## Multifractal models for asset prices\*

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#### Abstract

In this paper, we make a short overview of multifractal models of asset returns. All the proposed models rely upon the notion of random multiplicative cascades. We focus in more details on the simplest of such models namely the log-normal Multifractal Random Walk. This model can be seen as a stochastic volatility model where the (log-) volatility has a peculiar long-range correlated memory. We briefly address calibration issues of such models and their applications to volatility and VaR forecasting.

Since Mandelbrot first work on the fluctuations of cotton price in early sixties, it is well known that market price variations are poorly described by the standard geometric Brownian motion [15]: Extreme events are more probable than in a Gaussian world and volatility fluctuations are well known to be of intermittent and correlated nature. As we shall discuss along this article, multifractal analysis has provided new concepts and tools to analyze market fluctuations and inspired a particularly elegant family of models that

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accounts for the main observed empirical properties in a parcimonious way. These models while capturing the "heteroskedastic" nature of return fluctuations, still preserve, in some sense, the nice stability property accross time scales of the Brownian motion. This sharply contrasts with classical econometric models that can hardly be controlled as the time-scale is changed.

#### Multifractal scaling

Let p(t) be the price of some asset at time t and  $\delta_{\ell}X(t)$  the variation of the continuous compound return  $X(t) = \ln[p(t)]$  between times t and  $t + \ell$ :  $\delta_{\ell}X(t) = X(t + \ell) - X(t)^{1}$ . X(t) will be called a multifractal process if the order q absolute moment of  $\delta_{\ell}X(t)$  behaves as a power-law when  $\ell$  varies

$$M(q,\ell) = \mathbb{E}\left[|\delta_{\ell}X|^{q}\right] \sim C_{q}\ell^{\zeta(q)}, \ l \le T, \tag{1}$$

where  $\mathbb{E}[.]$  stands for the mathematical expectation, the exponent  $\zeta(q)$  is a non-linear concave function of q and T is a large characteristic time scale referred to as the *integral scale*.

The multifractal nature of market data is illustrated on the S&P500 index in Fig. 1 using traded prices sampled every 10mm. In perfect agreemnt with (1), log-log plots of M(q, l) versus  $\ell$  show linear behavior up to a scale of the order of  $T \simeq 1$  year and the so-obtained  $\zeta(q)$  spectrum (using linear regression) turns out to be well fitted by a parabolic shape. The S&P500 future index can thus be considered, at least at this description level, as a multifractal signal. During the last decade, a lot of works have studied various market data wihin the framework of multifractal scaling analysis: on FX rates, commodity markets, stock markets, future markets, emerging markets (see [3] for a review). From all these studies, it appears that multifractality

<sup>&</sup>lt;sup>1</sup>In this paper, we do not address the modeling of intraday modulation of the data. This modulation can be removed following [14]. In this paper, we shall consider that  $\delta_{\ell}X(t)$  is stationary and has zero mean.

is an universal feature of financial time series. <FIG1 HERE>

The scaling of return moments as a function of the time scale  $\ell$ , is in fact intimately related to some self-similarity property of the process X(t). A process X(t) is called *self-similar* of exponent H if it has stationary increments and if  $\forall s > 0$ ,

$$\delta_{s\ell}X(st) = s^H \delta_\ell X(t) \tag{2}$$

This means that the original process and a dilated version of it are somehow undistinguishable. In that case,  $M(q,\ell) \sim \ell^{qH}$  and  $\zeta(q)$  is a linear function of q. In order to account for multifractality (i.e., non-linear  $\zeta(q)$ ), one can replace the deterministic factor  $s^H$  by a random one and get the stochastic self-similarity property:

$$\delta X_{s\ell}(st) = W_s \delta X_{\ell}(t) \tag{3}$$

where  $W_s = e^{\omega_s}$  is a positive random variable independent of the process X and which law is log-infinitely divisible and only depends on s. From this equation, It is easy to show that  $\zeta(q) = \ln \mathbb{E}\left[e^{-q\omega_s}\right]/\ln s$ . The multifractality strength of the process is generally characterized by the curvature of  $\zeta(q)$ , i.e., by the so-called intermittency coefficient  $\lambda^2 = -\zeta''(0)$ . The simplest non linear case is the log-normal model that corresponds to a parabolic  $\zeta(q)$  (i.e.,  $\omega_s$  is Gaussian).

