varnothing, X, \{b\}\}$

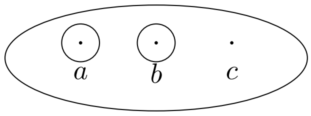
(iii)



$\mathcal{T} = \{\varnothing, X, \{a, b\}, \{b, c\}, \{b\}\}$

(iv) $\mathcal{T}$= all subsets of X.

non-example:



(2) X a set

  (i) $\mathcal{T} = \{\varnothing, X\}$:the trivial (平凡) topology.

  (ii) $\mathcal{T} =$ all subsets of X: the discrete (离散) topology.

(3) X a set, $\mathcal{T}_f = \{U \subset X | X - U \text{ is finite or } \varnothing\} \cup \{\varnothing\}$.

Then $\mathcal{T}_f$ is a topology on X, called the finite complement topology (余有限拓扑, 有限补拓扑).

Check:

  (i) $X \in \mathcal{T}_f, \varnothing \in \mathcal{T}_f$.

  (ii) $U_\alpha \in \mathcal{T}_f$ a family of open sets.
  $X - \bigcup\limits_\alpha U_\alpha = \bigcap\limits_\alpha (X - U_\alpha)$ is finite.
  $\therefore \bigcup\limits_\alpha \in \mathcal{T}_f$.

  (iii) $U_1, \ldots, U_n \in \mathcal{T}_f$.
  $X - \bigcap\limits_{i=1}^n U_i = \bigcup\limits_{i=1}^n (X - U_i)$ is finite.
  $\therefore \bigcap\limits_{i=1}^n U_i \in \mathcal{T}_f$.

(4) X a set. $\mathcal{T}_c = \{U \subset X | X - U \text{ is countable or finite or } \varnothing\} \cup \{\varnothing\}$, then $\mathcal{T}_c$ is a topology on X.

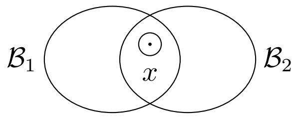
Definition 1.2. *Let $\mathcal{T}, \mathcal{T}'$ be two topologies on a set X. If $\mathcal{T}' \subset \mathcal{T}$, we say that $\mathcal{T}'$ is finer (细致) or larger than $\mathcal{T}$. $\mathcal{T}$ is coarser (粗糙) (or smaller) than $\mathcal{T}'$.$\mathcal{T}'$ is comparable with $\mathcal{T}$ if either $\mathcal{T}' \supset \mathcal{T}$ or $\mathcal{T} \subset \mathcal{T}'$.*

1.2 basis for a topology

Definition 1.3. *Let X be a set. A* <u>*basis for a topology*</u> *(拓扑基) on X is a collection $\mathscr{B}$ of subsets of X (called* <u>*basis elements*</u> *(基中的元素)) s.t.*

(1) for $\forall x \in X$, $\exists \mathcal{B} \in \mathscr{B}$ s.t. $x \in \mathcal{B}$.

(2) if $x \in \mathcal{B}_1 \cap \mathcal{B}_2$,$\mathcal{B}_1, \mathcal{B}_2 \in \mathscr{B}$,then $\exists \mathcal{B}_3 \in \mathscr{B}$ s.t. $x \in \mathcal{B}_3 \cap \mathcal{B}_1 \cap \mathcal{B}_2$.



Lemma 1.4. *If $\mathscr{B}$ satisfies these two conditions, let $\mathcal{T}$ be the collection of all unions of elements of $\mathscr{B}$,i.e.*

$$\mathcal{T} = \left\{ U \subset X \,\middle|\, U = \bigcup_\alpha \mathcal{B}_\alpha, \mathcal{B}_\alpha \in \mathscr{B} \right\}.$$

Then $\mathcal{T}$ is a topology on X, called the <u>*topology generated by $\mathscr{B}$*</u> *(由 $\mathscr{B}$ 生成的拓扑).*

Proof. (1) $\varnothing \in \mathcal{T}$,$X = \bigcup_{\mathcal{B} \in \mathscr{B}} \mathcal{B}$.

(2) if $U_\alpha \in \mathcal{T}$, then $\bigcup_\alpha U_\alpha \in \mathcal{T}$ by definition.

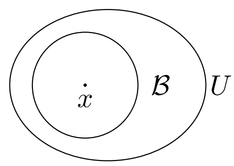
(3) If $U_1 = \bigcup_\alpha U_\alpha$,$U_2 = \bigcup_\beta \mathcal{B}'_\beta$, then

$$\begin{aligned} U_1 \cap U_2 &= \left(\bigcup_\alpha \mathcal{B}_\alpha \right) \bigcap \left(\bigcup_\beta \mathcal{B}'_\beta \right) \\ &= \bigcup_{\alpha,\beta} \underbrace{(\mathcal{B}_\alpha \cap \mathcal{B}'_\beta)}_{\substack{\parallel \\ \bigcup_\gamma \mathcal{B}''_\gamma}} \qquad \text{by (2)} \end{aligned}$$

$\therefore U_1 \cap U_2 \in \mathcal{T}$.

$\square$

An equivalent description of $\mathcal{T}$:



$$U \in \mathcal{T} \Leftrightarrow \forall x \in U, \exists \mathcal{B} \in \mathscr{B}, \text{s.t.} x \in \mathcal{B} \subset U.$$

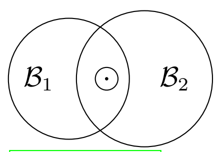
Examples:

(1) $\mathscr{B} = \{(a,b) \subset \mathbb{R} | a, b \in \mathbb{R}\}$

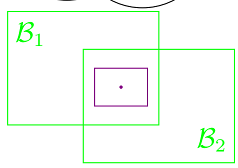
The topology generated by $\mathscr{B}$ is the standard topology on the real line $\mathbb{R}$.

(2)

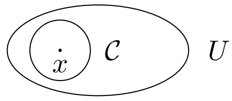
$\mathscr{B} = \{\text{all open discs in } \mathbb{R}^2\}$ generates the standard topology on $\mathbb{R}^2$



$\mathscr{B}' = \{\text{all open rectangular regions in } \mathbb{R}^2\}$ generates the standard topology on $\mathbb{R}^2$



Lemma 1.5. *$(X, \mathcal{T})$ a topological space, $\mathscr{C}$ a collection of open sets, s.t. for any open set U, and each $x \in U$, $\exists \mathcal{C} \in \mathscr{C}$, s.t. $x \in \mathcal{C} \subset U$. Then $\mathscr{C}$ is a basis for the topology $\mathcal{T}$ of X.*



Proof. (i) $\mathscr{C}$ is a basis:

(1) $\forall x \in X$, since X is open, $\exists \mathcal{C} \in \mathscr{C}$, s.t. $x \in \mathcal{C} \subset X$ $\quad \therefore \bigcup_{\mathcal{C} \in \mathscr{C}} \mathcal{C} = X$.

(2) Given $\mathcal{C}_1, \mathcal{C}_2 \in \mathscr{C}, \forall x \in \mathcal{C}_1 \cap \mathcal{C}_2$, since $\mathcal{C}_1 \cap \mathcal{C}_2$ is open, $\exists \mathcal{C} \in \mathscr{C}$ s.t. $s \in \mathcal{C} \subset \mathcal{C}_1 \cap \mathcal{C}_2$. $\therefore \mathscr{C}$ is a basis.

(ii) Let $\mathcal{T}'$ be the topology generated by $\mathscr{C}$, we need to show $\mathcal{T}' = \mathcal{T}$.

(1) For any open set $U \in \mathcal{T}, \forall x \in U, \exists C \in \mathscr{C}$, s.t. $x \in \mathcal{C} \subset U. \therefore U$ is a union of open sets $\mathcal{C}$ in $\mathscr{C}$, i.e. $U \in \mathcal{T}'$.

(2) Any open set $U \in \mathcal{T}'$ is a union of open sets $\mathcal{C}$ in $\mathscr{C}. \therefore U \in \mathcal{T}$.

$\square$

Lemma 1.6. *Let $\mathscr{B}$ and $\mathscr{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively, on X. Then the followings are equivalent.*

(1) $\mathcal{T}'$ is finer than $\mathcal{T}$, i.e. $\mathcal{T}' \supset \mathcal{T}$

(2) For any $\mathcal{B} \in \mathscr{B}, x \in \mathcal{B}, \exists \mathcal{B} \in \mathscr{B}'$ s.t. $x \in \mathcal{B}' \subset \mathcal{B}$.

Proof.
(2)⇒(1): $\forall U \in \mathcal{T}, \forall x \in U, \exists \mathcal{B} \in \mathscr{B}$ s.t. $x \in \mathcal{B} \subset U$.
$\therefore \exists \mathcal{B}' \in \mathscr{B}'$ s.t.$x \in \mathcal{B}' \subset U$. $\therefore U$ is a union of elements in $\mathcal{B}'$,i.e. $U \in \mathcal{T}'$.
(1)⇒(2): Given $\mathcal{B} \in \mathscr{B}, x \in \mathcal{B}$, now $\mathcal{B} \in \mathcal{T} \subset \mathcal{T}'$. $\therefore \mathcal{B} = \bigcup_\alpha \mathcal{B}'_\alpha$.
$\therefore x \in \mathcal{B}'_\alpha \subset \mathcal{B}$ for some $\mathcal{B}'_\alpha \in \mathscr{B}'$. $\square$

Examples:

(1) $\mathscr{B} = \{\text{open discs in } \mathbb{R}^2\}$, $\mathscr{B}' = \{\text{open rectangular regions in } \mathbb{R}^2\}$,

Then $\mathscr{B}$ and $\mathscr{B}'$ generate the same topology.

(2) $\mathscr{B} = \{(a,b) \subset \mathbb{R} | a, b \in \mathbb{R}\}$ generates $\mathcal{T}$, the standard topology on $\mathbb{R}$.

$\mathscr{B}' = \{[a,b) \subset \mathbb{R} | a, b \in \mathbb{R}\}$ generates $\mathcal{T}'$, the lower limit topology (下极限拓扑) on $\mathbb{R}$, denoted by $\mathbb{R}_l$.

$\mathscr{B}'' = \{(a,b) \subset \mathbb{R} | a, b \in \mathbb{R}\} \cup \{(a,b) - K : (a, b \in \mathbb{R})\}$ where $K = \left\{ \frac{1}{n} \middle| n \in \mathbb{Z}_+ \right\}$ generates $\mathcal{T}''$, the K-topology on $\mathbb{R}$,denoted by $\mathbb{R}_K$.

Lemma 1.7. *$\mathbb{R}_l$ and $\mathbb{R}_K$ are strictly finer than $\mathbb{R}$.*

$\mathbb{R}_l$ and $\mathbb{R}_K$ are not comparable with one another.

Lemma 1.8. *Let $\mathscr{S}$ be a collection of subsets of X s.t. the union of elements in $\mathscr{S}$ equals X. Let $\mathscr{B}$ be the collection of all finite intersections of elements of $\mathscr{S}$, i.e.*

$$\mathscr{B} = \left\{ \bigcup_{i=1}^{n} \mathcal{S}_i \middle| \mathcal{S}_i \in \mathscr{S}, n \in \mathbb{N}_+ \right\}$$

Then $\mathscr{B}$ is a basis, $\mathscr{S}$ is called a subbasis (子基,wikipedia 上翻译为"准基") of the topology $\mathcal{T}$ generated by $\mathscr{B}$.

Proof. (1) $\bigcup\limits_{\mathcal{S} \in \mathscr{S}} = X \Rightarrow \bigcup\limits_{\mathcal{B} \in \mathscr{B}} \mathcal{B} = X$.

(2) For any $\mathcal{B}_1, \mathcal{B}_2 \in \mathscr{B}, x \in \mathcal{B}_1 \cap \mathcal{B}_2$,assume $\mathcal{B}_1 = \bigcap\limits_{i=1}^{n} \mathcal{S}_i$,$\mathcal{B}_2 = \bigcap\limits_{j=1}^{m} \mathcal{S}'_j$, then $\mathcal{B}_1 \cap \mathcal{B}_2 = \bigcap_{i=1}^{n} \mathcal{S}_i \cap \bigcap\limits_{j=1}^{m} \mathcal{S}'_j \in \mathscr{B}$. □

Remark: Topology $\mathcal{T}$:

in terms of a basis $\mathscr{B}$: open set $U = \bigcup\limits_{\alpha} \mathcal{B}_\alpha$.

in terms of a subbasis $\mathscr{S}$: open set $U = \bigcup\limits_{\alpha} \left(\bigcap\limits_{i=1}^{n_\alpha} \mathcal{S}_i^\alpha \right)$.

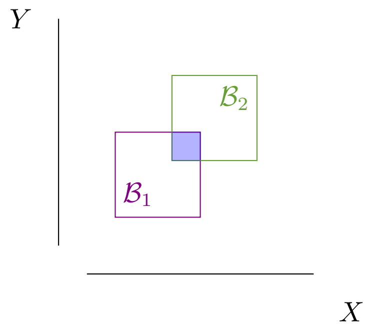
1.3 the product topology on $X \times Y$

Definition 1.9. *X, Y topological spaces. The product topology (乘积拓扑) on $X \times Y$ is the topology generated by the basis $\mathscr{B} = \{U \times V | U \subset X, V \subset Y \text{ open}\}$.*

Check: $\mathscr{B}$ is a basis.

(1) $\varnothing \in \mathscr{B}, X \times Y \in \mathscr{B}$.

(2) $\forall \mathcal{B}_1 = U_1 \times V_1, \mathcal{B}_2 = U_2 \times V_2 \in \mathscr{B}, \mathcal{B}_1 \cap \mathcal{B}_2 = (U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \times V_2) \in \mathscr{B}$.

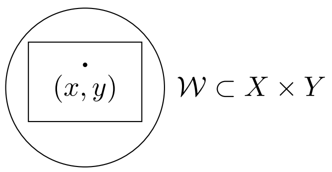


Theorem 1.10. *Let $\mathscr{B}_X, \mathscr{B}_Y$ be a basis for the topology on X and Y respectively. Then the collection*

$$\mathscr{D} = \{\mathcal{B}_1 \times \mathcal{B}_2 : \mathcal{B}_1 \in \mathscr{B}_X, \mathcal{B}_2 \in \mathscr{B}_2\}.$$

is a basis for the product topology of $X \times Y$.

Proof. Recall: $\mathscr{D}$ is a basis iff for any open set $\mathcal{W} \subset X \times Y$,$(x,y) \in \mathcal{W}$,$\exists \mathcal{B}_1, \mathcal{B}_2$ s.t. $(x,y) \in \mathcal{B}_1 \times \mathcal{B}_2 \subset \mathcal{W}$. $\because \mathcal{W}$ is open, $\therefore \exists\ U \subset X, V \subset Y$ open,



s.t. $(x,y) \in U \times V \subset \mathcal{W}$.$\exists \mathcal{B}_1 \in \mathscr{B}_X, \mathcal{B}_2 \in \mathscr{B}_Y$,s.t. $x \in \mathcal{B}_1 \subset U, y \in \mathcal{B}_2 \subset V$.$\therefore (x,y) \in \mathcal{B}_1 \times \mathcal{B}_2 \subset U \times V \subset \mathcal{W}$. □

Example: The standard topology on $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ is the product topology of $\mathbb{R}$ (with the standard topology) with itself. A basis is $\mathscr{B} = \{$open rectangular regions$\}$, another basis is $\mathscr{B}' = \{$open discs$\}$.

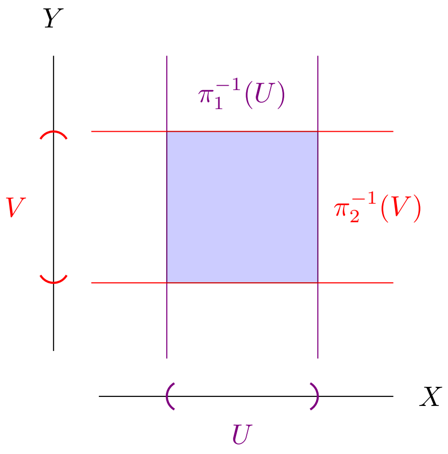
Let
$$\begin{aligned} \pi_1 :\ & X \times Y \to X, & \pi_2 :\ & X \times Y \to Y \\ & (x,y) \mapsto x & & (x,y) \mapsto y \end{aligned}$$

be the projections (投影) of $X \times Y$ onto its first and second factors, respectively. Let $U \subset X, V \subset Y$ be open sets, then $\pi_1^{-1}(U) = U \times Y$,$\pi_2^{-1}(V) = X \times V$ are open sets in $X \times Y$.$\pi_1^{-1}(U) \cap \pi_2^{-1}(V) = U \times V$.

Theorem 1.11. *The collection*

$$\mathcal{S} = \{\pi_1^{-1}(U) | U \text{ is open in } X\} \cup \{\pi_2^{-1}(V) | V \text{ is open in } Y\}$$

is a subbasis for the product topology on $X \times Y$.



Proof. Let $\mathcal{T}$ be the product topology on $X \times Y$,

$\mathcal{T}'$ be the topology generated by $\mathscr{S}$.

We need to show $\mathcal{T} = \mathcal{T}'$.

(i) every element of $\mathscr{S}$ belongs to $\mathcal{T}$,$\therefore$ open sets in $\mathcal{T}'$ belong to $\mathcal{T}$, i.e. $\mathcal{T}' \subset \mathcal{T}$.

(ii) $\underbrace{U \times V}_{\text{a basis element for } \mathcal{T}} = \pi_1^{-1}(U) \cap \pi_2^{-1}(V) \in \mathcal{T}'$, $\therefore \quad \mathcal{T} \subset \mathcal{T}'$. $\square$

1.4 the subspace topology

Definition 1.12. *Let X be a topological space with topology $\mathcal{T}$. If Y is a subset of X, the collection*

$$\mathcal{T}_Y = \{Y \cap U | U \in \mathcal{T}\}$$

is a topology on Y, called the <u>*subspace topology*</u> *(子空间拓扑). $(Y, \mathcal{T}_Y)$ is called a subspace of $(X, \mathcal{T})$.*

Check: $\mathcal{T}_Y$ is a topology on Y:

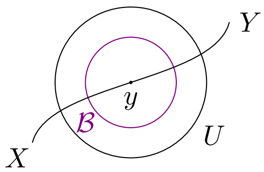
(1) $\varnothing, Y \in \mathcal{T}_Y$.

(2) $\bigcup_\alpha (Y \cap U_\alpha) = Y \cap \left(\bigcup_\alpha U_\alpha\right) \in \mathcal{T}_Y$.

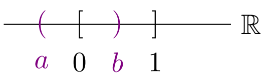
(3) $(Y \cap U_1) \cap (Y \cap U_2) = Y \cap (U_1 \cap U_2) \in \mathcal{T}_Y$.

Lemma 1.13. *If $\mathscr{B}$ is a basis for the topology of X, then the collection $\mathscr{B}_Y = \{\mathcal{B} \cap Y | \mathcal{B} \in \mathscr{B}\}$ is a basis for the subspace topology $\mathcal{T}_Y$.*

Proof. For any $y \in Y \cap U \in \mathcal{T}_Y$,$\exists \mathcal{B} \in \mathscr{B}$ s.t. $y \in \mathcal{B} \subset U$. $\therefore y \in \mathcal{B} \cap Y \subset U \cap Y$. □



Remark: An open set in Y is not necessarily open in X.e.g.$Y = [0, 1] \subset \mathbb{R}$ for $a < 0, b \in (0, 1)$.



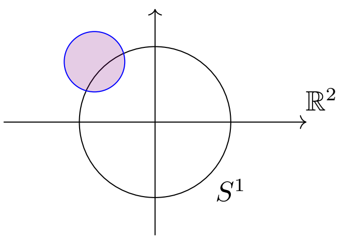
$(a, b) \cap Y = [0, b)$ not open in $\mathbb{R}$.

Lemma 1.14. *Let $Y \subset X$ be a subspace. If U is open in Y, and Y is open in X,then U is open in X.*

Proof. $U = V \cap Y$ for some $V \subset X$ open, and Y is open in X,$\therefore U = V \cap Y$ is open in X. □

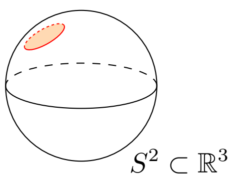
Examples:

(1) $S^1 = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\} \subset \mathbb{R}^2$ is the unit circle (1-sphere),with the subspace topology of $\mathbb{R}^2$(with the standard topology).



A basis of the topology is the collection of open arcs.

(2) the n-sphere $S^n (\subset \mathbb{R}^{n+1}) = \left\{ (x_1, x_2, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} \middle| \sum_{i=1}^{n+1} x_i^2 = 1 \right\}$.

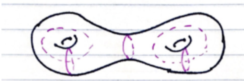


(3) the 2-torus  $T^2 \subset \mathbb{R}^3$

Question : Can you define the n-torus?

Tips: $nT^2 = \underbrace{T^2 \# T^2 \# \cdots \# T^2}_{nT^{2\prime}s}$

The closed orientable surface of genus 2

 $S_2 \subset \mathbb{R}^3$

Question : Can you define the surface of genus g?

Theorem 1.15. *Let $A \subset X, B \subset Y$ be subspaces. Then the product topology on $A \times B$ is the same as the topology $A \times B$ inherits as a subspace of $X \times Y$.*

Proof. Let $U \subset X, V \subset Y$ be open sets, then

$$\underbrace{\underset{\substack{\uparrow \\ \text{a basis element for } X \times Y}}{(U \times V) \cap (A \times B)}}_{\substack{\text{a basis element for} \\ \text{the subspace topology}}} = \underbrace{\underset{\substack{\uparrow \qquad\qquad \uparrow \\ \text{open sets in } A, B}}{(U \cap A) \times (V \cap B)}}_{\substack{\text{a basis element for} \\ \text{the product topology}}}$$

□

Theorem 1.16. *X, Y topological spaces, $y \in Y$. Identify $X \times \{y\}$ with X. Then the subspace topology on $X \times \{y\} \subset X \times Y$ is the same as the topology on X.*

1.5 closed sets and limit points

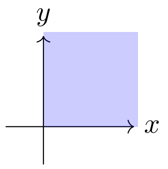
1.5.1 closed sets

Definition 1.17. *A subset A of a topological space X is said to be closed (闭) if $X - A$ is open.*

Examples:

(1) $[a, b] \subset \mathbb{R}$ is closed, since $\mathbb{R} - [a, b] = (-\infty, a) \cup (b, +\infty)$ is open.

(2) $\{(x, y) \in \mathbb{R}^2 | x \geqslant 0 \text{ and } y \geqslant 0\}$ is closed, since the complement $= (-\infty, 0) \times \mathbb{R} \cup \mathbb{R} \times (-\infty, 0)$.

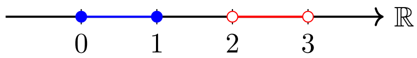


(3) X with the finite complement topology, closed sets=finite sets.

(4) X with the discrete topology, every set is closed.

(5) $Y = [0, 1] \cup (2, 3) \subset \mathbb{R}$,$[0, 1] \subset Y$ is open, $(2, 3) \subset Y$ is open.

$\therefore$ $[0, 1]$ and $(2, 3)$ are closed in Y.



Theorem 1.18 (equivalent definition of topology)**.** *Let X be a topological space. Then the following conditions hold.*

(1) $\varnothing$ and X are closed.

(2) Arbitrary intersections of closed sets are closed.

(3) Finite unions of closed sets are closed.

Proof. (1) $\varnothing = X - X, X = X - \varnothing, X, \varnothing$ open$\Rightarrow \varnothing, X$ closed.

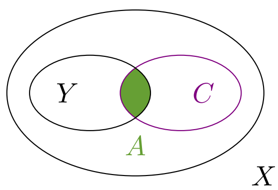
(2) $\{A_\alpha\}_{\alpha \in J}$ closed sets, then $X - \bigcap_{\alpha \in J} A_\alpha = \bigcup_{\alpha \in J} (X - A_\alpha)$ open.

(3) $A_1, A_2, \ldots, A_n$ closed sets, then $X - \bigcup_{i=1}^{n} A_i = \bigcap_{i=1}^{n} (X - A_i)$ open.

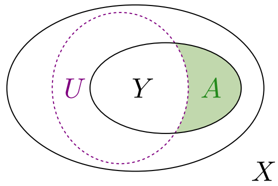
□

Theorem 1.19. *$Y \subset X$ a subspace. Then $A \subset Y$ is closed if and only if $A = C \cap Y$ for some closed set $C \subset X$.*

Proof. (i) Assume $A = C \cap Y, C \subset X$ closed. Then $Y - A = (X - C) \cap Y$, open in Y, since $X - C$ is open in X. $\therefore$ A is closed in Y.



(ii) Assume A is closed in Y,then $Y - A$ is open, i.e.$Y - A = U \cap Y$ for some $U \subset X$ open.$\therefore A = (X - U) \cap Y$, and $X - U$ is closed in X.



□

Theorem 1.20. *$Y \subset X$ a subspace, $A \subset Y$ closed,$Y \subset X$ closed. Then A is closed in X.*

The proof is left for exercise.

1.5.2 closure and interior

X a topological space, $A \subset X$ a subset.

Definition 1.21. *(i) The interior (内部) of A is the union of all open sets contained in A.*

$$\operatorname{int} A = \bigcup_{\substack{U \subset A \\ U \text{ open in } X}} U$$

This is the largest open set contained in A.

(ii) The closure (闭包) of A is the intersection of all closed sets containing A.

$$\overline{A} = \bigcap_{\substack{C \supset A \\ C \text{ closed in } X}} C$$

This is the smallest closed set containing A.

Remark:

$$\begin{array}{ccc} \operatorname{int} A & \subseteq A \subseteq & \overline{A} \\ \text{“=” iff} & & \text{“=” iff} \\ A \text{ is open} & & A \text{ is closed} \end{array}$$

Theorem 1.22. *$A \subset Y \subset X$, $\overline{A} =$ the closure of A in X. Then the closure of A in Y equals $\overline{A} \cap Y$.*

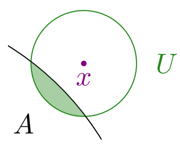
Proof. Let B be the closure of A in Y, we need to show $B = \overline{A} \cap Y$.

“$B \subset \overline{A} \cap Y$”: $\overline{A}$ is closed in X, $\therefore \overline{A} \cap Y$ is closed in Y, and $A \subset \overline{A} \cap Y$,$\therefore B \subset \overline{A} \cap Y$.

“$B \supset \overline{A} \cap Y$”: B is closed in Y, $\therefore B = C \cap Y$ for some $C \subset X$, C closed, and $A \subset C$. $\therefore \overline{A} \subset C$,$\therefore$ $\overline{A} \cap Y \subset C \cap Y = B$. ☐

Theorem 1.23. *Let $A \subset X$ be a subspace.*

(a) Then $x \in \overline{A}$ iff every open set U containing x intersects A.



(b) Let $\mathscr{B}$ be a basis for the topology on X, then $x \in \overline{A}$ iff every basis element containing x intersects A.

Proof. (a)

(i) If $x \in \overline{A}$, then $x \in U = X - \overline{A} \subset X - A$. U is open and $U \cap A = \varnothing$, i.e.$\exists$ an open set U containing x and $U \cap A = \varnothing$.

(ii) If there exists an open set U s.t. $x \in U$ and $U \cap A = \varnothing$, then $C' = X - U$ is closed, $A \subset C'$ and $x \notin C'$.

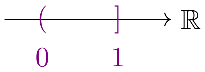
$$\therefore x \notin \overline{A} = \bigcap_{\substack{C \supset A \\ C \text{ closed}}} .$$

(b) is a consequence of (a). ☐

Definition 1.24. *An open set U containing x is a neighborhood (邻域) of x.*

Theorem 1.25. *$x \in \overline{A}$ iff every neighborhood of X intersects A.*

Examples:

(1) $A = (0,1] \subset \mathbb{R}, \overline{A} = [0,1]$ 

(2) $B = \left\{ \dfrac{1}{n} \middle| n \in \mathbb{N} \right\}, \overline{B} = \{0\} \cup B.$

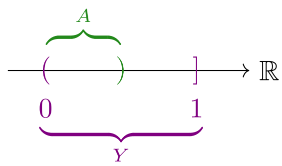
(3) $\mathbb{Q} = \{\text{rational numbers}\} \subset \mathbb{R}$,then $\overline{\mathbb{Q}} = \mathbb{R}$.

(4)

$Y = (0,1) \subset \mathbb{R}, A = \left(0, \dfrac{1}{2}\right).$

The closure of A in $\mathbb{R}$ is $\left[0, \dfrac{1}{2}\right]$.

The closure of A in Y is $\left[0, \dfrac{1}{2}\right] \cap Y = \left(0, \dfrac{1}{2}\right]$.



1.5.3 limit points

$A \subset X$ a subspace.

Definition 1.26. *A point $x \in X$ is a <u>limit point</u> (极限点) (or cluster point, point of accumulation, 聚点) of A if every neighborhood of x intersects A in some point <u>other than</u> x.*

In other words, x is a limit point of A iff $x \in \overline{A - \{x\}}$.

Examples:

(1) $A = (0, 1] \subset \mathbb{R}$, limit points of $A = [0, 1]$.

(2) $B = \left\{ \frac{1}{n} \middle| n \in \mathbb{N} \right\}$, then 0 is the only limit point of B.

(3) $\mathbb{Q} \subset \mathbb{R}$ every limit point of $\mathbb{R}$ is a limit point of $\mathbb{Q}$.

Theorem 1.27. *Let $A \subset X$ be a subspace, let A' be the set of all limit points of A. Then $\overline{A} = A \cup A'$.*

Proof. "$A \cup A' \subset \overline{A}$": $\forall x \in A'$,U a neighborhood of x. We have $U \cap A \neq \varnothing$. $\therefore x \in \overline{A}$.$\therefore A \cup A' \subset \overline{A}$.

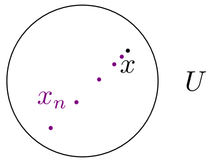
"$A \cup A' \supset \overline{A}$": For $\forall x \in \overline{A}$, if $x \in A$, then $x \in A \cup A'$; if $x \in \overline{A} - A$, then for any neighborhood U of x, $U \cap A \neq \varnothing$, and $x \notin U \cap A$. $\therefore x \in A'$. □

Corollary 1.28. *$A \subset X$ a subspace. A is closed iff A contains all its limit points.*

Proof. A is closed $\Leftrightarrow A = \overline{A} = A \cup A'$. $\therefore A$ is closed $\Leftrightarrow A' \subset A$. □

1.5.4 Hausdorff space

Definition 1.29. *A sequence of points $x_1, x_2, \cdots$ in X <u>converges</u> (收敛) to $x \in X$ if for each neighborhood U of x, $\exists N$ s.t. for all $n \geqslant N$,$x_n \in U$.*



Examples:

(i) $X = \mathbb{R}$

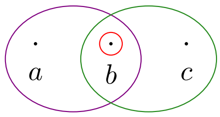
(1) one-point set $\{x_0\}$ is closed.

(2) A sequence cannot converge to more than one point.

(ii) $X = \{a, b, c\}$

(1) $\{b\}$ is not closed.

(2) $\{x_n = b\}_{n \geqslant 0}$ converges to a, b and c!



Definition 1.30. *A topological space X is called a <u>Hausdorff space</u> (豪斯多夫空间) if for each pair x_1, x_2 of distinct points of X, there exist neighborhoods U_1 of x_1, U_2 of x_2, s.t. $U_1 \cap U_2 = \varnothing$.*

Theorem 1.31. *Every finite set in a Hausdorff space X is closed.*

Proof. It suffices to show that for every $x_0 \in X$, $\{x_0\}$ is closed. For $\forall x \neq x_0$,$\exists$ neighborhoods U of x, mV of x_0,s.t. $U \cap V = \varnothing$.$\therefore x \in U \subset X - \{x_0\}$.$\therefore X - \{x_0\}$ is open, $\{x_0\}$ is closed. □

Theorem 1.32. *If X is a Hausdorff space, then a sequence of points of X converges to at most one point of X.*

Proof. Let $\{x_n\}_{n \geqslant 0}$ be a sequence converging to x. If $y \neq x$, then $\exists$ neighborhoods U of x,V of y,s.t.$U \cap V = \varnothing$.$\exists N$, s.t. for all $n \geqslant N, x_n \in U$.$\therefore \{x_n | n \geqslant N\} \cap V = \varnothing$.$\therefore \{x_n\}$ does not converge to y. □

Theorem 1.33. *(a) The product of two hausdorff spaces is a Hausdorff space.*

(b) A subspace of a Hausdorff space is a Hausdorff space.

The proof is left for exercise.

1.6 continuous maps(functions)

1.6.1 continuous maps

Definition 1.34. *Let X, Y be topological spaces. A* <u>*continuous map*</u> *(连续映射) is a map $f : X \to Y$ s.t. for each open set $V \subset Y$,$f^{-1}(V)$ is an open set of X.*

Remark:

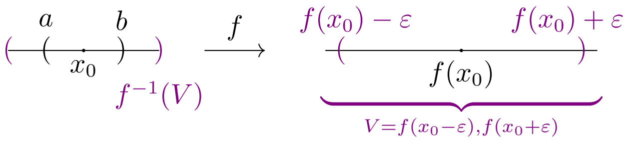
(1) If $\mathscr{B}$ is a basis for the topology on Y,then any open set $V = \bigcup\limits_{\alpha} \mathcal{B}_\alpha$ and $f^{-1}(V) = \bigcup\limits_{\alpha} f^{-1}(\mathcal{B}_\alpha)$.$\therefore$ f is continuous if $f^{-1}(\mathcal{B}_\alpha)$ is open for every basis element $\mathcal{B} \in \mathscr{B}$.

(2) If $\mathscr{S}$ is a subbasis for the topology on Y, then any basis element $\mathcal{B} = \mathcal{S}_1 \cap \cdots \cap \mathcal{S}_n$, and $f^{-1}(\mathcal{B}) = f^{-1}(\mathcal{S}_1) \cap \cdots \cap f^{-1}(\mathcal{S}_n)$.$\therefore$ f is continuous if $f^{-1}(\mathcal{S})$ is open for every element $\mathcal{S}$ in the subbasis $\mathscr{S}$.

Examples:

(1) $f : \mathbb{R} \to \mathbb{R}$ is continuous $\Leftrightarrow$ f is continuous under the "$\varepsilon - \delta$" definition.

"$\Rightarrow$"



$x_0 \in f^{-1}(V)$ open.

$\therefore$ $\exists a, b$ s.t. $x_0 \in (a, b) \subset f^{-1}(V)$. Choose δ small enough s.t. $(x_0 - \delta, x_0 + \delta) \subset (a, b)$. Then $f((x_0 - \delta, x_0 + \delta)) \subset (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$. $\square$

Proof of "$\Leftarrow$" part is left for exercise.

(2) (i) Continuous map $f : \mathbb{R} \to \mathbb{R}^n$ is called a continuous curve.

(ii) Continuous map $f : \mathbb{R}^n \to \mathbb{R}$ is called a continuous function.

(iii) Continuous map $f : \mathbb{R}^2 \to \mathbb{R}^2$ is called a continuous vector field.

(3) $\mathbb{R}_l$= the set of real numbers with lower limit topology, one of whose basis is $\mathscr{B} = \{[a, b) | a, b \in \mathbb{R}\}$. Then:

$f : \mathbb{R} \to \mathbb{R}_l, x \mapsto x$ is not continuous, since $f^{-1}([a, b)) = [a, b) \subset \mathbb{R}$ and the latter is not open.

$g : \mathbb{R}_l \to \mathbb{R}, x \mapsto x$ is continuous, since $g^{-1}((a, b)) = (a, b) = \bigcup\limits_{n=1}^{\infty} \left[a + \frac{1}{n}, b\right)$ and the latter is open.

Theorem 1.35. *X, Y topological spaces, $f : X \to Y$ a map. Then the followings are equivalent.*

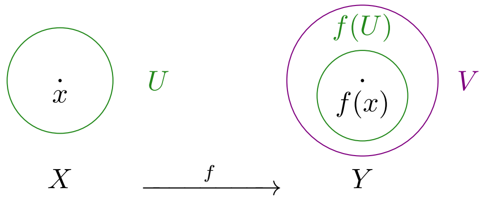
(1) f is continuous.

(2) For every subset $A \subset X$, $f(\overline{A}) \subset \overline{f(A)}$.

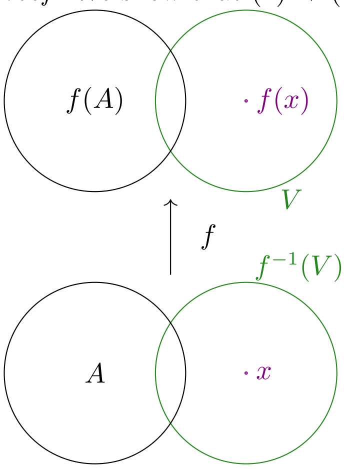
(3) For every closed set $\mathcal{B} \subset Y$, $f^{-1}(\mathcal{B})$ is closed in X.

(4) For each $x \in X$, and each neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subset V$.

If the condition (4) holds for the point $x \in X$, we say that f is <u>continuous at x</u> .



Proof. We show that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) and (1) $\Rightarrow$ (4) $\Rightarrow$ (1).



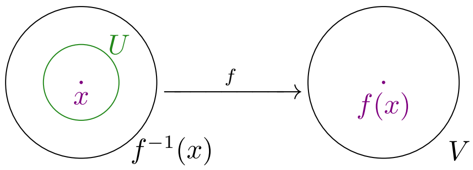
Assume that f is continuous, $A \subset X, X \in \overline{A}$, we need to show $f(x) \in \overline{f(A)}$. Let V be a neighborhood of $f(x)$, we need to show $f(A) \cap V \neq \varnothing$. It suffices to show that $f^{-1}(V) \cap A \neq \varnothing$. But $f^{-1}(V)$ is a neighborhood of x, $x \in \overline{A} \Rightarrow f^{-1}(V) \cap A \neq \varnothing$.

(2) $\Rightarrow$ (3): Let $\mathcal{B} \subset Y$ be a closed set, $A = f^{-1}(\mathcal{B})$. We need to show that $A = \overline{A}$. $f(A) = f(f^{-1}(\mathcal{B})) \subset \mathcal{B}$. If $x \in \overline{A}$,$f(x) \in f(\overline{A}) \subset \overline{f(A)} \subset \overline{\mathcal{B}} = \mathcal{B}$.$\therefore x \in f^{-1}(\mathcal{B}) = A$, i.e. $\overline{A} = A$.

(3) $\Rightarrow$ (1): Let $V \subset Y$ be an open set, $\mathcal{B} = Y - V$. Then $f^{-1}(\mathcal{B}) = X - f^{-1}(V)$ is closed. $\therefore f^{-1}(V)$ is open.

(1) $\Rightarrow$ (4): Let $x \in X$,V is a neighborhood of $f(x)$. Then $U = f^{-1}(V)$ is a nbhd of x, and $f(U) \subset V$.

(4) $\Rightarrow$ (1): Let $V \subset Y$ be an open set. We need to show for every $x \in f^{-1}(V)$,$\exists$ a neighborhood U of x and $U \subset f^{-1}(V)$.



Now V is a neighborhood of $f(x)$, by the hypothesis, $\exists$ a neighborhood U of x s.t. $f(U) \subset V$, i.e. $U \subset f^{-1}(V)$. $\square$

Remark: Let $C(X,Y) \subset \mathrm{Map}(X,Y)$ be the set of all continuous maps from X to Y. We will define a topology on $C(X,Y)$ later.

1.6.2 homeomorphisms

Definition 1.36. *A bijection $f: X \to Y$ between topological spaces is called a* <u>*homeomorphism*</u> *(同胚) if both f and f^{-1} are continuous.*

Notation: $f: X \xrightarrow{\cong} Y, X \cong Y$.

Remark: $f^{-1}: Y \to X$ is continuous means: $\forall U \subset X$ open, i.e. f maps open sets to open sets. Therefore a homeomorphism $f: X \to Y$ gives rise to a bijection between the topology $\mathcal{T}_X$ on X and the topology $\mathcal{T}_Y$ on Y:

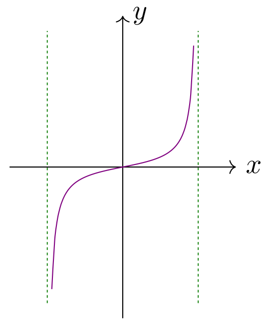
$$\mathcal{T}_x \underset{f^{-1}}{\overset{f}{\leftrightarrows}} \mathcal{T}_Y$$

A property of topological spaces which is invariant under homeomorphisms is called a <u>topological property</u> (拓扑性质).

Examples:

(1) $\begin{tikzcd} \mathbb{R} \arrow[r, bend left, "f"] & \mathbb{R} \arrow[l, bend left, "g"] \end{tikzcd}$,in which $f: x \mapsto 3x+1, g: y \mapsto \dfrac{y-1}{3}$. f is a homeomorphism.

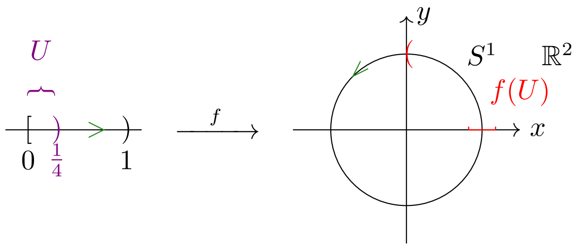
(2) $\begin{tikzcd} (-1,1) \arrow[r, bend left, "f"] & \mathbb{R} \arrow[l, bend left, "g"] \end{tikzcd}$, in which $f: x \mapsto \dfrac{x}{1-x^2}, g: y \mapsto \dfrac{2y}{1+\sqrt{1+4y^2}}$. f is a homeomorphism.



(3) $\begin{tikzcd} \mathbb{R}_l \arrow[r, bend left, "f"] & \mathbb{R} \arrow[l, bend left, "g"] \end{tikzcd}$ in which $f: x \mapsto x$ is continuous, $g: y \mapsto y$ is not continuous. In fact, $\mathbb{R}_l$ and $\mathbb{R}$ are <u>not homeomorphic</u> since the topology $\mathbb{R}_l$ is finer than $\mathbb{R}$.

(4)

$$\begin{array}{ccc} [0,1] \subset \mathbb{R} & \xrightarrow{f} & S^1 \subset \mathbb{R}^2 \\ t & \mapsto & (\cos 2\pi t, \sin 2\pi t) \end{array}$$



f is continuous and bijective, but f^{-1} is not continuous : $f\left(\left[0, \dfrac{1}{4}\right]\right) \subset S^1$ is not open. In fact, $[0,1)$ is not homeomorphic to S^1.

Let $f: X \to Y$ be an injective continuous map, $Z = f(X) \subset Y$, then $f': X \to Z$ is bijective.

Definition 1.37. *If $f': X \to Z = f(X)$ is a homeomorphism, then the map $f: X \to Y$ is called a* <u>*topological embedding*</u> *(拓扑嵌入) of X in Y.*

1.6.3 constructing continuous maps

Theorem 1.38. *Let X, Y, Z be topogical spaces*

(a) (constant map) the constant map $f : X \to Y, x \mapsto y_0$ is continuous for $\forall y_0 \in Y$.

(b) (inclusion) If $A \subset X$ is a subspace, then the inclusion map $i_A : A \to X$ is continuous.

(c) (composition) If $f : X \to Y, g : Y \to Z$ are continuous, then $g \circ f : X \to Z$ is continuous.

(d) (restricting the domain) If $f : X \to Y$ is continuous, $A \subset X$ a subspace, then the restricted map $f\big|_A : A \to Y$ is continuous.

(e) (restricting or expanding the range) If $f : X \to Y$ is continuous, $Z \subset Y$ a subspace s.t.$f(X) \subset Z$, then $g : X \to Z$(obtained by restricting the range of f) is continuous; If $Y \subset Z$ is a subspace, then $h : X \to Z$(obtained by expanding the range of f) is continuous.

(f) (local formulation of continuity) If $X = \bigcup\limits_{\alpha} U_\alpha$ is a union of open sets $\{U_\alpha\}$. Then $f : X \to Y$ is continuous iff $f\big|_{U_\alpha} : U_\alpha \to Y$ is continuous for all α.

Proof. (e) $\mathcal{B} \subset Z$ open, then $\mathcal{B} = Z \cap U$ for some $U \subset Y$ open, $g^{-1}(\mathcal{B}) = f^{-1}(U)$ open in X. (f) $V \subset Y$ open, $f^{-1}(V) \cap U_\alpha = \left(f\big|_{U_\alpha}\right)^{-1}(V)$, $f^{-1}(V) = \bigcup\limits_{\alpha}(f^{-1}(V) \cap U_\alpha)$. □

Theorem 1.39 (the pasting lemma)**.** *Let $X = A \cup B$, where A, B are closed. Let $f : A \to Y, g : B \to Y$ be continuous. If $f(x) = g(x)$ for all $x \in A \cap B$, then f and g combine to give a continuous map*

$$h : X \to Y, h(x) = \begin{cases} f(x) & x \in A \\ g(x) & x \in B \end{cases}$$

Proof. Let $C \subset Y$ be a closed set, then $h^{-1}(C) = f^{-1}(C) \cap g^{-1}(C)$ is a closed set, for both of the sets on the right side are closed. □

Examples:

(1) $h : \mathbb{R} \to \mathbb{R}, h(x) = \begin{cases} x & x \leqslant 0 \\ \dfrac{x}{2} & x \geqslant 0 \end{cases}$ is continuous.

(2) $f : \mathbb{R} \to \mathbb{R}, f(x) = \begin{cases} x-2 & x < 0 \\ x+2 & x \geqslant 0 \end{cases}$ is not continuous.

Theorem 1.40 (Maps into products)**.** *Let $f : A \to X \times Y, f(a) = (f_1(a), f_2(a))$ be a map. Then f is continuous iff the coordinate maps $f_1 : A \to X, f_2 : A \to Y$ are continuous.*

Proof. Let $\pi_1 : X \times Y \to X$,$\pi_2 : X \times Y \to Y$ be the projections, then π_1, π_2 are continuous. $f_i = \pi_i \circ f$.

f continuous $\Rightarrow$ f_i is continuous $i = 1, 2$

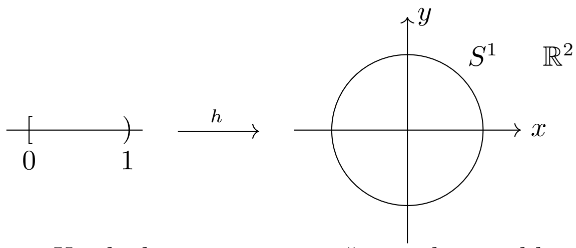
$$\begin{tikzcd} & Y \\ A \arrow[ur, "f_2"] \arrow[r, "f"] \arrow[dr, "f_1"'] & X \times Y \arrow[u, "\pi_2"'] \arrow[d, "\pi_1"] \\ & X \end{tikzcd}$$

Conversely, assume f_1, f_2 are continuous. A basis for the topology on $X \times Y$ os $\{U \times V | U \subset X, V \subset Y \text{ open }\}$. It suffices to show that $f^{-1}(U \times V)$ is open but $f^{-1}(U \times V) = f_1^{-1}(U) \cap f_2^{-1}(V)$. □

Example: A curve $\gamma : \mathbb{R} \to \mathbb{R}^2, x \mapsto (\gamma_1(x), \gamma_2(x))$ is continuous iff γ_1 and γ_2 are continuous functions.

e.g.

$$\begin{aligned} [0,1) &\xrightarrow{h} \quad S^1 \\ t &\mapsto \quad (\cos 2\pi t, \sin 2\pi t) \end{aligned}$$



Caveat: A map $f : A \times B \to X$ which is continuous "in each variable separately" is **not** necessarily continuous.

1.6.4 categories and functors

Definition 1.41. *A* <u>*category*</u> *(范畴) $\mathscr{C}$ consists of*

(i) a collection of objects (对象)

(ii) a set $\mathscr{C}(A,B)$ of morphsim (态射) between any two objects A,B in $\mathscr{C}$ satisfying

(1) a composition law $\begin{array}{ccccc} \mathscr{C}(B,C) & \times & \mathscr{C}(A,B) & \to & \mathscr{C}(A,C) \\ g & & f & \mapsto & g\circ f \end{array}$ *which is associative:* $h\circ(g\circ f)=(h\circ g)\circ f$*.*

(2) $\exists$ an identify morphism $\mathrm{id}_A\in\mathscr{C}(A,A)$ for any A in $\mathscr{C}$, s.t. $\mathrm{id}\circ f=f, f\circ\mathrm{id}=f$*.*

Examples:

(1) The category of sets: $\mathcal{S}$ets

objects=sets, morphisms=maps between sets.

(2) The category of vector spaces over $\mathbb{F}$:Vect$_{\mathbb{F}}$.

(3) The category of groups: $\mathcal{G}$rps.

(4) The category of abelian groups: $\mathcal{A}$b.

(5) The category of (left) R-modules: ${}_R$Mod.

(6) The category of topological spaces: $\mathcal{T}$op

objects=topological spaces

morphism=continuous maps

(7) If a category $\mathscr{C}$ consists of only one object $\cdot$, then $\mathscr{C}(\cdot,\cdot)$ is a <u>monoid</u> (半群).

A group G can be viewed as a category with one object and all morphisms are invertible.

