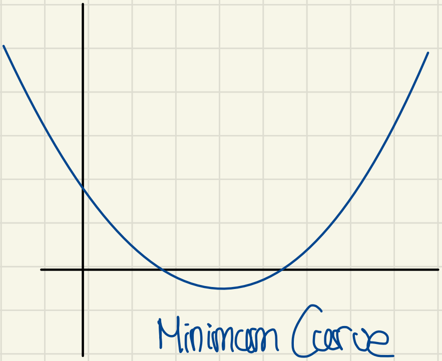
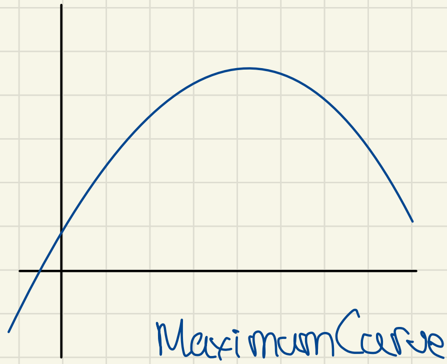
Quadratic Equations





What we need to know before sketching

1 = Nature (Minimum or Maximum
2 = Turning Point (h, k)
3 = Y intercept
4 = X intercept (may not occur)

① Nature

$$y = ax^2 + bx + c$$

$a > 0 \rightarrow$ Min

$a < 0 \rightarrow$ Max

② Turning Point

completing square $\longrightarrow$ $y = ax^2 + bx + c \rightarrow y = a(x-h)^2 + k$

T.P $\rightarrow (h, k)$

completing square → Identities; Conventional (same as math)

$y = -(x+1)^2 + 3$ T.P $(-1, 3)$

$-$ ↓ Tells the nature which is Max

Identities (comparing of coefficients)

$y = x^2 + 8x + 12$

$x^2 + 8x + 12 \equiv a(x-h)^2 + k \rightarrow \quad y = (x+4)^2 - 4$

$\equiv a(x^2 - 2hx + h^2) + k \quad$ T.P $\Rightarrow (-4, -4)$

$x^2 + 8x + 12 \equiv ax^2 - 2ahx + ah^2 + k$

compare the coefficient

$$\boxed{\begin{aligned} a &= 1 \\ h &= -4 \\ k &= -4 \end{aligned}}$$

$y = x^2 + 6x + 5 \longrightarrow y = (x+3)^2 - 4 \rightarrow$ T.P $= (-3, -4)$

Minimum curve

$x^2 + 6x + 5 \equiv ax^2 - 2ahx + ah^2 + k$

$a = 1$

$h = -3$

$k = -4$

$y = x^2 - 10x + 7$

$x^2 - 10x + 7 \Rightarrow ax^2 - 2ahx + ah^2 + k \rightarrow y = (x-5)^2 - 18$

$a = 1 \quad$ T.P $= (5, -18)$

$h = +5$

$k = -18$

Completing the square

① $y = x^2 + 8x + 12$

$y = x^2 + 8x + 4^2 + 12 - 4^2$

$y = (x+4)^2 + 12 - 16$

$y = (x+4)^2 - 4$

$(x+1)^2 = x^2 + 2x + 1^2$

$(x+2)^2 = x^2 + 4x + 2^2$

$(x+3)^2 = x^2 + 6x + 3^2$

$(x+4)^2 = x^2 + \underline{8}x + \underline{4}^2$

The coefficient of x is the double of the squared number.

