# Matrix Analysis

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**Abstract**

This manual provides an introduction to vectors and their properties, based on the question papers, year 2020, from Class 10 and 12, CBSE.

## 1 $2 \times 1$ VECTORS

1.1. Let

$$\mathbf{A} \equiv \overrightarrow{A} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \tag{1.1.1}$$

$$\equiv a_1 \overrightarrow{i} + a_2 \overrightarrow{j}, \tag{1.1.2}$$

$$\mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \tag{1.1.3}$$

be $2 \times 1$ vectors. Then, the determinant of the $2 \times 2$ matrix

$$\mathbf{M} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \end{pmatrix} \tag{1.1.4}$$

is defined as

$$\left|\mathbf{M}\right| = \left|\mathbf{A} \quad \mathbf{B}\right| \tag{1.1.5}$$

$$= \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1 \tag{1.1.6}$$

1.2. The value of the cross product of two vectors is given by (1.1.5).

1.3. The area of the triangle with vertices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ is given by the absolute value of

$$\frac{1}{2} \left|\mathbf{A} - \mathbf{B} \quad \mathbf{A} - \mathbf{C}\right| \tag{1.3.1}$$

1.4. The transpose of $\mathbf{A}$ is defined as

$$\mathbf{A}^\top = \begin{pmatrix} a_1 & a_2 \end{pmatrix} \tag{1.4.1}$$

1.5. The *inner product* or *dot product* is defined as

$$\mathbf{A}^\top \mathbf{B} \equiv \mathbf{A} \cdot \mathbf{B} \tag{1.5.1}$$

$$= \begin{pmatrix} a_1 & a_2 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = a_1 b_1 + a_2 b_2 \tag{1.5.2}$$

1.6. *norm* of $\mathbf{A}$ is defined as

$$\|A\| \equiv \left|\overrightarrow{A}\right| \tag{1.6.1}$$

$$= \sqrt{\mathbf{A}^\top \mathbf{A}} = \sqrt{a_1^2 + a_2^2} \tag{1.6.2}$$

Thus,

$$\|\lambda \mathbf{A}\| \equiv \left|\lambda \overrightarrow{A}\right| \tag{1.6.3}$$

$$= |\lambda| \, \|\mathbf{A}\| \tag{1.6.4}$$

1.7. The distance betwen the points $\mathbf{A}$ and $\mathbf{B}$ is given by

$$\|\mathbf{A} - \mathbf{B}\| \tag{1.7.1}$$

1.8. Let $\mathbf{x}$ be equidistant from the points $\mathbf{A}$ and $\mathbf{B}$. Then

$$\left(\mathbf{A}-\mathbf{B}\right)^{\top}\mathbf{x} = \frac{\left\|\mathbf{A}\right\|^2 - \left\|\mathbf{B}\right\|^2}{2} \tag{1.8.1}$$

**Solution:**

$$\left\|\mathbf{x}-\mathbf{A}\right\| = \left\|\mathbf{A}-\mathbf{B}\right\| \tag{1.8.2}$$

$$\implies \left\|\mathbf{x}-\mathbf{A}\right\|^2 = \left\|\mathbf{x}-\mathbf{B}\right\|^2 \tag{1.8.3}$$

which can be expressed as

$$\begin{aligned}&\left(\mathbf{x}-\mathbf{A}\right)^{\top}\left(\mathbf{x}-\mathbf{A}\right) = \left(\mathbf{x}-\mathbf{B}\right)^{\top}\left(\mathbf{x}-\mathbf{B}\right)\\ &\implies \left\|\mathbf{x}\right\|^2 - 2\mathbf{x}^{\top}\mathbf{A} + \left\|\mathbf{A}\right\|^2\\ &= \left\|\mathbf{x}\right\|^2 - 2\mathbf{x}^{\top}\mathbf{B} + \left\|\mathbf{B}\right\|^2\end{aligned} \tag{1.8.4}$$

which can be simplified to obtain (1.8.1).

1.9. If $\mathbf{x}$ lies on the $x$-axis and is equidistant from the points $\mathbf{A}$ and $\mathbf{B}$,

$$\mathbf{x} = x\mathbf{e}_1 \tag{1.9.1}$$

where

$$x = \frac{\left\|\mathbf{A}\right\|^2 - \left\|\mathbf{B}\right\|^2}{2\left(\mathbf{A}-\mathbf{B}\right)^{\top}\mathbf{e}_1} \tag{1.9.2}$$

**Solution:** From (1.8.1).

$$x\left(\mathbf{A}-\mathbf{B}\right)^{\top}\mathbf{e}_1 = \frac{\left\|\mathbf{A}\right\|^2 - \left\|\mathbf{B}\right\|^2}{2} \tag{1.9.3}$$

yielding (1.9.2).

1.10. The angle between two vectors is given by

$$\theta = \cos^{-1}\frac{\mathbf{A}^{\top}\mathbf{B}}{\left\|A\right\|\left\|B\right\|} \tag{1.10.1}$$

1.11. If two vectors are orthogonal (perpendicular),

$$\mathbf{A}^{\top}\mathbf{B} = 0 \tag{1.11.1}$$

1.12. The *direction vector* of the line joining two points $\mathbf{A}, \mathbf{B}$ is given by

$$\mathbf{m} = \mathbf{A} - \mathbf{B} \tag{1.12.1}$$

1.13. The unit vector in the direction of $\mathbf{m}$ is defined as

$$\frac{\mathbf{m}}{\left\|\mathbf{m}\right\|} \tag{1.13.1}$$

1.14. If the direction vector of a line is expressed as

$$\mathbf{m} = \begin{pmatrix}1\\m\end{pmatrix}, \tag{1.14.1}$$

the $m$ is defined to be the slope of the line.

1.15. The *normal vector* to $\mathbf{m}$ is defined by

$$\mathbf{m}^{\top}\mathbf{n} = 0 \tag{1.15.1}$$

1.16. The point $\mathbf{P}$ that divides the line segment $AB$ in the ratio $k:1$ is given by

$$\mathbf{P} = \frac{k\mathbf{B} + \mathbf{A}}{k+1} \tag{1.16.1}$$

1.17. The standard basis vectors are defined as

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{1.17.1}$$

$$\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{1.17.2}$$

## 2 $3 \times 1$ Vectors

2.1. Let

$$\mathbf{A} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \equiv a_1 \overrightarrow{i} + a_2 \overrightarrow{j} + a_3 \overrightarrow{j}, \tag{2.1.1}$$

$$\mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \tag{2.1.2}$$

and

$$\mathbf{A}_{ij} = \begin{pmatrix} a_i \\ a_j \end{pmatrix}, \tag{2.1.3}$$

$$\mathbf{B}_{ij} = \begin{pmatrix} b_i \\ b_j \end{pmatrix}. \tag{2.1.4}$$

2.2. The *cross product* or *vector product* of $\mathbf{A}, \mathbf{B}$ is defined as

$$\mathbf{A} \times \mathbf{B} = \begin{pmatrix} \begin{vmatrix} \mathbf{A}_{23} & \mathbf{B}_{23} \\ \mathbf{A}_{31} & \mathbf{B}_{31} \\ \mathbf{A}_{12} & \mathbf{B}_{12} \end{vmatrix} \end{pmatrix} \tag{2.2.1}$$

2.3. Verify that

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} \tag{2.3.1}$$

2.4. The area of a triangle is given by

$$\frac{1}{2} \left\| \mathbf{A} \times \mathbf{B} \right\| \tag{2.4.1}$$

## 3 Matrices

### *3.1 Eigenvalues and Eigenvectors*

3.1.1. The eigenvalue $\lambda$ and the eigenvector $\mathbf{x}$ for a matrix $\mathbf{A}$ are defined as,

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x} \tag{3.1.1.1}$$

3.1.2. The eigenvalues are calculated by solving the equation

$$f(\lambda) = \left|\lambda\mathbf{I} - \mathbf{A}\right| = 0 \tag{3.1.2.1}$$

The above equation is known as the characteristic equation.

3.1.3. According to the Cayley-Hamilton theorem,

$$f(\lambda) = 0 \implies f(\mathbf{A}) = 0 \tag{3.1.3.1}$$

3.1.4. The trace of a square matrix is defined to be the sum of the diagonal elements.

$$\operatorname{tr}(\mathbf{A}) = \sum_{i=1}^{N} a_{ii}. \tag{3.1.4.1}$$

where $a_{ii}$ is the $i$th diagonal element of the matrix $\mathbf{A}$.

3.1.5. The trace of a matrix is equal to the sum of the eigenvalues

$$\operatorname{tr}(\mathbf{A}) = \sum_{i=1}^{N} \lambda_i \tag{3.1.5.1}$$

### *3.2 Determinants*

3.2.1. Let

$$\mathbf{A} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}. \tag{3.2.1.1}$$

be a $3 \times 3$ matrix. Then,

$$\left|\mathbf{A}\right| = a_1 \begin{pmatrix} b_2 & c_2 \\ b_3 & c_3 \end{pmatrix} - a_2 \begin{pmatrix} b_1 & c_1 \\ b_3 & c_3 \end{pmatrix} + a_3 \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}. \tag{3.2.1.2}$$

3.2.2. Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigenvalues of a matrix $\mathbf{A}$. Then, the product of the eigenvalues is equal to the determinant of $\mathbf{A}$.

$$\left|\mathbf{A}\right| = \prod_{i=1}^{n} \lambda_i \tag{3.2.2.1}$$

3.2.3.

$$\left|\mathbf{AB}\right| = \left|\mathbf{A}\right| \left|\mathbf{B}\right| \tag{3.2.3.1}$$

3.2.4. If $\mathbf{A}$ be an $n \times n$ matrix,

$$\left|k\mathbf{A}\right| = k^n \left|\mathbf{A}\right| \tag{3.2.4.1}$$

### *3.3 Rank of a Matrix*

3.3.1. The rank of a matrix is defined as the number of linearly independent rows. This is also known as the row rank.
3.3.2. Row rank = Column rank.
3.3.3. The rank of a matrix is obtained as the number of nonzero rows obtained after row reduction.
3.3.4. An $n \times n$ matrix is invertible if and only if its rank is $n$.

### 3.4 Inverse of a Matrix

3.4.1. For a $2 \times 2$ matrix

$$\mathbf{A} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}, \tag{3.4.1.1}$$

the inverse is given by

$$\mathbf{A}^{-1} = \frac{1}{\left|\mathbf{A}\right|} \begin{pmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{pmatrix}, \tag{3.4.1.2}$$

3.4.2. For higher order matrices, the inverse should be calculated using row operations.

### 3.5 Orthogonality

3.5.1. The rotation matrix is defined as

$$\mathbf{R}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad \theta \in [0, 2\pi] \tag{3.5.1.1}$$

3.5.2. The rotation matrix is *orthogonal*

$$\mathbf{R}_\theta^\top \mathbf{R}_\theta = \mathbf{R}_\theta \mathbf{R}_\theta^\top = \mathbf{I} \tag{3.5.2.1}$$

3.5.3.

$$\mathbf{m}^\top \mathbf{n} = 0 \implies \mathbf{n} = \mathbf{R}_{\frac{\pi}{2}} \mathbf{m} \tag{3.5.3.1}$$

3.5.4.

$$\mathbf{n}^\top \mathbf{h} = 1 \implies \mathbf{n} = \frac{\mathbf{e}_1}{\mathbf{e}_1^\top \mathbf{h}} + \mu \mathbf{R}_{\frac{\pi}{2}} \mathbf{h}, \quad \mu \in \mathbb{R}. \tag{3.5.4.1}$$

## 4 Linear Forms

### 4.1 Two Dimensions

4.1.1. The equation of a line is given by

$$\mathbf{n}^\top \mathbf{x} = c \tag{4.1.1.1}$$

where $\mathbf{n}$ is the normal vector of the line.

