

Random cascade model in the limit of infinite integral scale as the exponential of a nonstationary $1/f$ noise: Application to volatility fluctuations in stock markets

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In this paper we propose a new model for volatility fluctuations in financial time series. This model relies on a nonstationary Gaussian process that exhibits aging behavior. It turns out that its properties, over any finite time interval, are very close to continuous cascade models. These latter models are indeed well known to reproduce faithfully the main stylized facts of financial time series. However, it involves a large-scale parameter (the so-called “integral scale” where the cascade is initiated) that is hard to interpret in finance. Moreover, the empirical value of the integral scale is in general deeply correlated to the overall length of the sample. This feature is precisely predicted by our model, which, as illustrated by various examples from daily stock index data, quantitatively reproduces the empirical observations.

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I. INTRODUCTION

For several decades, random cascade models have been at the heart of a wide number of studies in mathematics as well as in applied sciences. They were introduced to account for the intermittency phenomenon in fully developed turbulence and are involved every time one observes a multifractal (or a multiscaling) behavior in the variations of statistical properties of some field across different scales. Multifractal scaling is generally associated with the existence of a random cascade by which small-scale structures are constructed from the splitting of larger ones and multiplication by a random factor. One clearly sees that such a scenario necessarily implies the existence of a large *integral scale* T where the cascade is initiated. As emphasized below (see Appendix A), in general, one can show that the moment multiscaling behavior of the increments associated with any multifractal field cannot hold over an infinite range of scales. It necessarily involves a large-scale T above which scaling properties becomes trivial. In turbulence this scale naturally corresponds to the injection scale, i.e., the time or space scale where kinetic energy is injected into the flow [1]. The main question addressed in this paper concerns the fields where a multifractal behavior is observed without the existence of any obvious integral scale. This is notably the case in empirical finance.

In quantitative finance, volatility is one of the most important risk measures since it corresponds to the conditional variance associated with price fluctuations at any time t [2]. A well-known stylized fact is that volatility fluctuations are organized into persistent clusters. A huge amount of the

econometrics literature is devoted to the modeling of this volatility persistence. Among all the proposed alternatives, the Generalized Autoregressive Conditional Heteroskedasticity (GARCH) models [3] and their extensions have been thoroughly studied. The major drawback of such models is that, on one hand, their aggregation properties are not easy to control, and on the other hand, they cannot account for the long-range nature of volatility correlations [4,5]. This last feature translates to the fact that GARCH parameters are often found to be at the borderline of the stability region. This is the so-called IGARCH effect [6].

Under the impetus of early studies of Mandelbrot and his collaborators [7], the notions of multifractals and random cascades have been proposed to account for the volatility dynamics in many studies of financial time series (see, e.g., [8–12]). The class of continuous random cascades [13], and in particular the Multifractal Random Walk (MRW) model, provides a parsimonious class of random processes that reproduces very well most of stylized facts that characterize the price return fluctuations [2,10]. Unlike GARCH models, these models are continuous time models (so they do not involve a discrete time step) and their aggregation properties are easy to handle since they possess some self-similarity properties. Within this framework, various empirical estimations reported so far indicate that the value of T can vary from few months [8,9] to several years [14,15] (see Fig. 9 below). Even if it is well admitted that a precise estimation of T can be hardly achieved [14,15], one can naturally wonder why one observes such a large range in the estimated integral scale values. Beyond the problem of the determination of T , a challenging question remains to understand the *meaning* of the integral scale in finance. Unlike turbulence, there is no natural large scale that would obviously appear to be associated with some “source of volatility.”

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The idea we propose in this paper is that such a scale does not exist (or is formally “infinite”) and that the volatility is a nonstationary process. Let us notice that, within standard econometric framework, many authors already proposed to explain the above mentioned IGARCH effect by the nonstationary nature of volatility fluctuations: These models include fractionally integrated GARCH [16], GARCH models with time-varying parameters [17,18], and stochastic volatility models with unit roots [19]. Our approach is original in the sense that we account for the nonstationary nature of volatility fluctuations while remaining within the framework of multifractal processes. Indeed, we show that our model is such that every single trajectory, for each finite time interval, can hardly be distinguished from the path of a multifractal process where the *integral scale is precisely the length of the time interval under consideration*. Our construction is written as the exponential of a nonstationary $1/f$ noise and is based on an extension of continuous random cascades based on infinitely scattered random measure as introduced in Refs. [13,20]. We show that this process is well defined in the sense of distributions and cannot be distinguished from a continuous cascade model (as the MRW process defined in Ref. [10]) over any finite time interval far from the time origin. We check and illustrate our results on some numerical simulations. We then consider applications to stock index market data that are shown to exhibit some “aging” behavior as precisely predicted by our model.

The paper is organized as follows: In Sec. II we make a brief overview of multifractal models as they have been proposed to account for the volatility fluctuations in financial time series. The construction of log-infinitely divisible continuous random cascades as introduced in Refs. [13,20] is also explained but we mainly focus on the log-normal case. In Sec. III we show how one can some extend the former cascade models in the formal limit $T \rightarrow +\infty$. The price to pay is that the model is no longer stationary. However, this revised model has appealing properties since, in some sense, it reduces to a multifractal model over any bounded time interval without involving any large-scale parameter. Our results are illustrated using numerical simulations. In Sec. IV, we address the problem of the model estimation using a single realization. We then show that our approach is pertinent to account for the observed volatility correlations from intraday to many-year time scales. In particular it allows one to understand the wide range of estimated integral scale values reported in the literature. We use the Dow-Jones daily data recorded over several decades to provide evidence against the stationarity of the volatility process. Concluding remarks and paths for future research are provided in Sec. V. Some technical results are reported in the Appendixes.

II. MULTIFRACTAL VOLATILITY MODELS: A BRIEF OVERVIEW

A. Multiscaling

As mentioned in the introductory section, multifractal models have provided a family of stochastic processes that accounts very well for the main statistical features of financial time series [12,21]. In this section we recall the main results concerning random cascade models and set the main

notations. We refer the reader to Refs. [10,13,20,22] for more mathematical details.

As first proposed by Mandelbrot *et al.* [7], multifractal processes $X(t)$ with zero mean and stationary increments can be constructed through an auxiliary nondecreasing multifractal measure $M(t)$ as

$$X(t) = B[M(t)], \quad (1)$$

where $B(t)$ is a self-similar process [i.e., such that $B(\lambda t) =_{\text{law}} \lambda^H B(t)$] in general chosen to be a standard Brownian motion ($H = 1/2$). It results that the increments of X and M are related by

$$\begin{aligned} X(t + \tau) - X(t) &= [M(t + \tau) - M(t)]^H X(1) \\ &=_{\text{law}} M(\tau)^H X(1). \end{aligned} \quad (2)$$

In other words, the variations of $M(t)$ are related to the local variance of a Brownian motion. In finance, $X(t)$ represents some asset price (or the logarithm of an asset price) whose increments are the so-called asset returns. In that case, the measure $M(t)$ is usually referred to as the “trading time” or the “volatility process” since its increments $M(t + \tau) - M(t) \geq 0$ simply correspond to the volatility (i.e., the local variance) between times t and $t + \tau$. Henceforth, most of our considerations will concern the “volatility” $M(t)$. All the results can be extended to the “price” process $X(t)$ in a straightforward manner using Eq. (2).

