Singularity spectrum of multifractal functions involving oscillating singularities

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Abstract

We give general mathematical results concerning oscillating singularities and we study examples of functions composed only of oscillating singularities. These functions are defined by explicit coefficients on an orthonormal wavelet basis. We compute their Hölder regularity and oscillation at every point and we deduce their spectrum of oscillating singularities.

1 Introduction

Our purpose in this paper is two-fold. First we discuss the mathematical notion of *oscillating singularity* (or chirp). There is a general agreement on an informal definition: A chirp of type (h, β) at x_* should oscillate as

$$f(x) = |x - x_*|^h \sin\left(\frac{1}{|x - x_*|^\beta}\right) \tag{1}$$

in the neighborhood of x_*. In signal analysis, this notion should cover functions whose *instantaneous frequency* increases fast at some time (see [12]). A precise mathematical definition of chirps (Definition 1 below) was given by Yves Meyer (see [10] for instance). In Section 2 we discuss the mathematical definition of an oscillating singularity and we propose another definition which has several advantages. It is a natural generalization of (1) and it is stable with respect to the addition of "smoth noise", a feature which is mandatory for using this notion in signal processing, and which is not shared by Definition 1, as we will see. Our definition is based on the computation of pointwise Hölder exponents. We will thus prove some general results concerning Hölder exponents. This will unable us to associate to any function $F : \mathbb{R}^d \to \mathbb{R}$ two *oscillating singularity exponents* (h, β) at every point x_*.

Our second purpose is to define a family of functions which are multifractal in the following sense: the dimension $D(h, \beta)$ of the set $\mathcal{E}_{h,\beta}$ of points x_* where the oscillating singularity exponents are (h, β) is a function of two variables which we determine exactly for a large family of functions in Section 4. These functions are defined in Section 3.1 by explicit coefficients on an orthonormal wavelet basis. The computation of their oscillating singularity exponents (h, β) is performed in Sections 3.2 and 3.3.

This case study serves several purposes; it is the first explicit example of functions that are multifractal with only "chirp-type" singularities. Therefore it also shows that a "chirp" behavior is not necessarily something exceptional which happens only at isolated points or (at worse) on a countable set (the only function that was known to have chirps on a dense set was Riemann's function $\sum n^{-2} \sin(\pi n^2 x)$, see [10], which has chirps on a dense subset of the rationals). Finally the maxima of the wavelet transform of the functions we study satisfy the renormalizations described in

[1]; therefore these functions provide a validation of the multifractal formalism developed there. The computation of the *spectrum of oscillating singularities* $D(h,\beta)$ of these functions is performed in Section 4.

2 Oscillating singularity exponents

2.1 Chirps and oscillating singularities

A remarkable property of the function defined by (1) is that its primitive of order n (denoted f_n) has Hölder regularity $h+n(\beta+1)$ at x_* (the gain in regularity is not 1 at each integration, as might be expected, but $\beta+1$). A *chirp at* x_* is a function which has the same kind of oscillatory behavior near x_*, and in [10], this property is used in order to define the type (h,β) of a chirp as follows.

Definition 1 *Let $h \geq 0$ and $\beta > 0$. A function F bounded in a neighborhood $[x_*-\delta, x_*+\delta]$ of x_* is a generalized chirp of type (h,β) if the iterated primitives $F_1,\dots,F_n$ of F (normalized by $F_k(x_*)=0$) satisfy*

$$|F_n(x)| \leq C_n|x-x_*|^{h+n(\beta+1)} \quad \textit{for} \quad -\delta \leq x-x_* \leq \delta. \tag{2}$$

When (2) holds, F can be written

$$F(x) = |x-x_*|^h g_\pm\left(\pm\frac{1}{|x-x_*|^\beta}\right)$$

(where $\pm$ stands for the sign of $x-x_*$); here g_+ and g_- are *indefinitely oscillating*, i.e. all their primitives are bounded.

Such functions can be characterized by size estimates of their continuous wavelet transform. We will need a wavelet ψ in the Schwartz class (i.e. C^∞ with all derivatives having fast decay:

$$\forall m,n \in \mathbb{N} \;\; |\psi^{(n)}(x)| \leq \frac{C_{m,n}}{(1+|x|)^m}$$

and all moments vanishing: $\forall n \in \mathbb{N}$, $\int \psi(x)x^n dx = 0$.

The continuous wavelet transform of the function F is

$$C(a,b) = \frac{1}{a}\int F(x)\psi(\frac{x-b}{a})dx.$$

If g_+ and g_- are $C^r(\mathbb{R})$, the functions satisfying Definition 1 are characterized by the following decay conditions of their wavelet transform in the neighborhood of x_*: $\forall m \geq 0$

$$\left.\begin{array}{lll} \text{If } |b-x_*| \leq a & & |C(a,b)| \leq C_m a^m \\ \text{If } |b-x_*|^{1+\beta} \leq a \leq |b-x_*| & & |C(a,b)| \leq C_m|b-x_*|^h\left(\dfrac{|b-x_*|^{1+\beta}}{a}\right)^m \\ \text{If } |b-x_*|^{1+\beta} \geq a & & |C(a,b)| \leq C|b-x_*|^h\left(\dfrac{a}{|b-x_*|^{1+\beta}}\right)^r \end{array}\right\} \tag{3}$$

One immediately meets some difficulties when using Definition 1 for experimental data. Indeed Definition 1 is not stable when one adds to F a function G which is arbitrarily smooth, but not C^∞. Thus $x\sin(1/x)$ is a chirp of type $(1,1)$ while $x\sin(1/x)+|x|^h$ (h real >1 and $h$is not an even integer) is no more a chirp, even if h is chosen arbitrarily large (or, more precisely, it is a chirp of type $(1,0)$). Furthermore, the characterization (3) requires ψ to be C^∞ with all moments vanishing, a mandatory requirement which is hard to meet in real data analysis. Let B be a brownian motion; another example is the function (see Fig. 1)

$$f(x) = x^{0.3}\sin(1/x) + B(x); \tag{4}$$

the strongest singularity at 0 is the chirp $x^{0.3}\sin(1/x)$, and one actually observes this oscillatory behavior after magnifying enough the graph near the origin. Nonetheless, the oscillations are not

