MATRIX ANALYSIS

Through Coordinate Geometry

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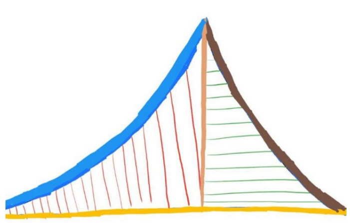

Contents

Introduction

This book links high school coordinate geometry to linear algebra and matrix analysis through solved problems.

Chapter 1

Quadrilaterals

1.1. Properties

1. The angles of quadrilateral are in the ratio 3:5:9:13. Find all the angles of the quadrilateral.

2. If diagonals of a parallelogram are equal then show that it is a rectangle.

   Solution: See Fig. 1.1. From (A.1.18),

$$\mathbf{B} - \mathbf{A} = \mathbf{C} - \mathbf{D} \tag{1.1}$$

$$\implies \mathbf{B} - \mathbf{C} = \mathbf{A} - \mathbf{D} \tag{1.2}$$

   Also, it is given that the diagonals of $ABCD$ are equal. Hence,

$$\|\mathbf{C} - \mathbf{A}\|^2 = \|\mathbf{D} - \mathbf{B}\|^2 \tag{1.3}$$

$$\implies \|(\mathbf{C} - \mathbf{B}) + (\mathbf{B} - \mathbf{A})\|^2 = \|(\mathbf{D} - \mathbf{C}) + (\mathbf{C} - \mathbf{B}\|^2 \tag{1.4}$$

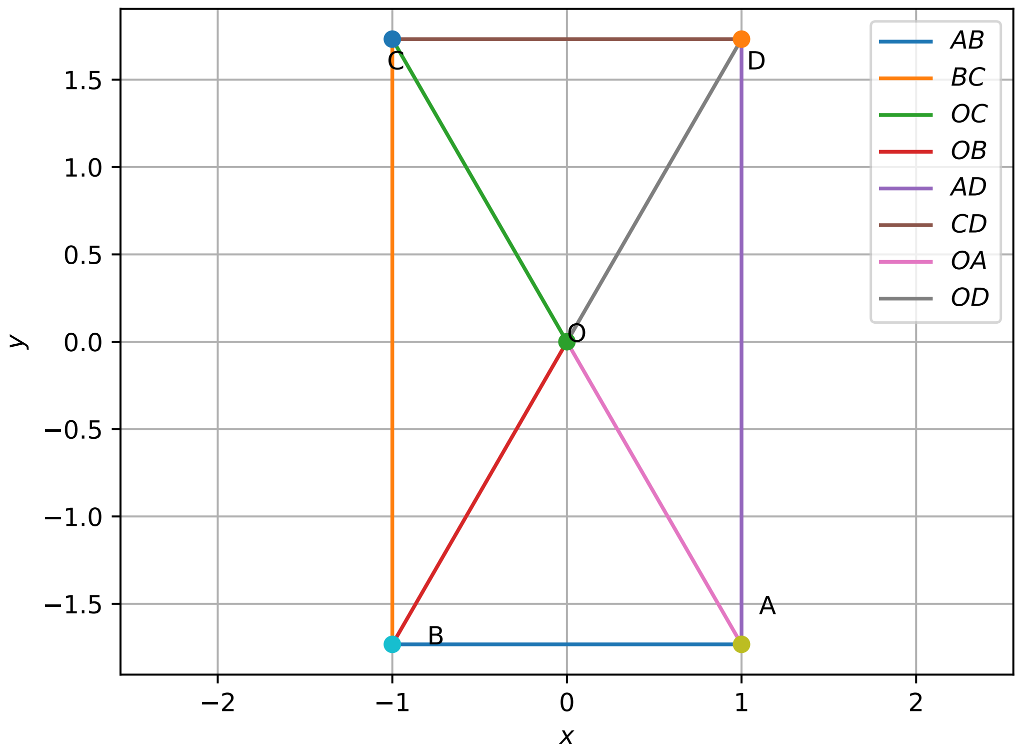

Figure 1.1:

which can be expressed as

$$\|\mathbf{C}-\mathbf{B}\|^2+\|\mathbf{B}-\mathbf{A}\|^2+2(\mathbf{C}-\mathbf{B})^\top(\mathbf{B}-\mathbf{A}) \\ =\|\mathbf{D}-\mathbf{C}\|^2+\|\mathbf{C}-\mathbf{B}\|^2+2(\mathbf{D}-\mathbf{C})^\top(\mathbf{C}-\mathbf{B}) \quad (1.5)$$

which, can be simplified to obtain

$$(\mathbf{C}-\mathbf{B})^\top(\mathbf{B}-\mathbf{A})=(\mathbf{D}-\mathbf{C})^\top(\mathbf{C}-\mathbf{B}) \quad (1.6)$$

since

$$\|\mathbf{D} - \mathbf{C}\| = \|\mathbf{B} - \mathbf{A}\| \tag{1.7}$$

yielding

$$(\mathbf{A} - \mathbf{B})^\top (\mathbf{B} - \mathbf{C}) = \mathbf{0} \tag{1.8}$$

from (1.1).

3. Show that if the diagonals of a quadrilateral bisect each other at right angles, then it is a rhombus.

   Solution: See Fig. 1.2. From the given information,

$$\frac{\mathbf{B} + \mathbf{D}}{2} = \frac{\mathbf{A} + \mathbf{C}}{2} \tag{1.9}$$

$$(\mathbf{B} - \mathbf{D})^\top (\mathbf{A} - \mathbf{C}) = 0 \tag{1.10}$$

   From (1.9),

$$\mathbf{B} - \mathbf{A} = \mathbf{C} - \mathbf{D} \tag{1.11}$$

   which, from (A.1.18), is the definition of a parallelogram. Further, substituting

$$\mathbf{B} - \mathbf{D} = (\mathbf{B} - \mathbf{A}) + (\mathbf{A} - \mathbf{D}) \tag{1.12}$$

$$\mathbf{A} - \mathbf{C} = (\mathbf{A} - \mathbf{B}) + (\mathbf{B} - \mathbf{C}) \tag{1.13}$$

