A multifractal random walk

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We introduce a class of multifractal processes, referred to as Multifractal Random Walks (MRWs). To our knowledge, it is the first multifractal processes with continuous dilation invariance properties and stationary increments. MRWs are very attractive alternative processes to classical cascade-like multifractal models since they do not involve any particular scale ratio. The MRWs are indexed by few parameters that are shown to control in a very direct way the multifractal spectrum and the correlation structure of the increments. We briefly explain how, in the same way, one can build stationary multifractal processes or positive random measures.

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Multifractal models have been used to account for scale invariance properties of various objects in very different domains ranging from the energy dissipation or the velocity field in turbulent flows [1] to financial data [2]. The scale invariance properties of a deterministic fractal function $f(t)$ are generally characterized by the exponents ζ_q which govern the power law scaling of the absolute moments of its fluctuations, i.e.,

$$m(q,l) = K_q l^{\zeta_q}, \tag{1}$$

where, for instance, one can choose $m(q,l) = \sum_t |f(t+l) - f(t)|^q$. When the exponents ζ_q are linear in q, a single scaling exponent H is involved. One has $\zeta_q = qH$ and $f(t)$ is said to be *monofractal*. If the function ζ_q is no longer linear in q, $f(t)$ is said to be *multifractal*. In the case of a stochastic process $X(t)$ with stationary increments, these definitions are naturally extended using

$$m(q,l) = E\left(|\delta_l X(t)|^q\right) = E\left(|X(t+l) - X(t)|^q\right), \tag{2}$$

where E stands for the expectation. Some very popular monofractal stochastic processes are the so-called *self-similar processes* [3]. They are defined as processes $X(t)$ which have stationary increments and which verify (in law)

$$\delta_{\lambda l} X(t) = \lambda^H \delta_l X(t), \quad \forall l, \lambda > 0. \tag{3}$$

Widely used examples of such processes are fractional Brownian motions (fBm) and Levy walks. One reason for their success is that, as it is generally the case in experimental time-series, they do not involve any particular scale ratio (i.e., there is no constraint on l or λ in Eq. (3)). In the same spirit, one can try to build multifractal processes which do not involve any particular scale ratio. A common approach originally proposed in the field of fully developed turbulence [1, 4–7], has been to describe such processes in terms of stochastic equations, in the scale domain, describing the cascading process that rules how the fluctuations evolves when going from coarse to fine scales. One can state that the fluctuations at scales l and λl ($\lambda < 1$) are related (for fixed t) through the cascading rule

$$\delta_{\lambda l} X(t) = W_\lambda \delta_l X(t) \tag{4}$$

where $\ln(W_\lambda)$ is a random variable which law G_λ depends only on λ. Let us note that this latter equation can be simply seen as a generalization of Eq. (3) with H being stochastic. Since Eq. (4) can be iterated, it implicitely imposes the random variable W_λ to have a log infinitely divisible law [8]. It is then easy to show that the iterative rule satisfied by W_λ implies the Fourier transform of G_λ can be written as $\hat{G}_\lambda(k) = \hat{G}^{\ln\lambda}(k)$. If follows that the q order absolute moments at scale l scales like

$$m(q,l) = \hat{G}_{l/L}(-iq) m(q,L) = m(q,L)\left(\frac{l}{L}\right)^{F(-iq)}, \tag{5}$$

where $F = \ln \hat{G}$ refers to the cumulant generating function of $\ln W$ [7, 10]. Thus, identifying this latter equation with Eq. (1), one finds $\zeta(q) = F(-iq)$. In the case of self-similar processes of exponent H, $\ln(W_\lambda)$ is non stochastic and has a dirac function law $G_\lambda(u) = \delta(u - H\ln(\lambda))$ leading to $\zeta_q = qH$. The simplest non-linear (i.e., multifractal) case is the so-called log-normal model that corresponds to a Normal shape for G_λ and thus to a parabolic ζ_q spectrum.

However, the previous "top to bottom" cascade construction is, to a large extend, only formal. According to our knowledge, nobody has succeeded in building effectively such processes yet, mainly because of the peculiar constraints in the time-scale half-plane. Indeed, the variables $\delta_\lambda X(t)$ cannot be chosen freely because they must satisfy $\delta_\lambda X(t) = \delta_\gamma X(t) + \delta_{\lambda-\gamma} X(t+\gamma)$ for all $\gamma \leq \lambda$. Multiplicative cascading processes [11–14] that consist in writing Eq. (4) starting from some "coarse" scale L and then iterating it towards finer scales has been constructed exclusively using an arbitrary fixed scale ratio (e.g., $\lambda = 1/2$). Such processes neither possess stationary increments nor continuous dilation invariance properties. Since they involve a particular arbitrary scale ratio, Eq (1) holds only for the discrete scales $l_n = \lambda^n L$.