4.1.2. The equation of a line with normal vector $\mathbf{n}$ and passing through a point $\mathbf{A}$ is given by

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{A}) = 0 \tag{4.1.2.1}$$

4.1.3. The equation of a line $L$ is also given by

$$\mathbf{n}^\top \mathbf{x} = \begin{cases} 0 & \mathbf{0} \in L \\ 1 & \text{otherwise} \end{cases} \tag{4.1.3.1}$$

4.1.4. The parametric equation of a line is given by

$$\mathbf{x} = \mathbf{A} + \lambda \mathbf{m} \tag{4.1.4.1}$$

where $\mathbf{m}$ is the direction vector of the line and $\mathbf{A}$ is any point on the line.

4.1.5. Let $\mathbf{A}$ and $\mathbf{B}$ be two points on a straight line and let $\mathbf{P} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$ be any point on it. If $p_2$ is known, then

$$\mathbf{P} = \mathbf{A} + \frac{p_2 - \mathbf{e}_2^\top \mathbf{A}}{\mathbf{e}_2^\top (\mathbf{B} - \mathbf{A})} (\mathbf{B} - \mathbf{A}) \tag{4.1.5.1}$$

**Solution:** The equation of the line can be expressed in parametric from as

$$\mathbf{x} = \mathbf{A} + \lambda \left(\mathbf{B} - \mathbf{A}\right) \tag{4.1.5.2}$$
$$\implies \mathbf{P} = \mathbf{A} + \lambda \left(\mathbf{B} - \mathbf{A}\right) \tag{4.1.5.3}$$
$$\implies \mathbf{e}_2^\top \mathbf{P} = \mathbf{e}_2^\top \mathbf{A} + \lambda \mathbf{e}_2^\top \left(\mathbf{B} - \mathbf{A}\right) \tag{4.1.5.4}$$
$$\implies p_2 = \mathbf{e}_2^\top \mathbf{A} + \lambda \mathbf{e}_2^\top \left(\mathbf{B} - \mathbf{A}\right) \tag{4.1.5.5}$$
$$\text{or, } \lambda = \frac{p_2 - \mathbf{e}_2^\top \mathbf{A}}{\mathbf{e}_2^\top \left(\mathbf{B} - \mathbf{A}\right)} \tag{4.1.5.6}$$

yielding (4.1.5.1).

4.1.6. The distance from a point $\mathbf{P}$ to the line in (4.1.1.1) is given by

$$d = \frac{\left|\mathbf{n}^\top \mathbf{P} - c\right|}{\left\|\mathbf{n}\right\|} \tag{4.1.6.1}$$

**Solution:** Without loss of generality, let $\mathbf{A}$ be the foot of the perpendicular from $\mathbf{P}$ to the line in (4.1.4.1). The equation of the normal to (4.1.1.1) can then be expressed as

$$\mathbf{x} = \mathbf{A} + \lambda \mathbf{n} \tag{4.1.6.2}$$
$$\implies \mathbf{P} - \mathbf{A} = \lambda \mathbf{n} \tag{4.1.6.3}$$

$\because$ $\mathbf{P}$ lies on (4.1.6.2). From the above, the desired distance can be expressed as

$$d = \left\|\mathbf{P} - \mathbf{A}\right\| = \left|\lambda\right| \left\|\mathbf{n}\right\| \tag{4.1.6.4}$$

From (4.1.6.3),

$$\mathbf{n}^\top \left(\mathbf{P} - \mathbf{A}\right) = \lambda \mathbf{n}^\top \mathbf{n} = \lambda \left\|\mathbf{n}\right\|^2 \tag{4.1.6.5}$$
$$\implies \left|\lambda\right| = \frac{\left|\mathbf{n}^\top \left(\mathbf{P} - \mathbf{A}\right)\right|}{\left\|\mathbf{n}\right\|^2} \tag{4.1.6.6}$$

Substituting the above in (4.1.6.4) and using the fact that

$$\mathbf{n}^\top \mathbf{A} = c \tag{4.1.6.7}$$

from (4.1.1.1), yields (4.1.6.1)

4.1.7. The distance from the origin to the line in (4.1.1.1) is given by

$$d = \frac{\left|c\right|}{\left\|\mathbf{n}\right\|} \tag{4.1.7.1}$$

4.1.8. The distance between the parallel lines

$$\begin{aligned} \mathbf{n}^\top \mathbf{x} &= c_1 \\ \mathbf{n}^\top \mathbf{x} &= c_2 \end{aligned} \tag{4.1.8.1}$$

is given by

$$d = \frac{\left|c_1 - c_2\right|}{\left\|\mathbf{n}\right\|} \tag{4.1.8.2}$$

4.1.9. The equation of the line perpendicular to (4.1.1.1) and passing through the point $\mathbf{P}$ is given by

$$\mathbf{m}^\top \left(\mathbf{x} - \mathbf{P}\right) = 0 \tag{4.1.9.1}$$

4.1.10. The foot of the perpendicular from $\mathbf{P}$ to the line in (4.1.1.1) is given by

$$\begin{pmatrix}\mathbf{m} & \mathbf{n}\end{pmatrix}^\top \mathbf{x} = \begin{pmatrix}\mathbf{m}^\top \mathbf{P} \\ c\end{pmatrix} \tag{4.1.10.1}$$

**Solution:** From (4.1.1.1) and (4.1.2.1) the foot of the perpendicular satisfies the equations

$$\mathbf{n}^\top \mathbf{x} = c \tag{4.1.10.2}$$

$$\mathbf{m}^\top (\mathbf{x} - \mathbf{P}) = 0 \tag{4.1.10.3}$$

where $\mathbf{m}$ is the direction vector of the given line. Combining the above into a matrix equation results in (4.1.10.1).

### *4.2 Three Dimensions*

4.2.1. The area of a triangle with vertices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ is given by

$$\frac{1}{2} \left\| (\mathbf{A} - \mathbf{B}) \times (\mathbf{A} - \mathbf{C}) \right\| \tag{4.2.1.1}$$

4.2.2. Points $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are on a line if

$$\text{rank} \begin{pmatrix}\mathbf{A} \\ \mathbf{B} \\ \mathbf{C}\end{pmatrix} = 1 \tag{4.2.2.1}$$

4.2.3. Points $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ form a paralelogram if

$$\text{rank} \begin{pmatrix}\mathbf{A} \\ \mathbf{B} \\ \mathbf{C} \\ \mathbf{D}\end{pmatrix} = 1, \text{rank} \begin{pmatrix}\mathbf{A} \\ \mathbf{B} \\ \mathbf{C}\end{pmatrix} = 2 \tag{4.2.3.1}$$

4.2.4. The equation of a line is given by (4.1.4.1)

4.2.5. The equation of a plane is given by (4.1.1.1)

4.2.6. The distance from the origin to the line in (4.1.1.1) is given by (4.1.7.1)

4.2.7. The distance from a point $\mathbf{P}$ to the line in (4.1.4.1) is given by

$$d = \left\| \mathbf{A} - \mathbf{P} \right\|^2 - \frac{\left\{ \mathbf{m}^\top (\mathbf{A} - \mathbf{P}) \right\}^2}{\left\| \mathbf{m} \right\|^2} \tag{4.2.7.1}$$

**Solution:**

$$d(\lambda) = \left\| \mathbf{A} + \lambda \mathbf{m} - \mathbf{P} \right\| \tag{4.2.7.2}$$

$$\implies d^2(\lambda) = \left\| \mathbf{A} + \lambda \mathbf{m} - \mathbf{P} \right\|^2 \tag{4.2.7.3}$$

which can be simplified to obtain

$$d^2(\lambda) = \lambda^2 \left\| \mathbf{m} \right\|^2 + 2\lambda \mathbf{m}^\top (\mathbf{A} - \mathbf{P}) + \left\| \mathbf{A} - \mathbf{P} \right\|^2 \tag{4.2.7.4}$$

which is of the form

$$d^2(\lambda) = a\lambda^2 + 2b\lambda + c \tag{4.2.7.5}$$

$$= a \left\{ \left( \lambda + \frac{b}{a} \right)^2 + \left[ \frac{c}{a} - \left( \frac{b}{a} \right)^2 \right] \right\} \tag{4.2.7.6}$$

with

$$a = \|\mathbf{m}\|^2, b = \mathbf{m}^\top (\mathbf{A} - \mathbf{P}), c = \|\mathbf{A} - \mathbf{P}\|^2 \tag{4.2.7.7}$$

which can be expressed as From the above, $d^2(\lambda)$ is smallest when upon substituting from (4.2.7.7)

$$\lambda + \frac{b}{2a} = 0 \implies \lambda = -\frac{b}{2a} \tag{4.2.7.8}$$

$$= -\frac{\mathbf{m}^\top (\mathbf{A} - \mathbf{P})}{\|\mathbf{m}\|^2} \tag{4.2.7.9}$$

and consequently,

$$d_{\min}(\lambda) = a\left(\frac{c}{a} - \left(\frac{b}{a}\right)^2\right) \tag{4.2.7.10}$$

$$= c - \frac{b^2}{a} \tag{4.2.7.11}$$

yielding (4.2.7.1) after substituting from (4.2.7.7).

4.2.8. The distance between the parallel planes (4.1.8.1) is given by (4.1.8.2).

4.2.9. The plane

$$\mathbf{n}^\top \mathbf{x} = c \tag{4.2.9.1}$$

contains the line

$$\mathbf{x} = \mathbf{A} + \lambda \mathbf{m} \tag{4.2.9.2}$$

if

$$\mathbf{m}^\top \mathbf{n} = 0 \tag{4.2.9.3}$$

**Solution:** Any point on the line (4.2.9.2) should also satisfy (4.2.9.1). Hence,

$$\mathbf{n}^\top (\mathbf{A} + \lambda \mathbf{m}) = \mathbf{n}^\top \mathbf{A} = c \tag{4.2.9.4}$$

which can be simplified to obtain (4.2.9.3)

4.2.10. The foot of the perpendicular from a point $\mathbf{P}$ to the plane

$$\mathbf{n}^\top \mathbf{x} = c \tag{4.2.10.1}$$

is given by

**Solution:** The equation of the line perpendicular to the given plane and passing through $\mathbf{P}$ is

$$\mathbf{x} = \mathbf{P} + \lambda \mathbf{n} \tag{4.2.10.2}$$

From (4.2.13.1), the intersection of the above line with the given plane is

$$\mathbf{x} = \mathbf{P} + \frac{c - \mathbf{n}^\top \mathbf{P}}{\|\mathbf{n}\|^2} \mathbf{n} \tag{4.2.10.3}$$

4.2.11. The image of a point $\mathbf{P}$ with respect to the plane

$$\mathbf{n}^\top \mathbf{x} = c \tag{4.2.11.1}$$

is given by

$$\mathbf{R} = \mathbf{P} + 2\frac{c - \mathbf{n}^\top \mathbf{P}}{\|\mathbf{n}\|^2} \tag{4.2.11.2}$$

**Solution:** Let $\mathbf{R}$ be the desired image. Then, subtituting the expression for the foot of the perpendicular from $\mathbf{P}$ to the given plane using (4.2.10.3),

$$\frac{\mathbf{P}+\mathbf{R}}{2} = \mathbf{P} + \frac{c - \mathbf{n}^\top \mathbf{P}}{\left\|\mathbf{n}\right\|^2} \tag{4.2.11.3}$$

4.2.12. Let a plane pass through the points $\mathbf{A}, \mathbf{B}$ and be perpendicular to the plane

$$\mathbf{n}^\top \mathbf{x} = c \tag{4.2.12.1}$$

Then the equation of this plane is given by

$$\mathbf{p}^\top \mathbf{x} = 1 \tag{4.2.12.2}$$

where

$$\mathbf{p} = \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{n} \end{pmatrix}^{-\top} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \tag{4.2.12.3}$$

**Solution:** From the given information,

$$\mathbf{p}^\top \mathbf{A} = d \tag{4.2.12.4}$$
$$\mathbf{p}^\top \mathbf{B} = d \tag{4.2.12.5}$$
$$\mathbf{p}^\top \mathbf{n} = 0 \tag{4.2.12.6}$$

$\because$ the normal vectors to the two planes will also be perpendicular. The system of equations in (4.2.12.6) can be expressed as the matrix equation

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{n} \end{pmatrix}^{\top} \mathbf{p} = d \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \tag{4.2.12.7}$$

which yields (4.2.12.3) upon normalising with $d$.