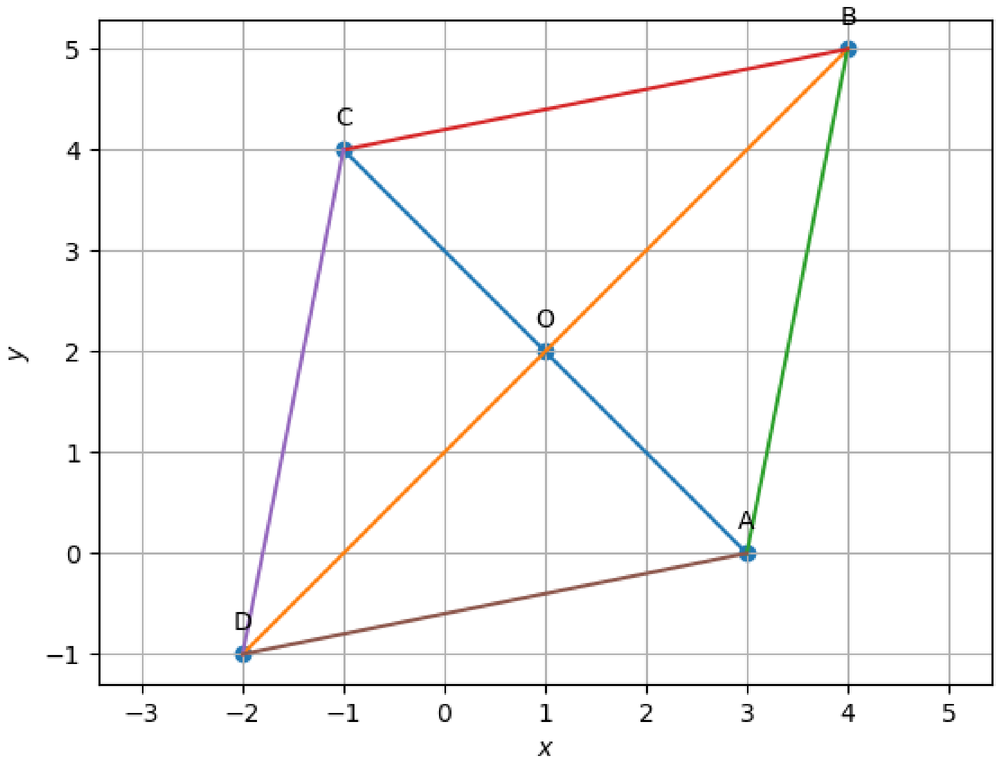

Figure 1.2: Rhombus

in (1.10),

$$
\begin{aligned}
&\left[(\mathbf{B}-\mathbf{A})+(\mathbf{A}-\mathbf{D})\right]^{\top}\left[(\mathbf{A}-\mathbf{B})+(\mathbf{B}-\mathbf{C})\right]=0\\
&\qquad\qquad\qquad\implies -\left\|\mathbf{B}-\mathbf{A}\right\|^{2}+(\mathbf{B}-\mathbf{A})^{\top}(\mathbf{B}-\mathbf{C})+\\
&\qquad\qquad\qquad\qquad (\mathbf{A}-\mathbf{D})^{\top}(\mathbf{A}-\mathbf{B})+(\mathbf{A}-\mathbf{D})^{\top}(\mathbf{B}-\mathbf{C})=0 \quad (1.14)
\end{aligned}
$$

From (1.11),

$$\mathbf{B} - \mathbf{C} = \mathbf{A} - \mathbf{D} \tag{1.15}$$

$$\implies (\mathbf{B} - \mathbf{A})^\top (\mathbf{B} - \mathbf{C}) + (\mathbf{A} - \mathbf{D})^\top (\mathbf{A} - \mathbf{B}) = \mathbf{0} \tag{1.16}$$

and

$$(\mathbf{A} - \mathbf{D})^\top (\mathbf{B} - \mathbf{C}) = \|\mathbf{B} - \mathbf{C}\|^2 \tag{1.17}$$

Substituting from

(1.16) and (1.17) in (1.14),

$$\|\mathbf{A} - \mathbf{B}\|^2 = \|\mathbf{B} - \mathbf{C}\|^2 \tag{1.18}$$

which means that the adjacent sides of the parallelogram are equal. Thus, the quadrilateral is a rhombus

4. Show that the diagonals of a square are equal and bisect each other at right angles.
   Solution: This is obvious from Problems (2) and (3).

5.

6. Diagonal AC of a parallelogram ABCD bisects $\angle A$ in Fig (1.3). Show that

   (a) it bisects $\angle C$ also

   (b) $ABCD$ is a rhombus

   Solution:

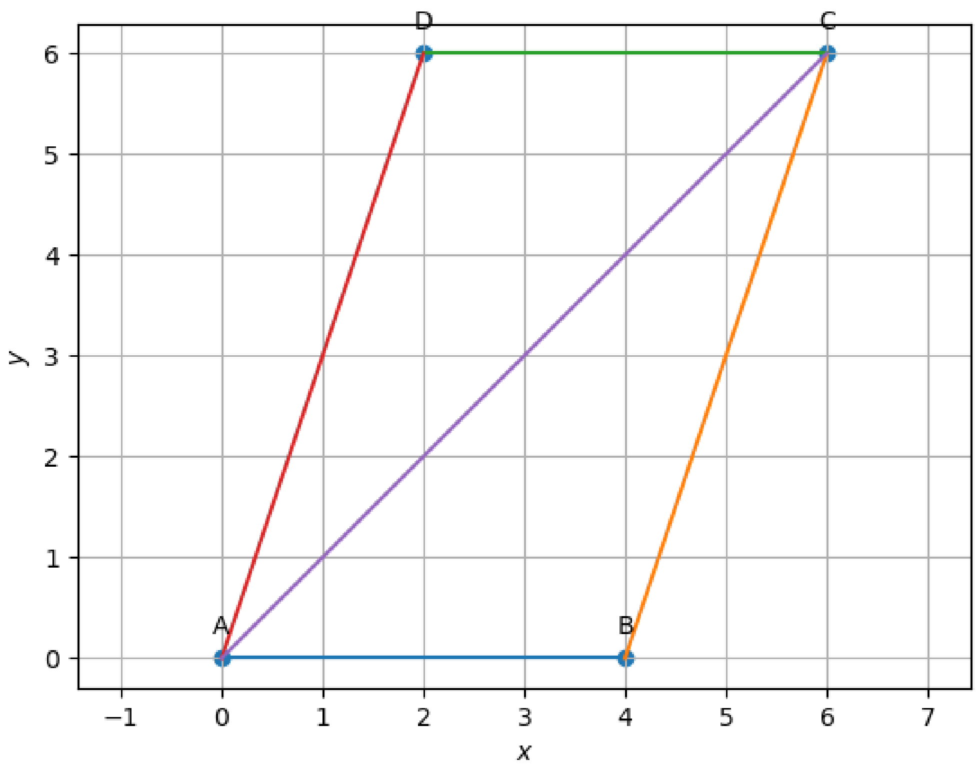

Figure 1.3:

(a) From (A.23),

$$\angle BAC = \angle DAC \tag{1.19}$$

$$\implies \frac{(\mathbf{A}-\mathbf{B})^T(\mathbf{A}-\mathbf{C})}{\|\mathbf{A}-\mathbf{B}\|\,\|\mathbf{A}-\mathbf{C}\|} = \frac{(\mathbf{A}-\mathbf{D})^T(\mathbf{A}-\mathbf{C})}{\|\mathbf{A}-\mathbf{D}\|\,\|\mathbf{A}-\mathbf{C}\|} \tag{1.20}$$