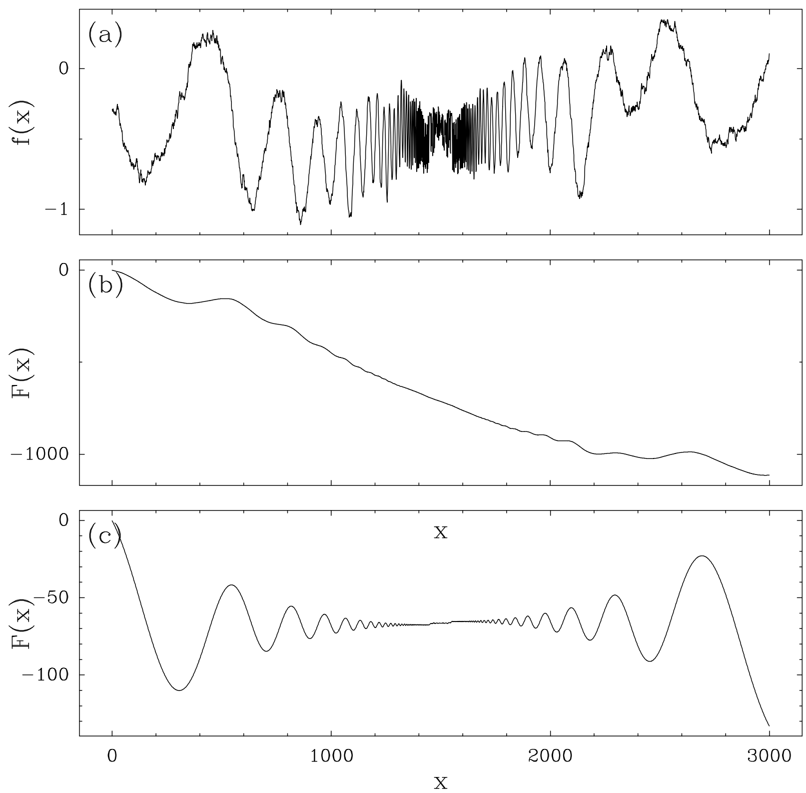

Figure 1: (a) Graph of the function $f(x)$ defined in (4). The chirp behavior appears clearly. (b) Graph of the primitive of $f(x)$. The chirp behavior has disappeared. Compare with (c) which corresponds to the graph of the primitive of $x^{0.3}\sin(1/x)$ (no noise added)

reflected in the chirp exponents since this function is a chirp of type $(0.3, 0)$ at the origin. These examples show that the oscillation exponent β, if defined as above, is a very instable quantity.

Let us now show how we can avoid this drawback, and introduce a slightly different definition of *oscillating singularities* which agrees with the definition of a chirp for functions such as (1), and has the required stability properties with respect to the addition of "smooth noise". Remark that if

$$F(x) = |x - x_*|^h g_\pm \left(\frac{1}{|x - x_*|^\beta} \right) + O\left(|x - x_*|^{h'} \right) \tag{5}$$

where $h' > h$, the first term describes the local behavior of F near x_*. In that case, we would like to say that the type of oscillating singularity at x_* is (h, β), i.e., the oscillation of F should be the oscillation of the lowest order term of its expansion. Clearly, we no more need to require that g_+ or g_- have more than one vanishing moment, since after one integration, the main term of the primitive of F may be the remaining term, which is $O(|x - x_*|^{h'+1})$. This last remark shows that, in sharp contrast with the definition of chirps, the oscillation exponent β of an oscillating singularity should not be determined using a large number of integrations, or even one integration.

Let $h_t(x_*)$ denote the Hölder exponent of the fractional primitive of order t at x_* of the function F defined by (5). More precisely, if F is a bounded function, we denote by $h_t(x_*)$ the Hölder exponent at x_* of the function

$$F_t = (Id - \Delta)^{-t/2}(\phi F) \tag{6}$$

where ϕ is a C^∞ compactly supported function satisfying $\phi(x_*) = 1$, and the operator $(Id - \Delta)^{-t/2}$ is the convolution operator which amounts to multiply the Fourier transform of the function with $(1 + |\xi|^2)^{-t/2}$. If one performs a fractional integration of order t small enough, i.e. such that

$$h + (1 + \beta)t < h' + t, \tag{7}$$

then $h_t(x_*) = h + (1+\beta)t$ (here h, h' and β are the exponents used in (5)). We see that the gain of pointwise Hölder regularity at x_* after a fractional integration of very small order t is $(1+\beta)t$; hence the following definition for exponents of oscillating singularities.

Definition 2 *Let $F : \mathbb{R}^d \to \mathbb{R}$ be a bounded function. The oscillating singularity exponents of F at a point x_* are defined by*

$$(h, \beta) = \left(h(x_*),\ \frac{\partial}{\partial t} h_t(x_*)\Big|_{t=0} - 1 \right) \tag{8}$$

These exponents belong to $[0, +\infty] \times [0, +\infty]$.

Remarks: This definition makes sense because, for a given x_*, the function $t \to h_t(x_*)$ is differentiable (with a possible derivative of $+\infty$), as we will show.
Note that if $h_t(x_*) = +\infty$ the exponent β is not defined.

2.2 Determination of oscillating exponents

Because of Definition 2, in order to determine the exponent of oscillating singularity at every point, we have to be able to compute the pointwise Hölder regularity of a function. This is made possible by the following proposition of [8], which relates pointwise regularity with decay conditions of the wavelet coefficients.

Proposition 1 *Let $F : \mathbb{R}^d \to \mathbb{R}$ be a bounded function and let $C(a,b)$ be the wavelet transform of F. If F is $C^h(x_*)$,*

$$|C(a,b)| \leq C a^h \left(1 + \frac{|b - x_*|}{a} \right)^h. \tag{9}$$

Conversely suppose that there exists $\varepsilon > 0$ such that $F \in C^\varepsilon(\mathbb{R}^d)$. If (9) holds, there exists a polynomial P of degree at most $[h]$ such that, if $|x - x_*| \leq 1/2$,

$$|F(x) - P(x - x_*)| \leq C|x - x_*|^h |\log(|x - x_*|)|. \tag{10}$$

In the following, the point x_* is fixed, and we suppose that there exists $\varepsilon > 0$ such that $F \in C^\varepsilon(\mathbb{R}^d)$.

