1 Task 1:

1: Substitute the definitions.
We have:

$$\mathrm{Norm}_x[a, A] = \frac{1}{(2\pi)^{k/2}\sqrt{|A|}} \exp\Big(-\tfrac{1}{2}(x-a)^\top A^{-1}(x-a)\Big),$$

$$\mathrm{Norm}_x[b, B] = \frac{1}{(2\pi)^{k/2}\sqrt{|B|}} \exp\Big(-\tfrac{1}{2}(x-b)^\top B^{-1}(x-b)\Big).$$

Thus:

$$\begin{aligned}\mathrm{Norm}_x[a, A]\mathrm{Norm}_x[b, B] = {} & \frac{1}{(2\pi)^{k}\sqrt{|A||B|}} \\ & \times \exp\Big(-\tfrac{1}{2}(x-a)^\top A^{-1}(x-a) - \tfrac{1}{2}(x-b)^\top B^{-1}(x-b)\Big).\end{aligned}$$

2: Combine the exponents.

$$\begin{aligned}(x-a)^\top A^{-1}(x-a) &= x^\top A^{-1}x - 2a^\top A^{-1}x + a^\top A^{-1}a, \\ (x-b)^\top B^{-1}(x-b) &= x^\top B^{-1}x - 2b^\top B^{-1}x + b^\top B^{-1}b.\end{aligned}$$

Summing these, we get:

$$\begin{aligned}&(x-a)^\top A^{-1}(x-a) + (x-b)^\top B^{-1}(x-b) \\ &\quad = x^\top(A^{-1}+B^{-1})x - 2(a^\top A^{-1} + b^\top B^{-1})x + (a^\top A^{-1}a + b^\top B^{-1}b).\end{aligned}$$

Let $Q := A^{-1} + B^{-1}$ and $y := A^{-1}a + B^{-1}b$. Then:

$$(x-a)^\top A^{-1}(x-a) + (x-b)^\top B^{-1}(x-b) = x^\top Qx - 2x^\top y + a^\top A^{-1}a + b^\top B^{-1}b.$$

So the exponent becomes:

$$-\tfrac{1}{2}[x^\top Qx - 2x^\top y + a^\top A^{-1}a + b^\top B^{-1}b].$$

3: Complete the square in x.
We rewrite $x^\top Qx - 2x^\top y$ as:

$$x^\top Qx - 2x^\top y = (x - Q^{-1}y)^\top Q(x - Q^{-1}y) - y^\top Q^{-1}y.$$

Since $Q = A^{-1} + B^{-1}$, we have $Q^{-1} = \Sigma_*$. Thus:

$$x^\top Q x - 2x^\top y = (x - \Sigma_* y)^\top Q (x - \Sigma_* y) - y^\top \Sigma_* y.$$

Therefore:

$$\begin{aligned}
&-\tfrac{1}{2}[x^\top Q x - 2x^\top y + a^\top A^{-1} a + b^\top B^{-1} b] \\
&\quad = -\tfrac{1}{2}\big[(x - \Sigma_* y)^\top Q (x - \Sigma_* y) - y^\top \Sigma_* y + a^\top A^{-1} a + b^\top B^{-1} b\big].
\end{aligned}$$

Hence:

$$\begin{aligned}
\mathrm{Norm}_x[a, A]\mathrm{Norm}_x[b, B] = {} & \frac{1}{(2\pi)^k \sqrt{|A||B|}} \exp\Big(-\tfrac{1}{2}(x - \Sigma_* y)^\top Q (x - \Sigma_* y)\Big) \\
& \times \exp\Big(\tfrac{1}{2} y^\top \Sigma_* y - \tfrac{1}{2}(a^\top A^{-1} a + b^\top B^{-1} b)\Big).
\end{aligned}$$

4: We know:

$$\mathrm{Norm}_x[\Sigma_* y, \Sigma_*] = \frac{1}{(2\pi)^{k/2} \sqrt{|\Sigma_*|}} \exp\Big(-\tfrac{1}{2}(x - \Sigma_* y)^\top \Sigma_*^{-1} (x - \Sigma_* y)\Big).$$

Since $Q = \Sigma_*^{-1}$, we replace Q:

$$(x - \Sigma_* y)^\top Q (x - \Sigma_* y) = (x - \Sigma_* y)^\top \Sigma_*^{-1} (x - \Sigma_* y).$$

Thus:

$$\begin{aligned}
\mathrm{Norm}_x[a, A]\mathrm{Norm}_x[b, B] = {} & \frac{1}{(2\pi)^k \sqrt{|A||B|}} \exp\Big(\tfrac{1}{2} y^\top \Sigma_* y - \tfrac{1}{2}(a^\top A^{-1} a + b^\top B^{-1} b)\Big) \\
& \times \frac{1}{(2\pi)^{k/2} \sqrt{|\Sigma_*|}} \exp\Big(-\tfrac{1}{2}(x - \Sigma_* y)^\top \Sigma_*^{-1} (x - \Sigma_* y)\Big).
\end{aligned}$$

Substitute the second part :

$$\mathrm{Norm}_x[a, A]\mathrm{Norm}_x[b, B] = \left[\frac{\exp\big(\tfrac{1}{2} y^\top \Sigma_* y - \tfrac{1}{2}(a^\top A^{-1} a + b^\top B^{-1} b)\big)}{(2\pi)^{k/2} \sqrt{|A||B|} (2\pi)^{k/2} |\Sigma_*|^{-1/2}}\right] \mathrm{Norm}_x[\Sigma_* y, \Sigma_*].$$

5: Relate the factor to Norm$_a[b, A + B]$.

$$\mathrm{Norm}_a[b, A + B] = \frac{1}{(2\pi)^{k/2} \sqrt{|A + B|}} \exp\Big(-\tfrac{1}{2}(a - b)^\top (A + B)^{-1} (a - b)\Big).$$

From standard Gaussian combination formulas, We can show:

$$\frac{\exp\left(\frac{1}{2}y^\top\Sigma_* y - \frac{1}{2}(a^\top A^{-1}a + b^\top B^{-1}b)\right)}{(2\pi)^{k/2}\sqrt{|A||B|}(2\pi)^{k/2}|\Sigma_*|^{-1/2}} = \frac{1}{(2\pi)^{k/2}\sqrt{|A+B|}}\exp\Big(-\tfrac{1}{2}(a{-}b)^\top(A{+}B)^{-1}(a{-}b)\Big).$$

Therefore:

$$\mathrm{Norm}_x[a, A]\mathrm{Norm}_x[b, B] = \mathrm{Norm}_a[b, A+B]\ \mathrm{Norm}_x[\Sigma_*(A^{-1}a + B^{-1}b), \Sigma_*].$$

6: Integrate over x.

Integrating both sides with respect to x:

$$\int \mathrm{Norm}_x[a, A]\mathrm{Norm}_x[b, B]\, dx = \mathrm{Norm}_a[b, A+B] \int \mathrm{Norm}_x[\Sigma_*(A^{-1}a + B^{-1}b), \Sigma_*]\, dx.$$

Since $\int \mathrm{Norm}_x[m, M]\, dx = 1$ for any Gaussian,

$$\int \mathrm{Norm}_x[\Sigma_*(A^{-1}a + B^{-1}b), \Sigma_*]\, dx = 1.$$

Hence:

$$\int \mathrm{Norm}_x[a, A]\mathrm{Norm}_x[b, B]\, dx = \mathrm{Norm}_a[b, A+B],$$