In a loose mathematical sense, a nondecreasing stochastic process $M(t)$ is called multifractal (or “multifractal measure”) if the moments of its increments (assumed to be stationary) $\delta_\tau M(t) = M(t + \tau) - M(t)$ verify the multiscaling properties:

$$\mathbb{E}[\delta_\tau M(t)^q] = \mathbb{E}[M(\tau)^q] \sim C_q \tau^{\zeta(q)}, \quad (3)$$

where $\zeta(q)$ is a nonlinear concave function of the moment order q . Notice that the multifractal nature is properly defined by the nonlinearity of $\zeta(q)$ as opposed to monofractal situations where $\zeta(q)$ is a linear function. In order to quantify the multifractality, one often defines the so-called *intermittency coefficient* λ^2 as the curvature of $\zeta(q)$ around $q = 0$:

$$\lambda^2 = -\zeta''(0) \geq 0. \quad (4)$$

The last inequality simply comes from the concavity of the $\zeta(q)$ spectrum. Indeed, the scaling behavior of Eq. (3) is generally interpreted in the limit of small time scales $\tau \rightarrow 0$. Accordingly, if one computes, for example, the kurtosis behavior,

$$\mathcal{F}(\tau) = \frac{\mathbb{E}[M(\tau)^4]}{\mathbb{E}[M(\tau)^2]^2} \sim \tau^{\zeta(4)-2\zeta(2)}, \quad (5)$$

one directly sees that because $\mathcal{F}(\tau) \geq 1$, one must have $\zeta(4) \leq 2\zeta(2)$. As shown in Appendix A, this kind of argument can be generalized (thanks to the Hölder inequality) to prove that $\zeta(q)$ is concave. Thanks to Eq. (2), one can conclude that the increment probability density functions (pdf) of $X(t)$ (the price returns in empirical finance) cannot keep a constant shape at different time scales τ (that would be Gaussian in the monofractal situation). It necessarily becomes more leptokurtic as $\tau \rightarrow 0$. Both multiscaling and increasing flatness at small scales are two well-known stylized facts

characterizing the return time series of many different financial markets [7,8,10].

Let us remark that the previous argument can also be used to show that the scaling (3) cannot hold for arbitrary large scales τ . Indeed, since $\mathcal{F}(\tau) > 1$, if $\zeta(4) - 2\zeta(2) < 0$ then Eq. (5) can be valid only on a bounded range of scales. Therefore, there exists an *integral scale* T below which multiscaling holds and beyond which one observes monofractal scaling properties (see Appendix A).

B. Continuous cascades

Explicit constructions of multifractal measures can be naturally obtained within the framework of random cascades. The picture of a random cascade comes from the physics of turbulence where kinetic energy injected in the flow at some large scale is transferred to the finest scales by successive steps of eddy fragmentation [1]. The large scale where the cascade “starts” corresponds precisely to the integral scale introduced previously. Accordingly, a discrete random cascade can be constructed as follows: One starts with an interval of length T where the measure $M(dt)$ is uniformly spread (meaning that the density is constant) and splits this interval in two equal parts: On each part, the density is multiplied by (positive) independent identically distributed (i.i.d.) random factors W . Each of the two subintervals is again cut in two equal parts and the process is repeated infinitely. Given the discrete and nonstationary nature of such constructions and the fact that they are only defined in a fixed bounded interval (of size T), more recently, continuous cascade constructions have been proposed. These models can be viewed as a “densification” of the discrete construction [20,21,23] where the multiplication along the dyadic tree associated with successive fragmentation steps,

$$dM = \prod_i W_i = e^{\sum_i \ln(W_i)},$$

is replaced by the exponential of an integral (instead of a discrete sum) of a white noise [instead of $\ln(W)$] over a conelike domain in the time-scale plane (instead of the tree-node set). More precisely, one defines [13,20]

$$dM_{\ell,T}(t) = M_{\ell,T}([t, t + dt]) = e^{\omega_{\ell,T}(t)} dt \quad (6)$$

with

$$\omega_{\ell,T}(t) = \mu_{\ell,T} + \int_{(u,s) \in C_{\ell,T}(t)} dW(u,s), \quad (7)$$

where $\mu_{\ell,T}$ is a constant such that $\mathbb{E}[e^{\omega_{\ell,T}(t)}] = 1$, $dW(u,s)$ is a white noise associated with some infinitely divisible law (more precisely an “independently scattered random measure” [13]), and $C_{\ell,T}(t)$ is the conelike domain:¹

$$(u,s) \in C_{\ell,T}(t) \iff \{s \geq \ell, t - \min(s,T) \leq u \leq t\}. \quad (8)$$

This construction is illustrated in Fig. 1. The final multifractal measure dM_T is then obtained as the weak limit of

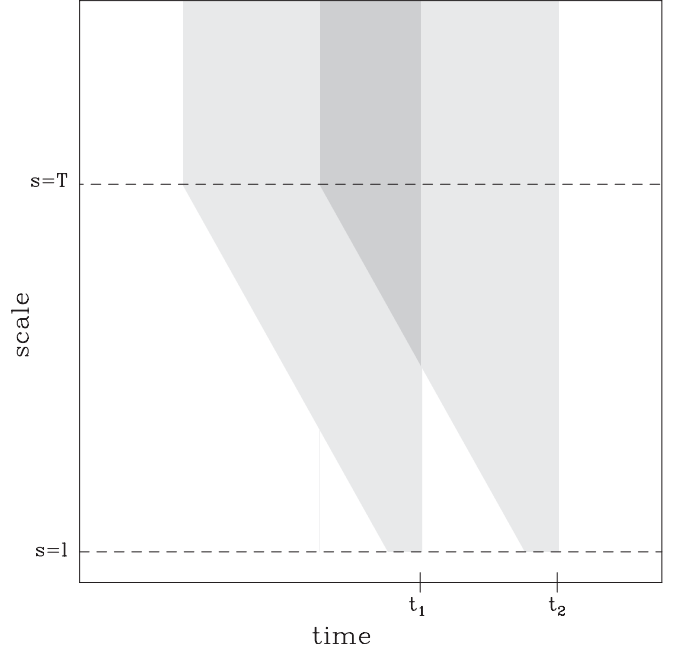


FIG. 1. Construction of a continuous cascade process: $\omega_{\ell,T}(t)$ is the integral of a white noise over a conelike domain $C_{\ell,T}(t)$ in the time-scale plane. The covariance of $\omega_{\ell,T}(t_1)$ and $\omega_{\ell,T}(t_2)$ corresponds to the area of the intersection $C_{\ell,T}(t_1) \cap C_{\ell,T}(t_2)$.