Also,

$$\cos\angle ACD = \frac{(\mathbf{C}-\mathbf{D})^T(\mathbf{C}-\mathbf{A})}{\|\mathbf{C}-\mathbf{D}\|\,\|\mathbf{C}-\mathbf{A}\|} \tag{1.21}$$

From Appendix A.1.18,

$$\mathbf{B}-\mathbf{A}=\mathbf{C}-\mathbf{D} \tag{1.22}$$

$$\implies \frac{(\mathbf{C}-\mathbf{D})^T(\mathbf{C}-\mathbf{A})}{\|\mathbf{C}-\mathbf{D}\|\,\|\mathbf{C}-\mathbf{A}\|}=\frac{(\mathbf{B}-\mathbf{A})^T(\mathbf{C}-\mathbf{A})}{\|\mathbf{B}-\mathbf{A}\|\,\|\mathbf{C}-\mathbf{A}\|} \tag{1.23}$$

upon substituting in (1.21). Thus, from (1.21) and (1.19),

$$\angle BAC=\angle DAC=\angle ACD \tag{1.24}$$

Similarly, it can be shown that

$$\angle ACD=\angle ACB \tag{1.25}$$

(b)

7. $ABCD$ is a rhombus. Show that the diagonal AC bisects angle A as well as angle C and diagonal BD bisects angle B as well as angle D.

Solution: For the rhombus in Fig. 1.4,

$$\begin{aligned}\|\mathbf{A}-\mathbf{B}\| &= \|\mathbf{A}-\mathbf{D}\|\\ \mathbf{A}-\mathbf{B} &= \mathbf{D}-\mathbf{C}\end{aligned} \tag{1.26}$$

From (A.23),

$$\begin{aligned}\cos\angle BAC &= \frac{(\mathbf{A}-\mathbf{B})^T(\mathbf{A}-\mathbf{C})}{\|\mathbf{A}-\mathbf{B}\|\,\|\mathbf{A}-\mathbf{C}\|}\\ \cos\angle DAC &= \frac{(\mathbf{C}-\mathbf{D})^T(\mathbf{C}-\mathbf{A})}{\|\mathbf{C}-\mathbf{D}\|\,\|\mathbf{C}-\mathbf{A}\|}\end{aligned} \tag{1.27}$$

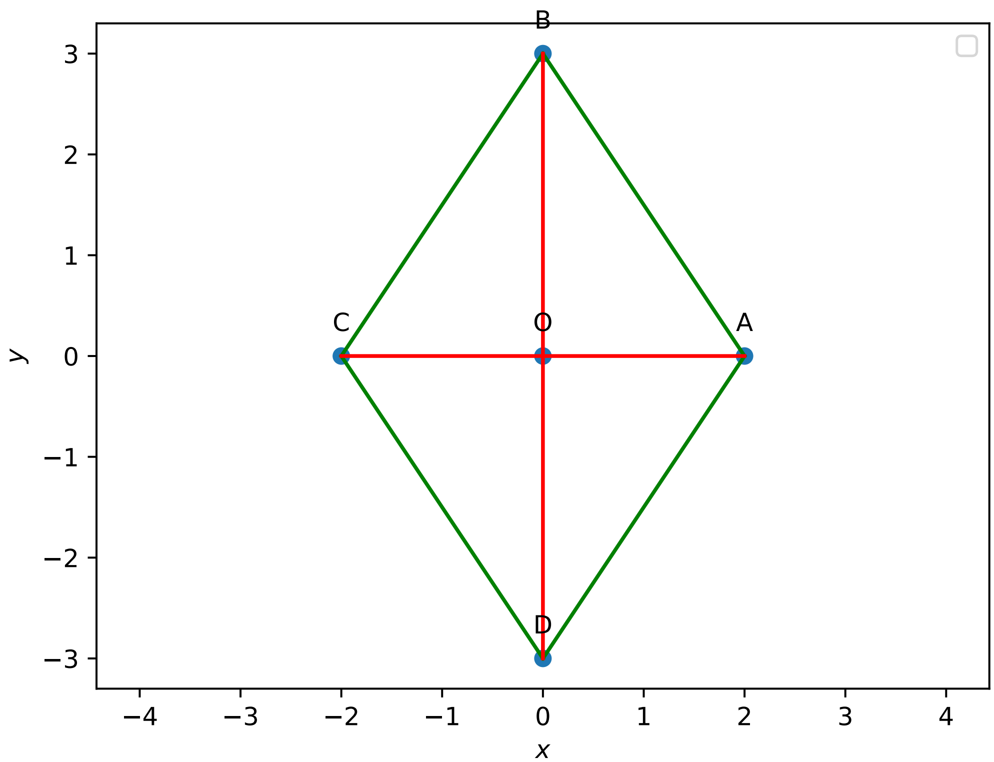

Figure 1.4:

From (1.26) and (1.27), we obtain

$$\cos \angle BAC = \cos \angle DAC \tag{1.28}$$

Thus, AC bisects $\angle A$. Similarly, the remaining results can be proved.

8.

9. In parallelogram $ABCD$, two points $\mathbf{P}$ and $\mathbf{Q}$ are taken on diagonal BD such that $DP = BQ$. Show that

   (a) $\Delta APD \cong \Delta CQB$

   (b) $AP = CQ$

   (c) $\Delta AQB \cong \Delta CPD$

   (d) $AQ = CP$

   (e) $APCQ$ is a parallelogram

   Solution: See Fig. 1.5.

