A multivariate multifractal model for return fluctuations

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Abstract

In this paper we briefly review the recently inrtroduced Multifractal Random Walk (MRW) that is able to reproduce most of recent empirical findings concerning financial time-series : no correlation between price variations, long-range volatility correlations and multifractal statistics. We then focus on its extension to a multivariate context in order to model portfolio behavior. Empirical estimations on real data suggest that this approach can be pertinent to account for the nature of both linear and non-linear correlation between stock returns at all time scales.

1 Introduction

Multifractal processes and the deeply connected mathematics of large deviations and multiplicative cascades have been widely used in many contexts to account for the time scale dependence of the statistical properties of a time-series. Recent empirical findings [1, 7, 4, 13] suggest that in finance, this framework is likely to be pertinent. The recently introduced Multifractal Random Walks (MRW) [2] are multifractal processes that have proved successful to model return fluctuations. They can be seen as simple "stochastic volatility" models (with stationary increments) whose statistical properties can be precisely controlled across the time scales using very few parameters. In that respect, they reproduce many features that characterize market price changes [10] including the decorrelation of the price increments, the long-range correlation of the volatility and the way the probability density function (pdf) of the price increments changes across time-scales, going from quasi Gaussian distributions at rather large time scales to fat tail distributions at fine scales.

In a recent work [11], Muzy et. al. have elaborated a "multivariate multifractal" framework that accounts for the time scale dependence of the mutual statistical properties of several time-series. They have shown that the statistical properties of financial time-series can be described within this framework. Though initially introduced for modelling single asset variations, the MRW models can be naturally extended in order to fit this new multivariate framework. Thus, the so-obtained Multivariate MRW (MMRW) can be used to reproduce precisely the statistical properties of several assets at any time-scale. This is of course particularly useful for modelling a portfolio behavior.

The goal of this paper is to explain how MMRW models are built and to show, using real data, that eventhough they involve very few parameters, they allow one to capture not only linear correlation between assets but also non linear correlation at all time scales. The paper is organized as follows. In section 2, after recalling briefly the main notations and definitions involved in the "classical" monovariate multifractal framework, we introduce the MRW model defined in [2], recall its main properties and show that it reproduces precisely, at any time-scale, the statistical properties of real financial time-series. In section 3, after presenting the multivariate multifractal framework defined in [11], we introduce the MMRW model and perform some analytical computations of its multifractal statistics. Numerical estimations of the parameters in the case of financial times-series are extensively discussed. Conclusions and prospects are reported in section 4.

2 The Multifractal Random Walk (MRW) model

2.1 Multifractal processes and cascade models

A multifractal process is a process wich has some scale invariance properties. These properties are generally characterized by the exponents ζ_q which govern the power law scaling of the absolute moments of its fluctuations, i.e.,

$$M(q,l) = K_q l^{\zeta_q}, \tag{1}$$

where

$$M(q,l) = E\left(|\delta_l X(t)|^q\right) = E\left(|X(t+l) - X(t)|^q\right),$$

where $X(t)$ is supposed to be a stochastisc process with stationary increments. Some very popular stochastic processes are the so-called *self-similar processes* [15]. They are defined as processes $X(t)$ which have stationary increments and which verify (in law)

$$\delta_l X(t) =^{law} (l/L)^H \delta_L X(t), \quad \forall l, L > 0.$$

For these processes, one easily gets $\zeta_q = qH$, i.e., the ζ_q spectrum is a linear function of q. Widely used examples of such processes are (fractional) Brownian motions (fBm) or Levy walks.