$dM_{\ell,T}$ when $\ell \rightarrow 0$, i.e.,

$$M_T(t) = \lim_{\ell \rightarrow 0} \int_0^t dM_{\ell,T}(t) = \lim_{\ell \rightarrow 0} \int_0^t e^{\omega_{\ell,T}(t)} dt. \quad (9)$$

For the sake of simplicity, we consider in this paper exclusively log-normal random cascades. All our results can be easily extended to arbitrary log-infinitely divisible laws within the framework introduced in Refs. [13,20]. In the log-normal case, $dW(t,s)$ is a two-dimensional (2D) Gaussian (Wiener) white noise of variance $\lambda^2 s^{-2} dt ds$ and it is easy to see (in Fig. 1) that the covariance of $\omega_{\ell,T}(t_1)$ and $\omega_{\ell,T}(t_2)$ is simply the area of $C_{\ell,T}(t_1) \cap C_{\ell,T}(t_2)$. Its expression reads

$$\text{Cov}(\omega_{\ell,T}(t), \omega_{\ell,T}(t + \tau)) = \begin{cases} \lambda^2 \ln\left(\frac{T}{\tau}\right) & \text{if } \ell \leq \tau \leq T, \\ \lambda^2 \left(\ln\left(\frac{T}{\ell}\right) + 1 - \frac{\tau}{\ell} \right) & \text{if } \tau \leq \ell, \\ 0 & \text{if } \tau > T. \end{cases} \quad (10)$$

In that respect the mean value of $\omega_{\ell,T}$ has to be chosen as

$$\mu_{\ell,T} = -\frac{\lambda^2}{2} \left[1 + \ln\left(\frac{T}{\ell}\right) \right]. \quad (11)$$

Notice that in the log-normal case, the intermittency coefficient λ^2 and the integral scale T are the only parameters that govern the multifractal statistics. The previous equation mainly says that the logarithm of a random log-normal multifractal measure is a Gaussian process which covariance decreases as a logarithmic function, $\log(T/\tau)$. This features has been shown to directly reflect the tree structure of discrete random cascades (see Refs. [9,24]).

¹Let us remark that the construction we consider here is a “causal version” of the original construction proposed in Refs. [13,20], where a symmetrical cone was used. All the results and computations remain unchanged for both constructions.

C. Stochastic self-similarity

All the (multi)scaling properties of $M(t)$ [and subsequently of $X(t)$] can be shown to result from the logarithmic nature of this covariance. Indeed, since $\omega_{\ell,T}(t)$ is a Gaussian process, one can directly infer from Eqs. (10) and (11) that, $\forall r < 1$, $\forall t \leq T$,

$$\omega_{r\ell,T}(rt) = \omega_{\ell,T}(t) + \Omega_r, \quad (12)$$

where Ω_r is a normal random variable of variance $-\lambda^2 \ln(r)$ and mean $\frac{\lambda^2}{2} \ln(r)$. From Eq. (12), the stochastic self-similarity property results [13,20] in

$$M_T(rt) = r e^{\Omega_r} M(t), \quad (13)$$

which directly proves the multiscaling [Eq. (3)] of the moments of $M(t)$ [and thus of $X(t)$] with a parabolic $\zeta(q)$ function:

$$\zeta(q) = q + \frac{\ln \mathbb{E}[e^{q\Omega_r}]}{\ln r} = q \left(1 + \frac{\lambda^2}{2}\right) - \frac{\lambda^2 q^2}{2}. \quad (14)$$

One can establish another self-similarity property [25] when one also rescales the integral scale. In that case, one has trivially from Eq. (10), $\forall r > 0$:

$$\omega_{r\ell,rT}(rt) = \omega_{\ell,T}(t), \quad (15)$$

$$M_{rT}(rt) = r M(t), \quad (16)$$

which means that a trivial scaling is obtained when the integral time T is rescaled with the time.

In the field of empirical finance, random cascades have allowed one to understand that the observed multiscaling properties of return moments and the long-range correlated nature of the volatility are the two faces of the same coin. The (log-normal) multifractal random walk model has proven to be a simple, parcimonious model that reproduces most of observed statistical properties of asset returns [2,10,21,26]. As far as statistical estimation issues are concerned, as shown in Ref. [27], intermittency exponent estimations based on Eq. (10) are far more reliable than those based on moment multiscaling (3) (see also Refs. [14,28] for additional results on the intermittency exponent estimation using Generalized Methods of Moments). Empirical evidence for the logarithmic nature of log-volatility correlations have been provided for different asset price time series over different markets [9,10,14,15,21]. All these results confirm the multifractal nature of asset return fluctuations with an intermittency coefficient $\lambda^2 \in [0.01, 0.03]$. However, the reported values of the integral scale T vary in a wide range of scales, between few months and several years. The main question we address in this paper concerns that point: What is the value of the integral scale in financial time series?

III. THE LIMIT OF INFINITE INTEGRAL SCALE: A NONSTATIONARY MODEL FOR LOG VOLATILITY

A. Definition of the model

The broad range of observed values of the integral scale in empirical studies leads us to question the interpretation of the integral scale value in financial markets. Unlike turbulence, there is no obvious large scale that could be singularized and associated with some “source” of randomness. Even if the

heterogeneity of agents and the wide range of time horizons used by market participants is a well-recognized fact, this can hardly be invoked to define a single scale that could be as large as several years.

A way to answer the previous remarks could be to consider the model introduced in Ref. [29] where the authors replaced the log-correlated $\omega_{\ell,T}(t)$ by a long-range (e.g., a fGn) correlated stationary Gaussian process. However, the continuous time limit of such a process is trivial (i.e., it necessarily involves a small scale cutoff) and its scaling properties are not exact and hard to handle. Another solution is to define a random cascade process in the limit $T \rightarrow \infty$. However, the definition of such a limit is not obvious, since, as emphasized in the previous section and shown in Appendix A, one cannot define any multiscaling behavior without involving a finite integral scale. As one can check in Eq. (10), by letting $T \rightarrow \infty$, one obtains an infinite value of the variance (and the mean) of $\omega_{\ell,T}$. In Ref. [26], the authors have considered the possibility of an infinite integral scale and provided an explicit prediction formula of $\omega_{\ell,T \rightarrow \infty}$ (that we denote as $\omega_{\ell,\infty}$). However, this process is not defined in a classical sense but only in some quotient space, namely a space of processes defined up to constant time functions. It has been shown that

$$\lim_{T \rightarrow \infty} \int \phi(u) \omega_{\ell,T}(t-u) du \quad (17)$$

is meaningful for a class of smooth functions ϕ provided it is of zero mean. We already know that the singularity of the covariance function at $\tau = 0$ when $\ell \rightarrow 0$ [Eq. (10)] means that the limit of $\omega_{\ell,T}$ [or $\exp(\omega_{\ell,T})$] has to be considered as a noise process and is well defined only when interpreted in the weak (distribution) sense. When $T \rightarrow +\infty$, Duchon *et al.* [26] show that $\omega_{\ell \rightarrow 0, \infty}$ can be still interpreted in a weak sense but only for test functions satisfying $\int \phi(t) dt = 0$. This process and notably its exponential $e^{\omega_{\ell,\infty}}$ are, however, hard to interpret and of unclear practical interest in quantitative finance.

In order to handle the low-frequency problem related to $T \rightarrow +\infty$, we propose in this paper an alternative solution that consists in considering a *nonstationary process*, where at time t the integral scale is precisely $T = t$. We define a process $\omega_\ell(t)$ as for standard cascade, from the integration over a conelike domain in a time-scale plane, where at time t the parameter T in Eq. (8) is replaced by t :

$$(u,s) \in C_\ell(t) \Leftrightarrow \{s \geq \ell, \max(0, t-s) \leq u \leq t\} \quad \text{if } t \geq \ell \quad (18)$$

$$C_\ell(t) = \emptyset \quad \text{otherwise.} \quad (19)$$

The process $\omega_\ell(t)$ is then defined by

$$\omega_\ell(t) = \mu_\ell(t) + \int_{(u,s) \in C_\ell(t)} dW(u,s), \quad (20)$$

where $\mu_\ell(t)$ is a deterministic mean value defined below and $dW(u,s)$ a Gaussian white noise of variance $\lambda^2 s^{-2} du ds$. This construction is illustrated in Fig. 2.