   From (A.23) and the given information,

$$\mathbf{A} - \mathbf{B} = \mathbf{D} - \mathbf{C} \tag{1.29}$$

$$\implies \mathbf{A} - \mathbf{D} = \mathbf{B} - \mathbf{C} \tag{1.30}$$

$$\mathbf{B} - \mathbf{Q} = \mathbf{P} - \mathbf{D} \tag{1.31}$$

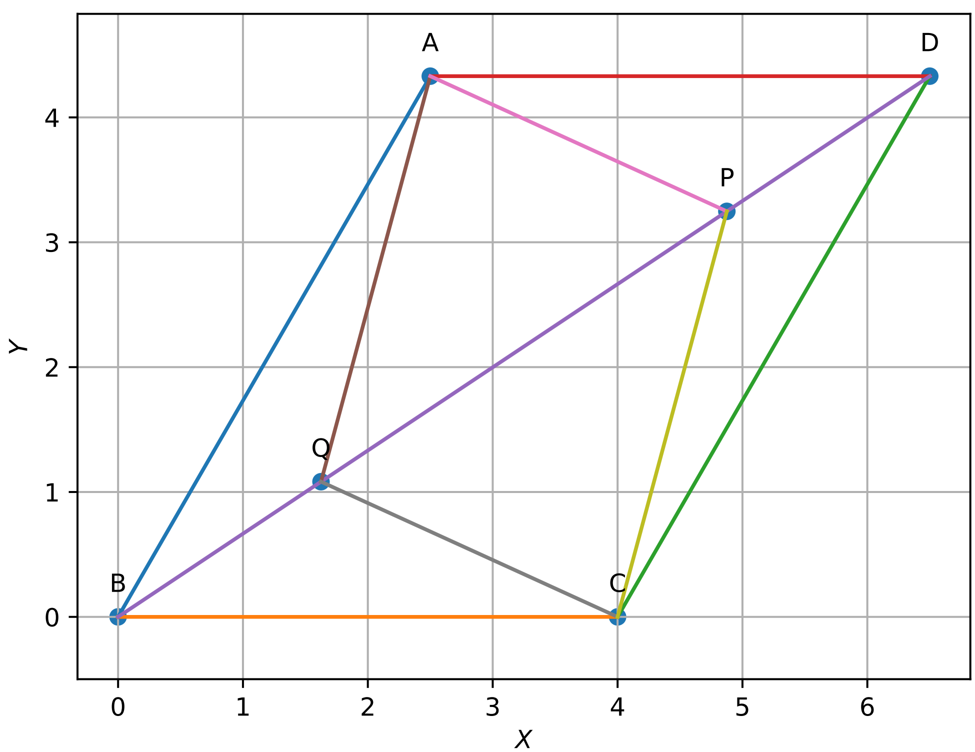

Figure 1.5:

From (1) and (3):

$$\begin{aligned}\mathbf{A}+\mathbf{C}&=\mathbf{B}+\mathbf{D}\\ \mathbf{P}+\mathbf{Q}&=\mathbf{B}+\mathbf{D}\end{aligned} \tag{1.32}$$

From (4)

$$\mathbf{A}+\mathbf{C}=\mathbf{P}+\mathbf{Q} \tag{1.33}$$

From (5)

$$\implies \mathbf{A}-\mathbf{Q}=\mathbf{P}-\mathbf{C} \tag{1.34}$$

$$\mathbf{A}-\mathbf{P}=\mathbf{Q}-\mathbf{C} \tag{1.35}$$

(a) From (2),(3) and (7)

$$\Delta APD \cong \Delta CQB \tag{1.36}$$

(b)

$$Equation(7) \implies AP=CQ \tag{1.37}$$

(c) From (1),(3) and (6)

$$\Delta AQB \cong \Delta CPD \tag{1.38}$$

(d)

$$Equation(6) \implies AQ = CP \tag{1.39}$$

(e) Equation (6) and (7) $\implies$ Quadrilateral APCQ is a parallelogram.

Chapter 2

This is Chapter Two Title

2.1. This is First Level Heading

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Appendix A

Vectors

A.1. 2×1 vectors

A.1.1. Let

$$\mathbf{A} \equiv \overrightarrow{A} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \tag{A.1}$$

$$\equiv a_1 \overrightarrow{i} + a_2 \overrightarrow{j}, \tag{A.2}$$

$$\mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \tag{A.3}$$

be 2×1 vectors. Then, the determinant of the 2×2 matrix

$$\mathbf{M} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \end{pmatrix} \tag{A.4}$$

is defined as

$$\begin{vmatrix}\mathbf{M}\end{vmatrix} = \begin{vmatrix}\mathbf{A} & \mathbf{B}\end{vmatrix} \tag{A.5}$$

$$= \begin{vmatrix}a_1 & b_1 \\ a_2 & b_2\end{vmatrix} = a_1 b_2 - a_2 b_1 \tag{A.6}$$

A.1.2. The value of the cross product of two vectors is given by (A.5).

A.1.3. The area of the triangle with vertices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ is given by the absolute value of

$$\frac{1}{2}\begin{vmatrix}\mathbf{A}-\mathbf{B} & \mathbf{A}-\mathbf{C}\end{vmatrix} \tag{A.7}$$

A.1.4. The transpose of $\mathbf{A}$ is defined as

$$\mathbf{A}^{\top} = \begin{pmatrix}a_1 & a_2\end{pmatrix} \tag{A.8}$$

A.1.5. The <u>inner product</u> or <u>dot product</u> is defined as

$$\mathbf{A}^{\top}\mathbf{B} \equiv \mathbf{A} \cdot \mathbf{B} \tag{A.9}$$

$$= \begin{pmatrix}a_1 & a_2\end{pmatrix}\begin{pmatrix}b_1 \\ b_2\end{pmatrix} = a_1 b_1 + a_2 b_2 \tag{A.10}$$

A.1.6. <u>norm</u> of $\mathbf{A}$ is defined as

$$\|A\| \equiv \left|\overrightarrow{A}\right| \tag{A.11}$$

$$= \sqrt{\mathbf{A}^{\top}\mathbf{A}} = \sqrt{a_1^2 + a_2^2} \tag{A.12}$$

Thus,

$$\|\lambda \mathbf{A}\| \equiv \left|\lambda \overrightarrow{A}\right| \tag{A.13}$$

$$= |\lambda| \, \|\mathbf{A}\| \tag{A.14}$$

A.1.7. The distance betwen the points $\mathbf{A}$ and $\mathbf{B}$ is given by

$$\|\mathbf{A} - \mathbf{B}\| \tag{A.15}$$

A.1.8. Let $\mathbf{x}$ be equidistant from the points $\mathbf{A}$ and $\mathbf{B}$. Then

$$(\mathbf{A} - \mathbf{B})^\top \mathbf{x} = \frac{\|\mathbf{A}\|^2 - \|\mathbf{B}\|^2}{2} \tag{A.16}$$

Solution:

$$\|\mathbf{x} - \mathbf{A}\| = \|\mathbf{A} - \mathbf{B}\| \tag{A.17}$$

$$\implies \|\mathbf{x} - \mathbf{A}\|^2 = \|\mathbf{x} - \mathbf{B}\|^2 \tag{A.18}$$

which can be expressed as

$$(\mathbf{x} - \mathbf{A})^\top (\mathbf{x} - \mathbf{A}) = (\mathbf{x} - \mathbf{B})^\top (\mathbf{x} - \mathbf{B})$$

$$\implies \|\mathbf{x}\|^2 - 2\mathbf{x}^\top \mathbf{A} + \|\mathbf{A}\|^2$$

$$= \|\mathbf{x}\|^2 - 2\mathbf{x}^\top \mathbf{B} + \|\mathbf{B}\|^2 \tag{A.19}$$

which can be simplified to obtain (A.16).

