Lab 3: implementation of the Weissenger method

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1 Introduction

The Weissenger method is based on the idea of describing the circulation distribution on a wing with a grid of horseshoes vortices distributed along the wing span and the profile chords. Beyond the same assumption of incompressible and irrotational flow adopted for the Hess-Smith method, it is also assumed that the wing is thin enough and that sweep and dihedral angles are small.

2 Circulation distribution

Let assume that the wing is discretized in $2M$ longitudinal sectors (along its span), each with N panels (along the chord of the $2M$ wing sections corresponding to the longitudinal sectors). In this way, there is a total of $2M \times N$ (2D) panels, each with a vortex with intensity Γ_k. The non-penetration condition is imposed for each panel, obtaining $2M \times N$:

$$\left(\sum_{j=1}^{2M \times N} \boldsymbol{v}_{jk} + \boldsymbol{U}_\infty \right) \cdot \hat{\boldsymbol{n}}_k = 0 \,. \tag{1}$$

The Kutta condition is imposed only approximately, with an appropriate choice of the vortex and control point for each panel, which are at $1/4$ and $3/4$ of the panel chord, respectively, as illustrated in Fig. 1.

3 Velocity distribution

Similarly to what done for the Hess-Smith method, it is now necessary to write the velocity induced by the generic panel. This is the integral of the velocity induced by the elements of each filament:

$$\mathrm{d}\boldsymbol{v}_C = \frac{\Gamma}{4\pi} \frac{\mathrm{d}\boldsymbol{l} \times \boldsymbol{r}}{|\boldsymbol{r}|^3} \quad \Rightarrow \quad \boldsymbol{v}_C = \frac{\Gamma}{4\pi} \int_1^2 \frac{\mathrm{d}\boldsymbol{l} \times \boldsymbol{r}}{|\boldsymbol{r}|^3} \,. \tag{2}$$

In this expression, $\mathrm{d}\boldsymbol{l}$ and $\boldsymbol{r}$ are a short element of the vortex line and the distance from the control point, respectively. We now introduce the angle β between the direction $\boldsymbol{l}$ and
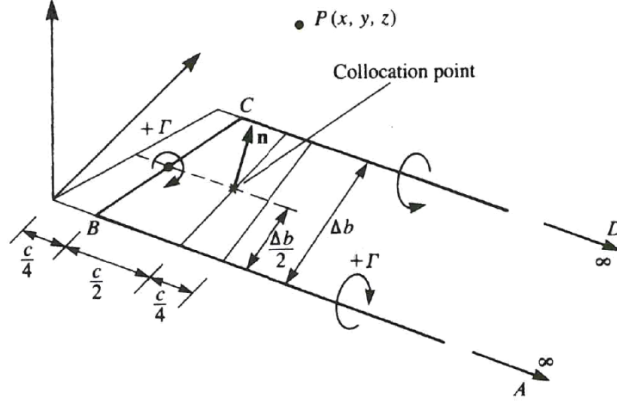
Figure 1: Vortices and control point for a wing element in the Weissenger method.
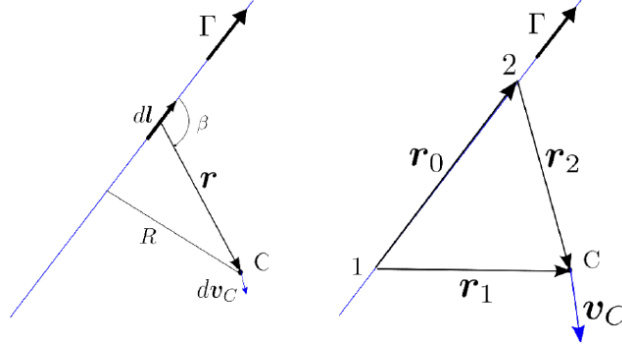

Figure 2: Notation for induced velocity computed in a point.

$\boldsymbol{r}$; and the distance between the extreme of the segment and the control point, $\boldsymbol{r}_1$ e $\boldsymbol{r}_2$, as shown in Fig. 2. We can then write the vector product in the integral as:

$$\boldsymbol{v}_C = \left(\frac{\Gamma}{4\pi} \int_1^2 \frac{\sin\beta}{|\boldsymbol{r}|^2} \mathrm{d}l \right) \hat{\boldsymbol{e}}_C, \quad \text{where:} \quad \hat{\boldsymbol{e}}_C = \frac{\boldsymbol{r}_1 \times \boldsymbol{r}_2}{|\boldsymbol{r}_1 \times \boldsymbol{r}_2|}. \tag{3}$$

To solve these integrals, we write $\mathrm{d}l$ and the distance module as function of β:

$$\mathrm{d}l = \frac{R}{\sin^2\beta} \mathrm{d}\beta \qquad |r| = \frac{R}{\sin\beta}, \tag{4}$$

from which follows:

$$\boldsymbol{v}_C = \frac{\Gamma}{4\pi R} (\cos\beta_1 - \cos\beta_2) \hat{\boldsymbol{e}}_C. \tag{5}$$

