9 - Multiobjective optimization

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Definition

A multiobjective optimization problem is defined by:

$$\begin{cases} \min \ f(x) = (f_1(x), f_2(x), \dots, f_s(x)) \\ x \in X \end{cases} \tag{P}$$

where $f : \mathbb{R}^n \to \mathbb{R}^s$, $f(x) = (f_1(x), f_2(x), \dots, f_s(x))$, $X \subseteq \mathbb{R}^n$.

- $f(x)$ is a vector in $\mathbb{R}^s$, i.e., there are several objectives to be simultaneously optimized.
- We need to define an order in $\mathbb{R}^s$.

Given $x, y \in \mathbb{R}^s$, we say that

$$x \geq y \quad \Longleftrightarrow \quad x_i \geq y_i \quad \text{for any } i = 1, \ldots, s.$$

This relation is a partial order in $\mathbb{R}^s$: it is

- reflexive: $x \geq x$
- asymmetric: if $x \geq y$ and $y \geq x$ then $x = y$
- transitive: if $x \geq y$ and $y \geq z$ then $x \geq z$

but it is not a total order: if $x = (1, 5)$ and $y = (4, 2)$ then $x \not\geq y$ and $y \not\geq x$

Definition

Given a subset $A \subseteq \mathbb{R}^s$, we say that

- $\bar{x} \in A$ is a Pareto ideal minimum (or ideal efficient point) of A if $y \geq \bar{x}$ for any $y \in A$.
- $\bar{x} \in A$ is a Pareto minimum (or efficient point) of A if there is no $y \in A$, $y \neq \bar{x}$, such that $\bar{x} \geq y$ (or, equivalently, there is no $y \in A$ such that $\bar{x} \geq y$ and $\bar{x}_j > y_j$, for some $j \in \{1, .., s\}$).
- $\bar{x} \in A$ is a Pareto weak minimum (or weakly efficient point) of A if there is no $y \in A$ such that $\bar{x} > y$, i.e., $\bar{x}_i > y_i$ for any $i = 1, \ldots, s$.

$IMin(A)$, $Min(A)$ and $WMin(A)$ denote the set of ideal minima, minima, weak minima of A, respectively.

Remark

$$IMin(A) \subseteq Min(A) \subseteq WMin(A).$$

Equivalent definitions of minimum points for a set of vectors

- $\bar{x} \in A$ is a Pareto ideal minimum if

$$A \subseteq (\bar{x} + \mathbb{R}^s_+)$$

- $\bar{x} \in A$ is a Pareto minimum of A if

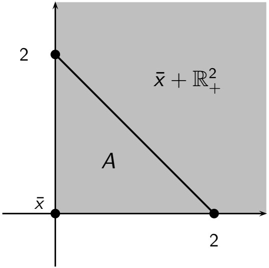
$$A \cap (\bar{x} - \mathbb{R}^s_+) = \{\bar{x}\}$$

- $\bar{x} \in A$ is a Pareto weak minimum of A if

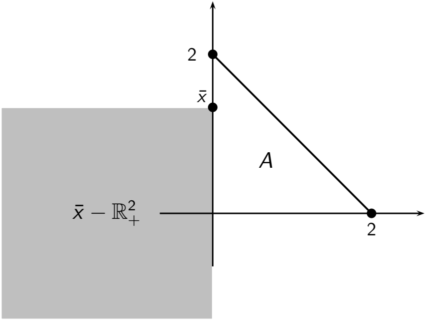
$$A \cap (\bar{x} - int(\mathbb{R}^s_+)) = \emptyset$$

Example 1

$$A = \{x \in \mathbb{R}^2 : \; x_1 \geq 0, \; x_2 \geq 0, \; x_1 + x_2 \leq 2\}.$$



Let $\bar{x} = (0, 0)$.



Let $\bar{x} = (0, a)$, $0 < a \leq 2$

$$IMin(A) = Min(A) = \{(0,0)\}, \ \ WMin(A) = \{x \in A: \ x_1 = 0 \text{ or } x_2 = 0\}.$$

Proposition

If $IMin(A) \neq \emptyset$, then $IMin(A) = Min(A) = \{\bar{x}\}$.

Proof. Let $x^1 \in IMin(A)$ and assume that there exists $x^2 \in Min(A)$, with $x^2 \neq x^1$.
We notice that:

- since $x^1 \in IMin(A)$ then $x_1 \leq x_2$ and
- since $x^2 \in Min(A)$, then $x_1 = x_2$.

Example 2

$B = \{x \in \mathbb{R}^2 : \; 0 \leq x_1 \leq 3, \; 0 \leq x_2 \leq 3, \; x_1 + x_2 \geq 2\}$.

- $IMin(B) = \emptyset$,
- $Min(B) = \{x \in B : x_1 + x_2 = 2\}$,
- $WMin(B) = \{x \in B : x_1 = 0 \text{ or } x_2 = 0 \text{ or } x_1 + x_2 = 2\}$.

We have the following fundamental existence result.

Theorem 1

If there exists $\hat{x} \in A$ such that the set $A \cap (\hat{x} - \mathbb{R}^s_+)$ is compact then $Min(A) \neq \emptyset$.

Definition

Given a multiobjective optimization problem

$$\begin{cases} \min \ f(x) = (f_1(x), f_2(x), \dots, f_s(x)) \\ x \in X \end{cases} \tag{P}$$

- $x^* \in X$ is a Pareto ideal minimum of (P) if $f(x^*)$ is a Pareto ideal minimum of $f(X)$, i.e., $f(x) \geq f(x^*)$ for any $x \in X$.
- $x^* \in X$ is a Pareto minimum of (P) if $f(x^*)$ is a Pareto minimum of $f(X)$, i.e., if there is no $x \in X$ such that

$$\begin{array}{ll} f_i(x^*) \geq f_i(x) & \text{for any } i = 1, \dots, s, \\ f_j(x^*) > f_j(x) & \text{for some } j \in \{1, \dots, s\}. \end{array}$$

- $x^* \in X$ is a Pareto weak minimum of (P) if $f(x^*)$ is a Pareto weak minimum of $f(X)$, i.e., if there is no $x \in X$ such that

$$f_i(x^*) > f_i(x) \qquad \text{for any } i = 1, \dots, s.$$

Example 3

$$\begin{cases} \min\ (x_1 - x_2,\ -2x_1 + x_2) \\ x_1 \leq 1 \\ -x_1 \leq 0 \\ -x_1 + x_2 \leq 2 \\ 2x_1 - x_2 \leq 0 \end{cases} \tag{P}$$

The image $f(X) = \{(y_1, y_2) :\ y_1 = x_1 - x_2,\ y_2 = -2x_1 + x_2,\ x \in X\}$.

