Lecture 3: Planning by Dynamic Programming

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Overview

- Last lecture: MDPs as a formal description of the RL problem
- This lecture: Solutions by dynamic programming
  - When the environment MDP is given explicitly
  - ...and the MDP is not too large
- Next lecture: Solutions for prediction problems by learning
  - Adapting from experience (without needing the MDP)

Outline

Reference: Sutton & Barto, chapter 4

What is Dynamic Programming?

Dynamic sequential or temporal component to the problem

Programming optimising a "program", i.e. a policy

- c.f. linear programming

- A method for solving complex problems
- By breaking them down into subproblems
  - Solve the subproblems
  - Combine solutions to subproblems

Requirements for Dynamic Programming

Dynamic Programming is a very general solution method for problems which have two properties:

- Optimal substructures
  - Optimal solution to a problem composed from optimal solutions to subproblems
- Overlapping subproblems
  - Subproblems recur many times
  - Solutions can be cached and reused
- Markov decision processes satisfy both properties
  - Bellman equation gives recursive decomposition
  - Value function stores and reuses solutions

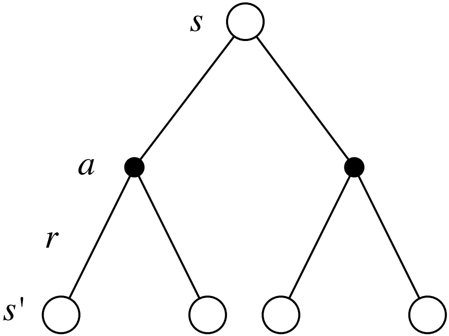
Planning by Dynamic Programming

- Dynamic programming assumes full knowledge of the MDP
- It is used for *planning* in an MDP
- For prediction:
  - Input: MDP $\langle \mathcal{S}, \mathcal{A}, \mathcal{P}, \mathcal{R}, \gamma \rangle$ and policy π
  - or: MRP $\langle \mathcal{S}, \mathcal{P}^\pi, \mathcal{R}^\pi, \gamma \rangle$
  - Output: value function v^π
- Or for control:
  - Input: MDP $\langle \mathcal{S}, \mathcal{A}, \mathcal{P}, \mathcal{R}, \gamma \rangle$
  - Output: optimal value function v^*
  - and: optimal policy π^*
- Many other uses of dynamic programming
  - Scheduling, string matching, shortest paths, Viterbi algorithm, ...

Iterative Policy Evaluation

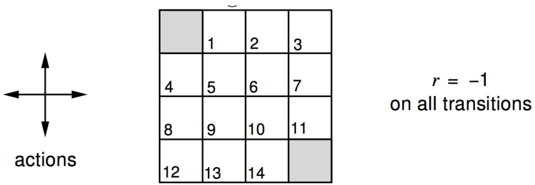
- Problem: evaluate a given fixed policy π
- Solution: iterative application of Bellman expectation backup
- $V_1 \rightarrow V_2 \rightarrow ... \rightarrow v^\pi$
- Using *synchronous* backups,
  - At each iteration $k+1$
  - For all states $s \in \mathcal{S}$
  - Update $V_{k+1}(s)$ from $V_k(s')$
  - where s' is a successor state of s
- Convergence to v^π will be proven at the end of the lecture

Iterative Policy Evaluation (2)



$$V_{k+1}(s) = \sum_{a \in \mathcal{A}} \pi(s, a) \left(\mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a V_k(s') \right)$$

Small Gridworld



- Undiscounted episodic MDP
  - $\gamma = 1$
  - All episodes terminate in absorbing terminal state
- Nonterminal states $1, ..., 14$
- One terminal state (shown twice as shaded squares)
- Actions that would take agent off the grid leave state unchanged
- Reward is -1 until the terminal state is reached

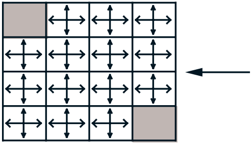
Iterative Policy Evaluation in Small Gridworld

V_k for the Random Policy

Greedy Policy w.r.t. V_k

$k = 0$

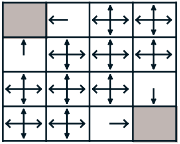
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0



random policy

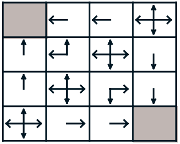
$k = 1$

0.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	0.0



$k = 2$

0.0	-1.7	-2.0	-2.0
-1.7	-2.0	-2.0	-2.0
-2.0	-2.0	-2.0	-1.7
-2.0	-2.0	-1.7	0.0



Iterative Policy Evaluation in Small Gridworld (2)