We introduce some notations that are fitted to our study. The first one is a weak form of the $\mathcal{O}$ notation of Landau, and the second one expresses the fact that two functions are of the same order of magnitude, disregarding "logarithmic corrections".

Definition 3 *If F and G are two functions, $F = \overline{\mathcal{O}}(G)$ if*

$$\liminf \frac{\log |F|}{\log |G|} \geq 1,$$

and $F \sim G$ if

$$\lim \frac{\log |F|}{\log |G|} = 1.$$

Let $h = h_0(x_*)$; the following corollary is a direct consequence of the definition of the Hölder exponent and of Proposition 1.

Corollary 1 *Suppose that $F \in C^\varepsilon(\mathbb{R}^d)$ for an $\varepsilon > 0$. The Hölder exponent of F at x_* is h if and only if the two following conditions hold:*

- *In the neighborhood of $(a,b) = (0, x_*)$*

$$|C(a,b)| = \overline{\mathcal{O}}(a^h + |b - x_*|^h). \tag{11}$$

- *There exists a sequence $(a_n, b_n) \to (0, x_*)$ satisfying*

$$|C(a_n, b_n)| \sim a_n^h + |b_n - x_*|^h. \tag{12}$$

We will call such a sequence (a_n, b_n) a *minimizing sequence* for F at x_*. Define the *oscillation exponent*

$$\beta(t) = \sup(0, \inf\{\beta : \exists \text{ a minimizing sequence for } F^t \text{ in the domain } a \geq |b - x_*|^{1+\beta}\})$$

(recall that $\inf \emptyset = +\infty$).

Proposition 2 *Let $x_* \in \mathbb{R}^d$. The function $t \to h_t(x_*)$ is concave and its right derivative at 0 is $1 + \beta(0)$.*

This proposition wil be proved in several steps.

Lemma 1 *If $t' \geq t$,*

$$h_{t'}(x_*) \leq h_t(x_*) + (1 + \beta(t))(t' - t). \tag{13}$$

Proof of Lemma 1: Let $C_t(a, b)$ be the wavelet transform of F_t. Using a Fourier transform of the formula which defines the continuous wavelet transform, we see that a wavelet transform of $F_{t'}$ is $a^{t'-t} C_t(a, b)$ (but using the new wavelet $\psi^{t'-t}$, where $\psi^u = (-\Delta)^{u/2}\psi$). We can suppose $\beta(t) > 0$ because if $\beta(t) = 0$, one immediately checks that

$$h_{t'}(x_*) = h_t(x_*) + (t' - t). \tag{14}$$

Thus there exists a minimizing sequence such that

$$a_n \sim |b_n - x_*|^{1+\beta(t)} \text{ and } C_t(a_n, b_n) \sim |b_n - x_*|^{h_t(x_*)}.$$

Thus

$$a_n^{t'-t} C_t(a_n, b_n) \sim |b_n - x_*|^{h_t(x_*)+(1+\beta(t))(t'-t)}$$

hence Lemma 1.

Lemma 2 *The function $t \to \beta(t)$ is decreasing.*

Proof of Lemma 2: As before, let $C_t(a, b)$ be the wavelet transform of F_t. Since $a^{t'-t} C_t(a, b)$ is a wavelet transform of $F_{t'}$, we pick a sequence (a_n, b_n) minimizing for $F_{t'}$ such that

$$\left.\begin{aligned} a_n &\sim |b_n - x_*|^{1+\beta(t')} \\ a_n^{t'-t} |C_t(a_n, b_n)| &\sim |b_n - x_*|^{h_{t'}(x_*)} \end{aligned}\right\} \tag{15}$$

If $\beta(t') > \beta(t)$, the sequence (a_n, b_n) is not minimizing for F_t so that

$$\exists \varepsilon > 0 : \quad |C_t(a_n, b_n)| \leq |b_n - x_*|^{h_t(x_*)+\varepsilon}$$

which implies

$$a_n^{t'-t} |C_t(a_n, b_n)| \leq |b_n - x_*|^{h_t(x_*)+\varepsilon+(1+\beta(t')))(t'-t)}$$

so that

$$h_t(x_*) + \varepsilon + (1 + \beta(t'))(t' - t) \leq h_t(x_*) + (1 + \beta(t))(t' - t).$$

which, together with (13) implies

$$h_t(x_*) + \varepsilon + (1 + \beta(t'))(t' - t) \leq h_t(x_*) + (1 + \beta(t))(t' - t);$$

hence a contradiction.

Lemma 3 *The following limit holds*

$$\lim_{t \to 0} \frac{h_t(x_*) - h_0(x_*)}{t} = \beta(0) + 1.$$

Proof of Lemma 3: Lemma 1 implies that the limsup of the left-hand side is at most $\beta(0)+1$.

If $\beta(0)=0$ we have nothing to prove because of (14). Suppose that $\beta(0)>0$ and let $0<\gamma<\beta(0)$. By definition of $\beta(0)$, $\exists\varepsilon>0$ such that the wavelet transform $C(a,b)$ of F satisfies

$$|C(a,b)| \le C a^{h_0(x_*)+\varepsilon}\left(1+\frac{|b-x_*|}{a}\right)^{h_0(x_*)+\varepsilon}$$

in the domain $D = a \ge |b-x_*|^{1+\gamma}$ (else there would exist a minimizing sequence in D). Outside this domain,

$$|C(a,b)| = \overline{\mathcal{O}}(|b-x_*|^{h_0(x_*)}).$$

Thus $a^t C(a,b)$ satisfies in D

$$\begin{aligned} a^t C(a,b) &\le C a^{h_0(x_*)+\varepsilon+t}\left(1+\frac{|b-x_*|}{a}\right)^{h_0(x_*)+\varepsilon} \\ &\le C a^{h_0(x_*)+\varepsilon+t}\left(1+\frac{|b-x_*|}{a}\right)^{h_0(x_*)+\varepsilon+t} \end{aligned}$$

and outside D

$$a^t C(a,b) = \overline{\mathcal{O}}(a^t|b-x_*|^{h_0(x_*)}) = \overline{\mathcal{O}}(|b-x_*|^{h_0(x_*)+t(1+\gamma)})$$

so that

$$h_t(x_*) \ge h_0(x_*) + \inf(t(1+\gamma), t+\varepsilon);$$

thus, if t is small enough,

$$h_t(x_*) \ge h_0(x_*) + t(1+\gamma);$$

hence the lower bound in Lemma 3.