We obtain $x_1 = -y_1 - y_2$ and $x_2 = -2y_1 - y_2$, hence

$$f(X) = \{(y_1, y_2) :\ -y_1 - y_2 \leq 1,\ y_1 + y_2 \leq 0, -y_1 \leq 2,\ -y_2 \leq 0\}.$$

$IMin(f(X)) = \emptyset$. $Min(f(X)) = \{y \in f(X) :\ -y_1 - y_2 = 1\}$, thus

$$\{\text{minima of (P)}\} = \{x \in X :\ -x_1 + x_2 + 2x_1 - x_2 = 1\} = \{x \in X :\ x_1 = 1\}.$$

$WMin(f(X)) = \{y \in f(X) :\ -y_1 - y_2 = 1 \text{ or } y_1 = -2 \text{ or } y_2 = 0\}$, thus

$$\{\text{weak minima of (P)}\} = \{x \in X :\ x_1 = 1 \text{ or } x_1 - x_2 = -2 \text{ or } -2x_1 + x_2 = 0\}.$$

We explicitly obtain

$$\text{minima of (P)} = (x_1, x_2) : \begin{cases} x_1 = 1 \\ -x_1 \leq 0 \\ -x_1 + x_2 \leq 2 \\ 2x_1 - x_2 \leq 0 \end{cases}$$

Weak minima of (P)=

$$= (x_1, x_2) : \begin{cases} x_1 = 1 \\ -x_1 \leq 0 \\ -x_1 + x_2 \leq 2 \\ 2x_1 - x_2 \leq 0 \end{cases} \cup \begin{cases} x_1 \leq 1 \\ -x_1 \leq 0 \\ -x_1 + x_2 = 2 \\ 2x_1 - x_2 \leq 0 \end{cases} \cup \begin{cases} x_1 \leq 1 \\ -x_1 \leq 0 \\ -x_1 + x_2 \leq 2 \\ 2x_1 - x_2 = 0 \end{cases}$$

Remark

If (P) is a multiobjective linear problem, then $Min(P)$ and $WMin(P)$ are union of faces of the polyhedron X.

Theorem 2

If f_i is continuous for any $i = 1 \ldots, s$ and X is compact, then there exists a minimum of (P).

Proof. It is an immediate consequence of Theorem 1. Indeed, since f is continuous and X is compact, then $f(X)$ is a compact set.

Theorem 3

If f_i is continuous for any $i = 1 \ldots, s$, X is closed and there exist $v \in \mathbb{R}$ and $j \in \{1, \ldots, s\}$ such that the sublevel set

$$\{x \in X : \; f_j(x) \leq v\}$$

is nonempty and bounded, then there exists a minimum of (P).

Proof. It is a further consequence of Theorem 1. We need to prove that there exists $\hat{y} \in f(X)$ such that

$$S_{\hat{y}} := f(X) \cap (\hat{y} - \mathbb{R}^s_+)$$

is compact, so that $Min(f(X)) \neq \emptyset$.

Set $\hat{y}_j = f_j(x)$ for some x in the level set $\{x \in X : f_j(x) \leq v\}$ and $\hat{y}_i = f_i(x)$, for $i \neq j$. Consider the subset B of X such that $f(B) = S_{\hat{y}}$, i.e.,

$$B := \{x \in X : f(x) \in S_{\hat{y}}\} = \{x \in X : f(x) \leq \hat{y}\}$$

or, equivalently, the solution set of the system

$$\begin{cases} f_1(x) \leq \hat{y}_1 \\ \dots\dots\dots\dots \\ \dots\dots\dots\dots \\ f_s(x) \leq \hat{y}_s \\ x \in X \end{cases}$$

By the continuity and compactness assumptions, the closed subset $\{x \in X : f_j(x) \leq \hat{y}_j\} \subseteq \{x \in X : f_j(x) \leq v\}$ is compact.
Moreover, since f is continuous then B is compact too, being a closed subset of the compact set $\{x \in X : f_j(x) \leq \hat{y}_j\}$ and consequently $f(B) = S_{\hat{y}}$ is compact, which completes the proof.

Corollary 1

If f_i is continuous for any $i = 1 \ldots, s$, X is closed and f_j is coercive for some $j \in \{1, \ldots, s\}$, then there exists a minimum of (P).

Example 4

Consider the multiobjective problem

$$\begin{cases} \min \ (x_1 + x_2^2, \ (x_1 - 1)^2 + (x_2 - 1)^2) \\ x \in X := \mathbb{R}^2_+ \end{cases}$$

Theorem 4

$x^* \in X$ is a minimum of (P) if and only if the auxiliary optimization problem

$$\begin{cases} \max \sum_{i=1}^{s} \varepsilon_i \\ f_i(x) + \varepsilon_i \leq f_i(x^*) \qquad \forall\ i = 1, \dots, s \\ x \in X \\ \varepsilon \geq 0 \end{cases}$$

has optimal value equal to 0.

Proof. Let $(\bar{x}, \bar{\varepsilon})$ be an optimal solution of the auxiliary problem. Assume that x^* is a minimum of (P) and the optimal value $\sum_{i=1}^{s} \bar{\varepsilon}_i > 0$.

Then, there exists $j \in \{1, \dots, s\}$ such that $\bar{\varepsilon}_j > 0$ and

$$\begin{aligned} & f_i(x^*) \geq f_i(\bar{x}) \qquad \text{for any } i = 1, \dots, s, \\ & f_j(x^*) \geq f_j(\bar{x}) + \bar{\varepsilon}_j > f_j(\bar{x}). \end{aligned}$$

which contradicts x^* is a minimum of (P).

Conversely, assume that the optimal value $\sum_{i=1}^{s} \bar{\varepsilon}_i = 0$ and x^* is not a minimum of (P).

Then for some $x \in X$

$$\begin{array}{ll} f_i(x^*) \geq f_i(x) & \text{for any } i = 1, \ldots, s, \\ f_j(x^*) > f_j(x) & \text{for some } j \in \{1, \ldots, s\}. \end{array}$$

Setting $\varepsilon_j = f_j(x^*) - f_j(x) > 0$, the solution (x, ε), with $\varepsilon_i = 0, i \neq j$ is feasible for the auxiliary problem and $\sum_{i=1}^{s} \bar{\varepsilon}_i > 0$ which contradicts that the optimal value is zero.

Theorem 5

$x^* \in X$ is a weak minimum of (P) if and only if the auxiliary optimization problem

$$\begin{cases} \max \ v \\ v \leq \varepsilon_i & \forall \ i = 1, \ldots, s \\ f_i(x) + \varepsilon_i \leq f_i(x^*) & \forall \ i = 1, \ldots, s \\ x \in X \\ \varepsilon \geq 0 \end{cases}$$

has optimal value equal to 0.

Example 5

Consider the linear multiobjective problem

$$\begin{cases} \min \ (x_1 + 2x_2 - 3x_3 \ , \ -x_1 - x_2 - x_3 \ , \ -4x_1 - 2x_2 + x_3) \\ x_1 + x_2 + x_3 \leq 10 \\ x_3 \leq 5 \\ x_1, x_2, x_3 \geq 0 \end{cases}$$

(a) Check if the point $x^* = (5, 0, 5)$, is a weak minimum by solving the corresponding auxiliary problem.
(b) Check if $x^* = (5, 0, 5)$, is a minimum by solving the corresponding auxiliary problem.

