

Convex optimization exercise sheet

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Notation

For a linear operator A , its adjoint is A^* and its Moore-Penrose pseudoinverse is A^\dagger . \mathcal{X} and \mathcal{Y} are two Hilbert spaces. Number of coffee cups (☕) indicates difficulty of exercise.

1 Convexity

Exercise 1.1. ☕ Show that local minimizers of convex functions are global minimizers.

Exercise 1.2 (Pointwise supremum preserves convexity). ☕ Let $(f_i)_I$ be a family of convex functions (not necessarily countable). Show that $x \mapsto \sup_{i \in I} f_i(x)$ is convex.

Exercise 1.3 (Precomposition by linear operator preserves convexity). ☕ Let $f : \mathcal{Y} \rightarrow \mathbb{R}$ be a convex function and $A : \mathcal{X} \rightarrow \mathcal{Y}$ a linear operator. Show that $f(A \cdot)$ is convex (on \mathcal{X}).

Exercise 1.4 (Misconceptions on existence of minimizers). ☕ Provide an example of convex function which does not admit a minimizer.

What if the function is continuous and lower bounded?

Exercise 1.5. ☕ Show that a strictly convex function has at most one minimizer.

Exercise 1.6 (Jensen's inequality). ☕ Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be convex. Let $n \in \mathbb{N}$, $x_1, \dots, x_n \in \mathbb{R}^d$, and let $\lambda_1, \dots, \lambda_n$ be positive scalars summing to 1.

Show that $f(\sum_{i=1}^n \lambda_i x_i) \leq \sum_{i=1}^n \lambda_i f(x_i)$.

Exercise 1.7. ☕ Show that the sublevel sets of a convex function are convex. Find a function with convex sublevel sets which is not convex.

Exercise 1.8 (First order characterization of convex functions). ☕☕ Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be a differentiable function. Show that the following are equivalent:

1. f is convex
2. f lies above its tangents: $\forall x, y \in \mathbb{R}^d, f(x) \geq f(y) + \langle \nabla f(y), x - y \rangle$
3. ∇f is monotone: $\forall x, y \in \mathbb{R}^d, \langle \nabla f(x) - \nabla f(y), x - y \rangle \geq 0$

Exercise 1.9 (Continuity). ☕☕☕ Show that a convex function is locally Lipschitz (hence continuous) on the interior of its domain.

Exercise 1.10 (Characterization of strongly convex functions in the Euclidean case). ☕ Show that f is μ -strongly convex with respect to the Euclidean norm if and only if $f - \frac{\mu}{2} \|\cdot\|^2$ is convex.

Exercise 1.11. ☕ Show that a strongly convex function admits exactly one minimizer.

2 Least squares

Exercise 2.1. 🍷 Let $A \in \mathbb{R}^{n \times d}$. Show that $\text{Ker } A = \text{Ker } A^*A$.

Show that for any $b \in \mathbb{R}^n$ there exist a solution to $A^*Ax = A^*b$.

🍷🍷 Show that there does not always exist a solution to $A^*Ax = A^*b$ in the infinite dimensional case (when A is a bounded linear operator between infinite dimensional Hilbert spaces.)

Exercise 2.2. 🍷 Let $A \in \mathbb{R}^{n \times d}$, $b \in \mathbb{R}^n$. Show that solving Ordinary Least Squares:

$$\min \frac{1}{2} \|Ax - b\|^2, \quad (2.1)$$

amounts to solving $A^*Ax = A^*b$ (aka the normal equations).

Show that the set of solutions is:

$$A^\dagger b + \text{Ker } A.$$

Exercise 2.3. 🍷 When is $x \mapsto \|Ax - b\|^2$ strictly convex? Strongly convex?

Exercise 2.4 (Least squares with intercept). 🍷🍷 An intercept x_0 is a constant scalar term in the linear prediction function, that becomes $a \mapsto a^\top x + x_0$. Fitting an intercept can be done by adding a column of 1s to A . Alternatively, show that the solution of least squares with intercept,

$$(\hat{x}, \hat{x}_0) \in \underset{x \in \mathbb{R}^d, x_0 \in \mathbb{R}}{\text{argmin}} \frac{1}{2} \|Ax - b - x_0 \mathbf{1}\|^2 \quad (2.2)$$

is given by:

$$\hat{x} = \hat{x}_c, \quad (2.3)$$

$$\hat{x}_0 = \frac{1}{n} \sum_{i=1}^n (a_i^\top \hat{x} - b_i), \quad (2.4)$$

where \hat{x}_c is the solution of least squares without intercept on centered data A_c and b_c (versions of A and b where the rowwise mean has been subtracted).

