An Optimization Approach to Chord Arc Constant Calculation

For a planar polygon or a polygonal arc, $\Gamma \subset \mathbb{R}^2$ is the set of all points on the shape's edges. The Chord Arc Constant of a planar polygon or a polygonal arc is defined as

$$C = \sup_{\mathbf{s},\mathbf{t}\in\Gamma} \left\{ \frac{\| \mathbf{s} - \mathbf{t} \|_{graph}}{\| \mathbf{s} - \mathbf{t} \|} \right\},$$

where $\| \mathbf{s} - \mathbf{t} \|_{graph}$ is the graph distance (arc length) between $\mathbf{s}$ and $\mathbf{t}$.

For any polygon or polygonal arc P, we define

$$\gamma_P : [0, l_P) \to \Gamma_P,$$

where l_P is the total length of P's edges. One can imagine a particle that travels at the unit speed along the edges of P. The particle can start from any point on one of P's edges and that starting point is just $\gamma_P(0)$. Suppose the distance that particle has travelled is x, then the particle's current position is $\gamma_p(x)$.

1 Polygonal Arc

Let $a < b$, $\gamma(a) = \mathbf{s}$, and $\gamma(b) = \mathbf{t}$, then a polygonal arc's Chord Arc Constant can be written as

$$\begin{aligned} C_{arc} &= \sup_{\substack{a,b\in[0,l) \\ \mathbf{s},\mathbf{t}\in\Gamma}} \left\{ \frac{b-a}{\| \mathbf{s} - \mathbf{t} \|} \right\} \\ &= \sup_{a,b\in[0,l)} \left\{ \frac{b-a}{\|\gamma(b) - \gamma(a)\|} \right\}. \end{aligned}$$

Now the optimization variables have been reduced from two vectors $\mathbf{s}$ and $\mathbf{t}$ to only one vector $(a, b)^T$. We can rewrite a polygonal arc's Chord Arc Constant as an optimization problem:

$$\begin{array}{ll} \max & \dfrac{b-a}{\|\gamma(b) - \gamma(a)\|} \\ \text{s.t.} & 0 \le a < b < l \end{array} \quad \iff \quad \begin{array}{ll} \min & \dfrac{a-b}{\|\gamma(b) - \gamma(a)\|} \\ \text{s.t.} & 0 \le a < b < l \end{array} \tag{1}$$

2 Polygon

Polygons are a bit more complicated than polygonal arcs because there are always 2 graph paths between 2 points $\mathbf{s}$ and $\mathbf{t}$, and the length of the shorter graph path is defined as the graph length between $\mathbf{s}$ and $\mathbf{t}$.

Intuitively a polygon's Chord Arc Constant is attained when $\mathbf{s}$ and $\mathbf{t}$ are close to each other on the plane but the graph length between them is long.

Let $a < b$, $\gamma(a) = s$, and $\gamma(b) = t$, then a polygon's Chord Arc Constant can be written as

$$\begin{aligned} C_{polygon} &= \sup_{\substack{a,b\in[0,l) \\ s,t\in\Gamma}} \left\{ \frac{\min(b-a, l-b+a)}{\|s - t\|} \right\} \\ &= \sup_{a,b\in[0,l)} \left\{ \frac{\min(b-a, l-b+a)}{\|\gamma(b) - \gamma(a)\|} \right\}. \end{aligned}$$

Similar to what we did in (1) for polygonal arc, we can rewrite a polygon's Chord Arc Constant as an optimization problem:

$$\begin{aligned} \max \quad & \frac{\min(b-a, l-b+a)}{\|\gamma(b)-\gamma(a)\|} \\ \text{s.t.} \quad & 0 \le a < b < l \end{aligned}$$

We can easily turn it into a minimization problem by negating the objective function.

$$\begin{aligned} \min \quad & \frac{-\min(b-a, l-b+a)}{\|\gamma(b)-\gamma(a)\|} \\ \text{s.t.} \quad & 0 \le a < b < l \end{aligned} \iff \begin{aligned} \min \quad & \frac{\max(-b+a, -l+b-a)}{\|\gamma(b)-\gamma(a)\|} \\ \text{s.t.} \quad & 0 \le a < b < l \end{aligned} \tag{2}$$

3 Practical Aspects of the Optimization Approach

The optimization problems (1) and (2) for calculating Chord Arc Constant of polygonal arcs and polygons, respectively, have the following features/advantages comparing to the traditional approaches:

1. The Chord Arc Constant's 4-dimensional domain $\mathcal{D} \ni (\mathbf{s}_x, \mathbf{s}_y, \mathbf{t}_x, \mathbf{t}_y)^T$ is now reduced to (1)'s and (2)'s 2-dimensional domain $\mathcal{S} \ni (a, b)^T$. Now the region of feasibility (or the search space) is simply a triangle (although not closed on 2 of the 3 edges) in $\mathbb{R}^2$. So (1) might even be simple enough to somehow be solved by hand.

2. The optimization problems can be easily implemented using Python or Matlab. Furthermore, we can parallelize the optimization process, making it many times faster.

