Proof. Let $x, a, b \in \mathbb{R}^k$ and A, B be positive-definite. Define

$$\mathrm{Norm}_x[m, M] = \frac{1}{(2\pi)^{k/2}\sqrt{|M|}} \exp\left(-\tfrac{1}{2}(x-m)^\top M^{-1}(x-m)\right).$$

We show:

$$\int \mathrm{Norm}_x[a, A]\mathrm{Norm}_x[b, B]\, dx = \mathrm{Norm}_a[b, A+B] \int \mathrm{Norm}_x[\Sigma_*(A^{-1}a+B^{-1}b), \Sigma_*]\, dx,$$

with $\Sigma_* = (A^{-1} + B^{-1})^{-1}$.
Substitute definitions:

$$\mathrm{Norm}_x[a, A]\mathrm{Norm}_x[b, B] = \frac{1}{(2\pi)^k\sqrt{|A||B|}} \exp\left(-\tfrac{1}{2}(x-a)^\top A^{-1}(x-a) - \tfrac{1}{2}(x-b)^\top B^{-1}(x-b)\right).$$

Combine exponents:

$$\begin{aligned}(x-a)^\top A^{-1}(x-a) + (x-b)^\top B^{-1}(x-b) = x^\top (A^{-1} + B^{-1})x - 2x^\top (A^{-1}a + B^{-1}b)\\ + a^\top A^{-1}a + b^\top B^{-1}b.\end{aligned}$$

Set $Q = A^{-1} + B^{-1}$, $y = A^{-1}a + B^{-1}b$. Complete the square:

$$x^\top Qx - 2x^\top y = (x - Q^{-1}y)^\top Q(x - Q^{-1}y) - y^\top Q^{-1}y.$$

With $Q^{-1} = \Sigma_*$:

$$\begin{aligned}\mathrm{Norm}_x[a, A]\mathrm{Norm}_x[b, B] = &\frac{\exp\left(\frac{1}{2}y^\top \Sigma_* y - \frac{1}{2}(a^\top A^{-1}a + b^\top B^{-1}b)\right)}{(2\pi)^k\sqrt{|A||B|}}\\ &\times \exp\left(-\tfrac{1}{2}(x - \Sigma_* y)^\top \Sigma_*^{-1}(x - \Sigma_* y)\right).\end{aligned}$$

Recognize the x-part as $\mathrm{Norm}_x[\Sigma_* y, \Sigma_*]$. Known identities yield:

$$\frac{\exp\left(\frac{1}{2}y^\top \Sigma_* y - \frac{1}{2}(a^\top A^{-1}a + b^\top B^{-1}b)\right)}{(2\pi)^k\sqrt{|A||B|}|\Sigma_*|^{-1/2}} = \frac{1}{(2\pi)^{k/2}\sqrt{|A+B|}} \exp\left(-\tfrac{1}{2}(a-b)^\top (A+B)^{-1}(a-b)\right).$$

Thus:

$$\mathrm{Norm}_x[a, A]\mathrm{Norm}_x[b, B] = \mathrm{Norm}_a[b, A+B]\ \mathrm{Norm}_x[\Sigma_*(A^{-1}a + B^{-1}b), \Sigma_*].$$

Integrating over x:

$$\int \mathrm{Norm}_x[a, A]\mathrm{Norm}_x[b, B]\, dx = \mathrm{Norm}_a[b, A+B],$$

since the integral of the latter norm is 1.

□