Convex optimization exercise sheet

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Notation

For a linear operator A, its adjoint is A^* and its Moore-Penrose pseudoinverse is $A^\dagger$. $\mathcal{X}$ and $\mathcal{Y}$ are two Hilbert spaces. Number of coffee cups (☕) indicates difficulty of exercise.

1 Convexity

Exercise 1.1. ☕ *Show that local minimizers of convex functions are global minimizers.*

Exercise 1.2 (Pointwise supremum preserves convexity)**.** ☕ *Let $(f_i)_I$ be a family of convex functions (not necessarily countable). Show that $x \mapsto \sup_{i \in I} f_i(x)$ is convex.*

Exercise 1.3 (Precomposition by linear operator preserves convexity)**.** ☕ *Let $f : \mathcal{Y} \to \mathbb{R}$ be a convex function and $A : \mathcal{X} \to \mathcal{Y}$ a linear operator. Show that $f(A\cdot)$ is convex (on $\mathcal{X}$).*

Exercise 1.4 (Misconceptions on existence of minimizers)**.** ☕ *Provide an example of convex function which does not admit a minimizer.*
What if the function is continuous and lower bounded?

Exercise 1.5. ☕ *Show that a strictly convex function has at most one minimizer.*

Exercise 1.6 (Jensen's inequality)**.** ☕ *Let $f : \mathbb{R}^d \to \mathbb{R}$ be convex. Let $n \in \mathbb{N}$, $x_1, \ldots, x_n \in \mathbb{R}^d$, and let $\lambda_1, \ldots, \lambda_n$ be positive scalars summing to 1.*
Show that $f(\sum_{i=1}^n \lambda_i x_i) \leq \sum_{i=1}^n \lambda_i f(x_i)$.

Exercise 1.7. ☕ *Show that a function is lower semicontinuous if its sublevel sets are closed.*

Exercise 1.8. ☕ *Show that the sublevel sets of a convex function are convex. Find a function with convex sublevel sets which is not convex.*

Exercise 1.9 (First order characterization of convex functions)**.** ☕☕ *Let $f : \mathbb{R}^d \to \mathbb{R}$ be a differentiable function. Show that the following are equivalent:*

1. *f is convex*
2. *f lies above its tangents: $\forall x, y \in \mathbb{R}^d, f(x) \geq f(y) + \langle \nabla f(y), x - y \rangle$*
3. *∇f is monotone: $\forall x, y \in \mathbb{R}^d, \langle \nabla f(x) - \nabla f(y), x - y \rangle \geq 0$*

Exercise 1.10 (Continuity)**.** ☕☕☕ *Show that a convex function is locally Lipschitz (hence continuous) on the interior of its domain.*

Exercise 1.11 (Characterization of strongly convex functions in the Euclidean case)**.** ☕ *Show that f is μ-strongly convex with respect to the Euclidean norm if and only if $f - \frac{\mu}{2}\| \cdot \|^2$ is convex.*

Exercise 1.12. ☕ *Show that a strongly convex function admits exactly one minimizer.*

2 Least squares

Exercise 2.1. ☕ *Let $A \in \mathbb{R}^{n \times d}$. Show that* $\operatorname{Ker} A = \operatorname{Ker} A^* A$.
Show that for any $b \in \mathbb{R}^n$ there exist a solution to $A^* A x = A^* b$.
☕☕ *Show that there does not always exist a solution to $A^* A x = A^* b$ in the infinite dimensional case (when A is a bounded linear operator between infinite dimensional Hilbert spaces.)*

Exercise 2.2. ☕ *Let $A \in \mathbb{R}^{n \times d}$, $b \in \mathbb{R}^n$. Show that solving Ordinary Least Squares:*

$$\min \frac{1}{2} \|Ax - b\|^2 \ , \tag{1}$$

amounts to solving $A^* A x = A^* b$ (aka the normal equations*).*
Show that the set of solutions is:

$$A^\dagger b + \operatorname{Ker} A \ .$$

Exercise 2.3. ☕ *When is $x \mapsto \|Ax - b\|^2$ strictly convex? Strongly convex?*