Let us point out that the stochastic self-similarity implies (i) that the smaller  $\ell$  is, the *fatter* the tails of the probability distribution function (pdf) of  $\delta_{\ell}X$  are and (ii) that the covariance function of  $\ln |\delta X_{\ell}|$  is slowly decreasing (up to scale T) as the logarithm of the lag. These two features are indeed observed on most financial data as illustrated on S&P 500 future data in Fig. 2 and Fig. 3. <FIGURE 2 AND 3 HERE >

#### Random cascades models

Let us first note that, following an idea of Mandelbrot [15], if  $\theta(t)$  (referred to as the *trading time*) is a non-decreasing multifractal process satisfying (3) and B(t) a Brownian motion that is independent of  $\theta(t)$ , then the process:

$$X(t) = B\left[\theta(t)\right],\,$$

is also multifractal with  $\zeta_X(q) = \zeta_{\theta}(q/2)$ . If we write  $\theta(t) = \int_0^t d\theta(t)$ , the multifractal random measure  $d\theta(t)$  can be seen as the instantaneous stochastic volatility. Consequently, multifractal models of asset returns reduce to multifractal models of volatility measures.

Discrete multiplicative cascades were first introduced in the field of empirical finance by the pioneering work of Calvet, Fisher and Mandelbrot on exchange rates volatility [16]. Actually, such cascades were originally proposed by russian physicists for modelling highly turbulent flows and have been the object of a lot of mathematical studies [11, 17]. Their construction is a direct translation of the self-similarity property (3). In the simplest case, the construction of  $\theta(t)$  involves a dyadic tree in a time-scale (t, s) half plane. The level n lies at scale  $s_n = T2^{-n}$  and has  $2^n$  nodes  $\{p_{n,k}\}_{0 \leq k < 2^n}$  which are uniformly distributed on [0,T] leading to a partition made of  $2^n$  intervals  $I_{n,k}$  of size  $s_n$ . Moreover i.i.d. (positive) log-infinitely divisible random factors  $W_{n,k}$  are associated with the nodes  $p_{n,k}$ . The cascading process starts at the integral scale  $s_0 = T$  where a measure is uniformly spread on [0, T]:  $d\theta_0(t) = dt$ . At step n = 1, the measure  $d\theta_1(t)$  is obtained from  $d\theta_0(t)$ by multiplying its density on the interval  $I_{1,0}$  (resp.  $I_{1,1}$ ) by the corresponding factors  $W_{1,0}$  (resp.  $W_{1,1}$ ). At step n,  $d\theta_n(t)$  is obtained by multiplying  $d\theta_{n-1}(t)$  on each  $I_{n-1,k}$  by  $W_{n-1,k}$ . Leading to the representation

$$d\theta_n(u) = 2^{-n} e^{\sum_{i=1}^n \delta\omega_{i,u_k}} dt \tag{4}$$

where  $W_{i,u_k} = e^{\delta \omega_{i,u_k}}$  and  $u_k$  corresponds to the integer part of  $u2^n$ . Finally, one gets  $d\theta(t) = \lim_{n \to +\infty} d\theta_n(t)$ .

Grid bounded cascades, though simple, do not provide a satisfying solution to model volatility fluctuations: (i) they involve an arbitrary fixed scale ratio, (ii) they are not built in a causal way and (iii) they are not stationary. In their "Poisson multifractal model", Calvet and Fisher [8], get rid of (ii) and (iii), by basically replacing, for a fixed n, the points  $\{p_{n,k}\}_k$  by random Poisson points with intensity  $r(s_n) = s_n^{-1}$ . The "Multifractal Product of Cylindrical Pulses" model of Barral and Mandelbrot [6] goes one step further (getting rid of (i)), by replacing the points  $\{p_{n,k}\}_{n,k}$  and their associated weights  $\{W_{n,k}\}_{n,k}$  using a non-homogeneous compound Poisson, with the compound law W and the intensity  $r(s,t) = s^{-1}$ . This construction can be generalized by replacing the compound Poisson process by some arbitrary infinitely divisible random noise  $d\omega(t,s)$  in the half-plane (t,s) (with density  $dtds/s^2$ ). Eq. (4) is replaced by

$$d\theta_s(t) = e^{\int_{(s',t')\in C_s(t)} d\omega(t',s')} dt, \tag{5}$$

where  $C_s(t)$  is a cone-like domain (pointing at (0,t) and truncated below scale s). This construction has been proposed by Bacry and Muzy [4] and corresponds to the fully continuous generalization of (4). The limiting measure  $d\theta(t) = \lim_{s\to 0} d\theta_s(t)$  is shown to be stochastic self-similar, with exact scaling  $(M(q, l) = C_q l^{\zeta(q)}, \forall l \leq T)$  and having none of the drawbacks (i)-(iii). Of course, the corresponding  $X(t) = B(\theta(t))$  process shares the exact same properties, it is referred to as a "Multifractal Random Walk" (MRW).