Of course, replacing F by F^t, Lemma 3 implies that $h_t(x_*)$ is for each t right-differentiable of derivative $1+\beta(t)$.

This lemma shows that the two definitions for $\beta(t)$ that we gave in (8) and in this section coincide; furthermore, Lemma 3 and Lemma 2 together imply that $h(t)$ is a concave function, hence Proposition 2. The following proposition gives a characterization of the oscillating singularity exponents at x_* of a function $F \in C^\varepsilon$. It is a direct consequence of the previous results.

Proposition 3 *Let $F \in C^\varepsilon(\mathbb{R}^d)$ for an $\varepsilon > 0$. The oscillating singularity exponents at x_* are (h,β) if and only if the wavelet transform of F satisfies the following conditions.*

- $|C(a,b)| = \overline{\mathcal{O}}(a^h + |b-x_*|^h)$ *in the neighborhood of* $(a,b)=(0,x_*)$
- *there exists a sequence* $(a_n,b_n) \to (0,x_*)$ *such that*

$$(a_n + |b_n - x_*|)^{1+\beta} \sim a_n \quad \text{and} \quad |C(a_n,b_n)| \sim a_n^h + |b_n - x_*|^h \tag{16}$$

- β *is the smallest number such that (16) holds.*

We denote by $\mathcal{E}_{h,\beta}$ the set of points where the oscillating singularity exponents are (h,β), and by $D(h,\beta)$ the Hausdorff dimension of $\mathcal{E}_{h,\beta}$.

3 Construction and analysis of functions with oscillating singularities

3.1 Construction of the multifractal function F

We now construct a family of functions which have oscillating singularities on fractal sets, and we determine their oscillating singularity exponents.

This construction is performed using an orthonormal wavelet basis. We define

$$\psi_{j,k}(x) = \psi(2^j x - k),\ j,k \in \mathbf{Z}$$

where the wavelet ψ is C^∞ and in the Schwartz class (see [11]). Note that we use an L^∞ normalization for wavelets, which is convenient to study Hölder regularity.

We denote by B $(= B_{j,k})$ the dyadic interval $[k2^{-j}, (k+1)2^{-j})$ where the wavelet is centered. We will use these intervals to index the wavelets, so that ψ_B is the wavelet "centered " on the interval B. Because of the normalization we chose for wavelets, the wavelet coefficients are given by

$$C_B = 2^j \int F\psi_B(x)dx.$$

Consider now the usual construction of a Cantor set:

- At level 0, we start with the interval $A_\emptyset = [0,1]$
- At level 1, we construct s disjoint intervals $A_1, \ldots, A_s$ included in $[0,1]$, where $A_k = [x_k, y_k)$ has length $|A_k| = y_k - x_k = \nu_k$, and $x_1 = 0$ and $y_s = 1$.
- At level n, we have s^n subintervals A_i indexed as usual by an n-tuple $i = (i_1, \ldots, i_n) \in \{1, \ldots s\}^n$; $A_i = [x_i, y_i)$ and the length of A_i is

$$|A_i| = \nu_i = \nu_{i_1} \ldots \nu_{i_n}.$$

We denote by K the Cantor set

$$K = \bigcap_{n\in\mathbb{N}} \left(\bigcup_{i\in\{1,\ldots s\}^n} A_i \right).$$

We pick s ratios for the scales $\lambda_1, \ldots, \lambda_s$ of the intervals indexing the wavelets such that

$$\forall k = 1, \ldots, s, \quad \lambda_k \le \nu_k;$$

then, we construct on each interval A_i a dyadic subinterval B_i of length

$$|B_i| = \lambda_i = C(B_i)\lambda_{i_1} \ldots \lambda_{i_n}$$

where $C(B_i) \in [1/2, 1)$ is chosen such that λ_i is a power of 2.

We still have to define the exact position of the wavelets and the size of the wavelet coefficients. For that we pick $\eta \in [0,1]$ and the interval B_i is the unique dyadic interval of length λ_i satisfying the condition

$$x_i + \eta\nu_i \in B_i$$

(it is thus a subinterval of A_i). In order to define the wavelet coefficients, we pick s ratios $\mu_1, \ldots, \mu_s$ in $[0,1)$, and the corresponding wavelet coefficient of the wavelet indexed by B_i is

$$\mu_i = \mu_{i_1} \ldots \mu_{i_n}.$$

The function F is completely determined by setting to zero all other wavelet coefficients.

The determination of the uniform Hölder regularity of F is immediate: Since all wavelet coefficients μ_i satisfy

$$\frac{\log \mu_i}{\log \lambda_i} \ge \inf_{l=1,\ldots,s} \frac{\log \mu_l}{\log \lambda_l},$$

and this estimate is best possible, F is $C^\varepsilon(\mathbb{R})$ with

$$\varepsilon = \inf_{l=1,\ldots,s} \frac{\log \mu_l}{\log \lambda_l}.$$

And ε is the best possible uniform regularity Hölder exponent.

