Carpenter's Rule Problem & Chord-arc Constant Algorithms

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Background

Carpenter's Rule Problem: Can a polygon be expanded into a convex shape without creating intersections or distorting lengths? It turns out, yes!

Let $\mathbf{p}_i$ be the position of vertex i, and $\mathbf{v}_i$ be its velocity. The desired motion is a solution to the optimization problem below, where E is the set of connected pairs and S disconnected pairs:

$$\min \sum_{i \in V} \|\mathbf{v}_i\|^2 + \sum_{(i,j)\in S} \frac{1}{\langle \mathbf{v}_i - \mathbf{v}_j, \mathbf{p}_i - \mathbf{p}_j \rangle - \|\mathbf{p}_i - \mathbf{p}_j\|}$$

$$\langle \mathbf{v}_i - \mathbf{v}_j,\ \mathbf{p}_i - \mathbf{p}_j \rangle > \|\mathbf{p}_i - \mathbf{p}_j\| \qquad \forall (i,\ j) \in S$$
$$\langle \mathbf{v}_i - \mathbf{v}_j,\ \mathbf{p}_i - \mathbf{p}_j \rangle = 0 \qquad \forall (i,\ j) \in E$$

The **Chord-arc Constant** of a curve Γ is the largest ratio between the arc distance $\ell_\Gamma(\mathbf{x}, \mathbf{y})$ and Euclidean distance:

$$\sup_{\mathbf{x},\,\mathbf{y} \in \Gamma} \frac{\ell_\Gamma(\mathbf{x}, \mathbf{y})}{\|\ \mathbf{x} - \mathbf{y}\ \|}.$$

These are two polygons we generated, with example Chord-arc distances shown in red.