4.2.13. The intersection of the line represented by (4.1.4.1) with the plane represented by (4.1.1.1) is given by

$$\mathbf{x} = \mathbf{A} + \frac{c - \mathbf{n}^\top \mathbf{A}}{\mathbf{n}^\top \mathbf{m}} \mathbf{m} \tag{4.2.13.1}$$

**Solution:** From (4.1.4.1) and (4.1.1.1),

$$\mathbf{x} = \mathbf{A} + \lambda \mathbf{m} \tag{4.2.13.2}$$
$$\mathbf{n}^\top \mathbf{x} = c \tag{4.2.13.3}$$
$$\implies \mathbf{n}^\top \left(\mathbf{A} + \lambda \mathbf{m}\right) = c \tag{4.2.13.4}$$

which can be simplified to obtain

$$\mathbf{n}^\top \mathbf{A} + \lambda \mathbf{n}^\top \mathbf{m} = c \tag{4.2.13.5}$$
$$\implies \lambda = \frac{c - \mathbf{n}^\top \mathbf{A}}{\mathbf{n}^\top \mathbf{m}} \tag{4.2.13.6}$$

Substituting the above in (4.2.13.4) yields (4.2.13.1).

4.2.14. The foot of the perpendicular from the point $\mathbf{P}$ to the line represented by (4.1.4.1) is given by

$$\mathbf{x} = \mathbf{A} + \frac{\mathbf{m}^\top \left(\mathbf{P} - \mathbf{A}\right)}{\left\|\mathbf{m}\right\|^2} \mathbf{m} \tag{4.2.14.1}$$

**Solution:** Let the equation of the line be

$$\mathbf{x} = \mathbf{A} + \lambda \mathbf{m} \tag{4.2.14.2}$$

The equation of the plane perpendicular to the given line passing through $\mathbf{P}$ is given by

$$\mathbf{m}^\top (\mathbf{x} - \mathbf{P}) = 0 \tag{4.2.14.3}$$

$$\implies \mathbf{m}^\top \mathbf{x} = \mathbf{m}^\top \mathbf{P} \tag{4.2.14.4}$$

The desired foot of the perpendicular is the intersection of (4.2.14.2) with (4.2.14.3) which can be obtained from (4.2.13.1) as (4.2.14.1)

4.2.15. The foot of the perpendicular from a point $\mathbf{P}$ to a plane is $\mathbf{Q}$. The equation of the plane is given by

$$(\mathbf{P} - \mathbf{Q})^\top (\mathbf{x} - \mathbf{Q}) = 0 \tag{4.2.15.1}$$

**Solution:** The normal vector to the plane is given by

$$\mathbf{n} = \mathbf{P} - \mathbf{Q} \tag{4.2.15.2}$$

Hence, the equation of the plane is (4.2.15.1).

4.2.16. Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be points on a plane. The equation of the plane is then given by

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix}^\top \mathbf{n} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{4.2.16.1}$$

**Solution:** Let the equation of the plane be

$$\mathbf{n}^\top \mathbf{x} = 1 \tag{4.2.16.2}$$

Then

$$\mathbf{n}^\top \mathbf{A} = 1 \tag{4.2.16.3}$$

$$\mathbf{n}^\top \mathbf{B} = 1 \tag{4.2.16.4}$$

$$\mathbf{n}^\top \mathbf{C} = 1 \tag{4.2.16.5}$$

which can be combined to obtain (4.2.16.1).

4.2.17. (Parallelogram Law) Let $\mathbf{A}, \mathbf{B}, \mathbf{D}$ be three vertices of a parallelogram. Then the vertex $\mathbf{C}$ is given by

$$\mathbf{C} = \mathbf{B} + \mathbf{C} - \mathbf{A} \tag{4.2.17.1}$$

**Solution:** Shifting $\mathbf{A}$ to the origin, we obtain a parallelogram with corresponding vertices

$$\mathbf{0}, \mathbf{B} - \mathbf{A}, \mathbf{D} - \mathbf{A} \tag{4.2.17.2}$$

The fourth vertex of this parallelogram is then obtained as

$$(\mathbf{B} - \mathbf{A}) + (\mathbf{D} - \mathbf{A}) = \mathbf{D} + \mathbf{B} - 2\mathbf{A} \tag{4.2.17.3}$$

Shifting the origin to $\mathbf{A}$, the fourth vertex is obtained as

$$\mathbf{C} = \mathbf{D} + \mathbf{B} - 2\mathbf{A} + \mathbf{A} \tag{4.2.17.4}$$

$$= \mathbf{D} + \mathbf{B} - \mathbf{A} \tag{4.2.17.5}$$

4.2.18. (Affine Transformation) Let $\mathbf{A}, \mathbf{C}$, be opposite vertices of a square. The other two points can be obtained as

$$\mathbf{B} = \frac{\left\|\mathbf{A} - \mathbf{C}\right\|}{\sqrt{2}}\mathbf{P}\mathbf{e}_1 + \mathbf{A} \tag{4.2.18.1}$$

$$\mathbf{D} = \frac{\left\|\mathbf{A} - \mathbf{C}\right\|}{\sqrt{2}}\mathbf{P}\mathbf{e}_2 + \mathbf{A} \tag{4.2.18.2}$$

where

$$\mathbf{P} = \begin{pmatrix} \cos\left(\theta - \frac{\pi}{4}\right) & \sin\left(\theta - \frac{\pi}{4}\right) \\ \sin\left(\theta - \frac{\pi}{4}\right) & \cos\left(\theta - \frac{\pi}{4}\right) \end{pmatrix} \tag{4.2.18.3}$$

and

$$\cos\theta = \frac{\left(\mathbf{C} - \mathbf{A}\right)^\top \mathbf{e}_1}{\left\|\mathbf{A} - \mathbf{C}\right\| \left\|\mathbf{e}_1\right\|} \tag{4.2.18.4}$$

# 5 Quadratic Forms

## 5.1 Definitions

**Definition 1.** *The* affine *transformation is given by*

$$\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c} \quad \textit{(Affine Transformation)} \tag{5.1.1}$$

*where $\mathbf{P}$ is invertible.*

**Definition 2.** *The eigenvalue decomposition of a symmetric matrix $\mathbf{V}$ is given by*

$$\mathbf{P}^\top \mathbf{V} \mathbf{P} = \mathbf{D}. \quad \textit{(Eigenvalue Decomposition)} \tag{5.1.2}$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \tag{5.1.3}$$

$$\mathbf{P} = \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix}, \quad \mathbf{P}^\top = \mathbf{P}^{-1}, \tag{5.1.4}$$

## 5.2 The Quadratic Form

**Definition 3.** *Let $\mathbf{q}$ be a point such that the ratio of its distance from a fixed point $\mathbf{F}$ and the distance $(d)$ from a fixed line*

$$L : \mathbf{n}^\top \mathbf{x} = c \tag{5.2.1}$$

*is constant, given by*

$$\frac{\left\|\mathbf{q} - \mathbf{F}\right\|}{d} = e \tag{5.2.2}$$

*The locus of $\mathbf{q}$ is known as a conic section. The line $L$ is known as the directrix and the point $\mathbf{F}$ is the focus. $e$ is defined to be the eccentricity of the conic.*

1) *For $e = 1$, the conic is a parabola*
2) *For $e < 1$, the conic is an ellipse*
3) *For $e > 1$, the conic is a hyperbola*

**Theorem 5.1.** *The equation of a conic with directrix $\mathbf{n}^\top \mathbf{x} = c$, eccentricity $e$ and focus $\mathbf{F}$ is given by*

$$\mathbf{x}^\top \mathbf{V} \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{5.2.1}$$

*where*

$$\mathbf{V} = \|\mathbf{n}\|^2 \mathbf{I} - e^2\mathbf{n}\mathbf{n}^\top, \tag{5.2.2}$$
$$\mathbf{u} = ce^2\mathbf{n} - \|\mathbf{n}\|^2 \mathbf{F}, \tag{5.2.3}$$
$$f = \|\mathbf{n}\|^2 \|\mathbf{F}\|^2 - c^2e^2 \tag{5.2.4}$$

*Proof.* Using Definition 3 and Lemma 4.1.6.1, for any point $\mathbf{x}$ on the conic,

$$\|\mathbf{x} - \mathbf{F}\|^2 = e^2\frac{\left(\mathbf{n}^\top\mathbf{x} - c\right)^2}{\|\mathbf{n}\|^2} \tag{5.2.5}$$
$$\implies \|\mathbf{n}\|^2 (\mathbf{x} - \mathbf{F})^\top (\mathbf{x} - \mathbf{F}) = e^2\left(\mathbf{n}^\top\mathbf{x} - c\right)^2 \tag{5.2.6}$$
$$\implies \|\mathbf{n}\|^2 \left(\mathbf{x}^\top\mathbf{x} - 2\mathbf{F}^\top\mathbf{x} + \|\mathbf{F}\|^2\right) = e^2\left(c^2 + \left(\mathbf{n}^\top\mathbf{x}\right)^2 - 2c\mathbf{n}^\top\mathbf{x}\right) \tag{5.2.7}$$
$$= e^2\left(c^2 + \left(\mathbf{x}^\top\mathbf{n}\mathbf{n}^\top\mathbf{x}\right) - 2c\mathbf{n}^\top\mathbf{x}\right) \tag{5.2.8}$$

which can be expressed as (5.2.1) after simplification.

□

**Theorem 5.2.** *The eccentricity, directrices and foci of* (5.2.1) *are given by*

$$e = \sqrt{1 - \frac{\lambda_1}{\lambda_2}} \tag{5.2.9}$$
$$\mathbf{n} = \sqrt{\lambda_2}\mathbf{p}_1,$$
$$c = \begin{cases} \frac{e\mathbf{u}^\top\mathbf{n} \pm \sqrt{e^2\left(\mathbf{u}^\top\mathbf{n}\right)^2 - \lambda_2(e^2-1)\left(\|\mathbf{u}\|^2 - \lambda_2 f\right)}}{\lambda_2 e(e^2-1)} & e \neq 1 \\ \frac{\|\mathbf{u}\|^2 - \lambda_2 f}{2\mathbf{u}^\top\mathbf{n}} & e = 1 \end{cases} \tag{5.2.10}$$
$$\mathbf{F} = \frac{ce^2\mathbf{n} - \mathbf{u}}{\lambda_2} \tag{5.2.11}$$

*Proof.* See Appendix A □

**Theorem 5.3.** (5.2.1) *represents*

1) *a parabola for* $\left|\mathbf{V}\right| = 0$,
2) *ellipse for* $\left|\mathbf{V}\right| > 0$ *and*
3) *hyperbola for* $\left|\mathbf{V}\right| < 0$.