A.1.9. If $\mathbf{x}$ lies on the x-axis and is equidistant from the points $\mathbf{A}$ and $\mathbf{B}$,

$$\mathbf{x} = x\mathbf{e}_1 \tag{A.20}$$

where

$$x = \frac{\|\mathbf{A}\|^2 - \|\mathbf{B}\|^2}{2\left(\mathbf{A} - \mathbf{B}\right)^\top \mathbf{e}_1} \tag{A.21}$$

Solution: From (A.16).

$$x\left(\mathbf{A} - \mathbf{B}\right)^\top \mathbf{e}_1 = \frac{\|\mathbf{A}\|^2 - \|\mathbf{B}\|^2}{2} \tag{A.22}$$

yielding (A.21).

A.1.10. The angle between two vectors is given by

$$\theta = \cos^{-1} \frac{\mathbf{A}^\top \mathbf{B}}{\|A\| \|B\|} \tag{A.23}$$

A.1.11. If two vectors are orthogonal (perpendicular),

$$\mathbf{A}^\top \mathbf{B} = 0 \tag{A.24}$$

A.1.12. The direction vector of the line joining two points $\mathbf{A}, \mathbf{B}$ is given by

$$\mathbf{m} = \mathbf{A} - \mathbf{B} \tag{A.25}$$

A.1.13. The unit vector in the direction of $\mathbf{m}$ is defined as

$$\frac{\mathbf{m}}{\|\mathbf{m}\|} \tag{A.26}$$

A.1.14. If the direction vector of a line is expressed as

$$\mathbf{m} = \begin{pmatrix} 1 \\ m \end{pmatrix}, \tag{A.27}$$

the m is defined to be the slope of the line.

A.1.15. The normal vector to $\mathbf{m}$ is defined by

$$\mathbf{m}^\top \mathbf{n} = 0 \tag{A.28}$$

A.1.16. The point $\mathbf{P}$ that divides the line segment AB in the ratio $k : 1$ is given by

$$\mathbf{P} = \frac{k\mathbf{B} + \mathbf{A}}{k+1} \tag{A.29}$$

A.1.17. The standard basis vectors are defined as

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{A.30}$$

$$\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{A.31}$$

A.1.18. If $ABCD$ be a parallelogram,

$$\mathbf{B} - \mathbf{A} = \mathbf{C} - \mathbf{D} \tag{A.32}$$

A.2. 3×1 vectors

A.2.1. Let

$$\mathbf{A} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \equiv a_1 \overrightarrow{i} + a_2 \overrightarrow{j} + a_3 \overrightarrow{j}, \tag{A.33}$$

$$\mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \tag{A.34}$$

and

$$\mathbf{A}_{ij} = \begin{pmatrix} a_i \\ a_j \end{pmatrix}, \tag{A.35}$$

$$\mathbf{B}_{ij} = \begin{pmatrix} b_i \\ b_j \end{pmatrix}. \tag{A.36}$$

A.2.2. The cross product or vector product of $\mathbf{A}, \mathbf{B}$ is defined as

$$\mathbf{A} \times \mathbf{B} = \begin{pmatrix} \begin{vmatrix} \mathbf{A}_{23} & \mathbf{B}_{23} \\ \\ \mathbf{A}_{31} & \mathbf{B}_{31} \\ \\ \mathbf{A}_{12} & \mathbf{B}_{12} \end{vmatrix} \end{pmatrix} \tag{A.37}$$

A.2.3. Verify that

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} \tag{A.38}$$

A.2.4. The area of a triangle is given by

$$\frac{1}{2} \|\mathbf{A} \times \mathbf{B}\| \tag{A.39}$$

Appendix B

Matrices

B.1. Eigenvalues and Eigenvectors

B.1.1. The eigenvalue λ and the eigenvector $\mathbf{x}$ for a matrix $\mathbf{A}$ are defined as,

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x} \tag{B.1}$$

B.1.2. The eigenvalues are calculated by solving the equation

$$f(\lambda) = \left|\lambda\mathbf{I} - \mathbf{A}\right| = 0 \tag{B.2}$$

The above equation is known as the characteristic equation.

B.1.3. According to the Cayley-Hamilton theorem,

$$f(\lambda) = 0 \implies f(\mathbf{A}) = 0 \tag{B.3}$$

B.1.4. The trace of a square matrix is defined to be the sum of the diagonal elements.

$$\mathrm{tr}(\mathbf{A}) = \sum_{i=1}^{N} a_{ii}. \tag{B.4}$$

where a_{ii} is the ith diagonal element of the matrix $\mathbf{A}$.

B.1.5. The trace of a matrix is equal to the sum of the eigenvalues

$$\mathrm{tr}\left(\mathbf{A}\right) = \sum_{i=1}^{N} \lambda_i \tag{B.5}$$

B.2. Determinants

B.2.1. Let

$$\mathbf{A} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}. \tag{B.6}$$

be a 3×3 matrix. Then,

$$\left|\mathbf{A}\right| = a_1 \begin{pmatrix} b_2 & c_2 \\ b_3 & c_3 \end{pmatrix} - a_2 \begin{pmatrix} b_1 & c_1 \\ b_3 & c_3 \end{pmatrix} + a_3 \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}. \tag{B.7}$$

B.2.2. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of a matrix $\mathbf{A}$. Then, the product of the eigenvalues is equal to the determinant of $\mathbf{A}$.

$$\left|\mathbf{A}\right| = \prod_{i=1}^{n} \lambda_i \tag{B.8}$$

B.2.3.

$$\left|\mathbf{AB}\right| = \left|\mathbf{A}\right|\left|\mathbf{B}\right| \tag{B.9}$$

B.2.4. If $\mathbf{A}$ be an $n \times n$ matrix,

$$\left|k\mathbf{A}\right| = k^n \left|\mathbf{A}\right| \tag{B.10}$$

B.3. Rank of a Matrix

B.3.1. The rank of a matrix is defined as the number of linearly independent rows. This is also known as the row rank.

B.3.2. Row rank = Column rank.

B.3.3. The rank of a matrix is obtained as the number of nonzero rows obtained after row reduction.

B.3.4. An $n \times n$ matrix is invertible if and only if its rank is n.

B.3.5. Points $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are on a line if

$$\text{rank}\begin{pmatrix}\mathbf{A}\\ \mathbf{B}\\ \mathbf{C}\end{pmatrix} = 1 \tag{B.11}$$

B.3.6. Points $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ form a paralelogram if

$$\operatorname{rank}\begin{pmatrix}\mathbf{A}\\\mathbf{B}\\\mathbf{C}\\\mathbf{D}\end{pmatrix} = 1, \operatorname{rank}\begin{pmatrix}\mathbf{A}\\\mathbf{B}\\\mathbf{C}\end{pmatrix} = 2 \tag{B.12}$$

B.4. Inverse of a Matrix

B.4.1. For a 2×2 matrix

$$\mathbf{A} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}, \tag{B.13}$$

the inverse is given by

$$\mathbf{A}^{-1} = \frac{1}{\left|\mathbf{A}\right|}\begin{pmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{pmatrix}, \tag{B.14}$$

B.4.2. For higher order matrices, the inverse should be calculated using row operations.