Exercise 2.4 (Least squares with intercept)**.** ☕☕ *An* intercept *x_0 is a constant scalar term in the linear prediction function, that becomes $a \mapsto a^\top x + x_0$. Fitting an intercept can be done by adding a column of 1s to A. Alternatively, show that the solution of least squares with intercept,*

$$(\hat{x}, \hat{x}_0) \in \operatorname*{argmin}_{x \in \mathbb{R}^d, x_0 \in \mathbb{R}} \frac{1}{2} \|Ax - b - x_0 \mathbf{1}\|^2 \tag{2}$$

is given by:

$$\hat{x} = \hat{x}_c \ , \tag{3}$$

$$\hat{x}_0 = \frac{1}{n} \sum_{1}^{n} (a_i^\top \hat{x} - b_i) \ , \tag{4}$$

where $\hat{x}_c$ is the solution of least squares without intercept on centered data A_c and b_c (versions of A and b where the rowwise mean has been subtracted).

Exercise 2.5 (Gradient descent on isotropic parabola)**.** ☕ *Let $A \in \mathbb{R}^{n \times d}$ be such that the condition number of $A^\top A$*[1] *is equal to 1. Show that gradient descent with stepsize $1/L$ converges in a single iteration for the problem* $\min \frac{1}{2} \|Ax - b\|^2$.

3 Gradient

Exercise 3.1. ☕ *Provide an example of setting where the gradient is not equal to the vector of partial derivatives.*

Exercise 3.2. ☕ *Show that the gradient of a function is orthogonal to the level lines of that function.*

Exercise 3.3. ☕ *Compute the gradients and Hessians of $x \mapsto \|x\|^2$, $x \mapsto \|x\|$, $x \mapsto a^\top x$. Is it true that the gradient of $x \mapsto \frac{1}{2} x^\top A x$ is equal to Ax?*

Exercise 3.4. ☕☕ *Compute the gradient of the logdet function, $M \mapsto \log\det(M)$.*

Exercise 3.5. ☕ *Let $A \in \mathbb{R}^{n \times d}$, $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^d \to \mathbb{R}$ defined as $g(x) = f(Ax)$ for all $x \in \mathbb{R}^d$. Show that*

$$\nabla g(x) = A^* \nabla f(Ax) \ ,$$
$$\nabla^2 g(x) = A^* \nabla^2 f(Ax) A \ .$$

[1] i.e. the ratio between the largest and the smallest eigenvalues of $A^\top A$.

4 Convexity inequalities

Exercise 4.1. ☕☕ *Let f be a convex and differentiable function. Let $L > 0$. Show that the following properties are equivalent:*

1. $\forall x, y,\ \|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\|$
2. $\forall x, y,\ f(x) \leq f(y) + \langle \nabla f(y), x - y\rangle + \frac{L}{2}\|x - y\|^2$
3. $\forall x, y,\ \frac{1}{L}\|\nabla f(x) - \nabla f(y)\|^2 \leq \langle x - y, \nabla f(x) - \nabla f(y)\rangle$

Exercise 4.2. ☕ *Let f be a twice differentiable L-smooth function. Show that for all $x \in \mathbb{R}^d$, $\nabla^2 f(x) \preccurlyeq L\,\mathrm{Id}$.*

Exercise 4.3. ☕ *Let f be a differentiable function. Show that the following properties are equivalent:*

1. *f is μ-strongly convex*
2. $\forall x, y,\ f(x) \geq f(y) + \langle \nabla f(y), x - y\rangle + \frac{\mu}{2}\|x - y\|^2$
3. $\forall x, y,\ \mu\|x - y\|^2 \leq \langle x - y, \nabla f(x) - \nabla f(y)\rangle$

Exercise 4.4. ☕ *Let f be a twice differentiable μ-strongly convex function. Show that for all $x \in \mathbb{R}^d$, $\mu\,\mathrm{Id} \preccurlyeq \nabla^2 f(x)$.*

Exercise 4.5 (Polyak-Łojasiewicz inequality)**.** ☕☕ *Let f be a μ-strongly-convex and differentiable function. Let $x^* = \operatorname{argmin} f(x)$. Show that f satisfies the Polyak-Lojasiewicz inequality:*

$$\mu(f(x) - f(x^*)) \leq \frac{1}{2}\|\nabla f(x)\|^2 \ .$$

Provide an example of function which is not strongly convex, but satisfies the inequality.