*Proof.* From (5.2.9),

$$\frac{\lambda_1}{\lambda_2} = 1 - e^2 \tag{5.2.1}$$

Also,

$$\left|\mathbf{V}\right| = \lambda_1\lambda_2 \tag{5.2.2}$$

yielding Table 3 □

| Eccentricity | Conic | Eigenvalue | Determinant |
|---|---|---|---|
| $e = 1$ | Parabola | $\lambda_1 = 0$ | $\lvert\mathbf{V}\rvert = 0$ |
| $e < 1$ | Ellipse | $\lambda_1 > 0, \lambda_2 > 0$ | $\lvert\mathbf{V}\rvert > 0$ |
| $e > 1$ | Hyperbola | $\lambda_1 < 0, \lambda_2 > 0$ | $\lvert\mathbf{V}\rvert < 0$ |

TABLE 3

### *5.3 Standard Form*

**Theorem 5.4.** *Using the affine transformation in* (5.1.1)*, the conic in* (5.2.1) *can be expressed in standard form as*

$$\mathbf{y}^\top \left(\frac{\mathbf{D}}{f_0}\right) \mathbf{y} = 1 \qquad |\mathbf{V}| \neq 0 \tag{5.3.1}$$

$$\mathbf{y}^\top \mathbf{D} \mathbf{y} = -\eta \mathbf{e}_1^\top \mathbf{y} \qquad |\mathbf{V}| = 0 \tag{5.3.2}$$

*where*

$$f_0 = \mathbf{u}^\top \mathbf{V}^{-1} \mathbf{u} - f \neq 0 \tag{5.3.3}$$

$$\eta = 2\mathbf{u}^\top \mathbf{p}_1 \tag{5.3.4}$$

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{5.3.5}$$

*Proof.* See Appendix B. □

**Corollary 5.5.** *For the standard conic,*

$$\mathbf{P} = \mathbf{I} \tag{5.3.6}$$

$$\mathbf{u} = \begin{cases} 0 & e \neq 1 \\ \frac{\eta}{2}\mathbf{e}_1 & e = 1 \end{cases} \tag{5.3.7}$$

$$\lambda_1 \begin{cases} = 0 & e = 1 \\ \neq 0 & e \neq 1 \end{cases} \tag{5.3.8}$$

*where*

$$\mathbf{I} = \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 \end{pmatrix} \tag{5.3.9}$$

*is the identity matrix.*

**Theorem 5.6.**

1) *The directrices for the standard conic are given by*

$$\mathbf{e}_1^\top \mathbf{y} = \pm \frac{1}{e} \sqrt{\frac{|f_0|}{\lambda_2 \left(1 - e^2\right)}} \qquad e \neq 1 \tag{5.3.1}$$

$$\mathbf{e}_1^\top \mathbf{y} = \frac{\eta}{2\lambda_2} \qquad e = 1 \tag{5.3.2}$$

2) *The foci of the standard ellipse and hyperbola are given by*

$$\mathbf{F} = \begin{cases} \pm e \sqrt{\frac{|f_0|}{\lambda_2 (1 - e^2)}} \mathbf{e}_1 & e \neq 1 \\ -\frac{\eta}{4\lambda_2} \mathbf{e}_1 & e = 1 \end{cases} \tag{5.3.1}$$

*Proof.* See Appendix C. □

### 5.4 Corollaries-Standard Form

**Corollary 5.7.** *The center of the standard ellipse/hyperbola, defined to be the mid point of the line joining the foci, is the origin.*

**Corollary 5.8.** *The principal (major) axis of the standard ellipse/hyperbola, defined to be the line joining the two foci is the $x$-axis.*

*Proof.* From (5.3.1), it is obvious that the line joining the foci passes through the origin. Also, the direction vector of this line is $\mathbf{e}_1$. Thus, the principal axis is the $x$-axis. □

**Corollary 5.9.** *The minor axis of the standard ellipse/hyperbola, defined to be the line orthogonal to the $x$-axis is the $y$-axis.*

**Corollary 5.10.** *The axis of symmetry of the standard parabola, defined to be the line perpendicular to the directrix and passing through the focus, is the $x$- axis.*

*Proof.* From (3.0.2.4) and (5.3.1), the axis of the parabola can be expressed using (4.1.2.1) as

$$\mathbf{e}_2^\top \left(\mathbf{y} + \frac{\eta}{4\lambda_2}\mathbf{e}_1\right) = 0 \tag{5.4.1}$$

$$\implies \mathbf{e}_2^\top \mathbf{y} = 0, \tag{5.4.2}$$

which is the equation of the $x$-axis. □

**Corollary 5.11.** *The point where the parabola intersects its axis of symmetry is called the vertex. For the standard parabola, the vertex is the origin.*

*Proof.* (5.4.2) can be expressed as

$$\mathbf{y} = \alpha \mathbf{e}_1, \tag{5.4.3}$$

using (4.1.2.1). Substituting (5.4.3) in (5.3.2),

$$\alpha^2 \mathbf{e}_1^\top \mathbf{D} \mathbf{e}_1 = -\eta\alpha \mathbf{e}_1^\top \mathbf{e}_1 \tag{5.4.4}$$

$$\implies \alpha = 0, \text{ or, } \mathbf{y} = \mathbf{0}. \tag{5.4.5}$$

□

**Corollary 5.12.** *The* focal length *of the standard parabola, , defined to be the distance between the vertex and the focus, measured along the axis of symmetry, is* $\left|\frac{\eta}{4\lambda_2}\right|$

### 5.5 Corollaries-Quadratic Form

**Corollary 5.13.** *The center/vertex of a conic section are given by*

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \qquad \left|\mathbf{V}\right| \neq 0 \tag{5.5.1}$$

$$\begin{pmatrix} \mathbf{u}^\top + \frac{\eta}{2}\mathbf{p}_1^\top \\ \mathbf{V} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -f \\ \frac{\eta}{2}\mathbf{p}_1 - \mathbf{u} \end{pmatrix} \qquad \left|\mathbf{V}\right| = 0 \tag{5.5.2}$$

*Proof.* In (5.1.1), substituting $\mathbf{y} = \mathbf{0}$, the center/vertex for the quadratic form is obtained as

$$\mathbf{x} = \mathbf{c}, \tag{5.5.3}$$

where $\mathbf{c}$ is derived as (5.5.1) and (5.5.2) in Appendix B. □

**Corollary 5.14.** *The equation of the minor and major axes for the ellipse/hyperbola are respectively given by*

$$\mathbf{p}_i^\top (\mathbf{x} - \mathbf{c}) = 0, i = 1, 2 \tag{5.5.4}$$

*The axis of symmetry for the parabola is also given by* (5.5.4).

*Proof.* From (5.8), the major/symmetry axis for the hyperbola/ellipse/parabola can be expressed using (5.1.1) as

$$\mathbf{e}_2^\top \mathbf{P}^\top (\mathbf{x} - \mathbf{c}) = 0 \tag{5.5.5}$$

$$\implies (\mathbf{P}\mathbf{e}_2)^\top (\mathbf{x} - \mathbf{c}) = 0 \tag{5.5.6}$$

yielding (5.5.4), and the proof for the minor axis is similar. □

*5.6 Pair of Straight Lines*

**Lemma 5.1** (Asymptotes)**.** *The asymptotes of the hyperbola in* (5.3.1)*, defined to be the lines that do not intersect the hyperbola, are given by*

$$\left(\sqrt{|\lambda_1|} \quad \pm\sqrt{|\lambda_2|}\right) \mathbf{y} = 0 \tag{5.6.1}$$

*Proof.* From (5.3.1), it is obvious that the pair of lines represented by

$$\mathbf{y}^\top \mathbf{D} \mathbf{y} = 0 \tag{5.6.2}$$

do not intersect the conic

$$\mathbf{y}^\top \mathbf{D} \mathbf{y} = f_0 \tag{5.6.3}$$

Thus, (5.6.2) represents the asysmptotes of the hyperbola in (5.3.1) and can be expressed as

$$\lambda_1 y_1^2 + \lambda_2 y_1^2 = 0, \tag{5.6.4}$$

which can then be simplified to obtain (5.6.1).

□

**Corollary 5.15.** (5.2.1) *represents a pair of straight lines if*

$$\mathbf{u}^\top \mathbf{V}^{-1} \mathbf{u} - f = 0 \tag{5.6.5}$$

**Theorem 5.16.** (5.2.1) *represents a pair of straight lines if the matrix*

$$\begin{pmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u} & f \end{pmatrix} \tag{5.6.6}$$

*is singular.*

*Proof.* Let

$$\begin{pmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u} & f \end{pmatrix} \mathbf{x} = \mathbf{0} \tag{5.6.7}$$

Expressing

$$\mathbf{x} = \begin{pmatrix} \mathbf{y} \\ y_3 \end{pmatrix}, \tag{5.6.8}$$

$$\begin{pmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^\top & f \end{pmatrix} \begin{pmatrix} \mathbf{y} \\ y_3 \end{pmatrix} = \mathbf{0} \tag{5.6.9}$$

$$\implies \mathbf{V}\mathbf{y} + y_3 \mathbf{u} = \mathbf{0} \quad \text{and} \tag{5.6.10}$$

$$\mathbf{u}^\top \mathbf{y} + f y_3 = 0 \tag{5.6.11}$$

From (5.6.10) we obtain,

$$\mathbf{y}^\top \mathbf{V}\mathbf{y} + y_3 \mathbf{y}^\top \mathbf{u} = \mathbf{0} \tag{5.6.12}$$

$$\implies \mathbf{y}^\top \mathbf{V}\mathbf{y} + y_3 \mathbf{u}^\top \mathbf{y} = \mathbf{0} \tag{5.6.13}$$

yielding (5.6.5) upon substituting from (5.6.11). □

**Corollary 5.17.** *Using the affine transformation,* (5.6.1) *can be expressed as the lines*

$$\begin{pmatrix}\sqrt{|\lambda_1|} & \pm\sqrt{|\lambda_2|}\end{pmatrix} \mathbf{P}^\top (\mathbf{x} - \mathbf{c}) = 0 \tag{5.6.14}$$

**Corollary 5.18.** *The angle between the asymptotes can be expressed as*

$$\cos\theta = \frac{|\lambda_1| - |\lambda_2|}{|\lambda_1| + |\lambda_2|} \tag{5.6.15}$$

*Proof.* The normal vectors of the lines in (5.6.14) are

$$\begin{aligned}\mathbf{n}_1 &= \mathbf{P}\begin{pmatrix}\sqrt{|\lambda_1|}\\ \sqrt{|\lambda_2|}\end{pmatrix}\\ \mathbf{n}_2 &= \mathbf{P}\begin{pmatrix}\sqrt{|\lambda_1|}\\ -\sqrt{|\lambda_2|}\end{pmatrix}\end{aligned} \tag{5.6.16}$$

The angle between the asymptotes is given by

$$\cos\theta = \frac{\mathbf{n_1}^\top \mathbf{n_2}}{\|\mathbf{n_1}\|\,\|\mathbf{n_2}\|} \tag{5.6.17}$$

The orthogonal matrix $\mathbf{P}$ preserves the norm, i.e.

$$\|\mathbf{n_1}\| = \left\|\mathbf{P}\begin{pmatrix}\sqrt{|\lambda_1|}\\ \sqrt{|\lambda_2|}\end{pmatrix}\right\| = \left\|\begin{pmatrix}\sqrt{|\lambda_1|}\\ \sqrt{|\lambda_2|}\end{pmatrix}\right\| \tag{5.6.18}$$

$$= \sqrt{|\lambda_1| + |\lambda_2|} = \|\mathbf{n_2}\| \tag{5.6.19}$$

It is easy to verify that

$$\mathbf{n_1}^\top \mathbf{n_2} = |\lambda_1| - |\lambda_2| \tag{5.6.20}$$

Thus, the angle between the asymptotes is obtained from (5.6.17) as (5.6.15). □

## *5.7 Chords of a Conic*

**Theorem 5.19** (Chord)**.** *The points of intersection of the line*

$$L: \quad \mathbf{x} = \mathbf{q} + \mu\mathbf{m} \quad \mu \in \mathbb{R} \tag{5.7.1}$$

