Long time behavior for the partition function of multiplicative cascades

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Abstract. In this note, we present results on the behavior for the partition function of multiplicative cascades in the case where the total time of observation is large, compared to the scale of decay for the correlation of the cascade process.

1 Introduction

Multifractal processes have been used successfully in many applications which involve series with invariance scaling properties. In finance, they have been shown to reproduce very accurately the major "stylized facts" of time-series (see Bouchaud & Potters 2003, Bacry *et al* 2007). Most of the information on the scaling properties of a process X is contained in the behavior at high frequency, $\delta_n^{-1} \to \infty$, of its partition function:

$$n^{-1} \sum_{k=0}^{n-1} \left(X_{(k+1)\delta_n} - X_{k\delta_n} \right)^p .$$

This quantity is also known to be related with singularities of the path of X via the Frisch-Parisi conjecture.

Molchan (1996) and Resnick *et al.* (2003) study the behavior of the partition function when X is the sample path of a multiplicative cascade on $[0,1]$ and $n\delta_n = 1$. However modeling data (X_t) with a cascade process on $[0,1]$

has the drawback that increments are highly correlated throughout the whole path. This situation might not be desirable in many cases.

Here, to relax this constraint, we define X by pasting independent cascade processes. Then, we characterize the behavior of the partition function as $\delta_n \to 0$, while the total time of observation $n\delta_n$ tends to infinity. It appears that the results are significantly changed.

The note is structured as follows. In Section 2 we recall basic facts on multifractal analysis. In Section 3 we present the standard construction of the cascade process on $[0, 1]$ and the initial result of Molchan (1996). Our results in the case of mixed asymptotics are presented in Section 4.

2   Partition function and multifractal formalism

In this section we recall standard facts about multifractal analysis. Let $f : [0, 1] \mapsto \mathbb{R}$ be a function with some minimal global Hölder exponent ϵ i.e. $|f(x) - f(x')| \leq c|x - x'|^\epsilon$. We define, the boundary of the Besov domain of f as the function $s_f : (0, \infty) \to [\epsilon, \infty]$ given by

$$s_f(1/p) = \sup\left\{ \sigma \geq 0 \mid \sup_{j \geq 0} 2^{j\sigma} \left(2^{-j} \sum_{k=0}^{2^j - 1} |f(\frac{k+1}{2^j}) - f(\frac{k}{2^j})|^p \right)^{1/p} < \infty \right\}.$$

This quantity is widely used for studying multifractal functions. If one replace in the definition above the increments of f by wavelet coefficients then the connection with a classical definition of Besov spaces becomes apparent (see Cohen 2000). The following simple proposition (see Jaffard 2000) gives a recap of the basic properties of s_f:

Proposition 1. *The function s_f is an increasing, concave function, with a derivative bounded by 1.*

Proof. Clearly $p \mapsto \left(2^{-j} \sum_{k=0}^{2^j-1} |f_{\frac{k+1}{2^j}} - f_{\frac{k}{2^j}}|^p \right)^{1/p} := \|\Delta f\|_p$ is increasing as a function of p, and thus s_f is increasing. The fact that s_f is concave is a direct consequence of the following interpolation relation:

$\|\Delta f\|_{p_3} \leq \|\Delta f\|_{p_1}^\alpha \|\Delta f\|_{p_2}^{1-\alpha}, \quad$ with $1/p_3 = \alpha/p_1 + (1-\alpha)/p_2, \quad \alpha \in [0, 1]$.

Finally, using for $p_2 \geq p_1$, $\|\Delta f\|_{p_2} \leq 2^{-j(\frac{1}{p_2} - \frac{1}{p_1})} \|\Delta f\|_{p_1}$ we obtain that

$$s(1/p_2) \geq s(1/p_1) + 1/p_2 - 1/p_1, \tag{1}$$

which in turn gives $s'_f \leq 1$. Let us stress that (1) may be seen as a form of the Sobolev embedding.

A important fact is that the local smoothness properties of the function f can be related to the boundary of its Besov domain. To see this, one need to define the space of function $\mathcal{C}^\alpha(x_0)$ for $x_0 \in [0,1]$ and $\alpha > 0$, as the set of function f such that the following control holds in a neighborhood of x_0:

$$|f(x) - P_{x_0}(x)| \leq c|x - x_0|^\alpha,$$

for some constant $c > 0$ and a polynomial P_{x_0} of degree at most $[\alpha]$. Then, for any $H > 0$, one define the set of points where the function is exactly local H-Hölder as

$$S_f(H) := \{x \in [0,1],\ \sup\{\alpha > 0,\ f \in \mathcal{C}^\alpha(x_0)\} = H\}, \tag{2}$$

and the spectrum of singularities of f is defined as:

$$d_f(H) := \dim S_f(H),$$

where possibly $\dim(H) = -\infty$ if $S_f(H) = \emptyset$.