The goal of this paper is to build a multifractal process $X(t)$, referred to as a Multifractal Random Walk (MRW), with stationary increments and such that Eq. (1) holds for all $l \leq L$. We first build a discretized version $X_{\Delta t}(t = k\Delta t)$ of this process. Let us note that the theoretical issue whether the limit process $X(t) = \lim_{\Delta t \to 0} X_{\Delta t}(t)$ is well defined will be addressed in a forthcoming paper. In this paper, we explain how it is built and prove that different quantities (q order moments, increment correlation,...) converge, when $\Delta t \to 0$.

Writing Eq. (4) at the smallest scale suggests that a good candidate might be such that $\delta_{\Delta t} X_{\Delta t}(k\Delta t) = \epsilon_{\Delta t}[k] W_{\Delta t}[k]$ where $\epsilon_{\Delta t}$ is a Gaussian variable and $W_{\Delta t}[k] = e^{\omega_{\Delta t}[k]}$ is a log normal variable, i.e.,

$$X_{\Delta t}(t) = \sum_{k=1}^{t/\Delta t} \delta_{\Delta t} X_{\Delta t}(t) = \sum_{k=1}^{t/\Delta t} \epsilon_{\Delta t}[k] e^{\omega_{\Delta t}[k]}, \quad (6)$$

with $X_{\Delta t}(0) = 0$ and $t = k\Delta t$. Moreover, we will choose $\epsilon_{\Delta t}$ and $\omega_{\Delta t}$ to be decorrelated and $\epsilon_{\Delta t}$ to be a white noise of variance $\sigma^2 \Delta t$. Obviously, we need to correlate the $\omega_{\Delta t}[k]$'s otherwise $X_{\Delta t}$ would simply converge towards a Brownian motion. Since, in the case of cascade-like processes, it has been shown [14–16] that the covariance of the logarithm of the increments decreases logarithmically, it seems natural to choose

$$Cov(\omega_{\Delta t}[k_1], \omega_{\Delta t}[k_2]) = \lambda^2 \ln \rho_{\Delta t}[|k_1 - k_2|], \quad (7)$$

with

$$\rho_{\Delta t}[k] = \begin{cases} \frac{L}{(|k|+1)\Delta t} & \text{for } |k| \leq L/\Delta t - 1 \\ 1 & \text{otherwise} \end{cases}, \quad (8)$$

i.e., the $\omega_{\Delta t}$ are correlated up to a distance of L and their variance $\lambda^2 \ln(L/\Delta t)$ goes to $+\infty$ when Δt goes to 0 [18]. For the variance of $X_{\Delta t}$ to converge, a quick computation shows that we need to choose

$$E(\omega_{\Delta t}[k]) = -rVar(\omega_{\Delta t}[k]) = -r\lambda^2 \ln(L/\Delta t), \quad (9)$$

with $r = 1$ (this value will be changed later) for which we find $Var(X(t)) = \sigma^2 t$.

Let us compute the moments of the increments of the MRW $X(t)$. Using the definition of $X_{\Delta t}(t)$ one gets

$$E(X_{\Delta t}(t_1) ... X_{\Delta t}(t_m)) = \sum_{k_1=1}^{t_1/\Delta t} ... \sum_{k_m=1}^{t_m/\Delta t} E(\epsilon_{\Delta t}[k_1] ... \epsilon_{\Delta t}[k_m]) E\left(e^{\omega_{\Delta t}[k_1] + ... + \omega_{\Delta t}[k_m]}\right).$$

Since $\epsilon_{\Delta t}$ is a 0 mean Gaussian process, this expression is 0 if m is odd. Let $m = 2p$. Since the $\epsilon_{\delta t}[k]$'s are δ-correlated Gaussian variables, one shows that the previous expression reduces to

$$\frac{\sigma^{2p}}{2^p p!} \sum_{\mathcal{S} \in S_{2p}} \sum_{k_1=1}^{(t_{\mathcal{S}(1)} \wedge t_{\mathcal{S}(2)})/\Delta t} .... \sum_{k_p=1}^{(t_{\mathcal{S}(2p-1)} \wedge t_{\mathcal{S}(2p)})/\Delta t} E\left(e^{2\sum_{j=1}^p \omega_{\Delta t}[k_j]}\right) \Delta t^p,$$