*with the conic section in* (5.2.1) *are given by*

$$\mathbf{x}_i = \mathbf{q} + \mu_i \mathbf{m} \tag{5.7.2}$$

*where*

$$\mu_i = \frac{1}{\mathbf{m}^\top \mathbf{V}\mathbf{m}} \Big(-\mathbf{m}^\top (\mathbf{V}\mathbf{q} + \mathbf{u}) \pm \sqrt{\left[\mathbf{m}^\top (\mathbf{V}\mathbf{q} + \mathbf{u})\right]^2 - \left(\mathbf{q}^\top \mathbf{V}\mathbf{q} + 2\mathbf{u}^\top \mathbf{q} + f\right)\left(\mathbf{m}^\top \mathbf{V}\mathbf{m}\right)}\Big) \tag{5.7.3}$$

*Proof.* Substituting (5.7.1) in (5.2.1),

$$(\mathbf{q}+\mu\mathbf{m})^\top \mathbf{V}(\mathbf{q}+\mu\mathbf{m}) + 2\mathbf{u}^\top(\mathbf{q}+\mu\mathbf{m}) + f = 0 \tag{5.7.4}$$

$$\implies \mu^2\mathbf{m}^\top\mathbf{V}\mathbf{m} + 2\mu\mathbf{m}^\top(\mathbf{V}\mathbf{q}+\mathbf{u}) + \mathbf{q}^\top\mathbf{V}\mathbf{q} + 2\mathbf{u}^\top\mathbf{q} + f = 0 \tag{5.7.5}$$

Solving the above quadratic in (5.7.5) yields (5.7.3). □

**Corollary 5.20.** *If $L$ in* (5.7.1) *touches* (5.2.1) *at exactly one point* $\mathbf{q}$*,*

$$\mathbf{m}^\top(\mathbf{V}\mathbf{q}+\mathbf{u}) = 0 \tag{5.7.6}$$

*Proof.* In this case, (5.7.5) has exactly one root. Hence, in (5.7.3)

$$\left[\mathbf{m}^\top(\mathbf{V}\mathbf{q}+\mathbf{u})\right]^2 - \left(\mathbf{m}^\top\mathbf{V}\mathbf{m}\right)\left(\mathbf{q}^\top\mathbf{V}\mathbf{q}+2\mathbf{u}^\top\mathbf{q}+f\right) = 0 \tag{5.7.7}$$

$\because$ $\mathbf{q}$ is the point of contact, $\mathbf{q}$ satisfies (5.2.1) and

$$\mathbf{q}^\top\mathbf{V}\mathbf{q}+2\mathbf{u}^\top\mathbf{q}+f = 0 \tag{5.7.8}$$

Substituting (5.7.8) in (5.7.7) and simplifying, we obtain (5.7.6). □

**Theorem 5.21.** *The length of the chord in* (5.7.1) *is given by*

$$\frac{2\sqrt{\left[\mathbf{m}^\top(\mathbf{V}\mathbf{q}+\mathbf{u})\right]^2 - \left(\mathbf{q}^\top\mathbf{V}\mathbf{q}+2\mathbf{u}^\top\mathbf{q}+f\right)\left(\mathbf{m}^\top\mathbf{V}\mathbf{m}\right)}}{\mathbf{m}^\top\mathbf{V}\mathbf{m}}\left\|\mathbf{m}\right\| \tag{5.7.9}$$

*Proof.* The distance between the points in (5.7.2) is given by

$$\left\|\mathbf{x}_1-\mathbf{x}_2\right\| = \left|\mu_1-\mu_2\right|\left\|\mathbf{m}\right\| \tag{5.7.10}$$

Substituing $\mu_i$ from (5.7.3) in (5.7.10) yields (5.7.9). □

**Theorem 5.22.** *The affine transform for the conic section, preserves the norm. This implies that the length of any chord of a conic is invariant to translation and/or rotation.*

*Proof.* Let

$$\mathbf{x}_i = \mathbf{P}\mathbf{y}_i + \mathbf{c} \tag{5.7.11}$$

be any two points on the conic. Then the distance between the points is given by

$$\left\|\mathbf{x}_1-\mathbf{x}_2\right\| = \left\|\mathbf{P}(\mathbf{y}_1-\mathbf{y}_2)\right\| \tag{5.7.12}$$

which can be expressed as

$$\left\|\mathbf{x}_1-\mathbf{x}_2\right\|^2 = (\mathbf{y}_1-\mathbf{y}_2)^\top\mathbf{P}^\top\mathbf{P}(\mathbf{y}_1-\mathbf{y}_2) \tag{5.7.13}$$

$$= \left\|\mathbf{y}_1-\mathbf{y}_2\right\|^2 \tag{5.7.14}$$

since

$$\mathbf{P}^\top\mathbf{P} = \mathbf{I} \tag{5.7.15}$$

□

**Corollary 5.23.** *For the standard hyperbola/ellipse, the length of the major axis is*

$$2\sqrt{\left|\frac{f_0}{\lambda_1}\right|} \tag{5.7.16}$$

*and the minor axis is*

$$2\sqrt{\left|\frac{f_0}{\lambda_2}\right|} \tag{5.7.17}$$

*Proof.* See Appendix D □

**Theorem 5.24** (latus rectum)**.** *The latus rectum of a conic section is the chord that passes through the focus and is perpendicular to the major axis. The length of the latus rectum for a conic is given by*

$$l = \begin{cases} 2\frac{\sqrt{|f_0\lambda_1|}}{\lambda_2} & e \neq 1 \\ \frac{\eta}{\lambda_2} & e = 1 \end{cases} \tag{5.7.18}$$

*Proof.* See Appendix E. □

### *5.8 Tangent and Normal*

**Theorem 5.25** (Tangent)**.** *Given the point of contact* $\mathbf{q}$*, the equation of a tangent to* (5.2.1) *is*

$$\left(\mathbf{V}\mathbf{q}+\mathbf{u}\right)^\top \mathbf{x} + \mathbf{u}^\top\mathbf{q} + f = 0 \tag{5.8.1}$$

*Proof.* The normal vector is obtained from (5.7.6) and (1.15.1) as

$$\kappa\mathbf{n} = \mathbf{V}\mathbf{q}+\mathbf{u}, \kappa \in \mathbb{R} \tag{5.8.2}$$

From (5.8.2) and (4.1.2.1), the equation of the tangent is

$$\left(\mathbf{V}\mathbf{q}+\mathbf{u}\right)^\top \left(\mathbf{x}-\mathbf{q}\right) = 0 \tag{5.8.3}$$

$$\implies \left(\mathbf{V}\mathbf{q}+\mathbf{u}\right)^\top \mathbf{x} - \mathbf{q}^\top\mathbf{V}\mathbf{q} - \mathbf{u}^\top\mathbf{q} = 0 \tag{5.8.4}$$

which, upon substituting from (5.7.8) and simplifying yields (5.8.1) □

**Theorem 5.26.** *If* $\mathbf{V}^{-1}$ *exists, given the normal vector* $\mathbf{n}$*, the tangent points of contact to* (5.2.1) *are given by*

$$\begin{aligned} \mathbf{q}_i &= \mathbf{V}^{-1}\left(\kappa_i\mathbf{n}-\mathbf{u}\right), i = 1, 2 \\ \textit{where } \kappa_i &= \pm\sqrt{\frac{f_0}{\mathbf{n}^\top\mathbf{V}^{-1}\mathbf{n}}} \end{aligned} \tag{5.8.5}$$

*Proof.* From (5.8.2),

$$\mathbf{q} = \mathbf{V}^{-1}\left(\kappa\mathbf{n}-\mathbf{u}\right), \quad \kappa \in \mathbb{R} \tag{5.8.6}$$

Substituting (5.8.6) in (5.7.8),

$$\left(\kappa\mathbf{n}-\mathbf{u}\right)^\top \mathbf{V}^{-1}\left(\kappa\mathbf{n}-\mathbf{u}\right) + 2\mathbf{u}^\top\mathbf{V}^{-1}\left(\kappa\mathbf{n}-\mathbf{u}\right) + f = 0 \tag{5.8.7}$$

$$\implies \kappa^2\mathbf{n}^\top\mathbf{V}^{-1}\mathbf{n} - \mathbf{u}^\top\mathbf{V}^{-1}\mathbf{u} + f = 0 \tag{5.8.8}$$

$$\text{or, } \kappa = \pm\sqrt{\frac{f_0}{\mathbf{n}^\top\mathbf{V}^{-1}\mathbf{n}}} \tag{5.8.9}$$

Substituting (5.8.9) in (5.8.6) yields (5.8.5). □

**Theorem 5.27.** *If* $\mathbf{V}$ *is not invertible, given the normal vector* $\mathbf{n}$*, the point of contact to* (5.2.1) *is given by the matrix equation*

$$\begin{pmatrix} \left(\mathbf{u}+\kappa\mathbf{n}\right)^\top \\ \mathbf{V} \end{pmatrix} \mathbf{q} = \begin{pmatrix} -f \\ \kappa\mathbf{n}-\mathbf{u} \end{pmatrix} \tag{5.8.10}$$

$$\textit{where } \kappa = \frac{\mathbf{p}_1^\top\mathbf{u}}{\mathbf{p}_1^\top\mathbf{n}}, \quad \mathbf{V}\mathbf{p}_1 = 0 \tag{5.8.11}$$

*Proof.* If $\mathbf{V}$ is non-invertible, it has a zero eigenvalue. If the corresponding eigenvector is $\mathbf{p}_1$, then,

$$\mathbf{V}\mathbf{p}_1 = 0 \tag{5.8.12}$$

From (5.8.2),

$$\kappa\mathbf{n} = \mathbf{V}\mathbf{q} + \mathbf{u}, \quad \kappa \in \mathbb{R} \tag{5.8.13}$$

$$\implies \kappa\mathbf{p}_1^\top\mathbf{n} = \mathbf{p}_1^\top\mathbf{V}\mathbf{q} + \mathbf{p}_1^\top\mathbf{u} \tag{5.8.14}$$

$$\text{or, } \kappa\mathbf{p}_1^\top\mathbf{n} = \mathbf{p}_1^\top\mathbf{u}, \quad \because \mathbf{p}_1^\top\mathbf{V} = 0, \quad (\text{ from (5.8.12)}) \tag{5.8.15}$$

yielding $\kappa$ in (5.8.11). From (5.8.13),

$$\kappa\mathbf{q}^\top\mathbf{n} = \mathbf{q}^\top\mathbf{V}\mathbf{q} + \mathbf{q}^\top\mathbf{u} \tag{5.8.16}$$

$$\implies \kappa\mathbf{q}^\top\mathbf{n} = -f - \mathbf{q}^\top\mathbf{u} \quad \text{from (5.7.8)}, \tag{5.8.17}$$

$$\text{or, } \left(\kappa\mathbf{n} + \mathbf{u}\right)^\top\mathbf{q} = -f \tag{5.8.18}$$

(5.8.13) can be expressed as

$$\mathbf{V}\mathbf{q} = \kappa\mathbf{n} - \mathbf{u}. \tag{5.8.19}$$

(5.8.18) and (5.8.19) clubbed together result in (5.8.10). □

**Theorem 5.28.** *The normal vectors of the tangents from a point* $\mathbf{h}$ *to the conic in* (5.2.1) *are given by*

$$\begin{aligned}\mathbf{n}_1 &= \mathbf{P}\begin{pmatrix}\sqrt{|\lambda_1|}\\ \sqrt{|\lambda_2|}\end{pmatrix}\\ \mathbf{n}_2 &= \mathbf{P}\begin{pmatrix}\sqrt{|\lambda_1|}\\ -\sqrt{|\lambda_2|}\end{pmatrix}\end{aligned} \tag{5.8.20}$$

*where* $\lambda_i, \mathbf{P}$ *are the eigenparameters of*

$$\mathbf{\Sigma} = \left(\mathbf{V}\mathbf{h} + \mathbf{u}\right)\left(\mathbf{V}\mathbf{h} + \mathbf{u}\right)^\top - \mathbf{V}\left(\mathbf{h}^\top\mathbf{V}\mathbf{h} + 2\mathbf{u}^\top\mathbf{h} + f\right). \tag{5.8.21}$$