B.5. Orthogonality

B.5.1. The rotation matrix is defined as

$$\mathbf{R}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad \theta \in [0, 2\pi] \tag{B.15}$$

B.5.2. The rotation matrix is orthogonal

$$\mathbf{R}_\theta^\top \mathbf{R}_\theta = \mathbf{R}_\theta \mathbf{R}_\theta^\top = \mathbf{I} \tag{B.16}$$

B.5.3.

$$\mathbf{m}^\top \mathbf{n} = 0 \implies \mathbf{n} = \mathbf{R}_{\frac{\pi}{2}} \mathbf{m} \tag{B.17}$$

B.5.4.

$$\mathbf{n}^\top \mathbf{h} = 1 \implies \mathbf{n} = \frac{\mathbf{e}_1}{\mathbf{e}_1^\top \mathbf{h}} + \mu \mathbf{R}_{\frac{\pi}{2}} \mathbf{h}, \quad \mu \in \mathbb{R}. \tag{B.18}$$

B.5.5. The affine transformation is given by

$$\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c} \quad \text{(Affine Transformation)} \tag{B.19}$$

where $\mathbf{P}$ is invertible.

B.5.6. The eigenvalue decomposition of a symmetric matrix $\mathbf{V}$ is given by

$$\mathbf{P}^\top \mathbf{V} \mathbf{P} = \mathbf{D}. \quad \text{(Eigenvalue Decomposition)} \tag{B.20}$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \tag{B.21}$$

$$\mathbf{P} = \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix}, \quad \mathbf{P}^\top = \mathbf{P}^{-1}, \tag{B.22}$$

Appendix C

Linear Forms

C.1. Two Dimensions

C.1.1. The equation of a line is given by

$$\mathbf{n}^\top \mathbf{x} = c \tag{C.1}$$

where $\mathbf{n}$ is the normal vector of the line.

C.1.2. The equation of a line with normal vector $\mathbf{n}$ and passing through a point $\mathbf{A}$ is given by

$$\mathbf{n}^\top \left(\mathbf{x} - \mathbf{A}\right) = 0 \tag{C.2}$$

C.1.3. The equation of a line L is also given by

$$\mathbf{n}^\top \mathbf{x} = \begin{cases} 0 & \mathbf{0} \in L \\ 1 & \text{otherwise} \end{cases} \tag{C.3}$$

C.1.4. The parametric equation of a line is given by

$$\mathbf{x} = \mathbf{A} + \lambda \mathbf{m} \tag{C.4}$$

where $\mathbf{m}$ is the direction vector of the line and $\mathbf{A}$ is any point on the line.

C.1.5. Let $\mathbf{A}$ and $\mathbf{B}$ be two points on a straight line and let $\mathbf{P} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$ be any point on it. If p_2 is known, then

$$\mathbf{P} = \mathbf{A} + \frac{p_2 - \mathbf{e}_2^\top \mathbf{A}}{\mathbf{e}_2^\top (\mathbf{B} - \mathbf{A})} (\mathbf{B} - \mathbf{A}) \tag{C.5}$$

Solution: The equation of the line can be expressed in parametric from as

$$\mathbf{x} = \mathbf{A} + \lambda (\mathbf{B} - \mathbf{A}) \tag{C.6}$$

$$\implies \mathbf{P} = \mathbf{A} + \lambda (\mathbf{B} - \mathbf{A}) \tag{C.7}$$

$$\implies \mathbf{e}_2^\top \mathbf{P} = \mathbf{e}_2^\top \mathbf{A} + \lambda \mathbf{e}_2^\top (\mathbf{B} - \mathbf{A}) \tag{C.8}$$

$$\implies p_2 = \mathbf{e}_2^\top \mathbf{A} + \lambda \mathbf{e}_2^\top (\mathbf{B} - \mathbf{A}) \tag{C.9}$$

$$\text{or, } \lambda = \frac{p_2 - \mathbf{e}_2^\top \mathbf{A}}{\mathbf{e}_2^\top (\mathbf{B} - \mathbf{A})} \tag{C.10}$$

yielding (C.5).

C.1.6. The distance from a point $\mathbf{P}$ to the line in (C.1) is given by

$$d = \frac{\left|\mathbf{n}^\top \mathbf{P} - c\right|}{\|\mathbf{n}\|} \tag{C.11}$$

Solution: Without loss of generality, let $\mathbf{A}$ be the foot of the perpendicular from $\mathbf{P}$

to the line in (C.4). The equation of the normal to (C.1) can then be expressed as

$$\mathbf{x} = \mathbf{A} + \lambda \mathbf{n} \tag{C.12}$$

$$\implies \mathbf{P} - \mathbf{A} = \lambda \mathbf{n} \tag{C.13}$$

$\because$ **P** lies on (C.12). From the above, the desired distance can be expressed as

$$d = \|\mathbf{P} - \mathbf{A}\| = |\lambda| \, \|\mathbf{n}\| \tag{C.14}$$

From (C.13),

$$\mathbf{n}^\top (\mathbf{P} - \mathbf{A}) = \lambda \mathbf{n}^\top \mathbf{n} = \lambda \|\mathbf{n}\|^2 \tag{C.15}$$

$$\implies |\lambda| = \frac{\left|\mathbf{n}^\top (\mathbf{P} - \mathbf{A})\right|}{\|\mathbf{n}\|^2} \tag{C.16}$$

Substituting the above in (C.14) and using the fact that

$$\mathbf{n}^\top \mathbf{A} = c \tag{C.17}$$

from (C.1), yields (C.11)

C.1.7. The distance from the origin to the line in (C.1) is given by

$$d = \frac{|c|}{\|\mathbf{n}\|} \tag{C.18}$$

C.1.8. The distance between the parallel lines

$$\begin{aligned} \mathbf{n}^\top \mathbf{x} &= c_1 \\ \mathbf{n}^\top \mathbf{x} &= c_2 \end{aligned} \tag{C.19}$$

is given by

$$d = \frac{|c_1 - c_2|}{\|\mathbf{n}\|} \tag{C.20}$$

C.1.9. The equation of the line perpendicular to (C.1) and passing through the point $\mathbf{P}$ is given by

$$\mathbf{m}^\top (\mathbf{x} - \mathbf{P}) = 0 \tag{C.21}$$

C.1.10. The foot of the perpendicular from $\mathbf{P}$ to the line in (C.1) is given by

$$\begin{pmatrix} \mathbf{m} & \mathbf{n} \end{pmatrix}^\top \mathbf{x} = \begin{pmatrix} \mathbf{m}^\top \mathbf{P} \\ c \end{pmatrix} \tag{C.22}$$

Solution: From (C.1) and (C.2) the foot of the perpendicular satisfies the equations

$$\mathbf{n}^\top \mathbf{x} = c \tag{C.23}$$

$$\mathbf{m}^\top (\mathbf{x} - \mathbf{P}) = 0 \tag{C.24}$$

where $\mathbf{m}$ is the direction vector of the given line. Combining the above into a matrix equation results in (C.22).