② $y = x^2 - 7x + 5$

$y = x^2 + 6x + 3^2 + 5 - 3^2$

$\boxed{y = (x+3)^2 - 3}$

$y = 2x^2 + 10x + 3$

completing the square

Identities

$y = 2x^2 + 10x + 3$

$y = 2(x^2 + 5x) + 3$

$y = 2[x^2 + 5x + (2.5)^2 - 2.5^2] + 3$

$y = 2[(x+2.5)^2 - 6.25] + 3$

$y = 2(x+2.5)^2 - 12.5 - 3$

$\boxed{y = 2(x+2.5)^2 - 15.5}$

$y = 2x^2 + 10x + 3$

$2x^2 + 10x + 3 \equiv ax^2 - 2ahx + ah^2 + K$

$a = 2$

$h = -2.5$

$K = -9.5$

$\boxed{2(x+2.5)^2 - 9.5}$

3 = Intercepts

a) x intercepts

$y = 0$

$y = x^2 + 6x + 8$

$0 = x^2 + 6x + 8$

$x^2 + 4x + 2x + 8 = 0$

$x(x+4) + 2(x+4) = 0$

$(x+4)(x+2) = 0$

$\boxed{x = -4 \quad x = -2}$

b) y intercept

$x = 0$

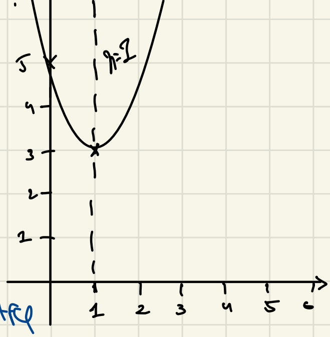
$\boxed{y = 8}$

Examples

① $y = 2(x-1)^2 + 3$

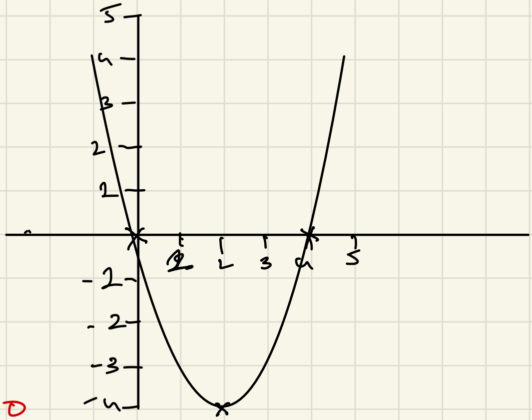
T.P $(1, 3)$

$y = 5$



Equation of line of symmetry

$x = h$



2. $y = x^2 - 4x$

$y = x^2 - 4x + 2^2 - 2^2$

$y = (x-2)^2 - 4$

T.P $= (2, -4)$

$x^2 - 4x = 0$

$x(x-4) = 0$

$x = 0, x = 4$

$y = 0$

③ $y = 9 - x^2$

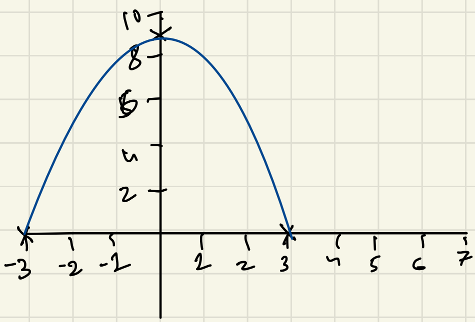
$y = -x^2 + 9$

$y = -(x+0)^2 + 9$

$y = 9$

$x = 3 \quad x = -3$

(T.P) $(0, 9)$



→ Nature of Roots

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$ → value(s) of roots

for nature we use → $b^2 - 4ac$

↳ also known as discriminant.

① $y = x^2 - 6x + 8$

$a = 1 \quad b = -6 \quad c = 8$

$$x = \frac{+6 \pm \sqrt{(-6)^2 - 4(1)(8)}}{2}$$

$$x = \frac{6 \pm \sqrt{4}}{2}$$

$\frac{6+4}{2} \rightarrow 5$

$\frac{6-4}{2} \rightarrow 1$

→ Roots

Nature : Real and distinct

↳ because they are real numbers and different.

when $b^2 - 4ac > 0$

① Real & distinct roots
② 2 Real & distinct roots
③ Curve will cut the x axis at 2 point

(2) $y = x^2 - 4x + 4$

$a = 1 \quad b = -4 \quad c = 4$

$x = \dfrac{+4 \pm \sqrt{(-4)^2 - 4(1)(4)}}{2}$

$b^2 - 4ac = 0$

$x = \dfrac{4 \pm \sqrt{0}}{2} \Rightarrow 2 \rightarrow$ Real and equal Roots

Repeating Roots

Curve is tangent to the x axis.