The "simplest" MRW model corresponds to the case W is log-normal. In that case  $d\omega(t',s')$  is a Gaussian white noise, and it can be shown that  $\theta(t)$  can be obtained directly by

$$\theta(t) = \lim_{\Delta \to 0} \int_0^t e^{2\omega_{\Delta}(u)} du \tag{6}$$

where the magnitude process  $\omega_{\Delta}(u)$  is a stationary Gaussian process fully defined by :

$$\mathbb{E}\left[\omega_{\Delta}(t)\right] = -\lambda^2 \ln\left(\frac{T}{\Delta}\right) \tag{7}$$

$$R_{\Delta}(\tau) = \begin{cases} \lambda^2 \left( \ln \left( \frac{T}{\Delta} \right) + 1 - \frac{\tau}{\Delta} \right), & \tau \leq \Delta \\ \lambda^2 \ln \left( \frac{T}{\tau} \right), & \tau \in [\Delta, T] \\ 0, & \tau > T \end{cases}$$

where  $R_{\Delta}(\tau) = \mathbb{C}\text{ov}(\omega_{\Delta}(t), \omega_{\Delta}(t+\tau)).$ 

Very few mathematical studies address statistical issues such as parameter estimation (see however [20]). Calvet and Fisher [8] introduced a Markov-Switching version of their cascade model that is amenable to Maximum Likelihood Estimation. They were the first to propose a Generalized Method of Moments (GMM) to estimate the parameters of their model (see also [12, 13] for discrete cascades). The log-normal MRW model is fully defined by 3 parameters:  $\sigma^2$  the variance of the Brownian motion, T the integral scale (decorrelation time-scale of the volatility) and  $\lambda^2$  the intermittency factor, which is involved in the non-linear part of  $\zeta_q$  (Eq. (8)). Since, as briefly discussed below, many of statistical moments associated with the MRW model can be analytically computed, GMM can be simply devised and all these parameters can be easily estimated [1]. We observed that for both daily and intraday financial data, the correlation time T is very large (greater than 1 year) and the parameter  $\lambda^2$  lies in [0.02, 0.05].

## MRW main properties and risk forecasting

For proofs and details sees refs [4, 3]. As far as multifractal properties are concerned, X(t) verifies the exact stochastic self-similarity property as defined by (3) and therefore its moments behave exactly as a power-law of the time scale ( $\ell \leq T$ ) with a parabolic multifractal spectrum

$$\zeta(q) = q(1/2 + \lambda^2) - \frac{\lambda^2}{2}q^2$$
 (8)

Many other scaling laws can be computed and notably those of "p-volatility" correlations  $C_p(\ell, \tau) \equiv \mathbb{E}\left[|\delta_\ell X(\tau)|^p |\delta_\ell X(0)|^p\right]$ . In refs. [19] it is shown that

$$C_p(\ell, \tau) \sim K_p^2 \left(\frac{l}{T}\right)^{2\zeta_p} \left(\frac{\tau}{T}\right)^{-\lambda^2 p^2}$$
 (9)

where the constant  $K_p$  can be computed analytically. Although they did not explicitly refer to the framework of multifractal analysis, Ding and Granger [9] were the first to observe a non trivial dependence of  $C_p(\ell, \tau)$  as a function of p. Notice that when  $p \to 0$ , one recovers the logarithmic slowly decreasing of the log-volatility covariance  $C_{\omega}(\ell, \tau)$  as illustrated in figure 3.

Eq. (9) can be directly used to predict volatility. In ref. [8], Calvet and Fisher have already shown that their cascade model provides better volatility forecasts as compared to GARCH(1,1) or Markov-Switching GARCH (see also [13]). Bacry and Muzy [3, 5] have shown that linear volatility forecast provided by the MRW model outperforms GARCH(1,1) models.

