DL : VA continues et discrètes

Exercice 1

1) Loi log-normale.

- $E(X) = \int_0^{+\infty} x \, p_X(x) \, dx = \frac{1}{\sigma\sqrt{2\pi}} \int_0^{+\infty} \exp\left(-\frac{(\ln(x)-\mu)^2}{2\sigma^2}\right)$

on pose $u = \ln(x)$ donc $x = e^u$ et $dx = e^u du$

donc $E(X) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left(-\frac{(u-\mu)^2}{2\sigma^2} + u\right) du$

On a $\frac{-(u-\mu)^2}{2\sigma^2} + u = -\frac{1}{2\sigma^2}\left(u^2 - 2u\mu - 2\sigma^2 u + \mu^2\right)$

$= -\frac{1}{2\sigma^2}\left(\left[u-(\mu+\sigma^2)\right]^2 - (\mu+\sigma^2)^2 + \mu^2\right)$

donc $E(X) = \frac{\exp\left((\mu+\sigma^2)^2-\mu^2\right)}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2\sigma^2}\left(u-(\mu+\sigma^2)\right)^2\right) du$

$= \exp\left(\frac{(\mu+\sigma^2)^2-\mu^2}{2\sigma^2}\right) \Rightarrow E(X) = e^{\mu+\frac{\sigma^2}{2}}$

- On a $V(X) = E(X^2) - (E(X))^2$

$E(X^2) = \frac{1}{\sigma\sqrt{2\pi}} \int_0^{+\infty} \exp\left(-\frac{(\ln(x^2)-\mu)^2}{2\sigma^2}\right) dx$

On prend $u = \ln(x^2)$ donc $dx = e^u du$

donc $E(X^2) = \exp\left(\frac{(\mu+2\sigma^2)^2-\mu^2}{2\sigma^2}\right)$

$= e^{2(\mu+\sigma^2)}$

d'où $V(X) = e^{2(\mu+\sigma^2)} - e^{2(\mu+\frac{\sigma^2}{2})}$

$= (e^{\sigma^2}-1)\, e^{2\mu+\sigma^2}$

2) Loi de Rayleigh

- $E(X) = \int_0^{+\infty} \frac{x^2}{\sigma^2} \exp\left(-\frac{x^2}{2\sigma^2}\right) dx = \frac{\sqrt{2\pi}}{\sigma} \underbrace{\int_0^{+\infty} \frac{x^2}{\sqrt{2\pi}\sigma} \exp\left(-\frac{x^2}{2\sigma^2}\right) dx}_{=\frac{\sigma^2}{2}}$ (loi de Gauss)

donc $E(X) = \sigma\sqrt{\frac{\pi}{2}}$

- $E(X^2) = \int_0^{+\infty} \frac{x^3}{\sigma^2} \exp\left(-\frac{x^2}{2\sigma^2}\right) dx$

On utilise une intégration par partie $\begin{cases} u = x^2 \\ u' = 2x \end{cases}$ $v' = \frac{x}{\sigma^2}\exp\left(-\frac{x^2}{2\sigma^2}\right)$ $v = 2\exp(-x^2/2\sigma^2)$

donc $E(X^2) = 0 + 2\int_0^{+\infty} x \exp\left(-\frac{x^2}{2\sigma^2}\right) dx$

$= 2\sigma^2$ d'où $V(X) = 2\sigma^2 - \sigma^2\frac{\pi}{2} = \boxed{\frac{4-\pi}{2}\sigma^2}$

3) Loi exponentielle

- $E(X) = \int_0^{+\infty} x \lambda e^{-\lambda x} dx$

on fait intégration par partie $\begin{cases} u = x \\ u' = 1 \end{cases}$ $v' = \lambda e^{-\lambda x}$ $v = e^{-\lambda x}$

$E(X) = 0 + \int_0^{+\infty} e^{-\lambda x} dx = \boxed{\frac{1}{\lambda}}$

- $E(X^2) = \int_0^{+\infty} x^2 \lambda e^{-\lambda x} dx$

On a $\begin{cases} u = x^2 \\ u' = 2x \end{cases}$ $v' = \lambda e^{-\lambda x}$ $v = -e^{-\lambda x}$

donc $E(X^2) = 0 + 2\int_0^{+\infty} x e^{-\lambda x} dx = 0 + \frac{2}{\lambda}\int_0^{+\infty} e^{-\lambda x} = 2/\lambda^2$

donc $V(X) = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \boxed{\frac{1}{\lambda^2}}$

4) Loi Gamma:

- $E(X) = \frac{\lambda^n}{\Gamma(n)} \int_0^{+\infty} x^n e^{-\lambda x} dx = \frac{\lambda^n}{\lambda\Gamma(n)} \int_0^{+\infty} e^{-y} \left(\frac{y}{\lambda}\right)^n dy$

$= \frac{\lambda^n}{\Gamma(n)} \frac{\Gamma(n+1)}{\lambda^{n+1}} = \boxed{\frac{n}{\lambda}}$

- $E(X^2) = \frac{\lambda^n}{\lambda^{n+2}\Gamma(n)} \int_0^{+\infty} e^{-y} y^{n+1} dy = \frac{n(n+1)}{\lambda^2}$

donc $V(X) = \frac{n(n+1)}{\lambda^2} - \frac{n^2}{\lambda^2} = \boxed{\frac{n}{\lambda^2}}$

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5) Loi géométrique.

- $E(X) = \sum_{k=1}^{+\infty} k p (1-p)^{k-1} = p \sum_{k=1}^{+\infty} k (1-p)^{k-1}$

on a $\forall t \in]0,1[\quad \frac{1}{1-t} = \sum_{k=0}^{+\infty} t^k$

alors $\forall t \in]0,1[\quad \sum_{k=1}^{+\infty} k t^{k-1} = \frac{1}{(1-t)^2}$

donc $E(X) = \frac{p}{(1-(1-p))^2} \Longrightarrow \boxed{E(X) = \frac{1}{p}}$

- $E(X^2) = \sum_{k=1}^{+\infty} k^2 p(1-p)^{k-1} = \sum_{k=1}^{+\infty} k(k-1) p(1-p)^{k-1} + \sum_{k=1}^{+\infty} k p (1-p)^{k-1}$

$= p(1-p) \sum_{2}^{+\infty} k(k-1)(1-p)^{k-2} + \frac{1}{p} = \frac{2p(1-p)}{(1-(1-p))^3} + \frac{1}{p}$

donc $V(X) = \frac{2p(1-p)}{(1-(1-p))^3} + \frac{1}{p} - \frac{1}{p^2} = \boxed{\frac{1-p}{p^2}}$

6) Loi de Poisson.

- $E(X) = \sum_{k=0}^{+\infty} k \frac{\lambda^k}{k!} e^{-\lambda} = \sum_{k=1}^{+\infty} \frac{\lambda^k}{(k-1)!} e^{-\lambda} = \lambda e^{-\lambda} \sum_{1}^{+\infty} \frac{\lambda^{k-1}}{(k-1)!}$

$= \lambda e^{-\lambda} e^{\lambda} = \boxed{\lambda}$

- $E(X^2) = \sum_{k=0}^{+\infty} k^2 \frac{\lambda^k}{k!} e^{-\lambda} = \sum k(k-1) \frac{\lambda^k}{k!} e^{-\lambda} + \sum k \frac{\lambda^k}{k!} e^{-\lambda}$

$= \lambda^2 e^{-\lambda} \sum \frac{\lambda^{k-2}}{(k-2)!} + \lambda = \lambda^2 + \lambda$

donc $V(X) = \boxed{\lambda}$