In Fig. 3 we have plotted a sample of $\omega_\ell(t)$ generated at rate $\Delta t = \ell = 1$ over 500 points. As one can see, the nonstationary nature of $\omega_\ell(t)$ is not obvious (see below).

We can compute the covariance of $\omega_\ell(t)$ that corresponds to the domain C_ℓ intersection areas (see Fig. 2). For $t_1 \leq t_2 =$

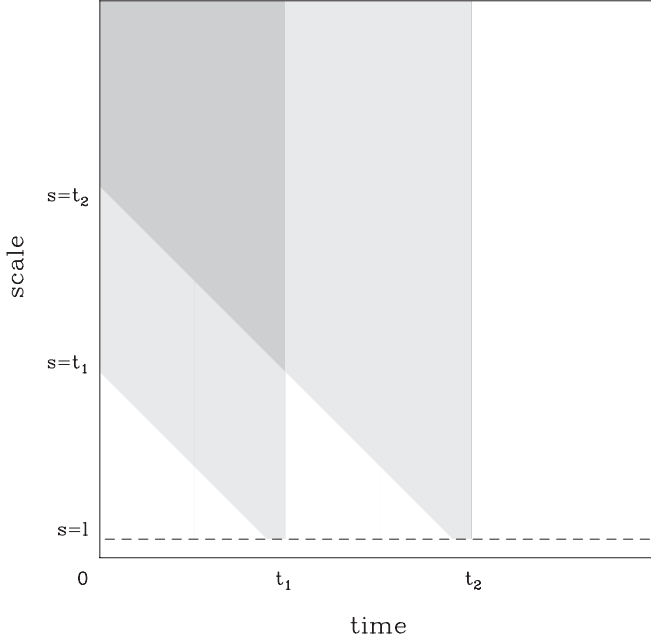


FIG. 2. Construction of the nonstationary $\omega_\ell(t)$ process as the integral of a white noise over a conelike domain $C_\ell(t)$ in the time-scale plane. The covariance of $\omega_\ell(t_1)$ and $\omega_\ell(t_2)$ is simply the area of the intersection $C_\ell(t_1) \cap C_\ell(t_2)$.

$t_1 + \tau$, its expression reads

$$\text{Cov}(\omega_\ell(t_1), \omega_\ell(t_2)) = \begin{cases} \lambda^2 \ln\left(\frac{t_2}{\tau}\right) & \text{if } \tau > l, \\ \lambda^2 \left(\ln\left(\frac{t_2}{l}\right) + 1 - \frac{\tau}{l}\right) & \text{if } \tau \leq l, \\ 0 & \text{if } t < l. \end{cases} \quad (21)$$

This equation implies notably that

$$\text{Var}[\omega_\ell(t)] = \lambda^2 \left[1 + \ln\left(\frac{t}{\ell}\right) \right]. \quad (22)$$

One clearly sees that $\omega_\ell(t)$ is a nonstationary Gaussian process but its covariance has striking similarities with the stationary situation [Eq. (10)], where the integral scale has been replaced by the current time t [or $\max(t_1, t_2)$ in the covariance expression].

The nonstationary behavior of the covariance is illustrated in Fig. 4, where we have plotted the estimated as well as analytical $\text{Cov}(\omega_\ell(t_1), \omega_\ell(t_2))$ as a function of $\ln(\tau) = \ln(t_2 - t_1)$ for different values of t_2 . This kind of nonstationarity

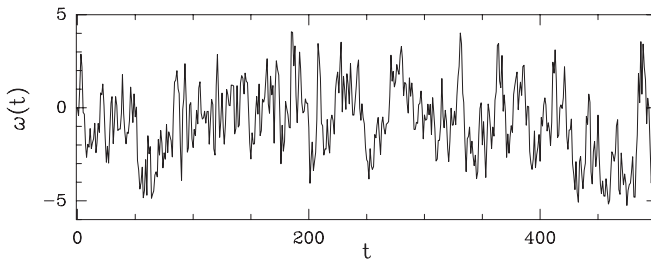


FIG. 3. An example of a path of $\omega_\ell(t)$ where the numerical construction has been performed by sampling both space and scale in Eq. (20).

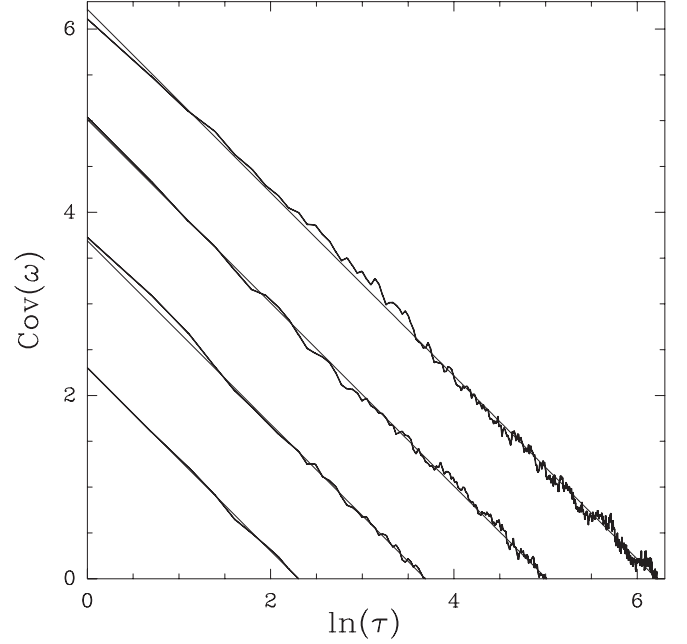


FIG. 4. Covariance function $\text{Cov}(\omega_\ell(t_1), \omega_\ell(t_2))$ as a function of $\ln(\tau)$, with $\tau = |t_2 - t_1|$ and $t_2 = 10, 40, 150, 500$ (from bottom to top curves). The bold lines correspond to numerical estimates using 500 samples of $\omega_\ell(t)$ while the thin lines correspond to the analytical expressions [Eq. (21)]. We have chosen $l = 1$ and $\lambda^2 = 1$.

is reminiscent of an *aging* behavior as observed in off-equilibrium relaxing systems [30,31] where the “age” of the process t_2 controls the characteristic correlation length. The logarithmic behavior of the variance is illustrated in Fig. 5.

