Online Convex Optimization
Regularized Follow The Leader and its equivalent formulation

Badis Jaouani
Sorbonne Université

1

badis.jaouani@student-cs.fr

Introduction

In an Online Convex Optimization setting, we have seen that the Follow The Leader fails, since it predicts a new point based on all past observations, see (Wintenberger, 2020) for a detailed example showing the failure of FTL. The aim of this short document is to make use of a regularized version of FTL, in order to get sublinear regret bounds. The first section introduces the concept of Online Regularization which leads in a generic algorithm. An equivalent formulation, the Stochastic Mirror Descent is obtained in section 2. The last section introduces the specific quadratic regularization.

1. Online Regularization

Let $\mathcal{K}$ be a convex set, and R be a strongly convex function, twice continuously differentiable. Instead of the FTL formulation

$$x_t^* = argmin_{x \in \mathcal{K}} \sum_{k=1}^{t} f_k(x)$$

the new x_{t+1} now satisfies

$$x_{t+1} = argmin_{x \in \mathcal{K}} \sum_{k=1}^{t} \nabla f_k(x_k)^T x + \frac{1}{\eta} R(x) \tag{1}$$

where the gradient trick is used to derive the scalar product. The aim is to obtain a regularized, thus more stable version of FTL. This results in the following algorithm :

Algorithm 1 Regularized Follow The Leader

Require: Regularization function R, step size $\eta > 0$
Initialization initial prediction $x_1 \in \mathcal{K}$
For each iteration $t \geq 1$
Predict x_t
Incur $f_t(x_t)$
Observe $\nabla f_t(x_t)$
Recursion : update

$$x_{t+1} = argmin_{x \in \mathcal{K}} \sum_{k=1}^{t} \nabla f_k(x_k)^T x + \frac{1}{\eta} R(x)$$

This algorithm defines a class of OCO algorithms, such as Exponentiated Gradient, or Adagrad. Specific properties are obtained for each regularization function R.

2. Online Mirror Descent

Alternatively, we can get a more explicit formulation by introducing the convex duality.

Définition 2.1 (Convex conjugate)**.** *Let g be a function defined on the convex set $\mathcal{K}$ then its convex conjugate g^* is defined on the dual space $\mathcal{K}^*$ as*

$$g^*(x^*) = max_{x \in \mathcal{K}}\{x^t x^* - g(x)\} \tag{2}$$

Taking $g = R$, we get

$$\sum_{t=1}^{T} \nabla f_t(x_t)^T x \leq R^* \left(\sum_{t=1}^{T} \nabla f_t(x_t) \right) + R(x)$$

Online Mirror Descent allows obtaining a bound over $R^* \left(\sum_{t=1}^{T} \nabla f_t(x_t) \right)$. The important point to highlight is that OMD performs gradient descent in the dual space $\mathcal{K}^* = \nabla R(x)^T, x \in \mathcal{K}$. This is done through the regularization function R. Doing so, we need to get back to the primal space $\mathcal{K}$, through the Bergman divergence.

Définition 2.2 (Bergman divergence)**.** *The Bergman divergence associated to the regularization function R is defined as*

$$B_R(y \mid\mid x) = R(y) - R(x) - \nabla R(x)^T (y - x) \tag{3}$$

We obtain the following algorithm, which can be referred to as the lazy version : The

---

Algorithm 2 Online Mirror Descent

Require: Regularization function R, step size $\eta > 0$

Initialization initial prediction $x_1 = argmin_{x \in \mathcal{K}} B_R(x \mid\mid y_1)$ where $y_1 \in \mathbf{R}^d$ s.t $\nabla R(y_1) = 0$

For each iteration $t \geq 1$

Predict x_t

Incur $f_t(x_t)$

Observe $\nabla f_t(x_t)$

Recursion : update

$$\nabla R(y_{t+1}) = \nabla R(y_t) - \eta \nabla f_t(x_t)$$
$$x_{t+1} = argmin_{x \in \mathcal{K}} B_R(x \mid\mid y_{t+1})$$

---

following theorem states the equivalence between RFTL and OMD.

Theorem 2.1 *The Online Mirror Descent algorithm is equivalent to Regularized Follow The Leader.*

Proof.
We show that the minimization steps in both algorithms are equal.