Let us check if $x^* = (5, 0, 5)$ is a weak minimum. Then, $f(x^*) = (-10, -10, -15,)^T$ and the corresponding auxiliary problem is given by

$$\begin{cases} \max \ v \\ v \leq \varepsilon_i, \quad i = 1, 2, 3 \\ x_1 + 2x_2 - 3x_3 + \varepsilon_1 \leq -10 \\ -x_1 - x_2 - x_3 + \varepsilon_2 \leq -10 \\ -4x_1 - 2x_2 + x_3 + \varepsilon_3 \leq -15 \\ x_1 + x_2 + x_3 \leq 10 \\ x_3 \leq 5 \\ x_1, x_2, x_3 \geq 0, \ \varepsilon_1, \varepsilon_2, \varepsilon_3 \geq 0 \end{cases}$$

Let us solve the auxiliary problem by Matlab. In matrix form the problem can be written as:

$$\begin{cases} -\min \ -v \\ A \begin{pmatrix} x \\ \varepsilon \\ v \end{pmatrix} \leq b \\ x \geq 0, \ \varepsilon \geq 0 \end{cases} \tag{1}$$

where

$$A = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 1 & 2 & -3 & 1 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 & 1 & 0 & 0 \\ -4 & -2 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad b = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -10 \\ -10 \\ -15 \\ 10 \\ 5 \end{pmatrix}$$

MATLAB COMMANDS

object. funct.	`c=[0, 0, 0, 0, 0, 0, -1]'`

constr.	`A=[ 0 0 0 -1 0 0 1; 0 0 0 0 -1 0 1 ; 0 0 0 0 0 -1 1; 1 2 -3 1 0 0 0 ; -1 -1 -1 0 1 0 0; -4 -2 1 0 0 1 0; 1 1 1 0 0 0 0; 0 0 1 0 0 0 0]`
	`b= [0;0;0;-10; -10;-15;10;5]`
	`Aeq=[ ];`
	`beq=[ ];`
	`lb= [zeros(6,1); -Inf]`
	`ub= [ ];`

Solut. Command	`[x,fval]=linprog(c, A, b,[ ],[ ],lb,ub)`

Solution

Optimal solution	(5,0,5,0,0,0,0)
Optimal value	0

Let us check if $x^* = (5, 0, 5)$ is a minimum. The corresponding auxiliary problem is:

$$\begin{cases} \max \ \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \\ x_1 + 2x_2 - 3x_3 + \varepsilon_1 \leq -10 \\ -x_1 - x_2 - x_3 + \varepsilon_2 \leq -10 \\ -4x_1 - 2x_2 + x_3 + \varepsilon_3 \leq -15 \\ x_1 + x_2 + x_3 \leq 10 \\ x_3 \leq 5 \\ x_1, x_2, x_3 \geq 0, \ \varepsilon_1, \varepsilon_2, \varepsilon_3 \geq 0 \end{cases}$$

In matrix form the problem can be written as:

$$\begin{cases} -\min \ -\varepsilon_1 - \varepsilon_2 - \varepsilon_3 \\ A \begin{pmatrix} x \\ \varepsilon \end{pmatrix} \leq b \\ x \geq 0, \ \varepsilon \geq 0 \end{cases} \tag{2}$$

where

$$A = \begin{pmatrix} 1 & 2 & -3 & 1 & 0 & 0 \\ -1 & -1 & -1 & 0 & 1 & 0 \\ -4 & -2 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \quad b = \begin{pmatrix} -10 \\ -10 \\ -15 \\ 10 \\ 5 \end{pmatrix}$$

MATLAB COMMANDS

object. funct.	`c=[0, 0, 0, -1, -1, -1]'`

constr.	`A=[1 2 -3 1 0 0 ; -1 -1 -1 0 1 0 ; -4 -2 1 0 0 1 ; 1 1 1 0 0 0; 0 0 1 0 0 0]`
	`b= [-10; -10;-15;10;5]`
	`Aeq=[ ];`
	`beq=[ ];`
	`lb= zeros(6,1)`
	`ub= [ ];`

Solut. Command	`[x,fval]=linprog(c, A, b,[ ],[ ],lb,ub)`

Solution

Optimal solution	(5,0,5,0,0,0)
Optimal value	0

Exercise

Consider the linear multiobjective problem defined in Example 5,

$$\begin{cases} \min\ (x_1 + 2x_2 - 3x_3\ ,\ -x_1 - x_2 - x_3\ ,\ -4x_1 - 2x_2 + x_3) \\ x_1 + x_2 + x_3 \leq 10 \\ x_3 \leq 5 \\ x_1, x_2, x_3 \geq 0 \end{cases}$$

- Prove that a Pareto minimum point exists;
- Check if the point $x^* = (3, 3, 4)$, is a weak minimum or a minimum by solving the corresponding auxiliary problems.

First-order optimality conditions: unconstrained problems

Consider an unconstrained multiobjective problem

$$\begin{cases} \min \ f(x) = (f_1(x), f_2(x), \dots, f_s(x)) \\ x \in \mathbb{R}^n \end{cases} \qquad (P_u)$$

where f_i is continuously differentiable for any $i = 1, \dots, s$.

Remark.

If x^* is a weak minimum of (P_u), then the system

$$\begin{cases} \nabla f_i(x^*)^\top d < 0, \quad i = 1, .., s, \\ d \in \mathbb{R}^n \end{cases} \qquad \text{(S1)}$$

is impossible.

Proposition 2 (Necessary optimality condition)

If x^* is a weak minimum of (P_u), then there exists $\theta^* \in \mathbb{R}^s$ such that (x^*, θ^*) is a solution of the system

$$\begin{cases} \sum_{i=1}^{s} \theta_i \nabla f_i(x) = 0 \\ \theta \geq 0, \quad \sum_{i=1}^{s} \theta_i = 1, \\ x \in \mathbb{R}^n \end{cases} \tag{S}$$

Proof. By the previous remark the system (S1) is impossible. Let $\Gamma := \{u \in \mathbb{R}^s : u_i = \nabla f_i(x^*)^\top d,\ d \in \mathbb{R}^n, \quad i = 1, ..., s\}$. Then the impossibility of (S1) is equivalent to:

$$\Gamma \cap (-int(\mathbb{R}^s_+)) = \emptyset.$$

Since Γ and $-int(\mathbb{R}^s_+)$ are disjoint convex sets then there exists an hyperplane of equation $\langle \theta, u \rangle = 0$, $\theta \in \mathbb{R}^s_+$, $\theta \neq 0$, which separates them, i.e.,

$$\langle \theta, u \rangle \geq 0, \quad \forall u \in \Gamma, \quad \langle \theta, u \rangle \leq 0, \quad \forall u \in (-int(\mathbb{R}^s_+)).$$

The first inequality can be written as

$$\sum_{i=1}^{s} \theta_i \nabla f_i(x^*)^T d \geq 0, \quad \forall d \in \mathbb{R}^n.$$

Since v is arbitrary, we have:

$$\sum_{i=1}^{s} \theta_i \nabla f_i(x^*) = 0$$

and setting

$$\theta^* = \frac{\theta}{\sum_{i=1}^{s} \theta_i}$$

we obtain that system (S) is fulfilled.

Proposition 3 (Sufficient optimality condition)

Assume that the problem (P_u) is convex, i.e., f_i is convex for any $i = 1, \ldots, s$, and (x^*, θ^*) is a solution of the system (S). Then:

- x^* is a weak minimum of (P_u).
- If, additionally, $\theta^* > 0$, then x^* is a minimum of (P_u).

Proof. Consider the function $L(\theta, x) := \sum_{i=1}^{s} \theta_i f_i(x)$, with $\theta \in \mathbb{R}^s_+$.

Since f is convex then $L(\theta, \cdot)$ is convex, and

$$\sum_{i=1}^{s} \theta_i^* \nabla f_i(x^*) = 0 \quad \Longrightarrow \quad L(\theta^*, x^*) \leq L(\theta^*, x), \quad \forall x \in \mathbb{R}^n,$$

i.e.,

$$\sum_{i=1}^{s} \theta_i^* (f_i(x^*) - f_i(x)) \leq 0, \quad \forall x \in \mathbb{R}^n. \tag{3}$$

As, $\theta^* \in \mathbb{R}^s_+$ and $\theta^* \neq 0$, the system

$$f(x^*) - f(x) > 0, \quad x \in \mathbb{R}^n,$$

is impossible,

in fact, if not, we would have:

$$\sum_{i=1}^{s} \theta_i^* (f_i(x^*) - f_i(x)) > 0, \quad \text{for some } x \in \mathbb{R}^n,$$

which contradicts (3). Therefore, x^* is a weak minimum of (P_u).