Let us show that one can choose a function $\mu_\ell(t)$ in Eq. (20) such that one can define the limit $\ell \rightarrow 0$ of e^{ω_ℓ} in the weak

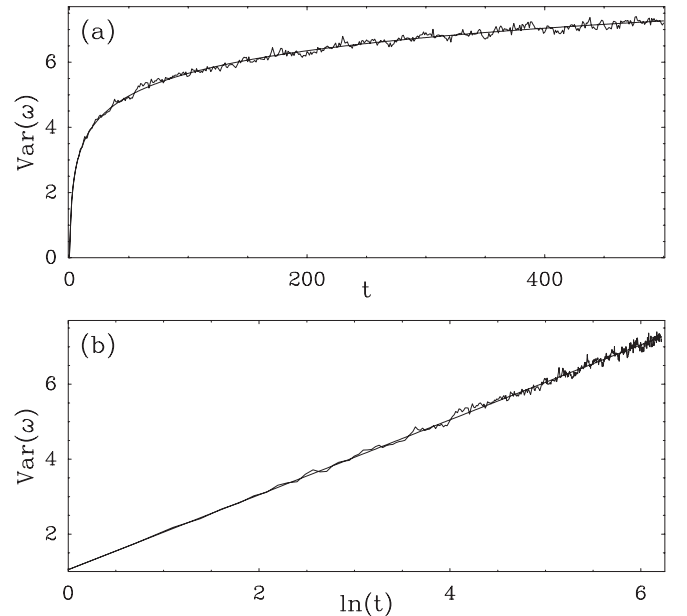


FIG. 5. Variance $\text{Var} \omega_\ell(t)$ as a function of t (a) and $\ln(t)$ (b). We have superimposed to the expected analytical expressions (22), the estimated variance using 500 Monte Carlo samples of $\omega_\ell(t)$ with $l = 1$ and $\lambda^2 = 1$.

sense, i.e.,

$$M(t) = \lim_{\ell \rightarrow 0} \int_0^t e^{\omega_\ell(u)} du. \quad (23)$$

In fact, as for continuous stationary cascades, one can use a general argument on positive martingales (as, e.g., in Ref. [13]) if, for all time intervals I , $\int_I e^{\omega_\ell(t)} dt$ is a martingale as a function of ℓ . This is precisely the case provided, $\forall t$,

$$\mathbb{E}[e^{\omega_\ell(t)}] = 1,$$

a condition equivalent, in the log-normal case, to

$$\mu_\ell(t) = -\frac{1}{2} \text{Var}[\omega_\ell(t)] = -\frac{\lambda^2}{2} \left[1 + \ln \left(\frac{t}{\ell} \right) \right]. \quad (24)$$

In Appendix B, we provide an alternative direct proof of mean square convergence.

Notice that this equation also guarantees that

$$\mathbb{E}[M(t)] = \text{Var}[X(t)] = \sigma^2 t \quad (25)$$

[recall that $X(t) = B[M(t)]$ with $B(t)$, a standard Brownian motion].

B. Scaling and self-similarity properties

Let us remark that increments of $\omega_\ell(t)$, $\delta_h \omega_\ell(t) = \omega_\ell(t+h) - \omega_\ell(t)$ ($h > \ell$) have a time-dependent variance so they are not stationary. However, for $\tau > h$, their covariance depends only on the lag τ . After a little algebra, their expression reads

$$\text{Cov}(\delta_h \omega_\ell(t), \delta_h \omega_\ell(t+\tau)) = \lambda^2 \ln \left(1 - \frac{h^2}{\tau^2} \right). \quad (26)$$

This covariance function corresponds to a power spectrum such that $P_{\delta_h \omega}(f) \sim |f|$ when $f \ll h^{-1}$. Since $P_{\omega_\ell}(f) \sim f^{-2} P_{\delta_h \omega_\ell}(f)$, $\lim_{\ell \rightarrow 0} \omega_\ell(t)$ can be associated with a $1/f$ power spectrum. Let us mention that in Ref. [31] the author has already raised the possibility of an “aging” nonstationary model in order to handle the low-frequency problem of $1/f$ noise. In that respect, $\omega_{\ell \rightarrow 0}(t)$ can be interpreted as the limit $H \rightarrow 0$ of a fractional Brownian motion (fBm) $B_H(t)$ of Hurst parameter H [32].

This interpretation of $\omega_\ell(t)$ can also be suggested from its self-similarity properties. Indeed, from the covariance expression (21), one can establish the following invariance relationship for $\omega_\ell(t)$:

$$\omega_{r\ell}(rt) =_{\text{law}} \omega_\ell(t). \quad (27)$$

This equality extends to $H = 0$ the standard fBm self-similarity $B_H(rt) =_{\text{law}} r^H B_H(t)$.² It is noteworthy that $\omega_{\ell \rightarrow 0}(t)$ has the drawbacks of both fractional Gaussian noises and fractional Brownian motions since it exists only in the sense of distributions and it is a nonstationary process. From the definition (23) and thanks to previous equality, one deduces

the simple self-similarity property of the volatility process $M(t)$:

$$M(rt) =_{\text{law}} r M(t). \quad (28)$$

Let us remark that relation (27) is different from Eq. (12) but can be understood as reminiscent of Eq. (15) where one allows the integral scale to become infinite (i.e., $T \rightarrow \infty$).

When one compares the self-similarity of M and M_T [Eqs. (28) and (13)], one sees that in the former case there is no stochastic factor e^{Ω_r} and the scaling of the moments of M (and therefore of the return process $X(t)$) becomes trivial:

$$\mathbb{E}[M(\tau)^q] = C_q \tau^q \Rightarrow \mathbb{E}[|X(\tau)|^q] = K_q \tau^{\frac{q}{2}}. \quad (29)$$

In the sense of Eq. (3), it thus appears that $M(t)$ [or $X(t)$] is not a multifractal process. However, one must carefully interpret the previous equation since $M(t)$ [and then $X(t)$] has no stationary increments. It results that there is no reason that the moments $\mathbb{E}[M(\tau)^q]$ and $\mathbb{E}[|M(t+\tau) - M(t)|^q]$ behave in the same way. Let us make the explicit computation for $q = 2$. In that case,

$$\begin{aligned} \mathbb{E}[M(\tau)^2] &= \lim_{\ell \rightarrow 0} \int_0^\tau \int_0^\tau \mathbb{E}[e^{\omega_\ell(u) + \omega_\ell(v)}] du dv \\ &= \lim_{\ell \rightarrow 0} \int_0^\tau \int_0^\tau e^{\text{Cov}(\omega_\ell(u), \omega_\ell(v))} du dv \\ &= \int_0^\tau \int_0^\tau \left(\frac{\max(u, v)}{|u - v|} \right)^{\lambda^2} du dv \\ &= \tau^2 \int_0^1 \int_0^1 \left(\frac{\max(u, v)}{|u - v|} \right)^{\lambda^2} du dv = C_2 \tau^2, \end{aligned}$$

whereas

$$\begin{aligned} \mathbb{E}[|M(t+\tau) - M(t)|^2] &= \lim_{\ell \rightarrow 0} \int_t^{t+\tau} \int_t^{t+\tau} \mathbb{E}[e^{\omega_\ell(u) + \omega_\ell(v)}] du dv \\ &= \int_t^{t+\tau} \int_t^{t+\tau} \left(\frac{\max(u, v)}{|u - v|} \right)^{\lambda^2} du dv \\ &= t^2 \int_1^{1+\frac{\tau}{t}} \int_1^{1+\frac{\tau}{t}} \left(\frac{\max(u, v)}{|u - v|} \right)^{\lambda^2} du dv \\ &= t^2 \int_0^{\frac{\tau}{t}} \int_0^{\frac{\tau}{t}} \left(\frac{1 + \max(u, v)}{|u - v|} \right)^{\lambda^2} du dv. \end{aligned}$$