*Proof.* From (5.7.3), and (5.7.7)

$$\left[\mathbf{m}^\top\left(\mathbf{V}\mathbf{h} + \mathbf{u}\right)\right]^2 - \left(\mathbf{m}^\top\mathbf{V}\mathbf{m}\right)\left(\mathbf{h}^\top\mathbf{V}\mathbf{h} + 2\mathbf{u}^\top\mathbf{h} + f\right) = 0 \tag{5.8.22}$$

$$\implies \mathbf{m}^\top\left[\left(\mathbf{V}\mathbf{h} + \mathbf{u}\right)\left(\mathbf{V}\mathbf{h} + \mathbf{u}\right)^\top - \mathbf{V}\left(\mathbf{h}^\top\mathbf{V}\mathbf{h} + 2\mathbf{u}^\top\mathbf{h} + f\right)\right]\mathbf{m} = 0 \tag{5.8.23}$$

yielding (5.8.21). Consequently, from (5.6.16), (5.8.20) can be obtained. □

**Theorem 5.29.** *The normal vectors of the tangents from a point* $\mathbf{h}$ *to the conic in* (5.2.1) *are given by*

$$\begin{aligned}\mathbf{n}_1 &= \mathbf{P}\begin{pmatrix}\sqrt{|\lambda_1|}\\ \sqrt{|\lambda_2|}\end{pmatrix}\\ \mathbf{n}_2 &= \mathbf{P}\begin{pmatrix}\sqrt{|\lambda_1|}\\ -\sqrt{|\lambda_2|}\end{pmatrix}\end{aligned} \tag{5.8.24}$$

*where* $\lambda_i, \mathbf{P}$ *are the eigenparameters of*

$$\mathbf{\Sigma} = \left(\mathbf{V}\mathbf{h} + \mathbf{u}\right)\left(\mathbf{V}\mathbf{h} + \mathbf{u}\right)^\top - \mathbf{V}\left(\mathbf{h}^\top\mathbf{V}\mathbf{h} + 2\mathbf{u}^\top\mathbf{h} + f\right). \tag{5.8.25}$$

*Proof.* From (5.7.6), the normal vector to the tangent at $\mathbf{q}$ can be expressed as

$$\mathbf{n} = \mathbf{V}\mathbf{q} + \mathbf{u} \tag{5.8.26}$$

$$\implies \mathbf{q} = \mathbf{V}^{-1}\left(\mathbf{n} - \mathbf{u}\right) \tag{5.8.27}$$

which upon substituting in (5.2.1) yields

$$(\mathbf{n}-\mathbf{u})^\top \mathbf{V}^{-1}\mathbf{V}\mathbf{V}^{-1}(\mathbf{n}-\mathbf{u}) + 2\mathbf{u}^\top\mathbf{V}^{-1}(\mathbf{n}-\mathbf{u}) + f = 0 \tag{5.8.28}$$

$$\implies \mathbf{n}^\top\mathbf{V}^{-1}\mathbf{n} - \mathbf{u}^\top\mathbf{V}^{-1}\mathbf{u} + f = 0 \tag{5.8.29}$$

□

### *5.9 Intersection of Conics*

**Lemma 5.2.** *Let*

$$\mathbf{x}^\top\mathbf{V}_i\mathbf{x} + 2\mathbf{u}_i^\top\mathbf{x} + f_i = 0, \quad i = 1, 2 \tag{5.9.1}$$

*be the equation of two conics. The locus of their intersection is a pair of straight lines if*

$$\begin{vmatrix} \mathbf{V}_1 + \mu\mathbf{V}_2 & \mathbf{u}_1 + \mu\mathbf{u}_2 \\ (\mathbf{u}_1 + \mu\mathbf{u}_2)^\top & f \end{vmatrix} = 0, \left|\mathbf{V}_1 + \mu\mathbf{V}_2\right| < 0 \tag{5.9.2}$$

*Proof.* The intersection of the conics in (5.9.1) is given by the curve

$$\mathbf{x}^\top(\mathbf{V}_1 + \mu\mathbf{V}_2)\mathbf{x} + 2(\mathbf{u}_1 + \mu\mathbf{u}_2)^\top\mathbf{x} + f_1 + \mu f_2 = 0, \tag{5.9.3}$$

which, from Theorem 5.16 represents a pair of straight lines if (5.9.2) is satisfied. □

**Corollary 5.30.** *The points of intersection of the conics in* (5.9.1) *are the points of the intersection of the lines in* (5.9.3).

## 6 Examples

### *6.1 Loney*

**Example 6.1** (parabola)**.** *To show that*

$$9x^2 - 24xy + 16y^2 - 18x - 101y + 19 = 0 \tag{6.1.1}$$

*is the equation of a parabola with latus rectum of length 3, vertex*

$$\frac{1}{25}\begin{pmatrix} -29 \\ 25 \end{pmatrix} \tag{6.1.2}$$

*and axis*

$$3x - 4y + 7 = 0 \tag{6.1.3}$$

**Solution:** Comparing (6.1.1) with (5.2.1),

$$\mathbf{V} = \begin{pmatrix} 9 & -12 \\ -12 & 16 \end{pmatrix} \tag{6.1.4}$$

$$\mathbf{u} = -\frac{1}{2}\begin{pmatrix} 18 \\ 101 \end{pmatrix} \tag{6.1.5}$$

$$f = 19 \tag{6.1.6}$$

The eigenvalues of $\mathbf{V}$ are obtained as

$$\left|\lambda\mathbf{I} - \mathbf{V}\right| = 0 \tag{6.1.7}$$

$$\implies \lambda_1 = 0, \lambda_2 = 25 \tag{6.1.8}$$

Since the $\mathbf{V}$ matrix has a 0 eigenvalue, (6.1.1) is a parabola. The eigenvector corresponding to the 0 eigenalue is given by

$$\begin{pmatrix} 9 & -12 \\ -12 & 16 \end{pmatrix} \mathbf{p}_1 = \mathbf{0} \tag{6.1.9}$$

yielding

$$\mathbf{p}_1 = \frac{1}{5}\begin{pmatrix} 4 \\ 3 \end{pmatrix} \tag{6.1.10}$$

Substituting from (6.1.4), (6.1.5), (6.1.6), (6.1.10) and (6.1.8) in (5.3.3), (5.3.4) and (5.7.18) the latus rectum is obtained as

$$\eta = \frac{\left|2\mathbf{u}^\top \mathbf{p}_1\right|}{\lambda_2} = 3 \tag{6.1.11}$$

The vertex of the parabola is obtained from (5.5.2) as

$$\begin{pmatrix} -39 & -73 \\ 9 & -12 \\ -12 & 16 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -19 \\ -21 \\ 28 \end{pmatrix} \tag{6.1.12}$$

yielding (6.1.2). The second eigenvector of $\mathbf{V}$ is orthogonal to $\mathbf{p}_1$ and obtained as

$$\mathbf{p}_2 = \frac{1}{5}\begin{pmatrix} 3 \\ -4 \end{pmatrix} \tag{6.1.13}$$

Substituting from (6.1.2) and (6.1.12) in (5.5.4), the equation of the axis of symmetry for the parabola can be expressed as (6.1.3).

**Example 6.2** (ellipse)**.** *To show that the equation*

$$14x^2 - 4xy + 11y^2 - 44x - 58y + 71 = 0 \tag{6.1.14}$$

*represents an ellipse with centre*

$$\mathbf{c} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \tag{6.1.15}$$

*and lengths of semi-axes*

$$\sqrt{6} \text{ and } 2 \tag{6.1.16}$$

**Solution:** The parameters for (6.1.14), are

$$\mathbf{V} = \begin{pmatrix} 14 & -2 \\ -2 & 11 \end{pmatrix} \tag{6.1.17}$$

$$\mathbf{u} = \begin{pmatrix} 22 \\ 29 \end{pmatrix} \tag{6.1.18}$$

$$f = 71 \tag{6.1.19}$$

Since

$$\left|\mathbf{V}\right| = 150 > 0, \tag{6.1.20}$$

(6.1.14) is an ellipse. Substituting from (6.1.17) and (6.1.18) in (5.5.1), the center of the ellipse is obtained as (6.1.15). Also, the eigenvalues of $\mathbf{V}$ are

$$\lambda_1 = 10, \lambda_2 = 15 \tag{6.1.21}$$

Substituting from (6.1.17), (6.1.18) and (6.1.19) in (5.5.4),

$$f_0 = 60 \tag{6.1.22}$$

Substituting from (6.1.21) and (6.1.22) in (5.7.16) and (5.7.17), the lengths of the semi-axes are obtained as (6.1.16).

**Example 6.3** (hyperbola)**.** *To show that the equation*

$$x^2 - 3xy + y^2 + 10x - 10y + 21 = 0 \tag{6.1.23}$$

represents a hyperbola with centre

$$\mathbf{c} = \begin{pmatrix} -2 \\ 2 \end{pmatrix} \tag{6.1.24}$$

and length of semi-axes

$$\sqrt{2} \text{ and } \sqrt{\frac{2}{5}} \tag{6.1.25}$$

**Solution:** The conic parameters for (6.1.23), are

$$\mathbf{V} = \frac{1}{2}\begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix} \tag{6.1.26}$$

$$\mathbf{u} = \begin{pmatrix} 5 \\ -5 \end{pmatrix} \tag{6.1.27}$$

$$f = 21 \tag{6.1.28}$$

Since

$$\left|\mathbf{V}\right| = -\frac{5}{4} < 0, \tag{6.1.29}$$

(6.1.23) is a hyperbola. Substituting from (6.1.26) and (6.1.26) in (5.5.1), the center of the hyperbola is obtained as (6.1.15). Also, the eigenvalues of $\mathbf{V}$ are

$$\lambda_1 = -\frac{1}{2}, \lambda_2 = \frac{5}{2} \tag{6.1.30}$$

From (5.5.4),

$$f_0 = -1 \tag{6.1.31}$$

Substituting from (6.1.30) and (6.1.31) in (5.7.16) and (5.7.17), the lengths of the semi-axes are then obtained as (6.1.25).

**Example 6.4** (tangents)**.** *To show that the tangents to the curve*

$$x^2 + 4xy + 3y^2 - 5x - 6y + 3 = 0 \tag{6.1.32}$$

*parallel to the line*

$$x + 4y + c = 0 \tag{6.1.33}$$

*are*

$$\begin{aligned} x + 4y - 5 &= 0 \\ x + 4y - 8 &= 0 \end{aligned} \tag{6.1.34}$$

**Solution:** The conic parameters for (6.1.32) are

$$\mathbf{V} = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \tag{6.1.35}$$

$$\mathbf{u} = \begin{pmatrix} -\frac{5}{2} \\ -3 \end{pmatrix} \tag{6.1.36}$$

$$f = 3 \tag{6.1.37}$$

Since

$$\left|\mathbf{V}\right| = -1 < 0, \tag{6.1.38}$$

(6.1.32) is a hyperbola. From (6.1.33), the normal vector to the tangent is

$$\mathbf{n} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \tag{6.1.39}$$

The equation of the tangent can be expressed as

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{q}_i) = 0 \tag{6.1.40}$$

where $\mathbf{q}_i$ are the points of contact. Comparing the above with (6.1.33),

$$c = -\mathbf{n}^\top \mathbf{q}_i \tag{6.1.41}$$

which, upon substituting from (5.8.5) can be expressed as

$$c = -\mathbf{n}^\top \left\{\mathbf{V}^{-1} (\kappa_i \mathbf{n} - \mathbf{u})\right\} \tag{6.1.42}$$

$$= -\mathbf{n}^\top \left\{\mathbf{V}^{-1} \left(\pm\sqrt{\frac{f_0}{\mathbf{n}^\top \mathbf{V}^{-1} \mathbf{n}}}\mathbf{n} - \mathbf{u}\right)\right\} \tag{6.1.43}$$

$$= \pm\sqrt{f_0 \mathbf{n}^\top \mathbf{V}^{-1} \mathbf{n}} + \mathbf{n}^\top \mathbf{V}^{-1} \mathbf{u}. \tag{6.1.44}$$

Substituting from (6.1.35), (6.1.36) and (6.1.37) in (5.3.3),

$$\mathbf{f}_0 = -\frac{3}{4} \tag{6.1.45}$$

Substituing from (6.1.35), (6.1.36), (6.1.39) and (6.1.45) in (5.8.9),

$$c = -5 \text{ or } c = -8 \tag{6.1.46}$$

yielding (6.1.34).