$k = 3$

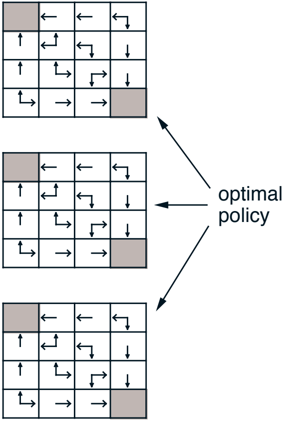
0.0	-2.4	-2.9	-3.0
-2.4	-2.9	-3.0	-2.9
-2.9	-3.0	-2.9	-2.4
-3.0	-2.9	-2.4	0.0

$k = 10$

0.0	-6.1	-8.4	-9.0
-6.1	-7.7	-8.4	-8.4
-8.4	-8.4	-7.7	-6.1
-9.0	-8.4	-6.1	0.0

$k = \infty$

0.0	-14.	-20.	-22.
-14.	-18.	-20.	-20.
-20.	-20.	-18.	-14.
-22.	-20.	-14.	0.0



Policy Improvement

- Consider a deterministic policy, $a = \pi(s)$
- We can *improve* the policy by acting greedily

$$\pi'(s) = \underset{a \in \mathcal{A}}{\operatorname{argmax}}\ q^\pi(s, a)$$

- This improves the value from any state s over one step,

$$q^\pi(s, \pi'(s)) = \max_{a \in \mathcal{A}} q^\pi(s, a) \geq q^\pi(s, \pi(s)) = v^\pi(s)$$

- It therefore improves the value function, $v^{\pi'}(s) \geq v^\pi(s)$

$$\begin{aligned}
v^\pi(s) \leq q^\pi(s, \pi'(s)) &= \mathbb{E}_{\pi'}\left[R_{t+1} + \gamma v^\pi(S_{t+1}) \mid S_t = s\right] \\
&\leq \mathbb{E}_{\pi'}\left[R_{t+1} + \gamma q^\pi(A_{t+1}, \pi'(S_{t+1})) \mid S_t = s\right] \\
&\leq \mathbb{E}_{\pi'}\left[R_{t+1} + \gamma R_{t+2} + \gamma^2 q^\pi(S_{t+2}, \pi'(S_{t+2})) \mid S_t = s\right] \\
&\leq \mathbb{E}_{\pi'}\left[R_{t+1} + \gamma R_{t+2} + ... \mid S_t = s\right] = v^{\pi'}(s)
\end{aligned}$$

Policy Improvement (2)

- If improvements stop,

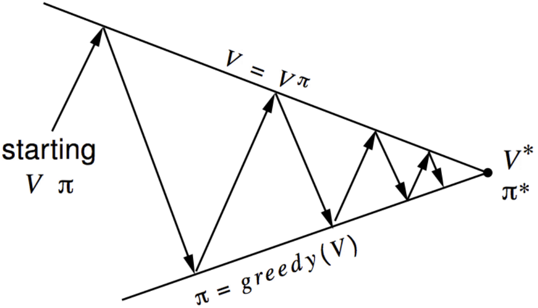
$$q^{\pi}(s, \pi'(s)) = \max_{a \in \mathcal{A}} q^{\pi}(s, a) = q^{\pi}(s, \pi(s)) = v^{\pi}(s)$$

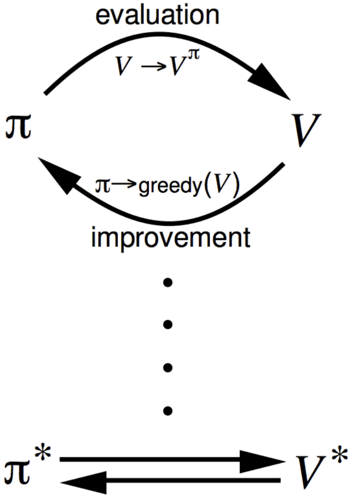
- Then the Bellman optimality equation has been satisfied

$$v^{\pi}(s) = \max_{a \in \mathcal{A}} q^{\pi}(s, a)$$

- Therefore $v^{\pi}(s) = v^{*}(s)$ for all $s \in \mathcal{S}$
- so π is an optimal policy