However, many empirical studies have shown that the ζ_q spectrum of return fluctuations is a non linear convex function. Let us note that, using a simple argument, it is easy to show that if ζ_q is a non-linear convex function the scaling behavior (1) cannot hold for all scales l but only for scales smaller than an arbitrary large scale T that is generally referred to as the *integral scale*. A very common approach originally proposed by several authors in the field of fully developed turbulence [12, 14, 8, 6, 5], has been to describe such processes in the scale domain, describing the *cascading* process that rules how the fluctuations evolves when going from coarse to fine scales. Basically, it amounts in stating that the fluctuations at the integral scale T are linked to the ones at a smaller scale $l < T$ using the cascading rule

$$\delta_l X(t) =^{law} W_{l/T} \delta_T X(t) \tag{2}$$

where $W_{l/T}$ is a log infinitely divisible stochastic variable which depends only on the ratio l/T. A straightforward computation [5] then shows that the pdf $P_l(\delta X)$ of $\delta_l X$ changes when varying the time-scale l according to the rule

$$P_l(\delta X) = \int G_{l/T}(u) e^{-u} P_T(e^{-u}\delta X) du, \tag{3}$$

where the *self-similarity kernel* $G_{l/T}$ is the pdf of $\ln W_{l/T}$. Since $W_{l/T}$ is a log infinitely divisible variable, the Fourier transform of $G_{l/T}$ is of the form

$$\hat{G}_{l/T}(k) = \hat{G}^{\ln l/T}(k). \tag{4}$$

From that equation, one easily gets the expression of the ζ_q spectrum

$$\zeta_q = \ln \hat{G}(-iq). \tag{5}$$

Thus, the simplest non-linear case is the so-called log-normal model that corresponds to a parabolic ζ_q and a Gaussian kernel.

Multiplicative cascading processes are examples of processes satisfying the cascading rule (2). However, they have fundamental drawbacks: they do not lead to stationary increments and they do not have continuous scale invariance properties, i.e., Eq. (2) and consequently Eq (1) only holds for discrete scales $l_n = \lambda^n$. To our knowledge, the MRW's are the only known multifractal processes with continuous dilation invariance properties and stationary increments.

2.2 Introducing the MRW model

An MRW process $X(t)$ is the limit process (when the time discretization step Δt goes to 0) of a standard random walk $X_{\Delta t}[k]$ with a stochastic variance (volatility), i.e.,

$$X(t) = \lim_{t \to 0} X_{\Delta t}(t),$$

with

$$X_{\Delta t}(t) = \sum_{k=1}^{t/\Delta t} \epsilon_{\Delta t}[k] e^{\omega_{\Delta t}[k]},$$

where $e^{\omega_{\Delta t}[k]}$ is the stochastic volatility and $\epsilon_{\Delta t}$ a gaussian white noise of variance $\sigma^2 \Delta t$ and which is independant of $\omega_{\Delta t}$. The choice for the process $\omega_{\Delta t}$ is simply dictated by the fact that we want the scaling (1) to be exact for all time scales $l \leq T$. Some long but straightforward computations [2] show that this is achieved if $\omega_{\Delta t}$ is a stationary Gaussian process such that $E(\omega_{\Delta t}[k]) = -\mathrm{Var}(\omega_{\Delta t}[k])$ and whose covariance is

$$\mathrm{Cov}(\omega_{\Delta t}[k], \omega_{\Delta t}[l]) = \lambda^2 \ln \rho_{\Delta t}[|k-l|]$$

where

$$\rho_{\Delta t}[k] = \begin{cases} \frac{T}{(|k|+1)\Delta t} & \text{for } |k| \leq T/\Delta t - 1 \\ 1 & \text{otherwise} \end{cases}$$

Let us note that it corresponds to a log-normal volatility which is correlated up to a time lag T.