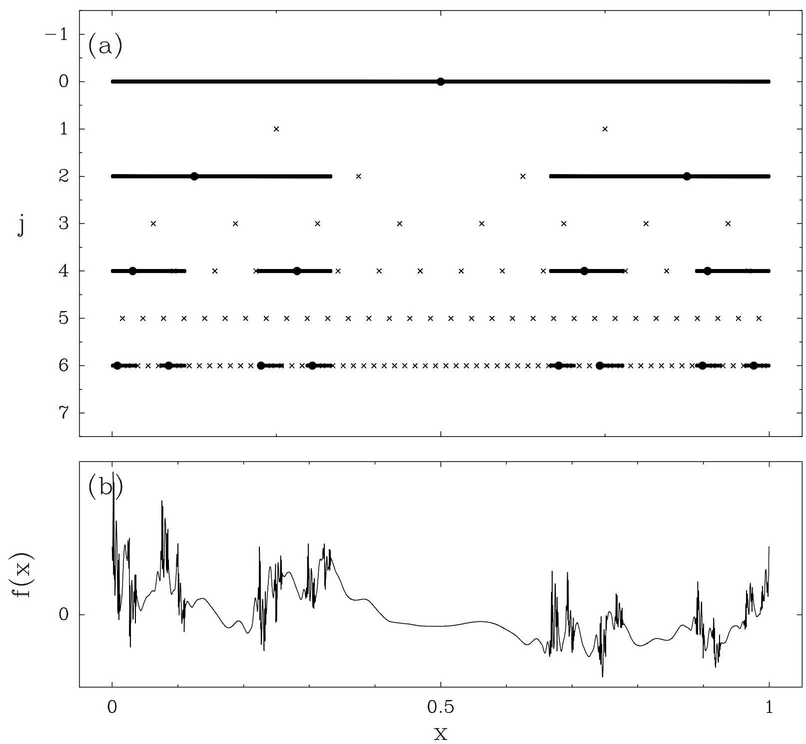

Figure 2: (a) Construction of the function F on the dyadic grid in the case $s = 2$, $\nu_1 = \nu_2 = 1/3$, $\lambda_1 = \lambda_2 = 1/4$, $\mu_1 = 0.7$ and $\mu_2 = 0.8$ (symbols ($\bullet$) are for non zero coefficients). In this case K is the triadic Cantor set. (b) Synthesis of the above decomposition using the Daubechies 9 wavelet.

3.2 Pointwise regularity and oscillation of F.

Note first that if $x_* \notin K$, there exists a neighborhood of x_* which includes no dyadic interval indexing a nonvanishing wavelet coefficient of scale smaller than a certain j_0; F is thus C^∞ in a neighborhood of x_*, and we only have to consider the points of the Cantor set K.

If x_* belongs to K it is indexed by an infinite sequence $i \in \{1, \ldots, s\}^{\mathbb{N}}$. Let us recall a few notations introduced in [9]. The set of indexes i is interpreted as a tree and a *branch above* x_* is a sequence i such that

$$dist(x_*, A_i) \leq 2\nu_i. \tag{17}$$

Using Proposition 1 , we see that we have to determine

$$h(x_*) = \liminf \frac{\log \mu_i}{\log[\lambda_i + dist(x_*, B_i)]}. \tag{18}$$

In the remaining of this section and in the next, we make the following hypothesis:

$$\textbf{(H)} \quad \eta \notin K \ .$$

Note that if $\eta \in K$ and $(\lambda_1, \ldots, \lambda_s) = (\nu_1, \ldots, \nu_s)$, the proofs will be the same and we will deduce the same results. However we won't consider this case here since it is exactly analogous to the construction already considered in [9].

Let $x_* \in K$ and let i be a finite sequence. Since $\eta \notin K$,

$$dist(x_*, B_i) \sim |A_{i'}| \tag{19}$$

where the sequence i' is defined as follows: If x_* is indexed by the sequence $(j_1, \ldots j_n, \ldots)$, let

$$l = \sup\{n : \ \forall m \leq n \ \ j_m = i_m\};$$

then

$$i' = (i_1, \dots, i_l). \tag{20}$$

Thus if Hypothesis (H) holds, $\lambda_i + dist(x_*, B_{i'}) \sim \nu_{i'}$ so that (18) becomes

$$h(x_*) = \liminf \frac{\log \mu_i}{\log \nu_{i'}} \tag{21}$$

We now distinguish two contributions in the liminf in (18): the first one corresponds to the wavelets indexed by a branch above x_*, denoted by $h_{1,t}$ (and we will show that this contribution is the main one), and the second is the contribution of all other wavelet coefficients, denoted by $h_{2,t}$.

The liminf in (18), when restricted to branches $i = (i_1, \dots i_n)$ above x_*, becomes

$$h_{1,t} = \liminf \frac{\log \mu_i}{\log \nu_i} \tag{22}$$

where the lim inf is taken on all branches above x_*.

After a fractional integration of order t, μ_i becomes $\lambda_i^t \mu_i$. More precisely, this is the wavelet coefficient of $(-\Delta)^{t/2} F$ on a biorthogonal wavelet basis, for which the criteria of pointwise Hölder regularity are the same as for orthonormal bases, see [10]. Thus

$$h_{1,t} = \liminf \frac{\log \mu_i + t \log \lambda_i}{\log \nu_i} \tag{23}$$

We now estimate the contribution of the other wavelet coefficients.

Let $i = (i_1, \dots, i_n)$ be an index which does not belong to the branch above x_*. We define the subindex $i' = (i_1, \dots, i_l)$ as in (20). Thus

$$\left.\begin{array}{l} dist(x_*, B_i) \sim |A_{i'}| \\ \\ \lambda_i = \lambda_{i'} \displaystyle\prod_{k=l}^{n} \lambda_{i_k}, \qquad \mu_i = \mu_{i'} \displaystyle\prod_{k=l}^{n} \mu_{i_k} \end{array}\right\} \tag{24}$$

so that

$$\frac{\log |\mu_i|}{\log dist(x_*, B_i)} \geq \frac{\log |\mu_{i'}|}{\log dist(x_*, B_{i'})}$$

and the contribution of the sequence i to the liminf in (18) is larger than the contribution of i'. Thus we have only to consider the contribution of $h_{1,t}$ in the computation of the oscillating singularity exponents.

In order to determine the value of $h_{1,t}$, we introduce the two following notions.