To obtain an expression of $\boldsymbol{v}_C$ that depends only on the distance from the estrema, we can define $\boldsymbol{r}_0 = \boldsymbol{r}_2 - \boldsymbol{r}_1$ and write:

$$\cos\beta_1 = \frac{\boldsymbol{r}_0 \cdot \boldsymbol{r}_1}{|\boldsymbol{r}_0||\boldsymbol{r}_1|}, \quad \cos\beta_2 = \frac{\boldsymbol{r}_0 \cdot \boldsymbol{r}_2}{|\boldsymbol{r}_0||\boldsymbol{r}_2|} \quad \& \quad R = \frac{|\boldsymbol{r}_1 \times \boldsymbol{r}_2|}{|\boldsymbol{r}_0|}. \tag{6}$$

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Substituting these expression in (5):

$$\boldsymbol{v}_C = \frac{\Gamma}{4\pi}\boldsymbol{r}_0 \cdot \left(\frac{\boldsymbol{r}_1}{|\boldsymbol{r}_1|} - \frac{\boldsymbol{r}_2}{|\boldsymbol{r}_2|}\right)\frac{\boldsymbol{r}_1 \times \boldsymbol{r}_2}{|\boldsymbol{r}_1 \times \boldsymbol{r}_2|^2}\,. \tag{7}$$

Note that this expression is also useful to compute the velocity of the vortex elements that extend towards the wake, using a length that is much larger than the wind size. We can then introduce $\boldsymbol{q}_j k$, defined as the velocity caused by panel j on the control point on panel k, and for $\Gamma = 1$. With this definition, we can write the system (1) as:

$$\left(\sum_{j=1}^{2M \times N} \boldsymbol{q}_{jk}\Gamma_k + U_\infty\right) \cdot \hat{\boldsymbol{n}}_k = 0\,, \quad \forall k\,, \tag{8}$$

Or, using the vector notation:

$$\boldsymbol{A\Gamma} = \boldsymbol{b}\,, \tag{9}$$

where:

$$A_{jk} = \boldsymbol{q}_{jk} \cdot \hat{\boldsymbol{n}}_k\,, \quad b_k = -\boldsymbol{U}_\infty \cdot \hat{\boldsymbol{n}}_k\,. \tag{10}$$

Solving this system we can find the list of vorticity intensities in all panel, Γ_k.

4 Aerodynamic loads

Lift if computed directly using the Kutta-Joukowsky theorem. The lift contribution for the i longitudinal sector of the wing is defined as:

$$L_{2D,i} = \sum_{j=1}^{N} \rho U_\infty \Gamma_{ij} \cdot \cos(\delta)\,, \tag{11}$$

where δ is the dihedral angle. The total lift is then:

$$L = \sum_{i=1}^{2M} \Delta b_i L_{2D,i}\,, \tag{12}$$

where Δb_i is the width of the longitudinal sector i. The (induced) drag coefficient is estimated using the wake contribution of each longitudinal section, and the its corresponding induced velocity, $\boldsymbol{v}_{ind,i}$, which is computed at $1/4$ of the chord of the entire longitudinal sector i. The angle of induced lift is then defined as:

$$\alpha_{ind,i} = \tan^{-1}\left(\frac{\boldsymbol{v}_{ind,i} \cdot \hat{\boldsymbol{e}}_i}{U_\infty}\right)\,, \tag{13}$$

where $\hat{\boldsymbol{e}}_i$ is the normal direction, and the induced drag for the section and the whole wing are, respectively:

$$D_{2D,i} = L_{2D,i}\sin(\alpha_{ind,i}) \tag{14}$$

and:

$$D = \sum_{i=1}^{2M} \Delta b_i D_{2D,i}\,. \tag{15}$$

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5   Summary

In the Weissinger method:

- We use $2M \times N$ panels to discretize the wing surface.

- Each panel is associated to a horse vortex of intensity Γ_k.

- The induced velocity from each vortex is evaluated with the Biot-Savart law.

- The $2M \times N$ values Γ_k can be evaluated imposing the non-penetration condition on the control point of each panel (the Kutta condition is approximately fulfilled, with this choice of discretization).

- The $2D$ lift distribution along the wing span is obtained from the Kutta-Jukowski theoreom.

- The induced angle of attack is computed considering only wake vortices associated with each panel.

It is then possible to estimate both lift and induced drag.

6   Airfoils and wing-tail interaction

We can study configurations with multiple lifting bodies, as in the case of the wing and tail plane of an aircraft, simply adding a new set of panels. Let us consider the case of a wing, discretized with M_w panels over the span and N_w panels over the profile chords, and a tail plane, discretized with M_t and M_t, over the span and the chord, respectively. The total number of panel is then:

$$2M_w \times N_w + 2M_t \times N_t \,, \tag{16}$$

corresponding to the two sets of unknowns, Γ^w and Γ^t, for the wing and the tail plane, respectively. As for the case of a single wing, the non-penetration condition must be imposed for all control points, and the induced angle of attack, α_i, is the result of the wake vortices of all panels.

