Hess-Smith method

November 2025

1 Introduction

In the hypothesis of incompressible and irrotational flows with negligible viscosity, the velocity field $u(x)$ is the gradient of the real potential, ϕ. Since ϕ is a solution the Laplace equation, the superposition principle holds and the we can write the potential as:

$$\phi = \sum_{k=1}^{N} c_k \phi_k \,, \tag{1}$$

where ϕ_k are potentials of simple known cases with a given intensity (chosen so that ϕ is suitable to represent a class of problem of interest), and c_k are unknown coefficient that must be found.

In the Hess-Smith method, the surface of a (2D) airfoil is made of a closed line discretized in segments, or *panels*. At each panel a constant distribution of irrotational vortices and sources is assigned. The intensity of the vortex distribution, γ, is uniform over the airfoil surface, while the intensity of each source, q_i, are independent. In this way, for N panels we can create a system of $N+1$ unknowns, which is written imposing the non-penetration condition for each panel and the Kutta condition at the trailing edge.

2 Linear system

Writing the linear system that must be solved means to write the generic velocity field that is associated with the potential ϕ and imposing the conditions that it must satisfy. The velocity field can be written as:

$$\boldsymbol{u}(x) = \boldsymbol{U}_\infty + \sum_{j=1}^{N} \boldsymbol{u}_j^s(x) q_j + \sum_{j=1}^{N} \boldsymbol{u}_j^v(x) \gamma \,. \tag{2}$$

In this expression, $\boldsymbol{U}_\infty$ is the velocity of the incoming flow and $\boldsymbol{u}_j^s(x)$ and $\boldsymbol{u}_j^v(x)$ denote the velocity associated with sources and vortices, respectively. Introducing the $\boldsymbol{n}_i$ array to each panel, the non-penetration condition for the N panels are written as:

$$\boldsymbol{u}(x_{c,i}) \cdot \boldsymbol{n}_i = 0 \qquad (i = 1, \ldots, N) \,, \tag{3}$$

where $x_{c,i}$ are the coordinates of a control point for each panel, which is typically its mid point (note that in this expression and all following ones there is no sum over repeated indexes, and i only denote the different equation for each N panel).

The Kutta condition requires that the velocity is continuous at the trailing edge. Assume that the panels are distributed clock-wise starting from the trailing edge, so that $i = 1$ and $i = N$ correspond to the first panel on the pressure side and the last panel on the suction side. In this way, these two panels have a common point, which corresponds to the trailing edge. The normal component of the velocity is already zero for both panels, to satisfy the non-penetration condition. The Kutta condition can thus me imposed imposing that the tangential velocity components are the same. If the tangent vectors for the two panel are denoted with τ_1 and τ_N, the Kutta condition is written as:

$$\boldsymbol{u}(\boldsymbol{x}_{c,1}) \cdot \tau_1 + \boldsymbol{u}(\boldsymbol{x}_{c,N}) \cdot \tau_N = 0\,. \tag{4}$$

The linear system that must be solved is then:

$$\begin{aligned}
&\sum_{j=1}^{N} \boldsymbol{u}_j^s(\boldsymbol{x}_i) \cdot \boldsymbol{n}_i q_j + \sum_{j=1}^{N} \boldsymbol{u}_j^v(\boldsymbol{x}_i) \cdot \boldsymbol{n}_i \gamma = -\boldsymbol{U}_\infty \cdot \boldsymbol{n}_i \\
&\sum_{j=1}^{N} (\boldsymbol{u}_j^s(\boldsymbol{x}_1) \cdot \boldsymbol{\tau}_1 + \boldsymbol{u}^s(\boldsymbol{x}_N) \cdot \boldsymbol{\tau}_N) q_1 + \sum_{j=1}^{N} (\boldsymbol{u}_j^v(\boldsymbol{x}_1) \cdot \tau_1 + \boldsymbol{u}_j^v(\boldsymbol{x}_N) \cdot \boldsymbol{\tau}_N)\gamma = -\boldsymbol{U}_\infty \cdot (\boldsymbol{\tau}_1 + \boldsymbol{\tau}_N)\,.
\end{aligned} \tag{5}$$