The connection between d_f and s_f is given by the Frisch-Parisi conjecture (also called multifractal formalism) which reads:

$$d_f(H) = \inf_p \{pH - ps_f(1/p) + 1\}. \tag{3}$$

An heuristic explanation of why (3) is plausible can be find in many references (e.g. see Jaffard 2000, Gloter & Hoffmann 2008). Several theoretical results discusses the validity of this relation both in the case of deterministic function f (Jaffard 1997, 2000), or when f is the realization of stochastic process (Molchan 1996, Barral 2004). Let us stress that not all the function f satisfies the multifractal formalism (3).

An interesting fact about (3), is that it links the condition $s'_f \leq 1$ with $d_f(H) \geq 0$, which is clearly necessary for the dimension of a non empty set.

3   Multifractal properties of multiplicative cascades

Multiplicative cascades were introduced in Mandelbrot (1974) and give rise to stochastic processes with multifractal sample paths. We briefly recall the construction of such processes. Let W be a positive random variable with $E[W] = 1$, admitting positive moments of any order. Consider a family of i.i.d. variable with law W:

$$\{W_r, r \in \{0,1\}^j, j \in \mathbb{N}^*\}.$$

Given a j-uplet $r = (r_1, \ldots, r_j)$, for all strictly positive integer $i \leq j$, we note $r|i$ the restriction of the j-uplet to its first i components, $r|i = (r_1, \ldots, r_i)$ and we define I_r as the dyadic interval

$$I_r = [\sum_{l=1}^j \frac{r_l}{2^l}, \sum_{l=1}^j \frac{r_l}{2^l} + \frac{1}{2^j}].$$

For $n \geq 1$, we define μ_n as the measure on $[0, 1]$ with piecewise constant density on dyadic interval of length 2^{-n} and

$$\mu_n(I_r) = 2^{-n} \prod_{l=1}^{n} W_{r|l}, \quad \text{for } r \in \{0, 1\}^n. \tag{4}$$

It is well know that a.s. the measures μ_n have a limit measure μ_∞ known as the 'cascade measure'. Kahane & Peyrière (1976) shows that the condition $E[W \log_2 W] < 1$ is equivalent to the non degeneracy of the limit: $E[\mu_\infty([0, 1])] = 1$. We recall the crucial result of Kahane & Peyrière (1976) about existence of moments for $\mu_\infty([0, 1])$.

Theorem 1. *Define $\tau(p) = p - \log_2 E[W^p]$ for $p > 0$. Then, $\tau(p) > 1$ implies $E[(\mu_\infty([0, 1]))^p] < \infty$.*

Let us remark that by (4), we have for $r \in \{0, 1\}^n$

$$\mu_\infty(I_r) = 2^{-n} \prod_{l=1}^{n} W_{r|l} \, \widetilde{\mu_\infty}([0, 1]),$$

where $\widetilde{\mu_\infty}([0, 1])$ is independent of the $W_{r|l}$, and is distributed as $\mu_\infty([0, 1])$.

Raising to the power p and taking expectation, we deduce the scaling law for moments: if $r \in \{0, 1\}^n$,

$$
\begin{aligned}
E[(\mu_\infty(I_r))^p] &= 2^{-np} E[W^p]^n E[(\mu_\infty([0, 1]))^p] \\
&= (2^{-n})^{\tau(p)} E[(\mu_\infty([0, 1]))^p]. \tag{5}
\end{aligned}
$$

To study the multifractal properties of the cascade process $t \mapsto \mu_\infty([0, t])$ we introduce its partition function as

$$S(j, p) = 2^{-j} \sum_{k=0}^{2^j - 1} \left(\mu_\infty([\frac{k}{2^j}, \frac{k+1}{2^j}]) \right)^p = 2^{-j} \sum_{r \in \{0, 1\}^j} \mu_\infty(I_r)^p.$$

From (5) we get a simple expression for the expectation of the partition function

$$E[S(j, p)] = (2^{-j})^{\tau(p)} E[(\mu_\infty([0, 1]))^p].$$

It is possible to precise the almost sure behavior of $S(j, p)$ (or equivalently of the boundary of the Besov domain for sample paths of the cascade). Define for $p > 0$, $\tilde{s}(1/p) = \tau(p)/p$. Then $\tilde{s}$ is a smooth concave, increasing function and let

$$p^\star = \sup\{p \mid \tilde{s}'(1/p) \leq 1\}. \tag{6}$$

The following result is proved in Molchan (1996).