Policy Iteration





Policy evaluation Estimate v^{π}

Iterative policy evaluation

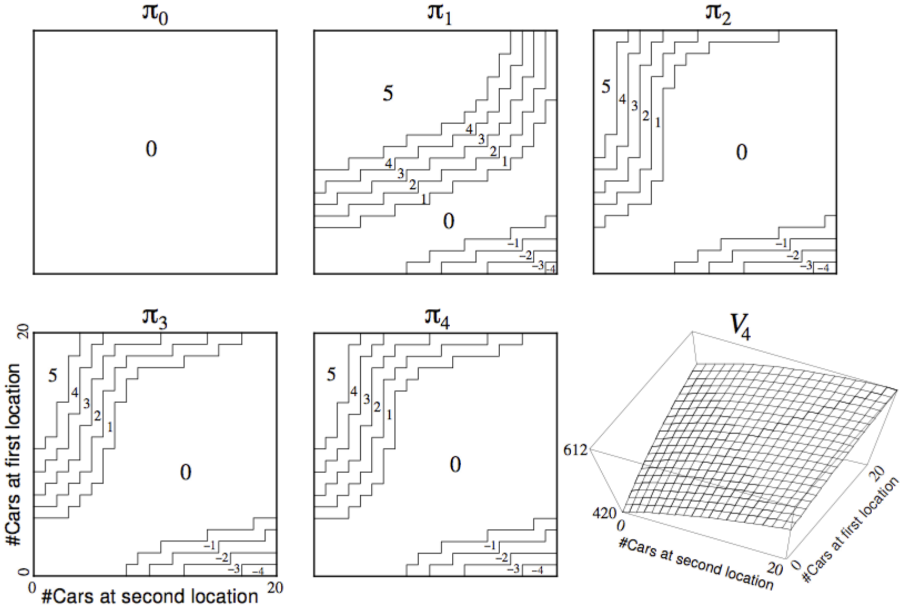
Policy improvement Generate $\pi' \geq \pi$

Greedy policy improvement

Jack's Car Rental

- States: Two locations, maximum of 20 cars at each
- Actions: Move up to 5 cars overnight (-$2 each)
- Reward: $10 for each car rented (must be available), $\gamma = 0.9$
- Transitions: Cars returned and requested randomly
  - Poisson distribution, n returns/requests with prob $\frac{\lambda^n}{n!}e^{-\lambda}$
  - 1st location: average requests $= 3$, average returns $= 3$
  - 2nd location: average requests $= 4$, average returns $= 2$

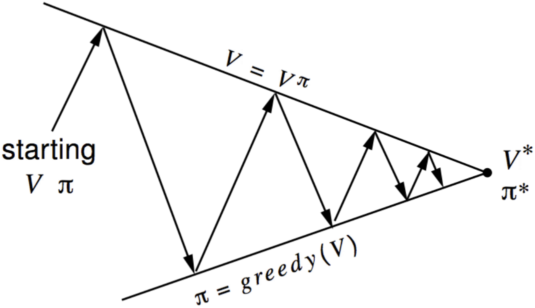
Policy Iteration in Jack's Car Rental

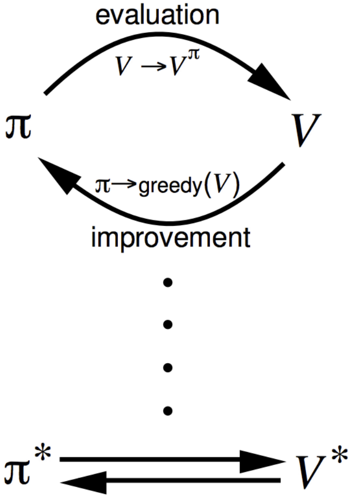


Modified Policy Iteration

- Does policy evaluation need to converge to v^{π}?
- Or should we introduce a stopping condition
  - e.g. ϵ-convergence of value function
- Or simply stop after k iterations of iterative policy evaluation?
- For example, in the small gridworld $k = 3$ was sufficient to achieve optimal policy
- Why not update policy every iteration? i.e. stop after $k = 1$
  - This is equivalent to *value iteration* (next section)