where $a \wedge b$ refers to the minimum of a and b and S_{2p} to the set of the permutations on $\{1, ..., 2p\}$. On the other hand, we have $E\left(e^{2\sum_{j=1}^p \omega_{\Delta t}[k_j]}\right) = \prod_{i<j} \rho[k_i - k_j]^{4\lambda^2}$. Then, when $\Delta t \to 0$, the general expression of the moments is

$$E(X(t_1) ... X(t_{2p})) = \frac{\sigma^{2p}}{2^p p!} \sum_{\mathcal{S} \in S_{2p}} \int_0^{t_{\mathcal{S}(1)} \wedge t_{\mathcal{S}(2)}} du_1 \\ ... \int_0^{t_{\mathcal{S}(2p-1)} \wedge t_{\mathcal{S}(2p)}} du_p \prod_{i<j} \rho(u_i - u_j)^{4\lambda^2}, \quad (10)$$

where $\rho(t) = \lim_{\Delta t \to 0} \rho_{\Delta t}[t/\Delta t]$. In the special case $t_1 = t_2 = ... = t_{2p} = l$, a simple scaling argument leads to the continuous dilation invariance property

$$m(2p, l) = K_{2p} \left(\frac{l}{L}\right)^{p - 2p(p-1)\lambda^2}, \quad \forall l \leq L, \quad (11)$$

where we have denoted the prefactor

$$K_{2p} = L^p \sigma^{2p} (2p-1)!! \int_0^1 du_1 ... \int_0^1 du_p \prod_{i<j} |u_i - u_j|^{-4\lambda^2}.$$

By analytical continuation, we thus obtain the following ζ_q spectrum

$$\zeta_q = (q - q(q-2)\lambda^2)/2. \quad (12)$$

We have illustrated this scaling behavior in fig. 1. Thus, the MRW $X(t)$ is a multifractal process with stationary increments and with continuous dilation invariance properties up to the scale L. Let us note that above this scale ($l >> L$), one gets from Eq. (10) that $\zeta_q = q/2$, i.e., the process scales like a simple Brownian motion, as if ω was not correlated, though, of course, $X(t)$ is not Gaussian. Indeed, K_{2p} is nothing but the moment of order $2p$ of the random variable $X(L)$ that is infinite for $p > 1 + \frac{1}{2\lambda^2}$ Consequently, the pdf of $X(L)$ has fat tails. In fig. 2, we illustate that the cascade picture of Eq. (4) accounts very well for the evolution of the pdf of the increments at different scales. One shows that the smaller the scale l, the fatter the tails of the pdf of $\delta_l X(t)$.

Let us study the correlation structure of the increments of $X(t)$. Since $\zeta_2 = 1$, one can prove that they are decorrelated (though not independant). Let

$$C_{2p}(l, \tau) = < |\delta_\tau X(l)|^{2p} |\delta_\tau X(0)|^{2p} >, \quad (13)$$

with $\tau < l$. Using the same kind of arguments as above, one can show that

$$C_{2p}(l, \tau) = (\sigma^{2p}(2p-1)!!)^2 \int_l^{l+\tau} du_1 ... \int_l^{l+\tau} du_p \\ \int_0^\tau du_{p+1} ... \int_0^\tau du_{2p} \prod_{1 \leq i < j \leq 2p} \rho(u_i - u_j)^{4\lambda^2}. \quad (14)$$

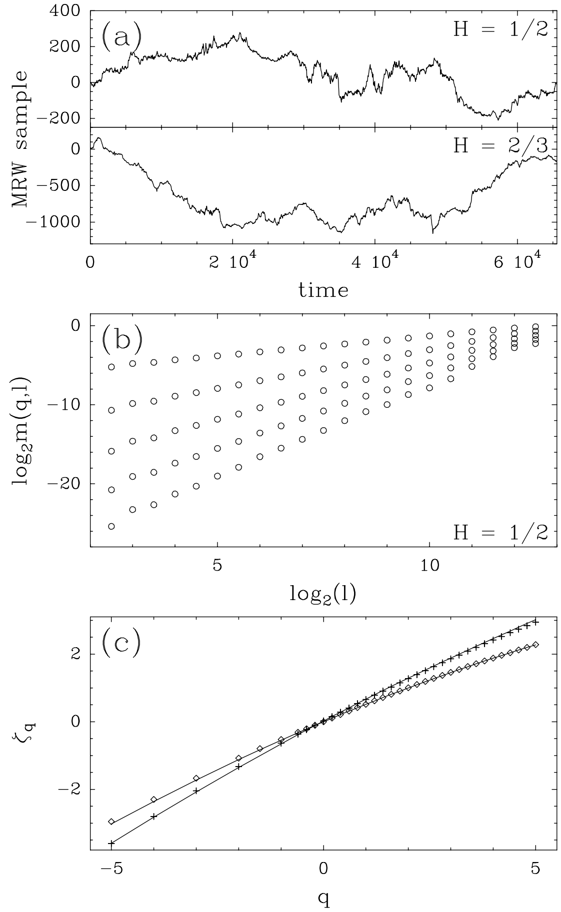

FIG. 1: (a) Plot of two realizations of 2^{17} samples of two MRWs with $\lambda^2 = 0.03$, $L = 2048$ and where $\epsilon_{\Delta t}$ is (top plot) a white noise or (bottom plot) a fGn (Eq. (16)) with $H = 2/3$. (b) Log-log plots of $m(q,l)$ of the MRW plotted in (a) (top plot) versus l for $q = 1, 2, 3, 4, 5$. (c) ($\circ$) (resp. (+)) : ζ_q spectrum estimation of the MRW plotted at the top (resp. bottom) in (a). These estimations (obtained using the WTMM method [17]) are in perfect agreement with the theoretical predictions (——) given by Eq. (12) (resp. Eq. (17)).

A straightforward argument then shows that

$$K_{2p}^2 \frac{(\tau/L)^{2\zeta_{2p}}}{((l+\tau)/L)^{4\lambda^2 p^2}} \leq C_{2p}(l,\tau) \leq K_{2p}^2 \frac{(\tau/L)^{2\zeta_{2p}}}{((l-\tau)/L)^{4\lambda^2 p^2}},$$

and consequently for $\tau << l$ fixed, using analytical continuation one expects $C_q(l,\tau)$ to scale like

$$C_q(l,\tau) \sim K_q^2 \left(\frac{\tau}{L}\right)^{2\zeta_q} \left(\frac{l}{L}\right)^{-\lambda^2 q^2}. \qquad (15)$$

This behavior is illustrated in fig. 3.

From the behavior of C_q when $q \to 0$, we can obtain using Eq. (15) that the covariance of the logarithm of the increments at scale τ and lag l behaves (for $\tau << l$) like $-\lambda^2 \ln\left(\frac{l}{L}\right)$. Thus, this correlation reflects the correlation of the $\omega_{\Delta t}$ process and is the same as observed in Refs [14–16] for the cascade models. This behavior is checked in fig. 4.

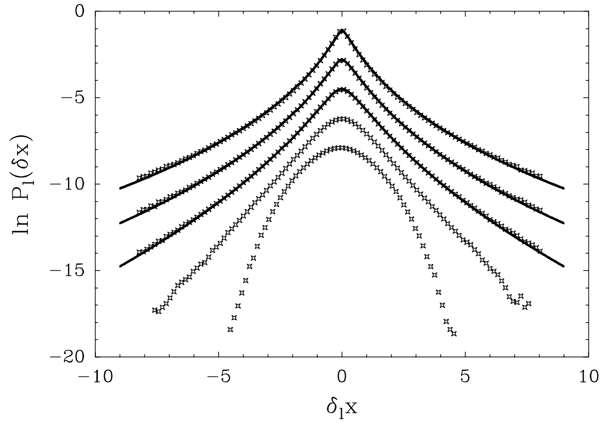

FIG. 2: (x) Standardized estimated pdf's of $\ln \delta_l X(t)$ for $l = 4, 32, 256, 2048$ and 4096 (from top to bottom). These estimations have been made on 500 realizations of 2^{17} samples each of a MRW with $\lambda^2 = 0.06$ and $L = 2048$. Plots have been arbitrarily shifted for illustration purpose. (——) theoretical prediction from the estimated pdf at the largest scale ($l = 2048$) using the cascade equation (4).

Finally, let us note that, one can built MRWs with correlated increments by just replacing the white noise $\epsilon_{\Delta t}$ by a fractional Gaussian noise (fGn)

$$\epsilon_{\Delta t}^{(H)}[k] = B_H((k+1)\Delta t) - B_H(k\Delta t), \qquad (16)$$

where $B_H(t)$ is a fBm with the scaling exponent H and of variance $\sigma^2 t^{2H}$, and choosing $r = 1/2$ in Eq. (9). Then, one can show (after tedious but straightforward computations) that the spectrum of the MRW $X^{(H)}(t)$ is

$$\zeta_q^{(H)} = qH - q(q-1)\lambda^2/2, \qquad (17)$$

($\zeta_q^{(H)} = qH$ at scales $>> L$) and consequently the MRW has correlated increments. Such a construction is illustrated in fig. 1 with $H = 2/3$. Since $H > 1/2$ it leads to a process which is more regular than the one previously built.