### *6.2 Miscellaneous*

6.2.1. Given unit basis vectors $\mathbf{a}, \mathbf{b}$, with angle $\theta$ between them, the locus of the coordinates of a unit vector $\mathbf{c}$ in the space spanned by $\mathbf{a}, \mathbf{b}$ is given by

$$\mathbf{x}^\top \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \mathbf{x} = 1 \tag{6.2.1.1}$$

with $\rho = \cos\theta$.
**Solution:** Let

$$\mathbf{c} = x_1 \mathbf{a} + x_2 \mathbf{b} = \begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix} \mathbf{x} \tag{6.2.1.2}$$

Then,

$$\|\mathbf{c}\|^2 = \mathbf{x}^\top \begin{pmatrix} \mathbf{a}^\top \\ \mathbf{b}^\top \end{pmatrix} \begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix} \mathbf{x} \tag{6.2.1.3}$$

$$= \mathbf{x}^\top \begin{pmatrix} 1 & \mathbf{a}^\top\mathbf{b} \\ \mathbf{a}^\top\mathbf{b} & 1 \end{pmatrix} \mathbf{x} \tag{6.2.1.4}$$

which can be expressed as (6.2.1.1).

6.2.2. Given the coordinates of $\mathbf{c}$, the angle $\theta$ between the basis vectors is given by

$$\rho = \frac{1 - \|\mathbf{x}\|^2}{\mathbf{x}^\top \mathbf{R} \mathbf{x}} \tag{6.2.2.1}$$

where

$$\mathbf{R} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{6.2.2.2}$$

**Solution:** Let

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{6.2.2.3}$$

For

$$\mathbf{V} = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}, \mathbf{u} = 0, f = -1 \tag{6.2.2.4}$$

in (5.2.1),
Since

$$\left|\mathbf{V}\right| = 1 - \rho^2, 0 < \left|\mathbf{V}\right| < 1, \tag{6.2.2.5}$$

(6.2.1.1) represents the equation of an ellipse. Using eigenvalue decomposition,

$$\mathbf{V} = \mathbf{P}^\top \mathbf{D} \mathbf{P} \tag{6.2.2.6}$$

where

$$\mathbf{P} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 1+\rho & 0 \\ 0 & 1-\rho \end{pmatrix} \tag{6.2.2.7}$$

Using the affine transformation,

$$\mathbf{x} = \mathbf{P}\mathbf{y} \tag{6.2.2.8}$$

(6.2.1.1) can be expressed as

$$\mathbf{y}^\top \mathbf{D} \mathbf{y} = 1 \implies y_1^2 (1+\rho) + y_2^2 (1-\rho) \qquad = 1 \tag{6.2.2.9}$$

which can be simplified to obtain

$$\rho = \frac{1 - y_1^2 - y_2^2}{y_1^2 - y_2^2} \tag{6.2.2.10}$$

$$= \frac{1 - \|\mathbf{y}\|^2}{\mathbf{y}^\top \mathbf{Q} \mathbf{y}} \tag{6.2.2.11}$$

where

$$\mathbf{Q} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{6.2.2.12}$$

From (6.2.2.8), (6.2.2.11) can be expressed as (6.2.2.1)

$$\because \|\mathbf{a}\| = \|\mathbf{b}\| = 1. \tag{6.2.2.13}$$

# APPENDIX A

From (5.2.2), using the fact that $\mathbf{V}$ is symmetric with $\mathbf{V} = \mathbf{V}^\top$,

$$\mathbf{V}^\top\mathbf{V} = \left(\|\mathbf{n}\|^2 \mathbf{I} - e^2\mathbf{n}\mathbf{n}^\top\right)^\top \left(\|\mathbf{n}\|^2 \mathbf{I} - e^2\mathbf{n}\mathbf{n}^\top\right) \tag{1.0.2.1}$$

$$\implies \mathbf{V}^2 = \|\mathbf{n}\|^4 \mathbf{I} + e^4\mathbf{n}\mathbf{n}^\top\mathbf{n}\mathbf{n}^\top - 2e^2 \|\mathbf{n}\|^2 \mathbf{n}\mathbf{n}^\top \tag{1.0.2.2}$$

$$= \|\mathbf{n}\|^4 \mathbf{I} + e^4 \|\mathbf{n}\|^2 \mathbf{n}\mathbf{n}^\top - 2e^2 \|\mathbf{n}\|^2 \mathbf{n}\mathbf{n}^\top \tag{1.0.2.3}$$

$$= \|\mathbf{n}\|^4 \mathbf{I} + e^2 \left(e^2 - 2\right) \|\mathbf{n}\|^2 \mathbf{n}\mathbf{n}^\top \tag{1.0.2.4}$$

$$= \|\mathbf{n}\|^4 \mathbf{I} + \left(e^2 - 2\right) \|\mathbf{n}\|^2 \left(\|\mathbf{n}\|^2 \mathbf{I} - \mathbf{V}\right) \tag{1.0.2.5}$$

which can be expressed as

$$\mathbf{V}^2 + \left(e^2 - 2\right) \|\mathbf{n}\|^2 \mathbf{V} - \left(e^2 - 1\right) \|\mathbf{n}\|^4 \mathbf{I} = 0 \tag{1.0.2.6}$$

Using the Cayley-Hamilton theorem, (1.0.2.6) results in the characteristic equation,

$$\lambda^2 - \left(2 - e^2\right) \|\mathbf{n}\|^2 \lambda + \left(1 - e^2\right) \|\mathbf{n}\|^4 = 0 \tag{1.0.2.7}$$

which can be expressed as

$$\left(\frac{\lambda}{\|\mathbf{n}\|^2}\right)^2 - \left(2 - e^2\right)\left(\frac{\lambda}{\|\mathbf{n}\|^2}\right) + \left(1 - e^2\right) = 0 \tag{1.0.2.8}$$

$$\implies \frac{\lambda}{\|\mathbf{n}\|^2} = 1 - e^2, 1 \tag{1.0.2.9}$$

$$\text{or, } \lambda_2 = \|\mathbf{n}\|^2, \lambda_1 = \left(1 - e^2\right)\lambda_2 \tag{1.0.2.10}$$

From (1.0.2.10), the eccentricity of (5.2.1) is given by (5.2.9). Multiplying both sides of (5.2.2) by $\mathbf{n}$,

$$\mathbf{V}\mathbf{n} = \|\mathbf{n}\|^2 \mathbf{n} - e^2\mathbf{n}\mathbf{n}^\top\mathbf{n} \tag{1.0.2.11}$$

$$= \|\mathbf{n}\|^2 \left(1 - e^2\right) \mathbf{n} \tag{1.0.2.12}$$

$$= \lambda_1 \mathbf{n} \tag{1.0.2.13}$$

$$\tag{1.0.2.14}$$

from (1.0.2.10). Thus, $\lambda_1$ is the corresponding eigenvalue for $\mathbf{n}$. From (5.1.4) and (1.0.2.14), this implies that

$$\mathbf{p}_1 = \frac{\mathbf{n}}{\|\mathbf{n}\|} \tag{1.0.2.15}$$

$$\text{or, } \mathbf{n} = \|\mathbf{n}\| \mathbf{p}_1 = \sqrt{\lambda_2}\mathbf{p}_1 \tag{1.0.2.16}$$

from (1.0.2.10) . From (5.2.3) and (1.0.2.10),

$$\mathbf{F} = \frac{ce^2\mathbf{n} - \mathbf{u}}{\lambda_2} \tag{1.0.2.17}$$

$$\implies \|\mathbf{F}\|^2 = \frac{\left(ce^2\mathbf{n} - \mathbf{u}\right)^\top \left(ce^2\mathbf{n} - \mathbf{u}\right)}{\lambda_2^2} \tag{1.0.2.18}$$

$$\implies \lambda_2^2 \|\mathbf{F}\|^2 = c^2e^4\lambda_2 - 2ce^2\mathbf{u}^\top\mathbf{n} + \|\mathbf{u}\|^2 \tag{1.0.2.19}$$

Also, (5.2.4) can be expressed as

$$\lambda_2 \|\mathbf{F}\|^2 = f + c^2e^2 \tag{1.0.2.20}$$

From (1.0.2.19) and (1.0.2.20),

$$c^2e^4\lambda_2 - 2ce^2\mathbf{u}^\top\mathbf{n} + \|\mathbf{u}\|^2 = \lambda_2\left(f + c^2e^2\right) \tag{1.0.2.21}$$

$$\implies \lambda_2e^2\left(e^2-1\right)c^2 - 2ce^2\mathbf{u}^\top\mathbf{n} + \|\mathbf{u}\|^2 - \lambda_2 f = 0 \tag{1.0.2.22}$$

yielding (5.2.11).

## Appendix B

Using (5.1.1) (5.2.1) can be expressed as

$$\left(\mathbf{P}\mathbf{y}+\mathbf{c}\right)^\top\mathbf{V}\left(\mathbf{P}\mathbf{y}+\mathbf{c}\right) + 2\mathbf{u}^\top\left(\mathbf{P}\mathbf{y}+\mathbf{c}\right) + f = 0, \tag{2.0.2.1}$$

yielding

$$\mathbf{y}^\top\mathbf{P}^\top\mathbf{V}\mathbf{P}\mathbf{y} + 2\left(\mathbf{V}\mathbf{c}+\mathbf{u}\right)^\top\mathbf{P}\mathbf{y} + \mathbf{c}^\top\mathbf{V}\mathbf{c} + 2\mathbf{u}^\top\mathbf{c} + f = 0 \tag{2.0.2.2}$$

From (2.0.2.2) and (5.1.2),

$$\mathbf{y}^\top\mathbf{D}\mathbf{y} + 2\left(\mathbf{V}\mathbf{c}+\mathbf{u}\right)^\top\mathbf{P}\mathbf{y} + \mathbf{c}^\top\left(\mathbf{V}\mathbf{c}+\mathbf{u}\right) + \mathbf{u}^\top\mathbf{c} + f = 0 \tag{2.0.2.3}$$

When $\mathbf{V}^{-1}$ exists, choosing

$$\mathbf{V}\mathbf{c}+\mathbf{u} = \mathbf{0}, \quad \text{or}, \mathbf{c} = -\mathbf{V}^{-1}\mathbf{u}, \tag{2.0.2.4}$$

and substituting (2.0.2.4) in (2.0.2.3) yields (5.3.1).