Exercise 2.5 (Gradient descent on isotropic parabola). 🍷 Let $A \in \mathbb{R}^{n \times d}$ be such that the condition number¹ of $A^\top A$ is equal to 1. Show that gradient descent with stepsize $1/L$ converges in a single iteration for the problem $\min \frac{1}{2} \|Ax - b\|^2$.

3 Gradient

Exercise 3.1. 🍷 Provide an example of setting where the gradient is not equal to the vector of partial derivatives.

Exercise 3.2. 🍷 Show that the gradient of a function is orthogonal to the level lines of that function.

Exercise 3.3. 🍷 Compute the gradients and Hessians of $x \mapsto \|x\|^2$, $x \mapsto \|x\|$, $x \mapsto a^\top x$. Is it true that the gradient of $x \mapsto \frac{1}{2} x^\top A x$ is equal to Ax ?

Exercise 3.4. 🍷 Let $A \in \mathbb{R}^{n \times d}$, $f : \mathbb{R}^n \rightarrow \mathbb{R}$ and $g : \mathbb{R}^d \rightarrow \mathbb{R}$ defined as $g(x) = f(Ax)$ for all $x \in \mathbb{R}^d$. Show that

$$\begin{aligned} \nabla g(x) &= A^* \nabla f(Ax), \\ \nabla^2 g(x) &= A^* \nabla^2 f(Ax) A. \end{aligned}$$

Exercise 3.5. 🍷🍷 Compute the gradient of the logdet function, $M \mapsto \log \det(M)$.

¹i.e. the ratio between the largest and the smallest eigenvalues of $A^\top A$.

4 Convexity inequalities

Exercise 4.1. 🐼🐼 Let f be a convex and differentiable function. Let $L > 0$. Show that the following properties are equivalent:

1. $\forall x, y, \|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\|$
2. $\forall x, y, f(x) \leq f(y) + \langle \nabla f(y), x - y \rangle + \frac{L}{2}\|x - y\|^2$
3. $\forall x, y, \frac{1}{L}\|\nabla f(x) - \nabla f(y)\|^2 \leq \langle x - y, \nabla f(x) - \nabla f(y) \rangle$

Exercise 4.2. 🐼 Let f be a twice differentiable L -smooth function. Show that for all $x \in \mathbb{R}^d$, $\nabla^2 f(x) \preceq L \text{Id}$.

Exercise 4.3. 🐼 Let f be a differentiable function. Show that the following properties are equivalent:

1. f is μ -strongly convex
2. $\forall x, y, f(x) \geq f(y) + \langle \nabla f(y), x - y \rangle + \frac{\mu}{2}\|x - y\|^2$
3. $\forall x, y, \mu\|x - y\|^2 \leq \langle x - y, \nabla f(x) - \nabla f(y) \rangle$

Exercise 4.4. 🐼 Let f be a twice differentiable μ -strongly convex function. Show that for all $x \in \mathbb{R}^d$, $\mu \text{Id} \preceq \nabla^2 f(x)$.

Exercise 4.5 (Polyak-Łojasiewicz inequality). 🐼🐼 Let f be a μ -strongly-convex and differentiable function. Let $x^* = \text{argmin } f(x)$. Show that f satisfies the Polyak-Łojasiewicz inequality:

$$\mu(f(x) - f(x^*)) \leq \frac{1}{2}\|\nabla f(x)\|^2 .$$

Provide an example of function which is not strongly convex, but satisfies the inequality.

Exercise 4.6. 🐼🐼 Let f be a L -smooth μ -strongly convex function. Show that for any x, y ,

$$\frac{\mu L}{\mu + L}\|x - y\|^2 + \frac{1}{L + \mu}\|\nabla f(x) - \nabla f(y)\|^2 \leq \langle x - y, \nabla f(x) - \nabla f(y) \rangle .$$

Exercise 4.7 (3 point descent lemma). 🐼 Let f be convex and L -smooth. Show that for any triplet (x, y, z) ,

$$f(x) \leq f(y) + \langle \nabla f(z), x - y \rangle + \frac{L}{2}\|x - z\| .$$

5 Gradient descent

Exercise 5.1 (Exercise 8.2 in discrete time). 🐼🐼 Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be a convex, twice differentiable L -smooth function.