C.1.11. The equations of the angle bisectors of the lines

$$\mathbf{n}_1^\top \mathbf{x} = c_1 \tag{C.25}$$

$$\mathbf{n}_2^\top \mathbf{x} = c_2 \tag{C.26}$$

are given by

$$\frac{\mathbf{n}_1^\top \mathbf{x} - c_1}{\|\mathbf{n}_1\|} = \pm \frac{\mathbf{n}_2^\top \mathbf{x} - c_2}{\|\mathbf{n}_2\|} \tag{C.27}$$

Proof. Any point on the angle bisector is equidistant from the lines. □

C.2. Three Dimensions

C.2.1. Points $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are on a line if

$$\operatorname{rank} \begin{pmatrix} \mathbf{A} \\ \mathbf{B} \\ \mathbf{C} \end{pmatrix} = 1 \tag{C.28}$$

C.2.2. Points $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ form a paralelogram if

$$\operatorname{rank} \begin{pmatrix} \mathbf{A} \\ \mathbf{B} \\ \mathbf{C} \\ \mathbf{D} \end{pmatrix} = 1, \operatorname{rank} \begin{pmatrix} \mathbf{A} \\ \mathbf{B} \\ \mathbf{C} \end{pmatrix} = 2 \tag{C.29}$$

C.2.3. The equation of a line is given by (C.4)

C.2.4. The equation of a plane is given by (C.1)

C.2.5. The distance from the origin to the line in (C.1) is given by (C.18)

C.2.6. The distance from a point $\mathbf{P}$ to the line in (C.4) is given by

$$d = \|\mathbf{A} - \mathbf{P}\|^2 - \frac{\left\{\mathbf{m}^\top (\mathbf{A} - \mathbf{P})\right\}^2}{\|\mathbf{m}\|^2} \tag{C.30}$$

Solution:

$$d(\lambda) = \|\mathbf{A} + \lambda\mathbf{m} - \mathbf{P}\| \tag{C.31}$$

$$\implies d^2(\lambda) = \|\mathbf{A} + \lambda\mathbf{m} - \mathbf{P}\|^2 \tag{C.32}$$

which can be simplified to obtain

$$d^2(\lambda) = \lambda^2 \|\mathbf{m}\|^2 + 2\lambda\mathbf{m}^\top (\mathbf{A} - \mathbf{P}) \\ + \|\mathbf{A} - \mathbf{P}\|^2 \tag{C.33}$$

which is of the form

$$d^2(\lambda) = a\lambda^2 + 2b\lambda + c \tag{C.34}$$

$$= a\left\{\left(\lambda + \frac{b}{a}\right)^2 + \left[\frac{c}{a} - \left(\frac{b}{a}\right)^2\right]\right\} \tag{C.35}$$

with

$$a = \|\mathbf{m}\|^2, b = \mathbf{m}^\top (\mathbf{A} - \mathbf{P}), c = \|\mathbf{A} - \mathbf{P}\|^2 \tag{C.36}$$

which can be expressed as From the above, $d^2(\lambda)$ is smallest when upon substituting from (C.36)

$$\lambda + \frac{b}{2a} = 0 \implies \lambda = -\frac{b}{2a} \tag{C.37}$$
$$= -\frac{\mathbf{m}^\top (\mathbf{A} - \mathbf{P})}{\|\mathbf{m}\|^2} \tag{C.38}$$

and consequently,

$$d_{\min}(\lambda) = a\left(\frac{c}{a} - \left(\frac{b}{a}\right)^2\right) \tag{C.39}$$
$$= c - \frac{b^2}{a} \tag{C.40}$$

yielding (C.30) after substituting from (C.36).

C.2.7. The distance between the parallel planes (C.19) is given by (C.20).

C.2.8. The plane

$$\mathbf{n}^\top \mathbf{x} = c \tag{C.41}$$

contains the line

$$\mathbf{x} = \mathbf{A} + \lambda \mathbf{m} \tag{C.42}$$

if

$$\mathbf{m}^\top \mathbf{n} = 0 \tag{C.43}$$

Solution: Any point on the line (C.42) should also satisfy (C.41). Hence,

$$\mathbf{n}^\top (\mathbf{A} + \lambda \mathbf{m}) = \mathbf{n}^\top \mathbf{A} = c \tag{C.44}$$

which can be simplified to obtain (C.43)

C.2.9. The foot of the perpendicular from a point $\mathbf{P}$ to the plane

$$\mathbf{n}^\top \mathbf{x} = c \tag{C.45}$$

is given by

Solution: The equation of the line perpendicular to the given plane and passing through $\mathbf{P}$ is

$$\mathbf{x} = \mathbf{P} + \lambda \mathbf{n} \tag{C.46}$$

From (C.58), the intersection of the above line with the given plane is

$$\mathbf{x} = \mathbf{P} + \frac{c - \mathbf{n}^\top \mathbf{P}}{\left\|\mathbf{n}\right\|^2}\mathbf{n} \tag{C.47}$$

C.2.10. The image of a point $\mathbf{P}$ with respect to the plane

$$\mathbf{n}^\top \mathbf{x} = c \tag{C.48}$$

is given by

$$\mathbf{R} = \mathbf{P} + 2\frac{c - \mathbf{n}^\top \mathbf{P}}{\left\|\mathbf{n}\right\|^2} \tag{C.49}$$

Solution: Let $\mathbf{R}$ be the desired image. Then, subtituting the expression for the foot of the perpendicular from $\mathbf{P}$ to the given plane using (C.47),

$$\frac{\mathbf{P}+\mathbf{R}}{2} = \mathbf{P} + \frac{c - \mathbf{n}^\top \mathbf{P}}{\|\mathbf{n}\|^2} \tag{C.50}$$

C.2.11. Let a plane pass through the points $\mathbf{A}, \mathbf{B}$ and be perpendicular to the plane

$$\mathbf{n}^\top \mathbf{x} = c \tag{C.51}$$

Then the equation of this plane is given by

$$\mathbf{p}^\top \mathbf{x} = 1 \tag{C.52}$$

where

$$\mathbf{p} = \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{n} \end{pmatrix}^{-\top} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \tag{C.53}$$

Solution: From the given information,

$$\mathbf{p}^\top \mathbf{A} = d \tag{C.54}$$

$$\mathbf{p}^\top \mathbf{B} = d \tag{C.55}$$

$$\mathbf{p}^\top \mathbf{n} = 0 \tag{C.56}$$

$\because$ the normal vectors to the two planes will also be perpendicular. The system of equations in (C.56) can be expressed as the matrix equation

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{n} \end{pmatrix}^\top \mathbf{p} = d \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \tag{C.57}$$

which yields (C.53) upon normalising with d.