(3) $y = x^2 + 5x + 10$

$x^2 + 5x + 10 = 0$

$x = \dfrac{-5 \pm \sqrt{-15}}{2} \rightarrow$ no answer

$b^2 - 4ac < 0$

no real roots

2 Find the values of k for which $x^2 + kx + 9 = 0$ has two equal roots.

3 Find the values of k for which $kx^2 - 4x + 8 = 0$ has two distinct roots.

4 Find the values of k for which $3x^2 + 2x + k = 0$ has no real roots.

5 Find the values of k for which $(k+1)x^2 + kx - 2k = 0$ has two equal roots.

(2) $b^2 - 4ac = 0$

$k^2 - 4(1)(9) = 0$

$k^2 - 36 = 0$

$\boxed{k = \pm 6}$

(3) $kx^2 - 4x + 8$

$16 - 4(k)(8) > 0$

$16 - 32k > 0$

$16 > 32k$

$\boxed{\frac{1}{2} > k}$

(4) $3x^2 + 2x + k = 0$

$b^2 - 4ac < 0$

$4 - 12k < 0$

$4 < 12k$

$\boxed{\frac{1}{3} < k}$

(5) $(k+1)x^2 + kx - 2k = 0$

$k^2 - 4(k+1)(-2k) = 0$

$k^2 + 8k(k+1) = 0$

$k^2 + 8k^2 + 8k = 0$

$9k^2 + 8k = 0$

$k(9k + 8) = 0$

$\boxed{k = \frac{-8}{9} \quad k = 0}$

6 Find the values of k for which $kx^2 + 2(k+3)x + k = 0$ has two distinct roots.

7 Find the values of k for which $3x^2 - 4x + 5 - k = 0$ has two distinct roots.

8 Find the values of k for which $4x^2 - (k-2)x + 9 = 0$ has two equal roots.

9 Find the values of k for which $4x^2 + 4(k-2)x + k = 0$ has two equal roots.

10 Show that the roots of the equation $x^2 + (k-2)x - 2k = 0$ are real and distinct for all real values of k.

11 Show that the roots of the equation $kx^2 + 5x - 2k = 0$ are real and distinct for all real values of k.

8) $4n^2 - (k-2)n + 9 = 0$

$b^2 - 4ac = 0$

$\boxed{-(k-2)}^2 - 4(4)(9) = 0$

$(k-2)^2 - 144 = 0$

$(k-2)^2 = 144$

$k-2 = 12$ $\quad$ $k-2 = -12$

$\boxed{k = 14}$ $\quad$ $\boxed{k = -10}$

(6) $kn^2 + 2(k+3)n + k = 0$

$(2k+6)^2 + 4(k)(k) \geq 0$

$4k^2 + 24k + 36 + 4k^2 \geq 0$

$8k^2 + 24k + 36 \geq 0$

$2k^2 + 6k + 9 \geq 0$

$\boxed{n \geq -\frac{3}{2}}$

(10) $n^2 + (k-2)n - 2k = 0$ $\quad > 0$

$b^2 - 4ac > 0$

$(k-2)^2 - 4(1)(-2k) > 0$

$k^2 - 4k + 4 + 8k > 0$

$k^2 + 4k + 4 > 0$

$k^2 + 2k + 2k + 4 > 0$

$k(k+2) + 2(k+2) > 0$

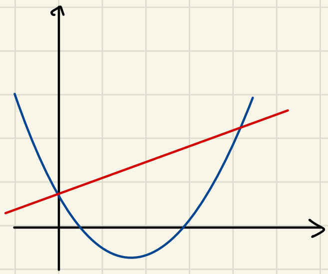
$(k+2)(k+2) > 0$

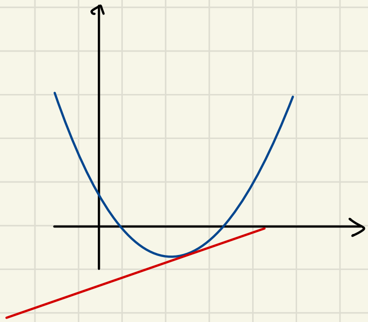
$(k+2)^2 > 0$

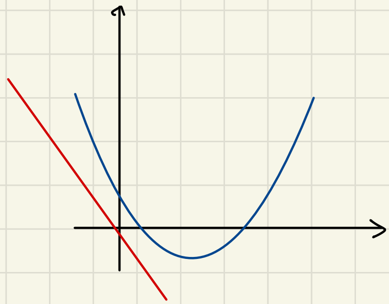
↳ Will always be greater than or equal to zero.