3. In the event of the optimizer not accepting constraints, we can rewrite (1) and (2) as

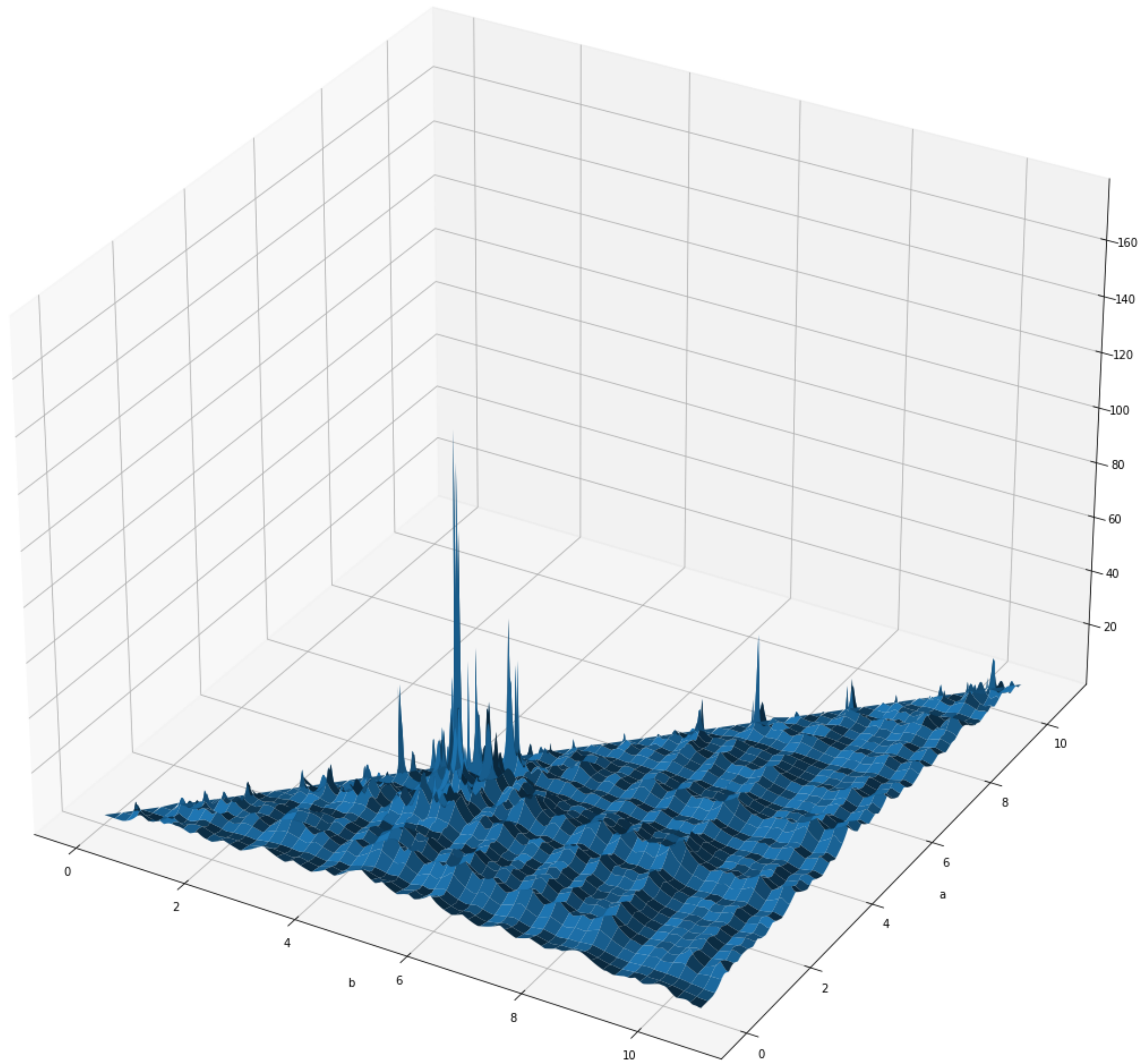
$$\begin{aligned} \min \quad & \frac{a-b}{\|\gamma(b)-\gamma(a)\|} \\ \text{s.t.} \quad & 0 \le a < b < l \end{aligned} \iff \begin{aligned} \min \quad & f(a,b) = \begin{cases} \frac{a-b}{\|\gamma(b)-\gamma(a)\|} & \text{if } a < b \\ +\infty & \text{if } a \ge b \end{cases} \\ \text{s.t.} \quad & 0 \le a < l \\ & 0 < b < l \end{aligned} \tag{1*}$$

   and

$$\begin{aligned} \min \quad & \frac{\max(-b+a, -l+b-a)}{\|\gamma(b)-\gamma(a)\|} \\ \text{s.t.} \quad & 0 \le a < b < l \end{aligned} \iff \begin{aligned} \min \quad & g(a,b) = \begin{cases} \frac{\max(-b+a, -l+b-a)}{\|\gamma(b)-\gamma(a)\|} & \text{if } a < b \\ +\infty & \text{if } a \ge b \end{cases} \\ \text{s.t.} \quad & 0 \le a < l \\ & 0 < b < l \end{aligned} \tag{2*}$$

   respectively.

4. Take a look at the surface plot of $-g(a, b)$ for a randomly generated polygon with 70 vertices on the next page. Admittedly, f's and g's behaviors are not nice at all for them to be objective functions since there are local min and max everywhere. Next, we will attempt to develop more efficient ways to compute/estimate Chord Arc Constant.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