With the notation that we adopted, the velocity field is expressed as:

$$\boldsymbol{u}(x,y) = \boldsymbol{U}_\infty + \sum_{j=1}^{2M_w \times N_w} \Gamma_j^w \boldsymbol{q}_j^w(x,y) + \sum_{j=1}^{2M_t \times N_t} \Gamma_j^t \boldsymbol{q}_j^t(x,y) \,, \tag{17}$$

and the set of non-penetration conditions are written as:

$$\begin{cases} \boldsymbol{u}(x_{c,i}^w, y_{c,i}^w) = \boldsymbol{b}_i^w \,, & i = 1, \ldots, 2M_w \times N_w \\ \boldsymbol{u}(x_{c,i}^t, y_{c,i}^t) = \boldsymbol{b}_i^t \,, & i = 1, \ldots, 2M_t \times N_t \end{cases} \tag{18}$$

Leading to the following linear system:

$$\begin{bmatrix} \boldsymbol{A}_{ww} & \boldsymbol{A}_{tw} \\ \boldsymbol{A}_{wt} & \boldsymbol{A}_{tt} \end{bmatrix} \begin{bmatrix} \boldsymbol{\Gamma}^w \\ \boldsymbol{\Gamma}^t \end{bmatrix} = \begin{bmatrix} \boldsymbol{b}^w \\ \boldsymbol{b}^t \end{bmatrix} , \tag{19}$$

where, using $N_w^* = 2M_w \times N_w$ e $N_t^* = 2M_t \times N_t$ for brevity:

$$A_{ww,ij} = \boldsymbol{q}_j^w \left(x_{c,i}^w, y_{c,i}^w \right) \cdot \boldsymbol{n}_i^w , \quad i = 1, \ldots, N_w^* \quad \& \quad j = 1, \ldots, N_w^* \tag{20}$$

$$A_{tw,ij} = \boldsymbol{q}_j^t \left(x_{c,i}^w, y_{c,i}^w \right) \cdot \boldsymbol{n}_i^w , \quad i = 1, \ldots, N_w^* \quad \& \quad j = 1, \ldots, N_t^* \tag{21}$$

$$A_{wt,ij} = \boldsymbol{q}_j^w \left(x_{c,i}^t, y_{c,i}^t \right) \cdot \boldsymbol{n}_i^t , \quad i = 1, \ldots, N_t^* \quad \& \quad j = 1, \ldots, N_w^* \tag{22}$$

$$A_{tt,ij} = \boldsymbol{q}_j^t \left(x_{c,i}^t, y_{c,i}^t \right) \cdot \boldsymbol{n}_i^t , \quad i = 1, \ldots, N_t^* \quad \& \quad j = 1, \ldots, N_t^* \tag{23}$$

$$b_i^w = - \boldsymbol{U}_\infty \cdot \boldsymbol{n}_i^w , \quad i = 1, \ldots, N_w^* \tag{24}$$

$$b_i^t = - \boldsymbol{U}_\infty \cdot \boldsymbol{n}_i^t , \quad i = 1, \ldots, N_t^* . \tag{25}$$

Once this system has been resolved, the distribution of Γ is known for both wing and tail planes, and aerodynamic loads are computed as for a single wing (keeping in mind that wake vortices of all panels contribute to all panels.

7 Ground effect

A similar method can be used to study the ground effect, adding an image system at the appropriate distance and with vortices of opposite intensities than the one for the real body or bodies. In this case, the velocity field at the control point of each panel are determined using all panels, and the non-penetration condition is imposed on the real wing: contrary to the addition of tail planes, the number of unknowns does not change in this case.

The new velocity is modified as:

$$\boldsymbol{u}(x,y) = \underbrace{\boldsymbol{U}_\infty}_{\text{Incoming flow}} + \underbrace{\sum_{j=1}^{2M \times N} \Gamma_j \boldsymbol{q}_j(x,y)}_{\text{Real system}} - \underbrace{\sum_{j=1}^{2M \times N} \Gamma_j \boldsymbol{q}_{m,j}(x,y)}_{\text{Image system}} , \tag{26}$$

the non-penetration conditions are:

$$\boldsymbol{u}(x_{c,i}, y_{c,i}) \cdot \boldsymbol{n}_i = \boldsymbol{b}_i , \tag{27}$$

and the linear system $\boldsymbol{A}\boldsymbol{\Gamma} = \boldsymbol{b}$ is modified as:

$$A_{ij} = \left(\boldsymbol{q}_j(x_{c,i}, y_{c,i}) - \boldsymbol{q}_{m,j}(x_{c,i}, y_{c,i}) \right) \cdot \boldsymbol{n}_i . \tag{28}$$