Definition 4 *Let (a_n) be a sequence of real numbers. A minimizing subsequence for (a_n) is a subsequence n_k such that*

$$\liminf_{n \to \infty} a_n = \lim_{n \to \infty} a_{n_k}. \tag{25}$$

We denote by $S(a_n)$ the set of all minimizing subsequences for (a_n); and if (a_n) and (b_n) are sequences, we define

$$\liminf_{(a_n)} b_n = \inf_{\{(n_k) \in S(a_n)\}} \liminf b_{n_k} \tag{26}$$

where the infimum is taken on all minimizing subsequences (n_k) for (a_n).

For instance if $a_n = (-1)^n$, $\liminf_{(a_n)} b_n = \liminf b_{2n+1}$.

Lemma 4 *The derivative at the origin of the function h_t is given by*

$$\lim_{t \to 0} \frac{h_t(x_*) - h_0(x_*)}{t} = \liminf_{(\log \mu_i / \log \nu_i)} \frac{\log \lambda_i}{\log \nu_i} (= \beta + 1). \tag{27}$$

Remarks:

- The difference with the determination of chirp parameters appears clearly here. If we were looking for the parameter β of the chirp at x_*, we would look for

$$\lim_{t\to\infty} \frac{h_t(x_*)}{t}$$

and for t large, the order of magnitude of (23) is

$$t \liminf \frac{\log \lambda_i}{\log \nu_i}$$

so that

$$\lim_{t\to\infty} \frac{h_t(x_*)}{t} = \liminf \frac{\log \lambda_i}{\log \nu_i}$$

which is, as expected, smaller than (27).

- We define the *regular points* of the Cantor set K as the points indexed by a sequence $i = (i_1, \dots, i_n, \dots)$ such that

$$\frac{Card\{i_l :\ l \le n \ \text{ and } \ i_l = k\}}{n}$$

has a limit for each $k = 1, \dots s$. Then for regular points the chirp parameters and the oscillating singularity parameters coincide.

Lemma 4 is a direct consequence of the following lemma.

Lemma 5 *Let (a_n) and (b_n) be two sequences in l^∞ and*

$$\phi(t) = \liminf (a_n + t b_n).$$

Near the origin $t = 0$,

$$\phi(t) = \liminf a_n + t \liminf_{(a_n)} b_n + o(t).$$

Proof of Lemma 5: Let $A = \liminf a_n$, $B = \liminf b_n$, $B^* = \liminf_{(a_n)} b_n$,

$$E^\varepsilon = [A - \varepsilon, A + \varepsilon]$$

and

$$B^\varepsilon = \liminf_{a_n \in E^\varepsilon} b_n.$$

Clearly

$$A + Bt \le \phi(t) \le A + B^* t$$

and

$$\forall \varepsilon > 0 \ \ B \le B^\varepsilon \le B^*$$

For all $\varepsilon > 0$ there exists $t(\varepsilon)$ such that if $t \le t(\varepsilon)$, there exists a minimizing subsequence n_k of $a_n + t b_n$ verifying $a_{n_k} \in E^\varepsilon$ for k large enough. Indeed, if it were not the case, $a_{n_k} \ge A + \varepsilon$ so that

$$a_{n_k} + t b_{n_k} \ge A + \varepsilon - t \parallel b \parallel_{l^\infty}$$

hence a contradiction as soon as $t < \frac{\varepsilon}{2} \parallel b \parallel_{l^\infty}$. Since $a_{n_k} \in E^\varepsilon$, we obtain that if $t \le t(\varepsilon)$,

$$\phi(t) \ge A + B^\varepsilon t.$$

Thus, in order to prove Lemma 5, we have to show that $B^\varepsilon \to B^*$ when $\varepsilon \to 0$. If it were not the case, there would exist $\alpha > 0$ such that $\forall n\ \exists \omega(n)$ such that $\forall m \ge \omega(n)$

$$b_m \in E^{1/n} \ \text{ and } \ b_m \le B^* - \alpha.$$

Finally, we pick an increasing sequence $\theta(n) \ge \omega(n)$; the sequence $\theta(n)$ is minimizing a_n and satisfies $b_{\theta(n)} \le B^* - \alpha$ hence a contradiction, and Lemma 5 is proved.

We have thus obtained the following proposition.

Proposition 4 *Under the Hypothesis (H), let $x_* \in K$; F has at x_* the oscillating singularity exponents*

$$(h, \beta) = \left(\liminf \left[\frac{\log \mu_i}{\log \nu_i} \right], \liminf_{(\log \mu_i / \log \nu_i)} \left[\frac{\log \lambda_i}{\log \nu_i} \right] - 1 \right) \tag{28}$$

where the liminf is taken on all branches above x_*.

4 Spectrum of oscillating singularities of F

Recall that $\mathcal{E}_{h,\beta}$ is the set of points where the oscillating singularity exponents of F are (h,β). We will use the following proposition (Proposition 4.9 of [6]) in order to compute Hausdorff dimensions of the $\mathcal{E}_{h,\beta}$. We denote by $B(x,r)$ the ball centered at x of radius r.

Proposition 5 *Let $\mathcal{H}^d$ be the d-dimensional Hausdorff measure. Let m be a probability measure on $\mathbb{R}$, $\mathcal{E} \subset \mathbb{R}$ and let C be a constant such that $0 < C < \infty$.*

- *If $\forall x \in \mathcal{E}$ $\limsup_{r\to 0} \dfrac{m(B(x,r))}{r^d} < C$ then $\mathcal{H}^d(\mathcal{E}) \geq \dfrac{m(F)}{C}$.*
- *If $\forall x \in \mathcal{E}$ $\limsup_{r\to 0} \dfrac{m(B(x,r))}{r^d} > C$ then $\mathcal{H}^d(\mathcal{E}) \leq \dfrac{2^d}{C}$.*

We define the function ρ by

$$\sum_{l=1}^{s} \lambda_l^a \mu_l^b \nu_l^{-\rho(a,b)} = 1. \tag{29}$$

We will prove the following theorem.