Show that the iterates of gradient descent on f with step size $0 < \alpha < 2/L$ have decreasing gradient norm.

Can you show it if f is not twice differentiable?

Is it still true when f is not convex?

Exercise 5.2. 🐼 Provide a finite dimensional example of convex L -smooth function f such that gradient descent with stepsize $< 2/L$ diverges.

Exercise 5.3 (Gradient descent on Lipschitz function without knowing the horizon). 🐼🐼 For gradient descent on a Lipschitz differentiable objective, the classical proof assumes that the total number of iterations t is known in advance, to set the fixed stepsize $\eta \propto 1/\sqrt{t}$.

Show that using a decreasing stepsize $\eta_k \propto 1/\sqrt{k}$ leads to a rate of order $\log k/\sqrt{k}$ on the ergodic or best iterate.

6 Constrained optimization

Exercise 6.1. 🍷 Show that the indicator function ι_C is convex (resp. lower semicontinuous, resp. proper) if C is convex (resp. closer, resp. nonempty).

Exercise 6.2 (Global optimality condition for constrained convex optimisation). 🍷 Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be a convex differentiable function, let C be a convex subset of \mathbb{R}^d . Show that $x^* \in \operatorname{argmin}_{x \in C} f(x)$ if and only if

$$\forall x \in C, \langle \nabla f(x^*), x - x^* \rangle \geq 0 .$$

Exercise 6.3. 🍷🍷 Let C be a non-empty closed convex set. Show that for all $x \in C, y \in \mathbb{R}^d$,

$$\|x - y\|^2 \geq \|x - \Pi_C(y)\|^2 + \|y - \Pi_C(y)\|^2 .$$

In particular this shows that $\|y - x\| \geq \|x - \Pi_C(y)\|$, which says that projection can only get you closer to optimum (taking $x = x^*$, if your current iterate is y).

7 Subdifferential and subgradients

Exercise 7.1. 🍷 Provide an example of convex function which has an empty subdifferential at some point of its domain.

Exercise 7.2. 🍷 Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be separable: $f(x) = \sum_1^d f_i(x_i)$ where the f_i 's are functions of the real variable.

Show that $\partial f(x) = \partial f_1(x_1) \times \dots \times \partial f_d(x_d)$.

Compute the subdifferential of the ℓ_1 -norm.

Exercise 7.3. 🍷 Show that the subdifferential of a function at a point writes as an intersection of half-spaces. Show that it is convex and closed.

Exercise 7.4. 🍷 Show that the subdifferential of ι_C at $x \in C$ is equal to the normal cone of C at x , that is:

$$\mathcal{N}_C(x) = \{u : \forall z \in C, \langle u, z - x \rangle \leq 0\}$$

Exercise 7.5. 🍷🍷 Compute the subdifferential of the Euclidean norm at any point in \mathbb{R}^d .

Exercise 7.6 (Exercise 6.2 revisited). 🍷🍷 Show that the first order optimality condition for differentiable convex constrained optimization rewrites:

$$x^* \in \operatorname{argmin}_C f(x) \Leftrightarrow -\nabla f(x^*) \in \mathcal{N}_C(x^*) .$$

Exercise 7.7 (Non emptiness of subdifferential). 🍷🍷🍷 Let $f : \mathbb{R}^d \rightarrow \overline{\mathbb{R}}$ be convex. Show that if $x \in \operatorname{int}(\operatorname{dom} f)$, $\partial f(x)$ is non empty and compact.

Show that if x lies on the boundary of $\operatorname{dom} f$, $\partial f(x)$ is either empty or unbounded.

Exercise 7.8. 🍷 This exercise is in dimension 2. Let C_1 and C_2 be two closed balls of strictly positive radius, that share a single point. Show that $\partial(\iota_{C_1} + \iota_{C_2})$ is a strict superset of $\partial\iota_{C_1} + \iota_{C_2}$ at this point.