Exercise 4.6. ☕☕ *Let f be a L-smooth μ-strongly convex function. Show that for any x, y,*

$$\frac{\mu L}{\mu + L}\|x - y\|^2 + \frac{1}{L + \mu}\|\nabla f(x) - \nabla f(y)\|^2 \leq \langle x - y, \nabla f(x) - \nabla f(y)\rangle \ .$$

Exercise 4.7 (3 point descent lemma)**.** ☕ *Let f be convex and L-smooth. Show that for any triplet (x, y, z),*

$$f(x) \leq f(y) + \langle \nabla f(z), x - y\rangle + \frac{L}{2}\|x - z\| \ .$$

5 Constrained optimization

Exercise 5.1. ☕ *Show that the indicator function ι_C is convex (resp. lower semicontinuous, resp. proper) if C is convex (resp. closer, resp. nonempty).*

Exercise 5.2 (Global optimality condition for constrained convex optimisation)**.** ☕ *Let $f : \mathbb{R}^d \to \mathbb{R}$ be a convex differentiable function, let C be a convex subset of $\mathbb{R}^d$. Show that $x^* \in \operatorname{argmin}_{x \in C} f(x)$ if and only if*

$$\forall x \in C,\ \langle \nabla f(x^*), x - x^*\rangle \geq 0 \ .$$

Exercise 5.3. ☕☕ *Let C be a non-empty closed convex set. Show that for all $x \in C, y \in \mathbb{R}^d$,*

$$\|x - y\|^2 \geq \|x - \Pi_C(y)\|^2 + \|y - \Pi_C(y)\|^2 \ .$$

In particular this shows that $\|y - x\| \geq \|x - \Pi_C(y)\|$, which says that projection can only get you closer to optimum (taking $x = x^*$, if your current iterate is y).

6 Subdifferential and subgradients

Exercise 6.1. ☕ *Provide an example of convex function which has an empty subdifferential at some point of its domain.*

Exercise 6.2. ☕ *Let $f : \mathbb{R}^d \to \mathbb{R}$ be separable: $f(x) = \sum_1^d f_i(x_i)$ where the f_i's are functions of the real variable.*
Show that $\partial f(x) = \partial f_1(x_1) \times \ldots \times f_d(x_d)$.
Compute the subdifferential of the ℓ_1-norm.

Exercise 6.3. ☕ *Show that the subdifferential of a function at a point writes as an intersection of half-spaces. Show that it is convex and closed.*

Exercise 6.4. ☕ *Show that the subdifferential of ι_C at $x \in C$ is equal to the* normal cone *of C at x, that is:*

$$\mathcal{N}_C(x) = \{u : \forall z \in C, \langle u, z - x\rangle \leq 0\}$$

Exercise 6.5. ☕☕ *Compute the subdifferential of the Euclidean norm at any point in $\mathbb{R}^d$.*

Exercise 6.6 (Exercise 5.2 revisited)**.** ☕☕ *Show that the first order optimality condition for differentiable convex constrained optimization rewrites:*

$$x^* \in \operatorname*{argmin}_C f(x) \Leftrightarrow -\nabla f(x^*) \in \mathcal{N}_C(x^*) \ .$$

Exercise 6.7 (Non emptiness of subdifferential)**.** ☕☕☕ *Let $f : \mathbb{R}^d \to \overline{\mathbb{R}}$ be convex. Show that if $x \in \operatorname{int}(\operatorname{dom} f)$, $\partial f(x)$ is non empty and compact.*
Show that if x lies on the boundary of $\operatorname{dom} f$, $\partial f(x)$ is either empty or unbounded.