Theorem 1 *The Hausdorff dimension $D(h,\beta)$ of the set $\mathcal{E}_{h,\beta}$ is given by*

$$D(h,\beta) = \inf_{a,b} \left(a(\beta+1) + bh - \rho(a,b) \right). \tag{30}$$

Proof of Theorem 1 : let $a, b \in \mathbb{R}$, $c = -\rho(a,b)$ and $P_l = \lambda_l^a \mu_l^b \nu_l^c$; thus $\sum_{l=1}^{s} P_l = 1$. We first consider on K a probability measure m such that

$$\forall (i_1, \dots, i_n),\ m(S_{i_1} \circ \dots \circ S_{i_n}(K)) = P_{i_1} \dots P_{i_n}. \tag{31}$$

where S_l is the linear mapping that maps the interval $[0,1]$ on the interval A_l. The construction of such a measure by induction is straightforward (see [4] or [5] for instance). Let $x \in K$, $r > 0$; then

$$\frac{m(B_r(x))}{r^d} \geq C \sup \frac{\lambda_i^a \mu_i^b \nu_i^c}{\nu_i^d}$$

where the sup is taken on all branches above x such that

$$\nu_{i_1} \dots \nu_{i_n} \leq r < \nu_{i_1} \dots \nu_{i_{n-1}}.$$

Suppose that the oscillating singularity exponents of F at x are (h,β). Then, using Proposition 4, there exists branches above x such that

$$\mu_i \sim \nu_i^h \quad \text{and} \quad \lambda_i \sim \nu_i^{\beta+1};$$

so that, for such branches

$$\nu_i^{a(\beta+1)+bh-\rho(a,b)-d} = \overline{\mathcal{O}} \left(\frac{m(B_r(x))}{r^d} \right)$$

and

$$\frac{m(B_r(x))}{r^d} \to +\infty$$

if $a(\beta+1) + bh - \rho(a,b) - d < 0$. Using Proposition 5, it implies that

$$D(h,\beta) \leq a(\beta+1) + bh - \rho(a,b)$$

hence the upper bound in Theorem 1.

In order to end the proof, we have to show that the infimum is reached ; using Proposition 5, it is sufficient to find a, b and c such that $m(\mathcal{E}_{h,\beta}) > 0$.

Suppose that a, b, c are solution of the following system

$$\left.\begin{array}{l} \displaystyle\sum_{l=1}^{s} \lambda_l^a \mu_l^b \nu_l^c = 1 \\[2ex] \displaystyle\frac{\sum P_l \log \mu_l}{\sum P_l \log \nu_l} = h, \quad \frac{\sum P_l \log \lambda_l}{\sum P_l \log \nu_l} = \beta + 1 \\[2ex] \text{where} \quad P_l = \lambda_l^a \mu_l^b \nu_l^c \end{array}\right\} \tag{32}$$

(note that there always exist couples (h, β) such that (32) has solutions).

If $(i_1, \ldots, i_n)$ is a branch above x, let $(n_j)_{j=1\ldots d}$ be the proportion of $j's$ in the sequence $i_1, \ldots, i_n$ and let $\mathcal{F}_{h,\beta}$ be the subset of K composed of the regular points x such that

$$n_j \longrightarrow P_j \tag{33}$$

(meaning here that $\forall \varepsilon > 0, \exists n : \ \forall n' \geq n$ if $(i_1, \ldots, i_{n'})$ is a branch above x, then $|n_j - p_j| \leq \varepsilon \quad \forall j = 1, \ldots, s$ for this branch).

Clearly, $\mathcal{F}_{h,\beta} \subset \mathcal{E}_{h,\beta}$. If the corresponding probability m defined by (31) is put on K, $m(\mathcal{F}_{h,\beta}) = 1$. This is true because we can interpret K as the set of symbolic sequences $(i_n, \ldots i_n, \ldots) \in \{1, \ldots s\}^{\mathbb{N}}$; under the probability m, the random variables i_n are i.i.d.. Thus, by the law of large numbers, under m, almost every sequence satisfies $n_j \rightarrow P_j$, which exactly means that $m(\mathcal{F}_{h,\beta}) = 1$. Thus, applying Proposition 5,

$$dim(\mathcal{E}_{h,\beta}) \geq dim F_{h,\beta} = a(\beta + 1) + bh - \rho(a, b)$$

for (a, b) solution of (32). Hence Theorem 1.

let us now give some precisions concerning the support of $D(h, \beta)$:

- If $s = 2$, there always exist a linear combination of h and β which is constant, so that the spectrum $D(h, \beta)$ is supported by a segment of line.
- Let S be the subset of $\mathbb{R}^s$ of s-uples $(p_1, \ldots, p_s)$ satisfying the second line of conditions in (32). If $s = 2$ or $s = 3$ the mapping $(a, b, c) \rightarrow (p_1, \ldots, p_s)$ will be generically onto S (except for some degenerate cases). In this case, S is then mapped on a triangle of the plane (h, β) (because segments are mapped on segments), and thus the spectrum $D(h, \beta)$ is supported by a triangle.
- If $s \geq 4$, we can only say that the spectrum is supported by a convex set (because the function $D(h, \beta)$ is concave).

In order to obtain the usual spectrum of singularities $f(h)$ ($f(h)$ is the Hausdorff dimension of the set of points where the Hölder exponent is h), we can reproduce exactly the same calculations as before but forgetting the oscillation index β. Theorem 1 becomes

$$f(h) = \inf_b bh - \omega(b)$$

where $\omega(b)$ is defined by

$$\sum \mu_l^b \nu_l^{-\omega(b)} = 1.$$

We can deduce the following corollary.

Corollary 2 *The spectrum of singularities $f(h)$ is given by*

$$f(h) = \sup_\beta D(h, \beta),$$

i.e., it is obtained by projecting the graph of $D(h, \beta)$ on the plane $\beta = 0$.

Proof of Corollary 2: The definition of $\omega(b)$ implies that

$$\omega(b) = \rho(0, b).$$

Let h be fixed, and consider the function $\beta \to D(h,\beta)$. This function can be written

$$D(h,\beta) = \inf_a \left(a(\beta+1) + \inf_b (bh - \rho(a,b)) \right) \tag{34}$$

so that it is the Legendre transform of the concave and C^1 function

$$a \to \sup_b (\rho(a,b) - bh).$$

Thus, it attains its maximum for $a = 0$; and we deduce from (34) that

$$\begin{aligned} \sup_\beta D(h,\beta) &= \inf_b (bh - \rho(0,b)) \\ &= \inf_b (bh - \omega(b)) \\ &= f(h). \end{aligned}$$

One of the interesting points about computing the whole spectrum of oscillating singularities $D(h,\beta)$ instead of its projection $f(h)$ is that its evolution when one applies to F a fractional integration or derivation is extremely simple to obtain.