One can then prove [2] the multifractal scaling property

$$M(q,l) = K_q l^{\zeta_q}, \quad \forall l \leq T, \tag{6}$$

with

$$\zeta_q = (q - q(q-2)\lambda^2)/2. \tag{7}$$

Since ζ_q is a parabolic function, it indicates that the self-similarity kernel $G_{l/T}$ which links the pdf's at different time scales (Eq. (3)) is Gaussian. Moreover one can show [2] that the *magnitude* correlation $C_\omega(l, \tau)$ defined by

$$C_\omega(\Delta t, l) = Cov\left(\ln|\delta_l X(t)|, \ln|\delta_l X(t+\tau)|\right), \tag{8}$$

behaves like

$$C_\omega(\Delta t, l) \sim -\lambda^2 \ln\left(\frac{\Delta t}{T}\right), \quad l < T\ . \tag{9}$$

2.3 Modelling return fluctuations using MRW

MRW processes can be used to model return fluctuations [10]. For this purpose 3 parameters need to be estimated : the variance σ, the integral scale T and the *intermittency parameter* λ (it is called this way since it controls the non linearity of the ζ_q spectrum and consequently it controls "how much stochastic" is the variable $W_{l/T}$). The variance σ can be estimated using the simple relation $Var(X(t)) = \sigma^2 t$. Both, the decorrelation scale T and the parameter λ can be obtained from the expression (9) of the magnitude correlation. Let us note that λ can be also estimated independantly from the ζ_q spectrum (Eq. (7)). The consistency between these two completly different estimators of λ is a very good test for the validity of the model.

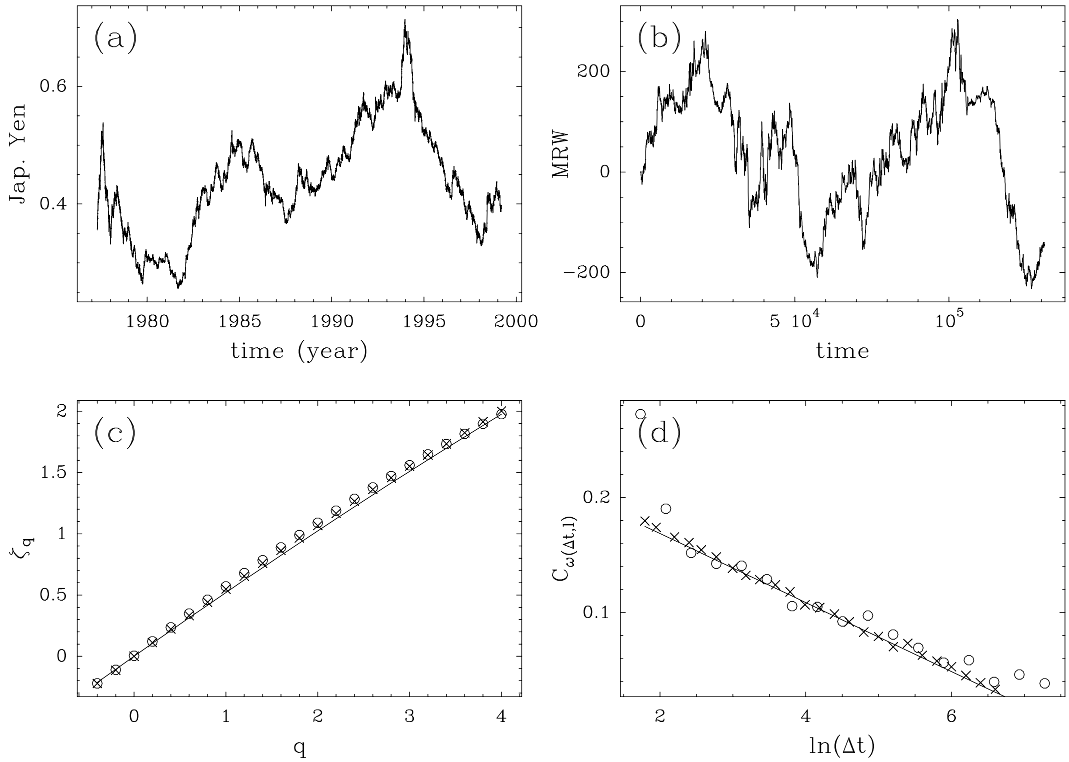

Figure 1: **Modelling intraday Japenese Yen futures using MRW**. The MRW is used to model the de-seasonalized logarithm of the time-series displayed in (a). The parameters have been estimated to $\sigma^2 = 4.10^{-6}$, $\lambda^2 = 0.03$ and $T = 1$ year. (a) Plot of the original index time-series : Japenese Yen futures from March 77 to February 99 (intraday tick by tick data). (b) Plot of a sample time series of length 2^{17} of the model. (c) ζ_q spectrum estimations for the Yen futures fluctuations ($\circ$) and for the MRW model ($\times$). The solid line corresponds to the theoretical prediction (Eq. (7)). (d) Magnitude correlation function estimations as defined in Eq. (8) (with $l \simeq 4$ days) for the Yen futures fluctuations ($\circ$) and for the MRW model ($\times$). The solid line corresponds to the theoretical prediction (Eq. (9)).

Parameter estimations have been made on financial data (japenese Yen futures). As shown in figure 1, the MRW reproduces very precisely both the parabolic ζ_q spectrum (which describes, through Eqs (5), (3), how the return fluctuation pdf evolves when going from one time scale to another) and the correlation structure of the magnitude.

Let us remark that one can show that $K_q = +\infty$ (in Eq. (6)) if $\zeta_q < 1$ and thus the pdf of $\delta_l X(t)$ has fat tails [2]. In order to control the order of the first divergent moment (without changing λ), one could simply choose for the $\epsilon_{\Delta t}$'s a law with fat tails (e.g. t-student laws).

3 The multivariate multifractal random walk (MMRW) model

3.1 The multivariate multifractal framework

In this section, we generalize in a very natural way the multifractal framework introduced in section 2.1 to multivariate processes. This generalization is inspired from [11]. It basically consists in rewriting the cascading rule (2) using multivariate processes. Thus if $\mathbf{X} = (X_1, \ldots, X_N)$ is a multivariate process, we will assume that it satisfies

$$\{\delta_l X_i(t)\}_{1 \le i \le N} =^{law} \left\{W_{i,l/T} \delta_T X_i(t)\right\}_{1 \le i \le N}$$

where $\mathbf{W} = \left\{W_{i,l/T}\right\}_{1\le i\le N}$ is a log infinitely divisible stochastic vector which depends only on the ratio l/T. A straightforward computation [5] then shows that the pdf $P_l(\delta\mathbf{X})$ of $\delta_l\mathbf{X}$ changes when varying the time-scale l according to the rule

$$P_l(\delta\mathbf{X}) = \int du_1 \ldots \int du_N G_{l/T}(\mathbf{u}) e^{-u_1-\ldots-u_N} P_T(e^{-\mathbf{u}} \otimes \delta\mathbf{X}), \tag{10}$$

where $\mathbf{u} = \{u_1, \ldots, u_N\}$ and where the $G_{l/T}$ is the pdf of $\ln \mathbf{W}_{l/T}$. (We used the notation $\mathbf{a}\otimes\mathbf{b} = (a_1b_1, \ldots, a_Nb_N)$). In the same way as in section 2.1, one can easily get the scaling law of the moments

$$M(q_1, \ldots, q_N, l) = E\left(|\delta_l X_1(t)|^{q_1} \ldots |\delta_l X_N(t)|^{q_N}\right) = K_{q_1,\ldots,q_N} l^{\zeta_{q_1,\ldots,q_N}}, \tag{11}$$

where the multifractal spectrum $\zeta_{q_1,\ldots,q_N}$ is linked to the self-similarity kernel through the relation

$$\zeta_{q_1,\ldots,q_N} = \hat{G}(-iq_1, \ldots, -iq_N),$$

*equation where $\hat{G}$ is defined as in Eq. (4).

