Modeling and Control of an Inverted Pendulum System Using Energy Shaping and LQR

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Abstract

This report presents a comprehensive study of the inverted pendulum on a cart: from deriving its nonlinear equations via Euler–Lagrange formalism, through implementing an energy-based swing-up controller with partial feedback linearization, to designing a Linear Quadratic Regulator (LQR) for local stabilization. Detailed Simulink implementations and 3D visualizations demonstrate the effectiveness of the two-stage hybrid control strategy across all operating regimes.

Contents

1 Introduction

The inverted pendulum is a classic example of an underactuated, inherently unstable system. Its control requires handling both *global* nonlinear behavior (to swing the pendulum up from the downward position) and *local* linear behavior (to stabilize it once near the upright equilibrium). This report integrates these approaches into a single Simulink model, enabling smooth transition between swing-up and balance modes.

2 System Description

2.1 Physical Setup

A pendulum of mass m and moment of inertia I is hinged on a cart of mass M_c, which moves horizontally under the influence of a DC motor via a pulley of radius r. The generalized coordinates are the cart displacement x and the pendulum angle θ.

2.2 Nominal Parameters

Parameter	Value
Pulley radius r	$0.006\,\mathrm{m}$
Cart mass M_c	$0.135\,\mathrm{kg}$
Pendulum mass m	$0.1\,\mathrm{kg}$
Pendulum CG distance l	$0.2\,\mathrm{m}$
Pendulum inertia I	$0.0007176\,\mathrm{kg\,m^2}$
Gravity g	$9.81\,\mathrm{m/s^2}$
Pivot damping b	$7.892 \times 10^{-5}\,\mathrm{N\,m\,s}$
Cart friction c	$0.63\,\mathrm{N\,s/m}$
Motor resistance R_m	$12.5\,\Omega$
$\mathrm{k}_b = \mathrm{k}_t$	0.031

Table 1: System physical parameters

3 Nonlinear Dynamics

3.1 Lagrangian Derivation

Define $q = [x, \theta]^\top$ and $\dot{q} = [\dot{x}, \dot{\theta}]^\top$. The kinetic energy T and potential energy U are:

$$\begin{aligned}
&T = \tfrac{1}{2}M_c\dot{x}^2 + \tfrac{1}{2}m\big(\dot{x}^2 + l^2\dot{\theta}^2 + 2l\dot{x}\dot{\theta}\cos\theta\big) + \tfrac{1}{2}I\dot{\theta}^2, \\
&U = mgl(1 - \cos\theta).
\end{aligned}$$

The Euler–Lagrange equations with damping forces yield the coupled dynamics:

$$(M_c + m)\ddot{x} + c\dot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta = F, \tag{1}$$

$$(I + ml^2)\ddot{\theta} + b\dot{\theta} + mgl\sin\theta - ml\ddot{x}\cos\theta = 0. \tag{2}$$

3.2 Simulink Implementation: Nonlinear Model

Figure 1 shows the Simulink realization of these dynamics in the `PendulumDynamics` subsystem.

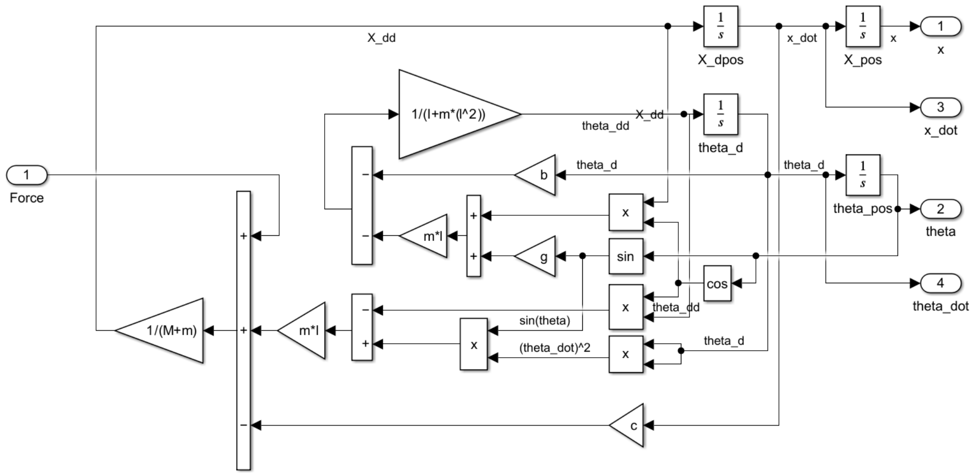

Figure 1: Nonlinear Simulink model of the inverted pendulum

4 Open-Loop Response

A step input in force is applied to the nonlinear model. Figure 2 plots the cart position and pendulum angle over time, illustrating the unstable open-loop behavior.