Definition 1.42. *A* <u>*functor*</u> *(函子) $F:\mathscr{C}\to\mathscr{D}$ between two categories $\mathscr{B},\mathscr{D}$ assigns an object $F(A)$ of $\mathscr{D}$ to each object A of $\mathscr{C}$, and a morphism $F(f):F(A)\to F(B)$ of $\mathscr{D}$ to each morphism $f:A\to B$ if $\mathscr{C}$, s.t.*

$$F(\mathrm{id}_A)=\mathrm{id}_{F(A)}, F(g\circ f)=F(g)\circ F(f)$$

$$\mathscr{C}\xrightarrow{F}\mathscr{D}$$

$$\begin{tikzcd} A \arrow[r, mapsto] \arrow[d, "f"] & F(A) \arrow[d, "F(f)"] \\ B \arrow[r, mapsto] & F(B) \end{tikzcd}$$

In algebraic topology one studies functors

$$\mathcal{T}\mathrm{op}\to\mathrm{Vect},\mathcal{G}\mathrm{rps},\mathcal{A}\mathrm{b}$$

1.7 the quotient topology

X a topological space, a disjoint union is $X = \coprod_{\alpha \in A} X_\alpha$. Then there is a surjective map $p : X \to A, X \mapsto a$ if $x \in X_a$.
Question: How to define a topology on A s.t. p is continious?
Answer: $U \subset A$ open iff $p^{-1}(U)$ is open in X.
Check: This is a topology on A

(1) $\varnothing, A \in \mathcal{T}_A$

(2) $p^{-1}\left(\bigcup_{\alpha} U_\alpha\right) = \bigcup_\alpha p^{-1}(U_\alpha)$ open.

(3) $p^{-1}(U_1 \cap U_2) = p^{-1}(U_1) \cap p^{-1}(U_2)$ open.

This is the largest topology on A s.t. p is continuous.

Definition 1.43. *X, Y topological spaces, $p : X \to Y$ a surjective map is said to be a quotient map (商映射) provided that a subset $U \subset Y$ is open iff $p^{-1}(U)$ is open in X.*

Remark: Equivalently, a surjective map $p : X \to Y$ is a quotient map if $\mathcal{B} \subset Y$ is closed iff $p^{-1}(\mathcal{B})$ is closed.(Since $p^{-1}(Y - \mathcal{B}) = X - p^{-1}(B)$).
Examples: A map $f : X \to Y$ an open map /closed map if the image of each open /closed set is open /closed . If $p : X \to Y$ is a surjective continuous map, and open or closed, then p is a quotient map.

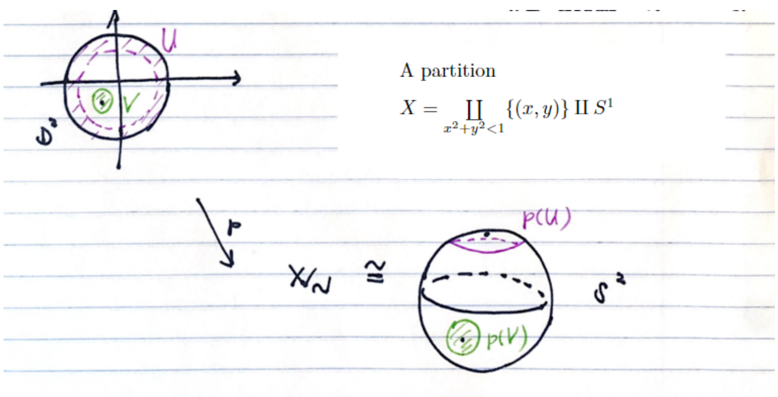
Definition 1.44. *X a topological space, A a set, $p : X \to A$ a surjective map. Then there exists exactly one topology on A relative to which p is a quotient map; it's called the quotient topology (商拓扑) induced by p.*

a partion of X,$X = \coprod_{a \in A} X_a \Leftrightarrow$ an equivalent relation $\sim$ on X, with $A = X/\sim$.

Definition 1.45. *$X/\sim$ with the quotient topology is called a quotient space (商空间) of X, or an identification space (等化空间).*

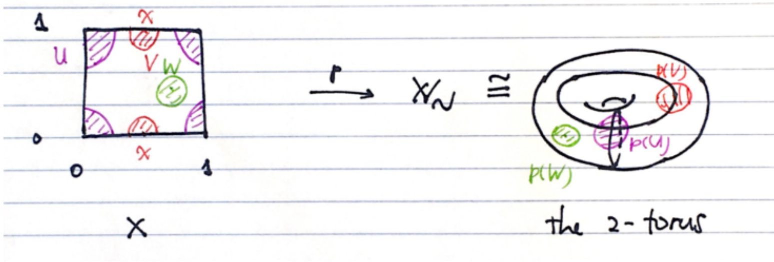
Examples:

(1) $X = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \leqslant 1\} = D^2 \subset \mathbb{R}^2$ the closed unit circle.



(2) $X = [0, 1] \times [0, 1]$ a partition.

$$X = \coprod_{\substack{0<x<1 \\ 0<y<1}} \{(x,y)\} \amalg \coprod_{0<x<1} \{(x,0),(x,1)\} \amalg \coprod_{0<y<1} \{(0,y),(1,y)\} \amalg \{(0,0),(0,1),(1,0),(1,1)\}$$



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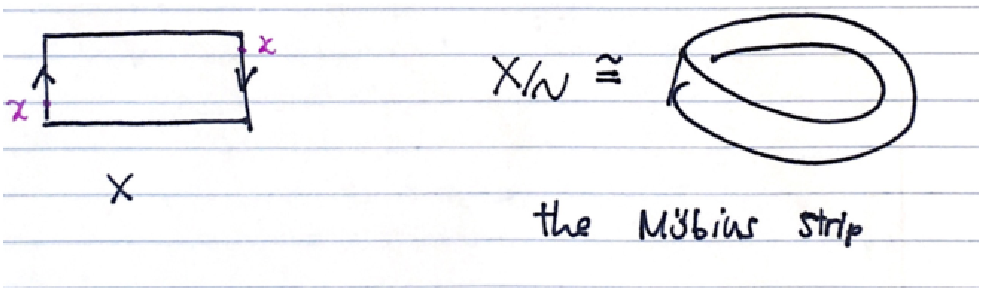

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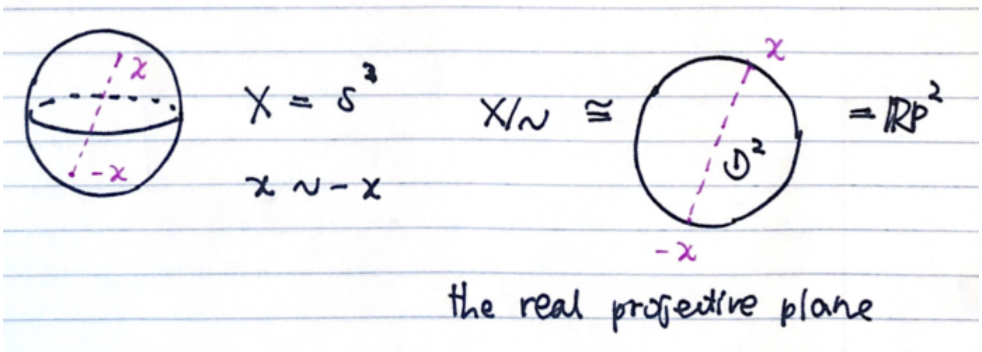

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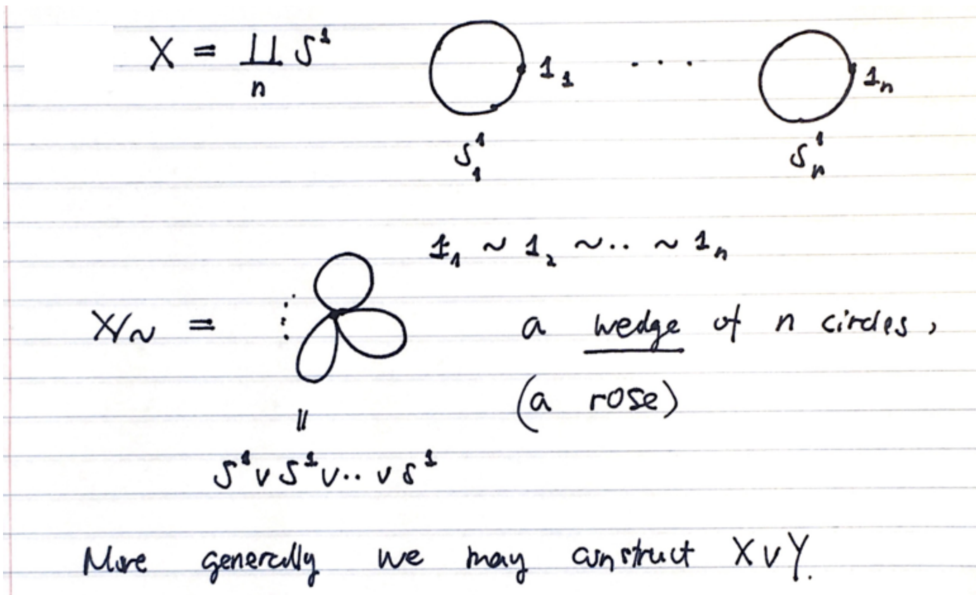


Theorem 1.46. *Let $p : X \to Y$ be a quotient map, $g : X \to Z$ be a map, that is constant on each $p^{-1}(y), \forall y \in Y$. Then g induces a map $f : Y \to Z$ s.t. $f \circ p = g$.*

Then

(1) f is continuous iff g is continuous.

(2) f is quotient map iff g is a quotient map.

$$\begin{tikzcd}
x \in \arrow[dr, mapsto] & X \arrow[r, "p"] \arrow[dr, "g"] & Y \arrow[d, dashed, "f"] & \ni y \arrow[d, mapsto] \\
& g(x) \in & Z & \ni f(y)
\end{tikzcd}$$

Proof. $\forall y \in Y$, for $x \in p^{-1}(y)$ defines $f(y) = g(y)$. This gives a well-defined map $f : X \to Z$ s.t. $g = f \circ p$.

(a) f continuous $\Rightarrow$ g continuous.

Now assume g is continuous. For any $V \subset Z$ open, $g^{-1}(V) \subset X$ is open. But $g^{-1}(V) = p^{-1}\left(f^{-1}(V)\right)$, and p is the quotient map.$\therefore$ $f^{-1}(V) \subset Y$ is open.

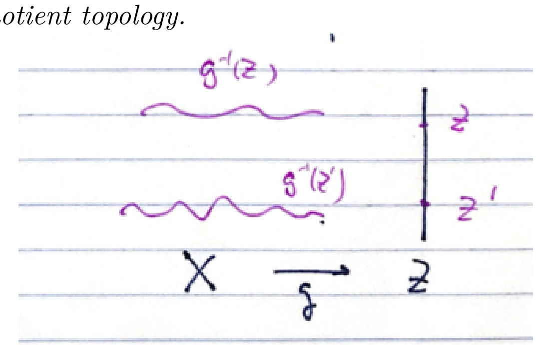
(b) f is a quotient map $\Rightarrow$ g is a quotient map. (Left for exercise)

Now assume g is a quotient map. Then f is surjective.

$V \subset Z$ open $\Leftrightarrow g^{-1}(V) \in X$ open $(g^{-1}(V) = p^{-1}\left(f^{-1}(V)\right)) \Leftrightarrow f^{-1}(V) \subset Y$ open . ∎

Corollary 1.47. *Let $g : X \to Z$ be a surjective continuous map. Let X^* be the following collection of subsets of X:$X^* = \{g^{-1}(Z) | z \in Z\}$. Give X^* a quotient topology.*

$$\begin{tikzcd}
\coprod_{X^*} g^{-1}(z) \arrow[r, equal] & X \arrow[r, "g"] \arrow[dr, "p"] & Z \\
& & X^* \arrow[u, dashed, "f"]
\end{tikzcd}$$



(a) The map g induces a bijective continuous map $f : X^* \to Z$, which is a homeomorphism iff g is a quotient map.

(b) If Z is Hausdorff, so is X^*.

Proof. (a) f is bijective, and continuous by the theorem.

(i) f is a homeomorphism $\Rightarrow$ f is a quotient map

$\Rightarrow$ g is a quotient map.

(ii) g is a quotient map $\Rightarrow$ f is a quotient map,since f is bijective $\Rightarrow$ f is a homeomorphism.

(b) For $x, y \in X^*, x \neq y \Rightarrow f(x) \neq f(y)$ in Z.

$\therefore$ $\exists$neighborhoods $U, V, f(x) \in U, f(y) \in V, U \cap V = \varnothing$.Then $f^{-1}(U)$ and $f^{-1}(V)$ are neighborhoods of x and y respectively, and $f^{-1}(U) \cap f^{-1}(V) = \varnothing$. ∎

Remark:

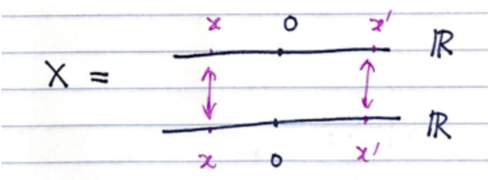
(1) Composites of quotient maps are quotient maps.

(2) The restriction of a quotient map $p : X \to Y$ to a subspace $A \subset X, p : A \to p(A)$, needs not to be a quotient map.

(3) Products of quotient maps are not necessarily quotient maps.

(4) The quotient space of a Hausdorff space is not necessarily Hausdorff.

e.g.



$x \times \{0\} \sim x \times \{1\}$ for $x \neq 0$

$X^* = X/\sim$, a line with two origins.

1.8 The product topology

Definition 1.48. *Let $\{X_\alpha\}_{\alpha\in J}$ be an indexed family if sets. The* <u>*cartesian product*</u> *(笛卡尔积)* $\coprod\limits_{\alpha\in J} X_\alpha = \left\{x : J \to \bigcup\limits_{\alpha\in J} X_\alpha \middle| x(\alpha) \in X_\alpha\right\}$ *is the set of all* <u>*J-tuples*</u> $(X_\alpha)_{\alpha\in J}$.

Theorem 1.49 (Axiom of choice)**.** *Given a collection of sets $\mathscr{A}$, there is a map $f : \mathscr{A} \to \bigcap\limits_{A\in\mathscr{A}} A$ s.t. $f(A) \in A, \forall A \in \mathscr{A}$.*

Definition 1.50. *Let $\{X_\alpha\}_{\alpha\in J}$ be an induced family of topological spaces. The topology on $\prod\limits_\alpha X_\alpha$ generates by a basis*

$$\mathscr{B} = \left\{\prod_{\alpha\in J} U_\alpha \middle| U_\alpha \subset X_\alpha \text{ open }\right\}$$

is called the <u>*box topology*</u> *(箱拓扑). The topology on $\prod\limits_\alpha X_\alpha$ generated by the basis*

$$\mathscr{B}' = \left\{\prod_{\alpha} U_\alpha \middle| U_\alpha \subset X_\alpha \text{ open for all but finitely many } \alpha' s\right\}$$

is called the <u>*product topology*</u> *. In this topology $\prod\limits_\alpha X_\alpha$ is called a* <u>*product space*</u> *.*

Remark: Let $\pi_\beta : \substack{\prod\limits_\alpha X_\alpha \to X_\beta \\ (x_\alpha)_J \mapsto x_\beta}$ be the projection to the β-th coordinate. Then the product topology has a subbasis

$$\mathcal{S} = \left\{\pi_\beta^{-1}(U_\beta) \middle| U_\beta \subset X_\beta \text{ open for some } \beta\right\}$$

Since $\pi_{\beta_1}^{-1}(U_{\beta_1}) \cap \cdots \cap \pi_{\beta_n}^{-1}(U_{\beta_n}) = \prod -\alpha U_\alpha$ where $U_\alpha = X_\alpha$ for $\alpha \neq \beta_1, \cdots, \beta_n$.

For finite products, "box topology"="product topology".

In general, the box topology is finer than the product topology.

Theorem 1.51. *Let $f : A \to \prod\limits_\alpha X_\alpha, f(a) = (f_\alpha(a)) - \alpha \in J$ be a map, where $f_\alpha : A \to X_\alpha$. Let $\prod\limits_\alpha X_\alpha$ have the product topology. Then f is continuous iff f_α is continuous for all α.*

Proof. (a) Note that the projection $\pi_\beta : \prod X_\alpha \to X\beta$ is continuous, since for any $U_\beta \subset X_\beta$ open, $\pi_\beta^{-1}(U_\beta)$ is open in $\prod X_\alpha$. $\therefore f_\beta = \pi_\beta \circ f$ is continuous for $\forall\beta$.

(b) Assume f_α is continuous for all α. We need to show f is continuous. It suffices to show for any element $\pi_\beta^{-1}(U_\beta)$ in the subbasis $\mathscr{S}$, $f^{-1}\left(\pi_\beta^{-1}(U_\beta)\right)$ is open. But $f^{-1}\left(\pi_\beta^{-1}(U_\beta)\right) = (\pi_\beta \circ f)^{-1}(U_\beta) = f_\beta^{-1}(U_\beta)$ open in A.

□

Example: $\mathbb{R}^\omega = \prod\limits_{i=1}^\infty \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}^\omega, t \mapsto (t, t, \cdots)$(the diagonal)

Then f is continuous if $\mathbb{R}^\omega$ is given the product topology, but not continuous if $\mathbb{R}^\omega$ is given the box topology.

Let $\mathcal{B} = (-1, 1) \times \left(-\frac{1}{2}, \frac{1}{2}\right) \times \left(-\frac{1}{3}, \frac{1}{3}\right) \times \cdots \times \left(-\frac{1}{n}, \frac{1}{n}\right) \times \cdots \subset \mathbb{R}^\omega$ be an open set in the box topology, then $f^{-1}(B) = \bigcap_{n=1}^\infty \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$.

Remark: (1) a family of topological spaces $Y_\lambda, \lambda \in \Lambda$, a family of maps $f_\lambda : X \to Y_\lambda$.

Q : What is the smallest topology on X s.t. f_λ is continuous for all $\lambda \in \Lambda$?

A :$\mathscr{S} = \left\{ f_\lambda^{-1}\left(U_\lambda\right) \middle| \lambda \in \Lambda, U_\lambda \subset Y_\lambda \text{ open } \right\}$ is a subbasis, generates the desired topology("weak topology").

e.g.

(i) subspace topology $A \subset X$

(ii) product topology $X = \prod\limits_\lambda X_\lambda$

(2) a family of topological spaces $X_\lambda, \lambda \in \Lambda$.

a family of maps $f_\lambda : X \to Y_\lambda$.

Q : What is the largest topology on X s.t. f_λ is continuous for all $\lambda \in \Lambda$?

A :$\mathcal{T} = \left\{ U \middle| f_\lambda^{-1}(U) \subset X_\lambda \text{ open for all } \lambda \in \Lambda \right\}$ ("the strong topology").

e.g. quotient map $f : X \to Y$.

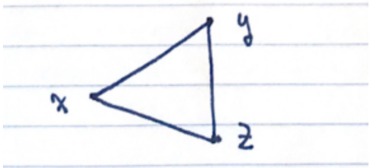
1.9 the metric topology

Definition 1.52. *A metric (度量) on a set X is a function $d : X\times X \to \mathbb{R}$ having the following properties:*

(1) $d(x, y) \geqslant 0$ for all $x, y \in X$; $d(x, y) = 0$ iff $x = y$.

(2) $d(x, y) = d(y, x)$ (symmetric)

(3) $d(x, y) + d(y, z) \geqslant d(x, z)$ $\forall x, y, z \in X$ (triangle inequality)

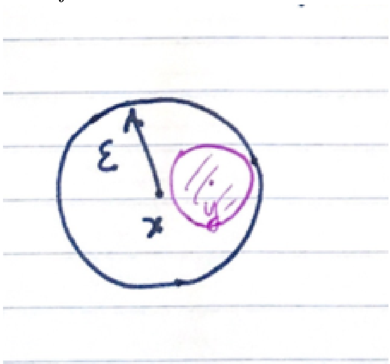


A set X together with a metric d is called a metric space (度量空间); $d(x, y)$ is called the distance between x and y; $B_d(x, \varepsilon) = \{y \in X|d(x, y) < \varepsilon\}$ is called the ε-ball centered at x.

Definition 1.53. *(X, d) a metric space,$\mathscr{B} = \{B_d(x, \varepsilon)|x \in X, \varepsilon > 0\}$ is a basis for a topology on X, called the metric topology induced by d.*

Check: Claim: for any $y \in B(x, \varepsilon), \exists\delta > 0$, s.t. $B(y, \delta) \subset B(x, \varepsilon)$.

Proof.



Take $\delta = \varepsilon - d(x, y) > 0$, then for any $z \in B(y, \delta)$,$d(x, z) \leqslant d(x, y) + d(y, z) < \varepsilon.\therefore$ $B(y, \delta) \subset B(x, \varepsilon)$.

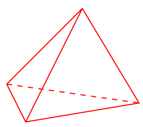
□

Given any balls B_1, B_2, and $y \in B_1 \cap B_2$,$\exists\delta_1, \delta_2$ s.t. $B(y_i, \delta_i) \subset B_i (i = 1, 2)$. Let $\delta = \min\{\delta_1, \delta_2\}$, then $B(y, \delta) \subset B_1 \cap B_2$.

Lemma 1.54. *A set U is open in the metric topology induced by d iff for each $y \in U$, $\exists\delta > 0$, s.t. $B_d(y, \delta) \subset U$.*

Examples:

(1) X a set, $d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$ is a metric on X, which induces the discrete topology.



(2) The standard metric on $\mathbb{R}$: $d(x, y) = |x - y|$. The induced topology is the standard topology on $\mathbb{R}$.

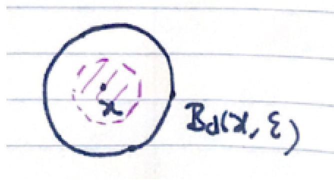
(3) The Euclidean metric on $\mathbb{R}^n$:

$$d(x,y) = \|x-y\| = \sqrt{(x_1-y_1)^2+\cdots+(x_n-y_n)^2}$$

The qurare metric on $\mathbb{R}^n$:

$$\rho(x,y) = \max\{|x_1-y_1|,\cdots,|x_n-y_n|\}$$

Lemma 1.55. *Let d and d' be two metrics on X, with induced topologies $\mathcal{T}$ and $\mathcal{T}'$ respectively. Then $\mathcal{T}'$ is the finer than $\mathcal{T}$ iff for $\forall x \in X, \varepsilon > 0, \exists \delta > 0, s.t. B_{d'}(x,\delta) \subset B_d(x,\delta)$.*



Proof. Suppose $\mathcal{T}'$ is finer than $\mathcal{T}$. Let $B_d(x,\varepsilon)$ be a basis element for $\mathcal{T}$, then $\exists \delta > 0$, s.t. $B_{d'}(x,\delta) \subset B(x,\varepsilon)$.

Conversely, suppose the condition holds. For any open set $U \in \mathcal{T}, \forall x \in U. \exists \varepsilon > 0$,s.t. $B_d(x,\varepsilon) \subset U$. Then $\exists \delta > 0$, s.t. $B_{d'}(x,\delta) \subset B_d(x,\varepsilon)$. $\therefore U$ is open in $\mathcal{T}'$. ☐

Theorem 1.56. *The topologies on $\mathbb{R}^n$ induced by the euclidean metric d and the square metric ρ are the same as the product topology on $\mathbb{R}^n$.*

Proof. (1)

$$\underbrace{\rho(x,y) \leqslant d(x}_{\substack{\Downarrow \\ B_d(x,\varepsilon)\subset B_\rho(x,\varepsilon)}}\underbrace{,y) \leqslant \sqrt{n}\rho(x,y)}_{\substack{\Downarrow \\ B_\rho\left(x,\frac{\varepsilon}{\sqrt{n}}\right)\subset B_d(x,\varepsilon)}}$$

$\therefore d$ and ρ induces the same topology.

(2)

(i) Let $\mathcal{B} = (a_1,b_1)\times(a_2,b_2)\times\cdots\times(a_n,b_n)$ be a basis element for the product topology, then for $\forall x = (x_1,\cdots,x_n) \in \mathcal{B}, \exists \varepsilon_i > 0$ s.t. $(x_i-\varepsilon_i, x_i+\varepsilon_i) \subset (a_i,b_i)$.Let $\varepsilon = \min\{\varepsilon_1,\cdots,\varepsilon_n\}$, then $B_\rho(x,\varepsilon) \subset B$. $\therefore$ the ρ-topology is finer than the product topology.

(ii) Conversely, $B_\rho(x,\varepsilon) = (x_1-\varepsilon, x_1+\varepsilon)\times\cdots\times(x_n-\varepsilon,x_n+\varepsilon)$ is a basis element for the product topology. $\therefore$ the product topology is finer than the ρ-topology.

☐

Definition 1.57. *A topological space X is metrizable if there exists a metric d on the set X Which induces the topology of X.*

Question :Is $\mathbb{R}^\omega = \prod_{n=1}^{\infty} \mathbb{R}$ metrizable?

Generalization of the d and ρ metrics:

(i) $d(x,y) = \left(\sum_{i=1}^{\infty}(x_i-y_i)^2\right)^{\frac{1}{2}}$,convergence problem $\to$ l^2-topology.

(ii) $\rho(x,y) = \sup\{|x_n-y_n| n \in \mathbb{N}\}$. The topology is not well-defined, so we modify the metric $\overline{\rho}(x,y) = \sup\left\{\overline{d}(x_n,y_n)|x \in \mathbb{N}\right\}$ where $\overline{d}(x,y) = \min\{|x-y|,1\} \to$ the uniform metric.

Theorem 1.58. *For $x, y \in \mathbb{R}^\omega$, define $D(x,y) = \sup_n \left\{ \frac{\overline{d}(x_n, y_n)}{n} \right\}$. Then D is a metric that induces the product topology on $\mathbb{R}^\omega$.*

Proof. (i) D is a metric:

$$\frac{\overline{d}(x_n, z_n)}{n} \leqslant \frac{\overline{d}(x_n, y_n)}{n} + \frac{\overline{d}(y_n, z_n)}{n} \leqslant D(x,y) + D(y,z)$$

$\therefore D(x,z) \leqslant D(x,y) + D(y,z)$.

(ii) $B_D(x,\varepsilon)$ is open in the product topology: Choose N large enough s.t. $\frac{1}{N} < \varepsilon$, let $V = (x_1 - \varepsilon, x_1 + \varepsilon) \times \cdots \times (x_N - \varepsilon, x_N + \varepsilon) \times \mathbb{R} \times \mathbb{R} \times \cdots$ be a neighborhood of x in the product topology.

Claim: $V \subset B_D(x,\varepsilon)$.

Proof. $y \in V$,$D(x,y) = \sup_n \left\{ \frac{\overline{d}(x_n, y_n)}{n} \right\}$ where $\begin{cases} \frac{\overline{d}(x_n, y_n)}{n} \leqslant \frac{1}{N} & \text{for } n \geqslant N \\ \overline{d}(x_n, y_n) < \varepsilon & \text{for } n < N \end{cases}$

$\therefore D(x,y) < \varepsilon, y \in B_D(x,\varepsilon)$. □

(iii) Let $U = \prod_{i=1}^\infty U_i$ be a basis element for the product topology, where $U_i = \mathbb{R}$ for $i \neq \alpha_1, \cdots, \alpha_n$. Given $x \in U$, choose $\varepsilon_i \leqslant 1$ s.t. $(x_i - \varepsilon_i, x_i + \varepsilon_i) \subset U_i$ for $i = \alpha_1, \cdots, \alpha_n$.Let $\varepsilon = \min \left\{ \frac{\varepsilon_i}{i} \middle| i = \alpha_1, \cdots, \alpha_n \right\}$.

Claim: $B_D(x,\varepsilon) \subset U$.

Proof. $\forall y \in B_D(x,\varepsilon)$, $\frac{\overline{d}(x_i, y_i)}{i} < \varepsilon \leqslant \frac{\varepsilon_i}{i}$ for $i = \alpha_1, \cdots, \alpha_n$.$\therefore \overline{d}(x_i, y_i) < \varepsilon_i \leqslant 1$.$d(x_i, y_i) = \overline{d}(x_i, y_i) < \varepsilon_i$.$\therefore y_i \in U_i, y \in U$. □

□

Remark:

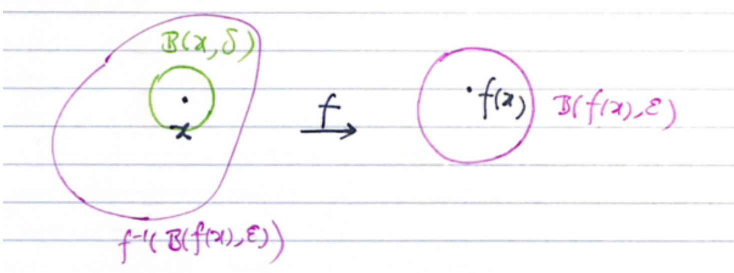
(1) $A \subset X$ a subspace of a metric space, then A, d is a metric space.

(2) A metric space (X, d) is a Hausdorff space.

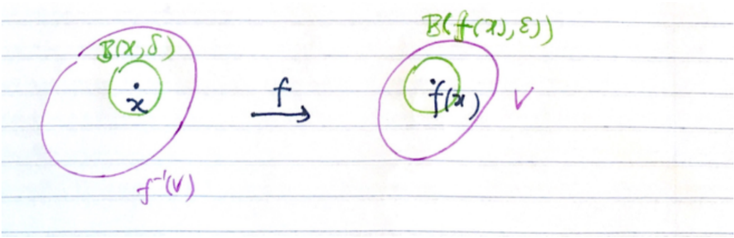
(3) A countable product $\prod_{i=1}^\infty X_i$ of metric spaces is metrizable.

Theorem 1.59. *$(X, d_X), (Y, d_Y)$ metric spaces. A map $f : X \to Y$ is continuous iff for $\forall x \in X, \varepsilon > 0, \exists \delta > 0$,s.t. $d_X(x,y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon$.*

Proof. (i) Assume f is continuous. Given $x \in X, \varepsilon > 0$, $f^{-1}(B(f(x),\varepsilon))$ is open.$\therefore \exists \delta > 0$, s.t. $B(x,\delta) \subset f^{-1}(B(f(x),\varepsilon))$.



(ii) Assume the ε, δ condition holds. Let $V \subset Y$ be open. For $\forall x \in f^{-1}(x), \exists \varepsilon > 0$, s.t. $B(f(x), \varepsilon) \subset V$. $\therefore \exists \delta > 0$, s.t.$B(x, \delta) \subset f^{-1}(B(f(x), \varepsilon)) \subset f^{-1}(V)$.$\therefore f^{-1}(V)$ is open.

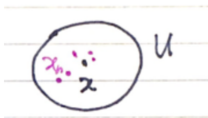


$\square$

Lemma 1.60 (The sequence lemma)**.** *X a topological space, $A \subset X$. If there is a sequence of points of A converging to x, then $x \in \overline{A}$; the converse holds if X is metrizable.*

Proof. (i) Suppose $x_n \to x$, where $x_n \to A$. Then any neighborhood U of x, $U \cap A \neq \varnothing$. $\therefore x \in \overline{A}$.

(ii) Suppose (X, d) is a metric space. For $x \in \overline{A}, B\left(x, \dfrac{1}{n}\right) \cap A \neq \varnothing$ for all $n \in \mathbb{N}$. Choose $x_n \in B\left(x, \dfrac{1}{n}\right) \cap A$. Then for any neighborhood U of x, $B\left(x, \dfrac{1}{n}\right) \subset U$ for n large. $\therefore x_n \in U$ for x large. i.e. $x_n \to x$.



$\square$

Theorem 1.61. *Let $f : X \to Y$. If f is continuous, then for every converging sequence $x_n \to x$ in X,$f(x_n)$ converges to $f(x)$; the converse holds if X is metrizable.*

Proof. (i) Assume f is continuous. Given $x_n \to x$, let V be a neighborhood of $f(x)$, then $f^{-1}(V)$ is a neighborhood of x. $\therefore \exists N$ s.t. $x_n \in f^{-1}(V)$ for $n \geqslant N$. Therefore, $f(x_n) \in V$ for $n \geqslant N$, i.e. $f(x_n) \to f(x)$.

(ii) Assume the convergent condition holds. Let $A \subset X$ be a subset, we show that $f(\overline{A}) \subset \overline{f(A)}$. For $x \in \overline{A}$, $\exists$ a sequence $x_n \to x, x_n \in A$. Then $f(x_n) \to f(x)$, and $f(x_n) \in f(A)$. $\therefore f(x) \in \overline{f(A)}$.$\therefore$ $f(\overline{A}) \subset \overline{f(A)}$.

$\square$

Theorem 1.62. *If $f, g : X \to \mathbb{R}$ are continuous functions, then $f \neq g, f \cdot g$ and f/g (If $g(x) \neq 0$ for all x) are continuous.*

Proof. $X \xrightarrow{f \times g} \mathbb{R} \times \mathbb{R} \xrightarrow{\cdot} \mathbb{R}$. $\square$

Definition 1.63. *Let $f_n : X \to Y$ be a sequence of maps from a set X to a metric space (Y, d). The sequence (f_n) <u>converges uniformly</u> (一致收敛) to the map $f : X \to Y$ if given $\varepsilon > 0, \exists$ an integer N s.t. $d\left(f_n(x), f(x)\right) < \varepsilon$ for all $n > N$ and all $x \in X$.*

Theorem 1.64 (Uniform limit theorem)**.** *Let $f_n : X \to Y$ be a sequence of continuous maps from a topological space to a metric space Y. If (f_n) converges uniformly to f, then f is continuous.*

Proof. Let $V \subset Y$ open, $x_0 \in f^{-1}(V), y_0 = f(x_0), B(y_0, \varepsilon) \subset V$. Choose N s.t. for all $n \geqslant N, x \in X, d\left(f_n(x), f(x)\right) < \dfrac{\varepsilon}{3}$. f_N is continuous $\Rightarrow f_N^{-1}\left(B\left(f_N(x_0), \dfrac{\varepsilon}{3}\right)\right) = U$ is a neighborhood of x_0.

Claim: $f(U) \subset B(y_0, \varepsilon) \subset V$.

Proof. $\forall x \in U$,

$$\begin{cases} d\left(f(x), f_N(x)\right) < \dfrac{\varepsilon}{3} \\ d\left(f_N(x), f_N(x_0)\right) < \dfrac{\varepsilon}{3} \\ d\left(f_N(x_0), f(x_0)\right) < \dfrac{\varepsilon}{3} \end{cases}$$

$\therefore d(f(x), f(x_0)) < \varepsilon$. □

□

Chapter 2

Tolological propertres

2.1 connected spaces and path connected spaces

Definition 2.1. *Let X be a topological space. A separation of X is a pair U,V of dispoint nonempty open subsets of X.whose union is X. The space X is said to be connected (连通) if there does not exist a separation of X.*

The following is an alternative definition.

Definition 2.2. *A space X is connected iff the only subsets of X that are both open and closed are $\varnothing$ and X.*

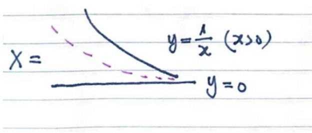
Examples:

(1) $X = \{a, b\}$ with the trivial topology is connected.

(2) $Y = [-1, 0) \cup (0, 1]$ is not connected.

(3) $Q \subset R$ with the subspace topology is not connected. Choose $a \in R - Q$,then $((-\infty, a) \cap Q) \cup ((a, +\infty) \cap Q) = Q$.

(4) There is a counter-example.



not connected

Lemma 2.3. *Let $X = C \cup D$ be a separation, $Y \subset X$ a connected subspace, then $Y \subset C$ or $Y \subset D$.*

Proof. $C, D \subset X$ are open, so $C \cap Y$,$D \cap Y$ open in Y. Y is connected $\Rightarrow C \cap Y$ or $D \cap Y = \varnothing$. ☐

Theorem 2.4. *The union of a collection of connected subspaces of X that have a point in common is connected.*

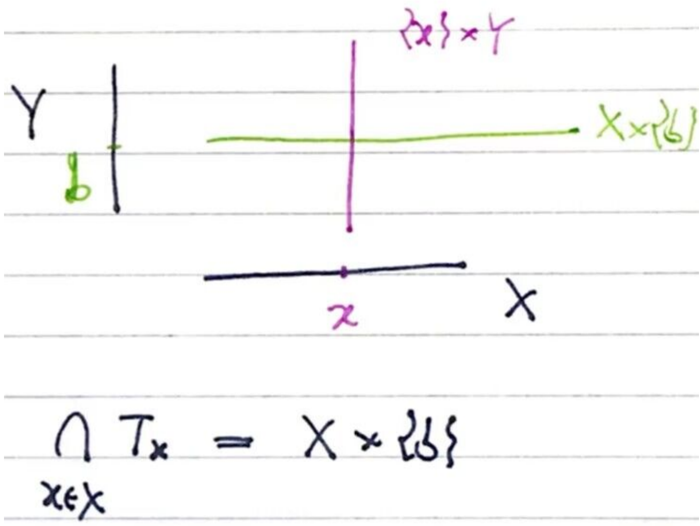
Proof. Let $\{A_\alpha\}$ be a collection of connected subspaces of X, $p \in \bigcap\limits_{\alpha} A_\alpha$,$Y = \bigcap\limits_{\alpha} A_\alpha$. we prove that Y is connected. Suppose $Y = C \cup D$ is a separation, $p \in C$ then $A_\alpha \subset C, \forall\alpha$, so $Y = C$. ☐

Theorem 2.5. *The image of a connected space under a continuous map is connected.*

Proof. Let $f : X \longrightarrow Y$ be a continuous map, $Z = f(x) \subset Y$. Let $Z = A \cup B$ be a separation, then $X = f^{-1}(A) \cup f^{-1}(B)$ is a separation, a contradrtion. □

Theorem 2.6. *A finite cartesian product of connected spaces is connected.*

Proof. Let X, Y be connected spaces. we shall show that $X \times Y$ is connected. Let $T_x = X \times \{b\} \cup \{x\} \times Y$, then T_x is connected $\forall x \in X$. Now $X \times Y = \bigcup_{x \in X} T_x$ and $\bigcap_{x \in X} T_x = X \times \{b\}$. So $X \times Y$ is connected.



□

Remark:

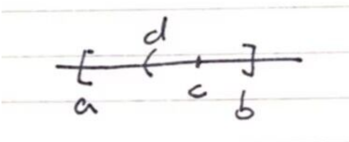
(i) $\mathbb{R}^\omega$ in the box topology is not connected.

(ii) An arbitrary product of connected spaces is connected in the product topology, e.g. $\mathbb{R}^\omega$.

Theorem 2.7. *The real line $\mathbb{R}$ is connected, and so are intervals and rays in $\mathbb{R}$.*

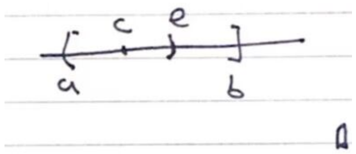
Proof. Assume $\mathbb{R} = A \cup B$ is a separation, $a \in A$, $b \in B$, $a < b$,$A_0 = A \cap [a, b]$, $B_0 = B \cap [a, b]$, $c = \sup A_0 \in [a, b]$

(i) if $c \in B_0$, then $c > a$(since A is open). B_0 is open in $[a, b] \Rightarrow \exists d < c$,s.t. $(d, c] \subset B_0$, then $\sup A_0 \leqslant d < c$.



(ii) if $c \in A_0$, then $c < b$ (since B is open).A_0 is open in $[a, b] \Rightarrow \exists e > c$, s.t. $[c, e) \subset A_0$. So $supA_0 \geqslant e > c$.

□



Theorem 2.8 (Intervediate value theorem)**.** *If $f : X \longrightarrow \mathbb{R}$ is a continuous function, X is a connected space. Let $a, b \in X$, $f(a) < f(b)$. Then for any $r \in (f(a), f(b))$, $\exists c \in X$, s.t. $f(c) = r$.*

Proof. Assume $r \notin f(x)$, let $A = f(x) \cap (-\infty, r)$. $B = f(x) \cap (r, +\infty)$, then $f(x) = A \cup B$ is a separation, a contradrtion since $f(x)$ is connected. □

Definition 2.9. *Define an equivalence relation $\sim$ on a topological space X: $x \sim y$ if $\exists$ a connected subspace of X containing both x and y. The equivalence classes are called connected components (连通分支) of X.*

Theorem 2.10. *The components of X are connected disjoint subspaces of X whose union is X, such that each nonempty connected subspace of X intersects only one of them.*

i.e. $X = \coprod_{\alpha} X_\alpha$, X_α connected components, $\forall A \subset X, \exists! \alpha$, s.t.$A \subset X_\alpha$.

Proof. (i) $X_\alpha \cap X_\beta = \varnothing, X = \bigcup_{\alpha} X_\alpha$ by definition.

(ii) $A \subset X$ connected, assume $\exists$ components C_1, C_2, s.t. $x_1 \in A \cap C_1, x_2 \in A \cap C_2$, then $x_1 \sim x_2$, so $C_1 = C_2$. So$\exists!$ one component C, s.t.$A \subset C$

(iii) we show esch component C is connected: choose $x_0 \in C$, $\forall x \in C$, $\exists A_x$ connected. s.t. $x_0, x \in A_x$ (since $x_0 \sim x$), and $A_x = C$. $C = \bigcup_{x \in C} A_x$ and $x_0 \in \bigcap_{x \in C} A_x$. So C is connected.

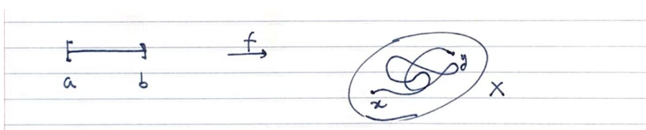
□

Remark: $X = \coprod_{\alpha} X_\alpha$, X_α connected components. Then $\overline{X}_\alpha = X_\alpha$(in general, $A \subset X$ connected $\to \overline{A}$ connected).

So each component is closed. if there are only finitely many components, then each component is also open.

e.g. $Q \subset \mathbb{R}$, each point is a component,closed but not open.

Definition 2.11. *A path (道路) in a topological space X from x to y is a continuous map $f : [a, b] \longrightarrow$ s.t.$f(a) = x, f(b) = y$.*



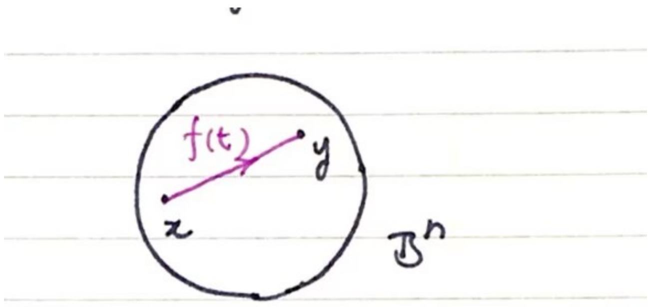
A space X is said to be path connected (道路连通) if every pair of in X can be joined by a path in X.

Remark: path-connected $\rightarrow$ connected, since $[a, b]$ is connected.

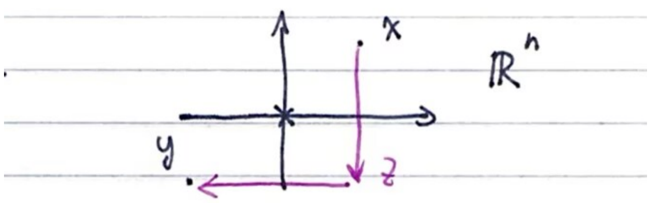
Examples:

(1) The unit ball $B^n = \left\{ x \in R^n \middle| \left(x_1^2 + x_2^2 + \cdots + x_n^2\right)^{1/2} \leqslant 1 \right\}$ is connected.$f(t) = (1-t)\,x + ty$,$t \in [0,1]$.

$\|f(t)\| \leqslant (1-t)\,\|x\| + t\,\|y\| \leqslant 1$



(2) the punctured euclidean space $\mathbb{R}^n - \{0\}$ is path-connected for $n > 1$.



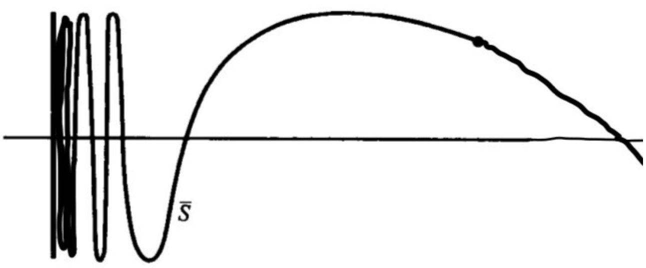
(3) the unit sphere $S^{n-1} = \{x \in \mathbb{R}^n | \|x\| = 1\} \subset \mathbb{R}^n$ is path-connected for $n > 1$.

$g : \mathbb{R}^n - \{0\} \longrightarrow S^{n-1}$, $x \mapsto \frac{x}{||x||}$ is continuous and surjective.

Remark: $\mathrm{GL}_n(\mathbb{R}) \subset M_n(\mathbb{R}) = \mathbb{R}^{n^2}$. Is $\mathrm{GL}_n(\mathbb{R})$ path-connected?

Examples:

(1) The topologists' sine curve



$S = \left\{ \left(x, \sin\frac{1}{x}\right) | 0 < x \leqslant 1 \right\} \subset \mathbb{R}^2$.

$\overline{S}$ = the closure of S in $\mathbb{R}^2$ is called the topologist's sine curve. $\overline{S} = S \cup \{0\} \times [-1, 1]$. S is connected $\rightarrow \overline{S}$ is connected.

Claim. $\overline{S}$ is not path-connected.

Proof. Let $f : [a, c] \longrightarrow \overline{S}$ be a path s.t. $f(a) = 0, f(c) \in S. f^{-1}(\{0\} \times [-1, 1]) \subset [a, c]$ is closed. So $\exists$ a largset element b.

Then $f : [a, c] \longrightarrow \overline{S}$ with $f(b) \in \{0\} \times [-1, 1]$, $f(t) \in S$ $(t > b)$. Assume b = 0, c = 1, then $f : [0, 1] \longrightarrow \overline{S}$ with $f(0) \in \{0\} \times [-1, 1]$, $f(t) \in S$, for $t > 0$. Let $f(t) = (x(t), y(t))$, then $x(0) = 0$, $x(t) > 0$ for $t > 0$. $y(t) = \dfrac{1}{\sin x(t)}$ for $t > 0$. We find a sequence t_n, $t_n \longrightarrow 0$ with $y(t_n) = (-1)^n$., as follows: take $0 < x_n < x(\frac{1}{n})$ s.t. $\sin \dfrac{1}{x_n} = (-1)^n . x : [0, 1] \longrightarrow \mathbb{R}$ a continuous function with $x(0) = 0$. Intervediate value theorem $\rightarrow \exists t_n \in (0, \dfrac{1}{n})$,s s.t. $x(t_n) = x_n$. So $y(t_n) = \dfrac{1}{\sin x_n} = (-1)^n$.

$\square$

Definition 2.12. *Define an equivalence relation $\sim$ on X: $x \sim y$ if $\exists$ a path in X from x to y . The equivalence classes are called the path components of X. (道路连通分支).*

Theorem 2.13. *The path components of X are path-connected disjoint subspaces of X whose union is X, such that each nonempty path-connected subspaces of X intersects only one of them.*

Definition 2.14. *A space X is said to be locally connected (局部连通) at x, if for every neighborhood μ of x, $\exists$ a connected neighborhood V of x, contained in U. If X is locally connected at each $x \in X$, it is said to be locally connected .*

Examples:

(1) the topologist's sine curve is connected but not locally connected.

(2) Q $\subset \mathbb{R}$ is not locally connected.

2.2 compactness

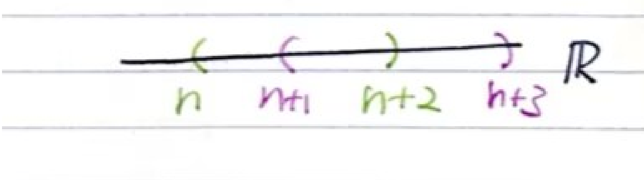
Definition 2.15. *A* <u>*covering*</u> *(覆盖) of a space X is a collection $\mathscr{A}$ of subspaces whose union is equal to X. It is called an* <u>*open covering*</u> *(开覆盖) if elements in $\mathscr{A}$ are open subsets.*

Definition 2.16. *A space X is called* <u>*compact*</u> *(紧致) if every open covering of X contains a finite subcollection that also covers X.*

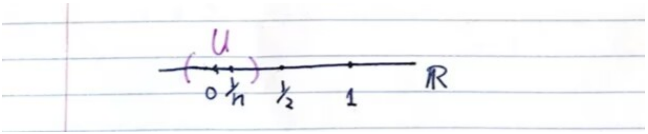
Examples:

(1) X= $\mathbb{R}$, $\mathscr{A}$= $\{(n, n+z)\,|n \in \mathbb{Z}\}$. $\nexists$ finite sub-covering.

So $\mathbb{R}$ is not compact.



(2) $X = \{0\} \cup \left\{\frac{1}{n}|n \in \mathbb{N}\right\} \subset \mathbb{R}$ is compact.



Theorem 2.17. *Every closed subspace of a compant space is compact.*

Proof. Let $Y \subset X$ be a closed subspace, X is compact. Let $\{A_\alpha\}_{\alpha\in J}$ be an open covering of Y, then $\exists$ open sets $U_\alpha \subset X$, s.t. $A_\alpha = U_\alpha \cap Y$. $\{U_\alpha|\alpha \in J\} \cup \{X - Y\}$ is an open covering of X, therefore $\exists U_{\alpha_1}, U_{\alpha_2} \ldots, U_{\alpha_n}$, s.t. $Y \subset \bigcup_{i=1}^n U_{\alpha_i}$. So $\bigcup_{i=1}^n A_{\alpha_i} = Y$. □

Theorem 2.18. *Every compact subspace of a Hausdorff space is closed.*

Proof. Let $Y \subset X$ be a compact subspace, X is Hausdorff. We shall show that $X - Y$ is open.