3.2 Introducing the MMRW model

In order to account for the fluctuations of financial portfolios and to consider management applications of our approach, it is important to build a multivariate version of the MRW model. Since only Gaussian random variables are involved in the construction of section 2.2, this generalization can be done in a very natural way [10]. The MMRW walk $\mathbf{X}(t)$ is defined as

$$\mathbf{X}(t) = \lim_{t\to 0} \mathbf{X}_{\Delta t}(t), = \lim_{t\to 0} \sum_{k=1}^{t/\Delta t} \epsilon_{\Delta t}[k] \otimes e^{\omega_{\Delta t}[k]}.$$

(We again used the notation $\mathbf{a}\otimes\mathbf{b} = (a_1b_1, \ldots, a_Nb_N)$). The process $\epsilon_{\Delta t}$ is Gaussian with zero mean and covariance $Cov(\epsilon_{i,\Delta t}(t), \epsilon_{j,\Delta t}(t+\tau)) = \delta(\tau)\mathbf{\Sigma}_{ij}\Delta t$. The matrix $\mathbf{\Sigma}$ quantifies the variance and the correlation of the different white noises involved in each component of $\mathbf{X}$. We will refer to this matrix as the "Markowitz matrix". The magnitude process $\omega_{\Delta t}$ is Gaussian with covariance $Cov(\omega_{i,\Delta t}(t), \omega_{j,\Delta \tau}) = \mathbf{\Lambda}_{ij} \ln(T_{ij}/(\Delta t + |\tau|))$ (for $\Delta t + |\tau| < T_{ij}$) and 0 elsewhere, where the matrix $\mathbf{\Lambda}$ controls the non-linearity of the multifractal spectrum so we will refer to it as the "multifractal matrix". Moreover, as in section 2.2, the mean of the process is chosen so that $E(\omega_{\mathbf{\Delta t}}) = -Var(\omega_{\mathbf{\Delta t}})$. Let us note that the previously defined coefficients σ^2 and λ^2 for an asset i correspond respectively to the diagonal elements $\mathbf{\Sigma}_{ii}$ and $\mathbf{\Lambda}_{ii}$.

In order to show that MMRW are multivariate multifractal processes (within the framework of the previous section), we would like now to compute the $\zeta_{q_1,\ldots,q_N}$ spectrum. There are 2 cases for which this computation is basically the same as for the regular MRW model : (i) the case were all the white noises are decorrelated (i.e., $\mathbf{\Sigma}$ is diagonal), (ii) the case where the stochastic variances of all the assets correspond to the same process, i.e., $\omega_{i,\Delta t} = \omega_{j,\Delta t}$, $\forall i,j$. In both cases, a straightforward computation shows that the scaling law (11) holds $\forall l \le \min_{ij}(T_{ij})$ and the spectrum is

$$\zeta_{q_1,\ldots,q_N} = \sum_{i=1}^{N} \zeta^i_{q_i} - \sum_{1\le i<j\le N} \mathbf{\Lambda}_{ij} q_i q_j, \tag{12}$$

where ζ^i_q refers to the spectrum of the X_i component of $\mathbf{X}$. The computation of the spectrum is trickier in the general case. However one can show that, in this case, all the extra terms (compared to the particular case (i)) that appear in the development of $M(q_1, \ldots, q_N, l)$ go to 0 when $\Delta t \to 0$ [3]. Consequently, the multifractal spectrum has the same expression.

Since the spectrum is a parabolic function, it indicates that the self-similarity kernel $G_{l/T}$ which links the pdf's at different time scales (Eq. (10)) is Gaussian. Moreover one can show [3] that the magnitude correlation behaves like

$$Cov\left(\ln|\delta_\tau X_i(t)|, \ln|\delta_\tau X_j(t+l)|\right) \sim -\mathbf{\Lambda}_{ij} \ln(l) + C\ . \tag{13}$$