To summarize, we have built the MRWs, a class of multifractal processes, with stationary increments and continuous dilation invariance. Such processes have been shown to satisfy, in a weak sense, the cascade equation 4. We do believe that they should be very helpful in all the fields where multiscaling is observed. Their construction, using aggregation of random variables, make them very interesting for the modelling of dynamical processes like Lagrangian turbulence or financial time series [19]. From a theoretical point of view, MRW can be seen as the simplest model that contains the main ingredients for multifractality, namely, the logarithms of amplitude fluctuations are nothing but a $1/f$ noise. Moreover, they involve very few parameters, mainly, the correlation length

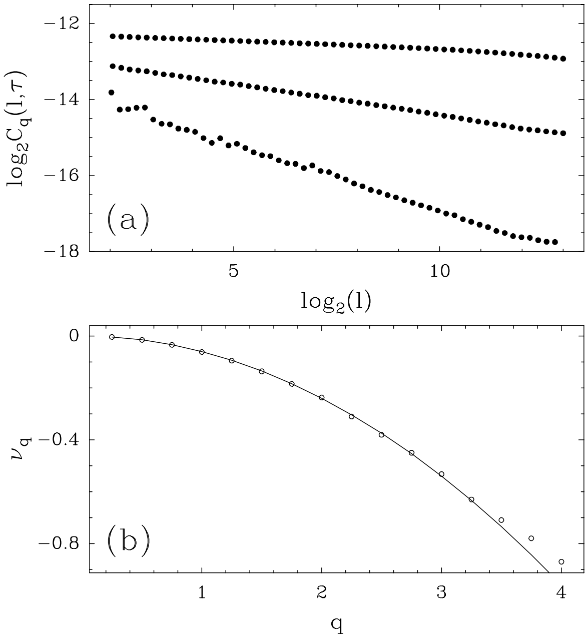

FIG. 3: (a) Log-log plots of $C_q(l,\tau)$ versus l for $q = 1,2,3$. (b) Estimation (∘) of the power law exponent $C_q(l,\tau) \sim l^{\nu_q}$. It is in perfect agreement with the theoretical prediction (Eq. (15)) $\nu_q = -\lambda^2 q^2$ (——).

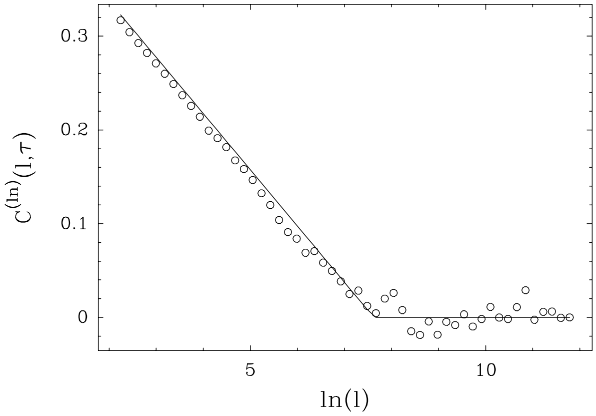

FIG. 4: Estimation (∘) of the covariance $C^{(\ln)}(l,\tau)$ of the logarithm of the increments of an MRW. It is in perfect agreement with the analytical expression $-\lambda^2 \ln\left(\frac{l}{L}\right)$ (——).

L, the intermittency parameter λ^2, the variance σ^2 and the roughness exponent H. They all can be easily estimated using the ζ_q spectrum and the increment correlations. The construction of MRWs, can be used as a general framework in which one can easily build other classes of processes in order to match some specific experimental situations. For instance, a stationary MRW can be obtained by just adding a friction $\gamma > 0$, i.e., $X_{\Delta_t}[k] = (1-\gamma)X_{\Delta_t}[k-1] + \epsilon_{\Delta t}[k]e^{\omega_{\Delta t}[k]}$. One can build a strictly increasing MRW (and consequently a stochastic positive multifractal measure) by just setting $\epsilon_{\Delta t} = \Delta t$ in Eq. (6) and use it as a multifractal time for subordinating a monofractal process (such as an fBm). One can also use other laws than the (log-)normal for ϵ and/or ω. Another interesting point concerns the problem of the existence of a limit ($\Delta t \to 0$) stochastic process and on the development of a new stochastic calculus associated to this process. All these prospects will be addressed in forthcoming studies.

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