Generalised Policy Iteration





Policy evaluation Estimate v^π

Any policy evaluation algorithm

Policy improvement Generate $\pi' \geq \pi$

Any policy improvement algorithm

Deterministic Value Iteration

- If we know the solution to subproblems $v^*(s')$
- Then it is easy to construct the solution to $v^*(s)$

$$v^*(s) \leftarrow \max_{a \in \mathcal{A}} \mathcal{R}_s^a + v^*(s')$$

- The idea of value iteration is to apply these updates iteratively
- e.g. Starting at the goal (horizon) and working backwards

Example: Shortest Path

g			

Problem

0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0

V_1

0	-1	-1	-1
-1	-1	-1	-1
-1	-1	-1	-1
-1	-1	-1	-1

V_2

0	-1	-2	-2
-1	-2	-2	-2
-2	-2	-2	-2
-2	-2	-2	-2

V_3

0	-1	-2	-3
-1	-2	-3	-3
-2	-3	-3	-3
-3	-3	-3	-3

V_4

0	-1	-2	-3
-1	-2	-3	-4
-2	-3	-4	-4
-3	-4	-4	-4

V_5

0	-1	-2	-3
-1	-2	-3	-4
-2	-3	-4	-5
-3	-4	-5	-5

V_6

0	-1	-2	-3
-1	-2	-3	-4
-2	-3	-4	-5
-3	-4	-5	-6

V_7

Value Iteration in MDPs

- MDPs don't usually have a finite horizon
- They are typically loopy
- So there is no "end" to work backwards from
- However, we can still propagate information backwards
- Using Bellman optimality equation to *backup* $V(s)$ from $V(s')$
- Each subproblem is "easier" due to discount factor γ
- Iterate until convergence

Optimality in MDPs

An optimal policy π^* must provide both

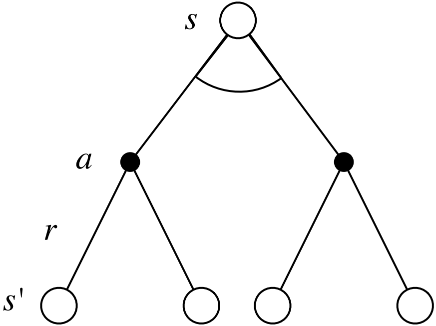
- An optimal first action a^* from any state s,
- Followed by an optimal policy from successor state s'

$$v^*(s) = \max_{a \in \mathcal{A}} \mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v^*(s')$$

Value Iteration

- Problem: find optimal policy π
- Solution: iterative application of Bellman optimality backup
- $V_1 \rightarrow V_2 \rightarrow ... \rightarrow v^*$
- Using synchronous backups
  - At each iteration $k+1$
  - For all states $s \in \mathcal{S}$
  - Update $V_{k+1}(s)$ from $V_k(s')$
- Convergence to v^* will be proven later
- Unlike policy iteration, there is no explicit policy
- Intermediate value functions may not correspond to any policy

Value Iteration (2)



$$V_{k+1}(s) = \max_{a \in \mathcal{A}} \left(\mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a V_k(s') \right)$$

Synchronous Dynamic Programming Algorithms

Problem	Bellman Equation	Algorithm
Prediction	Bellman Expectation Equation	Iterative Policy Evaluation
Control	Bellman Expectation Equation + Greedy Policy Improvement	Policy Iteration
Control	Bellman Optimality Equation	Value Iteration

- Algorithms are based on state-value function $v^{\pi}(s)$ or $v^{*}(s)$
- Complexity $O(mn^2)$ per iteration, for m actions and n states
- Could also apply to action-value function $q^{\pi}(s,a)$ or $q^{*}(s,a)$
- Complexity $O(m^2n^2)$ per iteration

Asynchronous Dynamic Programming

- DP methods described so far used *synchronous* backups
- i.e. all states are backed up in parallel
- *Asynchronous DP* backs up states individually, in any order
- For each selected state, apply the appropriate backup
- Can significantly reduce computation
- Guaranteed to converge if all states continue to be selected

Asynchronous Dynamic Programming

Three simple ideas for asynchronous dynamic programming:

- *In-place* dynamic programming
- *Prioritised sweeping*
- *Real-time* dynamic programming

In-Place Dynamic Programming

- Synchronous value iteration stores two copies of value function

for all s in $\mathcal{S}$

$$V_{new}(s) \leftarrow \max_{a \in \mathcal{A}} \left(\mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a V_{old}(s') \right)$$

$$V_{old} \leftarrow V_{new}$$

- In-place value iteration only stores one copy of value function

for all s in $\mathcal{S}$

$$V(s) \leftarrow \max_{a \in \mathcal{A}} \left(\mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a V(s') \right)$$