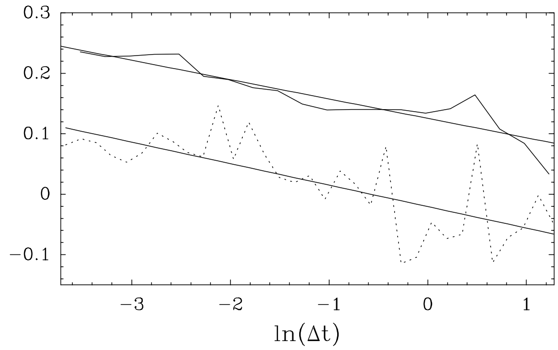

Figure 2: **Estimation on real data of a non diagonal term $\mathbf{\Lambda}_{ij}$ of the multifractal matrix $\mathbf{\Lambda}$.** $\mathbf{\Lambda}_{12}$ is estimated using daily data (from 1993 to 2000) where the first asset is AXA and the second one is TOTAL-FINA. As explained in the text, two estimators can be used. The solid curve corresponds to the $R_{12,q=1}$ estimator (Eq. (15)) which leads to the estimation $\mathbf{\Lambda}_{12} \simeq 0.032$. The dashed curve corresponds to the covariance estimator (Eq. (13)) which leads to the estimation $\mathbf{\Lambda}_{12} \simeq 0.036$. These two estimators lead to consistent estimation of $\mathbf{\Lambda}_{12}$.

3.3 Modelling return fluctuations using MMRW

For modelling a basket of assets using an MMRW, one needs to estimate the Markowitz matrix $\mathbf{\Sigma}$, the multifractal matrix $\mathbf{\Lambda}$ and the different integral scales T_{ij}. As for the MRW, the magnitude correlation can be used for estimating both the integral scales and the multifractal matrix. The Markowitz matrix $\mathbf{\Sigma}$ can be estimated, after having estimated $\mathbf{\Lambda}$, by using the simple relation

$$\mathrm{Cov}(X_i(l), X_j(l)) = \mathbf{\Sigma}_{ij} e^{\frac{1}{2}(\mathbf{\Lambda}_{ii}+\mathbf{\Lambda}_{jj}+2\mathbf{\Lambda}_{ij})} l \ . \tag{14}$$

Let us note that $\mathbf{\Lambda}$ can be also estimated independantly from the $\zeta_{q_1,\ldots,q_N}$ spectrum (Eq. (12)). Indeed, to estimate $\mathbf{\Lambda}_{ij}$ one could simply estimate, for instance, the exponent of the power law scaling

$$R_{ij,q}(l) = \frac{E(|X_i(l)|^q |X_j(l)|^q)}{E(|X_i(l)|^q) E(|X_j(l)|^q)} \sim l^{-\mathbf{\Lambda}_{ij} q^2} \tag{15}$$

The consistency between these two different estimators of $\mathbf{\Lambda}$ is a very good test for the validity of the model.

Parameter estimations have been made on some assets (daily data) of the cac40. Figure 2 shows the estimations of the non diagonal term $\mathbf{\Lambda}_{12}$ of the multifractal matrix $\mathbf{\Lambda}$ using both the magnitude correlation estimator (Eq. (13)) and the $R_{12,q}(\Delta t)$ estimator (Eq. (15)). These two different estimators lead to very close estimations of $\mathbf{\Lambda}_{12}$ which is consistent with the MMRW model.

On Figure 3 we have displayed the histograms of all the non diagonal terms $\mathbf{\Lambda}_{ij}$ (resp. $\mathbf{\Sigma}_{ij}$ and T_{ij}) for all the pair of assets in the cac40. Eventhough the histogram of the $\mathbf{\Lambda}_{ij}$ is pretty wide, its maximum is reached for $\mathbf{\Lambda}_{ij} \simeq 0.02$ which is the most common value found when estimating $\mathbf{\Lambda}_{ii} = \lambda_i^2$ in the monovariate case [10]. In the same way the histogram of the T_{ij} (resp. $\mathbf{\Sigma}_{ij}$) has a peak around 1-2 years (resp. 4.10^{-6}) which also corresponds to the most common value found when estimating T_i (resp. σ^2) in the monovariate case [10]. These results suggest that, as a first approximation, one could model these assets using an MMRW which shares the same magnitude process for all the assets, i.e., $\omega_i = \omega_j \ \forall \ i, j$. Though it is clearly not exactly the case, it simplifies the model a lot and allows to perform many analytic computations.