If one supposes that $\frac{\tau}{t} \ll 1$, then in the last integral the term $\max(u, v) \ll 1$ can be neglected and, using the change of variables $u' = ut/\tau$ and $v' = vt/\tau$, one gets

$$\mathbb{E}[|M(t+\tau) - M(t)|^2] \simeq C_2(t) \tau^{2-\lambda^2},$$

where the constant $C_2(t) \sim t^{-\lambda^2}$. The previous equation shows that in the limit $\tau \ll t$, the mean square of the increments of $M(t)$ behaves as the increment of the multifractal measure $M_T(t)$ (with the scaling exponent $\zeta(2) = 2 - \lambda^2$), where t plays precisely the role of the integral scale T . This behavior can be directly established from the expression of the covariance, Eq. (21): Indeed, let us consider two times t_1, t_2 in some interval $[t_0, t_0 + \Delta t]$. If $\Delta t \ll t_0$, then to the first order in $t_0/\Delta t$, we have $\text{Cov}(\omega_\ell(t_1), \omega_\ell(t_2)) = \lambda^2 \ln(t_0/|t_1 - t_2|)$, i.e.,

²Since fBm's are defined from both their self-similarity properties and the stationarity of their increments [32], in full rigor $\omega_{\ell \rightarrow 0}(t)$ cannot be identified to $B_H(t)$ with $H = 0$.

the same covariance as the process $\omega_{\ell,T}$ used to build an exact multifractal random measure with $T = t_0$. This means that the nonstationary process $M(t)$ defined in Eq. (23) cannot be distinguished from a (stationary) multifractal random measure $M_{t_0}(t)$ of integral scale $T = t_0$ over any interval $[t_0, t_0 + \Delta t]$ (to the first order in $t_0/\Delta t$).

IV. APPLICATION TO FINANCIAL DATA

As recalled in the introduction, various authors have suggested that most of stylized facts characterizing the volatility associated with asset prices in financial markets can be accounted for by multifractal measures. Let us illustrate how the model $M(t)$ introduced in this paper allows one to explain the large discrepancies of the reported integral scale values as a consequence of the nonstationary nature of log volatility. Since the model is nonstationary and since in practice it is not possible to have an ensemble of many independent samples, one has first to discuss which kind of estimation one can perform on a single realization of the volatility.

A. Pathwise properties and estimation issues

Let us suppose that one studies a multifractal random measure $M_T(t)$ (i.e., a classical random cascade with finite integral scale T) over an interval $[t_0, t_0 + \Delta t]$ (or, since M_T has stationary increments, over $[0, \Delta t]$) with $\Delta t < T$. Then from the self-similarity relations (12) and (15), one as, for all $r < \Delta t/T < 1$,

$$M_T(t) = r^{-1} M_{rT}(rt) = r^{-1} e^{\omega_r} M_{rT}(t). \quad (30)$$

Since the random variable ω_r is fixed on a single realization, this equality clearly means that one cannot distinguish over any interval $[t_0, t_0 + \Delta t]$ two multifractal measures $M_{T_1}(t)$ and $M_{T_2}(t)$ with $T_1 \neq T_2$ and $T_1, T_2 \geq \Delta t$. Estimating the integral scale on a single realization of $M_T(t)$ over an interval of length $\Delta t < T$ is thus impossible. The question is to which value an empirical estimation leads.

Empirically, as advocated, e.g., in Ref. [14], the correlation properties of $\omega_{\ell,T}(t)$ can be estimated using a proxy (called the “magnitude process”) of $\omega_{h,T}(t)$ estimated from the logarithm of the increments of $M_T(t)$: $\omega_{h,T} \simeq \ln \delta_h M_T(t)$. If $\omega_{h,T}$ is sampled at rate h over a time period of length Δt , the estimator of its covariance $\widehat{C}_{\Delta t}(\tau)$ at lag $\tau = nh$, reads

$$\widehat{C}_{\Delta t}(\tau) = (N - n)^{-1} \sum_{i=0}^{N-1-n} (\omega_{h,T}[ih] - \hat{\mu})(\omega_{h,T}[(i+n)h] - \hat{\mu}), \quad (31)$$

where $N = \frac{\Delta t}{h}$ is the sample size and $\hat{\mu}$ is the empirical mean: $\hat{\mu} = N^{-1} \sum_{k=0}^{N-1} \omega_{h,T}(kh)$. In Appendix C (see also Ref. [14] for a more technical approach) it is shown that

$$\mathbb{E}[\widehat{C}_{\Delta t}(\tau)] \simeq \lambda^2 \left[\ln \left(\frac{e^{-3/2} \Delta t}{\tau} \right) - \frac{\tau}{\Delta t} \right] + O \left(\frac{\tau^2}{\Delta t^2} \right). \quad (32)$$

This equation means that, over a sample of size Δt , the estimated autocovariance of the magnitude associated with a multifractal process of integral scale $T > \Delta t$ is the

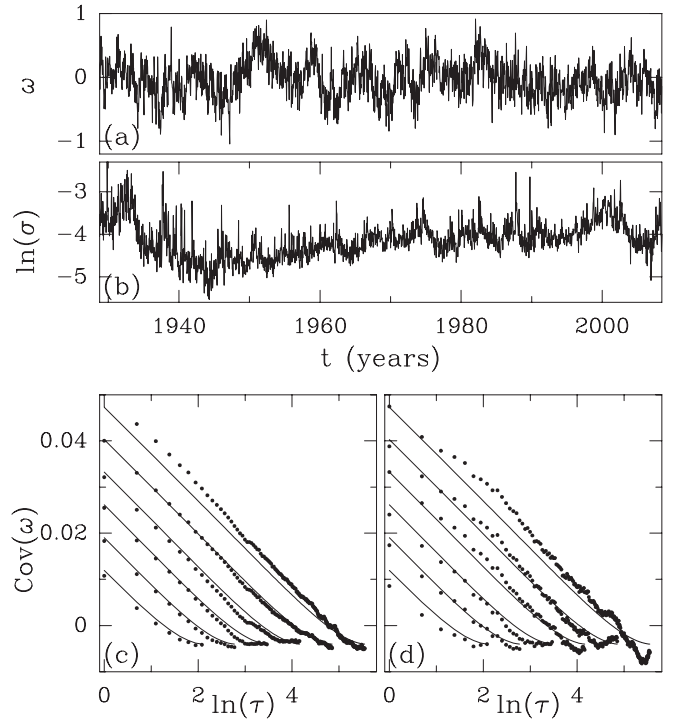


FIG. 6. (a) Sample path of $\omega_h(t)$ of length 2×10^4 where the numerical construction has been performed by sampling in both space and scale the conelike sets $C_\ell(t)$ for small ℓ in order to define $M(t)$. (b) Magnitude estimated as $\ln(\sigma(t))$ where $\sigma(t)$ is the daily range (difference between highest and lowest daily return values) associated with the Dow-Jones time series from 1929 to 2011. In panels (c) and (d) are reported the magnitude autocovariance estimation in semilogarithmic scale for different sample sizes $\Delta t = 16, 32, 64, 128, 256$, and 512 (from bottom to top). (c) Estimation from the model sample path in panel (a). (d) Estimation from the Dow-Jones daily data in panel (b). The solid lines represent the theoretical predictions from Eq. (32) with $\lambda^2 = 0.01$.

autocovariance of a multifractal process of integral scale $e^{-3/2} \Delta t$.