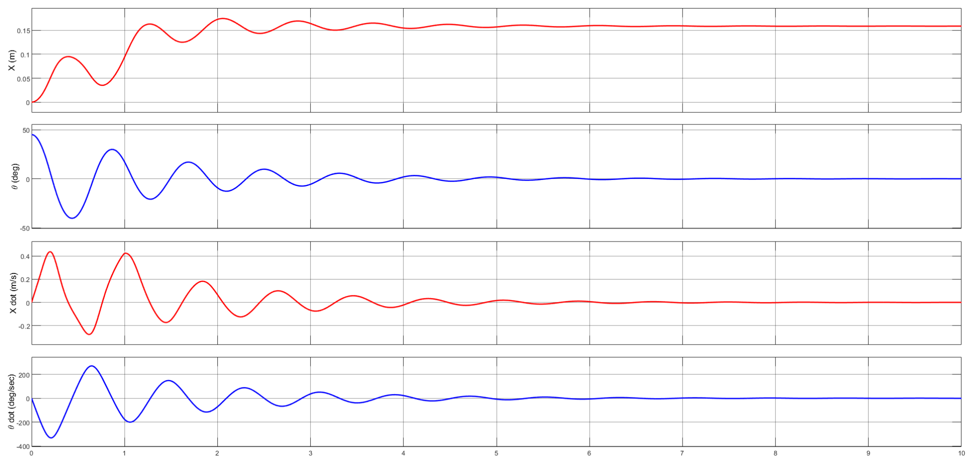

Figure 2: Open-loop step response of the nonlinear model

5 Linearization and State-Space Model

Linearize around the upright equilibrium $(x, \theta, \dot{x}, \dot{\theta}) = (0, \pi, 0, 0)$ via Taylor expansion. With state $X = [x, \theta - \pi, \dot{x}, \dot{\theta}]^\top$, the linearized model is:

$$\dot{X} = AX + Bu,$$

where u is the motor voltage. The matrices A, B are computed as detailed in Section **??**.

6 Control Design

6.1 Energy-Based Swing-Up and Partial Feedback Linearization

Define total pendulum energy

$$E = \tfrac{1}{2}(I + ml^2)\dot{\theta}^2 + mgl(1 - \cos\theta),$$

and target $E_r = 2mgl$. The energy-shaping control law is:

$$u_d = k\dot{\theta}\cos\theta\,(E - E_r),$$

implemented in the `SwingUpController` subsystem. Partial feedback linearization then computes the actual force input F to achieve the desired cart acceleration u_d.

6.2 LQR Stabilization

When $|\theta - \pi| < 25^\circ$, the `ModeSwitch` subsystem selects the LQR path, where $u = -KX$ with K designed for

$$J = \int_0^\infty (X^T Q X + u^2 R)\,dt,$$

using $Q = \mathrm{diag}(1200, 1500, 0, 0)$ and $R = 0.035$.

7 Full Simulink Model

Figure 3 presents the complete Simulink diagram, showing subsystem interconnections and mode switching logic.