Prioritised Sweeping

- Use magnitude of Bellman error to guide state selection, e.g.

$$\left| \max_{a \in \mathcal{A}} \left(\mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a V_k(s') \right) - V_k(s) \right|$$

- Backup the state with the largest remaining Bellman error
- Update Bellman error of affected states after each backup
- Requires knowledge of reverse dynamics (predecessor states)
- Can be implemented efficiently by maintaining a priority queue

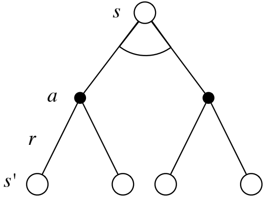
Real-Time Dynamic Programming

- Idea: only states that are relevant to agent
- Use agent's experience to guide the selection of states
- After each time-step S_t, A_t, R_{t+1}
- Backup the state S_t

$$V(S_t) \leftarrow \max_{a \in \mathcal{A}} \left(\mathcal{R}^a_{S_t} + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}^a_{S_t s'} V(s') \right)$$

Full-Width Backups

- DP uses *full-width* backups
- For each backup (sync or async)
  - Every successor state and action is considered
  - Using knowledge of the MDP transitions and reward function
- DP is effective for medium-sized problems (millions of states)
- For large problems DP suffers Bellman's *curse of dimensionality*
  - Number of states $n = |\mathcal{S}|$ grows exponentially with number of state variables
- Even one backup can be too expensive



Approximate Dynamic Programming

- Key idea: Approximate the value function
- Using a *function approximator* $V^{\theta}(s) = v(s;\theta)$, with a parameter vector $\theta \in \mathbb{R}^n$.
- The estimated value function at iteration k is $V_k = V^{\theta_k}$
- Use dynamic programming to compute $V^{\theta_{k+1}}$ from V^{θ_k}.
- e.g. Fitted Value Iteration repeats at each iteration k,
  - Sample states $\tilde{\mathcal{S}} \subseteq \mathcal{S}$ (more simply, let $\tilde{S} = S$)
  - For each sample state $s \in \tilde{\mathcal{S}}$, compute target value using Bellman optimality equation,

$$\tilde{V}_k(s) = \max_{a \in \mathcal{A}} \left(\mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a V^{\theta_k}(s') \right)$$

  - Train the next value function $V^{\theta_{k+1}}$ using the following loss.

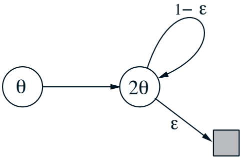
$$L(\theta; \tilde{\mathcal{S}}, \theta_k) = \sum_{s \in \tilde{\mathcal{S}}} \left(V^{\theta}(s) - \tilde{V}_k(s) \right)^2$$

Bootstrapping in Dynamic Programming

- A crucial part of dynamic programming is to improve the estimate of the value function at one state using the estimate of the value function at subsequent states.
- This idea is used frequently in RL, and it is called bootstrapping.
- There is a *theoretical* danger of divergence when combining three things we have considered.
  1. Bootstrapping
  2. General function approximation
  3. Updating values for a state distribution that doesn't match the transition dynamics of the MDP
- This theoretical danger is rarely encountered in practice.

Example of divergence with dynamic programming

- Tsitsiklis and Van Roy made an example where dynamic programming with linear function approximation can diverge. Consider the two state example below, where the rewards are all zero, there are no decisions, and there is a single parameter for estimating the value.



$$\begin{aligned}\theta_{k+1} &= \operatorname*{argmin}_{\theta} \sum_{s \in \mathcal{S}} (v^{\theta}(s) - \tilde{V}_k(s))^2 \\ &= \operatorname*{argmin}_{\theta} (\theta - \gamma 2\theta_k)^2 + (2\theta - \gamma(1-\epsilon)2\theta_k)^2 \\ &= \frac{6-4\epsilon}{5}\gamma\theta_k\end{aligned}$$

- What is $\lim_{k\to\infty}\theta_k$ when $\theta_0 = 1$, $\epsilon = \frac{1}{8}$, and $\gamma = 1$?