If we now go back to the nonstationary process $M(t)$, since we have shown that, over every interval $[t_0, t_0 + \Delta t]$, $M(t) \stackrel{\text{law}}{=} M_{t_0}(t)$, we can conclude that as soon as $t_0 > \Delta t$, the estimated autocovariance of $\omega_h(t) = \ln[M(t+h) - M(t)]$ will be provided by Eq. (32). In other words, for observations far from the time origin, the estimated integral scale is always (up to a constant factor) the overall sample size. This is illustrated in Fig. 6(c) where we have reported the estimation of the magnitude autocovariance for various sample lengths Δt . More precisely, we have generated a single large sample of the process $M(t)$ from which the magnitude time series $\omega_h(t)$ has been computed. This series (of overall size $L = 2 \cdot 10^4$) is displayed in Fig. 6(a). For each subinterval size $\Delta t = 16, 32, \dots, 512$, the sample is divided in $L/\Delta t$ subsamples of length Δt . The reported estimator $\widehat{C}_{\Delta t}(\tau)$ is the average of the obtained empirical covariances over all of the $L/\Delta t$ intervals. One can check in Fig. 6(c) that the theoretical predictions (32) (solid lines) are, for all Δt , in good agreement with the observations (\bullet) and one clearly observes an apparent integral scale that grows with Δt (as $e^{-3/2} \Delta t$).

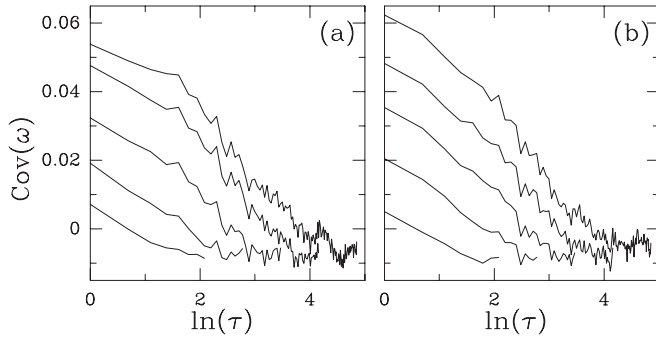


FIG. 7. Magnitude covariance estimation $\widehat{C}_{\Delta t}(\tau)$ as a function of $\ln(\tau)$ for respectively (a) CAC40 (5500 data points) and (b) FTSE 100 (7000 data points) stock index series. Each curve corresponds to a different sample size Δt used for the estimation ($\Delta t = 16, 32, 64, 128, 256$) and for each Δt , $\widehat{C}_{\Delta t}(\tau)$ has been obtained as the mean value over all available periods of size Δt . One clearly sees that the behavior is the same as that for the Dow-Jones index in Fig. 6(d): The integral scale (intercept) is growing as a function of Δt . The noise amplitude is greater because the overall sample sizes are smaller than for the Dow-Jones series.

B. Application to daily stock data

Let us now apply the previous analysis to real data. We report below the empirical results we obtained on three stock indices (namely the Dow-Jones, the CAC40, and the FTSE100 indices) over sufficient long time periods. In each case, $h = 1$ day and $\omega_h(k)$ at day k is estimated as $\omega_h(k) = \ln(\sigma(k))$, where $\sigma(k)$ is the relative range computed from highest and lowest stock values observed during the day k . The considered time periods are 1929–2011 for the Dow-Jones series (around 21 000 trading days), 1990 to 2011 for the CAC40 (around 5500 trading days), and 1984 to 2011 (around 7000 trading days) for the FTSE100.

In Fig. 6(b) is plotted the time series corresponding to the daily log volatility $\omega_h(k)$ of the Dow-Jones index. Very much like the model [Fig. 6(a)], this shows excursions away from

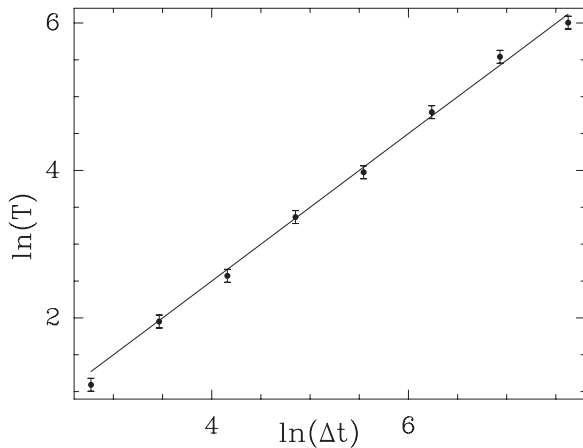


FIG. 8. Estimated integral scale T as a function of the sample size Δt (in log-log coordinates) for the Dow-Jones daily time series from 1929 to 2011. The solid line represents the value $e^{-3/2}\Delta t$ one expects theoretically. The reported error bars correspond to standard deviation of the empirical mean values estimated from the observed dispersion over all subintervals.

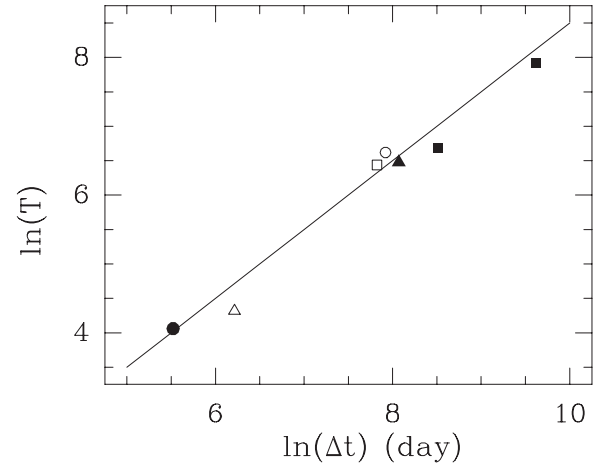


FIG. 9. Estimated integral scale T as a function of the sample length Δt (in log-log coordinates) gathered from several recent studies in the literature: (\bullet) from [8], (\circ) from [21], (Δ) from [9], (\blacktriangle) from [33], (\square) from [10], and (\blacksquare) from [15]. The solid line represents the model fit $\ln(T) = \ln(\Delta t) - 3/2$.

the mean value lasting for several years. For each of the three volatility series, we reproduced the same covariance estimation experiment we conducted for the model [Fig. 6(c)]. In Fig. 6(d) are reported the results obtained for the Dow-Jones index while in Fig. 7 are reported the results obtained for the CAC40 and the FTSE100 time series. Though these latter series have a smaller size and lead to more noisy results, it clearly appears in all cases that the empirical autocovariance functions are fairly well fitted by a multifractal logarithmic shape $\lambda^2 \ln(T/\tau)$ [i.e., they are linear as functions of $\ln(\tau)$] with a constant intermittency coefficient $\lambda^2 \simeq 0.01$. However, the apparent integral scale T (the intercept of each curve) appears to strongly depend on Δt . In Fig. 6(d), we see that the model predictions (solid lines), as described by Eq. (32), fit fairly well the data. This is confirmed in Fig. 8, where we have plotted (in log-log scale) the estimated integral scale as a function of the sample size Δt . One can see that the analytical prediction $T(\Delta t) = e^{-3/2}\Delta t$ (solid line) is in very good agreement with the empirical data. These results allow us to understand the origin of the wide range of integral scale values (from few months to several years) reported in the literature so far. This is illustrated in Fig. 9, where we have reported the estimated values of the integral scale T gathered from the recent literature [8–10,15,21,33]. Even if these studies concern various data sets at different time resolutions (intradays, daily, . . .) and different time periods and correspond to different asset classes (FX rates, stocks, . . .), we see that the reported values of T are spread closely around the theoretical curve (solid line in Fig. 9).