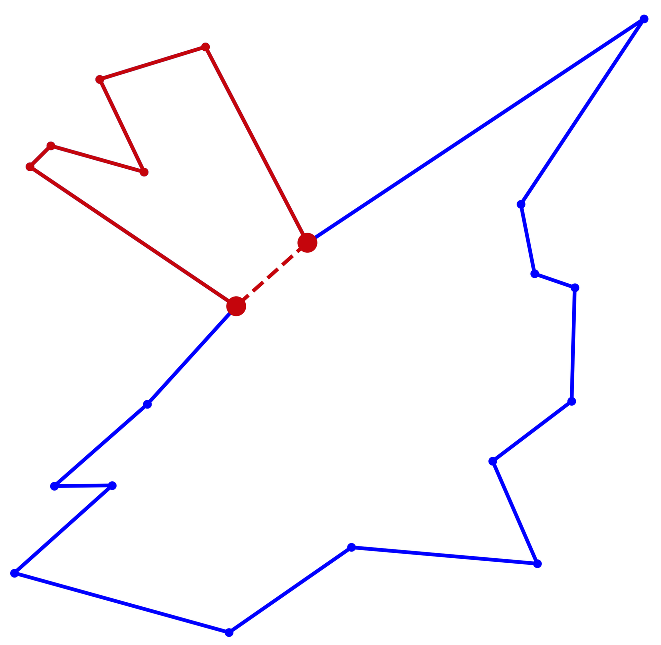


Figure 1a: Star-shaped

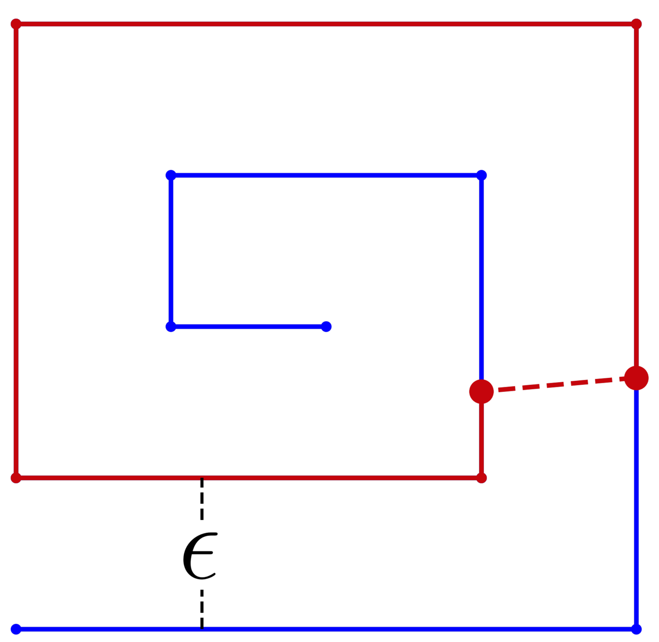


Figure 1b: $\epsilon = 1/4$-spiral

Main Question

How is the initial max speed of the vertices during expansion affected by the Chord-arc Constant?