Therefore roots will always be real

Discriminant applied to a line & a curve







$b^2-4ac>0$
Line intersects the curve at two points

$b^2-4ac=0$
Line touches the curve
Line is tangent to the curve

$b^2-4ac<0$
Line does not intersect the curve

Exercise 2.6

1 Find the values of k for which $y = kx + 1$ is a tangent to the curve $y = 2x^2 + x + 3$.

$y = kx + 1 \rightarrow$ line
$y = 2x^2 + x + 3 \rightarrow$ curve

① $2x^2 + x + 3 = kx + 1$
$2x^2 + x - kx + 3 - 1 = 0$
$2x^2 + x(1-k) + 2 = 0$
$2x^2 + (1-k)x + 2 = 0$

$b^2 - 4ac = 0$
$(1-k)^2 - 4(2)(2) = 0$
$1 - 2k + k^2 - 16 = 0$
$k^2 - 2k - 15 = 0$
$k^2 - 5k + 3k - 15 = 0$
$k(k-5) + 3(k-5) = 0$
$(k-5)(k+3)$
$k = 5$, $k = -3$

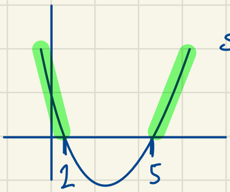
4 Find the set of values of k for which the line $y = 3x + 1$ cuts the curve $y = x^2 + kx + 2$ in two distinct points.

$y = 3x + 1 \Rightarrow$ line
$y = x^2 + kx + 2 \Rightarrow$ curve

$x^2 + kx + 2 - 3x - 1 = 0$
$x^2 + kx - 3x + 1 = 0$
$x^2 + (k-3)x + 1 = 0$
$b^2 - 4ac > 0$

because $b^2 - 4ac > 0$ so

$(k-3)^2 - 4(1)(1) > 0$
$k^2 - 6k + 9 - 4 > 0$
$k^2 - 6k + 5 > 0$
$k^2 - 5k - k + 5 > 0$
$k(k-5) - 1(k-5) > 0$
$(k-5)(k-1) > 0$
$k = 5$ OR $k = 1$



so the range would be

$1 > k < 5$

3 Find the values of the constant c for which the line $y = x + c$ is a tangent to the curve $y = 3x + \frac{2}{x}$.

$y = x + c$
$y = 3x + \frac{2}{x}$

$x + c = 3x + \frac{2}{x}$

$2x + \frac{2}{x} + c = 0$

$2x^2 + 2 = cx$
$2x^2 - cx + 2 = 0$

$b^2 - 4ac = 0$
$(-c)^2 - 4(2)(2) = 0$
$c^2 - 16 = 0$
$(c+4)(c-4) = 0$
$\boxed{c = 4 \quad c = -4}$

6 Find the set of values of k for which the line $y = k - x$ cuts the curve $y = x^2 - 7x + 4$ in two distinct points.

Qs6)

$y = k - x$

$y = x^2 - 7x + 4$

$x^2 - 6x + 4 - k = 0$

$b^2 - 4ac > 0$

$(-6)^2 - 4(2)(4-k) > 0$

$36 - 16 + 4k > 0$

$20 + 4k > 0$

$4k > -20$

$\boxed{k > -5}$

7 Find the values of k for which the line $y = kx - 10$ meets the curve $x^2 + y^2 = 10x$.

7)