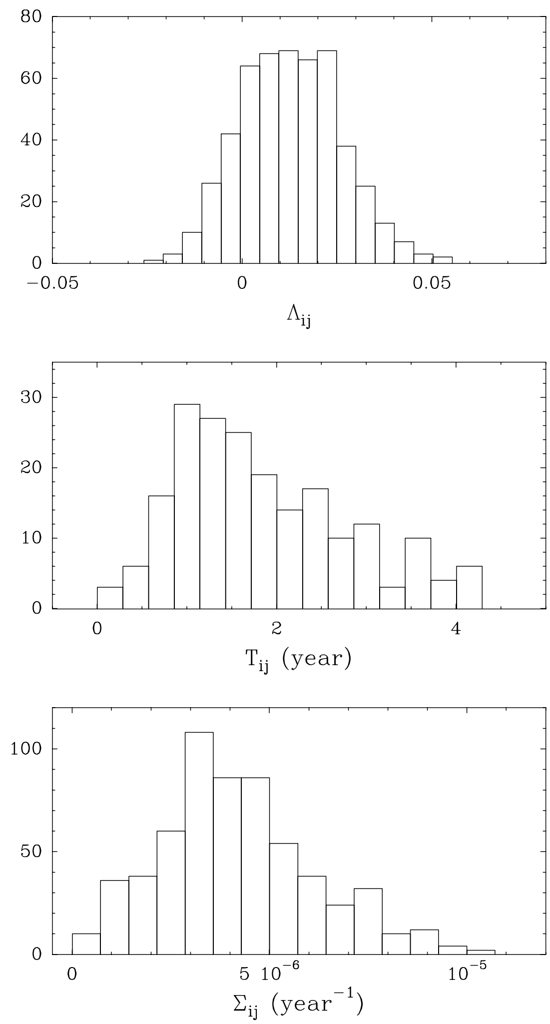

Figure 3: **Parameters estimations for a basket made of all the cac40 assets.** Histogram of the estimations, for all the pair of assets in the cac40, of the non diagonal terms (a) $\mathbf{\Lambda}_{ij}$, (b) T_{ij} and (c) $\mathbf{\Sigma}_{ij}$ (using Eq. 15).

4 Conclusion

In this paper, we have introduced a multivariate model for return fluctuations that is a definite step beyond standard correlation analysis. It basically corresponds to a multivariate random walk using a stochastic volatility. This model can be characterized using the recently introduced notion of multifractal multivariate that itself relies on the idea that the simplest way to describe the statistics of a process at all time scales is to assume some scale invariance properties. Consequently, the MMRW has a potentiality to capture the whole return joint law of a basket of assets at all time horizons. As shown in this paper, we are able to reproduce the main observed characteristics of financial time-series: no correlation between price variations, long-range volatility correlations, linear and non-linear correlation between assets and the price increment pdf and the way it changes when varying the time-scale. All of these features can be controlled using a few parameters : the multifractal matrix which controls the scale invrariance properties, the integral scales which controls the volatility correlation and the Markowitz matrix which controls the noise correlation. Moreover, as we have already pointed out, in good approximation, one can consider that all the assets share the same volatility process. Not only it reduces the number of parameters but it makes any analytical computation much easier. We are currently applying MMRW for portfolio management and risk control.

5 Acknowledgement

We acknowledge Matt Lee and Didier Sornette for the permission to use their financial data. We are also very grateful to Alain Arneodo and Didier Sornette for interesting discussions. All the computations in this paper have been made using the free GNU licensed sofware *LastWave* [9].

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