Note that $x_{c,i}$ has been substituted with x_i, for the sake of brevity. This system of the equation can be written in matrix form as follows:

$$\begin{bmatrix} \boldsymbol{A}^s & \boldsymbol{a}^v \\ (\boldsymbol{c}^s)^T & c^v \end{bmatrix} \begin{bmatrix} \boldsymbol{q} \\ \gamma \end{bmatrix} = \begin{bmatrix} \boldsymbol{b}^s \\ b^v \end{bmatrix}, \tag{6}$$

where:

$$A_{i,j}^s = \boldsymbol{u}_j^s(\boldsymbol{x}_{c,i}) \cdot \boldsymbol{n} \tag{7}$$

$$a_i^v = \sum_{j=1}^{N} \boldsymbol{u}_j^v(\boldsymbol{x}_{c,i}) \cdot \boldsymbol{n} \tag{8}$$

$$c_j^s = \boldsymbol{u}_j^s(\boldsymbol{x}_{c,1}) \cdot \boldsymbol{\tau}_1 + \boldsymbol{u}_j^s(\boldsymbol{x}_{c,N}) \cdot \boldsymbol{\tau}_N \tag{9}$$

$$c^v = \sum_{j=1}^{N} \boldsymbol{u}_j^v(\boldsymbol{x}_{c,1}) \cdot \boldsymbol{\tau}_1 + \boldsymbol{u}_j^v(\boldsymbol{x}_{c,N}) \cdot \boldsymbol{\tau}_N \tag{10}$$

$$b_i^s = -\boldsymbol{U}_\infty \cdot \boldsymbol{n}_i \tag{11}$$

$$b^v = -\boldsymbol{U}_\infty(\boldsymbol{\tau}_1 + \boldsymbol{\tau}_N)\,. \tag{12}$$

These coefficients must be evaluated for each case, so that it is possible to solve the linear system and obtained the velocity field and pressure distribution.

3 Compute the coefficient

We must now compute the coefficients that are required to solve the linear system. The potential for sources and irrotational vortices is:

$$\phi^s = \frac{Q}{2\pi}\log\left(\sqrt{x^2+y^2}\right) \qquad \phi^v = \frac{\Gamma}{2\pi}\arctan\frac{y}{x} \tag{13}$$

From these, the velocity components are written as:

$$u^s = \frac{\partial \phi^s}{\partial x} = \frac{Q}{2\pi}\frac{x}{x^2+y^2} \tag{14}$$

$$v^s = \frac{\partial \phi^s}{\partial y} = \frac{Q}{2\pi}\frac{y}{x^2+y^2} \tag{15}$$

$$u^v = \frac{\partial \phi^v}{\partial x} = -\frac{\Gamma}{2\pi}\frac{y}{x^2+y^2} \tag{16}$$

$$v^v = \frac{\partial \phi^v}{\partial y} = \frac{\Gamma}{2\pi}\frac{x}{x^2+y^2}\,. \tag{17}$$

These expressions can now be integrated for each j panel to find the velocity $\boldsymbol{u}_j^s(\boldsymbol{x}_{c,i})$ e $\boldsymbol{u}_j^v(\boldsymbol{x}_{c,i})$ at the control point of each i panel.

3.1 Contribution from a single panel

We must write the velocity field generated by the j panel in the control point of the i panel for unitary sources and vortex intensity. We use a reference frame with Cartesian axis (ξ, η) and origin that coincides with one of the extremes of panel j. The reference frame is inclined by an angle β with respect to the (x, y) system, so that the j panel lies on the ξ axis, as shown in Fig. 1. The velocity vectors $\tilde{\boldsymbol{u}}_j^s(\boldsymbol{\zeta}_{c,i})$ and $\tilde{\boldsymbol{u}}_j^v(\boldsymbol{\zeta}_{c,i})$ are then:

$$\tilde{u}_j^s(\xi_{c,i}, \eta_{c,i}) = \frac{1}{2\pi}\int_0^{I_j} \frac{\xi_{c,i}-t}{(\xi_{c,i}-t)^2+\eta_{c,i}^2}\,\mathrm{d}t = -\frac{1}{2\pi}\log\left(\frac{r_2}{r_1}\right) \tag{18}$$

$$\tilde{v}_j^s(\xi_{c,i}, \eta_{c,i}) = \frac{1}{2\pi}\int_0^{I_j} \frac{\eta_{c,i}}{(\xi_{c,i}-t)^2+\eta_{c,i}^2}\,\mathrm{d}t = \frac{\theta_2-\theta_1}{2\pi} \tag{19}$$

$$\tilde{u}_j^s(\xi_{c,i}, \eta_{c,i}) = \frac{1}{2\pi}\int_0^{I_j} \frac{\eta_{c,i}}{(\xi_{c,1}-t)^2+\eta_{c,i}^2}\,\mathrm{d}t = \frac{\theta_2-\theta_1}{2\pi} \tag{20}$$

$$\tilde{v}_j^s(\xi_{c,i}, \eta_{c,i}) = -\frac{1}{2\pi}\int_0^{I_j} \frac{\xi_{c,i}-t}{(\xi_{c,i}-t)^2+\eta_{c,i}^2}\,\mathrm{d}t = \frac{1}{2\pi}\log\left(\frac{r_2}{r_1}\right) \tag{21}$$