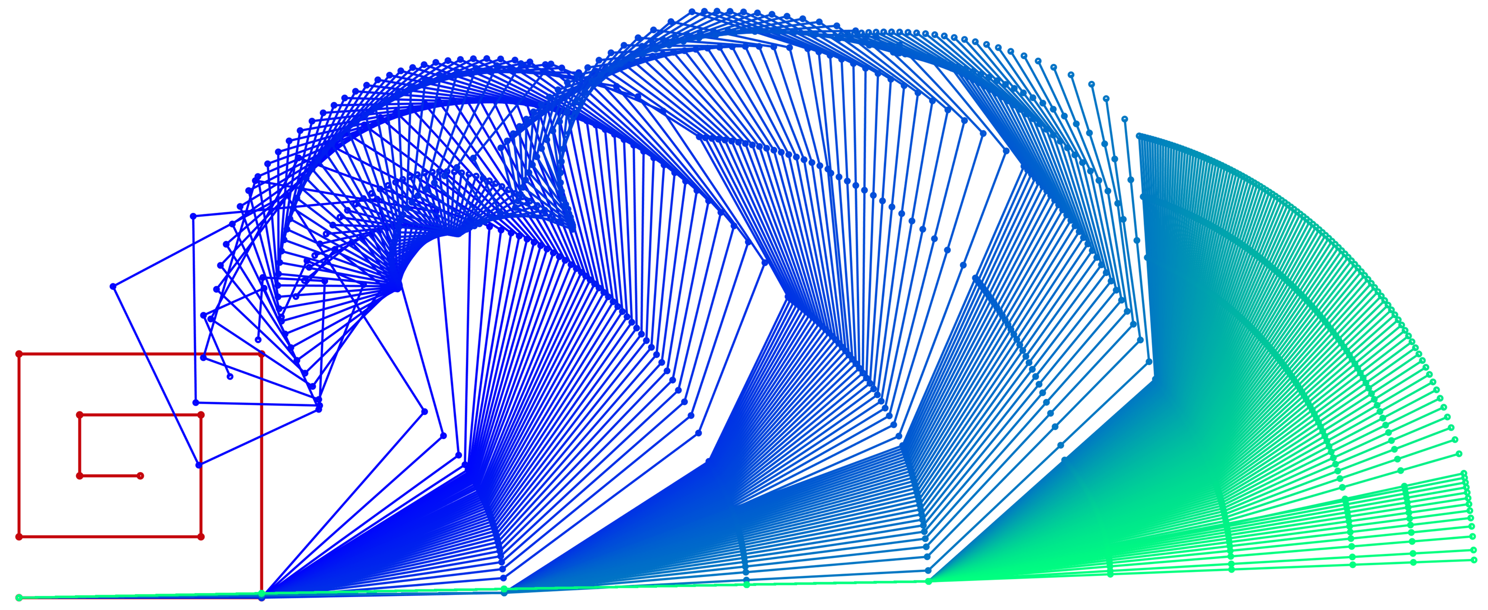


Figure 2a: $\epsilon = 1/4$-spiral Expansion

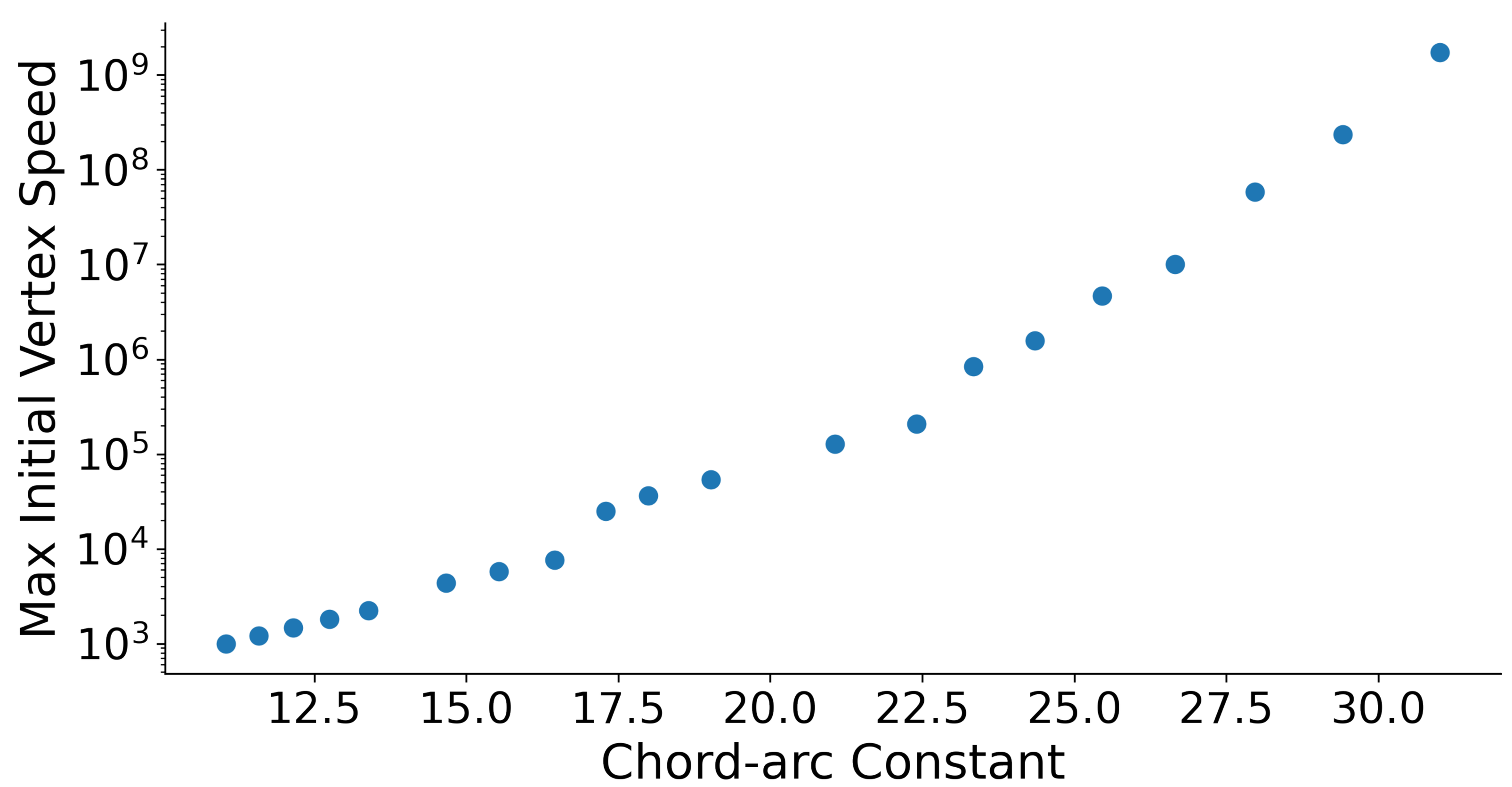


Figure 2b: Max Speed of ϵ-spirals vs Chord-arc Constant

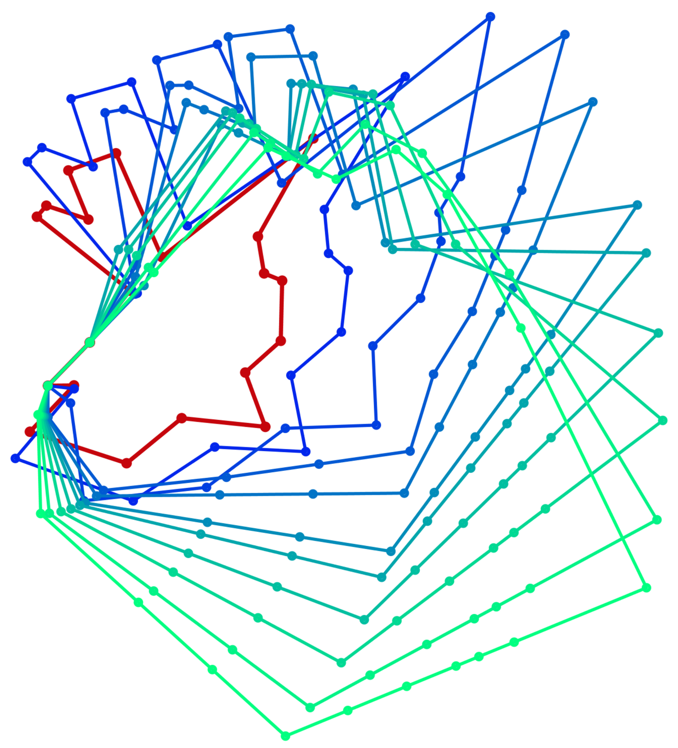
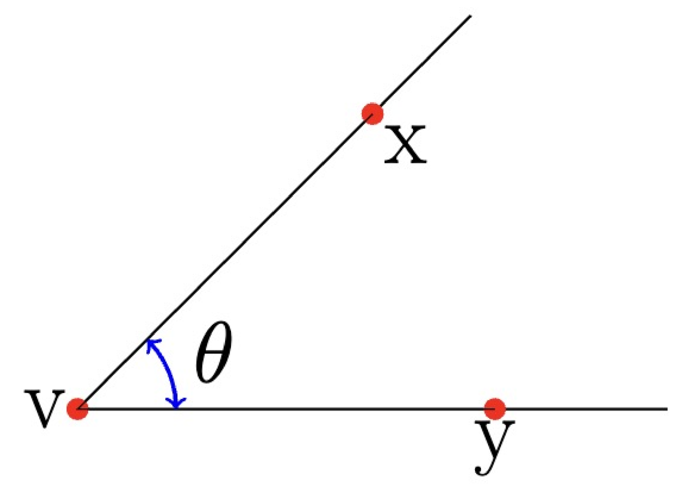


Figure 2c: Star-shaped Polygon Expansion

Related Questions

The Chord-arc Constant between two connected edges depends only on the angle θ:



Where x and y are equidistant to v.

$$C_{adj} = \sqrt{\frac{2}{1 - \cos(\theta)}}.$$

Elsewhere, finding the Chord-arc Constant is quite a computational challenge. We used optimization algorithms to find the maximizer $(a, b)^T$.