For $\forall x_0 \in X - Y$, $y \in Y$, $\exists$ nbhds U_y of x_0, and V_y of y, s.t. $U_y \cap V_y = \varnothing$. Now $\{V_y \cap Y|y \in Y\}$ is an open covering of Y. So $\exists$ finitely many $y_1, y_2 \ldots, y_n$, s.t. $(V_{y_1} \cap Y) \cup \cdots \cup (V_{y_n} \cap Y) = Y$. Let $V_{y_1} \cup \cdots \cup V_{y_n} = V$, then $Y \subset V$. let $U = U_{y_1} \cup \cdots \cup U_{y_n}$ be an open nbhd of x_0, then $U \cap V = \varnothing$. So $U \cap V = \varnothing$.

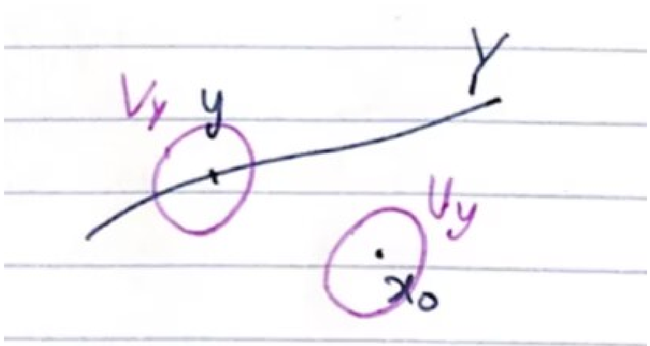
□

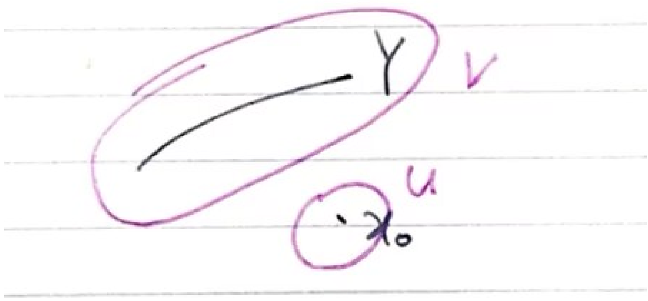
Lemma 2.19. *If Y is a compact subspace of a Hausdorff space X, and $x_0 \notin Y$. Then $\exists$ dispoint open sets U and V, of X, containing x_0 and Y, respectively.*

Example : $(a, b] \subset \mathbb{R}$ and $(a, b) \subset \mathbb{R}$ are not compact, since $\mathbb{R}$ is Hausdorff.

Remark: Given a ($\mathbb{R}$, finite complement topology). Then

(i) not Hausdorff.





(ii) every subset is compact.

(iii) proper closed subsets = finite sets.

Theorem 2.20. *The image od a compact space under a continuous map is compact.*

Proof. Let $f : X \longrightarrow Y$ be a continuous map, X is compact. Let $\mathscr{A}$ be an open covering of f(X), then $\{f^{-1}(A)|A \in \mathscr{A}\}$ is an open covering of X, So $\exists$ finitely many $A_1, \dots A_n$ s.t. $f^{-1}(A_1 \cup \dots\ f^{-1}(A_n)) = X$. So $A_1 \cup \cdots \cup A_n = f(x)$. □

Theorem 2.21. *Let $f : X \longrightarrow Y$ be a bijective continuous map, X is compact, Y is Hausdorff, then f is a homeomorphism.*

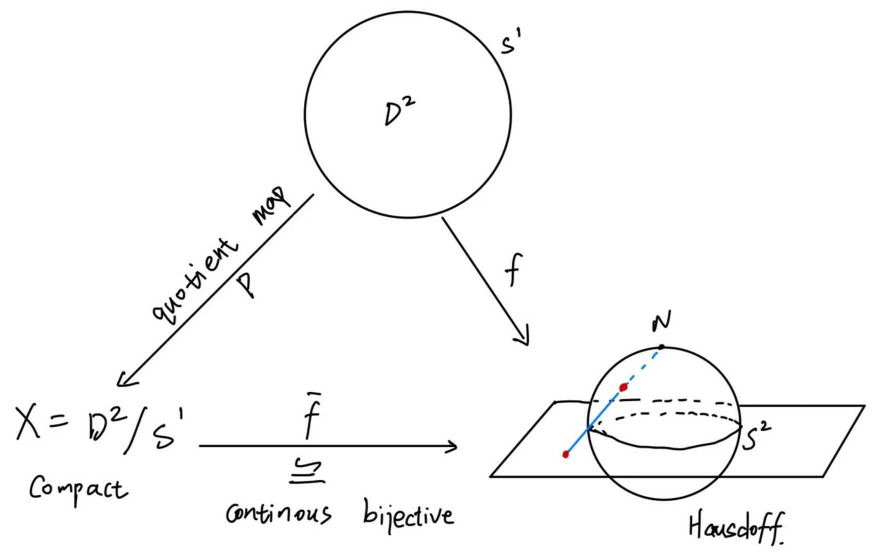
Proof. We need to show that inverse map $f^{-1} : Y \longrightarrow X$ is continuous. Let $A \subset X$ be closed subset, we want to show $(f^{-1})^{-1}(A) = f(A) \subset Y is closed$. X is compact and $A \subset X$ is closed $\Rightarrow$ A is compact. $\Rightarrow$ $f(A)$ is compact. Y is Hausdorff $\Rightarrow$ $f(A)$ is closed. □

example:

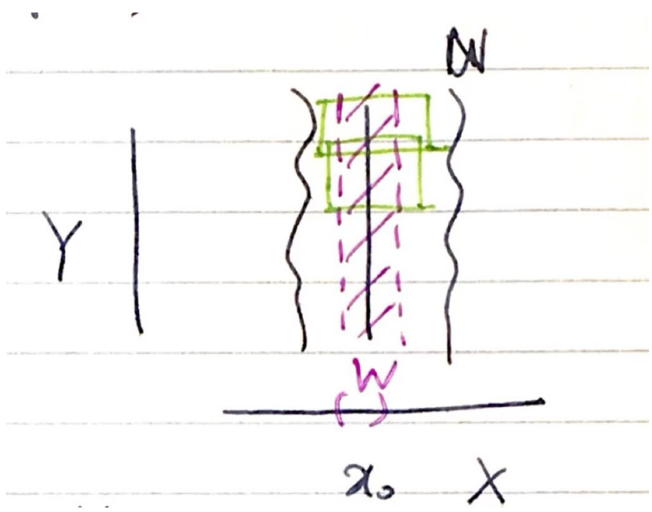
$$\begin{tikzcd}
S^2 = & N & \cup & S^2 - \{N\} \arrow[dr, "\cong"] & \\
 & & & & \mathbb{R}^2 \\
D^2 = & S^1 \arrow[uu] & \cup & D^2 - S^1 \arrow[uu, "f"] \arrow[ur, "\cong"'] &
\end{tikzcd}$$

Theorem 2.22. *The product of finitely many compact spaces is compact.*

Proof. Step 1: Let X, Y be compact, $N \subset X \times Y$ be an open nbhd of the "slice" $x_0 \times Y$. We prove that $\exists$ a nbhdW of $x_0 \in X$, s.t. $W \times Y \subset U$. $W \times Y$ is called a tube about $x_0 \times Y$. Basis



elements of product topology have the form $U \times V$, $U \subset X$, $V \subset Y$ open. $N \subset X \times Y$ open $\Rightarrow$ N is covered by $U \times V$ with $U \times V \subset N$. Since $x_0 \times Y$ is compact, $\exists$ finitely many such basis elements $U_1 \times V_1, \ldots, U_n \times V_n$ s.t. $x_0 \times Y \subset \bigcup_{i=1}^{n} (U_i \times V_i)$. and $x_0 \times Y \cap (U_i \times V_i) \neq \varnothing$. Define $W = U_1 \cap \cdots \cap U_n$.



<u>Claim</u> : $W \times Y \subset \bigcup_{i=1}^{n} (U_i \times V_i)$

$\forall\, (x, y) \in W \times Y$, $\exists i$ s.t. $(x_0, y) \in U_i \times V_i$

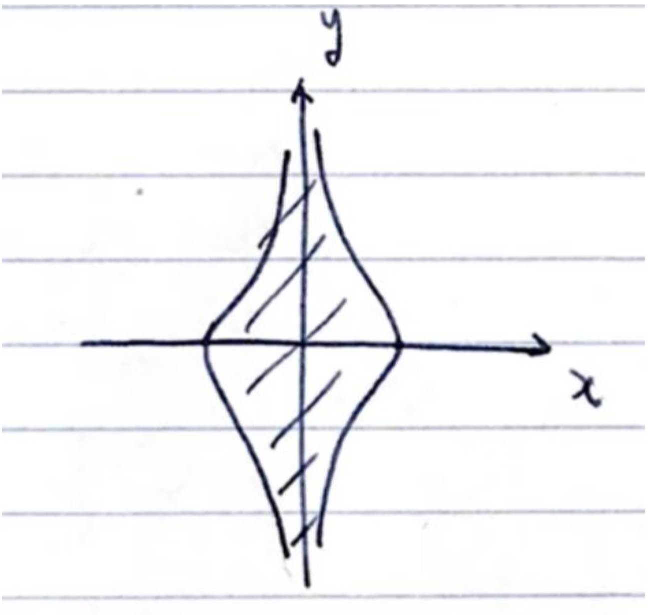
$x \in W \subset U_i$, So $(x, y) \in U_I \times V_i$.

So $W \times Y \subset \bigcup_{i=1}^{n} (U_i \times V_i) \subset N$. $\square$

Step 2: Let $\mathscr{A}$ be an open covering of $X \times Y$. For $\forall x \in X$, Since $X \times Y$ is compact, $\exists$ finitely many $A_1, \ldots A_m$ in $\mathscr{A}$ s.t. $Y \subset A_1 \cup \cdots \cup A_m = N$. By step 1, $\exists$ a tube $W_x \times Y$ s.t. $x \times Y \subset W_x \times Y \subset N$. $\{W_x | x \in X\}$ is an open covering of X, $\therefore \exists$ finitely many $W_1 \ldots W_k$, s.t. $\bigcup_{i=1}^{k} W_i = X$. $\therefore (W_1 \times Y) \cup \cdots \cup (W_k \times Y) = X \times Y$, Each $W_i \times Y$ is covered by finitely many elements in mA.

$\square$

Example :



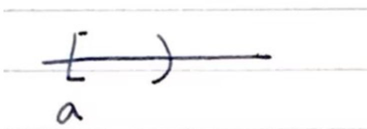
$Y = the\ y - axis$, $N = \left\{ \left(x, y||x| < \frac{1}{y^2+1} \right) \right\}$
Then $\nexists$ a tube for $0 \times Y$.

Theorem 2.23 (Tychonoff)**.** *An arbitrary product of compact spaces is compact in the product topology.*

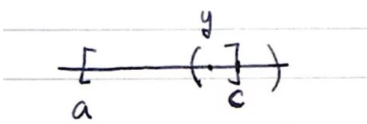
Theorem 2.24. *A closed interval* $[a, b] \subset \mathbb{R}$ *is compact.*

Proof. Proof. Let $\mathscr{A}$ be an open covering of $[a, b]$,
$C = \{y \in [a, b]|[a, b]$can be by finitely many elements in$\mathscr{A}\}$. Let $c = supC$, then we must have

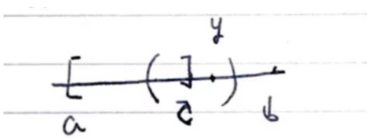
(i) c>a ,since



(ii) $c \in C$, since



(iii) c = b , since



$\square$

Theorem 2.25. *A subspace $A \in \mathbb{R}_n$ is compact iff it is closed and is bounded in the euclidean metric d or square metric ρ.*

Proof. $\rho(x,y) \leqslant d(x,y) \leqslant \sqrt{n}\rho(x,y)$ $\therefore$ it suflices to consider ρ.

(i) Assume A is compact. Then A is closed. Consider $\{B_\rho(0,m)|m \in \mathbb{N}\}$, $\bigcup_{m=1}^{\infty} = \mathbb{R}^n$. A is compact. $\therefore$ $\exists$ an M, s.t. $A \subset B_\rho(0,M)$, i,e, A is bounded.

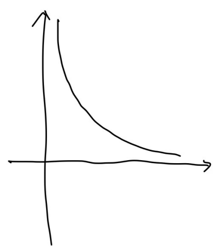
(ii) Assume A is closed and bounded, i.e. $\exists N$, s.t. $\rho(x,y) \leqslant N$ $\forall$ x,y $\in A$. Choose $x_0 \in A$, with $\rho(x_0,0) = b$. Then $\rho(x_0,0) \leqslant N+b\, \forall x \in A$. $\therefore A \subset [-(N+b), N+b]^n$ closed and $[-(N+b), N+b]^n$ is compact. $\therefore$ A is compact.

□

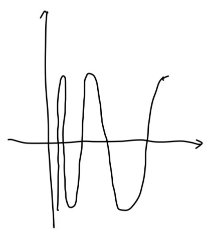
Examples:

(1) the unit sphere $S^{n-1} \subset \mathbb{R}^n$, unit ball $B^n \subset \mathbb{R}^n$are compact.

(2) $A = \left\{(x, \frac{1}{x})|0 < 1 \leqslant 1\right\} \subset \mathbb{R}^n$ closed but not bounded, hence not compact.



(3) $S = \left\{(x, sin\frac{1}{x})|0 < x \leqslant 1\right\}$ bounded but not closed hence not compact.

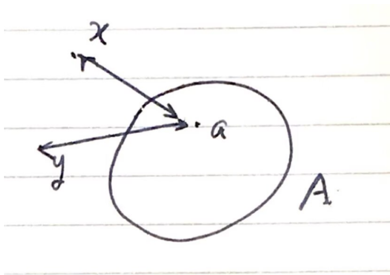


Theorem 2.26 (Extreme value theorem)**.** *$f : X \longrightarrow \mathbb{R}$ a continuous function, X is compact. Then $\exists c, d \in X$, s.t. $f(c) \leqslant f(x) \leqslant f(d)$ for $\forall x \in X$.*

Proof. $A = f(x) \subset \mathbb{R}$ is compact, hence is bounded and closed. bounded $\Rightarrow$ $infA = m > -\infty$, $supA = M < +\infty$. closed $\Rightarrow m \in A, M \in A$. □

Definition 2.27. *(X,d) is metric space, $A \subset X$ a non-empty subspace.*
Then for each $x \in X$, the distance from x to A (从 x 到 A 的距离) is $d(x,A) = \inf\{d(x,a)|a \in A\}$. This is a continuous function in X:

Check: $\forall x, y \in X$,



$d(x,A) \leqslant d(x,a) \leqslant d(x,y) + d(y,a) \therefore d(x,A) \leqslant d(x,y) + d(y,A)$, i.e. $|d(x,A) - d(y,A)| \leqslant d(x,y)$.

$A \subset X$ a bounded subspace, the diameter of A is

$$diam(A) = \sup\{d(a_1,a_2)|a_1,a_2 \in A\}$$

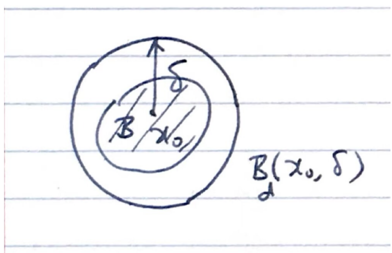
.

Lemma 2.28 (the Lebesgue number lemma). *Let $\mathscr{A}$ be an open covering of a metric space (X,d) . if X is compact, then $\exists$ a $\delta > 0$, s.t. for each subspace of diameter $< \delta$, there exists an elements of $\mathscr{A}$ containing it. The number δ is called a Lebesgue number of $\mathscr{A}$.*

Proof. Assume $X \notin \mathscr{A}$. Choose a finite collection $\{A_1,\ldots,A_n\}$ of $\mathscr{A}$ that covers X. Let $C_i = X - A_i$, define $f: X \longleftarrow \mathbb{R}$, $f(x) = \frac{1}{n}\sum_{i=1}^{n} d(x,C_i)$ a continuous function.

$\forall x \in X$, $\exists A_i$ s.t. $x \in A_i$, and A_i is open. $\therefore d(x,C_i) > 0$. $\therefore f(x) > 0$. Let $\delta = min\{f(x)|x \in X\} > 0$.

We show that δ is a Lebesgue number of $\mathscr{A}$. Let $B \in X$ with $diam(B) < \delta$, choose $x_0 \in B$, then $B \subset B_d(x_0,\delta)$. Now $\delta \leqslant f(x_0) = \frac{1}{n}\sum_{i=1}^{n} d(x_0,C_i) \leqslant \max_{i=1\ldots n} d(x_0,C_i) = d(x_0,C_m)$. Then $B_d(x_0,\delta) \subset X - C_m = A_m$.



□

Definition 2.29. *A topological space X is said to be sequentially compact (列紧) if every sequence of points of X has a convergent subsequence.*

Theorem 2.30. *If X is a metrizable space, then the following are equivalent.*

(1) X is compact.

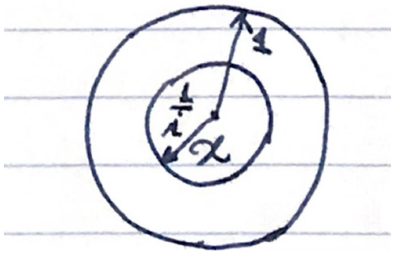
(2) X is sequentially compact.

Proof. (1) $\Rightarrow$ (2):

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X, $A = \{x_n|n \in \mathbb{N}\}$. If A is finite, then $\exists$ infinitely many $n \in \mathbb{N}$, s.t. $x_n = x$, giving a convergent subsequence. Now assume A is infinite.

(i) A has a limit point: if not, then $A = \bar{A}$ is closed ($recall that \bar{A} = A' \cup A$). For $\forall a \in A$, $\exists$ a nbhd U_a s.t. $U_a \cap A = a$. Now $\{U_a | a \in A\} \cup \{X - A\}$ is an open covering of X, and X is compact, $\therefore$ $\exists$ a finite subcovering , therefore A is finite.

(ii) Let x be a limit point of A, we will find a subsequence (x_{n_i}) converging to x ; Choose $x_{n_1 \in B(x,1)}$, inductivelly, since $B(x, \frac{1}{i}) \cap A$ is infinite, we may choose $n_i > n_{i-1}$ s.t. $x_{n_i} \in B(x, \frac{1}{i})$. The subsequence (x_{n_i}) converges to x.



□

(2) ⇒ (1):

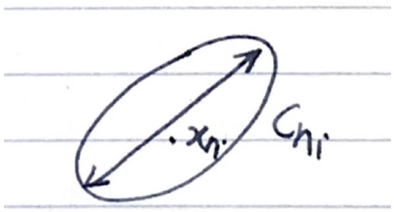
Step 1 : the Lebesgue number lemma holds for X, i.e. foa an open covering $\mathscr{A}$ of X, $\exists$ a $\delta > 0$, s.t. foa any $B \subset X$ with $diamB < \delta$, $\exists A \in \mathscr{A}$ s.t. $B \subset A$.

Proof. Assume there is no such δ. Then for $\forall n \in \mathbb{N}$, $\exists C_N \subset X$, s.t. $diamC_n < \frac{1}{n}$, but C_n is not contained in any $A \in \mathscr{A}$. Choose $x_n \in C_n$ for each n, then by assumption, $\exists$ a subsequence (x_{n_i}) of (x_n), converging to a. $\exists A \in \mathscr{A}$, s.t. $a \in A$. $\exists \varepsilon > 0$, s.t. $B(a, \varepsilon) \subset A$.

· if i is large enough s.t. $\dfrac{1}{n_i} < \dfrac{\varepsilon}{2}$, then $C_{n_i} \subset B(x_{n_i}, \dfrac{\varepsilon}{2})$

· i is large enough then $d(x_{n_i}, a) < \dfrac{\varepsilon}{2}$.

⇒ for i large enough $C_{n_i} \subset B(x_{n_i}, \dfrac{\varepsilon}{2}) \subset B(a, \varepsilon) \subset A$. □



Step 2 : given $\varepsilon > 0$, $\exists$ a finite covering of X by open $\varepsilon - balls$.

Proof. Assume $\exists$ an $\varepsilon > 0$, s.t. X cannot be covered by finitely many open $\varepsilon - balls$. Start from $x_1 \in X$, $\exists x_2 \notin B(x_1, \varepsilon)$, $x_3 \notin B(x_1, \varepsilon) \cup B(x_2, \varepsilon)$, and so on. Weget a sequence (x_n) s.t. $d(x_{n+1}, x_i) \geqslant \varepsilon$ for $i = 1, \ldots, n$. Therefore (x_n) has no convergent subsequence.

□

Step 3 : sequentially compact ⇒ compact.

Proof. Let $\mathscr{A}$ be an open covering of X, δ be a Lebesgue number of $\mathscr{A}$, $\varepsilon = \frac{\delta}{3}$. There is a finite covering of X by open $\varepsilon - balls$ $B_1, \ldots, B_n$. Now $diamB_i \leqslant \frac{2\delta}{3} < \delta$. $\therefore \exists A_i \in \mathscr{A}$ s.t. $B_i \subset A_i$. $\therefore \{A_1, \ldots, A_n\}$ is a finite covering of X. □

Definition 2.31. *If Y is a compact Hausdorff space and X is A proper subspace of Y whose closure equals Y, then Y is said to be a compactification (紧化) of X.*

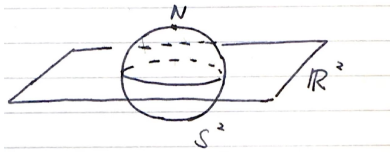
Examples:

(1) $X = (a, b)$, $Y = [a, b]$

(2) $X = \mathbb{R}$, $Y = S^1$

(3) $X = \overset{\circ}{D}{}^2$ the open 2-ball, $Y = D^2$ the losed 2-ball

(4) $X = \mathbb{R}^2$, $Y = S^2$



Definition 2.32. *A space X is said to be locally compact at X , if there is some compact subspace C of X that contains a nbhd of x. If X is locally compact at each x, X is said to be locally compact .*

Examples:

(1) $\mathbb{R}$ is locally compact, $x \in (a, b) \subset [a, b]$

(2) $\mathbb{R}^n$ is locally compact.

(3) $\mathbb{R}^\omega$ is not locally compact.

(4) $Q \subset \mathbb{R}$ is not locally compact.

Theorem 2.33. *Let X be a space. Then X is locally compact Hausdorff iff there exists a space Y satisfying the following conditions.*

(1) X is a subspace of Y.

(2) Y-X consists of a single point ∞.

(3) Y is a compact Hausdorff space.

if Y and Y' are two spaces satisfying these conditions, then there is a homeomorphism of Y with Y' that equals the identity map on X.

Remark: if X is compact, then $Y = X \cup \{\infty\}$.

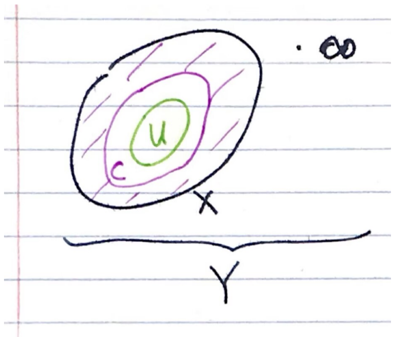
if X is not compact, then ∞ is a limit point of X and $\bar{X} = Y$. In this case Y is called the one-point compactification of X. (一点紧化)

e.g. $X = \mathbb{R}^n$, one-point compactification $= S^n$.

Proof. (1) uniqueness :

$$\begin{array}{ccc} Y & \xrightarrow{1:1} & Y', \quad \infty \mapsto \infty' \\ \cup & & \cup \\ X & \xrightarrow{\text{id}} & X \end{array}$$

(2) (i) construction of Y, $Y = X \cup \{\infty\}$ with topology $\mathcal{T} = \{U | U \subset X open\} \cup \{Y - C | C \subset X compact\}$
Check : intersection:



· $U_1 \cap U_2$ open

· $(Y - C_1) \cap (Y - C_2) = Y - (C_1 \cup C_2)$

· $U \cap (Y - C) = U \cap (X - C)$

C closed, ∴ X-C open.

union:

· $\bigcup_\alpha U_\alpha = U$ open

· $\bigcup_\beta (Y - C_\beta) = Y - \bigcap_\beta C_\beta = Y - C.$

· $(\bigcup_\alpha) U_\alpha \cup \bigcup_\beta (Y - C_\beta) = U \cup (Y - C) = Y - (C - U)$, C-U is compact.

(ii) X is a subspace of Y.

(iii) Y is compact: let $\mathscr{A}$ be an open covering of Y. Then $\exists A_1 \in \mathscr{A}$, $A_1 = Y - C$. $A' = \{A \in \mathscr{A} | A \neq A_1\}$ is an open covering of C. ∴ ∃ a finite subcovering.

(iv) Y is Hausdorff: to separate $x \in X$ and $y = \infty$, we need the local compactness of X.

(3) the converse is true.

□

Definition 2.34. *A subset A of a space X is said to be* <u>*dense*</u> *(稠密) in X if* $\bar{A} = X$.

2.3 countability

Definition 2.35. *If a space X has countable basis for its topology, then X is said to satisfy the Second countability axiom (第二可数性公理) or to be second-countable (第二可数的).*

Examples:

(1) $\mathbb{R} : \mathcal{B} = \{(a,b)|a,b \in \mathbb{Q}\}$

$\mathbb{R}^n : \mathcal{B} = \{B(x,r)|x \in \mathbb{Q}^n, r \in \mathbb{Q}\}$

or $\mathcal{B}' = \left\{ \prod_{i=1}^{n} [a_i, b_i] \middle| a_i, b_i \in \mathbb{Q} \right\}$.

(2) $\mathbb{R}_\omega : \mathcal{B} = \left\{ \prod_{n=1}^{\infty} U_n \middle| U_n = (a_n, b_n), a_n, b_n \in \mathbb{Q} \right\}$ for finitely many values of n; $U_n = \mathbb{R}$ for all other n.

Theorem 2.36. *A subset of a second-countable space is second-countable; a countable product of second countable spaces is second-countable.*

Proof. (i) Let $\mathcal{B}$ be a countable basis for X, $A \subset X$ a subspace,

then $\{A \cap B | B \in \mathcal{B}\}$ is a countable basis for A.

(ii) Let $\mathcal{B}_n$ be a countable basis for X_n, $n \in \mathbb{N}$.

Then

$\left\{ \prod_{n=1}^{\infty} U_n \middle| U_n \in \mathcal{B}_n \text{for finitely many values of n, and} U_n = X_n \text{for all other n} \right\}$.

□

Theorem 2.37. *Suppose that X is second-countable. Then*

(i) Every open covering of X contains a countable subcollection covering X.

(ii) There exists a countable subset of X that is dense in X.

Proof. Let $\mathcal{B} = \{\mathcal{B}_n\}$ be a countable basis of X.

(i) Let $\mathscr{A}$ be an open covering. For each $n \in \mathbb{N}$, if $\exists$ some $A \in \mathscr{A}$ s.t. $B_n \subset A$, then we choose such an element and denote it by A_n. $\mathscr{A}' = \{\text{all such } A_n\}$ is a countable subcollection of $\mathscr{A}$.

Claim : $\mathscr{A}'$ is a covering of X.

Proof. $\forall x \in X$, $\exists A \in \mathscr{A}$ s.t. $x \in A$. $\therefore \exists$ a basis element B_n s.t. $x \in B_N \subset A \therefore A_n$ is defined for this n and $x \in B_n \subset A_n$.

□

(ii) For each $n \in \mathbb{N}$, choose $x_n \in B_n$, let $D = \{x_n | n \in \mathbb{N}\}$. Then for $\forall x \in X$, any *neighborhood*U of X, $\exists B_n \subset U$ $x_n \in B_n \subset U \therefore D \cap U = \varnothing \therefore x \in \overline{D}$.

□

Remark:

· compact $\Longleftrightarrow$ $\exists$ finite sub-covering

· second-countable $\longrightarrow$ $\exists$ countable sub-covering (Lindelöf)

· paracompact $\Longleftrightarrow$ $\exists$ locally finite refinement.

2.4 separation axioms

Definition 2.38. *A space X is said to be Hausdorff if for each $x, y \in X$, $x \neq y$, there exist dispoint open sets containing x and y, respectively. (T_2 axiom)*

Remark: X Hausdorff $\Rightarrow$ one-point set are closed in X.

Definition 2.39. *Suppose that one-point sets are closed in X.*

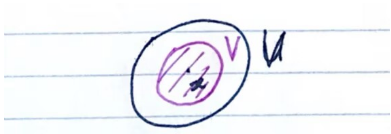
(1) X is said to be regular (正则) if for each point x, and closed set B, $x \notin B$, there exist dispoint open sets containing x and B, respectively. (T_3 axiom)

(2) X is said to be normal (正规) if for each pair of dispoint closed set A, B, there exist dispoint open sets containing A and B, respectively.(T_4 axiom)

Remark: normal $\Longrightarrow$ regular $\Longrightarrow$ Hausdorff.

Lemma 2.40. *Let X be a topological space, assume one-point sets in X are closed.*

(1) X is regular $\Longleftrightarrow$ for each $x \in X$ and a neighborhood U of x. $\exists$ a neighborhood V of x, s.t. $\overline{V} \subset U$.



(2) X is normal $\Longleftrightarrow$ for any closed set $A \subset X$ and a neighborhood U of A, $\exists$a a neighborhood V of A, s.t. $\overline{V} \subset U$.

Proof. (1) "$\Longrightarrow$": Given x, U, let $B = X - U$, then $\exists$ open sets V, W, s.t. $x \in V$, $B \subset W$, $V \cap W = \varnothing$. For $\forall y \in B$, W is a neighborhood of y and $W \cap V = \varnothing$ $\therefore y \notin \overline{V}$ $\therefore \overline{V} \cap B = \varnothing$ i.e. $\overline{V} \subset U$.

(2) "$\Longleftarrow$" : Given x and B, let $U = X - B$, a neighborhood of X. Then $\exists$ a neighborhood V of x s.t. $\overline{V} \subset U$. $X - \overline{V}$ is a neighborhood of B and $V \cap (X - \overline{V}) = \varnothing$, ☐

Theorem 2.41. *(1) A subspace of a Hausdorff space is Hausdorff; a product of Hausdorff spaces is Hausdorff.*

(2) A subspace of a regular space is regular; a product of regular spaces is regular.

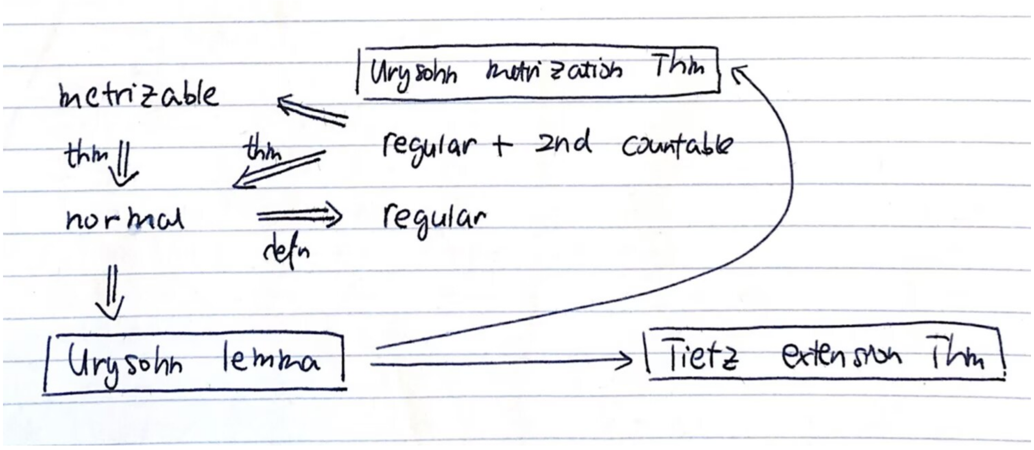
Proof. (2).

(i) $Y \subset X$ a subspace, then one-point sets are closed in Y. Let $x \in Y$, $B \subset Y$ closed, $x \notin B$. Let $\overline{B} =$ the closure of B in X, then $\overline{B} \cap Y = B$. Now $x \notin \overline{B}$, X is regular $\therefore \exists$ dispoint open sets U, V, $x \in U$, $\overline{B} \subset Y$. Then $U \cap Y$, $V \cap Y$ are dispoint neighborhoods of x and B in Y.

(ii) Let $\{X_\alpha\}$ be a family of regular spaces. $X = \prod\limits_\alpha X_\alpha$. Then X is Hausdorff, hence one-point sets are closed. Let $x = (x_\alpha) \in X$, U be a neighborhood of x in X. Choose a basis element $\prod\limits_\alpha U_\alpha \subset U$, then for $\forall$ α, $\exists$ a neighborhood V_α of x_α in X_α. Then $V = \prod\limits_\alpha V_\alpha$ is a neighborhood of x and $\overline{V} = \prod\limits_\alpha \overline{V}_\alpha$ (by the following theorem) $\subset U$

□

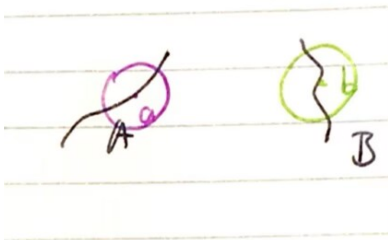
Theorem 2.42. *Let $\{X_\alpha\}$ be a family of spaces, $A_\alpha \subset X_\alpha$ be subspaces. Then $\prod_\alpha \overline{A}_\alpha = \overline{\prod_\alpha A_\alpha}$ (in the product or the box product).*



Theorem 2.43. *Every metrizable space is normal.*

Proof. metrizable $\Longrightarrow$ one-point sets are closed. Let X be a metric space with metric d, $A, B \subset X$ be dispoint closed sets.

$\forall\ a \in A$, $\exists \varepsilon_a$, s.t. $B(a, \varepsilon_a) \cap B = \varnothing$
$\forall\ b \in B$, $\exists\ \varepsilon_b$, s.t. $B(b, \varepsilon_b) \cap A = \varnothing$.



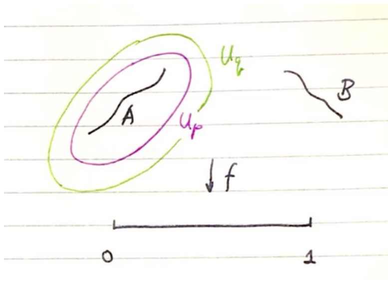
Define $U = \bigcup_{a \in A} B\left(a, \frac{\varepsilon_a}{2}\right)$, $V = \bigcup_{b \in B} B\left(b, \frac{\varepsilon_b}{2}\right)$ open neighborhoods of A and B, respectively.
Claim : $U \cap V = \varnothing$. proof: $\forall\ z \in U \cap V$, $\exists\ a \in A$, $b \in B$, s.t.$z \in B\left(a, \frac{\varepsilon_a}{2}\right) \cap B\left(b, \frac{\varepsilon_b}{2}\right)$
$\therefore d(a,b) \leqslant d(z,a) + d(z,b) < \frac{1}{2}(\varepsilon_a + \varepsilon_b) \leqslant \max\{\varepsilon_a, \varepsilon_b\}$. □

Theorem 2.44. *Every compact Hausdorff space is normal.*

Theorem 2.45 (Urysohn lemma)**.** *Let X be a normal space, $A, B \subset X$ be dispoint closed subsets. Then there exists a continuous function $f : X \longrightarrow [0,1]$ s.t. $f(A) = \{0\}$, $f(B) = \{1\}$.*

Proof. Step 1:

Let $P = \mathbb{Q} \cap [0,1]$, we shall define for each $p \in P$ an open set $U_p \subset X$, s.t. $p < q \Longrightarrow \overline{U_p} \subset U_q$. (*)



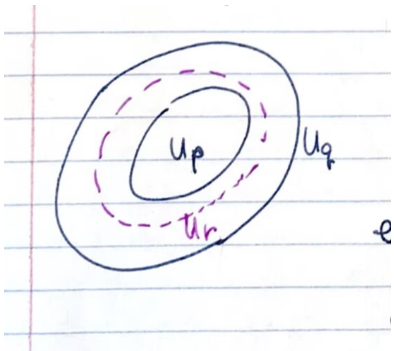
Let $U_1 = X - B$, choose U_0 an open neighborhood of A s.t. $\overline{U_0} \subset U_1$ (by the normality of X).

Induction: let p_n be the first n elements of P, suppose that U_p is defined for all $p \in P$ and $p < q \Longrightarrow U_p \subset U_q$. (e.g. $P_1 = \{0\}, P_2 = \{0, 1\}, P_3 = \dots$). Let r be the $(n+1)-st$ element in P, $P_{n+1} = \{p\} \cup P_n$,

p be the immediate predecessor of r;

q be the immediate successor of r.

normality of $X \Longrightarrow \exists$ an open set $U_r \subset X$, s.t. $\overline{U_p} \subset U_r$, $\overline{U_r} \subset U_q$.



Check the condition (*): $\forall\ s \in P_n$

either $s \leqslant p < r \Longrightarrow \overline{U_s} \subset U_p \subset U_r$ or $s \geqslant q > r \Longrightarrow \overline{U_r} \subset U_q \subset U_s$.

Step 2:

for $\forall\ p \in \mathbb{Q}$, define $U_P = \varnothing$ if $p < 0$, $U_p = X$ if $p > 1$. Then we still have $p < q \Longrightarrow \overline{U_p} \subset U_q$.

step 3:

Given $x \in X$, let $\mathbb{Q}(x) = \{p \in \mathbb{Q} | x \in U_p\}$, then $\mathbb{Q} \cap (1, +\infty) \subset \mathbb{Q}(x) \subset \mathbb{Q} \cap [0, +\infty)$. Define $f(x) = \inf \mathbb{Q}(x) = \inf \{p \in \mathbb{Q} | x \in U_p\}$.

step 4: f is the desired function.

(a) $x \in A \Longrightarrow x \in U_p$ for all $p \geqslant 0 \Longrightarrow f(x) = 0$

$x \in B \Longrightarrow x \notin U_p$ for all $p \leqslant 1 \Longrightarrow f(x) = 1$

(b) Fact

(1) $x \in \overline{U_r} \Longrightarrow f(x) \leqslant r$

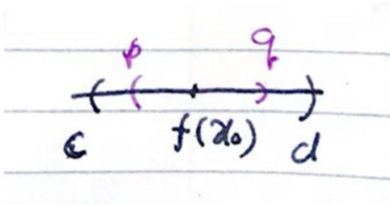
(2) $x \notin U_r \Longrightarrow f(x) \geqslant r$

Proof. (1) $x \in \overline{U_r} \Longrightarrow x \in U_s$ for all $s > r \therefore f(x) = \inf \mathbb{Q}(x) \leqslant r$

(2) $x \notin U_r \Longrightarrow x \notin U_s$ for all $s < r$. $\therefore f(x) = \inf \mathbb{Q}(x) \geqslant r$.

□

(c) f is continuous: Given $x_0 \in X$, and an open interval (c,d) s.t.$f(x_0) \in (c,d)$, We need to find a neighborhood U of x_0, s.t. $f(U \subset (c,d))$. Choose $p, q \in \mathbb{Q}$, s.t. $c < p < f(x_0) < q < d$.



Let $U = U_q - \overline{U}_p$

$$\text{(i)}\quad \left.\begin{array}{lcl} f(x_0) < q & \overset{(2)}{\Longrightarrow} & x_0 \in U_q \\ f(x_0) > p & \overset{(1)}{\Longrightarrow} & x_0 \notin \overline{U_p} \end{array}\right\} \Longrightarrow x_0 \in U$$

(ii) $\forall\ x \in U$, $x \in U_q \subset \overline{U_q} \Longrightarrow f(x) \leqslant q < d$

$x \notin \overline{U_p} \Longrightarrow x \notin U_r \Longrightarrow f(x) \geqslant p > c.$

□

Theorem 2.46 (Urysohn metrization theorem)**.** *Every regular space with a countable basis is metrizable,* e.g. $\mathbb{R}^\omega = \prod_{n=1}^{\infty} \mathbb{R}$ is metrizable.

Proof. (sketch)

step 1 :

$\exists$ continuous functions

$\{f_n : X \longrightarrow [0,1] | n \in \mathbb{N}\}$ s.t. $\forall\ x_0 \in X$ and a neighborhood U of x_0, $\exists\ n \in \mathbb{N}$, s.t. $f_n(x_0) > 0$, $f_n|_{X-U} = 0$. construction: let $\{B_n\}$ be a countable basis, for all $n, m \in \mathbb{N}$ s.t. $\overline{B_n} \subset B_m$, by the Urysohn lemma, $\exists\ g_{n,m} : X \longrightarrow [0,1]$ s.t. $g_{n,m}(\overline{B_n}) = \{1\}$, $g_{n,m}(X - B_m) = \{0\}$.

step 2:

define $F : X \Longrightarrow \mathbb{R}^\omega$, $x \mapsto (f_1(x), f_2(x) \dots)$

(i) F is continuous

(ii) F is injective

(iii) $F : X \longrightarrow Z \subset \mathbb{R}^\omega$ is an open map. $\therefore F : X \longrightarrow \mathbb{R}^\omega$ is an embedding, $\mathbb{R}^\omega$ is metrizable $\Longrightarrow X$ is metrizable.

□

Theorem 2.47 (Tietz extension theorem)**.** *Let X be a normal space, $A \subset X$ be a closed subspace.*

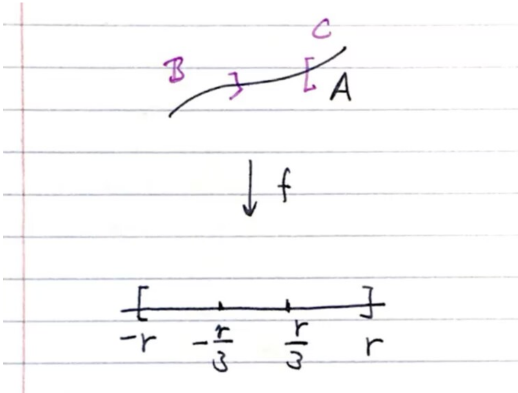
(a) Any continuous function of A into $[a,b]$ may be extended to a continuous function of X into $[a,b]$.

(b) Any continuous function of A into $\mathbb{R}$ may be extended to a continuous function of X into $\mathbb{R}$.

$$\begin{tikzcd} A \arrow[r, "f"] & {[a,b], \mathbb{R}} \\ \cap & \\ X \arrow[uur, dashed, "\exists \bar{f}?"'] & \end{tikzcd}$$

Proof. (sketch)

step 1 : Given a continuous function $f : A \longrightarrow [-r, r]$, we may construct a continuous function $g : X \longrightarrow \left[-\frac{1}{3}r, \frac{1}{3}r\right]$ s.t. $|g(a) - f(a)| \leqslant \frac{2}{3}r \ \forall\ a \in A$.



$I_1 = \left[-r, -\frac{1}{3}r\right]$
$I_2 = \left[-\frac{1}{3}r, \frac{1}{3}r\right]$
$I_3 = \left[\frac{1}{3}, r\right]$
$B = f^{-1}(I_1)$, $C = f^{-1}(I_3)$

Urysohn lemma $\Longrightarrow \exists$ a continuous function $g : X \longrightarrow \left[-\frac{1}{3}r, \frac{1}{3}r\right]$ s.t. $g(B) = -\frac{1}{3}r$, $g(C) = \frac{1}{3}r$.

step 2: Given $f : A \longrightarrow [-1, 1]$, we extend it to X:

(i) $r = 1$, $\exists\ g_1 : X \longrightarrow \left[-\frac{1}{3}, \frac{1}{3}\right]$ s.t. $|f(a) - g_1(a)| \leqslant \frac{2}{3} \ \forall\ a \in A$

(ii) $r = \frac{2}{3}$, $\exists\ g_2 : X \longrightarrow \left[-\frac{1}{3} \cdot \frac{2}{3}, \frac{1}{3} \cdot \frac{2}{3}\right]$ s.t. $|f(a) - g_1(a) - g_2(a)| \leqslant \left(\frac{2}{3}\right)^2 \ \forall\ a \in A$

Inductively, $\exists\ g_{n+1}$, s.t.

$|g_{n+1}(x)| \leqslant \frac{1}{3} \cdot \left(\frac{2}{3}\right)^2$, $|f(a) - g_1(a) \dots g_{n+1}(a)| \leqslant \left(\frac{2}{3}\right)^{n+1} \ \forall\ a \in A$. Let $S_k(x) = \sum_{n=1}^{k} g_n(x)$, continuous, converges uniformly to a continuous function $g : X \longrightarrow [-1, 1]$, and $g|_A = f$.

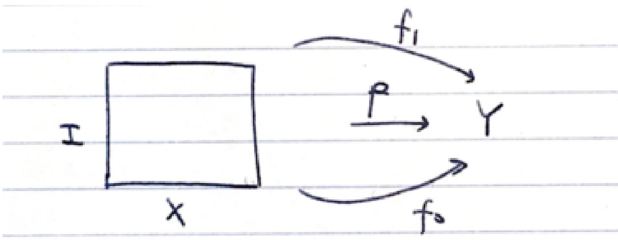
$\square$

Chapter 3

The Fundamental Group

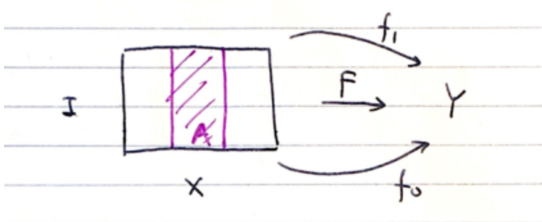
3.1 homotopy of paths

Definition 3.1. *Let $f_0, f_1 : X \to Y$ be continuous maps. A* homotopy *(同伦) between f_0 and f_1 is a continuous map $F : X \times [0,1] \to Y$ s.t.$F(x,0) = f_0(x), F(x,1) = f_1(x)$. f_0 and f_1 are said to be* homotopic *, denoted by $f_0 \simeq f_1$. If $f \simeq C$, a constant map, we say that f is* null-homotopic *(零伦的).*



Let $f : X \to Y$ be a continuous map. If there exists a continuous map $g : Y \to X$, s.t. $g \circ f \cong \mathrm{id}_X$ and $f \circ g \cong \mathrm{id}_Y$, then f is called a homotopy equivalence (同伦等价), g is a homotopy inverse of f, X and Y are homotopy equivalent .

Definition 3.2. *Let $A \subset X$ be a subspace, a* homotopy relative to A *is a continuous map $F : X \times I \to Y$, s.t. $F(a,t) = F(a,0)$ for $\forall a \in A, t \in I$. Then $f_0 = F(\cdot, 0)$ and $f_1 = F(\cdot, 1)$ are called* homotopic relative to A *, denoted by $f_0 \simeq f \mathrm{rel} A$.*

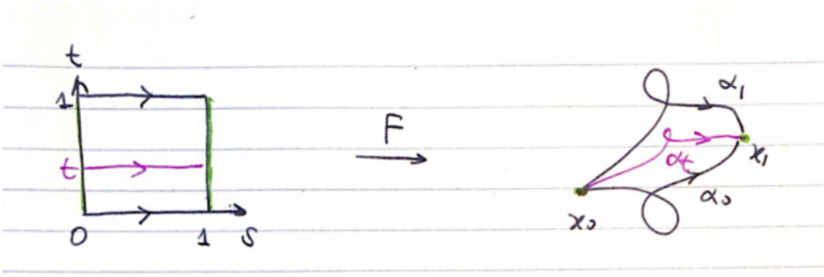


$$\mathrm{Map}(X,Y)/\simeq \; = \text{the set of path components of } \mathrm{Map}(X,Y)$$

Remark: $$= \pi_0 \mathrm{Map}(X,Y) = [X,Y] = ?$$

the homotopy set

Definition 3.3. *A path in X is a continuous map $\alpha : I \to X$. Two paths α_0 and α_1 are* path homotopic *, denoted by $\alpha_0 \simeq_p \alpha_1$, if $\alpha_0(0) = \alpha_1(0) = x_0, \alpha_0(1) = \alpha_1(1) = x_1$, and α_0 and α_1 are homotopic $\mathrm{rel}\{0,1\}$, i.e. $\exists$ a continuous map (a path homotopy) $F : X \times I \to X$ s.t. $F(s,0) = \alpha_0(s), F(s,1) = \alpha_1(s)$, $F(0,t) = x_0, F(1,t) = x_1$ for $\forall s \in I, t \in I$.*



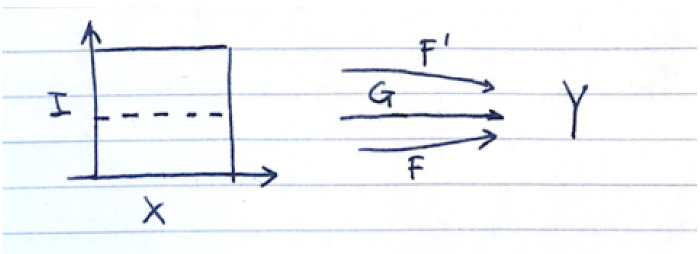
Lemma 3.4. *The relation $\simeq$ and $\simeq_p$ are equivalence relations.*

Proof. (i) $f \simeq f$ by $F(x,t) = f(x), \forall t \in I, x \in X$.