In these expressions, I_j denote the length of panel j, and the complex numbers $r_1 e^{i\theta_1} = \boldsymbol{x}_{c,i} - \boldsymbol{x}_{e,j}$ and $r_2 e^{i\theta_2} = \boldsymbol{x}_{c,i} - \boldsymbol{x}_{e,j+1}$ denote the distances between the control points of panel i and the two extrema of panel j. A rotation of $-\beta$ must now be used to go back to the reference frame (x, y):

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix} \begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix}\,. \tag{22}$$

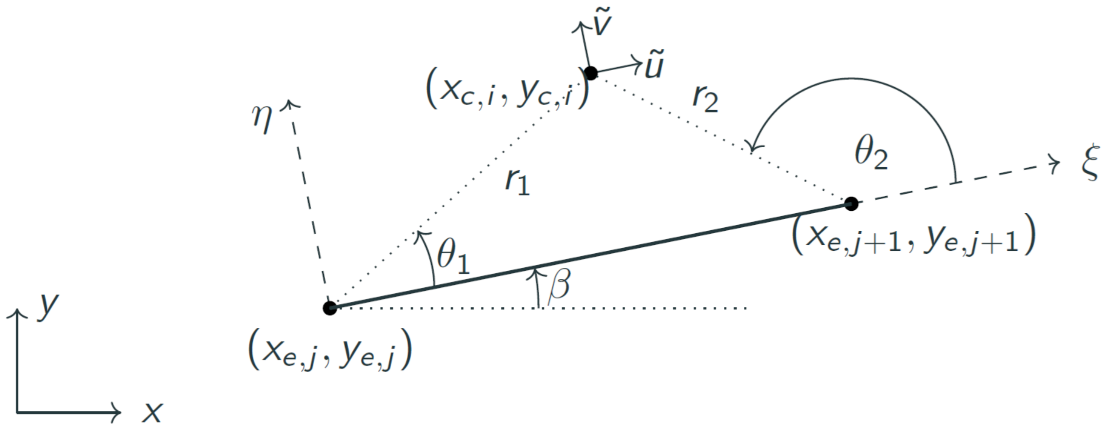

Figure 1: Reference frame to compute the field generate by panel j on the control point of panel i.

The velocity components obtained with these rotation can now be used for each panel to compute the known coefficients of the linear system, and the system can be solved.

4 Aerodynamic loads

After solving the linear system, the intensities γ and q_i becomes known quantities. The velocity field can thus be computed in any point of the domain and on the surface of the airfoil, and used to estimate aerodynamic loads within the limitation of the method.

The total circulation is the sum of the circulation created by the vortices in each panel, which are computed given the length of each panel and the intensity, γ:

$$\Gamma = \sum_{i=1}^{N} I_i \gamma \,. \tag{23}$$

The pressure coefficient is computed for the control point in each panel, using the velocity tangential component and the definition obtained for incompressible and irrotational flows with the Bernoulli theorem:

$$C_{p,i} = 1 - \frac{(\boldsymbol{u} \cdot \boldsymbol{\tau})^2}{|\boldsymbol{U}_\infty|^2} \,, \tag{24}$$

where:

$$\boldsymbol{u}_i \cdot \boldsymbol{\tau}_i = \left(\boldsymbol{U}_\infty + \sum_{j=1}^{N} \boldsymbol{u}_j^s(x_{c,i}, y_{c,i}) q_j + \gamma \sum_{j=1}^{N} \boldsymbol{u}_j^v(x_{c,i}, y_{c,i}) \right) \cdot \boldsymbol{\tau} \tag{25}$$

From these quantities, the lift coefficient can be computed both integrating the pressure distribution:

$$C_l = \sum_{i=1}^{N} C_{p,i} \frac{l_i}{c} \boldsymbol{n}_i \cdot \boldsymbol{n}_{U_\infty} \,, \tag{26}$$

or applying the Kutta-Joukowsky theorem:

$$L = \rho \boldsymbol{U}_\infty \times \Gamma \quad \Rightarrow \quad C_l = -2 \frac{\Gamma}{U_\infty} \tag{27}$$

Finally, the moment coefficient can be computed as:

$$C_{M,LE} = -\sum_{i=1}^{N} C_{p,i} \frac{l_i}{c} (\boldsymbol{r}_{c,i} \times \boldsymbol{n}_{c,i}) \cdot \hat{\boldsymbol{z}} \,. \tag{28}$$

5 Implementation

Summing up, implementing a code that uses the Hess-Smith method requires:

1. **To create a suitable panel distribution**, *i.e.*: to subdivide the line that describes the 2D geometry in panels, and find extrema and tangent vectors.
2. **To compute the velocity contribution from each segment**, *i.e.*: to create a set of functions that uses the double reference system to compute the velocity to each segment in generic points of the domain.
3. **To assemble the linear system and solve it**, *i.e.*: to compute the known coefficients of the system and then solve it, obtaining the intensities of sources and vertices.
4. **To compute velocity fields and forces**.