8 Continuous time

Exercise 8.1 (Preliminaries). 🍷 Let $f : \mathcal{X} \rightarrow \mathbb{R}$ be twice differentiable. Let $\theta : \mathbb{R}_+ \rightarrow \mathcal{X}$ be differentiable. Show that

$$\begin{aligned} \frac{d}{dt} f(\theta(t)) &= \langle \nabla f(\theta(t)), \dot{\theta}(t) \rangle , \\ \frac{d}{dt} \nabla f(\theta(t)) &= \nabla^2 f(\theta(t)) \dot{\theta}(t) . \end{aligned}$$

Exercise 8.2 (Everything decreases in gradient flow). 🐣 Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be a convex twice differentiable function. Let $x(t)$ be the gradient flow on f , defined as a solution of $\dot{x}(t) = -\nabla f(x(t))$. Show that $f(x(t))$ decreases. Show that $\|\nabla f(x(t))\|$ decreases. Show that $\|x(t) - x^*\|$ decreases for any minimizer x^* of f .

9 Bregmaneries

Exercise 9.1 (Some misconceptions). 🐣 Why is a Bregman divergence not “like a distance” in general? Is it “like a distance” in the Euclidean case?

Exercise 9.2 (KL is Bregman divergence of negative entropy). 🐣🐣 Show that the Bregman divergence associated to negative entropy $x \mapsto \sum_1^d x_i \log(x_i)$ is the Kullback-Leibler divergence:

$$D(x, y) = \sum_1^d y_i \log \left(\frac{x_i}{y_i} \right) .$$

Exercise 9.3. 🐣 Let $x, y, z \in \mathbb{R}^d$ and D be a Bregman divergence associated to a differentiable convex function ϕ . Show that

$$D(x, y) + D(z, x) - D(z, y) = \langle \nabla \phi(x) - \nabla \phi(y), x - z \rangle .$$

How is this a generalization of Pythagoras theorem?

Exercise 9.4 (Exercise 6.3 for Bregman projection). 🐣🐣 Let ϕ a differentiable convex function. Let D be the associated Bregman divergence. Let C be a non-empty closed and convex subset of \mathbb{R}^d , let $z \in \mathbb{R}^d$ and $y = \operatorname{argmin}_{y' \in C} D(y', z)$ its Bregman projection onto C . Show that for all $x \in C$:

$$D(x, z) \geq D(x, y) + D(y, z) .$$

This definitely shows that D is not “like a distance”.

Exercise 9.5 (Bregman projection onto the simplex). 🐣🐣 Show that the Bregman projection of $x \in \mathbb{R}_+^d$ onto the simplex Δ , when the potential is the negative entropy, is $(x_i / \sum_1^d x_i)_i$. Compare to the Euclidean projection.

10 Sparsity

Exercise 10.1. 🐣 It is often claimed that the ℓ_1 -norm is not differentiable at 0. Do you know a norm which is differentiable at 0?

Exercise 10.2 (Huber function). 🐣🐣 Compute $\|\cdot\|_1 \square \frac{\rho}{2} \|\cdot\|^2$. In which sense is it a smooth approximation of the ℓ_1 -norm?

11 Logistic regression

Exercise 11.1. 🐣 Let $b \in \{-1, 1\}^n$, $A \in \mathbb{R}^{n \times d}$. Let $f(x) = \sum_1^n \log(1 + \exp(-b_i a_i^\top x))$. Compute ∇f , $\nabla^2 f$.

Is f convex? Strictly convex? Strongly convex?

🐣🐣 Compute f^* .

12 Fixed point schemes

Exercise 12.1. ☹☹ Let $T : \mathcal{X} \rightarrow \mathcal{X}$ be a q -contractive operator, meaning that for all $x \in \mathcal{X}$,

$$\|Tx\| \leq q\|x\| ,$$

with $q < 1$.

Show that T admits at most one fixed point.

Show that the sequence defined by $x_0 \in \mathcal{X}$, $x_{k+1} = Tx_k$ converges, and that the limit is a fixed point.

Show that, denoting x^* this fixed point, the sequence x_k converges to x^* at linear speed.

Exercise 12.2. ☹ Show that the results of [Exercise 12.1](#) do not hold when $q = 1$.

13 Fenchel transforms

Exercise 13.1. Compute the Fenchel transform of the ℓ_p -norm for $p \in [1, +\infty]$.

Exercise 13.2. Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be separable: $f(x) = \sum_1^d f_i(x_i)$ where the f_i 's are functions of the real variable.

Show that $f^*(u) = (f_i^*(u_i))_{i \in [d]}$.

Exercise 13.3 (“Lipschitzing trick”). Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be convex.

Show that f is M -Lipschitz if and only if the domain of f^* is included in the Euclidean ball of center 0 and radius M .

Exercise 13.4 (Convolution smoothing). Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be convex.

Show that $f \square \frac{L}{2} \|\cdot\|^2$ is L -smooth.