$y = kx - 10$

$x^2 + y^2 = 10x$

$x^2 - 10x = y^2$

$x^2 - 10x = (kx - 10)^2$

$x^2 - 10x = k^2x^2 - 20kx + 100$

$x^2 - k^2x^2 - 10x - 20kx - 100 = 0$

$(1 - k^2)x^2 + (-10 - 20k)x - 100 = 0$

$(-10 - 20k)^2 - 4(1 - k^2)(-100) \geq 0$

$100 + 400k + 4\cancel{00}k^2 + 400 - 4\cancel{00}k^2 \geq 0$

$-500 \leq 400k$

$\boxed{-\frac{5}{4} \leq k}$

→ Disguised Quadratic Equations

① $x + 6\sqrt{x} + 5 = 0$

Substitution

$\sqrt{x} = a$

$x = a^2$

$a^2 + 6a + 5 = 0$

$a^2 + 5a + a + 5 = 0$

$a(a+5) + 1(a+5) = 0$

$(a+5)(a+1) = 0$

$a = -5 \quad , a = -1$

No solution

② $x - 8\sqrt{x} + 12 = 0$

$a = \sqrt{x}$

$a^2 - 8a + 12 = 0$

$a^2 - 6a - 2a + 12 = 0$

$a(a-6) - 2(a-6) = 0$

$(a-6)(a-2)$

$a = 6 \qquad a = 2$

$\sqrt{x} = 2 \qquad \sqrt{x} = 6$

$\boxed{x = 4 \qquad x = 36}$

Ans1)

$y = mx - 5$
$y = x^2 + 3x + 4$

$mx - 5 = x^2 + 3x + 4$
$0 = x^2 + 3x - mx + 9$

$(3-m)^2 - 4(1)(9) = 0$
$9 - 6m + m^2 - 36 = 0$
$m^2 - 6m - 27 = 0$
$m^2 - 9m + 3m - 27 = 0$
$m(m-9) + 3(m-9) = 0$
$(m-9)(m+3)$
$m = 9 \quad m = -3$

Ans2)

$y = 2x^2 + kx + 2k - 6$

$k^2 - 16k + 48 < 0$
$(k-4)(k-12) < 0$

$\boxed{4 < k < 12}$

Ans3)
(i) $8 + 7x - x^2$

$-x^2 + 7x + 8$
$-(x^2 - 7x - 8)$
$-\left(x^2 - 7x + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 - 8\right)$
$-\left[\left(x - \frac{3}{2}\right)^2 - 10.25\right] \Rightarrow \boxed{10.25 - \left(x - \frac{3}{2}\right)^2}$

(ii) $(1.5, 10.25)$

(iii) $8 + 7z^2 - z^4 = 0$ $\qquad z^2 = y$

$8 + 7y - y^2 = 0$

$8 = z^2$ $\qquad -1 = z^2$

$\boxed{\sqrt{8} = z}$ $\qquad$ not possible

$y^2 - 7y - 8 = 0$

$y^2 - 8y + y - 8 = 0$

$y(y-8) + 1(y-8) = 0$

$y = 8,\ y = -1$

4) $y = kx + 3$ $\qquad y = x^2 + 5x + 12$

$kx + 3 = x^2 + 5x + 12$

$0 = x^2 + (5-k)x + 9$

$(5-k)^2 - 4(1)(9) < 0$

$25 - 10k + k^2 - 36 < 0$

$k^2 - 10k - 11 < 0$

$k^2 - 11k + k - 11 < 0$

$k(k-11) + 1(k-11) < 0$

$(k-11)(k+1) < 0$

$\boxed{-1 < x < 11}$

5

(i) $y = 2x^2 - 4x - 7 \equiv a(x-h)^2 + k$

$2x^2 - 4x - 7 \equiv ax^2 - 2ahx + ah^2 + k$

$a = 2$ $\boxed{2(x-1)^2 - 9}$

$h = 1$

$k = -9$

(ii) $(1, -9)$

(iii) $2p - 4\sqrt{p} - 7 = 0$ $\sqrt{p} = x$

$2x^2 - 4x - 7 = 0$

$2(x-1)^2 - 9 = 0$

$(x-1)^2 = \pm 4.5$

$x - 1 = \pm \frac{3\sqrt{2}}{2}$

$x = 3.12$ $x = -1.12$

$\sqrt{p} = 3.12$ $\sqrt{p} = -1.12$

$\boxed{p = 9.73}$ $\boxed{p = -1.25}$