Corollary 3 *The spectrum of oscillating singularities $D_t(h,\beta)$ of F_t (which is the fractional integral of order t of F) is obtained from the spectrum of oscillating singularities of F by the following linear transformation of the coordinates*

$$D_t(h,\beta) = D(h - t(\beta+1), \beta).$$

Proof of Corollary 3: Suppose that we apply to F a fractional integration of order t ($t \geq 0$). This amounts to replacing in the definition of F μ_l by $\mu_l \lambda_l^t$, and to replacing the orthonormal wavelet basis by a biorthogonal system; but this last modification does not modify the criterium of pointwise regularity given by Proposition 1 (see [10]). Thus (30) becomes

$$D_t(h,\beta) = \inf_{a,b} (a(\beta+1) + bh - \rho_t(a,b))$$

where $\rho_t(a,b)$ is defined by

$$\sum_{l=1}^{s} \lambda_l^a (\mu_l \lambda_l^t)^b \nu^{-\rho_t(a,b)} = 1$$

(which is (29) after replacing μ_l by $\mu_l \lambda_l^t$). Thus

$$\rho_t(a,b) = \rho(a + bt, b)$$

and

$$\begin{aligned} D_t(h,\beta) &= \inf_{a,b} a(\beta+1) + bh - \rho(a+bt,b) \\ &= \inf_{a,b} (a - bt)(\beta+1) + bh - \rho(a,b) \\ &= \inf_{a,b} a(\beta+1) + b(h - t(\beta+1)) - \rho(a,b) \\ &= D(h - t(\beta+1), \beta) \end{aligned}$$

Note that if one performs a fractional derivative, the formula holds as long as we stay in the setting of the hypotheses,i.e., as long as

$$\forall l = 1, \ldots, s \qquad \mu_l (\lambda_l)^t < 1$$

In sharp contrast with $D(h,\beta)$, it is easy to check that the spectrum of singularities $f_t(h)$ of the fractional integrals of F cannot be deduced from the spectrum of F, for the functions we study in this paper. This remark can be pushed further and allows some constructions that are unexpected if one only considers the spectrum $f(h)$. For instance there exist functions which are monofractal (i.e., $f(h)$ is supported by one point, as in the Devil's staircase) and whose derivative is multifractal. Indeed, if for all l $\log \mu_l = h_0 \log \nu_l$, (29) and (30) show that $h = h_0$ (i.e., the spectrum $D(h,\beta)$ is supported on a segment of the line $h = h_0$) but as soon as one considers a fractional derivative or a fractional integral of F, this is no more the case, and the spectrum becomes multifractal. The explanation of the apparent anomaly is of course that F is in fact not monofractal since its spectrum of oscillating singularities is not supported by one point. This example shows again clearly the importance of computing the whole spectrum of oscillating singularities and not only the projection $f(h)$.

One of our motivations for considering oscillating singularities is that they can make the usual (one parameter) multifractal formalism (as described in [2], [3] or [7]) fails. It is therefore interesting to see what this multifractal formalism yields when applied to the functions we consider.

Recall that if

$$\theta(a) = \liminf_{j \to +\infty} \frac{\log_2(\sum_k |C_{j,k}|^a)}{j} \tag{35}$$

the multifractal formalism asserts that the function

$$\inf_a (ha - \theta(a)) \tag{36}$$

is the spectrum of singularities of the function F.

The functions (35) and (36) only depend on the parameters (λ_l) and (μ_l) (but do not depend on the (ν_l).) Thus the multifractal formalism as stated above fails for the functions studied in this paper. It is actually simple to to determine what (36) yields, since when $\forall l \ \lambda_l = \nu_l$, there are no more oscillating singularities and the methods developed for selfsimilar functions in [9] apply here: $\theta(a)$ is given by

$$\sum \mu_l^a \lambda_l^{-\theta(a)} = 1$$

and (36) is the Legendre transform of this function θ.

References

[1] A. Arneodo, E. Bacry, S. Jaffard and J.F. Muzy *Oscillating Singularities on Cantor sets. A grand-canonical multifractal formalism*, preprint (1996).

[2] A. Arneodo, E. Bacry and J.F.Muzy *The thermodynamics of fractals revisited with wavelets*, Physica A Vol.213, pp.232-275 (1995).

[3] E. Bacry A. Arneodo and J.F. Muzy *Singularity spectrum of fractal signals from wavelet analysis: exact results* , Jour. Stat. Phys. T.70, p.314 (1993).

[4] G. Brown, G. Michon and J. Peyrière *On the multifractal analysis of measures* , Jour. Stat. Phys. T.66, pp.775-790 (1992).

[5] P. Collet, J. Lebowitz and A. Porzio *The dimension spectrum of some dynamical systems* , Jour. Stat. Phys. T.47, pp.609-644 (1987).

[6] I. Falkoner *Fractal geometry,* John Wiley and sons (1990).

[7] U. Frisch and G. Parisi *Fully developped turbulence and intermittency* ., Proc. Int. Summer school Phys. Enrico Fermi, pp.84-88 North Holland (1985).

[8] S. Jaffard *Pointwise smoothness, two-microlocalization and wavelet coefficients*, Publicacions Matematiques Vol.35, pp.155-168 (1991).

[9] S. Jaffard *Multifractal formalism for functions Part I and Part II*, to appear in S.I.A.M. Journal of Mathematical Analysis.

[10] S. Jaffard and Y. Meyer *Wavelet methods for pointwise regularity and local oscillations of functions* , to appear in Memoirs of the A.M.S.

[11] Y. Meyer *Ondelettes et opérateurs*, Hermann (1990).

[12] Ph. Tchamitchian and B. Torrésani *Ridge and Skeleton extraction from the wavelet transform.*, Wavelets and their applications, ed. by Mary Beth Ruskai, Jones and Bartlett Publ. Boston (1992).