7 Continuous time

Exercise 7.1 (Preliminaries)**.** ☕ *Let $f : \mathcal{X} \to \mathbb{R}$ be twice differentiable. Let $\theta : \mathbb{R}_+ \to \mathcal{X}$ be differentiable. Show that*

$$\begin{aligned} \frac{d}{dt} f(\theta(t)) &= \langle \nabla f(\theta(t)), \dot\theta(t)\rangle \ , \\ \frac{d}{dt} \nabla f(\theta(t)) &= \nabla^2 f(\theta(t))\dot\theta(t) \ . \end{aligned}$$

Exercise 7.2 (Everything decreases in gradient flow)**.** ☕ *Let $f : \mathbb{R}^d \to \mathbb{R}$ be a convex twice differentiable function. Let $x(t)$ be the gradient flow on f, defined as a solution of $\dot x(t) = -\nabla f(x(t))$.*
Show that $f(x(t))$ decreases.
Show that $\|\nabla f(x(t))\|$ decreases.

8 Bregmaneries

Exercise 8.1 (Some misconceptions)**.** ☕ *Why is a Bregman divergence not "like a distance" in general? Is it "like a distance" in the Euclidean case?*

Exercise 8.2 (KL is Bregman divergence of negative entropy)**.** ☕☕ *Show that the Bregman divergence associated to negative entropy $x \mapsto \sum_1^d x_i \log(x_i)$ is the Kullback-Leibler divergence:*

$$D(x, y) = \sum_1^d y_i \log\left(\frac{x_i}{y_i}\right) \ .$$

Exercise 8.3. ☕ *Let $x, y, z \in \mathbb{R}^d$ and D be a Bregman divergence associated to a differentiable convex function ϕ.*
Show that

$$D(x, y) + D(z, x) - D(z, y) = \langle \nabla\phi(x) - \nabla\phi(y), x - z \rangle \ .$$

How is this a generalization of Pythagoras theorem?

Exercise 8.4 (Exercise 5.3 for Bregman projection)**.** ☕☕ *Let ϕ a differentiable convex function. Let D be the associated Bregman divergence. Let C be a non-empty closed and convex subset of $\mathbb{R}^d$, let $z \in \mathbb{R}^d$ and $y = \operatorname{argmin}_{y' \in C} D(y', z)$ its Bregman projection onto C.*
Show that for all $x \in C$:

$$D(x, z) \geq D(x, y) + D(y, z) \ .$$

This definitely shows that D is not "like a distance".

Exercise 8.5 (Bregman projection onto the simplex)**.** ☕☕ *Show that the Bregman projection of $x \in \mathbb{R}^d_+$ onto the simplex Δ, when the potential is the negative entropy, is $(x_i / \sum_1^d x_i)_i$.*
Compare to the Euclidean projection.

9 Sparsity

Exercise 9.1. ☕ *It is often claimed that the ℓ_1-norm is not differentiable at 0. Do you know a norm which is differentiable at 0?*

Exercise 9.2 (Huber function)**.** ☕☕ *Compute $\|\cdot\|_1 \,\square\, \frac{\rho}{2}\|\cdot\|^2$. In which sense is it a smooth approximation of the ℓ_1-norm?*

10 Logistic regression

Exercise 10.1. ☕ *Let $b \in \{-1, 1\}^n, A \in \mathbb{R}^{n \times d}$. Let $f(x) = \sum_1^n \log(1 + \exp(-b_i a_i^\top x))$.*
Compute ∇f, $\nabla^2 f$.
Is f convex? Strictly convex? Strongly convex?
☕☕ *Compute f^*.*

11 Fixed point schemes

Exercise 11.1. ☕☕ *Let $T : \mathcal{X} \to \mathcal{X}$ be a q-contractive operator, meaning that for all $x \in \mathcal{X}$,*

$$\|Tx\| \leq q\|x\| \ ,$$

with $q < 1$.
Show that T admits at most one fixed point.
Show that the sequence defined by $x_0 \in \mathcal{X}$, $x_{k+1} = Tx_k$ converges, and that the limit is a fixed point.
Show that, denoting x^* this fixed point, the sequence x_k converges to x^* at linear speed.

Exercise 11.2. ☕ *Show that the results of Exercise 11.1 do not hold when $q = 1$.*