By recursion, we have

$$\nabla R(y_t) = \nabla R(y_{t-1}) - \eta \nabla f_{t-1}(x_{t-1}) = -\eta \sum_{k=1}^{t-1} \nabla f_k(x_k)$$

On the other hand,

$$\begin{aligned} B_R(x \mid\mid y_t) &= R(x) - R(y_t) - \nabla R(y_t)^T (x - y_t) \\ &= R(x) - R(y_t) + \eta \sum_{k=1}^{t-1} \nabla f_k(x_k)^T (x - y_t) \end{aligned}$$

The x-dependent part of this equation is the same minimized by RFTL, which leads to the equivalence between both algorithms.

We are now interested in the regret bound satisfied by both RFTL and OMD. Only the the proof idea is given, since a short document is required.

Theorem 2.2 *OMD and RFTL satisfy the regret bound, of any $u \in K$,*

$$\sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(u) \leq \frac{\eta}{2} \sum_{t=1}^{T} \mid\mid \nabla f_t(x_t) \mid\mid_t^{*2} + \frac{R(u) - R(x_1)}{\eta} \tag{4}$$

where $\mid\mid . \mid\mid_t^{*2} = \mid\mid . \mid\mid_{\nabla^2 R^*(z_t^*)}^{*2}$ for R^* the convex conjugate of R and z_t^* a point in $\mathcal{K}^*$.

Proof.
The proof is based on the gradient trick as a first step.

$$\sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(u) \leq \sum_{t=1}^{T} \nabla f_t(x_t)^T (x_t - u)$$

The obtained bound is then derived by mirror analysis, noticing that $\theta_{t+1} = \theta_t - \eta \nabla f_t(x_t)$ where $\theta_t = \nabla R(y_t)$, we get

$$\sum_{t=1}^{T} \nabla f_t(x_t)^T (x_t - u) = -\frac{1}{\eta} \sum_{t=1}^{T} (\theta_{t+1} - \theta_t)^T \nabla R^*(\theta_t) + \frac{R(u) + R^*(\theta_{t+1})}{\eta}$$

The desired statement follows from recognising $B_{R^*}(\theta_{t+1} \mid\mid \theta_t) = \eta^2 \|\nabla f_t(x_t)\|_t^{*2}$.

3. Quadratic regularization

This setting corresponds to $R(x) = \frac{1}{2}\|x - x_1\|$ for an arbitrary $x_1 \in \mathcal{K}$. Then we can simply derive the equality $B_R(x \mid\mid y) = \frac{1}{2}\|x - y\|^2$ so that we have the following minimization step :

$$x_{t+1} = argmin_{x \in \mathcal{K}} B_R(x \mid\mid y_{t+1}) = \Pi_{\mathcal{K}}(y_{t+1}) \tag{5}$$

Also, since $\nabla R(x) = x$, we get $y_{t+1} = y_t - \eta \nabla f_t(x_t)$. Leading to an Online Gradient Descent algorithm projected on $\mathcal{K}$.

Algorithm 3 OMD for quadratic regularization

Require: Step size $\eta > 0$

Initialization initial prediction $x_1 = y_1 \in \mathcal{K}$

For each iteration $t \geq 1$

Predict x_t

Incur $f_t(x_t)$

Observe $\nabla f_t(x_t)$

Recursion : update

$$y_{t+1} = y_t - \eta \nabla f_t(x_t)$$
$$x_{t+1} = \Pi_{\mathcal{K}}(y_{t+1})$$

The following derivation starts from the statement of theorem 2.2 to get a specific regret bound for quadratic regularization.

$$\begin{aligned}
\sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(u) &\leq \frac{\eta}{4} \sum_{t=1}^{T} \|\nabla f_t(x_t)\|^2 + \frac{\|u - x_1\|^2}{2\eta} \\
&\leq \frac{1}{2}\left(\eta T/2G^2 + \frac{D^2}{\eta}\right) \\
&\leq GD\sqrt{T/2}
\end{aligned}$$

if $\eta = D/G\sqrt{T/2}$.

Implemented on linear SVM training, OMD for quadratic R performs better than Stochastic Gradient Descent, since it does not have an exploratory phase where the algorithm remains stable at an accuracy of around 0.1.

References

Wintenberger, Olivier (2020): *Online Convex Optimization*. , Sorbonne Université - Campus Jussieu. 4 Place Jussieu 75005 Paris.