V. CONCLUSION AND PROSPECTS

To conclude, we have introduced a model of stochastic measure as the exponential of a nonstationary Gaussian $1/f$ noise. We have shown that over any finite time interval, provided the considered time t is large enough, this model can be hardly distinguished from a multifractal random cascade with an integral scale that is equal to the sample length. Our approach can be very appealing to model all phenomena where

multiscaling properties are observed without the existence of any natural large “correlation” (or “injection”) scale in space or time. For example, in finance, the agreement of the model predictions with the observed behavior of log-volatility correlation in various stock indices is striking. These findings suggest a peculiar (aging) nonstationary nature of volatility fluctuations. The question of the meaning of the time origin and the possibility of estimating this time from empirical data will have to be considered in future works. On a more general ground, the explanation of such nonstationarity is an important question that will have to be addressed from the market dynamical properties at the microstructural level but also within the framework of agent-based approaches including behavioral finance or theory of self-referencing dynamics of market prices. On a mathematical ground, it will be interesting to study this model and its possible variants in relationship with fractional Brownian motion, since it offers the possibility of giving meaning to the limit $H \rightarrow 0$. Finally, our approach can shed a new light in the field of $1/f$ noise modeling.

APPENDIX A: PROOF OF THE CONCAVITY OF $\zeta(q)$ AND THE EXISTENCE OF AN INTEGRAL SCALE

Let us prove that if Eq. (3) holds in the limit of small time scales τ , then (i) $\zeta(q)$ is a concave function of q and (ii) it necessarily involves a bounded scale T below which it can no longer be valid. We start by assuming that the following scaling holds in some range of scales:

$$\mathbb{E}[|\delta_\tau X(t)|^q] \sim C_q \tau^{\zeta(q)}.$$

Let $F(q, \tau) = \ln(\mathbb{E}[|\delta_\tau X(t)|^q])$. Then by Hölder inequality, $F(q, \tau)$ is, for each τ , a convex function of q . If one assumes it is regular enough so that its second derivative exists, one thus has $\forall \tau > 0$:

$$F''(q, \tau) \geq 0. \quad (\text{A1})$$

If the previous scaling law holds, this can be written as

$$\frac{d^2 \ln C_q}{dq^2} + \zeta''(q) \ln(\tau) \geq 0. \quad (\text{A2})$$

One sees that if the scaling holds in the limit $\tau \rightarrow 0$, this inequality can be true only if $\zeta''(q) \leq 0$; i.e., $\zeta(q)$ must be a concave function of q . If the scaling is also valid at scale $\tau = 1$ (up to a redefinition of τ we can always assume it is the case), C_q is the order q moment of the random variable $\delta_1 X(t)$ and $c_q = \frac{d^2 \ln C_q}{dq^2} \geq 0$. This means that

$$\ln(\tau) \leq \frac{c_q}{-\zeta''(q)}. \quad (\text{A3})$$

In other words, if $\zeta(q)$ is strictly concave (multifractal case), the scaling can hold only in a limited range of scales and there exists an integral scale

$$T = \inf_q \left(e^{\frac{-c_q}{\zeta''(q)}} \right)$$

above which it is not valid.

APPENDIX B: PROOF OF THE MEAN-SQUARE CONVERGENCE OF $M_\ell(t)$

Let us provide a direct proof of the mean-square weak convergence of $M_\ell(t)$ [or $M_\ell(I) = \int_I e^{\omega_\ell(u)} du$ for any given time interval I] as defined in Eq. (23) when $\ell \rightarrow 0$. For that purpose let us show that

$$\lim_{\ell, \ell' \rightarrow 0} \mathbb{E}[(M_\ell(t) - M_{\ell'}(t))^2] = 0. \quad (\text{B1})$$

Without loss of generality, we assume in the sequel that $\ell' \geq \ell$. Since

$$\begin{aligned} \mathbb{E}[(M_\ell(t) - M_{\ell'}(t))^2] &= \mathbb{E} \left[\left(\int_0^t (e^{\omega_\ell(u)} - e^{\omega_{\ell'}(u)}) du \right)^2 \right] \\ &= \int_0^t \int_0^t du dv \mathbb{E}[e^{\omega_\ell(u) + \omega_\ell(v)} + e^{\omega_{\ell'}(u) + \omega_{\ell'}(v)} \\ &\quad - 2e^{\omega_\ell(u) + \omega_{\ell'}(v)}] \end{aligned}$$

and since $(\omega_\ell, \omega_{\ell'})$ is a vector of correlated Gaussian processes, thanks to Eq. (24), $\mathbb{E}[(M_\ell(t) - M_{\ell'}(t))^2]$ reduces to

$$\int_0^t \int_0^t du dv [e^{C_{\ell, \ell}(u, v)} - e^{C_{\ell', \ell'}(u, v)}], \quad (\text{B2})$$

where we denoted $C_{\ell, \ell'}(u, v) = \text{Cov}(\omega_\ell(u), \omega_{\ell'}(v))$ and used the obvious property $C_{\ell, \ell'}(u, v) = C_{\ell', \ell}(u, v)$ if $\ell' \geq \ell$. Let us split the integral in three domains:

$$\int_0^t \int_0^t = \iint_{|u-v| \leq \ell} + \iint_{\ell \leq |u-v| \leq \ell'} + \iint_{|u-v| \geq \ell'}.$$

It is clear that in the last interval, $C_{\ell, \ell}(u, v) = C_{\ell', \ell'}(u, v) = \lambda^2 \ln(\frac{\max(u, v)}{|u-v|})$. The corresponding integral in Eq. (B2) is thus zero. In interval $\ell \leq |u-v| \leq \ell'$, thanks to expression (21), one has

$$\iint_{\ell \leq |u-v| \leq \ell'} (e^{C_{\ell, \ell}(u, v)} - e^{C_{\ell', \ell'}(u, v)}) du dv = O(\ell^{1-\lambda^2}),$$

while in the last interval

$$\iint_{|u-v| \leq \ell} e^{C_{\ell, \ell}(u, v)} - e^{C_{\ell', \ell'}(u, v)} = O(\ell^{1-\lambda^2}),$$

proving the mean square convergence (B1).

APPENDIX C: MAGNITUDE COVARIANCE ESTIMATION

Let us establish Eq. (32). Let us denote $\hat{C}(n) = \hat{C}_{\Delta t}(\tau = nh)$ and $C(n) = \lambda^2 \ln(T/nh)$ as the theoretical covariance as given by Eq. (10) at lag $\tau = nh$. By taking the expectation of expression (31) after expanding the expression of $\hat{\mu}$, one finds

$$\mathbb{E}[\hat{C}(n)] = C(n) + K(0) - 2K(n), \quad (\text{C1})$$

where

$$K(n) = \frac{1}{N(N-n)} \sum_{i=0}^{N-n-1} \sum_{j=0}^{N-1} C(|i-j|).$$

If h is small enough, one can replace the double sum by its integral approximation:

$$K(n = \tau/h) = \frac{\lambda^2}{\Delta t^2 (1 - \frac{\tau}{\Delta t})} \int_0^{\Delta t} \int_0^{\Delta t - \tau} du dv \ln \left(\frac{T}{|u - v|} \right).$$

Evaluating this integral leads, to the first order in $\tau/\Delta t$, to the expression

$$K(n = \tau/h) = \lambda^2 \left[\ln \left(\frac{T e^{3/2}}{\Delta t} \right) - \frac{\tau}{2\Delta t} \right].$$

By inserting this expression in Eq. (C1), one gets Eq. (32).

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