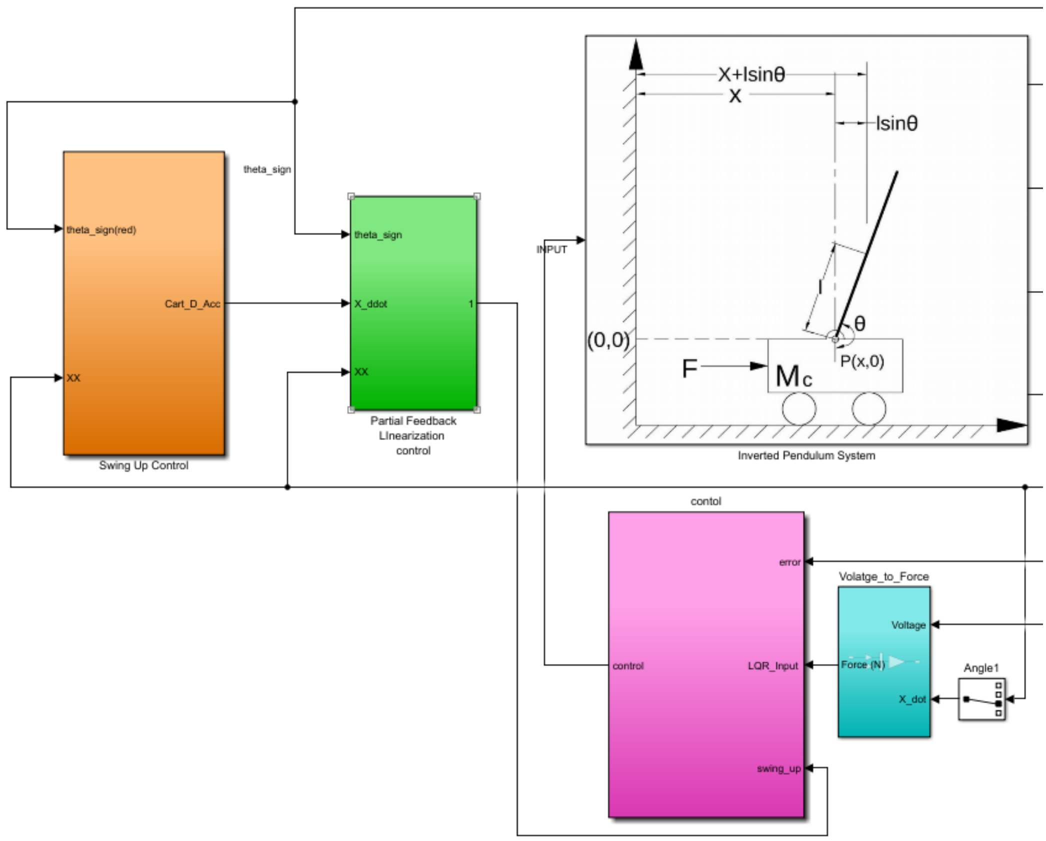

Figure 3: Complete Simulink model with energy shaping, PFL, and LQR

8 Closed-Loop Performance

Figure 4 overlays the swing-up phase and final stabilization under LQR control. The pendulum smoothly transitions to the upright equilibrium and remains balanced.

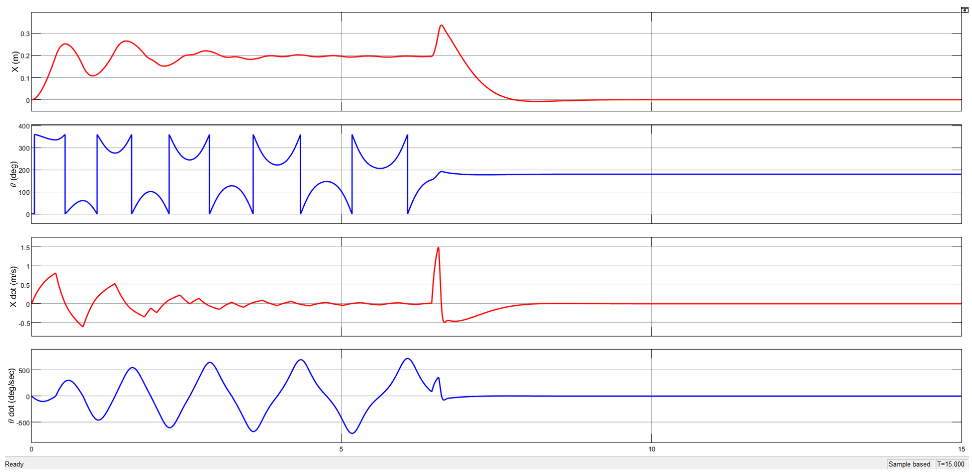

Figure 4: Closed-loop response: swing-up and stabilization

9 Conclusion

A two-stage control strategy—energy shaping with partial feedback linearization followed by LQR—successfully achieves global swing-up and local stabilization of the inverted pendulum. Future work will explore robust/adaptive extensions and hardware-in-the-loop validation.

References

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[2] G. F. Franklin, J. D. Powell, and M. L. Workman, *Feedback Control of Dynamic Systems*, 7th ed., Pearson, 2015. *Automatica*, 2008.

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