Some Technical Questions

- Consider the setting where we are not using function approximation
- How do we know that value iteration converges to v^*?
- Or that iterative policy evaluation converges to v^π?
- And therefore that policy iteration converges to v^*?
- Is the solution unique?
- How fast do these algorithms converge?
- These questions are resolved by *contraction mapping theorem*

Value Function Space

- *Consider the vector space $\mathcal{V}$ over value functions*
- There are $|\mathcal{S}|$ dimensions
- Each point in this space fully specifies a function $V(s)$ and vice versa
- What does a Bellman backup do to points in this space?
- We will show that it brings value functions *closer*
- And therefore the backups must converge on a unique solution

Value Function ∞-Norm

- We will measure distance between state-value functions U and V by the ∞-norm
- i.e. the largest difference between state values,

$$||U - V||_\infty = \max_{s \in \mathcal{S}} |U(s) - V(s)|$$

- Define the *Bellman expectation backup operator* T^π,

$$T^\pi(V) = \mathcal{R}^\pi + \gamma \mathcal{P}^\pi V$$

where $\mathcal{R}^\pi(s) = \sum_{a \in \mathcal{A}} \pi(a|s)\mathcal{R}_s^a$
and $(\mathcal{P}^\pi V)(s) = \sum_{a \in \mathcal{A}} \pi(a|s) \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a V(s')$.

Bellman Expectation Backup is a Contraction

- The Bellman expectation backup operator is a γ-contraction, i.e. it makes value functions closer by at least γ,

$$
\begin{aligned}
||T^\pi U - T^\pi V||_\infty &= \max_s |(\mathcal{R}^\pi + \gamma \mathcal{P}^\pi U)(s) - (\mathcal{R}^\pi + \gamma \mathcal{P}^\pi V)(s))| \\
&= \max_s |\gamma \textstyle\sum_a \pi(a|s) \sum_{s' \in \mathcal{S}} \mathcal{P}^a_{ss'}(U(s') - V(s'))| \\
&\leq \max_s \gamma \textstyle\sum_a \pi(a|s) \sum_{s' \in \mathcal{S}} \mathcal{P}^a_{ss'} |U(s') - V(s')| \\
&\leq \max_s \gamma \textstyle\sum_a \pi(a|s) \sum_{s' \in \mathcal{S}} \mathcal{P}^a_{ss'} ||U - V||_\infty \\
&\leq \gamma ||U - V||_\infty (\max_s \textstyle\sum_a \pi(a|s) \sum_{s' \in \mathcal{S}} \mathcal{P}^a_{ss'}) \\
&\leq \gamma ||U - V||_\infty
\end{aligned}
$$

Contraction Mapping Theorem

Theorem (Contraction Mapping Theorem)

For any metric space $\mathcal{V}$ that is complete (i.e. contains its limit points) under an operator $T(V)$, where T is a γ-contraction,

- *T converges to a unique fixed point*
- *At a linear convergence rate of γ*

Convergence of Iter. Policy Evaluation

- The Bellman expectation operator T^π has a unique fixed point
- v^π is a fixed point of T^π (by Bellman expectation equation)
- By contraction mapping theorem
- Iterative policy evaluation converges to v^π

Convergence of Policy Iteration

$$\pi_1 \longrightarrow v^{\pi_1} \longrightarrow \pi_2 \longrightarrow v^{\pi_2} \longrightarrow \pi_3 \longrightarrow v^{\pi_3} \longrightarrow \dots$$

- Policy iteration alternates between
  1. Iterative policy evaluation to compute v^{π_k} using π_k
  2. Policy improvement finds an improved deterministic policy $\pi_{k+1} \geq \pi_k$ using v^{π_k}
- Improvement only stops when policy iteration has converged to v^*
- As there are only a finite number of deterministic policies, this process must terminate at the optimal value function.

Bellman Optimality Backup is a Contraction

- Define the *Bellman optimality backup operator* T^*,

$$T^*(V) = \max_{a \in \mathcal{A}} \mathcal{R}^a + \gamma \mathcal{P}^a V$$

- This operator is a γ-contraction, i.e. it makes value functions closer by at least γ (similar to previous proof)

$$||T^*(U) - T^*(V)||_\infty \leq \gamma ||U - V||_\infty$$

Convergence of Value Iteration

- The Bellman optimality operator T^* has a unique fixed point
- v^* is a fixed point of T^* (by Bellman optimality equation)
- By contraction mapping theorem
- Value iteration converges on v^*