Similarly, we can prove that, if, additionally, $\theta^* > 0$, then x^* is a minimum of (P_u). Indeed, $x^* \in X$ is a minimum of (P_u) if the following system is impossible:

$$\begin{array}{ll} f_i(x^*) - f_i(x) \geq 0 & \text{for any } i = 1, \dots, s, \quad i \neq j \\ f_j(x^*) - f_j(x) > 0 & \text{for some } j \in \{1, \dots, s\}. \end{array}$$

By contradiction, assume that it is possible for some x. Since $\theta^* > 0$, then multiplying the inequality i by θ_i^* and summing all the inequalities we obtain:

$$\sum_{i=1}^{s} \theta_i^* (f_i(x^*) - f_i(x)) > 0$$

which contradicts (3). Hence, $x^* \in X$ is a minimum of (P_u).

Example 6

Let us determine the set of weak minima of the following nonlinear multiobjective problem (P_u) exploiting the first-order optimality conditions.

$$\begin{cases} \min \ (x_1^2 + x_2^2, \ (x_1 - 1)^2 + (x_2 - 1)^2) \\ x \in \mathbb{R}^2 \end{cases}$$

We preliminarily note the given problem is convex and differentiable: then system (S) provided a necessary and sufficient condition for a weak minimum.
In this case system (S) becomes:

$$\begin{cases} \theta_1(2x_1) + \theta_2 2(x_1 - 1) = 0 \\ \theta_1(2x_2) + \theta_2 2(x_2 - 1) = 0 \\ \theta_1, \theta_2 \geq 0, \ \theta_1 + \theta_2 = 1 \end{cases}$$

i.e.

$$\begin{cases} x_1(\theta_1 + \theta_2) - \theta_2 = 0 \\ x_2(\theta_1 + \theta_2) - \theta_2 = 0 \\ \theta_1, \theta_2 \geq 0, \ \theta_1 + \theta_2 = 1 \end{cases} \implies \begin{cases} x_1 = \theta_2 \\ x_2 = \theta_2 \\ 0 \leq \theta_2 \leq 1 \end{cases}$$

Therefore, the set of weak minima is given by

$$WMin(P_u) = \{(x_1, x_2) : x_1 = x_2,\ 0 \leq x_1 \leq 1\}$$

Exercise

Find the set of minima of the problem (P_u) defined in Example 6.

Notice that, by Proposition 3

$$\{(x_1, x_2) : x_1 = x_2,\ 0 < x_1 < 1\} \subseteq Min(P_u) \subseteq WMin(P_u)$$

We need only to check if the points $(0, 0)$ and $(1, 1)$ are minima for (P_u).

This can be done directly exploiting the definition of a minimum or by means of Theorem 4.

Consider a constrained multiobjective problem

$$\begin{cases} \min \ f(x) = (f_1(x), f_2(x), \dots, f_s(x)) \\ x \in X := \{x \in \mathbb{R}^n : g_j(x) \leq 0, \quad j = 1, \dots, m, h_k(x) = 0, \ k = 1, \dots, p\} \end{cases} \qquad \text{(P)}$$

where f_i, g_j and h_k are continuously differentiable for any i, j, k.

We briefly recall the Abadie constraint qualification introduced in the analysis of scalar optimization problems.

Recall that:

- The *Tangent cone* at $x^* \in X$, is defined by

$$T_X(x^*) = \left\{ d \in \mathbb{R}^n : \exists \ \{z_k\} \subset X, \ \exists \ \{t_k\} > 0, \ z_k \to x^*, \ t_k \to 0, \ \lim_{k \to \infty} \frac{z_k - x^*}{t_k} = d \right\}$$

- $\mathcal{A}(x^*) = \{j : \ g_j(x^*) = 0\}$ denotes the set of inequality constraints which are active at $x^* \in X$.
- The set

$$D(x^*) = \left\{ d \in \mathbb{R}^n : \begin{array}{ll} d^{\mathsf{T}} \nabla g_j(x^*) \leq 0 & \forall \ j \in \mathcal{A}(x^*), \\ d^{\mathsf{T}} \nabla h_k(x^*) = 0 & \forall \ k = 1, \dots, p \end{array} \right\}$$

  is the *first-order feasible direction cone* at $x^* \in X$.

Definition – Abadie constraint qualification (ACQ)

We say that the Abadie constraint qualification (ACQ) holds at a point $x^* \in X$, if $T_X(x^*) = D(x^*)$.

Theorem (Sufficient conditions for ACQ)

a) (*Affine constraints*)
If g_j and h_k are affine for all $j = 1, \dots, m$ and $k = 1, \dots, p$, then ACQ holds at any $x \in X$.

b) (*Slater condition for convex problems*)
If g_j are convex for all $j = 1, \dots, m$, h_k are affine for all $k = 1, \dots, p$ and there exists $\bar{x} \in X$ s.t. $g(\bar{x}) < 0$ and $h(\bar{x}) = 0$, then ACQ holds at any $x \in X$.

c) (*Linear independence of the gradients of active constraints*)
If $x^* \in X$ and the vectors

$$\begin{cases} \nabla g_j(x^*) & \text{for } j \in \mathcal{A}(x^*), \\ \nabla h_k(x^*) & \text{for } k = 1, \dots, p \end{cases}$$

are linearly independent, then ACQ holds at x^*.

Theorem (KKT necessary optimality conditions)

If x^* is a weak minimum of (P) and ACQ holds at x^*, then there exist $\theta^* \in \mathbb{R}^s$, $\lambda^* \in \mathbb{R}^m$ and $\mu^* \in \mathbb{R}^p$ such that $(x^*, \theta^*, \lambda^*, \mu^*)$ solves the KKT system

$$\begin{cases} \sum_{i=1}^{s} \theta_i \nabla f_i(x) + \sum_{j=1}^{m} \lambda_j \nabla g_j(x) + \sum_{k=1}^{p} \mu_k \nabla h_k(x) = 0 \\ \theta \geq 0, \quad \sum_{i=1}^{s} \theta_i = 1 \\ \lambda \geq 0 \\ \lambda_j \, g_j(x) = 0 \qquad \forall \; j = 1, \dots, m \\ g(x) \leq 0, \; h(x) = 0 \end{cases} \tag{4}$$

Remark

Notice that for an unconstrained problem, i.e. $X = \mathbb{R}^n$, then the KKT system (4) reduces to system (S).

Theorem

If x^* is a weak minimum of (P), then the system

$$\begin{cases} \nabla f_i(x^*)^{\mathsf{T}} d < 0, i = 1, ..., s \\ \quad d \in T_X(x^*). \end{cases}$$

has no solutions.

Proof. By contradiction, assume that there exists $d \in T_X(x^*)$ s.t. $\nabla f_i(x^*)^{\mathsf{T}} d < 0, i = 1, .., s$. Take the sequences $\{z_k\} \subseteq X$ and $\{t_k\} > 0$ s.t. $\lim\limits_{k\to\infty} (z_k - x^*)/t_k = d$. Then $z_k = x^* + t_k\, d + o(t_k)$, where $o(t_k)/t_k \to 0$. Let $i \in 1, ..., s$. The first order approximation of f_i gives

$$f_i(z_k) = f_i(x^*) + t_k\, \nabla f_i(x^*)^{\mathsf{T}} d + o(t_k),$$

thus there is $\bar{k} \in \mathbb{N}$ s.t.

$$\frac{f_i(z_k) - f_i(x^*)}{t_k} = \nabla f_i(x^*)^{\mathsf{T}} d + \frac{o(t_k)}{t_k} < 0 \qquad \forall\ k > \bar{k},\ \forall i = 1, .., s.$$

i.e. $f_i(z_k) < f_i(x^*)$ for all $k > \bar{k}$, and every $i = 1, .., s$, which is impossible because x^* is a weak minimum of (P). □

Corollary

If x^* is a weak minimum of (P) and ACQ holds at x^*, then the system

$$\begin{cases} v^{\mathsf{T}}\nabla f_i(x^*) < 0, i = 1, ..., s \\ v^{\mathsf{T}}\nabla g_j(x^*) \le 0,\ j \in \mathcal{A}(x^*), \\ v^{\mathsf{T}}\nabla h_k(x^*) = 0,\ k = 1, \dots, p, \\ v \in \mathbb{R}^n \end{cases} \tag{S1}$$

has no solutions.

Proof. It is enough to observe that, if ACQ holds at x^* then

$$T_X(x^*) = D(x^*) = \left\{ v \in \mathbb{R}^n : \begin{array}{ll} v^{\mathsf{T}}\nabla g_j(x^*) \le 0 & \forall\ j \in \mathcal{A}(x^*), \\ v^{\mathsf{T}}\nabla h_k(x^*) = 0 & \forall\ k = 1, \dots, p \end{array} \right\}$$

$\square$

Finally, by means of a Theorem of the alternative (similarly to the proof of Proposition 2), it is possible to show that the impossibility of system (S1) implies that the system KKT is possible, i.e., there exist $\theta^* \in \mathbb{R}^s$, $\lambda^* \in \mathbb{R}^m$ and $\mu^* \in \mathbb{R}^p$ such that $(x^*, \theta^*, \lambda^*, \mu^*)$ solves system (4).

If (P) is a convex problem then the KKT conditions are also sufficient for optimality.

Theorem

Assume that f_i and g_j are convex, $i = 1, .., s$, $j = 1, .., m$, h_k are affine $k = 1, .., p$.

- If $(x^*, \theta^*, \lambda^*, \mu^*)$ solves the KKT system, then x^* is a weak minimum of (P).
- If $(x^*, \theta^*, \lambda^*, \mu^*)$ solves the KKT system with $\theta^* > 0$, then x^* is a minimum of (P).

Proposition 4

If x^* is the unique global minimum of the function f_k on the set X for some $k \in \{1, ..., s\}$, then x^* is a minimum of (P).

Proof. It is enough to notice that $f_k(x^*) < f_k(x), \forall x \in X,\ x \neq x^*$, and that the previous inequality implies the impossibility of the system:

$$\begin{array}{ll} f_i(x^*) \geq f_i(x) & \text{for any } i = 1, \dots, s, \\ f_j(x^*) > f_j(x) & \text{for some } j \in \{1, \dots, s\} \\ x \in X & \end{array}$$

i.e., x^* is a minimum of (P).

Example 7

Consider the nonlinear multiobjective problem

$$\begin{cases} \min\ (x_1 + x_2,\ -x_1 + x_2) \\ x_1^2 + x_2^2 - 1 \leq 0 \end{cases}$$

(a) Find the set of weak minima by solving the KKT system.

(b) Find the set of minima.

(a) We preliminarly note that the given problem is convex and differentiable and ACQ holds at any $x \in X$; then the KKT system provides a necessary and sufficient condition for a weak minimum. KKT system is given by:

$$\begin{cases} \theta_1 - \theta_2 + 2\lambda x_1 = 0 \\ \theta_1 + \theta_2 + 2\lambda x_2 = 0 \\ \lambda(x_1^2 + x_2^2 - 1) = 0 \\ \theta_1, \theta_2 \geq 0,\ \theta_1 + \theta_2 = 1,\ \lambda \geq 0 \\ x_1^2 + x_2^2 - 1 \leq 0 \end{cases}$$

Consider the case $\lambda = 0$, then the system becomes:

$$\begin{cases} \theta_1 - \theta_2 = 0 \\ \theta_1 + \theta_2 = 0 \\ \theta_1, \theta_2 \geq 0, \ \theta_1 + \theta_2 = 1, \ \lambda \geq 0 \\ x_1^2 + x_2^2 - 1 \leq 0 \end{cases}$$

which is clearly impossible, since the first two equations imply $\theta_1 = \theta_2 = 0$, which contradicts $\theta_1 + \theta_2 = 1$.

Then $\lambda \neq 0$. The system becomes:

$$\begin{cases} \theta_1 - \theta_2 + 2\lambda x_1 = 0 \\ \theta_1 + \theta_2 + 2\lambda x_2 = 0 \\ x_1^2 + x_2^2 - 1 = 0 \\ \theta_1, \theta_2 \geq 0, \ \theta_1 + \theta_2 = 1, \ \lambda \geq 0 \end{cases}$$

Then

$$x_1 = \frac{\theta_2 - \theta_1}{2\lambda} = \frac{1 - 2\theta_1}{2\lambda}$$

$$x_2 = -\frac{\theta_1 + \theta_2}{2\lambda} = -\frac{1}{2\lambda}$$

Substituting x_1 and x_2 in the third equation yields:

$$(1 - 2\theta_1)^2 + 1 = 4\lambda^2$$

so that

$$\lambda = \frac{1}{2}\sqrt{(1 - 2\theta_1)^2 + 1}, \quad 0 \leq \theta_1 \leq 1$$

We obtain the following solutions.

Weak minima =

$$\{(x_1, x_2) : x_1 = \frac{1 - 2\theta_1}{\sqrt{(1 - 2\theta_1)^2 + 1}},\ x_2 = -\frac{1}{\sqrt{(1 - 2\theta_1)^2 + 1}}, \quad 0 \leq \theta_1 \leq 1\}.$$

(b) The subset of weak minima such that $0 < \theta_1 < 1$ is also a set of minima, since $\theta_1, \theta_2 > 0$.

We need to investigate the cases $\theta_1 = 0$, $\theta_1 = 1$ which correspond to the points

$$\bar{x} = (\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}), \quad \hat{x} = (-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}),$$

We note that in these cases the KKT conditions collapse to the necessary and sufficient optimality conditions for the problems

$$\min_{x \in X}(-x_1 + x_2) \qquad \min_{x \in X}(x_1 + x_2)$$

so that $\bar{x}$ is the unique global minimum point for the first problem and $\hat{x}$ is the unique global minimum point for the second one.

By Proposition 4, we obtain that $\bar{x}$ and $\hat{x}$ are also minima for the given problem.

Consider the vector optimization problem

$$\begin{cases} \min \quad f(x) = (f_1(x), .., f_s(x)) \\ x \in X \end{cases} \tag{P}$$

with the geometric constraint $x \in X$ and define a vector of weights associated to the objectives:

$$\alpha = (\alpha_1, \dots, \alpha_s) \geq 0 \quad \text{such that} \quad \sum_{i=1}^{s} \alpha_i = 1$$

We associate with (P) the following scalar optimization problem

$$\begin{cases} \min \displaystyle\sum_{i=1}^{s} \alpha_i f_i(x) \\ x \in X \end{cases} \tag{P_α}$$

Let S_α be the set of optimal solutions of (P_α).

Theorem

- $\bigcup_{\alpha \geq 0} S_\alpha \subseteq \{\text{weak minima of (P)}\}$
- $\bigcup_{\alpha > 0} S_\alpha \subseteq \{\text{minima of (P)}\}$

Proof. Consider the function $\psi(\alpha, x) = \sum_{i=1}^{s} \alpha_i f_i(x)$ and let $x^* \in S_\alpha$. Then,

$$\psi(\alpha, x^*) \leq \psi(\alpha, x), \quad \forall x \in X,$$

i.e.,

$$\sum_{i=1}^{s} \alpha_i (f_i(x^*) - f_i(x)) \leq 0, \quad \forall x \in X.$$

As, $\alpha \in \mathbb{R}^s_+$, $\alpha \neq 0$, the system

$$f_i(x^*) - f_i(x) > 0, \quad i = 1, .., s, \quad x \in X,$$

is impossible and x^* is a weak minimum of (P).

Similarly, we can prove that if, additionally, $\alpha > 0$, then x^* is a minimum of (P).

Solving (P_α) for any possible choice of α does not allow finding all the minima and weak minima.

Example 8

Consider the problem

$$\begin{cases} \min\ (x_1,\ x_2) \\ x_1^2 + x_2^2 - 4 \leq 0 \\ -x_1^2 - x_2^2 + 1 \leq 0 \\ x_1 \geq 0,\ x_2 \geq 0 \end{cases}$$

$$\bigcup_{\alpha \geq 0} S_\alpha = \{(0, x_2) :\ x_2 \in [1, 2]\} \cup \{(x_1, 0) :\ x_1 \in [1, 2]\},$$

while

$$\{\text{weak minima of (P)}\} = \{(0, x_2) :\ x_2 \in [1, 2]\} \cup \{(x_1, 0) :\ x_1 \in [1, 2]\} \cup \{x \in \mathbb{R}^2_+ :\ x_1^2 + x_2^2 = 1\}.$$

Furthermore,

$$\bigcup_{\alpha > 0} S_\alpha = \{(0, 1), (1, 0)\},$$

while

$$\{\text{minima of (P)}\} = \{x \in \mathbb{R}^2_+ :\ x_1^2 + x_2^2 = 1\}.$$

Theorem

Assume that X is a convex set and that f_i are convex on X for $i = 1, .., s$.
Then $\{\text{weak minima of (P)}\} = \bigcup_{\alpha \geq 0} S_\alpha$

Proof. By the previous theorem, we have only to prove the inclusion

$$\bigcup_{\alpha \geq 0} S_\alpha \supseteq \{\text{weak minima of (P)}\}.$$

Let x^* be a weak minimum of (P). Then, the system

$$f(x^*) - f(x) > 0, \quad x \in X,$$

is impossible, or ,equivalently,

$$(f(x^*) - f(X)) \cap int(\mathbb{R}^s_+) = \emptyset.$$

The previous condition can be proved to be equivalent to the following one:

$$(f(x^*) - (f(X) + \mathbb{R}^s_+)) \cap int(\mathbb{R}^s_+) = \emptyset.$$

Since f is convex and X is convex, then the set $f(X) + \mathbb{R}^s_+$ is proved to be convex and consequently, the set $\Gamma := f(x^*) - (f(X) + \mathbb{R}^s_+)$ is convex.

Since Γ and $int(\mathbb{R}^s_+)$ are disjoint convex sets then there exists an hyperplane of equation $\langle \theta, u \rangle = 0$, $\theta \in \mathbb{R}^s_+$, $\theta \neq 0$, which separates them, i.e.,

$$\langle \theta, u \rangle \leq 0, \quad \forall u \in \Gamma, \quad \langle \theta, u \rangle > 0, \quad \forall u \in int(\mathbb{R}^s_+).$$

In particular, the first inequality implies that

$$\langle \theta, f(x^*) - f(x) \rangle \leq 0, \quad \forall x \in X$$

and setting

$$\alpha = \frac{\theta}{\sum_{i=1}^{s} \theta_i}$$

we obtain that $x^* \in S_\alpha$.

Scalarization method: the linear case

Theorem

Let (P) be linear, i.e., f_i are linear for $i = 1, .., s$ and X is a polyhedron. Then,

- $\{\text{weak minima of (P)}\} = \bigcup_{\alpha \geq 0} S_\alpha$;
- $\{\text{minima of (P)}\} = \bigcup_{\alpha > 0} S_\alpha$.

Proof. The first assertion is a consequence of the previous theorem.

We omit the proof of the second assertion.

Next example shows that the second assertion of the previous theorem does not hold for a nonlinear convex problem.

Example 9

Consider the non linear convex multiobjective problem

$$\begin{cases} \min \left(x_1 \ , \ x_1^2 + x_2^2 - 4x_1\right) \\ (x_1, x_2) \in \mathbb{R}^2 \end{cases}$$

The scalarized problem P_α is given by:

$$\begin{cases} \min \ \alpha_1 x_1 + (1-\alpha_1)(x_1^2 + x_2^2 - 4x_1) =: \psi_\alpha(x) \\ (x_1, x_2) \in \mathbb{R}^2 \end{cases}$$

where $0 \leq \alpha_1 \leq 1$.
ψ_α is convex so that the optimal points coincide with the solutions of the system

$$\nabla \psi_\alpha(x_1, x_2) = \begin{pmatrix} 2x_1(1-\alpha_1) - 4 + 5\alpha_1 \\ 2x_2(1-\alpha_1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

i.e.,

$$(x_1, x_2) = \left(\frac{4-5\alpha_1}{2(1-\alpha_1)}, 0 \right), \quad 0 \leq \alpha_1 < 1$$

We obtain:

- the set of weak minima of (P) $= \{(x_1, x_2) : x_1 \leq 2, x_2 = 0\}$
- the set of minima of (P) $\supseteq \{(x_1, x_2) : x_1 < 2, x_2 = 0\} \quad (0 < \alpha_1 < 1)$

It remains to consider the case where $\alpha_1 = 0$ which corresponds to the point $(2, 0)$.

Notice that $(2,0)$ is the unique minimum point of the function $f_2(x_1,x_2) = x_1^2 + x_2^2 - 4x_1$. By the previous Proposition 4 we obtain that it is a minimum of (P).

Exercise 1

Consider the linear multiobjective problem

$$\begin{cases} \min\ (x_1 - x_2\ ,\ x_1 + x_2) \\ -2\,x_1 + x_2 \leq 0 \\ -x_1 - x_2 \leq 0 \\ 5\,x_1 - x_2 \leq 6 \end{cases}$$

Find the set of minima and weak minima by means of the scalarization method.

The scalarized problem P_α is given by

$$\begin{cases} \min\ \ \alpha_1(x_1 - x_2) + \alpha_2(x_1 + x_2) \\ -2\,x_1 + x_2 \leq 0 \\ -x_1 - x_2 \leq 0 \\ 5\,x_1 - x_2 \leq 6 \end{cases}$$

Recalling that $\alpha_1 + \alpha_2 = 1$, by eliminating α_2 we obtain that P_α is equivalent to the problem (P_{α_1})

$$\begin{cases} \min \ \alpha_1(x_1 - x_2) + (1-\alpha_1)(x_1 + x_2) = x_1 + (1 - 2\alpha_1)x_2 \\ -2\,x_1 + x_2 \leq 0 \\ -x_1 - x_2 \leq 0 \\ 5\,x_1 - x_2 \leq 6 \end{cases}$$

where $0 \leq \alpha_1 \leq 1$.

The previous problem can be solved by the Matlab function "linprog".

For $0 < \alpha_1 < 1$, we have that the optimal solutions of P_{α_1} are the minima of the given problem.

Recall that $\bigcup_{0<\alpha_1<1} Sol(P_{\alpha_1})$ is given by the union of faces of the polyhedron X.

Matlab solution

```
C = [1 -1; 1 1] ;

A =[ -2 1; -1 -1; 5 -1 ];

b = [0 0 6]';

% solve the scalarized problem with 0 < alfa1 < 1

MINIMA=[ ]; % First column: value of alfa1

LAMBDA=[ ]; % First column: value of alfa1

for alfa1 = 0.01 : 0.01 : 0.99
[x,fval,exitflag,output,lambda] = linprog(alfa1*C(1,:)+(1-alfa1)*C(2,:),A,b) ;
MINIMA=[MINIMA; alfa1, x'];
LAMBDA=[LAMBDA;alfa1,lambda.ineqlin'];
end
% solve the scalarized problem with alfa1 = 0 and alfa1 = 1
alfa1 = 0;
[xalfa0,f0,exitflag,output,lambda0] = linprog(alfa1*C(1,:)+(1-alfa1)*C(2,:),A,b) ;

alfa1 = 1;
[xalfa1,f1,exitflag,output,lambda1] = linprog(alfa1*C(1,:)+(1-alfa1)*C(2,:),A,b) ;
```

For $0 < \alpha_1 < 0.75$ we obtain the optimal solution of P_{α_1}: $x^* = (0, 0)$, with nondegenerate dual solution λ^*, so that x^* is the unique optimal solution of P_{α_1}.

For $\alpha_1 = 0.75$, the optimal solution of P_{α_1} is $x^* = (0, 0)$ with $\lambda^* = (0.5, 0, 0)$, which is degenerate (i.e., the number of strictly positive components of λ^* is less than the dimension n of the space where the problem is defined, in this case $n = 2$). .

By means of the KKT conditions we have that all the solutions of P_{α_1} solve the system

$$\begin{cases} \lambda_j^*(A_j x - b_j) = 0, \ j = 1, ..., m \\ Ax \leq b \end{cases}$$

where A_j denotes the $j - th$ row of A and λ^* is any dual solution of P_{α_1} which is given by linprog in the vector "lambda.ineqlin".

For $0.75 < \alpha_1 < 1$ we obtain the optimal solution of P_{α_1}: $x^* = (2, 4)$, with nondegenerate dual solution λ^*, so that x^* is the unique optimal solution of P_{α_1}. We obtain

$$\text{minima of (P)} = \bigcup_{0<\alpha_1<1} Sol(P_{\alpha_1}) = (x_1, x_2) : \begin{cases} -2\,x_1 + x_2 = 0 \\ -x_1 - x_2 \leq 0 \\ 5\,x_1 - x_2 \leq 6 \end{cases}$$

Considering the further particular cases $\alpha_1 = 0$ and $\alpha_1 = 1$ we have:

- For $\alpha_1 = 0$, the optimal solution of P_{α_1} is $x^* = (0, 0)$ with $\lambda^* = (0, 1, 0)$, which is degenerate.
- For $\alpha_1 = 1$, the optimal solution of P_{α_1} is $x^* = (2, 4)$ with $\lambda^* = (1.3333, 0, 0.3333)$, which is non degenerate.

We obtain that:

Weak minima of (P)$= \bigcup_{0 \le \alpha_1 \le 1} Sol(P_{\alpha_1})$

$$= (x_1, x_2) : \begin{cases} -2\,x_1 + x_2 = 0 \\ -x_1 - x_2 \le 0 \\ 5\,x_1 - x_2 \le 6 \end{cases} \cup \begin{cases} -2\,x_1 + x_2 \le 0 \\ -x_1 - x_2 = 0 \\ 5\,x_1 - x_2 \le 6 \end{cases}$$

The next sufficient condition is useful for detecting minima of (P) by means of a scalarized problem.

Proposition

If x^* is the unique global minimum of P_α for some α, then x^* is a minimum of (P).

Proof. Consider the function $L(\alpha, x) = \sum_{i=1}^{s} \alpha_i f_i(x)$ and let $x^* \in S_\alpha$. Then,

$$\sum_{i=1}^{s} \alpha_i (f(x^*) - f_i(x))\rangle < 0, \quad \forall x \in X, \quad x \neq x^*.$$

Ab absurdo, assume that x^* is not a minimum of (P). Then, the system:

$$\begin{array}{ll} f_i(x^*) \geq f_i(x) & \text{for any } i = 1, \dots, s, \;\; i \neq j \\ f_j(x^*) > f_j(x) & \text{for some } j \in \{1, \dots, s\} \\ x \in X & \end{array}$$

admits a solution $\hat{x} \neq x^*$.

Multiplying the i-th inequality by α_i and summing all the inequalities we obtain:

$$\sum_{i=1}^{s} \alpha_i f_i(x^*) \geq \sum_{i=1}^{s} \alpha_i f_i(\hat{x})$$

which contradicts that $L(\alpha, x^*) < L(\alpha, x), \quad \forall x \in X, \quad x \neq x^*$.

Therefore, x^* is a minimum of (P).

The previous proposition also allows us to obtain existence results for multiobjective optimization problems.

Exercise 2

Consider the nonlinear multiobjective problem (P)

$$\begin{cases} \min \, (x_1 \ , \ x_1^2 + x_2^2 - 2x_1) \\ -x_1 \leq 0 \\ x_1 + x_2 \leq 2 \end{cases}$$

a) Does a minimum point exists?

b) Find the set of weak minima by means of the scalarization method.

a) Consider the scalarized problem (P_{α_1}) where $\alpha_1 \neq 1$, i.e.

$$\begin{cases} \min \;\; \alpha_1 x_1 + (1 - \alpha_1)(x_1^2 + x_2^2 - 2x_1) =: \psi_{\alpha_1}(x) \\ -x_1 \leq 0 \\ x_1 + x_2 \leq 2 \end{cases}$$

with $0 \leq \alpha_1 < 1$.

ψ_{α_1} is strongly convex so that P_{α_1} admits a unique optimal solution which is a minimum of (P).

Exercise 3

Consider the nonlinear multiobjective problem

$$\begin{cases} \min\ (x_1^2 + x_2^2 + 2x_1 - 4x_2\ ,\ x_1^2 + x_2^2 - 6x_1 - 4x_2) \\ -x_2 \leq 0 \\ -2x_1 + x_2 \leq 0 \\ 2x_1 + x_2 \leq 4 \end{cases}$$

Find the set of minima and weak minima by means of the scalarization method.

The scalarized problem P_α is

$$\begin{cases} \min\ (\alpha_1(x_1^2 + x_2^2 + 2x_1 - 4x_2) + \alpha_2(x_1^2 + x_2^2 - 6x_1 - 4x_2)) \\ -x_2 \leq 0 \\ -2x_1 + x_2 \leq 0 \\ 2x_1 + x_2 \leq 4 \end{cases}$$

We note that the feasible set X is convex and the objective function of P_α is strongly convex for any $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2_+$ with $\alpha_1 + \alpha_2 = 1$ so that the set of minima and weak minima coincide.

Let us express the problem in matrix form.

The objectives are given by: $f_1(x_1, x_2) = \frac{1}{2}x^T Q_1 x + c1^T x$ where

$$Q_1 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \quad c1^T = (2, -4)$$

$f_2(x_1, x_2) = \frac{1}{2}x^T Q_2 x + c2^T x$ where

$$Q_2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \quad c2^T = (-6, -4)$$

The constraints are given by $Ax \leq b$ where

$$A = \begin{pmatrix} 0 & -1 \\ -2 & 1 \\ 2 & 1 \end{pmatrix} \quad b = (0, 0, 4)^T$$

The given problem becomes:

$$\begin{cases} \min \left(\frac{1}{2}x^T Q_1 x + c1^T x \ , \ \frac{1}{2}x^T Q_2 x + c2^T x\right) \\ Ax \leq b \end{cases}$$

The scalarized problem P_α becomes:

$$\begin{cases} \min \left(\frac{1}{2}x^T(\alpha_1 Q_1 + \alpha_2 Q_2)x + (\alpha_1 c1^T + \alpha_2 c2^T)x\right) \\ Ax \leq b \end{cases}$$

which can be solved by the Matlab function "quadprog".

Matlab solution

```
Q1 = [2 0; 0 2] ;

Q2 = [2 0; 0 2] ;

c1=[2;-4]; c2=[-6; -4]; A =[ 0 -1; -2 1; 2 1 ];

b = [0 0 4]';

% solve the scalarized problem with 0 ≤ alfa1 ≤ 1

MINIMA=[ ]; % First column: value of alfa1

LAMBDA=[ ]; % First column: value of alfa1

for alfa1 = 0 : 0.01 : 1
[x,fval,exitflag,output,lambda] =
quadprog(alfa1*Q1+(1-alfa1)*Q2,alfa1*c1+(1-alfa1)*c2,A,b) ;

MINIMA=[MINIMA; alfa1 x'];
LAMBDA=[LAMBDA;alfa1,lambda.ineqlin'];
end

plot(MINIMA(:,2),MINIMA(:,3))
```

We obtain:

Minima =Weak Minima= $AB \cup BC$

where

$$A = (0.6, 1.2), \quad B = (1, 2), \quad C = (1.4, 1.2)$$

Exercise 4

Consider the nonlinear multiobjective problem defined in Example 7:

$$\begin{cases} \min \ (x_1 + x_2, \ -x_1 + x_2) \\ x_1^2 + x_2^2 - 1 \leq 0 \end{cases}$$

(a) Find the set of weak minima by means of the scalarization method.

(a) Find a suitable subset of minima by means of the scalarization method.

The scalarized problem P_α is given by

$$\begin{cases} \min \ \alpha_1(x_1 + x_2) + \alpha_2(-x_1 + x_2) \\ x_1^2 + x_2^2 - 1 \leq 0 \end{cases}$$

Since $\alpha_1 + \alpha_2 = 1$, by eliminating α_2 we obtain that P_α is equivalent to the problem (P_{α_1})

$$\begin{cases} \min \ \alpha_1(x_1 + x_2) + (1 - \alpha_1)(-x_1 + x_2) = (2\alpha_1 - 1)x_1 + x_2 \\ x_1^2 + x_2^2 - 1 \leq 0 \end{cases}$$

where $0 \leq \alpha_1 \leq 1$.

The previous problem can be solved by the KKT conditions or by the Matlab function "fmincon".

The Matlab function "fmincon"

The function fmincon solves a problem of the form:

$$\begin{cases} \min \ f(x) \\ Ax \leq b \\ D\,x = e \\ l \leq x \leq u \\ c(x) \leq 0 \\ ceq(x) = 0 \end{cases} \tag{5}$$

where x, b, e, l, u are vectors, A, D are matrices, c and ceq are functions that return vectors and f is a scalar function.
The sintax of the function is the following:

```
fun=@(x) ..............

nonlcon=@(x) const(x)

[x,fval,exitflag,output,lambda]=fmincon(fun,x0,A,b,D,e,l,u,nonlcon)

function [c,ceq] = const(x)
c=[.........];
ceq= [.........];
end
```

Matlab solution

```
% solve the scalarized problem with 0 ≤ alfa1 ≤ 1

MINIMA=[ ]; % First column: value of alfa1

for alfa1 = 0 : 0.01 : 1
fun=@(x) (2*alfa1-1)*x(1)+x(2);
nonlcon= @(x) const(x);
x0=[0,0]';
[x,fval,exitflag,output,lambda] = fmincon(fun,x0,[ ],[ ],[ ],[ ],[ ],[ ],nonlcon) ;

MINIMA=[MINIMA; alfa1, x'];
end

plot(MINIMA(:,2),MINIMA(:,3))

function [C,Ceq]=const(x)
C=x(1)^2 +x(2)^2 -1;
Ceq=[ ];
end
```

In the objective space $\mathbb{R}^s$ define the ideal point z as

$$z_i = \min_{x \in X} \; f_i(x), \qquad \forall \; i = 1, \dots, s.$$

Since very often (P) has no ideal minimum, i.e., $z \notin f(X)$, we want to find the point of $f(X)$ which is as close as possible to z:

$$\begin{cases} \min \; \|f(x) - z\|_q \\ x \in X \end{cases} \qquad \text{with} \quad q \in [1, +\infty]. \tag{G}$$

Theorem

- If $q \in [1, +\infty)$, then any optimal solution of (G) is a minimum of (P).
- If $q = +\infty$, then any optimal solution of (G) is a weak minimum of (P).

Goal method: the linear case with 2-norm

Assume that (P) is a linear multiobjective optimization problem, i.e.,

$$\begin{cases} \min \ Cx \\ Ax \leq b \end{cases} \tag{P}$$

where C is a $s \times n$ matrix, A is a $m \times n$ matrix, $b \in \mathbb{R}^m$.

If $q = 2$, then (G) is equivalent to a quadratic programming problem:

$$\begin{cases} \min \ \frac{1}{2}\|Cx - z\|_2^2 = \frac{1}{2}x^\mathsf{T} C^\mathsf{T} Cx - x^\mathsf{T} C^\mathsf{T} z + \frac{1}{2}z^\mathsf{T} z \\ Ax \leq b \end{cases} \tag{G_2}$$

Matlab solution

data	`C=[.....];`
	`A=[.......];`
	`b=[.......];`
	`s=size(C,1);`

Ideal point	`z=zeros(s,1);`
	`for i=1:s`
	`[a,z(i)] = linprog(C(i,:)',A,b)`
	`end`

object. funct.	`H=C'*C` `f= -C'*z`
Solut. Command	`x=quadprog(H,f, A, b)`

Example 10

Consider the problem

$$\begin{cases} \min\ (x_1 - x_2\ ,\ x_1 + x_2) \\ -2\,x_1 + x_2 \leq 0 \\ -x_1 - x_2 \leq 0 \\ 5\,x_1 - x_2 \leq 6 \end{cases}$$

a) Find the ideal point.

b) Apply the goal method with norm $q = 2$.

Matlab solution

```
C = [1 -1; 1 1] ;

A =[ -2 1; -1 -1; 5 -1 ];

b = [0 0 6]';

% find the ideal point z

z=[0,0]';
for i = 1:2
[a,z(i)]=linprog(C(i,:)',A,b);
end

% solve the quadratic problem with norm q = 2

x = quadprog(C'*C,-C'*z,A,b);
```

a) The ideal point is $z = (-2, 0)$.

b) The optimal solution of (G_2) is $x^* = (0.2, 0.4)$.

Exercise 5

Consider the non linear convex multiobjective problem (defined in Example 9)

$$\begin{cases} \min \left(x_1 \ , \ x_1^2 + x_2^2 - 4x_1\right) \\ (x_1, x_2) \in \mathbb{R}^2 \end{cases}$$

Find, by making use of Matlab, the sets of weak minima and minima.

Exercise 6

Consider the linear multiobjective problem (defined in Example 5)

$$\begin{cases} \min \left(x_1 + 2x_2 - 3x_3 \ , \ -x_1 - x_2 - x_3 \ , \ -4x_1 - 2x_2 + x_3\right) \\ x_1 + x_2 + x_3 \leq 10 \\ x_3 \leq 5 \\ x_1, x_2, x_3 \geq 0 \end{cases}$$

Find, by making use of Matlab, the sets of weak minima and minima.