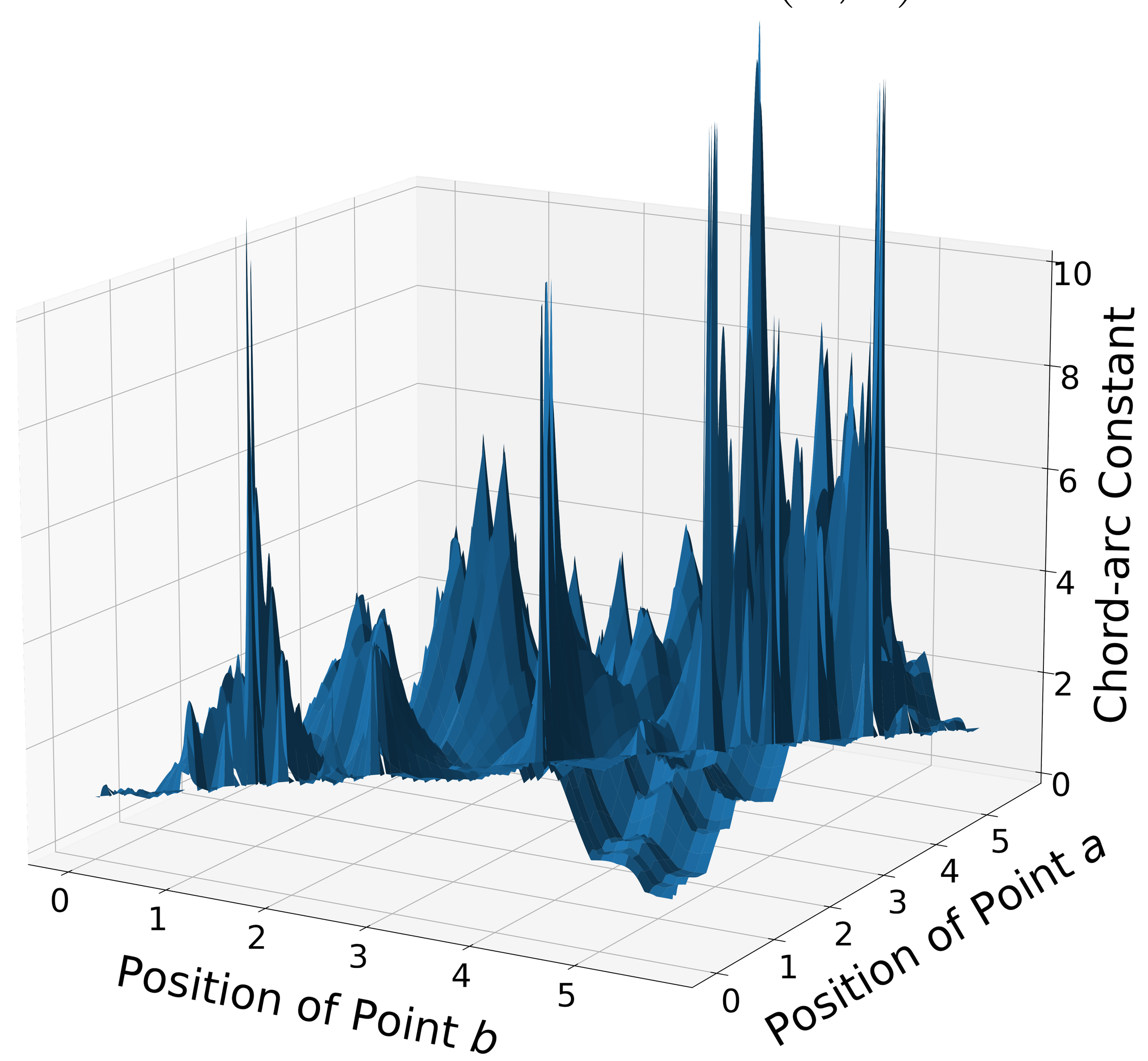


Figure 3: Chord-arc Constant Optimization

Future Explorations:

1. Efficiently calculating the Chord-arc Constant
2. Other Carpenter's Rule Problem approaches
3. Improving polygon generation algorithms

Connelly, Robert, Erik D. Demaine, and Günter Rote (Aug. 1, 2003). "Blowing Up Polygonal Linkages". In: *Discrete and Computational Geometry* 30.2, pp. 205–239. ISSN: 0179-5376, 1432-0444. DOI: 10.1007/s00454-003-0006-7.
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