### *B.1*

When $\left|\mathbf{V}\right| = 0, \lambda_1 = 0$ and

$$\mathbf{V}\mathbf{p}_1 = 0, \mathbf{V}\mathbf{p}_2 = \lambda_2\mathbf{p}_2. \tag{2.1.2.1}$$

where $\mathbf{p}_1, \mathbf{p}_2$ are the eigenvectors of $\mathbf{V}$ such that (5.1.2)

$$\mathbf{P} = \begin{pmatrix}\mathbf{p}_1 & \mathbf{p}_2\end{pmatrix}, \tag{2.1.2.2}$$

Substituting (2.1.2.2) in (2.0.2.3),

$$\mathbf{y}^\top\mathbf{D}\mathbf{y} + 2\left(\mathbf{c}^\top\mathbf{V}+\mathbf{u}^\top\right)\begin{pmatrix}\mathbf{p}_1 & \mathbf{p}_2\end{pmatrix}\mathbf{y} + \mathbf{c}^\top\left(\mathbf{V}\mathbf{c}+\mathbf{u}\right) + \mathbf{u}^\top\mathbf{c} + f = 0 \tag{2.1.2.3}$$

$$\implies \mathbf{y}^\top\mathbf{D}\mathbf{y} + 2\begin{pmatrix}\left(\mathbf{c}^\top\mathbf{V}+\mathbf{u}^\top\right)\mathbf{p}_1 & \left(\mathbf{c}^\top\mathbf{V}+\mathbf{u}^\top\right)\mathbf{p}_2\end{pmatrix}\mathbf{y} + \mathbf{c}^\top\left(\mathbf{V}\mathbf{c}+\mathbf{u}\right) + \mathbf{u}^\top\mathbf{c} + f = 0 \tag{2.1.2.4}$$

$$\implies \mathbf{y}^\top\mathbf{D}\mathbf{y} + 2\begin{pmatrix}\mathbf{u}^\top\mathbf{p}_1 & \left(\lambda_2\mathbf{c}^\top+\mathbf{u}^\top\right)\mathbf{p}_2\end{pmatrix}\mathbf{y} + \mathbf{c}^\top\left(\mathbf{V}\mathbf{c}+\mathbf{u}\right) + \mathbf{u}^\top\mathbf{c} + f = 0 \tag{2.1.2.5}$$

upon substituting from (2.1.2.1) yielding

$$\lambda_2 y_2^2 + 2\left(\mathbf{u}^\top\mathbf{p}_1\right)y_1 + 2y_2\left(\lambda_2\mathbf{c}+\mathbf{u}\right)^\top\mathbf{p}_2 + \mathbf{c}^\top\left(\mathbf{V}\mathbf{c}+\mathbf{u}\right) + \mathbf{u}^\top\mathbf{c} + f = 0 \tag{2.1.2.6}$$

Thus, (2.1.2.6) can be expressed as (5.3.2) by choosing

$$\eta = 2\mathbf{u}^\top\mathbf{p}_1 \tag{2.1.2.7}$$

and $\mathbf{c}$ in (2.0.2.3) such that

$$2\mathbf{P}^\top\left(\mathbf{V}\mathbf{c}+\mathbf{u}\right) = \eta\begin{pmatrix}1\\0\end{pmatrix} \tag{2.1.2.8}$$

$$\mathbf{c}^\top\left(\mathbf{V}\mathbf{c}+\mathbf{u}\right) + \mathbf{u}^\top\mathbf{c} + f = 0 \tag{2.1.2.9}$$

$\because \mathbf{P}^\top\mathbf{P} = \mathbf{I}$, multiplying (2.1.2.8) by $\mathbf{P}$ yields

$$\left(\mathbf{V}\mathbf{c}+\mathbf{u}\right) = \frac{\eta}{2}\mathbf{p}_1, \tag{2.1.2.10}$$

which, upon substituting in (2.1.2.9) results in

$$\frac{\eta}{2}\mathbf{c}^\top\mathbf{p}_1 + \mathbf{u}^\top\mathbf{c} + f = 0 \tag{2.1.2.11}$$

(2.1.2.10) and (2.1.2.11) can be clubbed together to obtain (5.5.2).

## Appendix C

a) For the standard hyperbola/ellipse in (5.3.1), from (5.3.6), (5.2.10) and (5.3.7),

$$\mathbf{n} = \sqrt{\frac{\lambda_2}{f_0}}\mathbf{e}_1 \tag{3.0.2.1}$$

$$c = \pm\frac{\sqrt{-\frac{\lambda_2}{f_0}\left(e^2-1\right)\left(\frac{\lambda_2}{f_0}\right)}}{\frac{\lambda_2}{f_0}e\left(e^2-1\right)} \tag{3.0.2.2}$$

$$= \pm\frac{1}{e\sqrt{1-e^2}} \tag{3.0.2.3}$$

yielding (5.3.1) upon substituting from (5.2.9) and simplifying. For the standard parabola in (5.3.2), from (5.3.6), (5.2.10) and (5.3.7), noting that $f = 0$,

$$\mathbf{n} = \sqrt{\lambda_2}\mathbf{e}_1 \tag{3.0.2.4}$$

$$c = \frac{\left\|\frac{\eta}{2}\mathbf{e}_1\right\|^2}{2\left(\frac{\eta}{\mathbf{2}}\right)\left(\mathbf{e_1}\right)^\top\mathbf{n}} \tag{3.0.2.5}$$

$$\tag{3.0.2.6}$$

$$= \frac{\eta}{4\sqrt{\lambda_2}} \tag{3.0.2.7}$$

yielding (5.3.2).

b) For the standard ellipse/hyperbola, substituting from (3.0.2.3), (3.0.2.1), (5.3.7) and (5.2.9) in (5.2.11),

$$\mathbf{F} = \pm\frac{\left(\frac{1}{e\sqrt{1-e^2}}\right)\left(e^2\right)\sqrt{\frac{\lambda_2}{f_0}}\mathbf{e}_1}{\frac{\lambda_2}{f_0}} \tag{3.0.2.8}$$

yielding (5.3.1) after simplification. For the standard parabola, substituting from (3.0.2.7), (3.0.2.4), (5.3.7) and (5.2.9) in (5.2.11),

$$\mathbf{F} = \frac{\left(\frac{\eta}{4\sqrt{\lambda_2}}\right)\sqrt{\lambda_2}\mathbf{e}_1 - \frac{\eta}{\mathbf{2}}\mathbf{e_1}}{\lambda_2} \tag{3.0.2.9}$$

$$\tag{3.0.2.10}$$

yielding (5.3.1) after simplification.

## Appendix D

Since the major axis passes through the origin,

$$\mathbf{q} = \mathbf{0} \tag{4.0.2.1}$$

Further, from Corollary (5.8),

$$\mathbf{m} = \mathbf{e}_2, \tag{4.0.2.2}$$

and from (5.3.1),

$$\mathbf{V} = \frac{\mathbf{D}}{f_0}, \mathbf{u} = 0, f = -1 \tag{4.0.2.3}$$

Substituting the above in (5.7.9),

$$\frac{2\sqrt{\mathbf{e}_1^\top \frac{\mathbf{D}}{f_0}\mathbf{e}_1}}{\mathbf{e}_1^\top \frac{\mathbf{D}}{f_0}\mathbf{e}_1}\|\mathbf{e}_1\| \tag{4.0.2.4}$$

yielding (5.7.16). Similarly, for the minor axis, the only different parameter is

$$\mathbf{m} = \mathbf{e}_2, \tag{4.0.2.5}$$

Substituting the above in (5.7.9),

$$\frac{2\sqrt{\mathbf{e}_2^\top \frac{\mathbf{D}}{f_0}\mathbf{e}_2}}{\mathbf{e}_2^\top \frac{\mathbf{D}}{f_0}\mathbf{e}_2}\|\mathbf{e}_2\| \tag{4.0.2.6}$$

yielding (5.7.17).

## APPENDIX E

The latus rectum is perpendicular to the major axis for the standard conic. Hence, from Corollary (5.8),

$$\mathbf{m} = \mathbf{e}_2, \tag{5.0.2.1}$$

Since it passes through the focus, from (5.3.1)

$$\mathbf{q} = \mathbf{F} = \pm e\sqrt{\frac{f_0}{\lambda_2\left(1-e^2\right)}}\mathbf{e}_1 \tag{5.0.2.2}$$

for the standard hyperbola/ellipse. Also, from (5.3.1),

$$\mathbf{V} = \frac{\mathbf{D}}{f_0}, \mathbf{u} = 0, f = -1 \tag{5.0.2.3}$$

Substituting the above in (5.7.9),

$$\frac{2\sqrt{\left[\mathbf{e}_2^\top\left(\frac{\mathbf{D}}{f_0}e\sqrt{\frac{f_0}{\lambda_2(1-e^2)}}\mathbf{e}_1\right)\right]^2 - \left(e\sqrt{\frac{f_0}{\lambda_2(1-e^2)}}\mathbf{e}_1^\top\frac{\mathbf{D}}{f_0}e\sqrt{\frac{f_0}{\lambda_2(1-e^2)}}\mathbf{e}_1 - 1\right)\left(\mathbf{e}_2^\top\frac{\mathbf{D}}{f_0}\mathbf{e}_2\right)}}{\mathbf{e}_2^\top\frac{\mathbf{D}}{f_0}\mathbf{e}_2}\|\mathbf{e}_2\| \tag{5.0.2.4}$$

Since

$$\mathbf{e}_2^\top\mathbf{D}\mathbf{e}_1 = 0, \mathbf{e}_1^\top\mathbf{D}\mathbf{e}_1 = \lambda_1, \mathbf{e}_1^\top\mathbf{e}_1 = 1, \|\mathbf{e}_2\| = 1, \mathbf{e}_2^\top\mathbf{D}\mathbf{e}_2 = \lambda_2, \tag{5.0.2.5}$$

(5.0.2.4) can be expressed as

$$\frac{2\sqrt{\left(1-\frac{\lambda_1 e^2}{\lambda_2(1-e^2)}\right)\left(\frac{\lambda_2}{f_0}\right)}}{\frac{\lambda_2}{f_0}} \tag{5.0.2.6}$$

$$= 2\frac{\sqrt{f_0\lambda_1}}{\lambda_2} \qquad \left(\because e^2 = 1-\frac{\lambda_1}{\lambda_2}\right) \tag{5.0.2.7}$$

For the standard parabola, the parameters in (5.7.9) are

$$\mathbf{q} = \mathbf{F} = -\frac{\eta}{4\lambda_2}\mathbf{e}_1, \mathbf{m} = \mathbf{e}_1, \mathbf{V} = \mathbf{D}, \mathbf{u} = \frac{\eta}{2}\mathbf{e}_1^\top, f = 0 \tag{5.0.2.8}$$

Substituting the above in (5.7.9), the length of the latus rectum can be expressed as

$$2\frac{\sqrt{\left[\mathbf{e}_2^\top\left(\mathbf{D}\left(-\frac{\eta}{4\lambda_2}\mathbf{e}_1\right)+\frac{\eta}{2}\mathbf{e}_1\right)\right]^2-\left(\left(-\frac{\eta}{4\lambda_2}\mathbf{e}_1\right)^\top\mathbf{D}\left(-\frac{\eta}{4\lambda_2}\mathbf{e}_1\right)+2\frac{\eta}{2}\mathbf{e}_1^\top\left(-\frac{\eta}{4\lambda_2}\mathbf{e}_1\right)\right)\left(\mathbf{e}_2^\top\mathbf{D}\mathbf{e}_2\right)}}{\mathbf{e}_2^\top\mathbf{D}\mathbf{e}_2}\left\|\mathbf{e}_2\right\| \tag{5.0.2.9}$$

Since

$$\mathbf{e}_2^\top\mathbf{D}\mathbf{e}_1=0,\mathbf{e}_2^\top\mathbf{e}_2=0,\mathbf{e}_1^\top\mathbf{D}\mathbf{e}_1=0,\mathbf{e}_1^\top\mathbf{e}_1=1,\left\|\mathbf{e}_1\right\|=1,\mathbf{e}_2^\top\mathbf{D}\mathbf{e}_2=\lambda_2, \tag{5.0.2.10}$$

(5.0.2.9) can be expressed as

$$2\frac{\sqrt{\frac{\eta^2}{4\lambda_2}\lambda_2}}{\lambda_2}=\frac{\eta}{\lambda_2} \tag{5.0.2.11}$$