Theorem 2. *If $p < p^\star$, then*

$$-\frac{1}{j}\frac{1}{p}\log_2\left(S_{j,p}\right) \xrightarrow{j\to\infty} \tilde{s}(\frac{1}{p}), \quad \text{almost surely.}$$

If $p \geq p^\star$, then

$$-\frac{1}{j}\frac{1}{p}\log_2\left(S_{j,p}\right) \xrightarrow{j\to\infty} \tilde{s}(\frac{1}{p^\star}) - \frac{1}{p^\star} + \frac{1}{p}, \quad \text{almost surely.}$$

The fact that as soon as $\tilde{s}'(1/p) = 1$ the boundary of the Besov domain is a straight line is consistent with Proposition 1. This result is known in the literature as a 'linearisation effect' and is consistent with many empirical observations (Mandelbrot 1990). However it happens in empirical works on turbulent data that the behavior of the partition function is sensitive to the ratio between the typical decorrelation time of the series and total time of observation. We analyze this situation in the next Section.

4   Mixed asymptotic

First we define a multifractal process on $\mathbb{R}_+$ in the following way. Consider $(\mu_\infty^{(m)})_{m\in\mathbb{N}}$ a sequence of i.i.d. cascades on $[0,1]$ as defined in the previous section. Then define the process $(X_t)_{t\geq 0}$ by

$$X_t = \sum_{m=0}^{k-1} \mu_\infty^{(m)}\left([0,1]\right) + \mu_\infty^{(k)}\left([k, t-k]\right), \quad \text{for } t \in [k, k+1].$$

It is clear that the typical decorrelation time for the increments of X is 1, meanwhile the local smoothness property of X are the same as for a cascade process on $[0,1]$.

We assume that the process X is observed on $[0,L]$ with a sampling step 2^{-j} and introduce the partition function of the observed signal (assuming L is an integer for simplicity):

$$S(j,L,p) = \frac{1}{2^j L} \sum_{k=0}^{L2^j - 1} (X_{\frac{k+1}{2^j}} - X_{\frac{k}{2^j}})^p.$$

Due to the normalization we have again $E[S(j,L,p)] = (2^{-j})^{\tau(p)} E\left[(\mu_\infty\left([0,1]\right))^p\right]$.

We make the assumption that the total time of observation L tends to infinity and the sampling step 2^{-j} to zero with the relation $L \sim 2^{j\chi}$ as $j \to \infty$ for a fixed $\chi > 0$.

Analogously to (6), we let

$$p^\star(\chi) = \sup\{p \mid \tilde{s}'(\frac{1}{p}) \leq 1 + \chi\}.$$

The following result, whose proof can be found in Bacry *et al* (2008), clarifies the behavior of the partition function under mixed asymptotics:

Theorem 3. *Assume $\tau(p^\star(\chi)) > 0$, then: if $p < p^\star(\chi)$, we have*

$$-\frac{1}{j}\frac{1}{p}\log_2\left(S_{j,p,L}\right) \xrightarrow{j\to\infty} \tilde{s}(\frac{1}{p}), \quad \text{almost surely.}$$

and if $p \geq p^\star(\chi)$, we have

$$-\frac{1}{j}\frac{1}{p}\log_2\left(S_{j,p,L}\right) \xrightarrow{j\to\infty} \tilde{s}(\frac{1}{p^\star(\chi)}) - \frac{1+\chi}{p^\star(\chi)} + \frac{1+\chi}{p}, \quad \text{almost surely.}$$

For financial data, the maximal variation of the observed series is an important quantity. As a by product of the proof of theorem 3 we obtain the following scaling law for the maximal variation of X (see Bacry *et al* (2008)):

$$-\frac{1}{j}\log\left(\sup_{0\leq k\leq L2^j-1}|X_{\frac{k+1}{2^j}} - X_{\frac{k}{2^j}}|\right) \xrightarrow{j\to\infty} \tilde{s}(\frac{1}{p^\star(\chi)}) - \frac{1+\chi}{p^\star(\chi)}.$$

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