(ii) If $F : f_0 \simeq f_1$ is a homotopy between f_0 and f_1. Then $G(x,t) = F(x, 1-t)$ is a homotopy between f_1 and f_0.

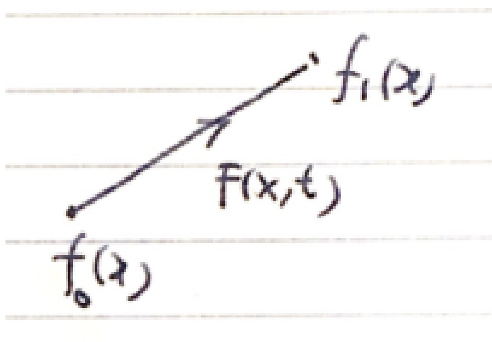
(iii) If $F : f_0 \simeq f_1, F' : f_1 \simeq f_2$ are homotopies, then $G : X \times [0,1] \to Y$, defined as $G(x,t) = \begin{cases} F(x, 2t) & t \in \left[0, \frac{1}{2}\right] \\ F'(x, 2t-1) & t \in \left[\frac{1}{2}, 1\right] \end{cases}$ is a homotopy between f_0 and f_2. (Continuity by the pasting lemma)

$\square$



Examples:

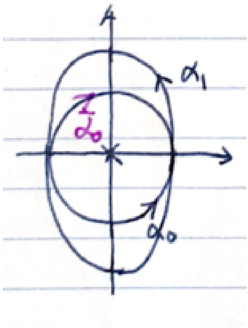
(1) $f_0, f_1 : X \to \mathbb{R}^2$ are always homotopic by the straight-line homotopy $F(x,t) = (1-t)f_0(x) + tf_1(x)$. The same holds for path-homotopy, the same holds if $\mathbb{R}^2$ is related by a convex subspace $A \subset \mathbb{R}^n$.



(2) $X = \mathbb{R}^2 - \{0\}$ The punctured plane, $\alpha_0, \alpha_1 : S^1 \to X, \alpha_0(s) = (\cos \pi s, \sin \pi s)$,

$\alpha_1(s) = (\cos \pi s, 2 \sin \pi s)$ are homotopic by a straight-line homotopy.

But α_0 is not homotopic to $\overline{\alpha}_0(s) = (\cos \pi s, -\sin \pi s)$ in X.



Definition 3.5. *If α_0 is a path in X from x_0 to x_1, α_1 is a path from x_1 to x_2, we define the* <u>*product*</u> *$\alpha_0 * \alpha_1$ to be the path* $\alpha_2(s) = \begin{cases} \alpha_0(2s) & s \in \left[0, \frac{1}{2}\right] \\ \alpha_1(2s-1) & s \in \left[\frac{1}{2}, 1\right] \end{cases}$ *from x_0 to x_2.*

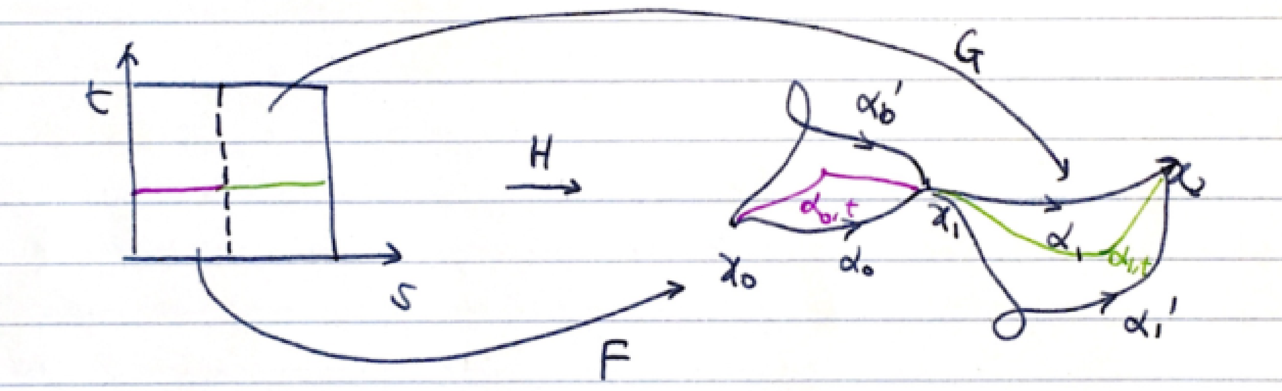
The product operation induces a well-defined operation on path-homotopy classes:

$$[\alpha_0] * [\alpha_1] := [\alpha_0 * \alpha_1].$$

Check: Given path-homotopies $F : \alpha_0 \simeq_F \alpha_0'$, $G : \alpha_1 \simeq_P \alpha_1'$. Define

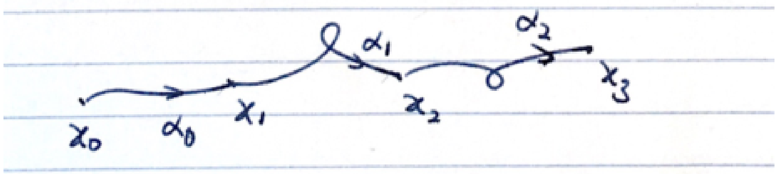
$$H(s,t) = \begin{cases} F(2s,t) & s \in \left[0, \frac{1}{2}\right] \\ G(2s-1,t) & s \in \left[\frac{1}{2}, 1\right] \end{cases}$$

a homotopy between $\alpha_0 * \alpha_1$ and $\alpha_0' * \alpha_1'$.



Theorem 3.6. *The operation $*$ has the following properties:*

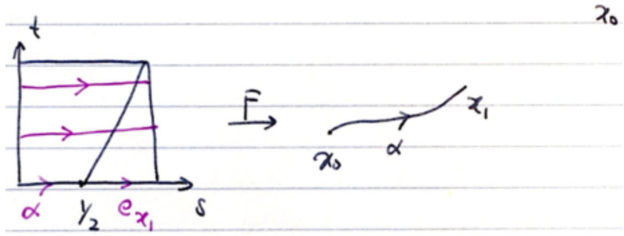
(1) (associativity) If $[\alpha_0] * ([\alpha_1] * [\alpha_2])$ is defined,
then so is $([\alpha_0] * [\alpha_1]) * [\alpha_2]$, and they are equal:



(2) (right and left identities) If α is a path from x_0 to x_1, then $[\alpha] * [e_{x_1}] = [\alpha], [e_{x_0}] * [\alpha] = [\alpha]$ where e_x denotes the constant path at $x \in X$.

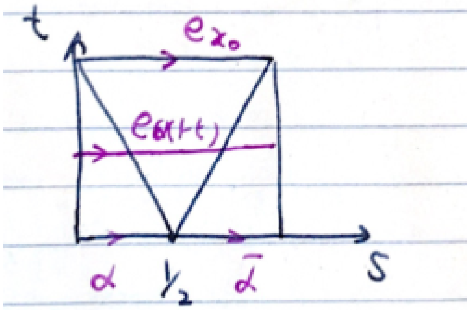
(3) (inverse) If α is a path from x_0 to x_1, let $\overline{\alpha}$ be the path $\overline{\alpha}(s) = \alpha(1-s)$. It is called the <u>*inverse*</u> *of α. Then $[\alpha] * [\overline{\alpha}] = [e_{x_0}], [\overline{\alpha} * [\alpha] = [e_{x_1}]]$.*

Proof. (2)



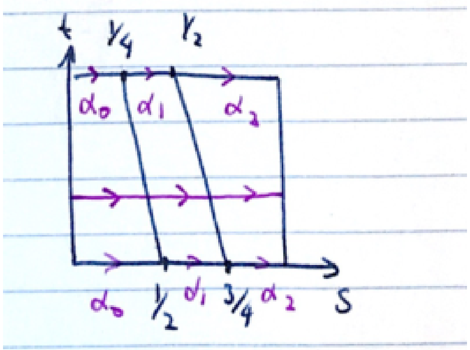
$$F(s,t)=\begin{cases}\alpha\left(\dfrac{2s}{1+t}\right) & s\in\left[0,\dfrac{1+t}{2}\right]\\ x_1 & s\in\left[\dfrac{1+t}{2},1\right]\end{cases}\quad \therefore \alpha * e_{x_1}\simeq_p \alpha.$$

(3)



$\therefore \alpha * \overline{\alpha} \simeq_p e_{x_0}$.

(1)



$\therefore \alpha_0 * (\alpha_1 * \alpha_2) \simeq_p (\alpha_0 * \alpha_1) * \alpha_2$.

□

Definition 3.7. *A* <u>*groupoid*</u> *is a category in which every morphism is an isomorphism.*

Examples:

(1) A group is a groupoid with one object.

(2) For any space X, the <u>fundamental groupoid</u> $\prod_(X)$ is a category whose objects are the points of X, and whose morphisms are path homotopy classes of paths.

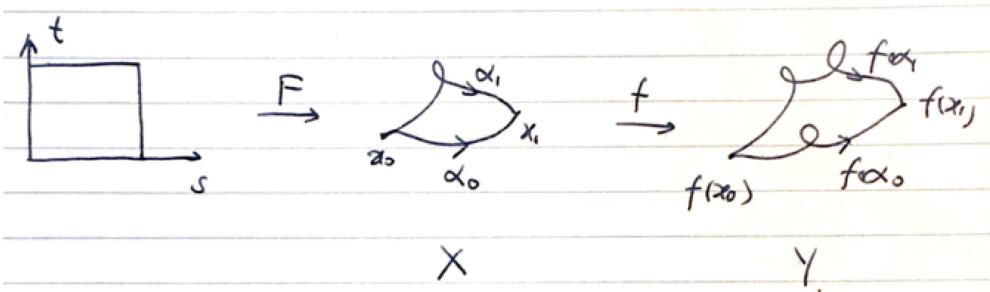
Naturality/functoriality:

Lemma 3.8. *Let $f: X \to Y$ be a continuous map.*

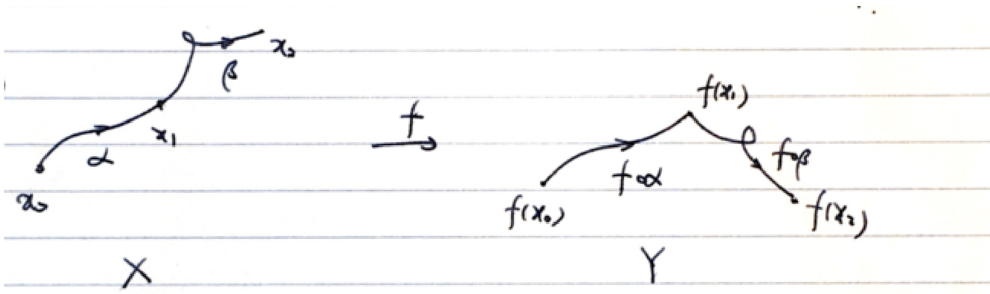
(1) If F is a path homotopy between the paths α_0 and α_1 in X, then $f\circ F$ is a path homotopy between the paths $f\circ\alpha_0$ and $f\circ\alpha_1$.

(2) Let α and β be paths in X, with $\alpha(1)=\beta(0)$, then $f\circ(\alpha*\beta)=(f\circ\alpha)*(f\circ\beta)$.

Proof. (1)



(2)



□

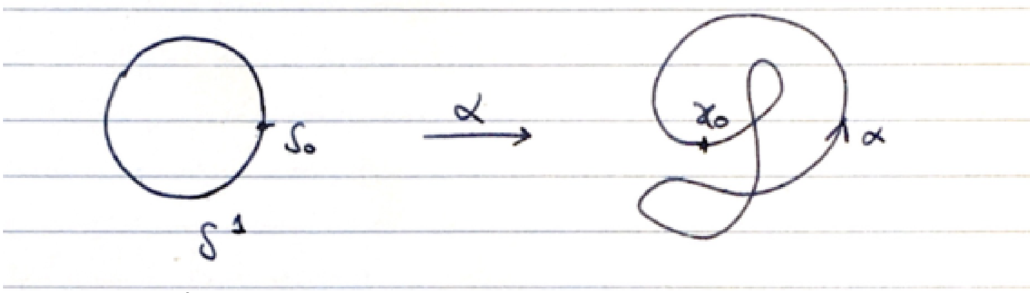
$$\begin{array}{lrl} \Pi_1 : & \mathcal{T}\text{op} & \to \mathcal{G}\text{roupoids} \\ \text{Therefore} & X & \mapsto \pi_1(X) \qquad \text{is a functor.} \\ & \left(X \xrightarrow{f} Y\right) & \mapsto f_* \pi_1(X) \end{array}$$

3.2 the fundamental group

Definition 3.9. *Let X be a space, $x_0 \in X$ be a point. A path α in X with $\alpha(0) = \alpha(1) = x_0$ is called a* *loop* *(环路) based at x_0. The set of path homotopy classes of loops at x_0 with the operation $*$, is called the* *fundamental group* *(基本群) (the first homotopy group, 一阶同伦群) of X, relative to the base point x_0, denoted by $\pi_1(X, x_0)$.*

Alternative definition:

$\pi_1(X, x_0) = \text{Map}\left((S^1, s_0), (X, x_0)\right) / \textit{based homotopy}$



$\pi_0(X, x_0) = \text{Map}\left((S^0, s_0), (X, x_0)\right) / \textit{based homotopy a set.}$

$\pi_0(X, x_0) = 0 \Leftrightarrow X$ *is path-connected.*

$\pi_n(X, x_0) = \text{Map}\left((S^n, s_0), (X, x_0)\right) / \textit{based homotopy an abelian group for } n \geqslant 2.$

Example:

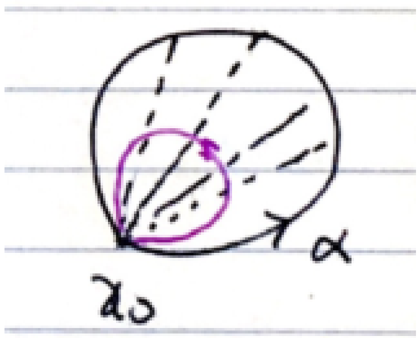
$\pi_1(\mathbb{R}^n, x_0) = \{e\}$

Straight-line homotopy

$F(s, t) = (1 - t)\alpha(s) + t x_0$

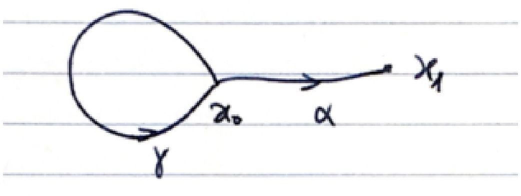
For any $A \subset \mathbb{R}^n$ convex,

$\pi_1(A, x_0) = \{e\}$



Definition 3.10. *Let α be a path in X from x_0 to x_1. We define a map*

$$\begin{aligned} \Phi_\alpha : \pi_1(X, x_0) &\to \pi_1(X, x_1) \\ [\gamma] &\mapsto [\overline{\alpha}] * [\gamma] * [\alpha] \end{aligned}$$



Theorem 3.11. *The map Φ_α is a group isomorphism.*

Proof. (i) Φ_α is a homomorphism:

$$\begin{aligned} &\Phi_\alpha([\gamma_1]) * \Phi_\alpha([\gamma_0]) \\ =&([\overline{\alpha}] * [\gamma_1] * [\alpha]) * ([\overline{\alpha}] * [\gamma_2] * [\alpha]) \\ =&[\overline{\alpha}] * [\gamma_1] * [\gamma_2] * [\alpha] \\ =&\Phi_\alpha([\gamma_1] * [\gamma_2]) \end{aligned}$$

(ii) Φ_α has an inverse homomorphism $\Phi_{\overline{\alpha}}$:

$$\Phi_\alpha \circ \Phi_{\overline{\alpha}}([\gamma]) = [\overline{\alpha}] * [\alpha] * [\gamma] * [\overline{\alpha}] * [\alpha] = [\gamma].$$

Similar for $\Phi_{\overline{\alpha}} \circ \Phi_\alpha$.

□

Corollary 3.12. *If X is path connected, then for any $x_0, x_1 \in X$, $\pi_1(X, x_0)$ is isomorphic to $\pi_1(X, x_1)$.*

Definition 3.13. *A space X is said to be simply connected (单连通) if it is path-connected and $\pi_1(X, x_0) =$ 0.*

Definition 3.14. *Let $f : X \to Y$ be a continuous map, $f(x_0) = y_0$. Define $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_1)$ by $f_*([\gamma]) = [f \circ \gamma]$.$f_*$ is called the homomotphism induced by f.*

Check:

(i) well-defined: $\gamma \underset{F}{\simeq_p} \gamma' \Rightarrow f\circ \underset{f \circ F}{\simeq_p} f \circ \gamma'$

(ii) homomorphism: $f \circ (\gamma_1 * \gamma_2) = (f \circ \gamma_1) * (f \circ \gamma_2)$

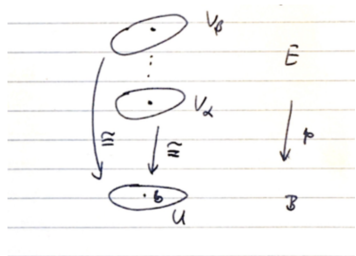
Theorem 3.15. *If $f : (X, x_0) \to (Y, y_0), g : (Y, y_0) \to (Z, z_0)$ are continuous maps, then $(g \circ f)_* = g_* \circ f_*$; if $\mathrm{id} : (X, x_0) \to (X, x_0)$ is the identity map, then id_* is the identity homomorphism.*

Proof. $(g \circ f)_*[\gamma] = [g \circ f \circ \gamma] = [g \circ (f \circ \gamma)] = g_*([f \circ \gamma]) = g_*(f_*[\gamma])$. □

Corollary 3.16. *If $h : X \to Y$ is a homeomorphism, $h(x_0) = y_0$, then $h_* : \pi_1(X, x_0) \to \pi(Y, y_0)$ is an isomorphism.*

3.3 covering spaces

Definition 3.17. *Let $p : E \to B$ be a continuous surjective map. If for every point $b \in B$, there exists a neighborhood U s.t. $p^{-1}(U)$ is a union of disjoint open sets V_α in E, $p^{-1}(U) = \coprod_\alpha V\alpha$ and for each $\alpha, p|_{V_\alpha} : V_\alpha \to U$ is a homeomorphism. Then p is called a* <u>*covering map*</u> *(覆/复叠/迭映射), and E is said to be a covering space (覆叠空间) of B.*



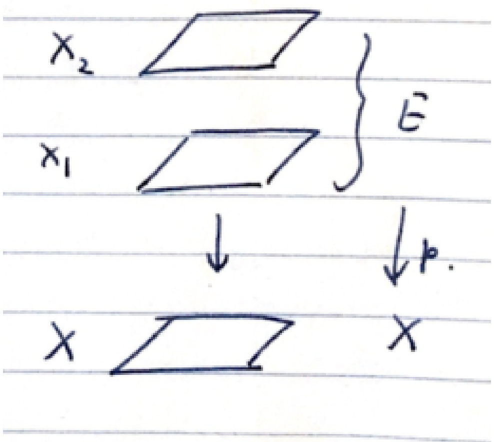
Remark:

(1) $p^{-1}(b)$ has discrete topology.

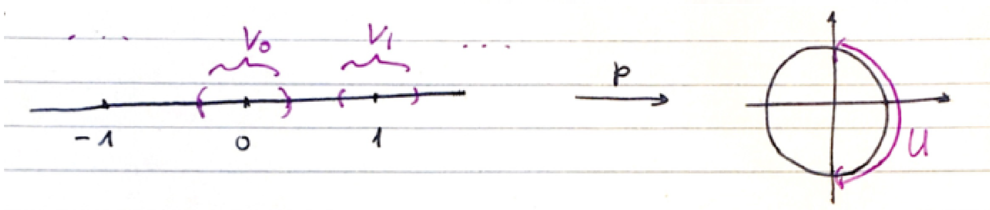
(2) p is an open map.

Examples:

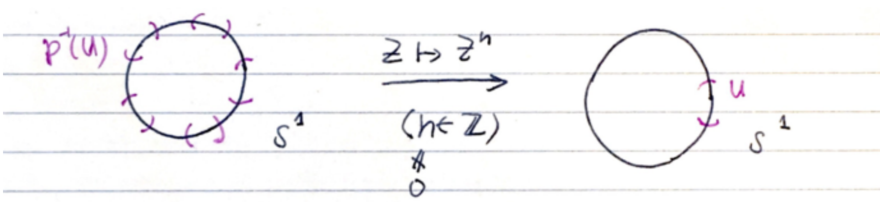
(1) $\begin{array}{ll} E = X\times\{1,\cdots,n\} & \xrightarrow{r} X \\ (x,i) & \mapsto x \end{array}$ a trivial covering map

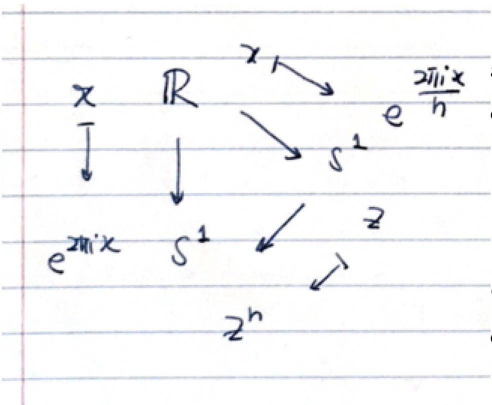


(2) $p : \mathbb{R} \to S^1 \subset \mathbb{C}$ is a covering map.
$x \mapsto (\cos 2\pi x), \sin(2\pi x) = e^{i2\pi x}$
$U = \{(x, y) \in S^1 | x > 0\}$
$p^{-1}(U) = \coprod_{n\in\mathbb{Z}} \left(n - \frac{1}{4}, n + \frac{1}{4}\right)$
$p : V_n \xrightarrow{\cong} U$.



(3) $p : S^1 \to S^1, z \mapsto z^n$ is a covering map.



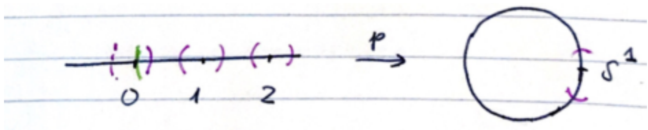


a commutative diagram of covering spaces.

Question: Is a composite of covering maps a covering map?

Remark: A covering map $p : E \to B$ is a local homomorphism, (i.e.$\forall e \in E, \exists$ a neighborhood V of e s.t. $p\big|_V$ is a homeomorphism). But the converse is not true:

$p : \mathbb{R}_+ \longrightarrow S^1 (\mathbb{R}_+ = \mathbb{R}_{>0})$



is a local homeomorphism, but not a covering map.

Theorem 3.18. *If $p : E \to B, p' : E' \to B'$ are covering maps, then $p \times p; : E \times E' \to B \times B'$ is a covering map.*

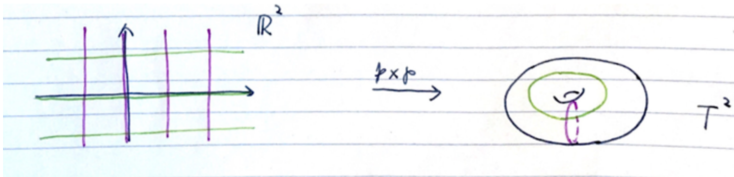
Proof.

$$(p \times p')^{-1}(U \times U') = p^{-1}(U) \times (p')^{-1}(U')$$
$$= \coprod_{\alpha} V_\alpha \times \coprod_{\beta} V'_\beta$$
$$= \coprod_{\alpha,\beta} (V_\alpha \times V'_\beta)$$

So $p \times p' : V_\alpha \times V'_\beta \to U \times U'$ is a homeomorphism. □

Examples:

(1) $p \times p : \mathbb{R}^2 = \mathbb{R} \times \mathbb{R} \to S^1 \times S^1 = T^2$ is a covering map.



(2) $B_0 = S^1 \times \{*\} \cup \{*\} \times S^1 \subset T^2$

the figure-eight space  $= S^1 \vee S^1$

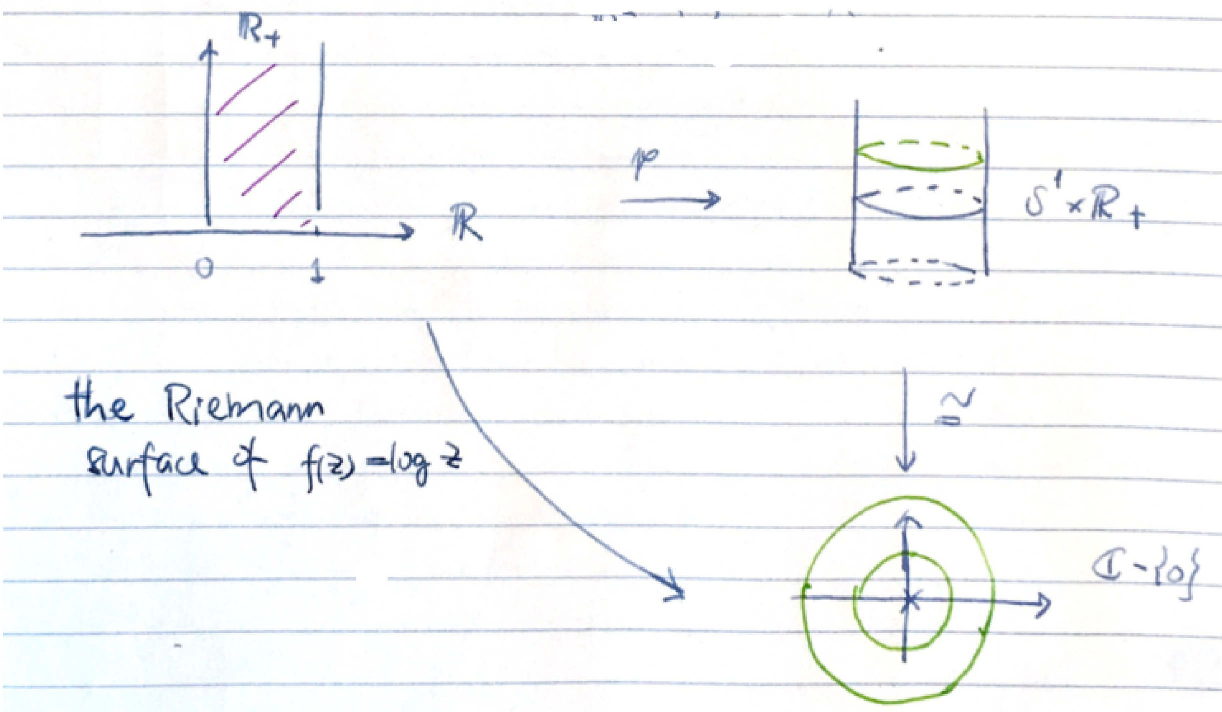
$E_0 = p^{-1}(B_0) = (\mathbb{R} \times \mathbb{Z}) \cup (\mathbb{Z} \times \mathbb{R})$

the "infinite grid" $E_0 \xrightarrow{p} B_0$ is a covering map.

Question: Can you find other covering spaces of $S^1 \vee S^1$?

(3)

$\times\mathrm{id} : \mathbb{R} \times \mathbb{R}_+ \to S^1 \times \mathbb{R}_+, (x, t) \mapsto (e^{2\pi i x}, t)$ is a covering map, in which $S^1 \times \mathbb{R}_+ \cong \mathbb{R}^2 - \{0\}$ by $\varphi : (z, t) \mapsto (t, z)$.



$\mathbb{R}_+$
$\mathbb{R}$
0
1
p
$S^1 \times \mathbb{R}_+$
$\cong$
the Riemann surface of $f(z) = \log z$
$\mathbb{C} - \{0\}$

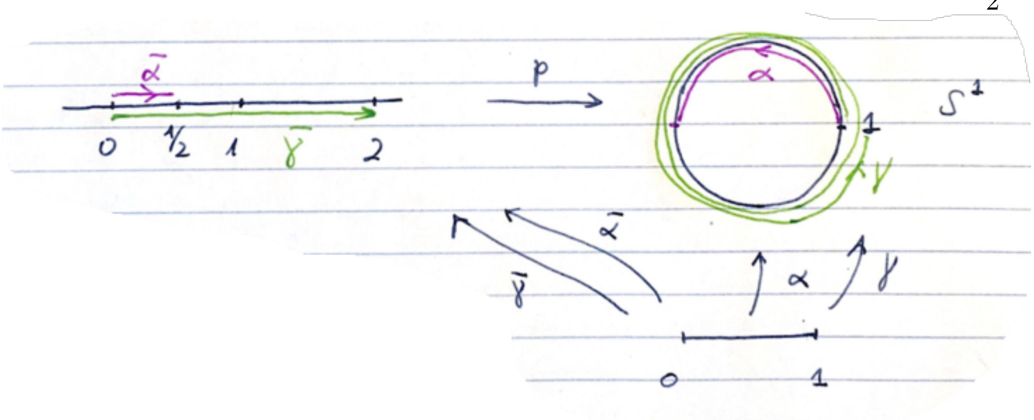
3.4 the fundamental group of the circle

Definition 3.19. *Let $p : E \to B$ be a map, $f : X \to B$ be a continuous map. A* <u>*lifting*</u> *(提升) of f is a map $\overline{f} X \to E$ s.t. $p \circ \overline{f} = f$.*

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, dashed, "\overline{f}"] \arrow[r, "f"] & B \end{tikzcd}$$

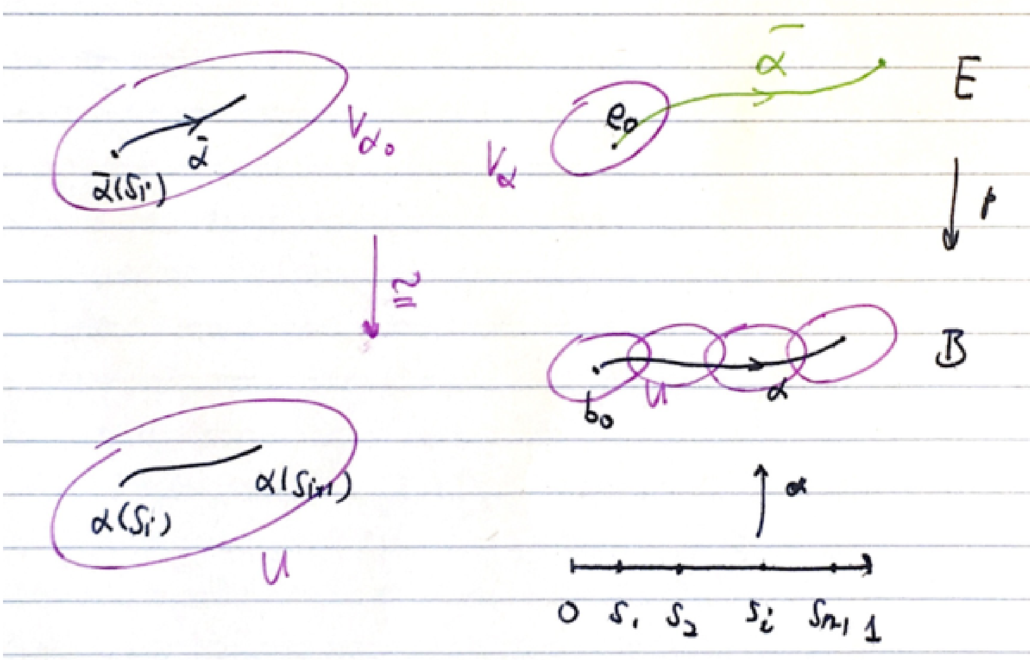
Example: $p : \mathbb{R} \to S^1$ the covering map.

The path $\alpha : [0,1] \to S^1, \alpha(s) = (\cos \pi s, \sin \pi s)$, lifts to the path $\overline{\alpha} : [0,1] \to \mathbb{R}, \alpha(s) = \frac{s}{2}$.



The loop $\gamma : [0,1] \to S^1, \gamma(s) = (\cos 4\pi s, \sin 4\pi s)$ lifts to the path $\overline{\gamma} : [0,1] \to \mathbb{R}, \overline{\gamma}(s) = 2s, \overline{\gamma}(0) = 0, \overline{\gamma}(1) = 2$.

Lemma 3.20. *Let $p : E \to B$ be a covering map,$p(e_0) = b_0$. Then any path $\alpha : [0,1] \to B$ beginning at b_0 has a unique lifting to a path $\overline{\alpha} : [0,1] \to E$ beginning at E_0.*



Proof. (1) Existence: cover B by evenly-covered open sets U, find a subdivision $O = s_0 < s_1 < \cdots < s_n = 1$ of $[0,1]$ s.t. $[s_i, s_{i+1}]$ lies in such an U (Lebesgue number lemma). Define the lifting $\overline{\alpha}$ inductively:

(i) $\overline{\alpha}(0) = e_0$

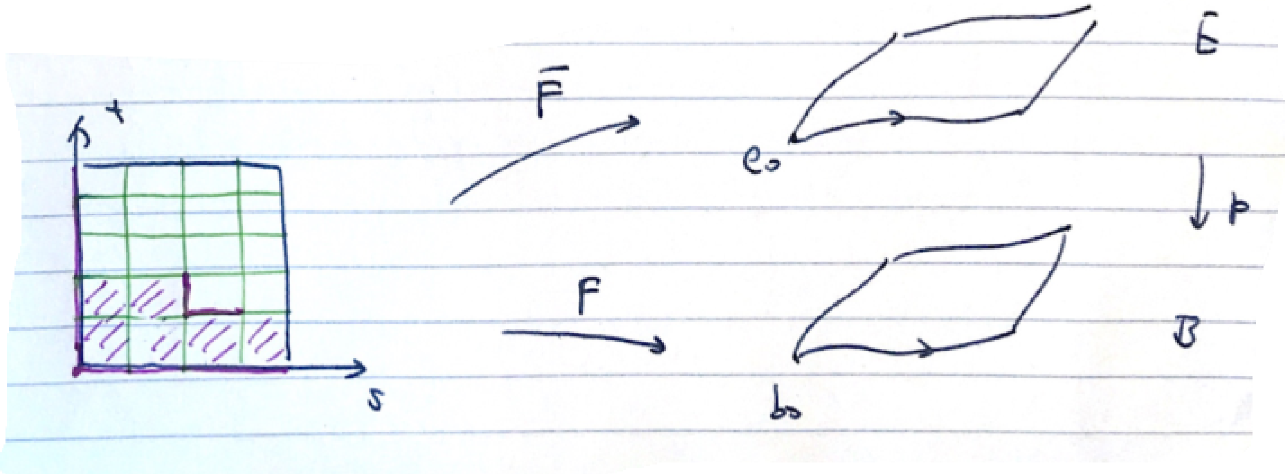
(ii) Suppose $\overline{\alpha}(s)$ is defined for $0 \leqslant s \leqslant s_i$,define $\overline{\alpha}$ on $[s_i, s_{i+1}]$ as follows: $f([s_i, s_{i+1}]) \subset U, p^{-1}(U) = \coprod\limits_{\alpha} V_\alpha$. Assume $\overline{\alpha}(s_i) \in V_{\alpha_0}$,$p\big|_{V_{\alpha_0}} : V_{\alpha_0} \xrightarrow{\cong} U$ a homeomorphism. $\therefore$ define $\overline{\alpha}(s) = \left(p\big|_{V_\alpha}\right)^{-1}(\alpha(s))$ for $s \in [s_i, s_{i+1}]$

(2) uniqueness: Assume $\overline{\alpha}'$ is another lifting of α with $\overline{\alpha}'(0) = e_0 = \overline{\alpha}(0)$. Assume $\overline{\alpha}'(s) = \overline{\alpha}(s)$ on $[0, s_i]$. $\overline{\alpha}'$ is a lifting of $\alpha \Rightarrow \overline{\alpha}'([s_i, s_{i+1}]) \subset \coprod\limits_{\alpha} V_\alpha$. $\overline{\alpha}'([s_i, s_{i+1}])$ is connected $\Rightarrow \overline{\alpha}'([s_i, s_{i+1}]) \subset V_{\alpha_0}$ since $\overline{\alpha}'(s_i) = \overline{\alpha}(s_i) \in V_{\alpha_0}$.

$p\big|_{V_{\alpha_0}} : V_{\alpha_0} \to U$ is a homeomorphism $\Rightarrow \overline{\alpha}' = \overline{\alpha}$ on $[s_i, s_{i+1}]$.

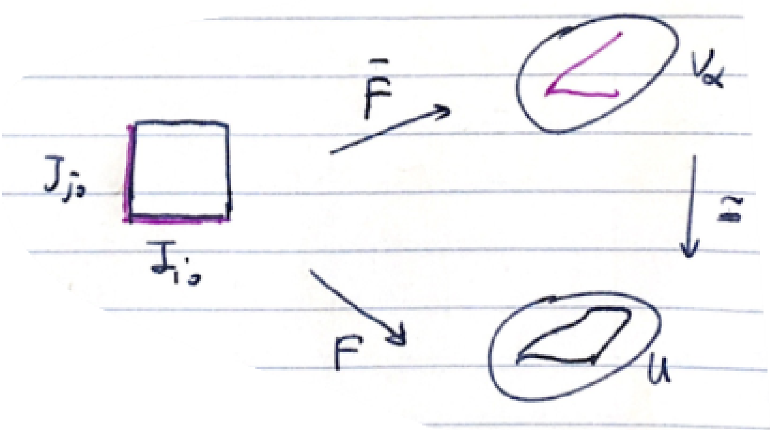
□

Lemma 3.21. *Let $p : E \to B$ be a covering map, $p(e_0) = b_0$. Let $F : I \times I$ be a continuous map, $F(0,0) = b_0$. Then there is a unique lifting of F to a continuous map $\overline{F} : I \times I \to E$ s.t. $\overline{F}(0,0) = e_0$. If F is a path homotopy, then so is $\overline{F}$.*



Proof. (1) Define $\overline{F}(0,0) = e_0$, extend to $\overline{F} : 0 \times I \cup I \times 0 \to E$ by the preceding lemma. Choose subdivisions $0 = s_0 < s_1 < \cdots < s_m = 1$,$0 = t_0 < t_1 < \cdots < t_n = 1$, with $I_i = [s_{i-1}, s_i]$,$J_j = [t_{i-1}, t_j]$, s.t. $F(I_i) \times J_j \subset U$ with $p^{-1}(U) = \coprod\limits_{\alpha}(V_\alpha)$,$p\big|_{V_\alpha} : V_\alpha \xrightarrow{\cong} U$ a homeomorphism.

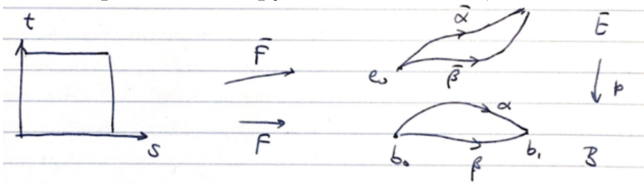
Extend $\overline{F}$ inductively to $I_1 \times J_1, I_2 \times J_1, \cdots, I_1 \times J_2, I_2 \times J_2, \cdots$ Assume $\overline{F}$ is defined on A, $I_{i_0} \times J_{j_0}$ is the next rectangle. Since $\overline{F}(A \cap (I_{i_0} \times J_{j_0}))$ is connected it is connected in some V_α, $p\big|_{V_\alpha}$: $V_\alpha \to U$ is a homeomorphism. $\therefore$ define $\overline{F}(x) = p^{-1}(F(x))$ for $\forall x \in I_{i_0} \times J_{j_0}$. And $\overline{F} : X \times I \to E$ is unique since in each step the extension is unique.



(2) If F is a path homotopy, i.e. $F(0 \times I) = b_0$, then $\overline{F}(0 \times I) \subset= e_0$. Similiar for $\overline{F}(1 \times I)$. ☐

Theorem 3.22. *Let $p : E \to B$ be a covering map, $p(e_0) = b_0$. If α and β are two paths in B from b_0 to b_1, let $\overline{\alpha}$ and $\overline{\beta}$ be their liftings to paths in E, $\overline{\alpha}(0) = \overline{\beta}(0) = e_0$. If α and β are path homotopic, then $\overline{\alpha}(1) = \overline{\beta}(1)$, and $\overline{\alpha}$ and $\overline{\beta}$ are path homotopic.*

Proof. Let $F : X \times I \to B$ be the path homotopy between α and β.



Let $\overline{F} : I \times I \to E$ be the lifting of F with $\overline{F}(0,0) = e_0$. By the preceding lemma, $\overline{F}(0 \times I) = e_0$,$\overline{F}(1 \times I) = e_1$,$\overline{F}(0, s) = \overline{\alpha}(s)$,$\overline{F}(1, s) = \overline{\beta}(s)$. ☐

Slogan: Paths have liftings; path-homotopic paths have path-homotopic liftings.

Definition 3.23. *Let $p : E \to B$ be a covering map, fix $b_0 \in B$ and $e_0 \in p^{-1}(b_0)$. Given an element $[\gamma] \in \pi_1(B, b_0)$, let $\overline{\gamma}$ be the lifting of γ in E with $\overline{\gamma}(0) = e_0$. Let $\phi([\gamma]) = \overline{\gamma}(1)$. Then ϕ is a well-defined map.*

$$\begin{aligned}\Phi : \pi_1(B, b_0) &\to p^{-1}(b_0)\\ [\gamma] &\mapsto \overline{\gamma}(1)\end{aligned}$$

(Since $\gamma \simeq_p \gamma' \Rightarrow \overline{\gamma}(1) = \overline{\gamma}'(1)$) called the <u>lifting correspondence</u> *, depending on the choice of e_0.*

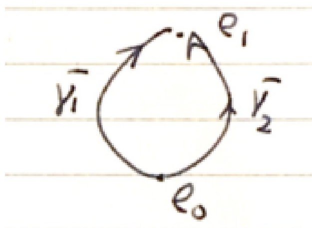
Theorem 3.24. *Let $p : E \to B$ be a covering map, $p(e_0) = b_0$. If E is path connected, then the lifting correspondence*

$$\phi : \pi_1(B, b_0) \to p^{-1}(b_0)$$

is surjective. If E is simply connected, it is bijective.

Proof. (1) If E is path connected, given $e_1 \in p^{-1}(b_0)$, let $\overline{\gamma}$ be a path from e_0 to e_1, then $\gamma = p \circ \overline{\gamma}$ is a loop at b_0 with $\phi([\gamma]) = \overline{\gamma}(1) = e_1$.

(2) Assume E is simply connected, $[\gamma_1], [\gamma_2] \in \pi_1(B, b_0)$ s.t. $\phi([\gamma_1]) = \phi([\gamma_2])$, $\overline{\gamma}_1, \overline{\gamma}_2$ be the liftings of γ_1 and γ_2, $\overline{\gamma}_1(0) = \overline{\gamma}_2(0) = e_0$. E is simply connected $\Rightarrow \overline{\gamma}_1 \simeq_p \overline{\gamma}_2 (\overline{\gamma}_1 \simeq_p \overline{\gamma}_2 \Leftrightarrow \overline{\gamma}_1 * (-\overline{\gamma}_2) \simeq_p e_{e_0} \Rightarrow \gamma_1 \simeq_p \gamma_2)$



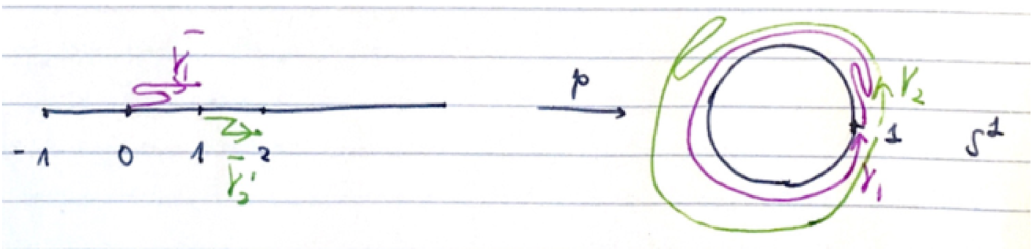
☐

Theorem 3.25. *The fundamental group of S^1 is isomorphic to the additive group of integers, $\pi_1(S^1) \simeq \mathbb{Z}$.*

Proof. Let $p : \mathbb{R} \to S^1, x \mapsto e^{i2\pi x}$ be the covering map. $e_0 = 0 \in \mathbb{R}$, $b_0 = 1 \in S^1 \subset \mathbb{C}$. Then $p^{-1}(b_0) = \mathbb{Z}$, $\mathbb{R}$ is 1-connected $\Rightarrow \phi : \pi_1(S^1, b_0) \to \mathbb{Z}$ is bijective. ☐

Claim: ϕ is a homomorphism.

Proof.



given $[\gamma_1], [\gamma_2] \in \pi_1(S^1, b_0)$, let $\overline{\gamma}_1, \overline{\gamma}_2$ be the liftings in $\mathbb{R}$ with $\overline{\gamma}_1(0) = \overline{\gamma}_2(0) = 0$. Let $\phi([\gamma_1]) = \overline{\gamma}_1(1) = n$,$\phi([\gamma_2]) = \overline{\gamma}_2(1) = m$. Let $\overline{\gamma}'2$ be the path $\overline{\gamma}_2'(s) = n + \overline{\gamma}_2(s)$, then $\overline{\gamma}_2'$ is a lifting of γ_2, with $\overline{\gamma}_2'(0) = n = \overline{\gamma}_1(1)$.

$\therefore \overline{\gamma}_1 * \overline{\gamma}_2'$ is a lifting of $\gamma_1 * \gamma_2$, begins at 0, and $\overline{\gamma}_2'(1) = n+m$. Therefore $\Phi([\gamma_1] * [\gamma_2]) = (\overline{\gamma}_1 * \overline{\gamma}_2')(1) = \overline{\gamma}_2'(1) = n + m$. □

Theorem 3.26. *Let $p : E \to B$ be a covering map, $p(e_0) = b_0$.*

(1) The homomorphism $\varphi_* : \pi_1(E, e_0) \to \pi_1(B, b_0)$ is a monomorphism.

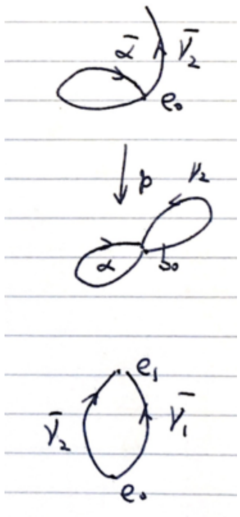
(2) Let $H = p_0(\pi_1(E, e_0))$ The lifting correspondence induces an injective map: $\Phi : H \backslash \pi_1(B, b_) \to p^{-1}(b_0)$ of the set of the right cosets of H into $p^{-1}(b_0)$.Φ is bijective if E is path connected.

(3) If γ is a loop in B based at b_0, then $[\gamma] \in H$ if and only if γ lifts to a loop based at e_0.

Proof. (1) Let $\overline{\gamma}$ be a loop in E at e_0 s.t. $p_*[\overline{\gamma}]$ is the identity element in $\pi_1(B, b_0)$. Let F be a path homotopy between $p \circ \overline{\gamma}$ and e_{b_0}.$\overline{F}$ be the lifting of F with $\overline{F}(0,0) = e_0$. Then $\overline{F}(0,s) = \overline{\gamma}(s)$.$\overline{F}(1,s) = e_{e_0}$,$\overline{F}$ is a path homotopy between $\overline{\gamma}$ and e_{e_0}.

(2) Let $[\gamma_1], [\gamma_2] \in \pi_1(B, b_0)$. Claim: $\phi([\gamma_1]) = \phi([\gamma_2]) \Leftrightarrow [\gamma_1] \in H * [\gamma_2]$.

Proof: Let $\overline{\gamma}_1, \overline{\gamma}_2$ be liftings of γ_1 and γ_2 at e_0.



(i) if $[\gamma_1] \in H * [\gamma_2]$, then $[\gamma_1] = [\alpha * \gamma_2]$ for some $[\alpha] \in H$. There exists a loop $\overline{\alpha}$ at e_0 s.t. $\alpha = p \circ \overline{\alpha}$. Now $\overline{\alpha} * \overline{\gamma}_2$ is a lifting of $\alpha * \gamma_2$ at e_0. $\therefore \overline{\gamma}_1$ and $\overline{\alpha} * \overline{\gamma}_2$ must have the same ending point, i.e. $\overline{\gamma}_1(1) = \overline{\gamma}_2(1)$.
(ii) If $\overline{\gamma}_1(1) = \overline{\gamma}_2(1)$, then $\overline{\alpha} = \overline{\gamma}_1 * (-\overline{\gamma}_2)$ is a loop at e_0 and $[\overline{\gamma}_1] = [\overline{\alpha} * \overline{\gamma}_2]$. Let $F : \overline{\gamma}_1 \simeq_p \overline{\alpha} * \overline{\gamma}_2$ be a path homotopy, then $F = p \circ \overline{F}$ is a path homotopy between γ_1 and $\alpha * \gamma_2$, where $\alpha = p \circ \overline{\alpha}$ is a loop at b_0, i.e. $[\gamma_1] = [\alpha] * [\gamma_2] \in H * [\gamma_2]$.

E path connected $\Rightarrow \phi$ is surjective $\Rightarrow \Phi$ is bijective.

(3) Since $\phi([\gamma_1]) = \phi([\gamma]) \Leftrightarrow [\gamma_1]*H*[\gamma_2]$, take $\gamma_1 = \gamma$,$\gamma_2 = e_{b_0}$, then $[\gamma] \in H \Leftrightarrow \phi([\gamma]) = \phi\left([e_{b_0}]\right) = e_0$. i.e. $[\gamma] \in H \Leftrightarrow \overline{\gamma}$ is a loop at e_0.

□

3.5 applications

Definition 3.27. *Let $A \subset X$ be a subspace. A <u>retraction</u> (收缩映射) of X onto A is a continuous map $\gamma : X \to A$ s.t. $r\big|_A \mathrm{id}_A$. If such a map r exists, we say that A is a <u>retract</u> (收缩核) of X.*

Lemma 3.28. *If A is a retract of X, $j : A \to X$ denote the inclusion map, then the induced homomorphism $j_* : \pi_1(A, a) \to \pi_1(X, a)$ is injective.*

Proof. Let $r : X \to A$ be a retraction, then $\mathrm{id}_A = r \circ j : A \xrightarrow{j} X \xrightarrow{r} A$. $\therefore$ $\mathrm{id} = r_* \circ j_* : \pi_1(A, a) \to \pi_1(X, a) \to \pi_1(A.a)$.$\therefore$ j_* is injective. $\square$

Theorem 3.29. *S^1 is not a retract of D^2.*

Proof. $j_* : \pi_1(S^1, b_0) \to \pi_1(D^2, b_0) = 0$ is not injective, in which $\pi_1(S^1, b_0) \cong \mathbb{Z}$. $\square$

Lemma 3.30. *Let $h : S^1 \to X$ be a continuous map. Then the followings are equivalent.*

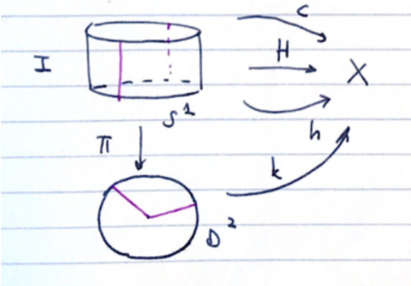
(1) h is nullhomotopic, i.e. $h \simeq C$.

(2) h extends to a continuous map $k : D^2 \to X$,i.e.

$$\begin{tikzcd} S^1 \arrow[r, "h"] \arrow[d, "i"'] & X \\ D^2 \arrow[ur, "k"'] & \end{tikzcd}$$

(3) $h_* : \pi_1(S^1, b_0) \to \pi_1(X, h(b_0))$ is the trivial homomorphism.

Proof. (1)⇒(2) Let $H : S^1 \times I \to X$ be a homotopy between h and a constant map c.



Let $\pi : S^1 \times I \to D^2, (x, t) \mapsto (1 - t)x$. Then π is continuous, closed and surjective, so it is a quotient map. $H\big|_{S^1 \times \{1\}} = C$. $\therefore$ H induces a continuous map $k : D^2 \to X, k\big|_{S^1} = h$.

(2)⇒(3)

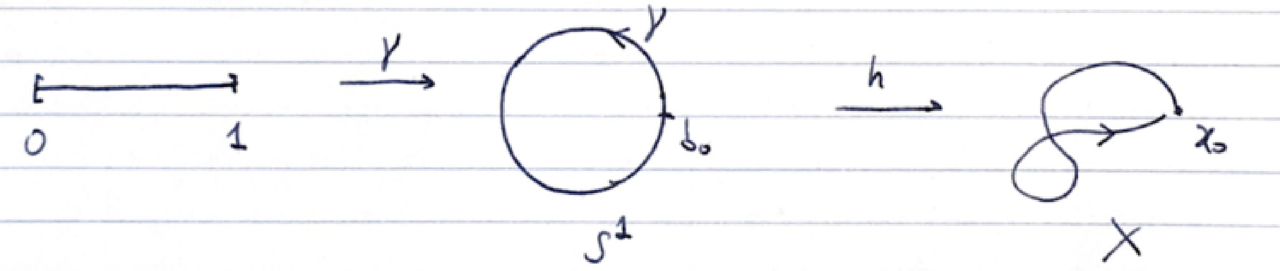
$$\begin{tikzcd} S^1 \arrow[r, "h"] \arrow[d, "i"'] & X \\ D^2 \arrow[ur, "k"'] & \end{tikzcd}$$

$\therefore h_* = k_* \circ j_*$.

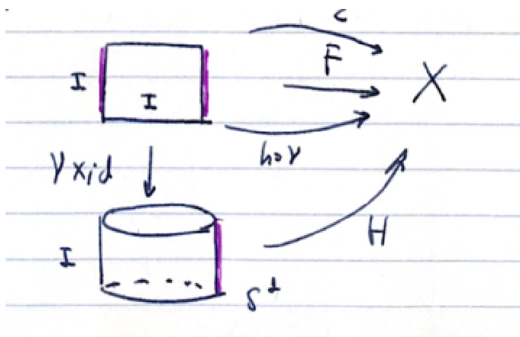
$$\pi_1(S^1, b_0) \to \pi_(D^2, b_0) \to \pi_1(X, h(b_0))$$
$$\| \quad 0$$

$\therefore$ h_* is the trivial homomorphism.

(3)⇒(1): A generator of $\pi_1(S^1, b_0) \cong \mathbb{Z}$ is represented by the loop $\gamma : [0, 1] \to S^1, x \mapsto e^{i2\pi x}$.



Let $h(b_1) = x_0$, if $h_* : \pi_1(S^1, b_0) \to \pi_1(X, x_0)$ is trivial, then $h \circ \gamma \simeq_p e_{x_0}$. Let $F : I \times I \to X$ be a path homotopy.

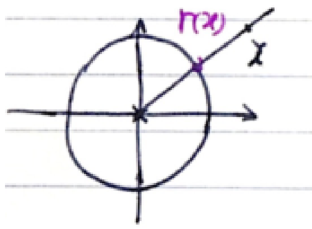


The map $\gamma \times \mathrm{id} : I \times I \times S^1 \times I, (x,t) \mapsto (e^{i2\pi x}, t)$ is a quotient map (since continuous, closed and surjective). $F(0 \times I) = x_0 = F(1 \times I)$. $\therefore F$ induces a continuous map $H : S^1 \times I \to X$, which is a homotopy between h and c.

□

Corollary 3.31. *The inclusion map $j : S^1 \to \mathbb{R}^2 = 0$ is not nullhomotopic. the identity map $\mathrm{id} : S^1 \to S^1$ is not nullhomotopic.*

Proof.



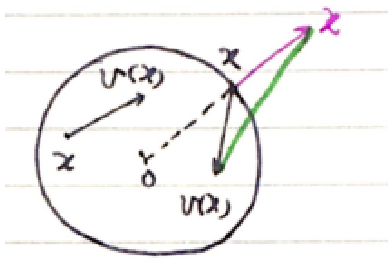
$\exists$ a retraction $r : \mathbb{R}^2 - 0 \to S^1, x \mapsto \dfrac{x}{|x|}$
$\therefore j_* : \pi_1(S^1, b_0) \to \pi_1(\mathbb{R}^2 - 0, b_0)$ is injective not trivial.
Similar for $\mathrm{id} : S^1 \to S^1$.

□

Theorem 3.32. *Given a nonvanishing continuous vector field $v : D^2 \to \mathbb{R}^2 - 0$ on D^2, there exists a point $x_1 \in S^1$ s.t. $v(x_1)$ points directly inward, and a point $x_2 \in S^1$ s.t. $v_*(x_2)$ points directly outward.*

Proof. Assume $v(x)$ does not point directly inward for $\forall x \in S^1$. Consider $h = v\big|_{S^1} : S^1 \to \mathbb{R}^2 - 0$.

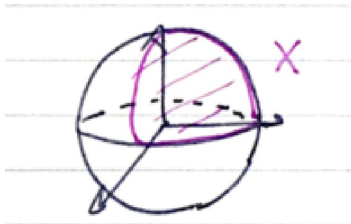
Claim: h is homotopic to the inclusion map $j : S^1 \hookrightarrow \mathbb{R}^2 - 0$, hence not nullhomotopic.



proof.
We have a straight-line homotopy $F : S^1 \times I \to \mathbb{R}^2 - 0, F(x,t) = (1-t)v(x) + tx$.
On the other hand, h extends to $v : D^2 \to \mathbb{R}^2 - 0$, hence is nullhomotopic, a contradiction.
For the directly put-pointing point x_2, consider $v'(x) = -v(x)$. □

Theorem 3.33 (Brouwer fixed-point theorem)**.** *If $f : D^n \to D^n$ is a continuous map, then there exists a point $x \in D^n$ s.t. $f(x) = x$.*

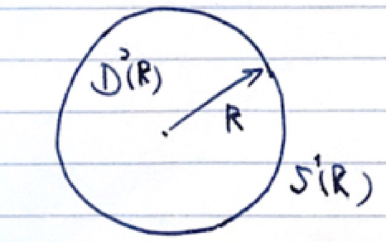
Proof. Let $X = S^{n-1} \cap \mathbb{R}^n_{\geqslant 0}$, then X is homeomorphic to D^{n-1}. $f : X \to X, x \mapsto Ax/|Ax|$ has a fixed point $x_0 \in X$, i.e. $Ax_0 = |Ax_0| \cdot x_0$.



$\square$

Theorem 3.34 (The fundamental theorem of algebra). *A polynomial $f(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0 \in \mathbb{C}[z](n > 0)$ has a root in $\mathbb{C}$.*

Proof. Assume $f(z) \neq 0$ for $\forall z \in \mathbb{C}$.
Let $D^2(R)$ be the closed disc of radius R. $\partial D^2(R) = S^1(R)$ the circle of radius R.

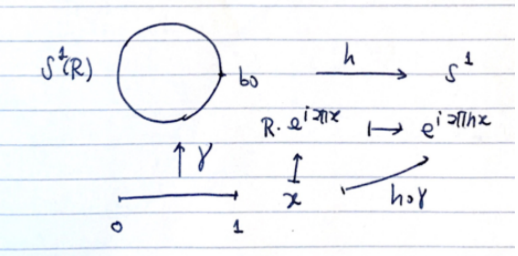


$\xrightarrow{f} \mathbb{C} - 0$

Let $\varphi = f\big|_{S^1(R)} : S^1(R) \to \mathbb{C} - 0$,
Claim: For R large enough, φ is homotopic to $g : S^1(R) \to \mathbb{C} - 0, g(z) = z^n$.
Proof. We have a straight-line homotopy between φ and g:

$$F : S^1(R) \times I \to \mathbb{C}, F(z,t) = (1-t)\varphi(z) + tz^n = z^n + (1-t)\left(a_{n-1}z^{n-1} + \cdots + a_0\right)$$

For R large enough, $|F(z,t)| \geqslant |z|^n - (|a_{n-1}|) > 0$.
$\therefore F : S^1(R) \times I \to \mathbb{C} - 0$.
φ extends to $f\big|_{D^2}(R) : D^2(R) \to \mathbb{C} - 0$, hence is nullhomotopic $\Rightarrow$ g is nullhomotopic for R large enough.
Claim: g is not nullhomotopic for all $R > 0$.
Proof.
$h : S^1(R) \xrightarrow{g} \mathbb{C} - 0 \xrightarrow{r} S^1$
$z \mapsto z^n, \quad z \mapsto z/|z|$



$$\begin{array}{ccc} h_* : \pi_1(S^1(R)) & \to & S^1 \\ \text{\rotatebox{90}{$\cong$}} & & \text{\rotatebox{90}{$\cong$}} \\ \mathbb{Z} & & \mathbb{Z} \\ 1 & \mapsto & n \end{array}$$

$\therefore h$ is not homotopic$\Rightarrow$ g is not nullhomotopic. $\square$

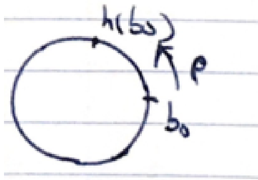
Theorem 3.35 (Borsuk-Ulam theorem). *Given a continuous map $f : S^n \to \mathbb{R}^n (n \geqslant 0)$, there is a point $x \in S^n$ s.t. $f(x) = f(-x)$.*

Definition 3.36. *The antipode of a point $x \in S^n$ is the point $-x$. A map $h : S^n \to S^n$ is antipode-preserving (保对径点的) if $h(-x) = -h(x)$ for all $x \in S^n$.*

Theorem 3.37. *If $h : S^1 \to S^1$ is a continuous antipode-preserving map, then h is not nullhomotopic.*

Proof. Let $b_0 = 1 \in S^1, \rho : S^1 \to S^1$ be a rotation s.t. $\rho(h(b_0)) = b_0$. Then $\rho \circ h$ is also antipode-preserving. $\rho \simeq \mathrm{id}_{S^1} \Rightarrow h \simeq C$ iff $\rho \circ h \simeq C$.

$\therefore$ It suffices to prove the theorem for those h with $h(b_0) = b_0$.



Consider the covering map $q : S^1 \to S^1, q(z) = z^2$.

$$\begin{array}{ccc} q_* : \pi_1(S^1, b_0) & \to & \pi_1(S^1, b_0) \\ \cong & & \cong \\ \mathbb{Z} & \xrightarrow{\cdot 2} & \mathbb{Z} \end{array}$$

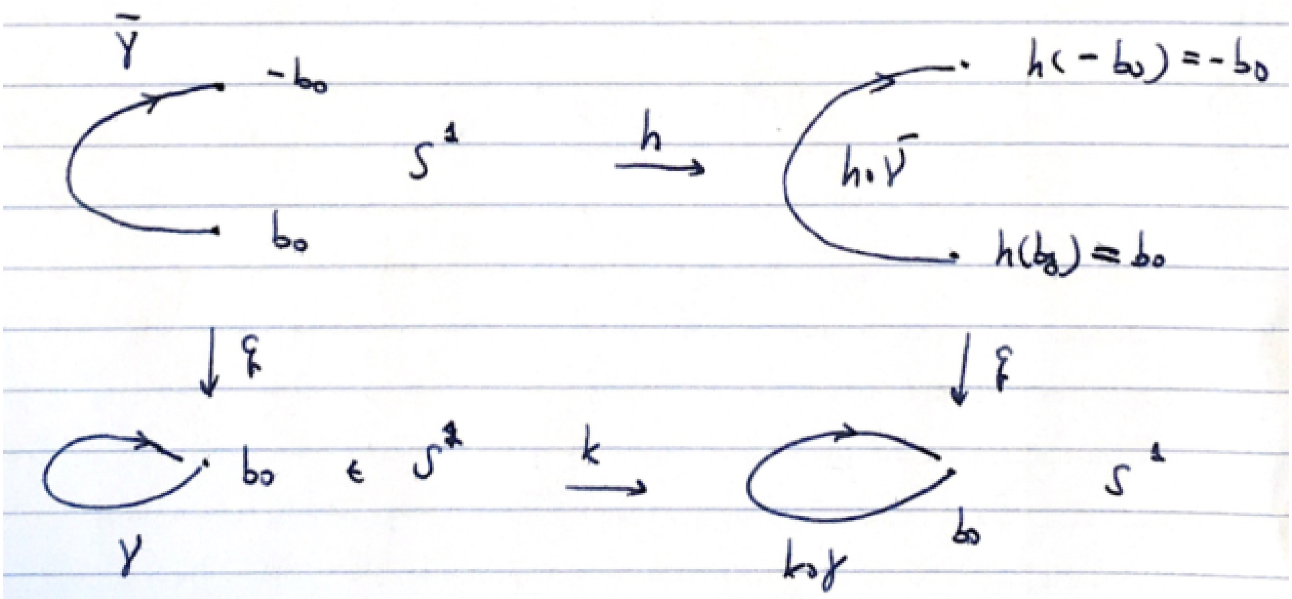
is a multiplication by 2.

Since $q(h(-z)) = q(-h(z)) = q(h(z))$, h induces a map $k : S^1 \to S^1$ s.t. $k \circ q = q \circ h$, k is continuous since q is a quotient map.

$$\begin{array}{ccc} S^1 & \xrightarrow{h} & S^1 \\ \downarrow q & & \downarrow q \\ S^1 & \xrightarrow{k} & S^1 \end{array}$$

Claim: $k_* : \pi_1(S^1, b_0) \to \pi_1(S^1, b_0)$ is nontrivial.

Proof.



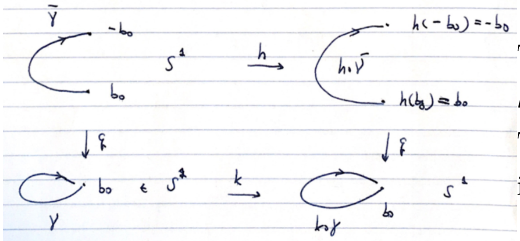
Let $\overline{\gamma}$ be a path in S^1 from b_0 to $-b_0$, then $\gamma = q \circ \overline{\gamma}$ is a loop at b_0 representing a nontrivial element $[\gamma] \in \pi_1(S^1, b_0)$. Now $h \circ \overline{\gamma}$ is a lifting of $k \circ \gamma$ with $[k \circ \gamma] = k_*[\gamma] \in \pi_1(S^1, b_0)$. Since $h \circ \overline{\gamma}$ is a path from b_0 to $-b_0$, $k_*[\gamma] \neq e$. Therefore $k_* : \pi_1(S^1, b_0) \to \pi_1(S^1, b_0)$ is nontrivial, hence injective. $k \circ q = q \circ h \Rightarrow k_* \circ q_* = q_* \circ h_* : \pi_1(S^1, b_0) \to \pi_1(S^1, b_0)$ h_* us injective, hence h is not nullhomotopic.

$$\begin{array}{ccc} \pi_1(S^1, b_0) & \xrightarrow{h_*} & \pi_1(S^1, b_0) \\ \downarrow q_* = \cdot 2 & & \downarrow q_* = \cdot 2 \\ \pi_1(S^1, b_0) & \xrightarrow{k_*} & \pi_1(S^1, b_0) \end{array}$$

□

Theorem 3.38. *There is no continuous antipode-preserving map $g : S^2 \to S^1$.*

Proof. Suppose $g : S^2 \to S^1$ is such a map.



Take S^1 to be the equator of S^2,
$h = g\big|_{S^1} : S^1 \to S^1$.
Then h is continuous and antipode- preserving, $\therefore$ h is not nullhomotopic.

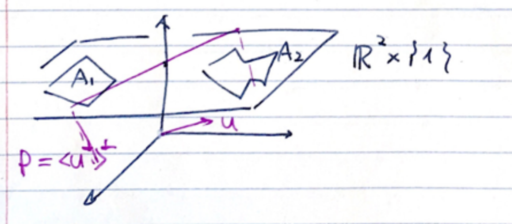
But g is an extension of h to $D^2_+ \cong D^2$, hence h is nullhomotopic, a contradiction. $\square$

Proof of the Borsuk-Ulam theorem for S^2

Let $f : S^2 \to \mathbb{R}^2$ be a continuous map s.t. $f(x) \neq f(-x)$ for all $x \in S^2$. Then the map $g : S^2 \to S^1, x \mapsto \dfrac{f(x) - f(-x)}{|f(x) - f(-x)|}$ is continuous and $g(-x) = -g(x), \forall x \in S^2$. $\square$

Theorem 3.39 (the bisection theorem)**.** *Given two bounded polygonal regions A_1 and A_2 in $\mathbb{R}^2$, there exists a line L in $\mathbb{R}^2$ that bisects each of them.*

Proof. Let $\mathbb{R}^2 = \mathbb{R}^2 \times \{1\} \subset \mathbb{R}^3$.



For $u \in S^2$, let $\rho = \langle u \rangle^\perp = \{x \in \mathbb{R}^3 | \langle x, u \rangle = 0\}$ a plane
$v = \{x \in \mathbb{R}^3 | \langle x, u \geqslant 0 \rangle\}$
$f_i(u) = \text{Area}(A_i \cap V)(i = 1, 2)$ a continuous function in U. Then $f_i(U) + f_i(-U) = \text{Area}(A_i)$.
$F : S^2 \to \mathbb{R}^2, F(u) = (f_1(u), f_2(u))$.
Borsuk-Ulam theorem $\Rightarrow$ $\exists u \in S^2$ s.t. $F(u) = F(-u)$.
$\therefore f_i(u) = f_i(-u) = \dfrac{1}{2}\text{Area}(A_i)$. The corresponding line $L = P \cap (\mathbb{R}^2 \times \{1\})$ bisects A_1 and A_2. $\square$

Theorem 3.40 (invariance of domain, 区域不变性)**.** *$U \subset \mathbb{R}^2$ an open set, $f : U \to \mathbb{R}^2$ continuous and injective. Then $f(U) \subset \mathbb{R}^2$ is open and the inverse function $f^{-1} : f(U) \to U$ is open and the inverse function $f^{-1} : f(U) \to U$ is continuous, i.e. f is a homeomorphism.*

A simple closed curve (简单闭曲线) C is a space homeomorphic to S^1.

Theorem 3.41 (the Jordan curve theorem)**.** *Let $C \subset S^2$ be a simple closed curve. Then C separates S^2 into two components W_1 and W_2, i.e. $S^2 - C = W_1 \sqcup W_2$. Each of W_i has C as its boundary, i.e. $\overline{W}_i - W_i = C$.*

Theorem 3.42 (Schöenflies theorem)**.** *Let $C \subset S^2$ be a simple closed curve with $S^2 - C = W_1 \bigcup W_2$. Then $\overline{W}_i$ is homeomorpgic to the closed disc D^2.*

Counter example in dimension 3: the Alexander horned sphere.

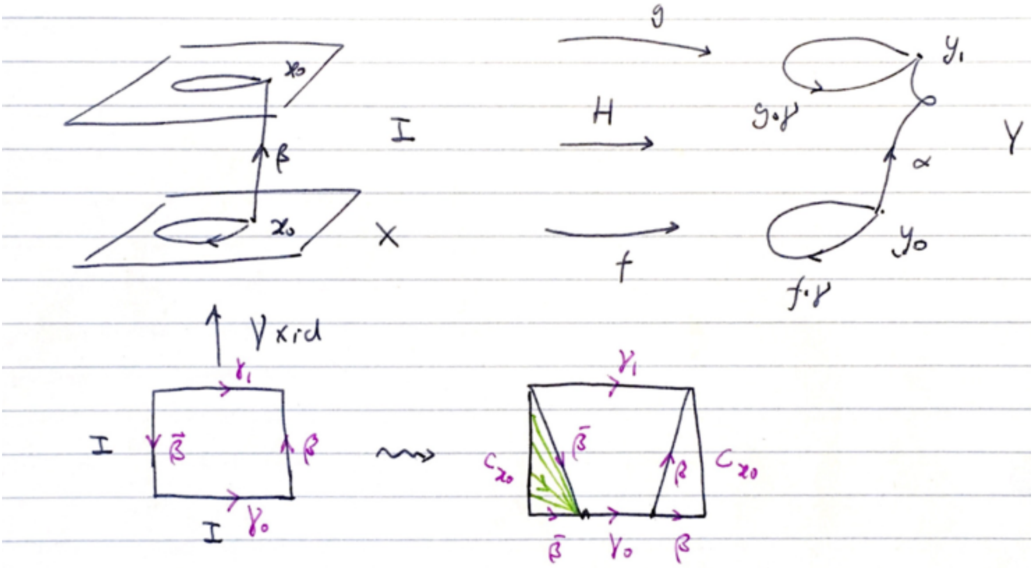
3.6 homotopy type

Theorem 3.43. *Let $f, g : X \to Y$ be continuous maps, $f(x_0) = y_0$,$g(x_0) = y_1$. Let $H : X \times I \to Y$ be a homotopy between f and g, $\alpha(t) = H(x_0, t)$ be the path in Y from y_0 to y_1. Then there is a commutative diagram*

$$\begin{tikzcd} \pi_1(X, x_0) \arrow[r, "f_*"] \arrow[dr, "g_*"'] & \pi_1(Y, y_0) \arrow[d, "\Phi_\alpha", "\cong"'] \\ & \pi_1(Y, y_1) \end{tikzcd}$$

Especially, if $f(x_0) = y_0 = g(x_0)$, H is a homotopy relative to x_0, then $f_* = g_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$.

Proof. Let $\gamma : I \to X$ be a loop at x_0, we need to show $f_*[\gamma] = \Phi_\alpha(f_*[\gamma])$, i.e. $g \circ \gamma \simeq_p \overline{\alpha} * (f \circ \gamma) * \alpha$. Consider $\gamma \times \mathrm{id} : I \times I \to X \times I$, let $\gamma_0(t) = (\gamma(t), 0), \gamma_1(t) = (\gamma(t), 1), \beta(t) = (x_0, t)$. Then $\gamma \times \mathrm{id}$ is a path homotopy between γ_1 and $\overline{\beta} * \gamma_0 \beta$, $H \circ (\gamma \times \mathrm{id})$ is a path homotopy between $g \circ \gamma$ and $\overline{\alpha} * (f \circ \gamma) * \alpha$.



□

Corollary 3.44. *If $f : X \to Y$ is nullhomotopic, then f_* is the trivial homomorphism.*

Theorem 3.45. *If $f : X \to Y$ is a homotopy equivalence, then $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is an isomorphism.*

Proof. Let $g : Y \to X$ be a homotopy inverse of f, $g \circ f \simeq \mathrm{id}_X \implies$

$$\begin{tikzcd} \pi_1(X, x_0) \arrow[r, "f_*"] \arrow[drr, bend right, "\mathrm{id}"'] & \pi_1(Y, f(x_0)) \arrow[r, "g_*"] & \pi_1(X, gf(x_0)) \arrow[d, "\Phi_\alpha", "\cong"'] \\ & & \pi_1(X, x_0) \end{tikzcd}$$

$\therefore (\Phi_{\alpha \circ g_*}) \cdot f_* = \mathrm{id} : \pi_1(X, x_0) \to \pi_1(X, x_0)$.

$f \circ g \simeq \mathrm{id}_Y \implies$

$$\begin{tikzcd} \pi_1(Y, f(x_0)) \arrow[r, "g_*"] \arrow[drr, bend right, "\mathrm{id}"'] & \pi_1(X, gf(x_0)) \arrow[r, "f_*"] & \pi_1(Y, fgf(x_0)) \arrow[d, "\Phi_beta"', "\cong"] \\ & & \pi_1(Y, f(x_0)) \end{tikzcd}$$

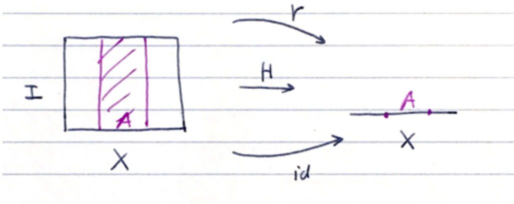
$\therefore$

$$\begin{tikzcd}
\pi_1(X,x_0) \arrow[r, "f_*"] & \pi_1(Y,f(x_0)) \\
\pi_1(X,gf(x_0)) \arrow[u, "\cong"', "\Phi_\alpha"] \arrow[r, "f_*"] & \pi_1(Y,fgf(x_0)) \arrow[u, "\cong"', "\Phi_{f\circ\alpha}"] \arrow[dl, "\Phi_\beta"] \\
\pi_1(Y,f(x_0)) \arrow[u, "g_*"] &
\end{tikzcd}$$

$\therefore f_* \circ \left(\Phi_\alpha \circ g_* \circ \Phi_\beta \circ (\Phi_{f\circ\alpha})^{-1}\right) = \mathrm{id} : \pi_1(Y,f(x_0)) \to \pi_1(Y,f(x_0)).$ $\square$

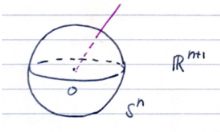
Slogan: homotopy equivalent spaces have isomorphic fundamental groups.

Definition 3.46. *Let $A \subset X$ be a subspace. A deformation retraction (形变收缩) is a homotopy $H : X \times I \to X$ s.t. $H(x,0) = x, H(x,1) \in A$ for all $x \in X$, and $H(a,t) = a$ for all $a \in A, t \in I$.$r : X \to A, r(x) = H(x,1)$ is a retraction. A is a deformation retract (形变收缩核) of X.*



Examples:

(1)

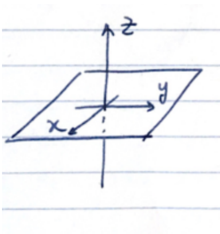


$S^n \subset \mathbb{R}^{n+1} - 0$

$r : \mathbb{R}^{n+1} - 0 \longrightarrow S^n, x \mapsto x/|x|$

$H(x,t) = (1-t)x + t \cdot \dfrac{x}{|x|}$

(2)



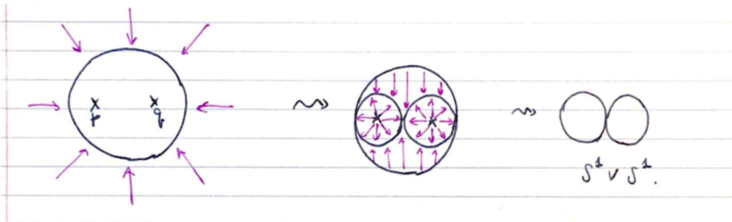
$X = \mathbb{R}^3 -$ the z-axis has $\mathbb{R}^2 - 0$ as a deformation retract.

$H(x,y,z,t) = (x,y,(1-t)z)$

$\therefore \mathbb{R}^3 -$ the z-axis $\simeq \mathbb{R}^2 - 0 \simeq S^1$

$\pi_1(X) \simeq \mathbb{Z}$.

(3) the double punctured plane $X = \mathbb{R}^2 - \{p,q\}$



3.7 the Seifert-van Kampen theorem

Theorem 3.47. *Let* $p_\alpha : \prod_\alpha X_\alpha \to X_\alpha$ *be the projection. Then the induced homomorphism* $\pi_{p_{\alpha *}} : \pi_1\left(\prod_\alpha X_\alpha\right) \to \prod_\alpha \pi_1(X_\alpha)$ *is an isomorphism.*

Proof.

$$\begin{aligned} \pi_{p_{\alpha *}} : \pi_1\left(\prod X_\alpha\right) &\to \prod_\alpha \pi_1(X_\alpha) \\ \text{surjective:} \quad (\gamma_\alpha(t)) &\mapsto (\gamma_\alpha(t)) \\ \text{injective:} \quad (H_\alpha(t)) &\mapsto (H_\alpha(t)) \end{aligned}$$

□

Example: $T^n = \underbrace{S^1 \times \cdots \times S^1}_{n \text{ fold}}$, the n-torus, $\pi_1(T^n) \cong \bigoplus_n \mathbb{Z} = \mathbb{Z}^n$.

Definition 3.48. *Let* $\{G_\alpha\}$ *be a family of groups. A* *word* *with letters in* $\{G_\alpha\}$ *is* $g_1 g_2 \cdots g_m$ *with* $m > 0$, $g_i \in G_{\alpha_i}, g_i \neq e$.

It is *reduced* *if* $\alpha_1 \neq \alpha_{i+1}$. *The* *free product* *of* $\{G_\alpha\}$ *is the group* $G = \underset{\alpha}{*} G_\alpha$, *where*

- $\underset{\alpha}{*} G_\alpha =$ *the set of reduced words with letters in* $\{G_\alpha\}$ *together with the empty word.*
- *Group operation=juxtaposition and reduction:*

  $(g_1 \cdots g_m) \cdot (h_1 \cdots h_n) = g_1 \cdots g_m h_1 \cdots h_n \rightsquigarrow$ *reduced word.*
- *inverse element:* $(g_1 \cdots g_m)^{-1} = g_m^{-1} \cdots g_1^{-1}$.
- *associativity law:*

$$\underbrace{g_1 \cdots g_m h_1 \cdots h_1 \cdots \overbrace{h_i \cdots h_j}^{}}_{\textit{reduction}}\!\!\overbrace{\cdots h_n k_1 \cdots k_l}^{\textit{reduction}}$$

(c.f. [Hatcher, p.41])

Examples:

(1) a free group (自由群) is a free porduct of any number of copies of $\mathbb{Z}$.

$F_n = \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{n \text{ fold}}$ the free group of rank n.

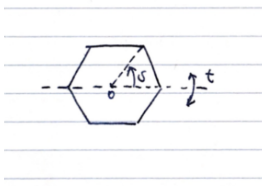
e.g. $F_2 = \{a^{m_1} b^{n_1} a^{m_2} b^{n_2} \cdots a^{m_k} b^{n_k} | m_i, n_i \in \mathbb{Z}, k \in \mathbb{N}\}$.

(2) $\mathbb{Z}/2 * \mathbb{Z}/2 = \{e, a, b, ab, ba, aba, bab, \cdots\}$.

$\exists$ a split short exact sequence:

$$\begin{tikzcd} 1 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d, equal] & \mathbb{Z}/2 * \mathbb{Z}/2 \arrow[r] & \mathbb{Z}/2 \arrow[r] \arrow[l, bend right, "a \leftarrow\!\shortmid 1"'] & 1 \\ & \{(ab)^n | \mathbb{Z}\} & w \arrow[r, mapsto] & \text{Length of } w \text{ mod } 2 & \end{tikzcd}$$

$\therefore \mathbb{Z}/2 * \mathbb{Z}/2 \cong \mathbb{Z} \rtimes \mathbb{Z}/2$, s semi-direct product.



the finite dihedral group

a regular n-gon

$D_{2n} = \langle s, t | s^n, t^2, tsts \rangle$

the dihedral group (二面体群)

$$1 \longrightarrow \mathbb{Z}/n \longrightarrow D_{2n} \longrightarrow \mathbb{Z}/2 \longrightarrow 1$$

There is an embedding $G_\alpha \xhookrightarrow{i_\alpha} \underset{\alpha}{*} G_\alpha$, $g \mapsto g$.

If there is a collection of homomorphisms $\{\varphi_\alpha : G_\alpha \to H\}$, then there is a unique extexsion $\varphi : \underset{\alpha}{*} G_\alpha \to H$ s.t. $\varphi_\alpha = i_\alpha \circ \varphi : G_\alpha \to H$ given by $\varphi(g_1 \cdots g_m) = \varphi_{\alpha_1}(g_1) \cdots \varphi_{\alpha_m}(g_m)$.

Remark: in the category of abelian groups, $\bigoplus_\alpha G_\alpha$ satisfies the property.

Especially, the family of homomorphisms $\{i_k : G_k \to G_1 \times \cdots \times G_n\}$ extends to a surjective homomorphism $\overset{n}{\underset{i=1}{*}} G_i \to G_1 \times \cdots \times G_n$.

Consider $F_n = \underset{n}{*} \mathbb{Z} \twoheadrightarrow \mathbb{Z}^n = \bigoplus_n \mathbb{Z}$. Any homomorphism $\varphi : F_n \to A$ to an abelian group A factors through $\overline{\varphi}\mathbb{Z}^n \to A$,

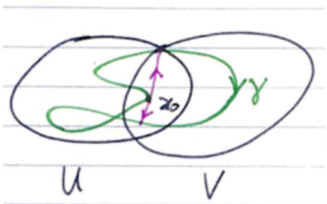
$$\begin{tikzcd} F_n \arrow[r, two heads] \arrow[d, "\varphi"'] & \mathbb{Z}^n \arrow[dl, dashed, "\overline{\varphi}"'] \\ A & \end{tikzcd} \quad \text{, since} \quad \begin{tikzcd} \mathbb{Z} \arrow[r, "i_k"] \arrow[dr, "i_k"'] & F_n \arrow[r, "\varphi"] \arrow[d, two heads] & A \\ & \mathbb{Z}^n \arrow[ur, "\overline{\varphi}"'] & \end{tikzcd}$$

$\therefore \mathbb{Z}^n$ is the abelianization of F_n, i.e. $F_n/[F_n, F_n]$. $\mathbb{Z}^n \ncong \mathbb{Z}^m$ if $m \neq n \Rightarrow F_n \ncong F_m$ if $m \neq n$.

Remark: A subgroup of $\mathbb{Z}^n$ is a free abelian group of rank $\leqslant n$. A subgroup of F_n is a free group, rank may be $> n$. e.g. $F_n < F_2, F_\infty < F_2$.

Let $X = U \cup V$, where U and V are open in X; assume that U, V and $U \cap V$ are path connected, $x_0 \in U \cap V$.

Then we have a commutative diagram
$$\begin{tikzcd} \pi_1(U \cap V) \arrow[r, "i_1"] \arrow[d, "i_2"] & \pi_1(U) \arrow[d, "j_1"] \\ \pi_1(V) \arrow[r, "j_2"] & \pi_1(X) \end{tikzcd}$$

X= 

Theorem 3.49 (Seifert-van Kampen)**.** *Under the above assumption, the homomorphism $\Phi : \pi_1(U) * \pi_1(V) \to \pi_1(X)$ is surjective, the kernel of Φ is the normal subgroup generated by all elements of the form $i_1(\alpha)i_2(\alpha)^{-1}, \alpha \in \pi_1(U \cap V)$.*

$\therefore \pi_1(X) \cong \pi_1(U) * \pi_1(V) / \langle i_1(\alpha)i_2(\alpha)^{-1} | \alpha \in \pi_1(U \cap V) \rangle_N$.

Corollary 3.50. *Under the same hypothesis,*

(1) if $U \cap V$ is simply connected, then

$$\pi_1(U) * \pi_1(V) \to \pi_1(X)$$

is an isomorphism.

(2) if V is simply connected, then

$$\pi_1(V) / \langle \mathrm{Im} i_* : \pi_1(U \cap V) \to \pi_1(U) \rangle_N$$

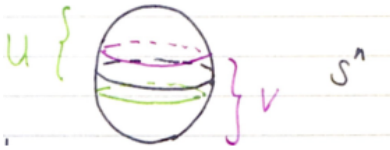
Examples:

(1)

$\pi_1(S^n) = 0, (n \geqslant 2)$

$S^n = U \cup V$

$U \cong \overset{\circ}{D} \cong V$



$U \cap V \cong S^{n-1} \times (-1, 1) \simeq S^{n-1}, \therefore \pi_1(S^n) = \pi_1(U) * \pi_1(V)/N = 0$

(2) (i)

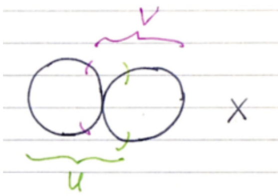
$X = S^1 \vee S^1$ "figure eight"

$X = U \cap V, U \simeq S^1 \simeq V$,

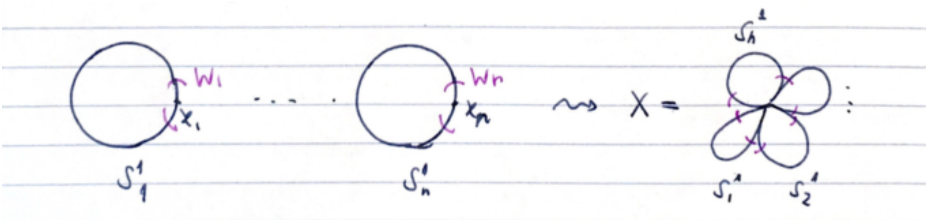
$U \cap V \simeq$ pt.

$\therefore \pi_1(S^1 \vee S^1) \cong \pi_1(S^1) * \pi_1(S^1) \cong \mathbb{Z} * \mathbb{Z} = F_2$.

(Compare with $\pi_1(S^1 \times S^1 \cong Z^2)$)



(ii) in general, let $(S^1_1, x_1), \cdots, (S^1_n, x_n)$ be n circles. $X = \bigvee_n S^1 = S^1_1 \cup \cdots \cup S^1/x_1 \sim x_2 \sim \cdots \sim x_n$ the quotient space is a wedge of n circles.



$U \subset X$ is open $\Leftrightarrow U \cap S^1_i$ is open in S^1_i for all i,

Let $W_i \subset S^1_i$ be an open arc containing x_i,

$U_i = S^1_i \vee \bigvee_{j \neq i} U_j$, then $X = \bigcup_{i=1}^{n} U_i, U_i \simeq S^1, U_i \cap U_j = W_i \cap W_j \simeq pt$. Inductively, $\pi_1(X) \cong \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{n \text{ fold}} = F_n$.

(iii) $X = \bigvee_\infty S^1$ a countably infinite wedge of circles. $\pi_1(X) = \pi_1\left(\lim\limits_{n\to\infty} \bigvee_n S^1\right) = \lim\limits_{n\to\infty} \pi_1\left(\bigvee_n S^1\right) = \lim\limits_{n\to\infty} F_n = F_\infty$.

"taking π_1 commutes with direct limits"

(iv) $C_n =$ the circle of radius $\dfrac{1}{n}$, center $\left(\dfrac{1}{n}, 0\right)$. $Y = \bigcup_{n=1}^{\infty} C_n \subset \mathbb{R}^2$ with the subspace topology.

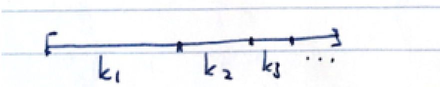
(the hawaii's earring)

$\exists$ a retraction $r_n : Y \to C_n$ collapsing all $C_i (i \neq n)$ to $0. \rho_n = (r_n)_* : \pi_1(Y) \to \pi_1(C_n) \rightsquigarrow \rho = \prod \rho_n : \pi_1(Y) \to \prod_{n=1}^{\infty} \pi_1(C_n) = \prod_\infty \mathbb{Z}$ a homomorphism.

ρ is surjective: $\forall (k_1, k_2, \cdots, k_n, \cdots) \in \prod_\infty \mathbb{Z}$,

$\gamma : [0,1] \to Y, \gamma\Big|_{\left[1-\frac{1}{n}, 1-\frac{1}{n+1}\right]} : \left[1 - \dfrac{1}{n}, 1 - \dfrac{1}{n+1}\right] \to C_n$

represents $k_n \in \pi_1(C_n) . \gamma$ is continuous.



$\prod_{\infty} \mathbb{Z}$ is uncountable$\Rightarrow$ $\pi_1(\gamma)$ is uncountable!

whereas $\pi_1\left(\bigvee_{\infty} S^1\right) = F_\infty$ is countable.

Consider the retraction $r: Y \to C_1 \cup \cdots \cup C_n$ collapsing all $C_i (i > n)$ to 0. Then $r_*: \pi_1(Y) \to \pi_1(C_1 \cup \cdots \cup C_n) = F_n$ is surjective. $\Rightarrow \pi_1(Y)$ is nonabelian.

Generators and relations:

G a group, $S \subset G$ is a set of generators if $\forall g \in G, g = S_1^{n_1} \cdots S_k^{n_k}, s_i \in S, n_i \in \mathbb{Z}$. Let $\{a_\alpha\}_{\alpha \in J}$ be a set of generators of G, $F \cong \mathop{*}_J \mathbb{Z}$ be the free group generated by $\{a_\alpha\}_{\alpha \in J}$, then there is a surjective homomorphism $h: F \to G, h(a_\alpha) = a_\alpha$. Let $N = \ker h$, called the relation subgroup, each element of N is called a relation. If $\{r_\beta\}$ is a set of N s.t. they normally generate N, i.e. N is the smallest normal subgroup of F that contains $\{r_\beta\}$, we call the family $\{r_\beta\}$ a complete set of relations for G.

Definition 3.51. *A presentation (表现) of a group G consists of a set of generators $\{a_\alpha\}$, along with a complete set of relations $\{r_\beta\}$. Notation: $G = \langle a_\alpha | r_\beta \rangle$. If $\{a_\alpha\}$ is finite, G is finitely generated (有限生成); if both $\{a_\alpha\}$ and $\{r_\beta\}$ are finite, G is finitely presented (有限表现), $\{a_\alpha\}$,$\{r_\beta\}$ form a finite presentation of G.*

Examples:

(1) $F_n = \langle a, \cdots, a_n \rangle$.

$\mathbb{Z}/n = \langle a | a^n \rangle, \mathbb{Z}^2 = \langle a, b | [a, b] \rangle$

$D_{2n} = \langle s, t | s^n, t^2, tsts \rangle$

$\mathbb{Z}/2 * \mathbb{Z}/2 = \left\langle a, b \middle| a^2, b^2 \right\rangle$

$= D_\infty = \mathbb{Z} \rtimes \mathbb{Z}/2 = \left\langle s, t \middle| t^2, tsts \right\rangle (s = ab, t = a)$

e.x. find a representation for $S_3, \mathrm{SL}_2(\mathbb{Z})$.

(2) $\mathbb{Q}$ is not finitely generated.

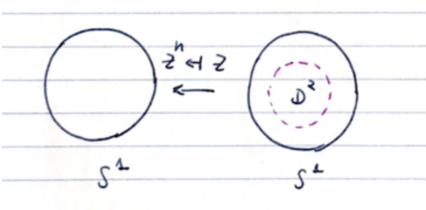
(3) If $G_1 = \langle a_\alpha | r_\beta \rangle$,$G_2 = \langle b_{\alpha'} | s_{\beta'} \rangle$, then $G_1 * G_2 = \langle a_\alpha, b_{\alpha'} | r_\beta, s_{\beta'} \rangle$.

Remark: A group has various presentations. Given two finite presentations $G_1 = \langle a_\alpha | r_\beta \rangle$ and $G_2 = \langle b_{\alpha'} | s_{\beta'} \rangle$, to determine if G_1 is isomorphic to G_2 is an unsolvable problem !

Examples:

(1) $f: S^1 \to S^1, z \mapsto z^n (n \geqslant 1)$

$X_n = D \sqcup S^1 / (z \sim f(z) \quad z \in S^1)$=$S^1 \cup_f D^2$ the quotient space (compact Hausdorff)



$X_1 \cong D^2$

X_n the n-fold dunce cap

$$X = \underset{\mathring{D}^2}{\underset{\|}{U}} \cup \underset{(D^2-0)\cup S^1/\sim \xrightarrow[\simeq]{r} S^1}{\underset{\|}{V}}$$

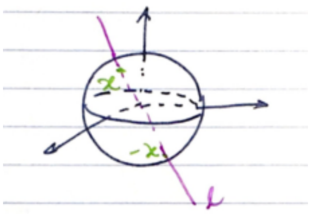
$$U \cap V = \mathring{D}^2 - 0 \simeq S^1\left(\frac{1}{2}\right).$$

$$\begin{aligned}\pi_1(X) =& \pi_1(V)/\left(\mathrm{Im}\left(\pi_1(U\cap V) \to \pi_1(V)\right)\right)_N \\ \cong& \pi_1(V)/\left(\mathrm{Im} f_*\right)_N \\ =& \mathbb{Z}/n\mathbb{Z} = \langle a | a^n \rangle\end{aligned}$$

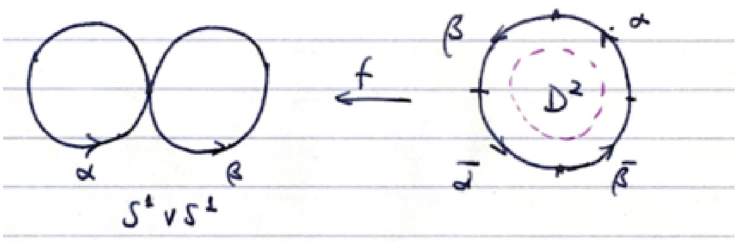
$X_2 = \mathbb{R}P^2 =$ the space of straight lines in $\mathbb{R}^3$ through O

the real projective plane (实射影平面)

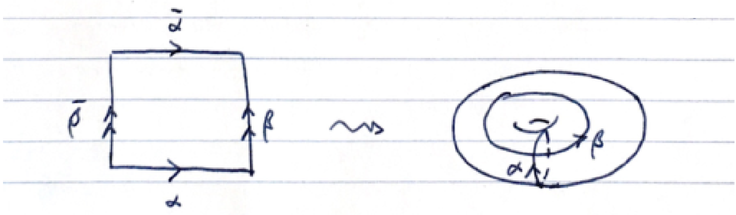
$$\begin{aligned}&= \mathbb{R}^3 - 0/\left(x \sim \lambda x | \lambda \neq 0\right) \\ &= S^2/_{x\sim -x} = D^2/_{(x\sim -x | x\in S^1)}\end{aligned}$$



(2) $f : S^1 \to S^1 \vee S^1$



$$\left((S^1\vee S^1)\cup D^2\right)/_{(x\sim f(x)|x\in S^1)} = (S^1\vee S^1)\cup_f D^2 = T^2$$



$$T^2 = \underset{\mathring{D}^2}{\underset{\|}{U}} \cup \underset{(S^1\vee S^1)\cup_f (D^2-0) \xrightarrow[\simeq]{r} S^1\vee S^1}{\underset{\|}{V}}$$

$$U \cap V = \mathring{D}^2 - O \simeq S^1\left(\frac{1}{2}\right)$$

$\therefore \pi_1(T^2) = \pi_1(V)/_{\langle \mathrm{Im} f_* \rangle_N} = \mathbb{Z} * \mathbb{Z}/_{\langle [a,b] \rangle_N} \cong \mathbb{Z}^2$

(Claim: $\langle [a,b] \rangle_N = [F_1, F_2]$

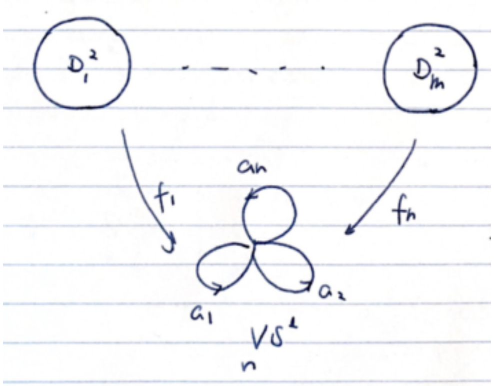
Proof. (i) $\langle [a,b] \rangle_N \subset [F_1, F_2]$

(ii) $F_2/_{\langle [a,b]\rangle_N}$ is generated by aN, bN where $N = \langle [a,b]\rangle_N$. But $aN \cdot bN = bN \cdot aN, \therefore F_2/_{\langle [a,b]\rangle_N}$ is abelian,$\therefore [F_1, F_2] \subset \langle [a,b]\rangle_N$)

(3) in general, given a finitely presented group

$G = \langle a_1, \cdots, a_n | r_1, \cdots, r_m\rangle$

$X = \left(\bigvee_n S^1 \cup \bigcup_{i=1}^m D_i^2\right) / \sim$ a 2-dim complex (复形)



$f_i : S_i^1 \to \bigvee_n S^1$ representing $r_i \in \pi_1\left(\bigvee_n S^1\right)$, then

$$\begin{aligned}\pi_1(X) \cong & F_n/_{\langle r_1, \cdots, r_m\rangle_N} \\ = & \langle a_1, \cdots, a_n | r_1, \cdots, r_m\rangle \\ = & G\end{aligned}$$

$\therefore$ the classification of finite 2-dim complexes is unsolvable.

e.g. $\exists$ a finite 2-dim complex X with $\pi_1(X) \cong D_{2n}, S_n, \mathrm{SL}_2(\mathbb{Z})$. What is your favourite f.p. group?

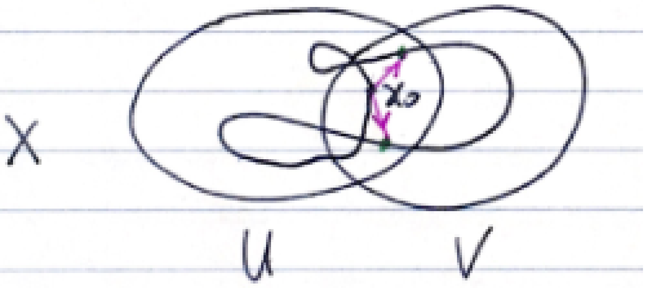
(4) more general, given $a \in \pi_1(X, x_0)$, represented by $\gamma : (S^1, b_0) \to (X, x_0)$. Let $X \cup_\gamma D^2 = X \cup D^2/_{(x \sim \gamma(x) | x \in S^1)}$. Then $\pi_1(X \cup_\gamma D^2) = \pi_1(X)/_{\langle a\rangle_N}$,i.e. a is "killed" in $X \cup_\gamma D^2$.

Proof of the Seifert-van Kanpen Theorem

(sketch)

(c.f.[基础拓扑学　尤承业])

(1) $\pi_1(U) * \pi_1(V) \xrightarrow{\Phi} \pi_1(X)$ is surjective:



Let $\gamma : [0,1] \to X$ be a loop at x_0.

$\exists$ a subdivision $0 = a_0 < a_1 < \cdots < a_n = 1$, s.t. $\gamma([a_1, a_{i+1}]) \subset U$ or V, $\gamma(a_i) \in U \cap V$.(Lebesgue number lemma).

Choose a path α_i from x_0 to $\gamma(a_i), i = 1, \cdots, n-1$.

Let $\gamma_i = \gamma\big|_{[a_{i-1},a_i]}$,$\beta_i = \alpha_{i-1} * \gamma_i * \overline{\alpha}_i$, a loop at x_0 in U or V. Then

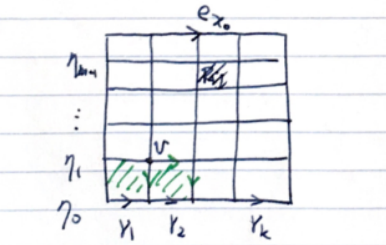
$$
\begin{aligned}
\gamma \simeq_p &(\alpha_0 * \gamma_1 * \overline{\alpha}_1) * (\alpha_1 * \gamma_2 * \overline{\alpha}_2) * \cdots * (\alpha_{n-1} * \gamma_n \overline{\alpha}_n) \\
= &\beta_1 * \cdots * \beta_n
\end{aligned}
$$

$\therefore [\gamma] = \Phi(\beta_1 \beta_2 \cdots \beta_n)$.

(2) let $\langle i_1(a) i_2(a)^{-1} | a \in \pi_1(U \cap V) \rangle_N = N$, clearly $N \subset \ker \Phi$. Need to show $\ker \Phi \subset N$, it suffices to show that $\pi = \pi_1(U) * \pi_1(V)/N \xrightarrow{\Phi} \pi_1(X)$ is injective.

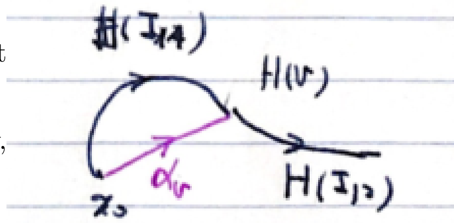
Let γ be a loop in U or V, $[\gamma]$ be the element in $\pi_1(U)$ or $\pi_1(V)$,$\langle \gamma \rangle$ be the element in π. If γ is a loop in $U \cap V$, $\langle \gamma \rangle$ is a well-defined element in π. For $w \in \pi, w = [\gamma_1] \cdots [\gamma_k] N = \langle \gamma_1 \rangle \cdots \langle \gamma_k \rangle$.

If $\Phi(w) = e \in \pi_1(X)$, then there is a path homotopy $H : X \times I \to X$.



Subdivide $I \times I$ into rectangles R_{ij} s.t. $H(R_{ij}) \subset U$ or V. For each vertice v of R_{ij}, choose a path α_γ from x_0 to $H(v)$ in U or V.

Consider the loop η_1 in X by inserting $\overline{\alpha}_v \alpha_v$, we have an element $\langle \eta_1^1 \rangle \cdots \langle \eta_1^k \rangle \in \pi$,
and $\langle \eta_1^1 \rangle \cdots \langle \eta_1^k \rangle = \langle \gamma_1 \rangle \cdots \langle \gamma_k \rangle$ using $H\big|_{R_{1i}}$. Inductively, $\langle \gamma_1 \rangle \cdots \langle \gamma_k \rangle = \langle e_{x_0} \rangle = e \in \pi$.

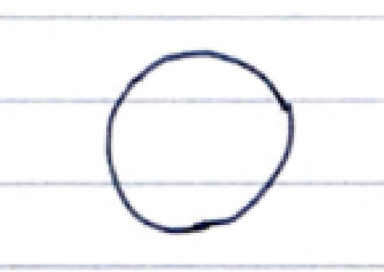


3.8 additional topics

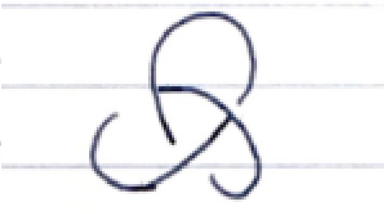
3.8.1 knots and links

(c.f.[Armstrong, Basic Topology])

Definition 3.52. *A knot (扭结) is a subspace of* $\mathbb{R}^3$ *or* S^3 *which is hoeomorphic to* S^1.



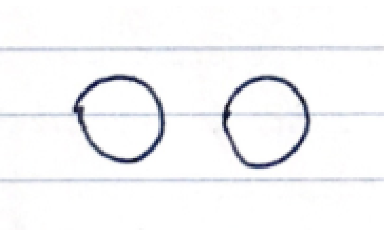
(a) the trivial knot



(b) the trefoil knot(三叶结)

(c) the figure eight knot (8 字结)

A link (链环) L is a disjoint union of knots.
$L = K_1 \cup K_2 \cup \cdots K_n (g \geqslant 2)$.



(a)

(b) the hopf link

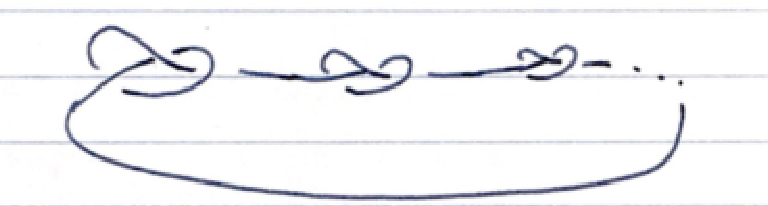
(c) the whitehead link

the Borromean ring

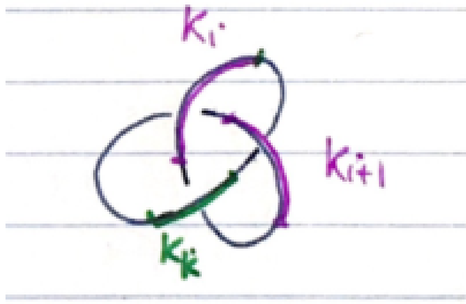
Two knots K_1 and K_2 are equivalent if $\exists$ a homeomorphism $f : \mathbb{R}^3 \to \mathbb{R}^3$ s.t. $f(K_1) = K_2$, and f is isotopic (同痕) to id. (i.e. $\exists$ a homotopy $H : \mathbb{R}^3 \times I \to \mathbb{R}^3$ between f and id, and $H(-, t) : \mathbb{R}^3 \to \mathbb{R}^3$ is a homeomorphism $\forall t \in I$).

(To avoid: )

A knot is tame if it is equivalent to a polygonal knot.

A wild knot: 

$\pi_1(\mathbb{R}^3 - K)$ is called the knot group. It is an invariant of the knot K.

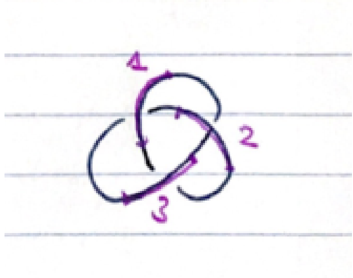


$$\mathbb{R}^3 - K = \big(\mathbb{R}^3_{\geqslant 0} - K\big) \cup (\mathbb{R}^3_{\leqslant 0} \quad -K)$$

$$\wr\!\mid$$
$$*$$

$$\pi_1(\mathbb{R}^3_{\geqslant 0} - K) \cong \langle x_1 \cdots x_n \rangle$$

Each underpass between K_i and K_{i+1},separated by K_k gives a relation $x_i x_k = x_k x_{i+1}$.
$\therefore \pi_1(\mathbb{R}^3 - K) \cong \big\langle x_1 \cdots x_n \big| x_i x_k x_{i+1}^{-1} x_k^{-1} i = 1, \cdots, n-1 \big\rangle$ (the wirtinger presentation)



e.g. $K =$ the trefoil knot
generators: $x_1, x_2, x_3,$
relations: $x_1 x_3 = x_3 x_2, x_2 x_1 = x_1 x_3$

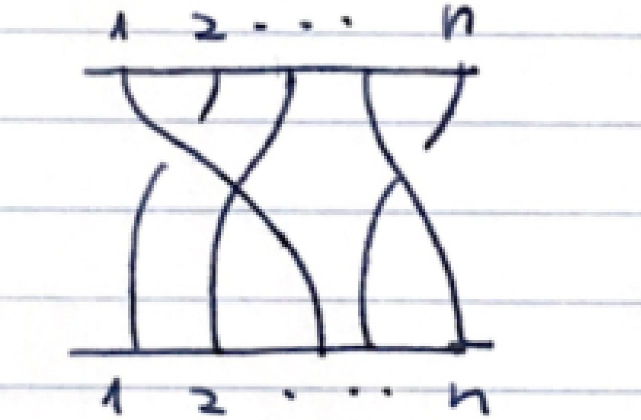
$$x_3 = x_1^{-1} x_2 x_1 \rightsquigarrow \pi_1(\mathbb{R}^3 - K) \cong \big\langle x_1, x_2 \big| x_2 x_1 = x_1^{-1} x_2 x_1 x_2 \big\rangle$$
$$\cong \langle x_1, x_2 | x_1 x_2 x_1 = x_2 x_1 x_2 \rangle$$

$$\begin{tikzcd} \pi_1(\mathbb{R}^3 - K) \arrow[d, two heads] & x_1 \arrow[d, mapsto] & x_2 \arrow[d, mapsto] \\ S_3 & (1\,2) & (2\,3) \end{tikzcd}$$

$\therefore \pi_1(\mathbb{R}^3 - K) \ncong \pi_1(\mathbb{R}^3 - \bigcirc) \cong \mathbb{Z}$.
$\therefore$ the trefoil knot is not trivial.

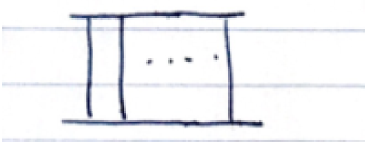
3.8.2 braids

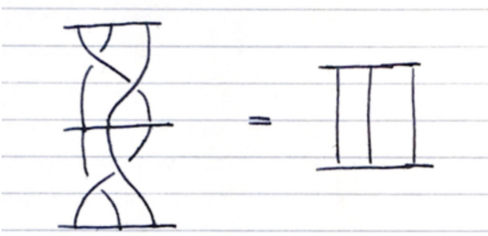
A braid (辫) of n strings is an object as follows:



$n = 5$

The set of (equivalences classes of) n-braids B_n has a group structure by concatenation:

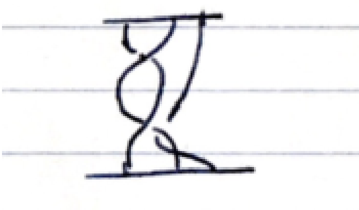
- neutral element = the trivial braid 
- inverse element = the reversed braid



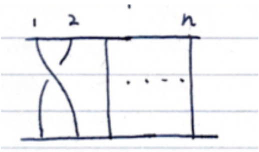
B_n is called the (Artin) braid group. (辫群)

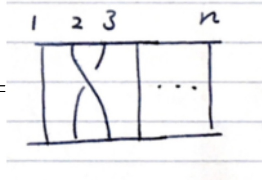
e.g. $B_2 \cong \mathbb{Z}$

$$1 \to \underset{\substack{\| \\ \text{the pure braid group}}}{PB_n} \to B_n \to S_n \to 1$$

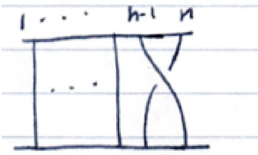


a presentation of B_n:

$\sigma_1=$ 

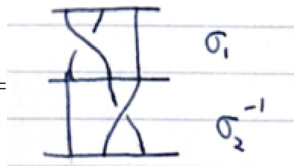
, $\sigma_2=$ 

$,\cdots,$

$\sigma_{n-1}=$ 

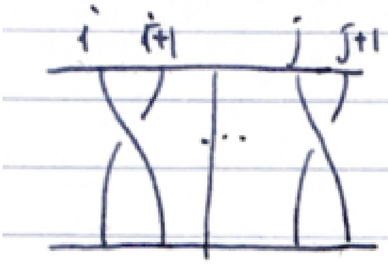
, Then B_n is generated by $\sigma_1,\cdots,\sigma_{n-1}$.

e.g. 

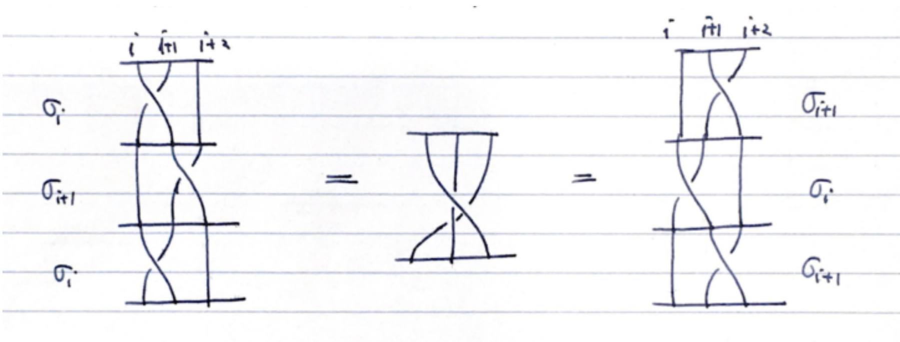
$=$ 

$=\sigma_1\cdot\sigma_2^{-1}$.

relations: (1)$\sigma_1\sigma_j = \sigma_j\sigma_i$ for $|i-j|\geqslant 2$



(2) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for all i.



$\therefore B_n = \langle\sigma_1\cdots\sigma_{n-1}|[\sigma_i,\sigma_j]|i-j|\geqslant 2, \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1},\forall i\rangle$.

Definition 3.53. *The unordered configuration space (构型空间) of n points in $\mathbb{R}^2$ is* $U\mathrm{conf}_n(\mathbb{R}^2) = \left\{(x_1,\cdots,x_n)\in(\mathbb{R}^2)^n | x_\varepsilon\neq x_j,\forall\varepsilon\neq j\right\}/_{S_n}$.

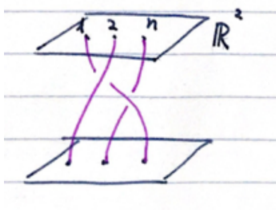
A loop $\gamma: I\to U\mathrm{conf}_n(\mathbb{R}^2)$ at the base point $(1,2,\cdots,n)\in(\mathbb{R}^2)^n$ corresponds to a braid.

$\therefore \pi_1(U\mathrm{conf}_n(\mathbb{R}^2))\cong B_n$

$\pi_1(\mathrm{conf}_n(\mathbb{R}^2))\cong PB_n$

$\quad\|$

the ordered configuration space.



3.8.3 pushout

(1) given sets and maps, let $\rho = X \sqcup Y/_{(}f(z) \sim g(z))|z \in Z$

$$\begin{tikzcd}
Z \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "i_1"] \arrow[ddr, bend left, "j_1"] & \\
Y \arrow[r, "i_2"] \arrow[drr, bend right, "j_2"'] & P \arrow[dr, dashed, "u"] & \\
& & Q
\end{tikzcd}$$

(P, i_1, i_2) satisfies the universal property:
for any $j_1 : X \to Q, j_2 : Y \to Q$,
s.t.$j_1 \circ f = j_2 \circ g$, $\exists!$ a map
$u : P \to Q$, s.t. $j_1 = i_1 \circ u, j_2 = i_2 \circ u$.

P, i_1, i_2 is unique up to isomorphism, called the pushout of the diagram

$$\left(\begin{tikzcd}
Z \arrow[r, "f"] \arrow[d, "g"] & X \\
Y &
\end{tikzcd}\right).$$

(2) topological spaces and continuous maps

$$\begin{tikzcd}
Z \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "i_1"] \arrow[ddr, bend left, "j_1"] & \\
Y \arrow[r, "i_2"] \arrow[drr, bend right, "j_2"'] & P \arrow[dr, dashed, "u"] & \\
& & Q
\end{tikzcd}$$

$\rho = X \sqcup Y/_{(}f(z) \sim g(z)|z \in Z)$
quotient space
satisfies the sane universal property in $\mathcal{T}$op.

(3) groups and homomorphisms

$$\begin{tikzcd}
H \arrow[r, "f_1"] \arrow[d, "f_2"'] & G_1 \arrow[d, "i_1"] \arrow[ddr, bend left, "j_1"] & \\
G_2 \arrow[r, "i_2"] \arrow[drr, bend right, "j_2"'] & P \arrow[dr, dashed, "u"] & \\
& & Q
\end{tikzcd}$$

$\rho = G_1 * G_2/_{\langle f_1(h)f_2(h)^{-1}|h\in H\rangle_N}$
satisfies the same universal
property in $\mathcal{G}$p.

If $H < G_1, H < G_2, P$ is denoted by $G_1 *_H G_2$, called the amalgamated product of G_1 and G_2 over H.

e.g.
$\mathrm{SL}_2(\mathbb{Z}) = \mathbb{Z}/_4 *_{\mathbb{Z}/_2} Z/_6$
$\mathcal{T}\mathrm{op}_* \xrightarrow{\pi_1} \mathcal{G}\mathrm{p}$

$$\begin{tikzcd}
U \cap V \arrow[r] \arrow[d] & U \arrow[d] \\
V \arrow[r] & X = U \cup V
\end{tikzcd}
\mapsto
\begin{tikzcd}
\pi_1(U \cap V) \arrow[r] \arrow[d] & \pi_1(U) \arrow[d] \\
\pi_1(V) \arrow[r] & \pi_1(X)
\end{tikzcd}$$

pushout

pushout
(Seifert-van Kampen theorem)

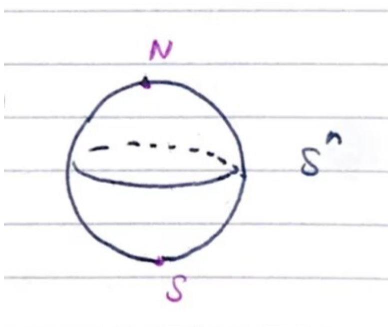
Chapter 4

Surfaces

4.1 introduction

Definition 4.1. *An n-dimensional* <u>*topological manifold*</u> *(拓扑流形) M^N is a second countable Hausdorff space which is locally homeomorphic to the euclidean space space $\mathbb{R}^n$, i.e. for $\forall\ x \in M$, $\exists$ an open neighborhood U of x, and a homeomorphism $h : U \xrightarrow{\cong} \mathbb{R}^n$.*

Examples.
$\mathbb{R}^n$, $S^n = (S^n - \{N\}) \cup (S^N - \{S\})$
$T^n = S^1 \times \cdots \times S^1$



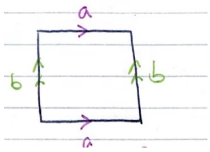
<u>Question</u> . What is a classification of an n-dimensional (conpact) manifolds?

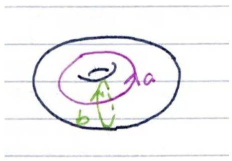
Theorem 4.2. *A compact connected 1-dimensional manifold is homeomorphic to S^1.*

4.2 the classification of closed Surfaces

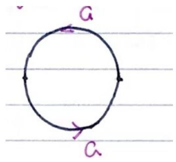
Examples of compact Surfaces (closed urfaces, 闭曲面).

(1) S^2 the 2-sphere

(2) $T^2 =$ 

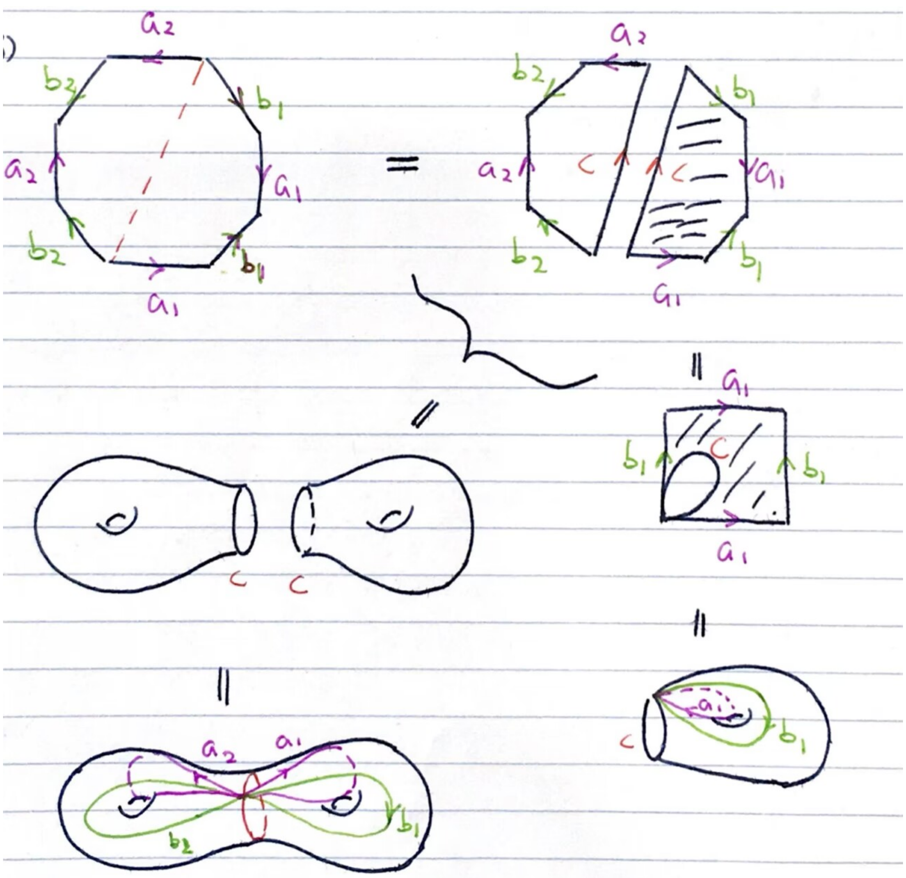
$=$ 

the torus.

$P^2 = \mathbb{R}P^2 =$ 

the projective plane

(3)

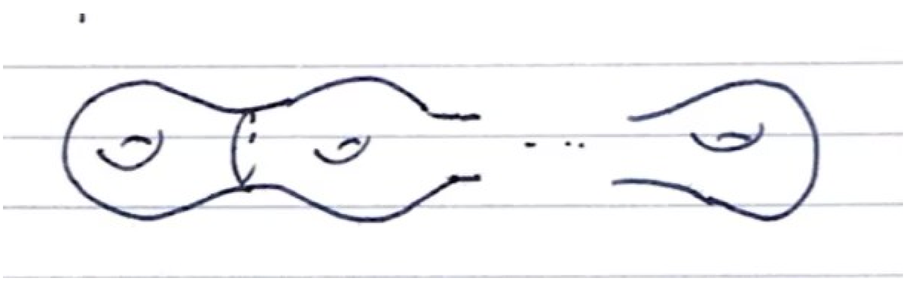


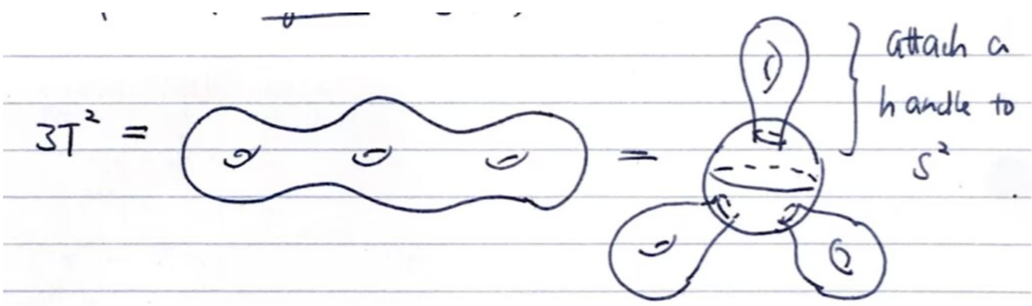
the 2-torus $2T^2 = T^2 \# T^2 = (T^2 - \mathring{D}^2) \cup_\partial (T^2 - \mathring{D}^2)$ the concected sum (连通和) of T^2 and T^2.

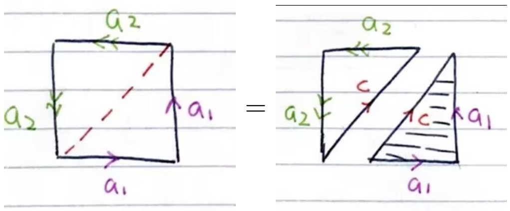
In general, take a regular $4n$-gon, label the edges $a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1}\dots a_nb_na_n^{-1}b_n^{-1}$, glue the corresponcling edges, we get a closed surface $nT^2 = T^2\#T^2\#\dots\#T^2$ (n-fold) called the n-torus, or the orientable (可定向) closed surface of genus (亏格) n.

Remark:

(4)

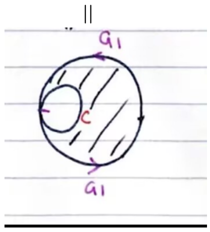






$$= \left(\mathbb{R}P^2 - \overset{\circ}{D}{}^2\right) \cup_\partial \left(\mathbb{R}P^2 - \overset{\circ}{D}{}^2\right)$$

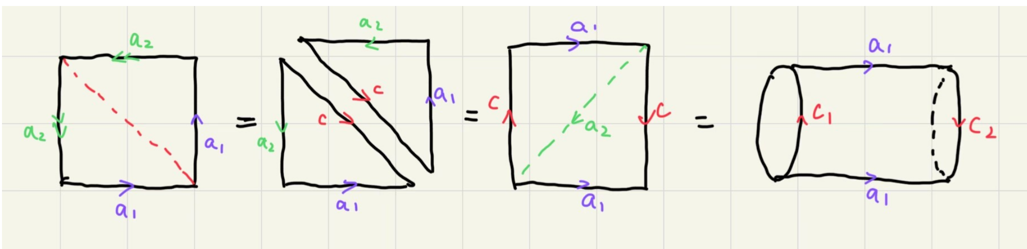
$\|$



$=2P^2 = P^2 \# P^2$

the connected sum of P^2 and P^2

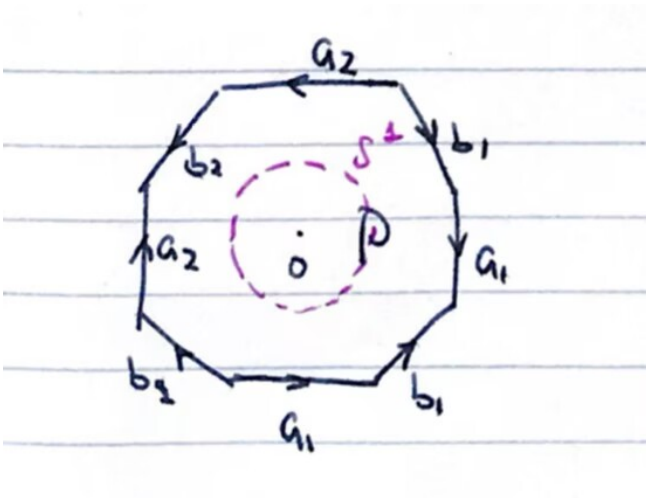
another point of view:



In general, take a regular $2m$-gon, label the edgs $a_1a_1a_2a_2\ldots a_ma_m$, glue the corresponcling edges, we get a closed surface $mP^2 = P^2\#\ldots\#P^2$ (m fold), called the non-orientable closed surface of genus m.

Theorem 4.3 (The classification of closed Surfaces)**.** *A closed surface is homeomorphic to exactly one of the surfaces nT^2 $(n \geqslant 0)$ or mP^2 $(m \geqslant 1)$.*

Proof. (1) the "models" nT^2 and mP^2 are not homeomorphic to each other.

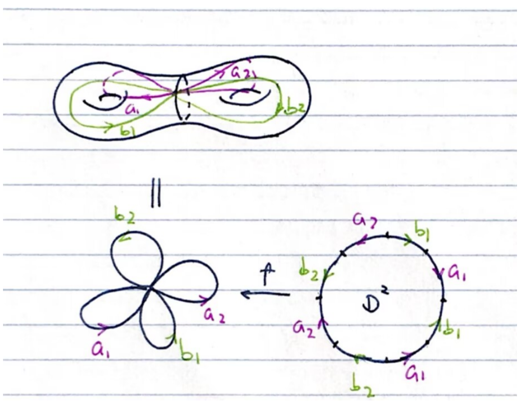


$$\begin{array}{cccc} nT^2 = & U & \cup & V \\ & \| & & \| \\ & \mathring{P} & & (P-0)/\sim \xrightarrow{p,\simeq} \bigvee_{2n} S^1 \end{array}$$

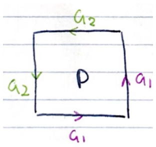
$U \cap V = \mathring{P} - 0 \simeq S^1$

$\therefore \pi_1(nT^2) \cong \pi_1(V)/_{\langle Im \pi_1(U\cap V) \longrightarrow \pi_1(V) \rangle_N}$

$$= \left\langle a_1, b_1, \ldots, a_n, b_n \middle| a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_n b_n a_n^{-n} b_n^{-n} = \prod_{i=1}^{n} [a_i, b_i] \right\rangle$$



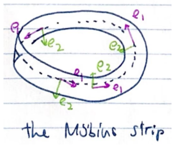
$$\pi_1(nT^2)_{ab} (\cong H_1(nT^2)) = \left\langle a_1, b_1 \ldots a_n, b_n \middle| \prod_{i=1}^{n} [a_i, b_i] \right\rangle_{ab}$$

similarly 

$\pi_1(mP^2) \cong \langle a_1, \ldots a_m | a_1^2 a_2^2 \ldots a_m^2 \rangle$

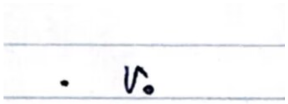
$$H_1(mP^2) = \pi_1(mP^2)_{ab} \cong \mathbb{Z}^m/_{2(e_1+\cdots+e_m)} \cong \mathbb{Z}/_2 \oplus \mathbb{Z}^{m-1}$$

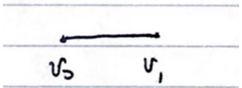
□

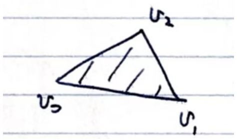
Remark: $P^2 =$  $\cup_\partial D^2$

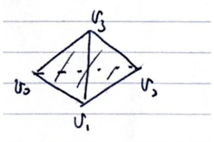
Any mP^2 contains a möbius strop, hence is not orientable.

Let $v_1, \ldots v_k$ be points in $\mathbb{R}^n$. They are in <u>general position</u> if $v_1 - v_0, \ldots, v_k - v_0$ are linearly independent. A <u>k-simplex</u> (k-常形) with vertices $v_0, \ldots v_k$ is $\sigma = \left\{ x = \sum_{i=0}^{k} \lambda_i k_i \in \mathbb{R}^n \middle| \sum_{i=0}^{k} \lambda_i = 1, \lambda_i \geqslant 0 \right\} \subset \mathbb{R}^n$

e.g. 0-simplex :  (point)

1-simplex :  (line segment)

2-simplex :  (triangle)

3-simplex :  (tetrahedron)

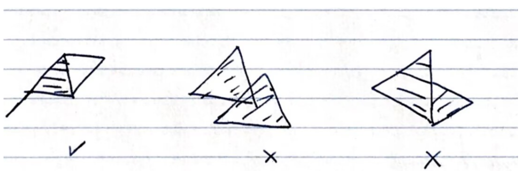
If σ, τ are simplices, and the vertices of τ form a subset of the vertices of σ, then we say that τ is a <u>face</u> of σ.

Definition 4.4. *A finite collection K of simplices in some euclidean space $\mathbb{R}^n$ is called a (finite) <u>simplcial complex</u> (单纯复形) If*

(i) for $\forall\ \sigma \in K$, all the faces of σ are elements in K

(ii) for $\forall\ \sigma, \tau \in K$, if $\sigma \cap \tau \neq \varnothing$, then σ and τ intersect in a common face.

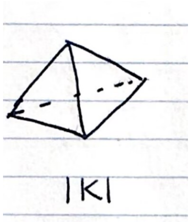
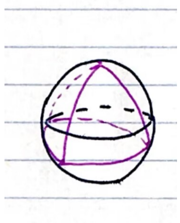
e.g.

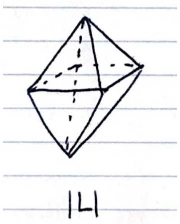
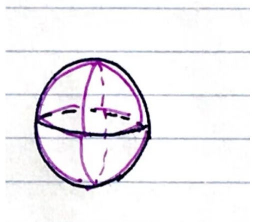


The <u>geometric realization</u> of a simplcial complex K is $|K| = \bigcup_{\sigma \in K} \sigma \subset \mathbb{R}^n$, a subspace of $\mathbb{R}^n$, a <u>polyhedron</u> (多面体).

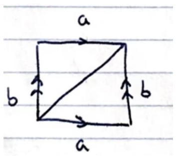
Definition 4.5. *A <u>triangulation</u> (单纯剖分/三角剖分) of a topological space X consists of a simplcial complex K and a homeomorphism $h : |K| \xrightarrow{\cong} X$. Such a space X is called <u>triangulable</u> (可三角剖分的).*

Examples

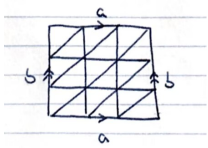
(1)  $\underset{\cong}{\xrightarrow{h}}$  S^2

 $\underset{\cong}{\xrightarrow{h'}}$ 

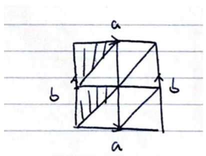
(2) T^2 :



not a triangulation.

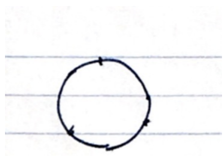


not a triangulation.



a triangulation.

Theorem 4.6. *A compact Topological manifold of dim$\leqslant$ 3 is triangulable. In each dimension $n \geqslant 4$, there exist non-triangulable topological manifolds.*

diam 1 : S^1 is triangulable 

diam 2 : closed surfaces are triangulable. (Radó 1925)

diam 3 : compact 3-manifolds are triangulable.(Möise 1950)

diam 4 : Freedman (1980)

diam$\geqslant$ 5 : Manolescu (2016)

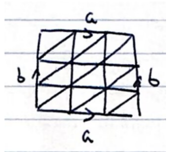
Definition 4.7. *Let K be a finite simplcial complex, denote the number of i-simplices of K by m_i, the Euler characteristic (欧拉示性数) of K is $\chi(K) = \sum_{i=1}^{n}(-1)^i m_i \in \mathbb{Z}$.*

Definition and theorem 4.8.

Let X be a triangulable space, the Euler characteristic of X is $\chi(X) = \chi(K)$, where K is a triangulation of X. $\chi(X)$ is independent of the choice of triangulations, it is a topological invariant.

Examples

(i) $\chi(S^2) = \chi($  $) = 4 - 6 + 4 = 2$

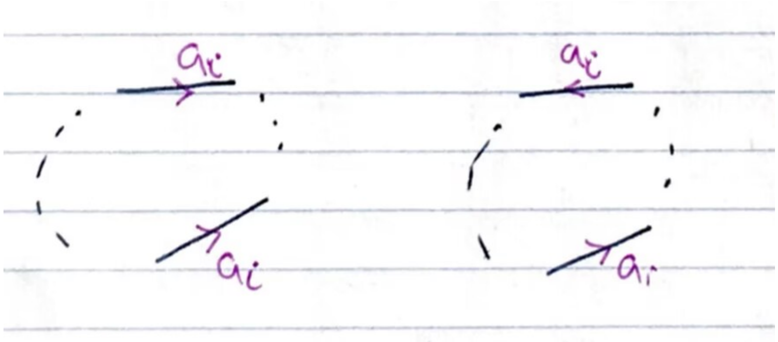
(ii) $\chi(T^2) = \chi($ 

$) = 0$

<u>proof of the classification theorem of surfaces</u>

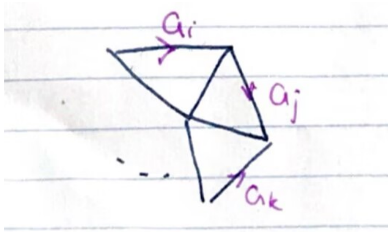
(2) every closed surface is homeomorphic to nT^2 or mP^2.

(i) Let P be a $2n$-gon in $\mathbb{R}^2$, E be the set of edges of P, a labelling of the edges of P is a 2-to-1 map $E \xrightarrow{l} \{a_1, \dots a_n\}$. Given an orientation of each edge of P. Identifying the edges of P according to the labelling and orientables, we get a quotient space $P/_\sim$.



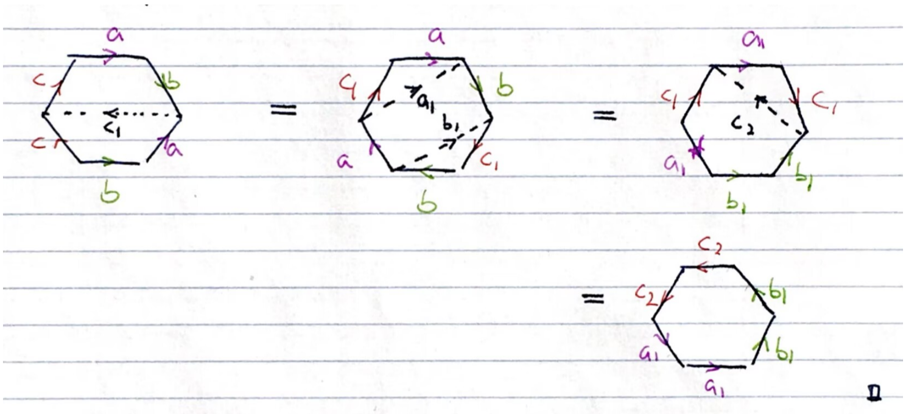
if $S \cong P/_\sim$, we say this is a polygonal presentation of S. As a corollary of the triangulabilrfy of closed surfaces;

Lemma 4.9. *Every colsed surface has a polygonal presentation.*



(ii) transform a polygonal presentation to the standard presentation of nT^2 or mP^2: $a_1b_1a_1^{-1}b_1^{-1}\dots$ or $a_1a_1a_2a_2\dots$

Example . $P^2\#T^2 \cong 3P^2 \cong P^2\#$ Klein bottle.

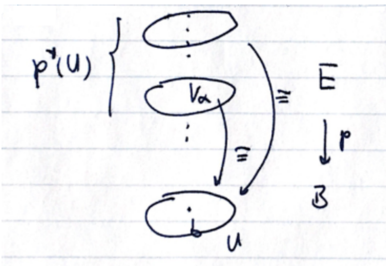


Chapter 5

Classification of covering spaces

5.1 revision

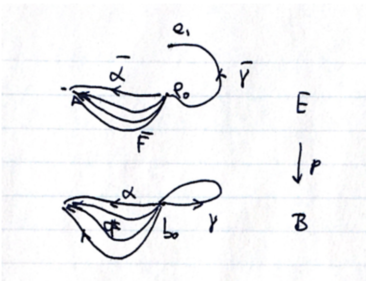
Recall a surjective continuous map $p : E \to B$ is a covering map, if for any $b \in B$, $\exists$ a neighborhood U of b, s.t. $p^{-1}(U) = \coprod_{\alpha} V_\alpha$, $V_\alpha \subset E$ open, and $p\big|_{V_\alpha} : V_\alpha \to U$ is a homomorphism.



Lifting properties (Chapter 3 §4) Let $p(e_0) = b_0$.

(1) A path $\alpha : [0,1] \to B$ with $\alpha_0 = b_0$ has a unique lifting $\overline{\alpha} : [0,1] \to E$ with $\overline{\alpha}(0) = e_0$.

(2) A (path) homotopy $F : I \times I \to B$ can be lifted to a (path) homotopy $\overline{F} : I \times I \to E$.



(3) lifting correspondence:$\pi_1(B, b_0) \to p^{-1}(b_0), [\gamma] \mapsto \overline{\gamma}(1)$

· if E is path connected.

· if E is simply connected.

Let $H_0 = p_*(\pi_1(E, e_0)) < \pi_1(B, b_0)$ a subgroup

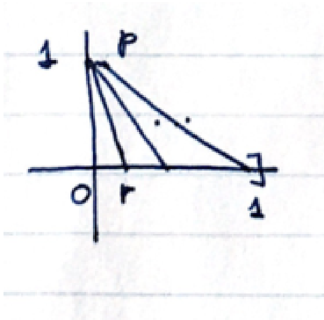
Question: Does H_0 determine (E, p)?

5.2 equivalence of covering spaces

Definition 5.1. *Let $p : E \to B, p' : E' \to B$ be covering maps. They are said to be equivalent if there exists a homeomorphism $h : E \to E'$ such that $p = p' \circ h$. The homeomorphism h is called an equivalence of covering maps/spaces .*

$$\begin{tikzcd} E \arrow[rr, "h", "\cong"'] \arrow[dr, "p"'] & & E' \arrow[dl, "p'"] \\ & B & \end{tikzcd}$$

Definition 5.2. *A space X is said to be locally path connected at x if for every neighborhood U of x, there exists a path-connected neighborhood V of x contained in U. If X is locally path connected at each $x \in X$, it is said to be locally path connected.*

e.g. 

T= the union of all the segments joining $p = (0, 1)$ to (0,r) for all $r \in \mathbb{Q} \cap [0, 1]$.
Then T is path connected, but not locally path connected except at p.

Theorem 5.3. *Let $p : E \to B, p' : E' \to B$ be covering maps, $p(e_0) = b_0 = p'(e'_0)$,$H_0 = p_*(\pi_1(E, e_0))$,$H'_0 = p'_*(\pi_1(E^1, e'_0))$. Then there is an equivalence $h : E \to E'$ s.t. $h(E_0) = e'_0$ iff $H_0 = H'_0$. If h exists, it is unique.(Assume all spaces are path connected and locally path connected.)*

Proof. "⇒": $H_0 = p_*(\pi_1(E, e_0)) = p'_*(h_*(\pi_1(E', e'_0)))$.

h a homeomorphism Ra $h_* : \pi_1(E, e_0) \to \pi_1(E', e'_0)$ an isomorphism.

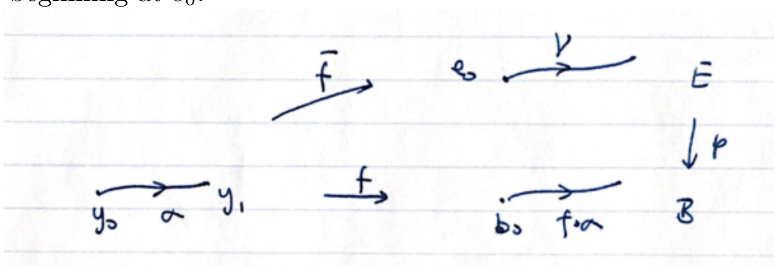
(to be continued) □

Lemma 5.4 (the general lifting lemma)**.** *Let $p : E \to B$ be a covering map, $p(e_0) = b_0$; $f : Y \to B$ be a continuous map, $f(y_0) = b_0$. Assume all spaces are path connected and locally path connected. Then the map f can be lifted to a map $\overline{f} : Y \to E$ s.t. $\overline{f}(y_0) = e_0$ iff $f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(E, e_0)) = H_0$. If such a lifting exists, it is unique.*

$$\begin{tikzcd} & E \ni e_0 \arrow[d, "p"] \\ y_0 \in Y \arrow[ur, "\overline{f}"] \arrow[r, "f"] & B \ni b_0 \end{tikzcd}$$

Proof. (1)"⇒": $f_*(\pi_1(Y, y_0)) = p_*(\overline{f}_*(\pi_1(Y, y_1))) \subset p_*(\pi_1(E, e_0))$.

(2) uniqueness: given $y_1 \in Y$, choose a path α in Y from y_0 to y_1, then the path $f \circ \alpha$ in B has a unique lifting γ in E beginning at e_0.



If there exists a lifting $\overline{f}$ of f, then $\overline{f} \circ \alpha$ is also $\overline{f} \circ \alpha$ is also a lifting of $f \circ \alpha$ in E beginning at e_0.

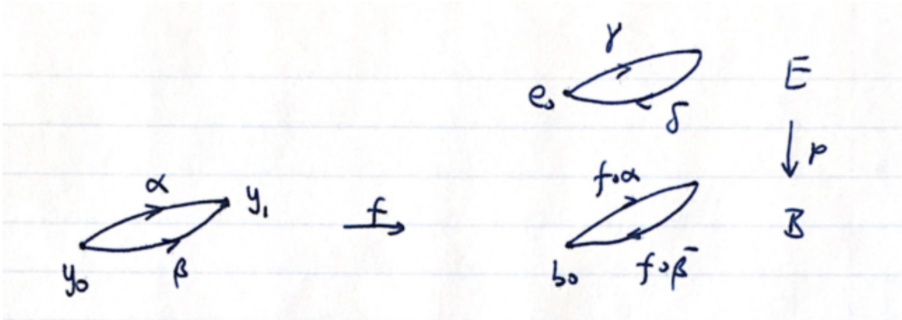
$\therefore \gamma(1) = \overline{f}(\alpha(1)) = \overline{f}(y_1)$.

(3) "$\Leftarrow$" We define the lifting $\overline{f}$ by the same construction, i.e. define $\overline{f}(y_1) = \gamma(1)$. We need to show

(i) $\overline{f}$ is well-defined (independent of the choice of α).

(ii) $\overline{f}$ is continuous.

(i)Let β be another path from y_0 to y_1.



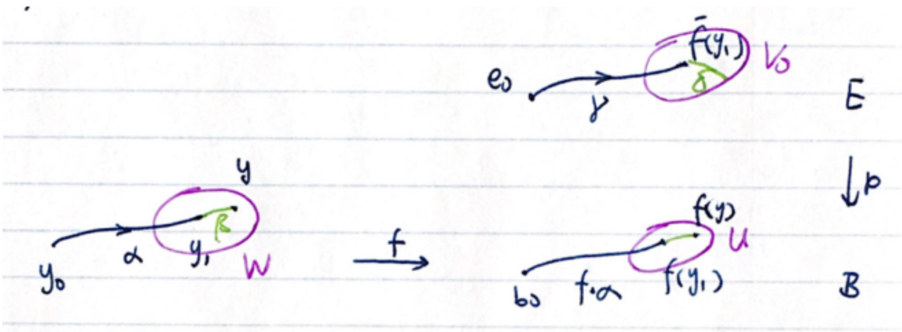
Let S be the lifting of $f \circ \overline{\beta}$ begining at $\gamma(1)$, then $\gamma * \delta$ is a lifting of the loop $f \circ (\alpha * \overline{\beta})$. By the hypothesis $f * (\pi_1(Y, y_0)) \subset p_*(\pi_1(E, e_0))$,$[f \circ (\alpha * \overline{\beta})] \in \mathrm{Im} p_*$.$\therefore \gamma * \delta$ is a loop in E. $\therefore \overline{\delta}$ is the lifting of $f \circ \beta$ beginning at e_0, $\gamma(1) = \overline{\delta}(1)$,$\therefore \overline{f}$ is well-defined.

(ii) For any $y_1 \in Y$, any neighborhood N of $\overline{f}(y_1)$, we find a neighborhood W of y_1 s.t. $\overline{f}(W) \subset N$ as follows:

Choose a path-connected neighborhood U of $f(y_1)$, s.t. $p^{-1}(U) = \coprod\limits_{\alpha} V_\alpha$, $\overline{f}(y_1) \in V_0$ and $V_0 \subset N, p_0 = p\big|_{V_0} : V_0 \xrightarrow{\cong} U$ a homeomorphism. f is continuous and Y is locally path connected $\Rightarrow \exists$ a path-connected neighborhood W of y, s.t. $f(W) \subset U$.

Claim: $\overline{f}(W) \subset V_0$

Proof.



$\forall y \in W$, choose a path β in W from y_1 to y, $f \circ \beta$ is a path in U from $f(y_1)$ to $f(y)$, and $\delta = p_0^{-1} \circ f \circ \beta$ is a path in V_0.

$\therefore \gamma * \delta$ is a lifting of $f \circ (\alpha * \beta)$, $\overline{f}(y) \in V_0$. $\square$

<u>Proof of theorem 5.3</u> (continued)

Proof. $\Leftarrow$: Assume $H_0 = H_0'$.

$$\begin{tikzcd} & E' \arrow[d, "p'"] \arrow[dl, "k"] \\ E \arrow[ur, bend left, "h"] \arrow[r, "p"'] & B \end{tikzcd}$$

$H_0 \subset H_0' \Rightarrow p$ has a lifting h s.t. $h(e_0) = e_0'$.
$H_0' \subset H_0 \Rightarrow p'$ has a lifting k s.t. $k(e_0') = e_0$

$$\begin{tikzcd} & E \arrow[d, "p"] \\ E \arrow[ur, bend left, "k\circ h"] \arrow[ur, bend right, "\mathrm{id}"'] \arrow[ur, phantom, "\shortparallel"] \arrow[r, bend right, "p"'] & B \end{tikzcd}$$

$k \circ h : E \to E$ is a lifting of p, by the uniqueness of liftings, $k \circ h = \mathrm{id}_E$. Similarly $h \circ k = \mathrm{id}_{E'}$.

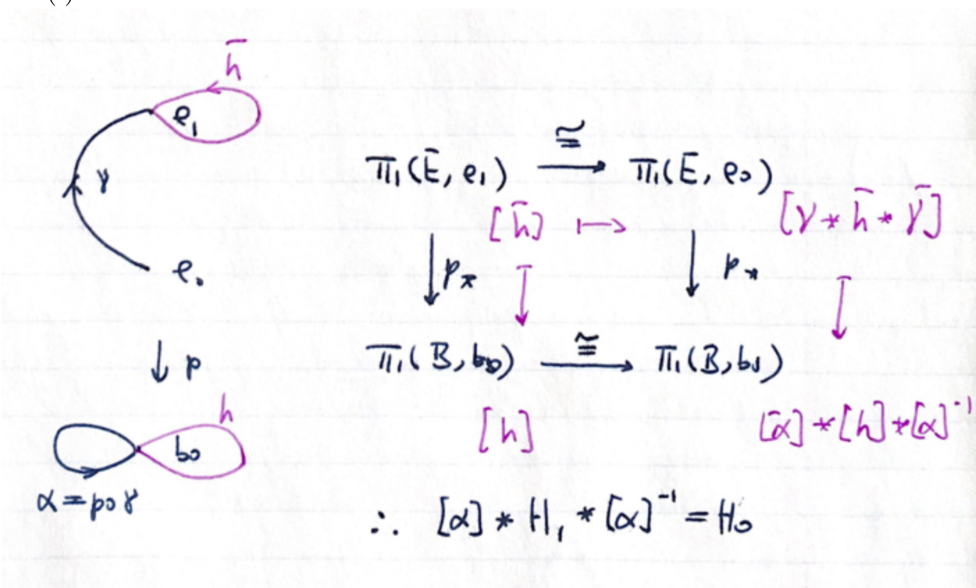
$\square$

Lemma 5.5. *Let $p : E \to B$ be a covering map, $e_0, e_1 \in p^{-1}(b_0)$, $H_i = p_*(\pi_1(E, e_i))$, $i = 0, 1$.*

(i) If γ is a path from e_0 to e_1, $\alpha = p \circ \gamma$ is the loop in B, then $[\alpha] * H_1 * [\alpha]^{-1} = H_0$, i.e. H_0 and H_1 are conjugate by $[\alpha]$.

(ii) Conversely, given e_0 and a subgroup H of $\pi_1(B, b_0)$ conjugate to H_0, there exists a point $e_1 \in p^{-1}(b_0)$ s.t. $H_1 = H$.

Proof. (i)



(ii) Assume $H_0 = [\alpha] * H * [\alpha]^{-1}$ for some $[\alpha] \in \pi_1(B, b_0)$. Let γ be the lifting of α in E beginning at e_0, $\gamma(1) = e_1$. Then by (i), $H_0 = [\overline{\alpha}] * H_1 * [\alpha]^{-1}$,i.e. $H = H_1$.

$\square$

Theorem 5.6. *Let $p : E \to B, p' : E' \to B$ be covering maps, $p(e_0) = b_0 = p'(e_0')$. Assume all spaces are path connected and locally path connected. Then the covering maps p and p' are equivalent if and only if $H_0 = p_*(\pi_1(E_0, e_0))$ and $H_0' = p_*'(\pi_1(E', e_0'))$ are conjugate in $\pi_1(B, b_0)$.*

Therefore we have an injective map

$$\left\{ \begin{array}{c} \textit{equivalence classes of} \\ \textit{covering spaces of } B \end{array} \right\} \xrightarrow[\textit{injective}]{} \left\{ \begin{array}{c} \textit{conjugacy classes of} \\ \textit{subgroups of } \pi_1(B, b_0) \end{array} \right\}$$

Proof. "$\Rightarrow$": If $h : E \to E'$ is an equivalence. Let $e_1' = h(e_0), H_1' = p_*(\pi_1(E', e_1'))$. Then $H_0 = H_1'$ and H_0 and H_1' are conjugate.

"$\Leftarrow$": If H_0 and H_0' are conjugate, then $\exists e_1' \in (p')^{-1}(b_0)$ s.t. $H_0 = H_1' = p_*'(\pi_1(E', e_1'))$. Then $\exists$ an equivalence $h : E \to E$ s.t. $h(e_0) = e_1'$. $\square$

Example: $B = S^1, \pi_1(S^1) \cong \mathbb{Z}$.(Conjugacy classes) of subgroups of $\mathbb{Z}$ are n$\mathbb{Z}, n \in \mathbb{N} \cup \{0\}$. They are realized by the covering maps: $S^1 \to S^1, z \mapsto z^n (n \geqslant 1)$ and $\mathbb{R} \to S^1, x \mapsto e^{i2\pi x}$. $\therefore$ These are all the equivalences classes of covering maps of S^1. Note that the trivial covering map $S^1 \times \{1, \ldots, n\} \to S^1$ is not considered.

Remark: If B is locally path connected, then $B = \coprod\limits_\alpha B_\alpha$ is open,in which B_α's are path components of B. $\therefore$ we may only condider the case B locally path connected and path connected. In this case, if $p : E \to B$ is a covering map, $E = \coprod\limits_\alpha E_\alpha$,$E_\alpha$'s are path components of E, then $p : E_\alpha \to B$ is a covering map, E_α is path connected and locally path connected.

Definition 5.7. *Let $p : E \to B$ be a covering map. If E is simply-connected, then E is called a <u>universal covering space</u> of B (万有覆叠/泛覆叠). Any two universal coverings of B are equivalent since $\pi_1(E) = 0 = \pi_1(E')$.*

$\therefore$ we speak of "the" universal covering of B.

$$\begin{tikzcd} E' \arrow[rr, "h", "\cong"'] \arrow[dr, "p'"'] & & E \arrow[dl, "p"] \\ & B & \end{tikzcd}$$

Theorem 5.8. *Let $p : E \to B$ be the universal covering space. Given any covering map $r : Y \to B$, there is a covering map $q : E \to Y$ s.t. $r \circ q = p$.*

$$\begin{tikzcd} E \arrow[dr, "q"] \arrow[dd, "\beta"'] & \\ & Y \arrow[dl, "r"] \\ B & \end{tikzcd}$$

Slogan: The universal covering space is a covering space of all covering spaces.

Proof.

$$\begin{tikzcd} & Y \ni y_0 \arrow[d, "r"] \\ e_0 \in E \arrow[ur, "q"] \arrow[r, "p"] & B \ni b_0 \end{tikzcd}$$

$p_*(\pi_1(E, e_0)) = 0 \subset r_*(\pi_1(Y, y_0))$
$\therefore \exists$ a lifting q of p.
by the following lemma, $q : E \to Y$ is a covering map. $\square$

Lemma 5.9. *In the following commutative diagram, p, q, r are continuous maps, X, Y, Z are path connected and locally path connected.*

$$\begin{tikzcd} X \arrow[dr, "q"] \arrow[dd, "p"'] & \\ & Y \arrow[dl, "r"] \\ Z & \end{tikzcd}$$

Then

(i) If p and r are covering maps, so is q.

(ii) If p and q are covering maps, so is r.

(compare with exercise 15.5)(We cannot find the original exercise, so we failed to provide more information about it. But as compensation, we put exercise1 of §80 in [Munkres] below.)

1. Let $q : X \to Y$ and $r : Y \to Z$ be maps; let $p = r \circ q$.

(a) Let q and r be covering maps. Show that if Z has a universal covering space, the p is a covering map.

$^*(b)$ *Give an example where q and r are covering maps but p is not.*

Key word for the proof: lifting of paths.
Examples: $\mathbb{R} \to S^1, \mathbb{R}^2 \to T^2, S^2 \to P^2$.
More examples from group actions.

5.3 covering transformations

Definition 5.10. *An equivalence of a covering map $p : E \to B$ with itself is called a covering transformation (覆叠变换).*

The set of all covering transformations forms a group, called the group of covering transformations (覆叠变换群), denoted by $C(E, p, B)$.

$$\begin{tikzcd} E \arrow[r, "h", "\cong"'] \arrow[dr, "p"'] & E \arrow[d, "p"] \\ & B \end{tikzcd}$$

Recall: the conjugacy class of $H_0 = p_*(\pi_1(E, e_0))$ determines the equivalence class of (E, p).

Question: Does H_0 determine the group $C(E, p, B)$?

Definition 5.11. *Let H be a subgroup of G, the normalizer (正规化子) of H in G is $N(H) = \{g \in G | gHg^{-1} = H\}$.$N(H)$ is a subgroup of G, H is a normal subgroup of $N(H)$, and $N(H)$ is the largest such group.*

Given a covering transformation $h : E \to E$, by the uniqueness of lifting, h is determined by $h(e_0) \in p^{-1}(b_0)$.

$\therefore$ define an injective map

$$\begin{aligned} C(E, p, B) &\xrightarrow{\Phi} \quad p^{-1}(b_0) = F \\ h &\mapsto \quad h(e_0) \end{aligned}$$

$$\begin{tikzcd} & E \ni e_0 \arrow[d, "p"] \\ e_0 \in E \arrow[ur, "h", "\cong"'] \arrow[r, "p"'] & B \ni b_0 \end{tikzcd}$$

Recall we also have the bijective lifting correspondence:

$$\begin{aligned} \pi_1(B, b_0)/_{H_0} &\xrightarrow[1:1]{\Phi} \quad p^{-1}(b_0) \\ [\alpha] &\mapsto \quad \overline{\alpha}(1) \end{aligned}$$

$$\begin{tikzcd} C(E, p, B) \arrow[r, "\Psi", "\text{inj}"'] & p^{-1}(b_0) & \overline{\alpha}(1) \\ h \mapsto h(e_0) & & \\ & \pi_1(B, b_0)/_{h_0} \ni [\alpha] \arrow[uu, "\Phi", "\cong"'] & {[\alpha]} \arrow[uu, mapsto] \end{tikzcd}$$

Lemma 5.12. $\mathrm{Im}\Psi = \Phi\left(N(H_0)/_{H_1}\right)$.

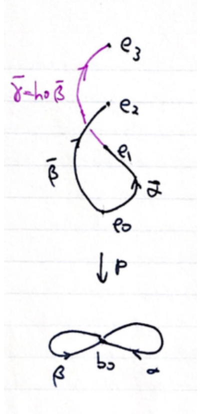
Proof. Given $e_1 = \overline{\alpha}(1) \in p^{-1}(b_0)$,$\exists$ a covering transformation $h : E \to E$ s.t. $h(e_0) = e_1$ iff $p_*(\pi_1(E, e_0)) = H_0 = H_1 = p_*(\pi_1(E, e_1))$ But we know $H_0 = [\alpha]{*}H_1{*}[\alpha]^{-1}$. $\therefore H_0 = H_1$ iff $[\alpha] \in N(H_0)$.

$$\begin{tikzcd}
 & E \ni e_0 \arrow[d, "p"] \\
e_0 \in E \arrow[ur, "h"] \arrow[r, "p"'] \arrow[ur, phantom, "\cong"', shift right=2] & B \ni b_0
\end{tikzcd}$$

$\square$

Theorem 5.13. *The bijection $\Phi^{-1} \cdot \Psi : C(E, p, B) \to N(H_0)/_{H_0}$ is an isomorphism of groups.*

Proof. We need to show $\Phi^{-1} \cdot \Psi$ is a homomorphism. Let $h, k : E \to E$ be covering transformations, $h(e_0) = e_1$,$k(e_0) = e_2$, then $\Psi(h) = e_1, \Psi(k) = e_2$. Choose path $\overline{\alpha}$ from e_0 to e_1, $\overline{\beta}$ from e_0 to e_2, $\alpha = p \circ \overline{\alpha}, \beta = p \circ \overline{\beta}$, then $\Phi^{-1} \circ \Psi(h) = [\alpha] \cdot H_0, \Phi^{-1} \circ \Psi(k) = [\beta]H_0$. Let $h(k(e_0)) = e_3$, then we need to show $\Phi([\alpha * \beta] \cdot H_0) = e_3$.



Let $\overline{\gamma} = h \circ \overline{\beta}$ be a path from e_1 to e_3.
Then $p \circ \overline{\gamma} = \beta$.
$\therefore \overline{\alpha} * \overline{\gamma}$ is a lifting of
$\alpha * \beta$ beginning at e_0.
$\therefore \Phi([\alpha * \beta] \cdot H_0) = \overline{\gamma}(1) = e_3$

$\square$

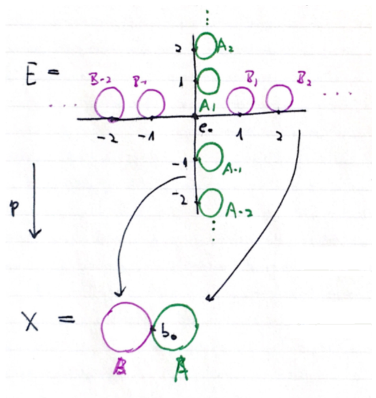
Corollary 5.14. *The group H_0 is a normal subgroup of $\pi_1(B, b_0)$ if and only if for every pair of points e_1, e_2 of $p^{-1}(b_0)$, there is a covering transformation $h : E \to E$ with $h(e_1) = e_2$. In this case there is an isomorphism $\Phi^{-1} \circ \Phi : C(E, p, B) \xrightarrow{\subseteq} \pi_1(B, b_0)/_{H_0}$. Such a covering map is called a* <u>*regular covering*</u> *(正规覆叠).*

Corollary 5.15. *Let $p : E \to B$ be the universal covering, then $C(E, p, B) \cong \pi_1(B, b_0)$.*

Examples:

(1) If $\pi_1(B)$ is abelian, then any covering map is regular e.g. $B = S^1, T^n$, or a topological group G.

(2)



the x-axis wraps around A.
the y-axis wraps around B.

The loop A lifts to

$$\begin{cases} \text{a loop in } E \text{ at base points } (0, n) & (n \neq 0) \\ \text{a line segment in } E \text{ at base points}(n, 0) & \forall n \in \mathbb{Z}. \end{cases}$$

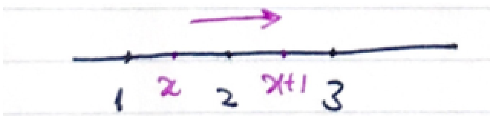
Similar for the loop B.

$\therefore$ a covering transformation h must fix e_0, $\therefore h = \mathrm{id}$.

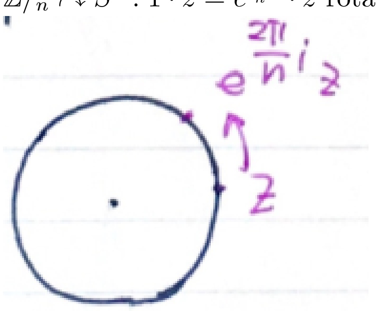
Definition 5.16. *Let X be an topological space,* $\mathrm{Homeo}(X)$ *be the group of self-homeomorphisms of X, G be a group. An action (作用) of G on X is a homomorphism $G \to \mathrm{Homeo}(X)$, i.e. we assign to each $g \in G$ a homeomorphism $g : X \xrightarrow{\cong} X$ s.t. $g(h(x)) = (gh)(x) \forall x \in X, g, h \in G$. (Notation:$G \curvearrowright X$) The action is effective if $G \to \mathrm{Homeo}(X)$ is injective. It suffices to consider effective actions.*

Examples:

(1) $\mathbb{Z} \curvearrowright \mathbb{R} : 1 \cdot x = x + 1$ translation



(2) $\mathbb{Z}/_n \curvearrowright S^1 : 1 \cdot z = e^{\frac{2\pi}{n} i} \cdot z$ rotation



(3) $\mathbb{Z} \curvearrowright S^1 : 1 \cdot z = e^{2\pi\alpha i} \cdot z$ rotation

- if $\alpha = m/n$, then this action factors through $\mathbb{Z}/n$:

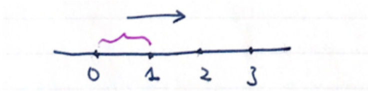
$$\begin{tikzcd} \mathbb{Z} \arrow[r] \arrow[dr] & \mathbb{Z}/_n \arrow[d] \\ & \mathrm{Homeo}(S^1) \end{tikzcd}$$

- if α is irrational, then this action is effective.

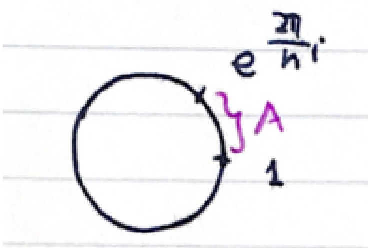
Definition 5.17. *The* <u>*orbit*</u> *(轨道) of $x \in X$ is $\mathrm{Orb}(x) = \{g(x) | g \in G\}$. The* <u>*orbit space*</u> *$X/_G$ is the quotient space obtained from X by the equivalence relation $x \sim g(x), \forall g \in G, x \in X$.*

Examples:

(1) $\mathbb{Z} \curvearrowright \mathbb{R}$ translation: $\mathbb{R}/_{\mathbb{Z}} = [0,1]/_{0\sim 1} \cong S^1$.

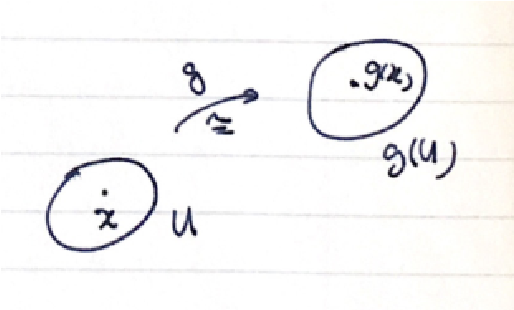


(2) $\mathbb{Z}/_n \curvearrowright S^1$ rotation: $S^1/_{\mathbb{Z}/_n} = A/_{1\sim e^{\frac{2\pi}{n} i}} \cong S^1$



(3) each orbit if the irrational rotation action $\mathbb{Z} \curvearrowright S^1$ is dense in S^1.

Definition 5.18. *An action $G \curvearrowright X$ is called* <u>*free*</u> *(自由) if $g(x) \neq x, \forall x \in X, g \neq e$. In other words, the map $G \to \mathrm{Orb}(x), g \mapsto g(x)$ is bijective for all $x \in X$. An action $G \curvearrowright X$ is called* <u>*properly discontinuous*</u> *(真不连续) if for every $x \in X$, $\exists$ a neighborhood U of x, s.t. $g(U) \cap U = \varnothing, \forall g \neq e$. (This implies $g_1(U) \cap g_2(U) = \varnothing, \forall g_1, g_2 \in G$).*



Remark:

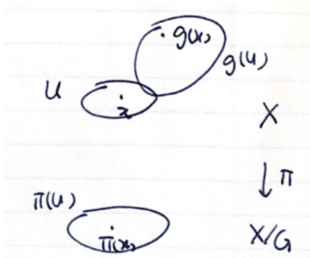
(i) properly discontinuous $\Rightarrow$ free

(ii) the irrational rotation action $\mathbb{Z} \curvearrowright S^1$ is free but not properly discontinuous.

(iii) if G is finite, then free $\Rightarrow$ properly discontinuous.(Assume X is Hausdorff)

Theorem 5.19. *Let X be path connected and locally path connected, $G \curvearrowright X$ be an action. The quotient map $X \xrightarrow{\pi} X_G$ is a covering map if and only of the action is properly discontinuous. In this case, the covering map π is regular and G is its group of covering transformations.*

Proof. (i) π is an open map:$X \xrightarrow{\pi} X/_G$. Let $U \subset X/_G$ is open. $\pi_1(U)$ open $\Leftrightarrow \pi^{-1}(\pi(U))$ open, and $\pi^{-1}(\pi(U)) = \bigcup_{g \in G} g(U)$ open.



(ii) Suppose the action is properly discontinuous, we show that π is a covering map: $\forall x \in X$, let U be a neighborhood of x s.t. $g(U) \cap U = \varnothing, \forall g \neq e$. Then $\pi(U)$ is an open neighborhood of $\pi(x)$, and $\pi^{-1}(\pi(U)) = \coprod_{g \in G} g(U)$ is a disjoint umion of open sets $G(U)$.$\pi : U \to \pi(U)$ is continuous, open and bijective, hence a homeomorphism.(If $\exists y, gy \in U$, then $gy \in U \cap g(U)$,lightening sign)

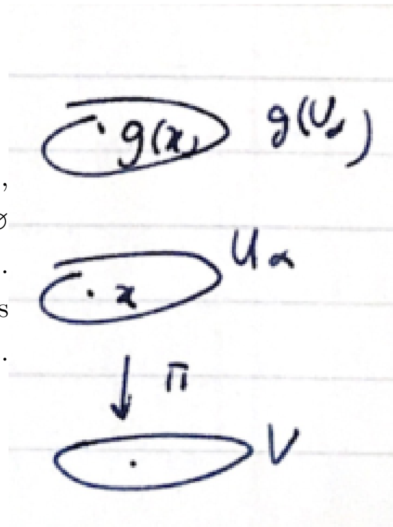
$$\begin{tikzcd} U \arrow[r, "g", "\cong"'] \arrow[dr, "\pi"'] & g(U) \arrow[d, "\pi"] \\ & \pi(U) \end{tikzcd}$$

(iii) Suppose $\pi : X \to X/_G$ is a covering map, we show that the action is properly discontinuous:

$$\begin{tikzcd} X \arrow[d, "\pi"'] & \ni x \arrow[d, mapsto] \\ X/_G & \ni \pi(X) \subset V \end{tikzcd}$$

Given $x \in X$, let V be a neighborhood of $\pi(x)$ s.t. $\pi^{-1}(V) = \coprod_\alpha U_\alpha$,$\pi : U_\alpha \xrightarrow{\cong} V$.

Assume $x \in U_\alpha$,
then $g(U_\alpha) \cap U_\alpha = \varnothing$
for $\forall g \neq e$.
$\therefore$ the action is
properly discontinuous.



(iv) Assume $\pi : X \to X/_{G}$ is a covering map, we show $G = C(X, \pi, X/_{G})$.

(1)
$$\begin{tikzcd} X \arrow[r, "g", "\cong"'] \arrow[dr, "\pi"'] & X \arrow[d, "\pi"] \\ & X/_{G} \end{tikzcd}$$

$\therefore g \in G$ is a covering transformation.

(2) given a covering transformation $\varphi : X \xrightarrow{\cong} X$. If $\varphi(x_1) = x_2$ then x_1 and x_2 are in the same orbit. $\therefore \exists g \in G$ s.t. $g(x_1) = x_2$. By the uniqueness of lifting, $\varphi = g . \therefore G = C(X, \pi, X/_{G})$.

(3) Since for any two points x_1, x_2 in the same orbit, $\exists g \in G = C(X, \pi, X/_{G})$ s.t. $g(x_1) = x_2$, the covering is regular. $(p.186 Corollary)$.

"regular covering=the group of covering transformations acts transitively"

$\square$

Theorem 5.20. *If $p : X \to B$ is a regular covering map and $G = C(X, p, B)$ is its group of covering transformations. Then there is a homeomorphism $k : X/_{G} \to B$ s.t.*

$$\begin{tikzcd} X \arrow[d, "\pi"'] \arrow[dr, "p"] & \\ X/_{G} \arrow[r, "k", "\cong"'] & B \end{tikzcd}$$

regular covering
Slogan: $\parallel$
properly discontinuous action

Proof. (1) π is a quotient map, p is constant on each orbit $\Rightarrow \exists$ a continuous map $k : X/_{G} \to B$.

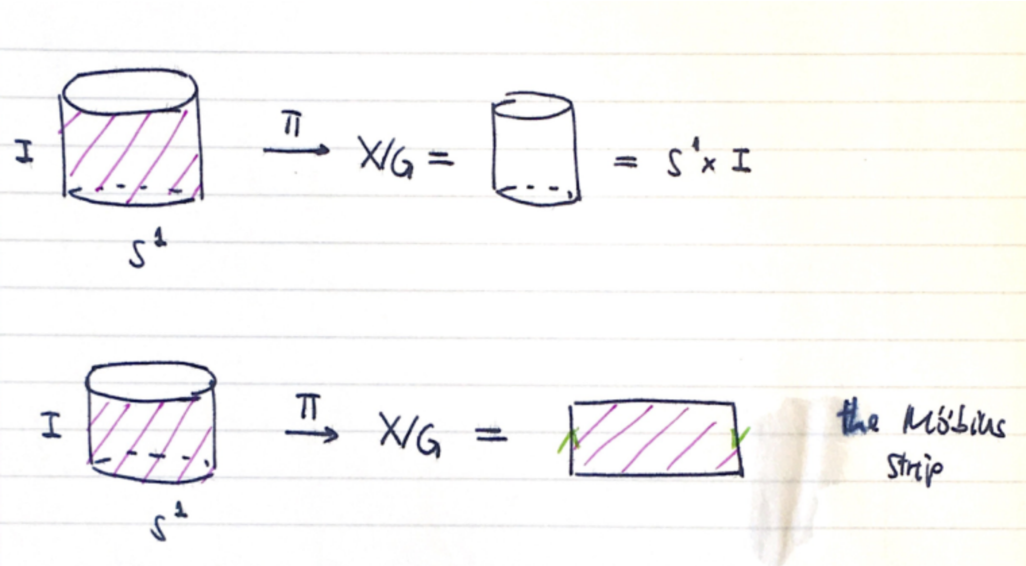
(2) p is a quotient map (continuous, surjective and open), π is constant on each $p^{-1}(b) \Rightarrow \exists$ a continuous map $B \to X/_{G}$, which is the inverse of k.

$\square$

Remark: $G \curvearrowright X$ properly discontinuous, there is a short exact sequence $1 \to \pi_1(X) \to \pi_1\left(X/_{G}\right) \to G \to 1$.

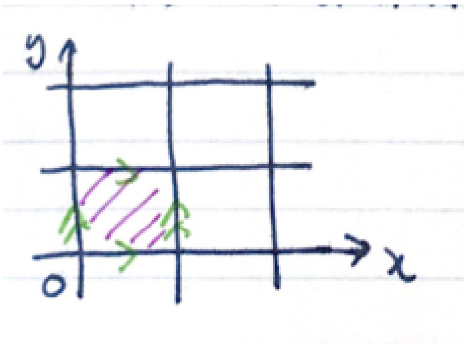
Example: $X = S^1 \times [0,1]$ two $G = \mathbb{Z}/_2$ actions on X:

$X \xrightarrow[\cong]{h} X, (x,t) \to (-x,t)$
$X \xrightarrow[\cong]{k} X, (x,t) \to (-x,1-t)$



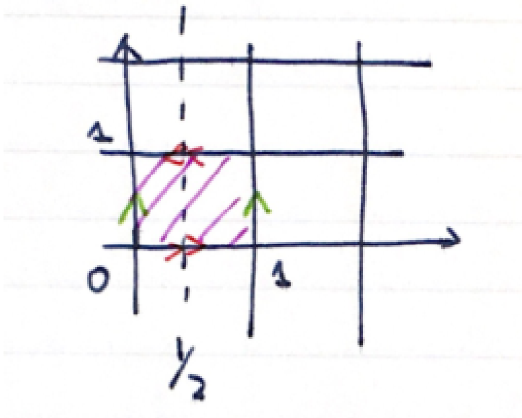
Example: X path connnected, locally path connected, simply connected, $G \curvearrowright X$ properly discontinuous, Then $X \xrightarrow{\pi} X/G$ is the universal covering.

(1)



$\mathbb{Z} \curvearrowright \mathbb{R}$ translations $\rightsquigarrow \mathbb{R} \xrightarrow{\pi} S^1$
$\mathbb{Z}^2 \curvearrowright \mathbb{R}^2$ translations $\rightsquigarrow \mathbb{R}^2 \xrightarrow{\pi} T^2$
In general,
$\mathbb{Z}^n \curvearrowright \mathbb{R}^n$ translations$\rightsquigarrow \mathbb{R}^n \xrightarrow{\pi} T^n$

(2)



$\mathbb{Z}^2$ generates a, b, $\mathbb{Z} \curvearrowright \mathbb{R}^2$ as follows:
$a : (x,y) \mapsto (x+1,y)$,
S$b : (x,y) \mapsto (1-x, y+1)$
$b \circ a : (x,y) \mapsto (x+1,y) \mapsto (-x,y+1)$
$a \circ b : (x,y) \mapsto (1-x,y+1) \mapsto (-x,y+1)$
$\therefore a \circ b = b \circ a$
$\mathbb{R}^2/_{\mathbb{Z}^2}$ = the Klein bottle

(3) $\mathbb{Z}/2 \curvearrowright S^2, x \mapsto -x, S^2/_{\mathbb{Z}/2} = p^2$
in general $\mathbb{Z}/2 \curvearrowright S^n, x \mapsto -x, S^n/_{\mathbb{Z}/2} = \mathbb{R}P^n$, the n-dim projective space.

5.4 existence of covering spaces

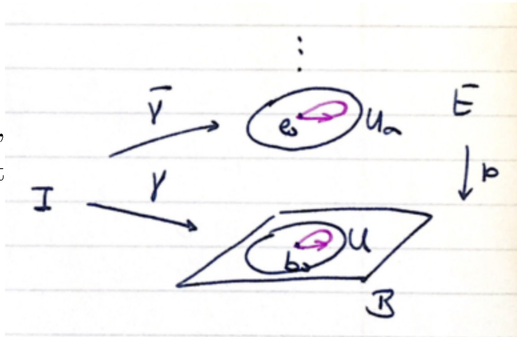
Recall: $\left\{\begin{array}{c}\text{equivalence classes of}\\ \text{covering spaces of } B\end{array}\right\} \xrightarrow{\text{inj}} \left\{\begin{array}{c}\text{conjugacy classes of}\\ \text{subgroups of } \pi_1(B,b_0)\end{array}\right\}$,
$(E \xrightarrow{p} B) \mapsto p_*(\pi_1(E,e_0)) = H$.
Question: when is the map surjective?

Lemma 5.21. *Let $p : E \to B$ be the universal covering, $p(e_0) = b_0$. Then there is a neighborhood U of b_0 s.t. the inclusion $i : U \to B$ induces the trivial homomorphism*

$$i_* : \pi_1(U, b_0) \to \pi_1(B, b_0)$$

Proof. Let U be a neighborhood of b_0 that is evenly covered by p, i.e. $\pi^{-1}(U) = \coprod_\alpha U_\alpha, p : U_\alpha \xrightarrow{\cong} U$. Let $e_0 \in U_\alpha$,

Let $\gamma : I \to U$ be a loop in U based at b_0, then $\overline{\gamma} = p^{-1} \circ \gamma$ is a loop in U_α based at e_0.



$$\begin{array}{ccc}\pi_1(E,e_0) & \xrightarrow{p_*} & \pi_1(B,b_0)\\ \Psi & & \Psi\\ [\overline{\gamma}] & \mapsto & [\gamma]\end{array}$$

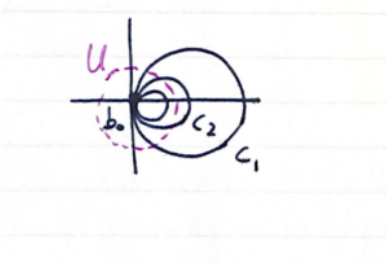
$\pi_1(E, e_0) = 0 \Rightarrow [\gamma] = 0$.

$\therefore$

$$\begin{array}{ccc}\pi_1(U,b_0) & \longrightarrow & \pi_1(B,b_0)\\ \Psi & & \Psi\\ [\gamma] & \mapsto & [\gamma]\end{array}$$

is trivial. $\square$

Example:
$X = \bigcup_{n=1}^{\infty} C_n$ the hawaiian earring
$C_n =$ the circle of radius $\frac{1}{n}$, centered at $\left(\frac{1}{n}, 0\right)$.



Claim: For any neighborhood U of b_0, $i_* : \pi_1(U, b_0) \to \pi_1(X, b_0)$ is nontrivial.

Proof. Choose n large s.t. $C_n \subset U$, consider the retraction $r : X \to C_n, r\big|_{C_n} = \mathrm{id}, r(C_m) = b_0 (m \neq n)$.

$$\begin{tikzcd} C_n \arrow[r, "j"] \arrow[rrr, bend right, "\mathrm{id}"] & U \arrow[r, "i"] & X \arrow[r, "r"] & C_n \end{tikzcd} \quad \text{induce}$$

$$\begin{tikzcd} \pi_1(C_n, b_0) \arrow[r, "j_*"] \arrow[rrr, bend right, "\mathrm{id}"] & \pi_1(U, b_0) \arrow[r, "i_*"] & \pi_1(X, b_0) \arrow[r, "r_*"] & \pi_1(C_n, b_0) \\ \mathbb{Z} \arrow[u, phantom, "\cong"] & & & \mathbb{Z} \arrow[u, phantom, "\cong"] \end{tikzcd}$$

$\therefore i_* : \pi_1(U, b_0) \to \pi_1(X, b_0)$ is nontrivial.

$\therefore X$ has no universal covering. ☐

Definition 5.22. *A space B is said to be semilocally simply connected if for each $b \in B$, there is a neighborhood U od b, s.t. the homomorphism $i_* : \pi_1(U, b) \to \pi_1(B, b)$ induced by the inclusion is trivial.*

e.g. B is a manifold or B is simply connected.

Theorem 5.23. *Let B be path connected, locally path and semilocally simply connected. Let $b_0 \in B$, given a subgroup H of $\pi_1(B, b_0)$, there exists a covering map $\varphi : E \to B$ (E is path connected) and a point $e_0 \in p^{-1}(b_0)$ s.t. $p_*\pi_1(E, e_0) = H$.*

Under the assumption on B

$$\left\{\begin{matrix}\text{equivalence classes of}\\ \text{covering spaces of } B\end{matrix}\right\} \xrightarrow{1:1} \left\{\begin{matrix}\text{conjugacy classes of}\\ \text{subgroups of } \pi_1(B, b_0)\end{matrix}\right\}$$

Corollary 5.24. *The space B has a universal covering space if and only if B is path connected, locally path connected and semilocally path connected.*

idea of the proof :

(1) recall if $E \xrightarrow{p} B$ is a covering map, E path connected. $\forall e \in E$, let $\overline{\alpha}$ be a path from e_0 to e, then $\alpha = p \circ \overline{\alpha}$ is a path in B from b_0 to $b = \alpha(1)$, and $\overline{\alpha}$ is a lifting of α. Take another path β from b_0 to b, let $\overline{\beta}$ be the lifting of β beginning at e_0, then

$$\overline{\alpha}(1) = \overline{\beta}(1) \Leftrightarrow [\alpha * (-\beta)] \in H = p_*\pi_1(E, e_0)$$

(2) construction of E: $\mathcal{P}$ = the set of all paths in B beginning at b_0. Define an equivalence relation on $\mathcal{P}$:$\alpha \sim \beta \Leftrightarrow \alpha(1) = \beta(1)$ and $[\alpha * (-\beta)] \in H$.

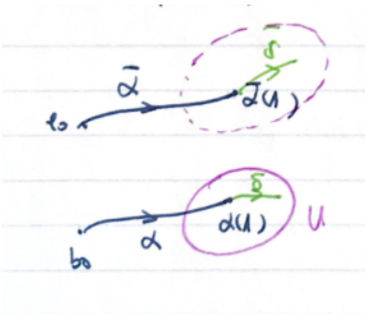
Let $E = \mathcal{P}/_\sim$ be the set of equivalence classes denote by the equivalence class of a path a path α by $\alpha^\#$, define $p : E \to B, \alpha^\# \mapsto \alpha(1)$.

We need to show E is a path connected topological space, p is a covering map.

(3) topology on E: there are two ways to topologize E.

(i) give $\mathcal{P}$ the compact-open topology and $E = \mathcal{P}/\sim$ the quotient topology.

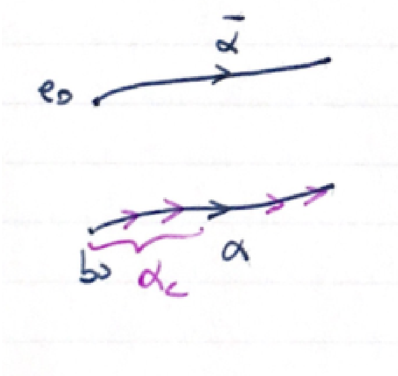
(ii) given $\alpha \in \mathcal{P}$, let U be any path connected neighborhood of $\alpha(1)$.
Define $B(U, \alpha) = \left\{ (\alpha * \delta)^\# \middle| \delta \text{ is a path in } U \text{ beginning at } \alpha(1) \right\}$.
Then $B(U, \alpha)$ is a basis of a topology on E.



(4) $p : E \to B$ is continuous, open and a covering map.

(5) lifting of path:

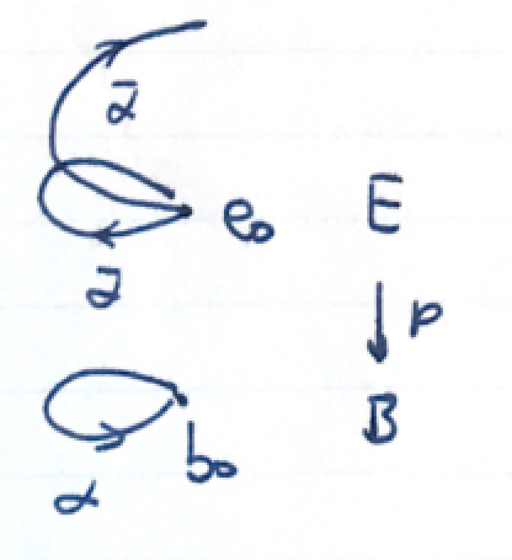
$e_0 = (\text{the constant path at } b_0)^{\#} \in E$,
then $p(B_0) = b_0$.
For any $e = \alpha^{\#} \in E, C \in [0, 1]$,
let $\alpha_C : I \to B, \alpha_C(t) = \alpha(tc)$.
Then $\overline{\alpha} : I \to E, c \mapsto (\alpha_C)^{\#}$ is a path in E from E_0 to e,
a lifting of α, $\overline{\alpha}(1) = \alpha^{\#} = e$.



Need to verify the continuity of $\overline{\alpha}$.

$\therefore E$ is path connected.

(6) compute $p_*\pi_1(E, e_0)$.



$$[\alpha] \in p_*\pi_1(E, e_0)$$
$$\Updownarrow$$
$$\overline{\alpha} \text{ is a loop in } E$$
$$\Updownarrow$$
$$\alpha^{\#} = \overline{\alpha}(1) = e_0 \iff [\alpha] \in H$$

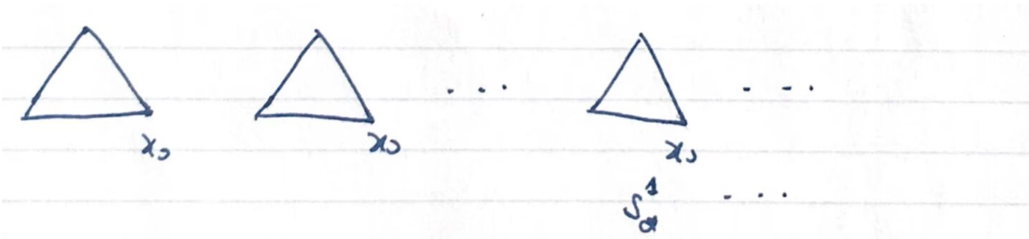
$\therefore p * \pi_1(E, e_0) = H$.

5.5 application to free groups

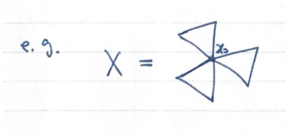
Theorem 5.25. *If H is a subgroup of a free group, then H is free.*

sketch of proof .

Let F be a free group with generators $\{\alpha | \alpha \in J\}$, $F = \underset{J}{*}\mathbb{Z}$. Let X be a wedge of circles $S^1_\alpha (\alpha \in J)$ $X = \bigvee_{\alpha \in J} S^1_\alpha$, $x_0 \in X$ be the common point, then $\pi_1(X, x_0) \cong \underset{J}{*} = F$.



X is a 1-dimensional simplcial complex.

e.g.  X is path connected, locally path connected and semilocally simply connected.

$\therefore$ There is a covering space $p : E \longrightarrow X$ s.t. $p_*(\pi_1(E, e_0)) = H$. $\therefore \pi_1(E, e_0) \cong H$.

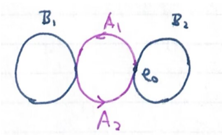
Fact .

(1) A covering space of a 1-dimensional simplcial complex is a 1-dimensional simplcial complex.

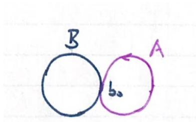
(2) the fundament group of a 1-dimensional simplcial complex K is a free group. (one can find a maximal tree T in K, $K \simeq K/T \simeq \vee S^1$.)

Examples

(1)



$\xrightarrow{p}$



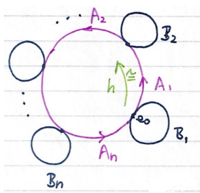
$\pi_1 \cong F_3 \xrightarrow{p_*} \pi_1 \cong F_2$

$[F_2 : p_*(F_3)] = \left| F_2/_{p_*(F_3)} \right| = p^{-1}(b_0) = 2$

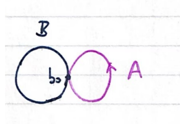
$\therefore F_3 \triangleleft F_2$ a noraml subgroup of index 2.

Question , if $F_2 = \langle a, b \rangle$, find generators of $p_* F_3$

(2) in general

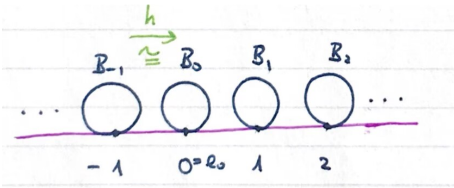


$\xrightarrow{p}$

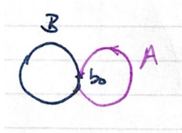


$\pi_1 \cong F_{n+1} \xrightarrow{p_*} \pi_1 \cong F_2$

$F_{n+1} \triangleleft F_2$ index = n. $F_2/F_{n+1} = \mathbb{Z}/n$



$\xrightarrow{p}$

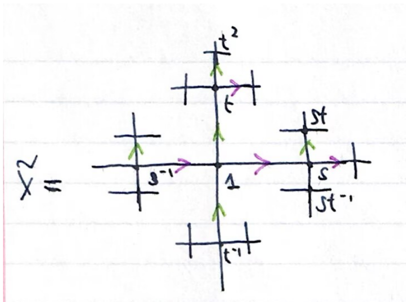


$\pi_1 \cong F_\infty = \underset{\infty}{*}\mathbb{Z} \quad \xrightarrow{p_*} \quad \pi_1 \cong F_2$

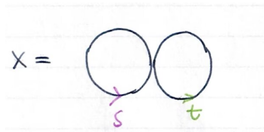
$\therefore F_\infty \triangleleft F_2 \quad F_2/_{F_\infty} \cong \mathbb{Z}$

(3) the universal covering space of $X = S^1 \vee S^1$.

$\widetilde{X}$ is a 1-dimensional(infinite) simplcial complex, vertices $V = \pi_1(X) \cong F_2 = \langle s, t\rangle$. edges $E = \{(g, gt), (g, gs) | g \in \pi_1(X)\}$



$\longrightarrow$



an infinite four-valent tree (antenna space) contractible. (Ref . [Browm, Cohomology of Groups]. [Office Hours with a Geometric Group Theorist] [包志强])

Additional topics

(1) the universal covering of closed surfaces.

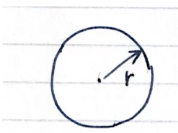
three geometries

(i) the educidean geometry $\mathbb{E}^2$: $\mathrm{Isom}(\mathbb{E}^2) = \mathbb{R}^2 \rtimes O(2)$

$$\begin{tikzcd} 1 \arrow[r] & \mathbb{R}^2 \arrow[r] & \mathrm{Isom}(\mathbb{E}^2) \arrow[r] & O(2) \arrow[r] \arrow[l, bend left] & 1 \end{tikzcd}$$

translations rotations, reflections

curvature = 0 (flat)



$L = 2\pi r$

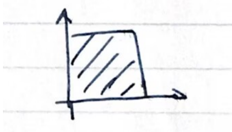
$\alpha + \beta + \gamma = \pi$.

- $\mathbb{Z}^2 \curvearrowright \mathbb{R}^2$ by translations, $\mathbb{R}^2/_{\mathbb{Z}^2} = T^2$

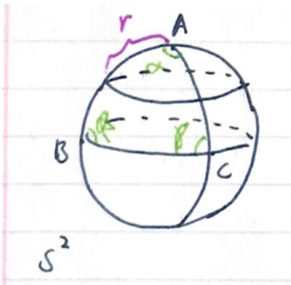
- $\mathbb{Z}^2 \curvearrowright \mathbb{R}^2$ by translations and reflections $\mathbb{R}^2/_{\mathbb{Z}^2}$ = the klein bottle K

$\therefore T^2, K$ inhent a geometry of curvature 0,

$\chi(T^2) = 0 = \chi(K)$.



(ii) the spherical geometry S^2 :

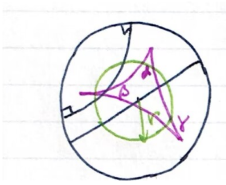


$$Isom(S^2) = O(3)$$
$$curvature = 1(positivelycurved)$$
$$L = 2\pi r, \pi < \alpha + \beta + \gamma < 3\pi$$

$\mathbb{Z}^2 \curvearrowright S^2$ by isometry, $S^2/_{\mathbb{Z}/2} = \rho^2$

$\chi(S^2) = 2, \quad \chi(\rho^2) = 1$

(iii) the hyperbolrc geometry $\mathbb{H}^2$ the poincaré disk model :
$\mathbb{H}^2 = \{z \in \mathbb{C} ||z| < 1\}$



$$ds^2 = 4 \cdot \frac{dx^2 + dy^2}{(1 - r^2)^2}$$
$$d(z_1, z_2) = cosh^{-1}(1 + \frac{2|z_1 - z_2|^2}{(1 - |z_1|^2)(1 - |z_2|^2)})$$

curvature $= -1$ (negativaly curved)

$L = 2\pi$sinhr, $\quad \alpha + \beta + \gamma < \pi$.

the poincaré upper half-palne nodel:

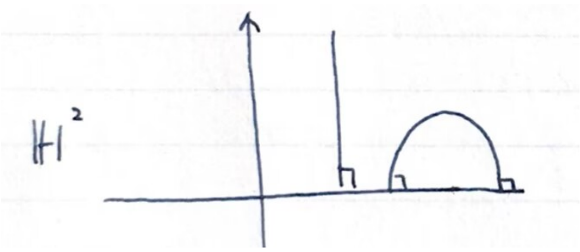
$Isom^+(\mathbb{H}^2) = PSL_2(\mathbb{R}) = SL_2(\mathbb{R})/_{(\pm I)}$

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}), \quad A \cdot z = \frac{az + b}{cz + d}$ (Möbius transformation) For any closed surfaces

$S = nT^2$ $(n \geqslant 2)$, mP^2 $(m \geqslant 3)$. $\exists\, G < Isom(\mathbb{H}^2)$,

$G \curvearrowright \mathbb{H}^2$ properly discontinuously. s.t. $\mathbb{H}^2/_G \cong S$.

$\therefore$ the universal covering space of S is $\mathbb{H}^2 \cong \mathbb{R}^2$. and S inhents a geometry of curvature-1.

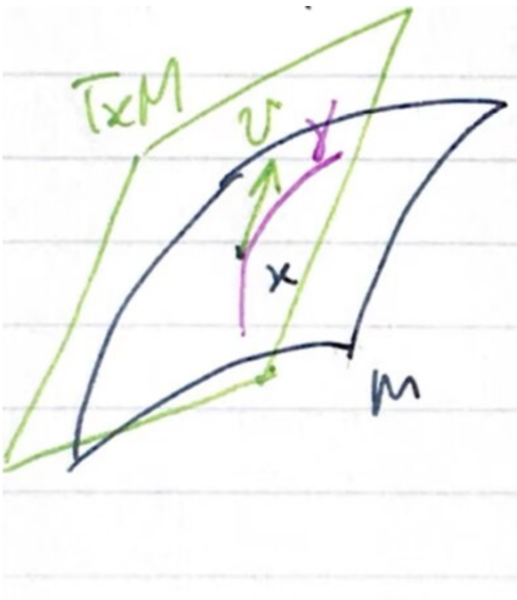


$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$

$$\{z \in \mathbb{C} | ReZ > 0\}$$

$\chi(nT^2) = 2 - 2n$, $\chi(mP^2 = 2 - m)$.

In general, if (M^n, g) is a closed Riemannian manifold of curvature $\leqslant 0$, then the exponential map.



exp:

$$\begin{array}{ccc} T_xM & \longrightarrow & M \\ v & \longmapsto & \gamma_v(1) \end{array}$$

is a covering map (cartan's Thm)

$\therefore \widetilde{M} \cong T_xM \cong \mathbb{R}^n$.

(2)

$$\begin{aligned}
&\mathrm{GL}_n^+(\mathbb{R}) = \{A \in \mathrm{M}_n(\mathbb{R}) | \det A \neq 0\} \quad \text{a Lie group}\\
&\quad \cup\\
&\mathrm{SO}(n) = \left\{A \in \mathrm{M}_n(\mathbb{R}) \middle| A \cdot A^t = I, \det A = 1\right\}
\end{aligned}$$

a maxiaml compact subgroup of $\mathrm{GL}_n(\mathbb{R})$

Then the space of left cosets

$\mathrm{GL}_n^+(\mathbb{R})/_{\mathrm{SO}(n)}$ = the space of positive definite quadrate forms on $\mathbb{R}^n$

$\cap|$ open, convex

$symm_n(\mathbb{R}) \cong \mathbb{R}^{\frac{n(n+1)}{2}}$

$\therefore \mathrm{GL}_n^+(\mathbb{R})/_{\mathrm{SO}(n)} \cong \mathbb{R}^{\frac{n(n+1)}{2}}$

Let $\Gamma < \mathrm{GL}_n^+(\mathbb{R})$ be a discrete group, the Γ acts on $\mathrm{GL}_n^+/_{\mathrm{SO}(n)}$ by left translation: $\gamma \cdot A\mathrm{SO}(n) = \gamma A \cdot \mathrm{SO}(n)$

Fact . if Γ is torsion free, then the action is properly discontinuous.

$\therefore {}_{r}\backslash \mathrm{GL}_{n^+(\mathbb{R})}/_{\mathrm{SO}(n)}$ has universal covering space $\mathbb{R}^{\frac{n(n+1)}{2}}$.

e.g. $\mathrm{GL}_n^+(\mathbb{Z}) = \mathrm{SL}_n(\mathbb{Z}) \subset \mathrm{GL}_n^+(\mathbb{R})$ has torsion elements.

$$\begin{array}{c}
1 \longrightarrow \Gamma(\mathbb{N}) \longrightarrow \mathrm{SL}_n(\mathbb{Z}) \to \mathrm{SL}_n(\mathbb{Z}/_N)\\
\qquad \| \\
\{A \in \mathrm{SL}_n(\mathbb{Z}) | A \equiv I \mod N\}
\end{array}$$

the princial congruente subgroup (同余子群) of level N.

(i) $[\mathrm{SL}_n(\mathbb{Z}) : \Gamma(N)] < \infty$ since $|\mathrm{SL}_n(\mathbb{Z}/_{\mathbb{N}})| < \infty$

(ii) For $N >\geqslant 3$, $\Gamma(N)$ is torsion free.

Ref . [Brown, Cohomology of Goups p.38.].

Chapter 6

the homology theory

The fundamental group functor

$\mathcal{T}\mathrm{op}_* \xrightarrow{\pi_1} \mathcal{G}\mathrm{p}, (X, x_0) \mapsto \pi_1(X, x_0)$

$\Big((X, x_0) \xrightarrow{f} (Y, y_0)\Big) \mapsto f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0).$

homotopy invariance, applications, computation: (Seifert-van Kampen Theorem).

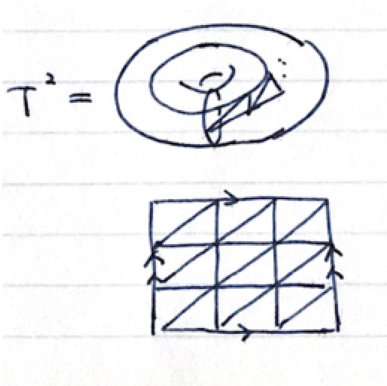
The homology group functors $n \geqslant 0$

$\mathcal{T}\mathrm{op} \xrightarrow{H_n} \mathcal{A}\mathrm{b}$ (the category of abelian groups)

$X \mapsto H_n(X)$

$\Big(X \xrightarrow{f} Y\Big) \mapsto f_* : H_n(X) \to H_n(Y)$

e.g.



$$\begin{tikzcd} & \mathbb{R}^2 \arrow[d, "\pi"] \\ S^2 \arrow[ur, "\overline{f}"] \arrow[r, "f"] & T^2 \end{tikzcd}$$

$\therefore f$ is nullhomotopic.

6.1 simplicial homology

Definition 6.1. *A* <u>*chain complex*</u> (C_*, ∂) *(链复形) of abelian groups consists of a sequence of abelian groups* $\{C_n | n \in \mathbb{Z}\}$ *(called chain groups(链群)) and homomorphisms* $\{\partial_n : C_n \to C_{n-1} | n \in \mathbb{Z}\}$ *(called* <u>*boundary homomorphisms*</u>*(边缘同态)) s.t.*

$\partial_{n-1} \cdot \partial_n = 0, \forall n \in \mathbb{Z}$

$$\to \cdots C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} C_{n-2} \to \cdots$$

$Z_n = \ker \partial_n$ *is called the group of* <u>*closed chains*</u> *(闭链群) or the group of n-cycles.* $B_n = \mathrm{Im}\partial_{n+1}$*(in which* $B_n \subseteq Z_n$*) is called the group of* <u>*boundary chains*</u> *(边缘链群).*

$$\partial_n \circ \partial_{n+1} = 0 \Rightarrow B_n \subseteq Z_{n+1}$$

For two cycles $z_1, z_2 \in Z_n$*, if* $z_1 - z_2 \in B_n$*, i.e.* $\exists C \in C_{n+1}$*, s.t.* $z_1 - z_2 = \partial_{n+1}(C)$*, then they are called homologous (同调的).*

$H_n(C_*, \partial) = Z_n/_{B_n} = \ker \partial_n / \mathrm{Im}\partial_{n+1}$ *is the n-dim* <u>*homology group*</u> *(同调群) of the chain complex* (C_*, ∂)*.*

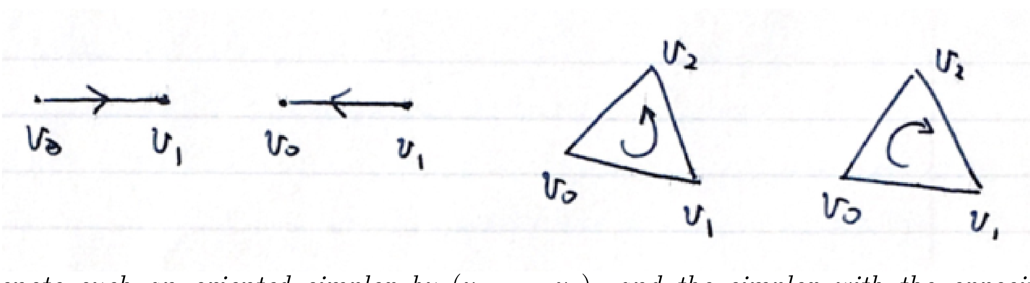
Let $v_0, \cdots, v_k$ *be* $(k+1)$ *points in* $\mathbb{R}^n$ *of general position, (i.e.* $v_1 - v_0, \cdots, v_k - v_0$ *are linearly indenpendent), the k-simplex with vertices* $v_0, \cdots, v_k$ *is*

$$\sigma = \left\{ \sum_{i=0}^{k} \lambda_i v_i \in \mathbb{R}^n \middle| \sum_{i=0}^{k} \lambda_i = 1, \lambda_i \geqslant 0 \right\}.$$

An <u>*orientation*</u> *(定向) of σ is an ordering of the vertices* $v_0, \cdots, v_k$*. Two orderings are equivalent if they differ by an even permutation.*

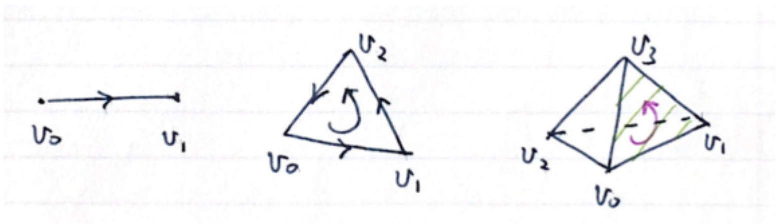
$\therefore$ *there are exactly two orientations on a k-simplex σ when* $k \geqslant 1$*.*

e.g.



We denote such an oriented simplex by $(v_0, \cdots, v_k)$*, and the simplex with the opposite orientation by -*$(v_0, \cdots, v_k)$*. An orientation on a k-simplex σ induces orientations on its $(k-1)$-faces by* $(-1)^i(v_0, \cdots, \hat{v}_i, \cdots, v_k)$

e.g.



The <u>*boundary*</u> *of an oriented simplex* $(v_0, \cdots, v_k)$ *is*

$$\partial(v_0, \cdots, v_k) = \sum_{i=0}^{k} (-1)^i (v_0, \cdots, \hat{v}_i, \cdots, v_k).$$

e.g. $\partial v_0 = 0, \quad \partial(v_0, v_1) = v_1 - v_0$.
$\partial(v_0, v_1, v_2) = (v_1, v_2) - (v_0, v_2) + (v_0, v_1)$.
Let K be a finite simplicial complex, (a collection of simplices s.t.

(i) if $\sigma \in K$, then all the faces of σ are in K.

(ii) any two simplices in K intersect properly: $\sigma \cap \tau$ is a common face of σ and τ.

)

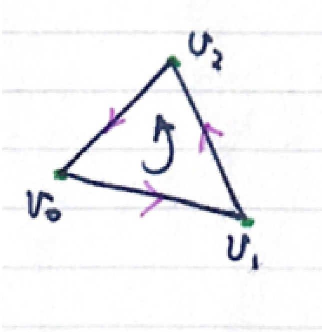
Fix an orientation of each simplex in K, let

$$C_p(K) = \left\{ \sum n_\sigma \cdot \sigma \middle| \sigma \in K \text{ an orientated } p\text{-simplex, } n_\sigma \in \mathbb{Z} \right\}$$

be the free abelian group generated by oriented p-simplices of K. Define the boundary homomorphism
$\partial_p : C_p(K) \to C_{p-1}(K)$,
$\sigma = (v_0, \cdots, v_p) \mapsto \partial_p(\sigma) = \sum\limits_{i=0}^{p} (-1)^i (v_0, \cdots, \hat{v}_i, \cdots, v_p)$

Lemma 6.2. *$\partial_p \cdot \partial_{p+1} : C_{p+1}(K) \to C_p(K) \to C_{p-1}(K)$ is the zero homomorphism.*

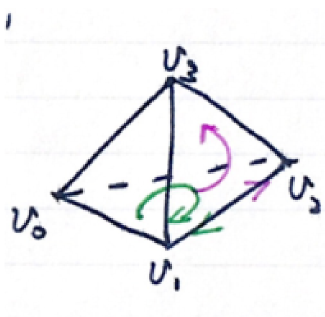
e.g.



$$\begin{aligned} &\partial_1 \cdot \partial_2(v_0, v_1, v_2) \\ =&\partial_1((v_1, v_2) - (v_0, v_2) + (v_0, v_1)) \\ =&v_2 - v_1 - (v_2 - v_0) + v_1 - v_0 = 0 \end{aligned}$$

Proof. Let $\sigma = (v_0, \cdots, v_{p+1})$ be an oriented $(p+1)$-simplex.

$$\begin{aligned} \partial^2(v_0, \cdots, v_{p+1}) =& \partial \left(\sum_{i=0}^{p+1} (-1)^i (v_0, \cdots, \hat{v}_i, \cdots, v_{p+1}) \right) \\ =& \sum_{i=0}^{p+1} (-1)^i \left(\sum_{j=0}^{i-1} (-1)^j (v_0, \cdots, \hat{v}_j, \cdots, \hat{v}_i, \cdots, v_{p+1}) \right. \\ &\left. + \sum_{j=i+1}^{p+1} (-1)^{j-1} (v_0, \cdots, \hat{v}_i, \cdots, \hat{v}_j, \cdots, v_{p+1}) \right) \\ =&0 \end{aligned}$$



$\therefore (C_*(K), \partial) = \{C_p(K), \partial_p | p \geqslant 0\}$ is a chain complex of abelian groups.

$C_p(K) \supseteq \ker \partial_p = Z_p(K) \supseteq \mathrm{Im}\partial_{p+1} = B_p(K)$ do not depend on the orientations on the simplicies. The p-th homology group (同调群) of the simplicies complex K is $H_p(K) = Z_p(K) / B_p(K)$.

If K is a finite simplicial complex, then $C_p(K)$ is a finitely generated free abelian group, so is $Z_p(K) = \ker \partial_p$. $\therefore$ $H_p(K)$ is a finitely generated abelian group. $\square$

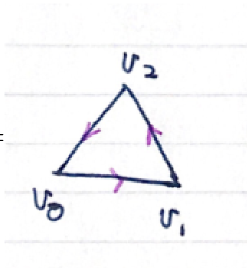
Theorem 6.3 (the structure of $f.g.$ abelian groups)**.** *Let A be a finitely generated abelian group, then $A \cong F \oplus T$, where $F \cong \mathbb{Z}^n$ is a free abelian group of rank r (the rank of A), T is the torsion subgroup of A, a finite abelian group.*

$$T = T_{p_i} \oplus \cdots \oplus T_{p_r}, p_1, \cdots, p_r \in \mathbb{P}$$

$$T_{p_i} = \mathbb{Z}/_{p_i^{\alpha_i}} \oplus \cdots \oplus \mathbb{Z}/_{p_i^{\alpha_{l_i}}}$$

Definition 6.4. *The rank of $H_p(K)$ is called the p-th Betti number of K, denoted by $\beta_p(K)$.*

Examples:

(1) $K =$ 

the boundary of the triangle $|K| \cong S^1$.

$$\begin{array}{ccccccc} \cdots \to C_2(K) & \xrightarrow{\partial_2} & C_1(K) & \xrightarrow{\partial_1} & C_0(K) & \xrightarrow{\partial_0} & 0 \\ = 0 & & \cong \mathbb{Z}^3 & & \cong \mathbb{Z}^3 & & \end{array}$$

basis of $C_1(K)$: $\begin{cases} e_1 = (v_0, v_1) \\ e_2 = (v_1, v_2) \\ e_3 = (v_2, v_0) \end{cases}$

basis of $C_0(K)$: v_0, v_1, v_2.

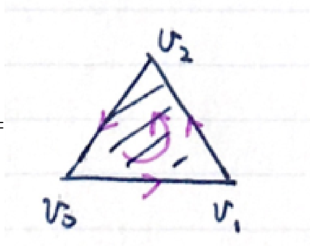
In which:

$$\begin{cases} e_1 = (v_0, v_1) & \mapsto (v_1 - v_0) \\ e_2 = (v_1, v_2) & \mapsto (v_2 - v_1) \\ e_3 = (v_2, v_0) & \mapsto (v_0 - v_2) \end{cases}$$

$H_0(K) = \dfrac{\ker \partial_0}{\mathrm{Im}\partial_1} = \dfrac{C_0(K)}{\mathrm{Im}\partial_1} \cong \mathbb{Z}$ generated by $[v_0](= [v_1] = [v_2])$

$H_1(K) = \dfrac{\ker \partial_1}{\mathrm{Im}\partial_2} = \ker \partial_1 = \cong \mathbb{Z}$ generated by $[e_1 + e_2 + e_3]$.

$H_p(K) = 0, p \geqslant 2$.

(2) $K =$ 

=the 2-simplex $\triangle^2$.

$$\begin{array}{ccccccccc} \cdots \to C_3(K) & \xrightarrow{\partial_3} & C_2(K) & \xrightarrow{\partial_2} & C_1(K) & \xrightarrow{\partial_1} & C_0(K) & \xrightarrow{\partial_0} & 0 \\ & & \cong Z & & \cong \mathbb{Z}^3 & & & & \end{array}$$

basis of $C_0(K)$: e_1, e_2, e_3.

basis of $C_2(K)$:$\sigma = (v_0, v_1, v_2) \mapsto e_1 + e_2 + e_3$

$$H_1(K) = \frac{\ker \partial_1}{\mathrm{Im}\partial_2} = 0$$

$$H_2(K) = \frac{\ker \partial_2}{\mathrm{Im}\partial_3} = \ker \partial_2 = 0.$$

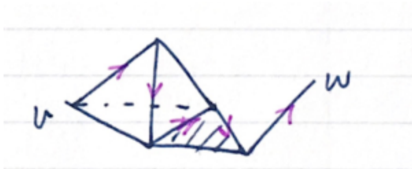
(3) Two vertices v, w of a simplicial complex K are in the same (path) component of $|K|$.(Remark:$|K|$ is locally path connected.)

$\Updownarrow$

v is joined to w by an edge path, i.e. $\exists$ vertices $v_1, \cdots, v_k$ s.t.

$(v, v_1), (v_1, v_2), \cdots, (v_k, w)$ are edges in K.

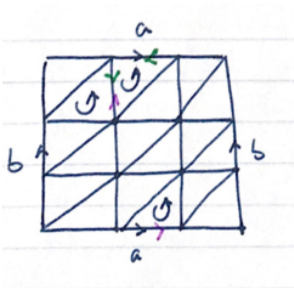
$\Updownarrow$



v and w are homologous as 0-cycles $\partial\left((v, v_1) + (v_1, v_2) + \cdots + (v_k, w)\right) = v - w$.

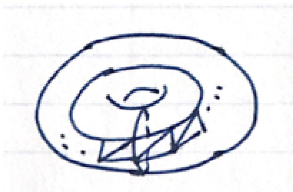
Theorem 6.5. *$H_0(K) \cong \mathbb{Z}^r, r =$ the number of path components of $|K|$.*

(4)



Let K be a triangulation of the torus T^2, there is a compatible way to orient the 2-simplices $\{\sigma_1, \cdots, \sigma_n\}$, s.t. the two induced orientations on an edge are opposite.

$$\cdots \to \underset{\overset{\|}{0}}{C_3(K)} \xrightarrow{\partial_3} C_2(K) \xrightarrow{\partial_2} C_1(K) \to$$



$\sigma = \sigma_1 + \cdots + \sigma_n \mapsto 0$

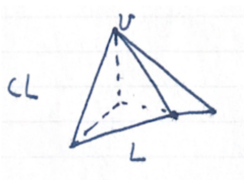
and $\ker \partial_2 \cong \mathbb{Z}$ is generated by σ.

$\therefore H_2(K) = \ker \partial_2 \cong \mathbb{Z}, \sigma = [K]$ called the fundamental class .

The same holds for any closed orientable surface $S = nT^2 (n \geqslant 0)$, but not for non-orientable surfaces.

(5) Let L be a finite simplicial complex, assume $L \subset \mathbb{R}^n$. let $v = (0, \cdots, 0, 1) \in \mathbb{R}^{n+1}$. The cone on L is the simplicial complex

$$K = CL = \{v, (v, v_0, \cdots, v_k), (v_0, \cdots, v_k) | (v_0, \cdots, v_k) \in L\}$$



$$|CL| = C|L| = |L| \times [0,1] / |L| \times \{1\}$$

Define a homomorphism $d : C_p(K) \to C_{p+1}(K)$,

$$\begin{cases} \sigma = (v_0, \cdots, v_p) \mapsto (v, v_0, \cdots, v_p) & \sigma \in L \\ \sigma \mapsto 0 & \sigma \notin L. \end{cases}$$

Then $\partial \mathrm{d}(\sigma) = \sigma - \mathrm{d}(\partial(\sigma))(p > 0)$

Check: if $\sigma \notin L$, then $\sigma = (v, v_0, \cdots, v_{p-1})$.

LHS $= 0$, $\partial\sigma = (v_0, \cdots, v_{p-1}) + \sum\limits_{i=0}^{p-1} (-1)^{i+1}(v, v_0, \cdots, \hat{v}_i, \cdots, v_{p-1})$.$\therefore$ $\mathrm{d}(\partial\sigma) = (v, v_0, \cdots, v_{p-1}) = \sigma$.

$\therefore$ RHS $= \sigma - \mathrm{d}\partial(\sigma) = 0$

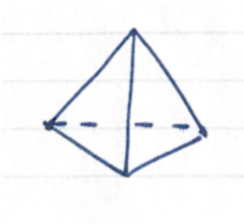
$\therefore$ For any p-cycle z, we have $\partial \mathrm{d}(z) = z - \mathrm{d}\partial(z) = z$.

$\therefore$ z is null homologous. $\therefore$ $H_p(K) = 0 . \forall p > 0$.

And $h_0(K) \cong \mathbb{Z}$ since $|K| = C|L|$ is path connected.

(6) Let $\triangle^{n+1}$ be the $(n+1)$-simplex,$n > 0$. $\triangle^{n+1}$ is the cone on $\triangle^n$. $\therefore$ $H_p(\triangle^{n+1}) = 0, \forall p > 0$.

Let Σ^n be the simplicial complex consisting of all faces of $\triangle^{n+1}$ of dimension $\leqslant n$,(the "boundary" of $\triangle^{n+1}$), then $|\Sigma^n| \cong S^n$.



And according to the diagram below,

$H_p(\Sigma^n) \cong H_p(\triangle^{n+1}), 0 \leqslant p \leqslant n-1$.

$H_n(\Sigma^n) = Z_n(\Sigma^n) = \ker \partial_n = \ker \partial'_n \underset{(a)}{\longrightarrow} \mathrm{Im}\partial'_{n+1} \underset{[(b)]}{\cong} \mathbb{Z}$

$(a) : H_n(\triangle^{n+1}) = \dfrac{\ker \partial'_n}{\mathrm{Im}\partial'_{n+1}} = 0$

$(b) : 0 = H_{n+1}(\triangle^{n+1}) = \ker \partial'_{n+1}$

$$
\begin{tikzcd}
 & 0 \arrow[r] & C_n(\Sigma^n) \arrow[r, "\partial_n"] \arrow[d, "="] & C_m(\Sigma^n) \arrow[r] \arrow[d, "="] & \cdots \arrow[r] & C_0(\Sigma^n) \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & C_{n+1}(\triangle^{n+1})_{\cong\mathbb{Z}} \arrow[r, "\partial'_{n+1}"] & C_n(\triangle^{n+1}) \arrow[r, "\partial'_n"] & C_{n-1}(\triangle^{n+1}) \arrow[r] & \cdots \arrow[r] & C_0(\triangle^{n+1}) \arrow[r] & 0
\end{tikzcd}
$$

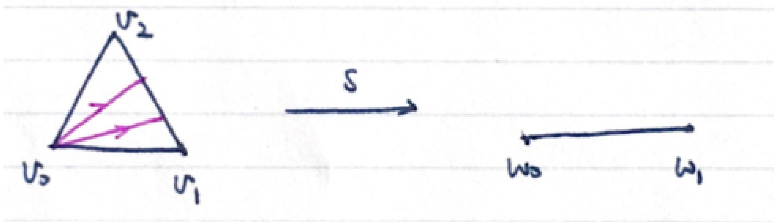
6.2 topological invariance

Definition 6.6. *Let K,L be simplicial complexes,$|K|, |L|$ be their geometric realizations. A map $s : |K| \mapsto |L|$ is called simplicial if it takes simplices of K linearly onto simplices of L.*

Explanation:

$$\begin{aligned} \text{vertices} \to & \quad \text{vertices} \\ \text{edges} \to & \quad \text{vertices, or edges} \\ \text{triangles} \to & \quad \text{vertices,edges or triangles} \\ \vdots \quad & \end{aligned}$$

If $\sigma = (v_0, \cdots, v_k)$ is a k-simplex of K, $x = \sum_{i=0}^{k} \lambda_i v_i \in \sigma$. then $s(x) = \sum_{i=0}^{k} \lambda_i s(v_i)$. We say s is non-degenerate on σ if $\dim s(\sigma) = \dim \sigma$.

e.g.


$s(v_0) = w_0, s(v_1) = s(v_2) = w_1$.

Remark: A simplicial map is continuous, since

(i) continuous on each simplex;

(ii) the glueing lemma.

Let $s : |K| \to |L|$ be a simplicial map, if induces a homomorphism between the chain groups

$$S_p : C_p(K) \to C_p(L), \sigma = (v_0, \cdots, v_p) \mapsto \begin{cases} (s(v_0), \cdots, s(v_p)) & \text{if } s \text{ is non-deg on } \sigma \\ 0 & \text{otherwise} \end{cases}$$

Lemma 6.7. *The following diagram commutes.*

$$\begin{array}{ccc} C_p(K) & \xrightarrow{\partial_p} & C_{p-1}(K) \\ \downarrow S_p & & \downarrow S_{p-1} \\ C_p(L) & \xrightarrow{\partial_p} & C_{p-1}(L) \end{array}$$

Proof. let $\sigma = (v_0, \cdots, v_p)$, we need to show $\partial S_p(\sigma) = S_{p-1}(\partial \sigma)$

(i) if s is non-degenerate on σ, then

$$\begin{aligned} \text{LHS} =& \partial(s(v_0), \cdots, s(v_p)) \\ =& \sum_{i=0}^{p} (-1)^i (s(v_0), \cdots, \widehat{s(v_i)}, \cdots, s(v_p)) \\ \text{RHS} =& s_{p-1} \left(\sum_{i=0}^{p} (-1)^i (v_0, \cdots, \widehat{v_i}, \cdots, v_p) \right) \\ =& \sum_{i=0}^{p} (-1)^i (s(v_0), \cdots, \widehat{s(v_i)}, \cdots, s(v_p)) \end{aligned}$$

(ii) if s is degenerate on σ, suppose $s(v_j) = s(v_k), j < k$. Then LHS $= 0$,

$$
\begin{aligned}
\text{RHS} =& s\left(\sum_{i=0}^{p}(-1)^i\,(v_0,\cdots,\widehat{v_i},\cdots,v_p)\right)\\
=&(-1)^j s(v_0,\cdots,\widehat{v_j},\cdots,v_p) + (-1)^k s(v_0,\cdots,\widehat{v_k},\cdots,v_p)\\
=&\begin{cases}
0 & \text{if } s \text{ is generated on } (v_0,\cdots,\widehat{v_j},\cdots,v_p)\\
 & \text{or} \cdots\cdots\cdots\cdots\cdots (v_0,\cdots,\widehat{v_i},\cdots,v_p)\\
(-1)^j s(v_0,\cdots,\widehat{v_j},\cdots,v_p) & \\
+(-1)^k s(v_0,\cdots,\widehat{v_k},\cdots,v_p) & \text{otherwise}
\end{cases}
\end{aligned}
$$

But $s(v_0,\cdots,\widehat{v_j},\cdots,v_p) = (-1)^{j-k-1} s(v_0,\cdots,\widehat{v_k},\cdots,v_p)$. $\square$

$\{S_p : C_p(K) \to C_p(L) | p \geqslant 0\}$ is called a chain map (链映射) between the chain complexes (C_*K, ∂) and $(C_*(L), \partial)$.

$$
\begin{tikzcd}
\cdots \arrow[r] & C_{p+1}(K) \arrow[r, "\partial"] \arrow[d, "S_{p+1}"] \arrow[dr, phantom, "\circlearrowright"] & C_p(K) \arrow[r, "\partial"] \arrow[d, "S_p"] \arrow[dr, phantom, "\circlearrowright"] & C_{p-1}(K) \arrow[r] \arrow[d, "S_{p-1}"] & \cdots \\
\cdots \arrow[r] & C_{p+1}(L) \arrow[r, "\partial"] & C_p(L) \arrow[r, "\partial"] & C_{p-1}(L) \arrow[r] & \cdots
\end{tikzcd}
$$

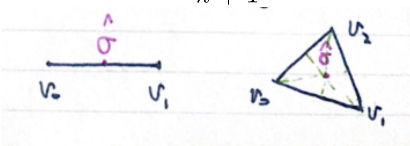
(i) $\forall$ cycle $z \in Z_p(K)$,$\partial s(z) = s(\partial(z)) = 0$, $\therefore s(z) \in Z_p(L)$.

(ii) $\forall C \in C_{p+1}(K), \partial C \in B_p(K)$,$s(\partial C) = \partial s(C) \in B_p(L)$,$\therefore$

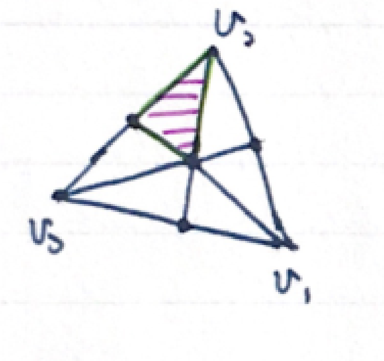
$$
\begin{tikzcd}
C_p(K) \supset \arrow[d, "S_p"] & Z_p(K) \supset \arrow[d, "S_p"] & B_p(K) \arrow[d, "S_p"] \\
C_p(L) \supset & Z_p(L) \supset & B_p(L)
\end{tikzcd}
$$

$\therefore$ the chain map induces a homomorphism between the homology groups $S_{*_p} : H_p(K) \to H_p(L), \forall p \geqslant 0$.

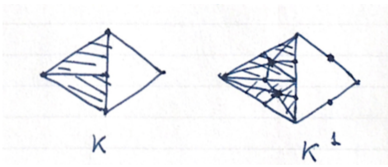
Let $\sigma = (v_0, \cdots, v_k)$ be a k-simplex, its barycenter (重心) is $\widehat{\sigma} = \dfrac{1}{k+1}(v_0 + \cdots + v_k)$.



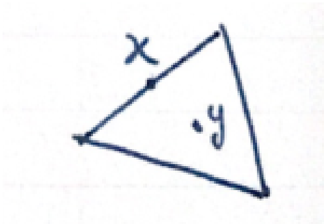
The barycentric subdivision (重心重分) of σ is a simplicial complex consisting of the simplices $(\widehat{\sigma}_0, \cdots, \widehat{\sigma}_l)$, where $\sigma_0, \cdots, \sigma_l$ are faces of σ, $\dim \sigma_0 < \dim \sigma_1 < \cdots < \dim \sigma_l$.



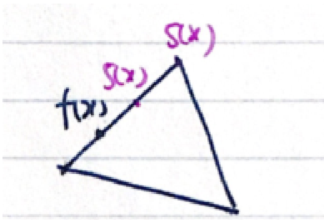
Let K be a simplicial complex, the barycentric subdivision of K is the union of the barycentric subdivisions of all its simplices, denoted by K^1. Inductively we define the m-th barycentric subdivision of K by $K^m = (K^{m-1})^1$. Clearly $|K|^m = |K|$.



Definition 6.8. *Let K be a simplicial complex, for any $x \in |K|$, there is a unique simplex σ of K s.t. x is the interior of σ. σ is called the carrier of x.*



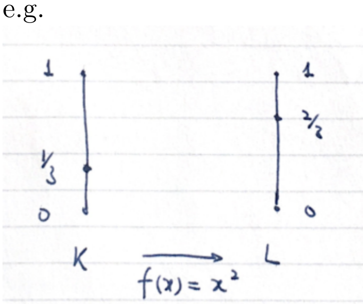
Definition 6.9. *A simplicial map $s : |K| \to |L|$ is a simplicial approximation (单纯逼近) of a continuous map $f : |K| \to |L|$ if $s(x)$ lies in the carrier of $f(x)$ for each $x \in |K|$.*



Simplicial approximation theorem

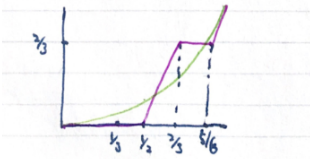
Let $f : |K| \to |L|$ be a continuous map, then $\exists m \in \mathbb{N}$ s.t. $f : |K^m| \to |L|$ has a simplicial approximation $s : |K^m| \to |L|$.

e.g.



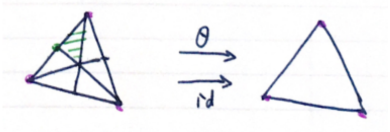
There is no simplicial approximation of f.

$\exists$ a simplicial approximation $s : |K^2| \to |L|$



Let K^1 be the barycentric subdivision of K, we want to show $H_p(K^1) \cong H_p(K)(\forall p \geqslant 0)$. Define a chain map $\{\eta_p : C_p(K) \to C_p(K^1) | p \geqslant 0\}$, $\eta_p : C_p(K) \to C_p(K^1), \sigma \mapsto \sum\limits_i \sigma_(i)$, the subdivision map.

There is a chain map $\{\theta_p : C_p(K^1) \to C_p(K) | p \geqslant 0\}$ induced by the simplicial approximation $\theta : |K^1| \to |K|$ of $\mathrm{id} : |K^1| \to |K|$.



Theorem 6.10. *$\eta_{p_*} H_p(K^1) \to H_p(K)$ is an isomorphism, $(\eta_{p_*})^{-1} = \theta_{p_*}$.*

$\therefore$ barycentric subdivision does not change the homology groups of a simplicial complex.

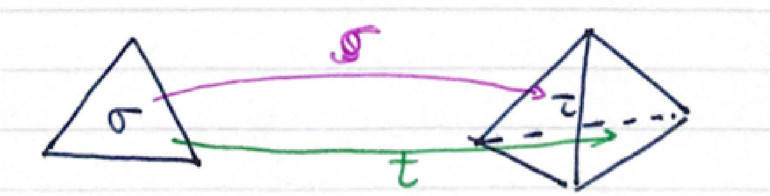
Now give a continuous map $f : |K| \to |L|$, let $s : |K^m| \to |L|$ be a simplicial approximation of f, we define $f_{*p} : H_p(K) \xrightarrow[\cong]{\eta^m} H_p(K^m) \xrightarrow{S_*} H_p(L)$ to be the homomorphism between homology groups induced by f. We need to show:

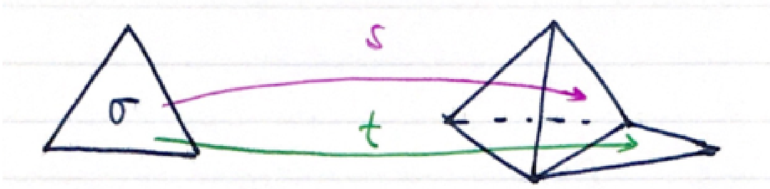
(i) f_* does not depend on the simplicial approximation

(ii) If f and g are homotopic, $f \simeq g : |K| \to |L|$, then $f_* = g_* : H_p(K) \to H_p(L), \forall p \geqslant 0$.

Facts:

(1) If $s, t : |K| \to |L|$ are "close" simplicial maps,(i.e. for each simplex $\sigma \in K, \exists$ a simplex $\tau \in L$,s.t. $s\sigma$ and $t\sigma$ are faces of τ).

e.g.  close

 not close

Then $s_* = t_* : H_p(K) \to H_p(L)$ for all p.

(2) If $f, g : |K| \to |L|$ are homotopic maps, we may find a barycentric subdivision K^m and a sequence of simplicial maps $s_1, \cdots, s_n : |K^m| \to |L|$, s.t. s_1 is a simplicial approximation of f, s_i and s_{i+1} are close for $1 \leqslant n_1$, and s_n is a simplicial approximation of g.

Theorem 6.11. *(1) Any continuous map $f : |K| \to |L|$ induces a homomorphism $f_* : H_p(K) \to H_p(L) (\forall p \geqslant 0)$.*

(2)

(i) $(\mathrm{id})_* : H_p(K) \to H_p(K)$ is the identity homomorphism.

(ii) Given $f : |K| \to |L|, g : |L| \to |M|$, then $(g \circ f)_* = g_* \circ f_* : H_p(K) \to H_p(|M|), \forall p \geqslant 0$.

(3) If $f, g : |K| \to |L|$ are homotopic maps, then $f_* = g_* : H_p(K) \to H_p(L)$.

Corollary 6.12. *If K and $|L|$ are homotopy equivalent, then $H_p(K) \cong H_p(L)$ for all p.*

Proof. Let $f : |K| \to |L|$ be a homotopy equivalence, $g : |L| \to |K|$ be a homotopy inverse of F, then

$$\begin{tikzcd} H_p(K) \arrow[r, "f_*"] \arrow[rr, bend right, "\mathrm{id}"'] & H_p(L) \arrow[r, "g_*"] & H_p(K) \end{tikzcd}$$

$$\begin{tikzcd} H_p(L) \arrow[r, "g_*"] \arrow[rr, bend right, "\mathrm{id}"'] & H_p(K) \arrow[r, "f_*"] & H_p(L) \end{tikzcd}$$

$\therefore f_* : H_p(K) \to H_p(L)$ is an isomorphism. ☐

<u>Proof of the theorem</u> :

(1) let $s : |K^{m+1}| \to |L|, t : |K^n| \to |L|$ be simplicial approximations of $f : |K| \to |L|, n \geqslant m$, $\theta : |K^n| \to |K^m|$ be a (interated) simplicial approximation of id.

$$\begin{tikzcd} {|K^m|} \arrow[r] & {|L|} \\ {|K^n|} \arrow[u, "\theta"] \arrow[ur, "t"] & \end{tikzcd}$$

Then both $s \circ \theta$ and t are simplicial approximations of f, since $s(x) \in$ the carrier of $f(x) \mapsto s(\theta(x)) \in$ the carrier of $f(x)$. $\therefore s \circ \theta$ and t are close ($\forall \theta \in K^n$,let $x \in \sigma, \tau \in L$ be the largest simplex of L containing the carrier of $f(x)$ as a face, then $s\theta(\sigma)$ and $t(\sigma)$ are both faces of τ).

$\Rightarrow t_* = s_* \cdot \theta_* : H_p(K^n) \xrightarrow{\theta_*} H_p(K^m) \xrightarrow{S_*} H_p(L)$.

$$\begin{tikzcd} H_p(K) \arrow[r, "\eta^m"] \arrow[d, equal] & H_p(K^m) \arrow[r, "S_*"] \arrow[d, shift right, "\eta^{n-m}"'] & H_p(L) \\ H_p(K) \arrow[r, "\eta^n"] \arrow[ur, phantom, "\circlearrowright"] & H_p(K^n) \arrow[u, shift right, "\theta"'] \arrow[ur, "t_*"'] & \end{tikzcd}$$

(2) (i) By definition.

(ii) Let $t : |L^n| \to |M|$ be a simplicial approximation of g, $s : |K^m| \to |L^n|$ be a simplicial approximation of $f : |K^m| \to |L^n|, \theta : |L^n| \to |L|$ be a (iterated) simplicial approximation of id, then $\theta \circ s$ is a simplicial approximation of $f : |K^m| \to |L|$,$t \circ s : |K^m| \to |M|$ is a simplicial approximation of $g \circ f : |K^m| \to |M|$.

$$\begin{tikzcd} & H_p(K^m) \arrow[r, "s_*"] & H_p(L^n) \arrow[d, shift right, "\theta_*"'] \arrow[dr, "t_*"] & \\ (g \circ f)_* : & H_p(K) \arrow[u, "\eta^m"] \arrow[r, "f_*"] & H_p(L) \arrow[u, shift right, "\eta'^n"'] \arrow[r, "g_*"] & H_p(M) \end{tikzcd}$$

i.e. $(g \circ f)_* = g_* \circ f_*$.

(3) By the above facts, we have simplicial maps

$s_1 \cdots s_n : |K^m| \to |L|$, inducing

$$H_p(K) \xrightarrow{\eta^m} H_p(K^m) \xrightarrow{\overset{S_{n*}}{\overset{\|}{\overset{\vdots}{\overset{\|}{S_{1*}}}}}} H_p(L)$$

$\therefore f_* = s_{1*} \circ \eta^m = s_{n*} \circ \eta^m = g_*$.

Applications:

(1)

Theorem 6.13 (invariance of dimension)**.** *$\mathbb{R}^m$ is homeomorphic to $\mathbb{R}^n$ if and only if $m = n$.*

Proof. Let $h : \mathbb{R}^m \to \mathbb{R}^n$ be a homeomorphism, then $S^{m-1} \simeq \mathbb{R}^m - \{0\} \cong r^n - \{h(0)\} \simeq S^{n-1}$.

But $H_p(S^k) \cong \begin{cases} \mathbb{Z} & p = 0, k \\ o & \text{otherwise} \end{cases}$ □

(2)Brower fixed-point Theorem

A continuous map $f : D^n \to D^n$ has a fixed point.

Proof. If there is a fixed-point free map $f : D^n \to D^n$, then we may construct a retraction $r : D^n \to S^{n-1}$, $\text{id} : S^{n-1} \xrightarrow{i} D^n \xrightarrow{r} S^{n-1}$ induce

$$\text{id} : \underset{\cong \mathbb{Z}}{H_{n-1}(S^{n-1})} \xrightarrow{i_*} \underset{= 0}{H_{n-1}(D^n)} \xrightarrow{r_*} \underset{\cong \mathbb{Z}}{H_{n-1}(S^{n-1})}$$

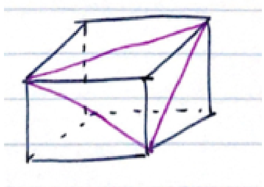
a contradiction. □

(3) Let K be a finite simplicial complex, the Euler characteristic

$\chi(K) = \sum\limits_{q=0}^{n} (-1)^q(\#$ of q-simplices of $K)$.

Euler-Poincaré Formula : $\chi(K) = \sum\limits_{q=0}^{n} (-1)^q \beta_q$, where $\beta_q = \text{rank} H_q(K)$ is the q-th Betti number of K.

Corollary 6.14. *Let $p \subset \mathbb{R}^3$ be a convex polytope, $V = \#$ of vertices, $E = \#$ of edges, $F = \#$ of faces, then $V_E + F = 2$.*



Proof.

$P \cong S^2, \chi(S^2) = 2$.

$C_p(K; \mathbb{Q}) =$ the $\mathbb{Q}$-vector space generated by oriented p-simplices of K.

$\partial_p : C_p(K; \mathbb{Q}) \to C_{p-1}(K; \mathbb{Q}), \sigma = (v_0, \cdots, v_p) \mapsto \sum\limits_{i=0}^{p} (-1)^i (v_0, \cdots, \widehat{v}_i, \cdots, v_p)$.

Then $\{(C_p(K; \mathbb{Q}), \partial_p) | p \geqslant 0\}$ is a chain complex of $\mathbb{Q}$-vector spaces, its gomology group $H_p(K; \mathbb{Q}) = \ker \partial_p / \text{Im} \partial_{p+1}$ is called the p-th homopogy group of K with rational coefficients, a $\mathbb{Q}$-vector space.

$\dim C_p(K; \mathbb{Q}) = \#$ of p-simplicies of K.

A standard fact in linear algebra $\Rightarrow$

$$\sum_{p=0}^{n} (-1)^p \dim_{\mathbb{Q}} C_p(K; \mathbb{Q}) = \sum_{p=0}^{n} (-1)^p \dim \mathbb{Q} H_p(K, \mathbb{Q})$$

. □

Lemma 6.15. $\dim_{\mathbb{Q}} H_p(K; \mathbb{Q}) = \text{rank} H_p(K)$.

Proof. Let A be an abelian group, $A \otimes_Z \mathbb{Q} = \{\sum a_i \otimes r_i | a_i \in A, r_i \in \mathbb{Q}\}$ with relations:

$$\begin{cases} na \otimes r = a \otimes nr, n \in \mathbb{Z} \\ a \otimes r + a' \otimes r = (a + a') \otimes r, a \otimes r + a \otimes r' = a_{\otimes}(r + r') \end{cases}$$

□

Fact: $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$, a short exact sequence of abelian group, then $0 \to A \otimes \mathbb{Q} \xrightarrow{f\otimes \mathrm{id}} B \otimes \mathbb{Q} \xrightarrow{g\otimes \mathrm{id}} C \otimes \mathbb{Q} \to 0$ is a short exact sequence.($- \otimes \mathbb{Q}$ is an exact functor)

Proof. Assume $f\left(\sum\limits_i a_i \otimes r_i\right) = 0$, i.e. $\sum\limits_i f(a_i) \otimes r_i = 0$.

Let $r_i = \dfrac{m_i}{n_i}$,$\prod\limits_j \dfrac{n_j}{n_i} = l_i$, then $\sum\limits_i f(a_i) \otimes \dfrac{m_i}{n_i} = 0 \Rightarrow \sum\limits_i m_i l_i f(a_i) \otimes 1 = 0$.

$\therefore f\left(\sum\limits_i m_i l_i a_i\right) = 0$,$\therefore \sum\limits_i m_i l_i a_i = 0 \Leftrightarrow \sum\limits_i a_i \otimes \dfrac{m_i}{n_i} = 0$

$\therefore f \otimes \mathrm{id}$ is injective.

Now from $0 \to Z_p(K) \to C_p(K) \xrightarrow{\partial p} B_{n-1}(K) \to 0$, we have $0 \to Z_p(K) \otimes \mathbb{Q} \to C_p(K) \otimes \mathbb{Q} \xrightarrow{\partial p \otimes \mathrm{id}} B_{n-1}(K) \otimes \mathbb{Q} \to 0$.

$\therefore Z_p(K) \otimes \mathbb{Q} = \ker(\partial_p \otimes \mathrm{id})$,$B_{p-1}(K) \otimes \mathbb{Q} = \mathrm{Im}(\partial_p \otimes \mathrm{id})$.

From $0 \to B_p(K) \to Z_p(K) \to H_p(K) \to 0$, we have

$$\begin{array}{l} 0 \to B_p(K) \otimes \mathbb{Q} \longrightarrow Z_p(K) \otimes \mathbb{Q} \to H_p(K) \otimes \mathbb{Q} \to 0 \\ \qquad = \mathrm{Im}(\partial_{p+1} \otimes \mathrm{id}) \qquad\qquad = \ker(\partial_p \otimes \mathrm{id}) \end{array}$$

$\therefore H_p(K) \otimes \mathbb{Q} \cong \ker(\partial_p \otimes \mathrm{id}) / (\partial_{p+1} \otimes \mathrm{id}) = H_p(k; \mathbb{Q})$,

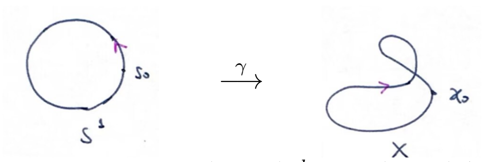
at the same time RHS $\cong (\mathbb{Z}^\beta \oplus T) \otimes \mathbb{Q} \cong \mathbb{Q}^{\beta_p}$ $\square$

(4) the Hurewicz homomorphism

$h : \pi_1(X, x_0) \longrightarrow H_1(X)$

$[\gamma : (S^1, s_0) \longrightarrow (X, x_0)] \longrightarrow \gamma_*[S^1]$

where $[S^1] \in H_1(S^1) \cong \mathbb{Z}$ a generator



h factors through $\pi_1(X, x_0)^{ab} = \pi_1(X,_{x_0}) / [\pi_1(X, x_0), \pi_1(X, x_0)]$

i.e.

$$\begin{tikzcd} \pi_1(X, x_0) \arrow[r, "h"] \arrow[d] & H_1(X) \\ \pi_{X,x_0} / [\pi_1(X,x_0), \pi_1(X,x_1)])) \arrow[ur, "\bar{h}"] & \end{tikzcd}$$

Theorem 6.16. $\bar{h} : \pi_1(X, x_0) / [\pi_1(X.x_0, \pi_1(X, x_1))] \longrightarrow H_1(X)$ *is an isomophism.*

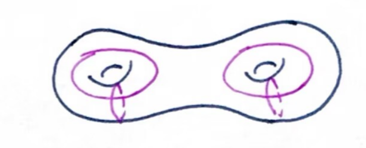
<u>Example</u>

$S = gT^2$ closed orientable surface of genus g

$\pi_1(S) \cong \left\langle a_1, b_1, \dots a_g, b_g \,\middle|\, \prod\limits_{i=1}^{g} [a_i, b_i] = 1 \right\rangle$

$H_1(S) \cong \pi_1(S)^{ab} = \mathbb{Z}^{2g}$ with basis $\{a_1, b_1, \dots a_g, b_g\}$.

p	0	1	2
$H_p(S)$	$\mathbb{Z}$	$\mathbb{Z}^{2g}$	$\mathbb{Z}$



$\chi(S) = 1 - 2g + 1 = 2 - 2g$

$S = mP^2$ closed non-orientable surface of genus m.

$\pi_1(S) \cong \left\langle a_1, \cdots, a_m \middle| \prod_{i=1}^{m} a_i^2 = 1 \right\rangle$

$H_1(S) \cong \pi_1(S)^{ab} = \mathbb{Z}^{m-1} \oplus \mathbb{Z}/_2$

p	0	1	2
$H_p(S)$	$\mathbb{Z}$	$\mathbb{Z}^{m-1} \oplus \mathbb{Z}/_2$	$\mathbb{Z}$

$\chi(S) = 1 - (m-1) = 2 - m$