Actually, the MRW model allows to estimate the full (conditional) return probability density function at all time horizons and scales. Very much like standard Brownian motion which is stable with respect to time aggregation, the self-similarity properties of the MRW process X(t) allows to control how the return probability law changes when varying the time-scale. Indeed, it can be shown that, in a good approximation, when  $\lambda^2 \ll 1$ , the returns of the MRW process, can be written, in law, as

$$\delta_{\ell}X(t) \underset{Law}{\simeq} \epsilon_{\ell}(t)e^{\Omega_{\ell}(t)}$$
 (10)

(see [5] for details and for the precise meaning of the symbol  $\simeq$ ) for all  $\ell \leq T$ , where  $\epsilon_{\ell}$  is a Gaussian white noise of variance  $\sigma^2\ell$  and the process  $\Omega_{\ell}(t)$  is the renormalized magnitude. It is a stationnary Gaussian process whose mean and covariance are very similar to (7). In fig. 2, we see that, as observed for S&P 500 data, the MRW returns pdf strongly depends on the time scale and evolves from a "quasi-Gaussian" shape at large scales ( $\ell \simeq T$ ) to pdf's with high kurtosis at small scales. Even if the precise shape of pdf tails returns is

a matter of debate, the most commonly admitted form is a power-law with a tail exponent  $\mu$  within the interval [3,5] (see e.g., refs [7, 10]). The MRW model allowed to point out [18, 2] that this behavior is strongly related to the choice of the asymptotics in the estimation process. The "high-frequency asymptotics" (sampling frequency going to infinity) has to be distinguished from the "long duration asymptotics" (duration of the considered time period going to infinity).

All previous considerations can be extented to conditional laws and are thus of great practical interest for Value at Risk (VaR) forecasting. It has shown in [3, 5] that the MRW predictions of the VaR at any time-scales and time-horizons are much more reliable than e.g. GARCH(1,1) (normal or t-Students) predictions.

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Fig 1: Multifractal Analysis of S&P500 future index. Traded prices have been sampled every 10mn during the period 1988-1999. Top: Log-log plots of M(q, l) versus  $\ell$  for q = 1, 2, 3, 4, 5. The time scales  $\ell$  range from 10 minutes to 6 years. They display linear behavior up to a scale  $T \simeq 1$  year. Bottom:  $\zeta(q)$  spectrum obtained by linear regression of plots on the left. The plot in the inset is the parabolic nonlinear part of  $\zeta(q)$ .

Fig 2: Continuous deformation of increment pdf's  $\delta_{\ell}X$  across scales. Standardized pdf's (in logarithm scale) at large (bottom) to small (top) scales. Plots have been arbitrarily shifted along the vertical axis to illustrate the tails getting fatter as the scale is going smaller. Bottom: same S&P 500 data as in Fig.1. The scales range from 10 min to 1 month. Top: MRW Model with  $\lambda^2 = 0.03$ ,  $\Delta = 1/16$  and T = 8192. The scales range from 16 sampling size to 2 integral scales. Estimation ( $\circ$ ) made from 500 MRW realizations of  $2^{17}$  sampled points. The solid line corresponds to the prediction from the largest scale using equation (10).

Fig 3: Log-volatility covariance  $C_{\omega}(\ell,\tau)$  of the S&P 500 future. Top:  $C_{\omega}(\ell,\tau)$  versus  $\tau$  for  $\tau=10$  min. The solid line represents a fit according to the MRW model logarithmic expression. Bottom:  $C_{\omega}(\ell,\tau)$  versus  $\ln(\tau)$ . The MRW model predicts a linear behavior that crosses the y-axis at  $\ln(\tau) = \ln(T)$ .

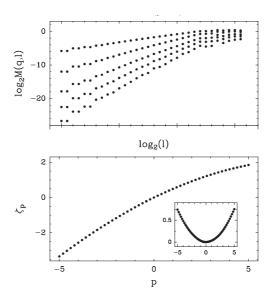


Figure 1:

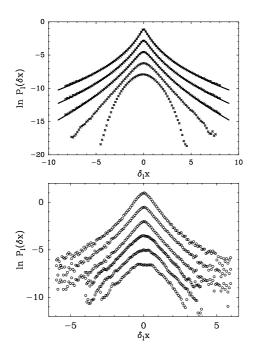


Figure 2:

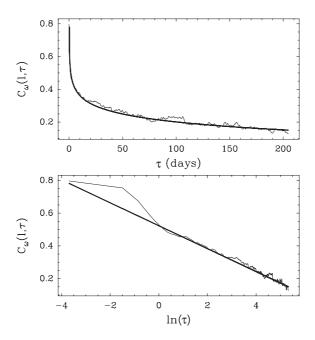


Figure 3: