Mathematical modeling
Individual Project

Option pricing with stochastic volatility
Calibration and simulation of Heston model

Abstract

The goal of this project is to go deeper in option pricing models. In our previous project about option pricing, volatility was always constant while in real life this is not true. Therefore, in real life, traders may use different models to estimate volatility over time and Heston model is one of the most famous. It assumes that volatility follows a stochastic process as well as stock price. As this model has parameters, we need first to estimate the parameters of this model with historical data and the maximum likelihood estimation method. Then we will estimate the price of the option with Monte-Carlo method and compare the performance of the model with our 3 previous models: Black-Scholes, binomial and Monte-Carlo model with constant volatility.

1. Introduction

As a reminder about options, an option is a financial contract that gives the holder the right, but not the obligation, to buy or sell an underlying asset at a specified price (known as the strike price) on or before a predetermined date (known as the expiration or maturity date).In this project, the underlying asset will be a stock There are two main types of options: call options and put options. A call option gives the right to the buyer of the option to purchase the underlying asset at a specific strike price on or before the maturity date. A put option gives the right to the buyer of the option to sell the underlying asset at a specific strike price on or before the maturity date. Basically, buying a call is a bet where you believe underlying stock price will go up, and buying a put is a bet where you believe underlying stock price will go down. Options are much used in finance as they allow traders to invest with very high leverage compared to directly buying the underlying stock.

The volatility is the standard deviation of the asset return. If S_t is the asset price at time t, the asset return over the period [t,t+1] is $\frac{S_{t+1}}{S_t}$. So you can notice here that we have to choose a period for the return. This period can be a day, then we talk about daily-return and daily volatility.

Having said that, as you can see below with the VIX which is an index that tracks the volatility of S&P 500 over time, the assumption of constant volatility is not true.

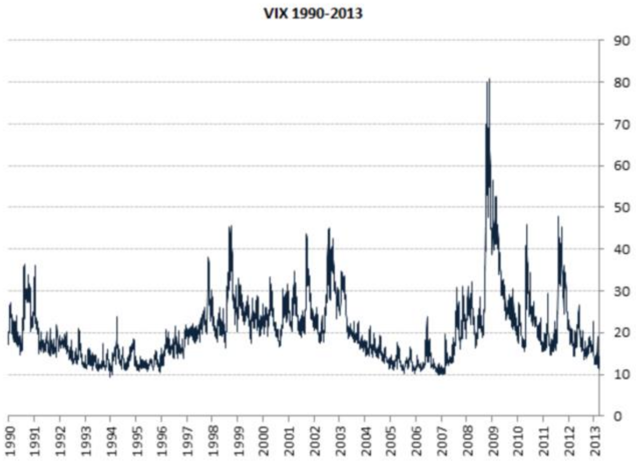

Figure 1 : Volatility Index S&P 500

However, we made that assumption for Black-Scholes, binomial and Monte-Carlo models. Indeed, we supposed the following stochastic process for stock price:

$$dS_t = rS_t dt + \sigma S_t \epsilon \sqrt{dt}$$

Where :
- S_t : Stock price at time t
- r : Risk-free rate
- σ : Volatility (constant)
- dt : indefinitely small positive time increment
- ϵ : Standard normal distribution variable ($\epsilon\sqrt{dt} = dW_t$)

The goal of this project is to relax the assumption of constant volatility with Heston model and observe the performance of the model for 4 companies' stocks: Tesla, Microsoft, Apple and Meta. So, we will observe the performance of the Heston model at predicting option prices with different strike prices for those companies.

2. <u>Description of the model</u>

In 1993, Heston proposed the following formula for asset prices where both asset's price and volatility follow Brownian motion processes:

$$dS_t = rS_t dt + \sigma S_t dW_{1t}$$

$$dV_t = k(\theta - V_t)dt + \sigma\sqrt{V_t}dW_{2t}$$

Where the variables of the system are:

- S_t : Stock price at time t
- r : Risk-free rate
- $\sqrt{V_t}$: volatility (standard deviation) of the asset price
- σ : Volatility of the volatility $\sqrt{V_t}$

- θ : long-term price variance
- dt : indefinitely small positive time increment
- W_{1t} : Brownian motion of the asset price
- W_{2t} : Brownian motion of the asset's price variance
- ρ : Correlation coefficient for W_{1t} and W_{2t}

The Heston model is widely used by banks as it allows to capture a phenomenon in financial markets named the volatility skew, which is the relationship between strike price and implied volatility observed in the market. The following curve shows that traders overpay for out-of-the-money put and in-the-money call, and they underpay for in-the-money put and out-of-the-money call.

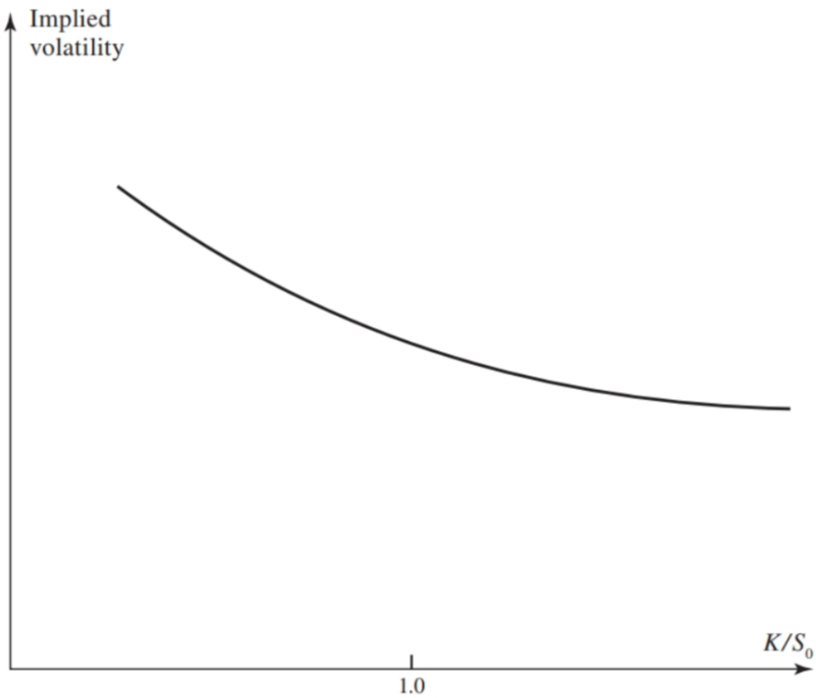

Figure 2 : Volatility smile for equity options (K : strike price, S0 : current stock price)

3. Discretization of the model and implementation

As we have daily data for stock prices, we need to discretize the model to fit it with our data and then obtain parameters for each company's daily stock prices. With Euler technique of mid-point, we get the following formula for the equation system (you can see all steps in the article "Estimating Option Prices with Heston's Stochastic Model"):

$$Q_{t+1} = 1 + r + \sqrt{V_t}(\rho Z_1 + \sqrt{1-\rho^2}\, Z_2)$$

$$V_{t+1} = V_t + k(\theta - V_t) + \sigma\sqrt{V_t} Z_1$$

Where $Z_1, Z_2 \sim N(0,1)$ and $Q_{t+1} = \frac{S_{t+1}}{S_t}$

This formula allows us to fit our daily stock prices with the model. Then we need to estimate the parameters $k, \theta, \sigma, r, \rho$. For this, we use data fitting method, particularly the maximum likelihood estimation as we have seen it in class. But according to the papers, this is not the most accurate method as the method of moments is more accurate for instance. I decided to use Nelder-Mead method to find the parameters that maximize the log-likelihood as this is the method I know the best. As the negative log-likelihood function is not globally convex, it has a lot of local minimums, so to increase the chance to converge to the global minimum, I run the Nelder-Mead algorithm with several random values for initial guesses and then I take the set of parameters that give the minimum value for the negative log-likelihood function among all minimum points found.

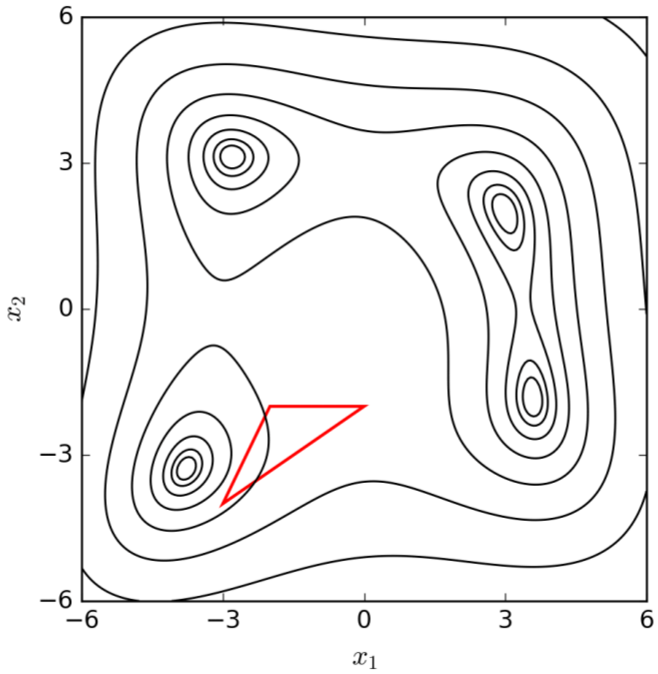

Figure 3: Nelder-Mead applied to a function having 4 local minimum

$$
\begin{aligned}
\ell(r, k, \theta, \sigma, \rho) \quad = \quad & \sum_{t=1}^{n} \Bigg(-\log(2\pi) - \log(\sigma) - \log(V_t) - \frac{1}{2}\log(1-\rho^2) \\
& -\frac{(Q_{t+1} - 1 - r)^2}{2V_t(1-\rho^2)} + \frac{\rho(Q_{t+1} - 1 - r)(V_{t+1} - V_t - \theta k + kV_t)}{V_t \sigma (1-\rho^2)} \\
& -\frac{(V_{t+1} - V_t - \theta k + kV_t)^2}{2\sigma^2 V_t (1-\rho^2)} \Bigg).
\end{aligned}
$$

Figure 4 : Log-Likelihood function

In my code, I have created 2 classes:

- Company : This class contain all information about a company stock prices, volatility, option prices and it retrieve historical data from yahoo finance
- Option : This class contain all information about an option, whether it is a call or a put. It retrieves historical data from yahoo finance

4. Results

Firstly, I computed the log-likelihood function for each company under study, namely Tesla, Apple, Microsoft and Meta and I estimate the 5 parameters $k, \theta, \sigma, r, \rho$ for each company.

```
Parameters order : rho, sigma, theta, kappa, r
Apple parameters : [0.9426639999163873, 0.049436615110588465, 0.018142663549486023, 0.052461284453659704, -0.0020786263482190995]
Microsoft parameters : [0.4150047991933562, 0.005450957238421386, 0.020858594570104187, 0.06769625582276259, 0.0015522958787570033]
Tesla parameters : [0.21241396681062755, 0.003980419105441439, 0.03715447419864505, 0.011463933191124999, -0.0008072323227129021]
Meta parameters : [-0.4303676809184748, 0.00916516910472501, 0.03672271738662268, 0.054215595064590824, 0.0021863331595050103]
```

Figure 5 : Parameters for each company

Then, with these parameters and the discrete equation system, we perform Monte-Carlo simulation to obtain the path of each stock from today to maturity. For maturity, we take the date 2023-08-25 as option price data are available for this date for all companies. Actually, there is a formula for call prices with Heston model that you can find in the resources, but the formula is very complicated as it first need to work in complex space before integrating the real part of complex numbers. For simplicity, I have chosen Monte-Carlo.

```
Prices at maturity with Monte Carlo
Microsoft : $328.33
Apple : $191.67
Tesla : $255.76
Meta : $312.99
```

Figure 6 : Prices at maturity with Monte-Carlo applied on Heston model

Then, by definition, the price of an option is the expected payoff of the option actualized with the continuous compounded risk-free rate:

$$e^{-rT}E(S_T - K) \text{ for a call option}$$

$$e^{-rT}E(K - S_T) \text{ for a put option}$$

Where:

- r is the risk-free rate. We get it from the US Treasury bond rate https://home.treasury.gov/resource-center/data-chart-center/interest-rates/TextView?type=daily_treasury_yield_curve&field_tdr_date_value_month=202308

- T is the time to maturity (weekdays between today and maturity date)
- K is the strike price
- S_T is the stock price at maturity

With those previous formula, we can compute the price of call and put options for each company for 21 different strike prices. Then we plot on the same graph the price of options in function of strike prices for real prices and for prices obtained from our model. The dotted green line shows the current stock price on the abscise. It indicates if the option is in or out-of-the money either.

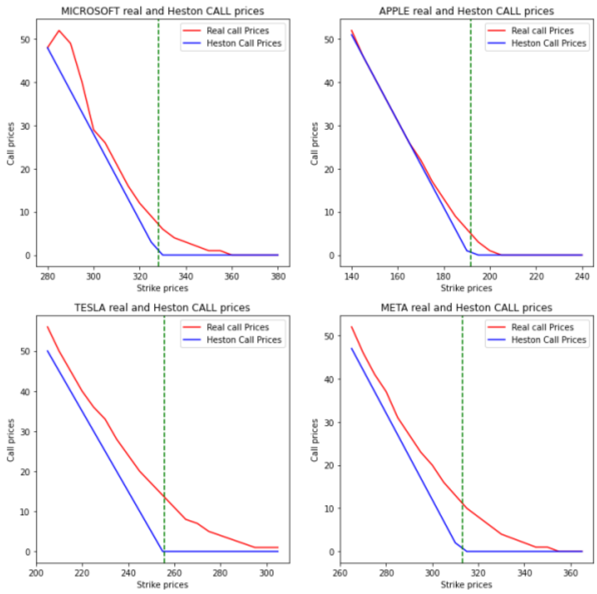

Figure 7 : Real and Heston call prices for different maturities

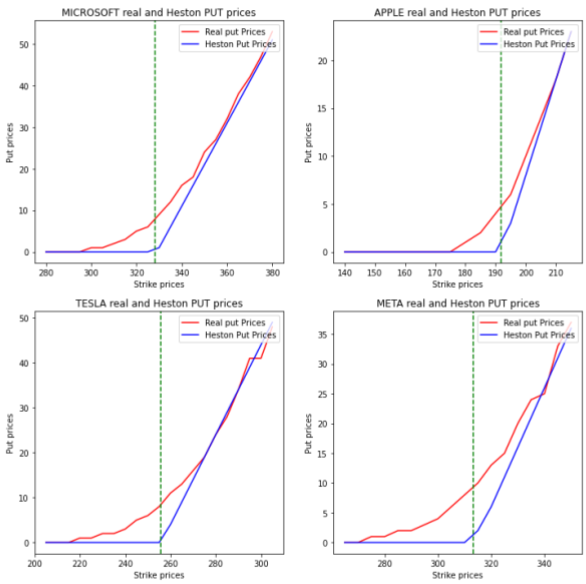

Figure 8 : Real and Heston put prices for different maturities

Then we do the same for Black-Scholes, binomial and Monte-Carlo with constant volatility.

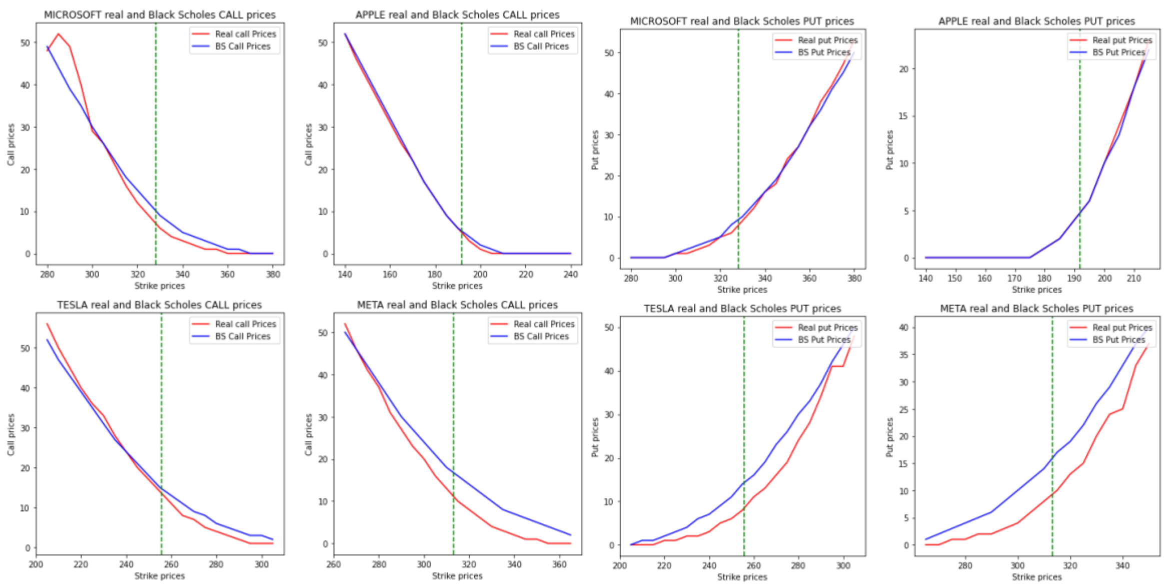

Figure 9 : Real and Black-Scholes call and put prices for different maturities

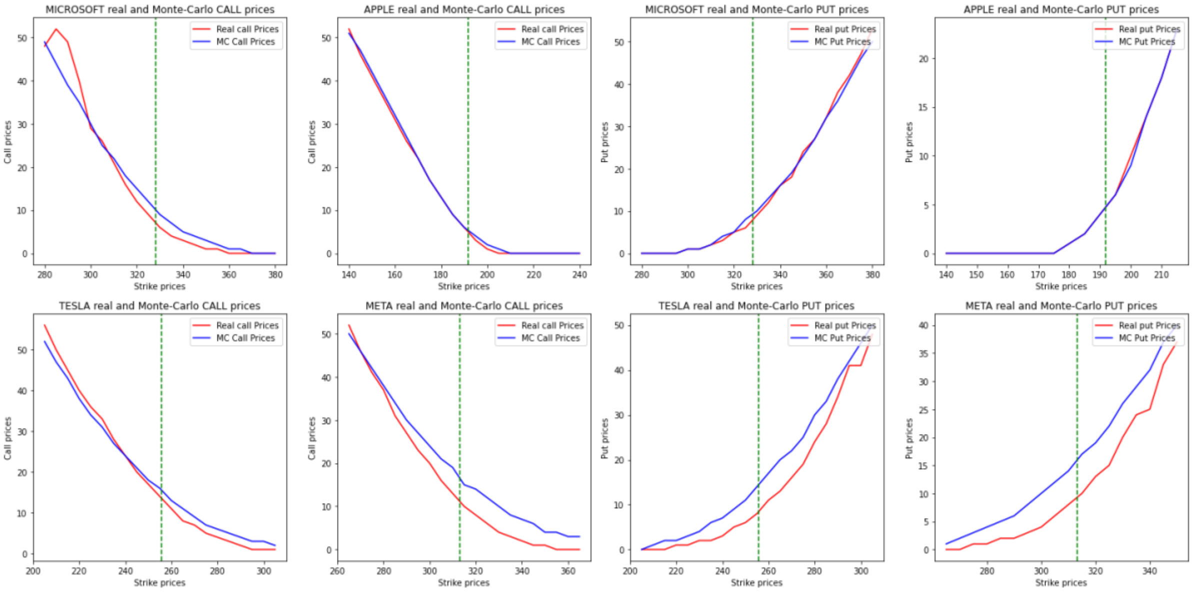

Figure 10 : Real and Monte-Carlo call and put prices for different maturities

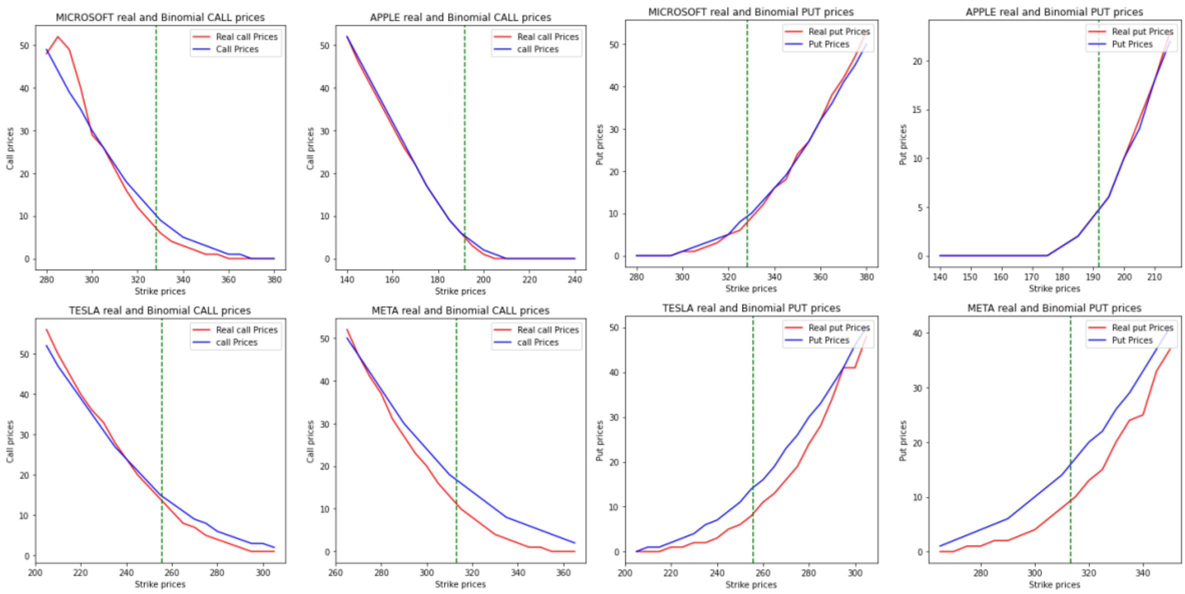

Figure 11 : Real and Binomial call and put prices for different maturities

5. Analysis and conclusion

From the graphs, it seems that, depending on the stock, Heston model is outperformed by other binomial, Black-Scholes and Monte-Carlo with constant volatility model. This is not how it should work according to the resources. Indeed, the cumulative error is lower with Heston model than with other model accord to the paper “Estimating Option Prices with Heston’s Stochastic Model”. So the reason for such a difference is the fact that the maximum likelihood estimation did not perform well in finding the parameters of the model. Therefore, it must be better to use the method of moments which seems to perform way better for data fitting purpose in our case. More accurately, the stochastic model with maximum likelihood estimation work better just in the case of in-the-money put option. And to go deeper in the reasons for the maximum likelihood estimation to not work, the issue is that in does not converge well to a global minimum, instead it converges to local minimum. We can notice it with the value of r (risk-free rate) that the maximum likelihood estimation gives, it is not the value we expected as it is far from what we can observe in the US Treasury bond yield. Moreover, with my computer, I can't compute many initial guesses to converge to the global minimum as it is computationally very demanding.

Finally, the maximum likelihood estimation is not a very good estimator of the parameters.

6. Ressources

- "Estimating Option Prices with Heston's Stochastic Model" by Robin Dunn, Paloma Hauser, Tom Seibold and Hugh Gong
- "Euler and Milstein Discretization" by Fabrice Douglas Rouah