C.2.12. The intersection of the line represented by (C.4) with the plane represented by (C.1) is given by

$$\mathbf{x} = \mathbf{A} + \frac{c - \mathbf{n}^\top \mathbf{A}}{\mathbf{n}^\top \mathbf{m}} \mathbf{m} \tag{C.58}$$

Solution: From (C.4) and (C.1),

$$\mathbf{x} = \mathbf{A} + \lambda \mathbf{m} \tag{C.59}$$

$$\mathbf{n}^\top \mathbf{x} = c \tag{C.60}$$

$$\implies \mathbf{n}^\top \left(\mathbf{A} + \lambda \mathbf{m} \right) = c \tag{C.61}$$

which can be simplified to obtain

$$\mathbf{n}^\top \mathbf{A} + \lambda \mathbf{n}^\top \mathbf{m} = c \tag{C.62}$$

$$\implies \lambda = \frac{c - \mathbf{n}^\top \mathbf{A}}{\mathbf{n}^\top \mathbf{m}} \tag{C.63}$$

Substituting the above in (C.61) yields (C.58).

C.2.13. The foot of the perpendicular from the point $\mathbf{P}$ to the line represented by (C.4) is given by

$$\mathbf{x} = \mathbf{A} + \frac{\mathbf{m}^\top (\mathbf{P} - \mathbf{A})}{\|\mathbf{m}\|^2}\mathbf{m} \tag{C.64}$$

Solution: Let the equation of the line be

$$\mathbf{x} = \mathbf{A} + \lambda \mathbf{m} \tag{C.65}$$

The equation of the plane perpendicular to the given line passing through **P** is given by

$$\mathbf{m}^\top (\mathbf{x} - \mathbf{P}) = 0 \tag{C.66}$$

$$\implies \mathbf{m}^\top \mathbf{x} = \mathbf{m}^\top \mathbf{P} \tag{C.67}$$

The desired foot of the perpendicular is the intersection of (C.65) with (C.66) which can be obtained from (C.58) as (C.64)

C.2.14. The foot of the perpendicular from a point **P** to a plane is **Q**. The equation of the plane is given by

$$(\mathbf{P} - \mathbf{Q})^\top (\mathbf{x} - \mathbf{Q}) = 0 \tag{C.68}$$

Solution: The normal vector to the plane is given by

$$\mathbf{n} = \mathbf{P} - \mathbf{Q} \tag{C.69}$$

Hence, the equation of the plane is (C.68).

C.2.15. Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be points on a plane. The equation of the plane is then given by

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix}^{\top} \mathbf{n} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{C.70}$$

Solution: Let the equation of the plane be

$$\mathbf{n}^{\top} \mathbf{x} = 1 \tag{C.71}$$

Then

$$\mathbf{n}^{\top} \mathbf{A} = 1 \tag{C.72}$$

$$\mathbf{n}^{\top} \mathbf{B} = 1 \tag{C.73}$$

$$\mathbf{n}^{\top} \mathbf{C} = 1 \tag{C.74}$$

which can be combined to obtain (C.70).

C.2.16. (Parallelogram Law) Let $\mathbf{A}, \mathbf{B}, \mathbf{D}$ be three vertices of a parallelogram. Then the vertex $\mathbf{C}$ is given by

$$\mathbf{C} = \mathbf{B} + \mathbf{C} - \mathbf{A} \tag{C.75}$$

Solution: Shifting $\mathbf{A}$ to the origin, we obtain a parallelogram with corresponding vertices

$$\mathbf{0}, \mathbf{B} - \mathbf{A}, \mathbf{D} - \mathbf{A} \tag{C.76}$$

The fourth vertex of this parallelogram is then obtained as

$$(\mathbf{B}-\mathbf{A})+(\mathbf{D}-\mathbf{A})=\mathbf{D}+\mathbf{B}-2\mathbf{A} \tag{C.77}$$

Shifting the origin to $\mathbf{A}$, the fourth vertex is obtained as

$$\mathbf{C}=\mathbf{D}+\mathbf{B}-2\mathbf{A}+\mathbf{A} \tag{C.78}$$

$$=\mathbf{D}+\mathbf{B}-\mathbf{A} \tag{C.79}$$

C.2.17. (Affine Transformation) Let $\mathbf{A}, \mathbf{C}$, be opposite vertices of a square. The other two points can be obtained as

$$\mathbf{B}=\frac{\|\mathbf{A}-\mathbf{C}\|}{\sqrt{2}}\mathbf{P}\mathbf{e}_1+\mathbf{A} \tag{C.80}$$

$$\mathbf{D}=\frac{\|\mathbf{A}-\mathbf{C}\|}{\sqrt{2}}\mathbf{P}\mathbf{e}_2+\mathbf{A} \tag{C.81}$$

where

$$\mathbf{P}=\begin{pmatrix}\cos\left(\theta-\frac{\pi}{4}\right) & \sin\left(\theta-\frac{\pi}{4}\right)\\ \sin\left(\theta-\frac{\pi}{4}\right) & \cos\left(\theta-\frac{\pi}{4}\right)\end{pmatrix} \tag{C.82}$$

and

$$\cos\theta=\frac{\left(\mathbf{C}-\mathbf{A}\right)^{\top}\mathbf{e}_1}{\|\mathbf{A}-\mathbf{C}\|\,\|\mathbf{e}_1\|} \tag{C.83}$$

Appendix D

Quadratic Forms

D.1. Conic equation

D.1.1. Let $\mathbf{q}$ be a point such that the ratio of its distance from a fixed point $\mathbf{F}$ and the distance (d) from a fixed line

$$L : \mathbf{n}^\top \mathbf{x} = c \tag{D.1}$$

is constant, given by

$$\frac{\|\mathbf{q} - \mathbf{F}\|}{d} = e \tag{D.2}$$

The locus of $\mathbf{q}$ is known as a conic section. The line L is known as the directrix and the point $\mathbf{F}$ is the focus. e is defined to be the eccentricity of the conic.

(a) For $e = 1$, the conic is a parabola

(b) For $e < 1$, the conic is an ellipse

(c) For $e > 1$, the conic is a hyperbola

D.1.2. The equation of a conic with directrix $\mathbf{n}^\top \mathbf{x} = c$, eccentricity e and focus $\mathbf{F}$ is given by

$$\mathbf{x}^\top \mathbf{V} \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{D.3}$$

where

$$\mathbf{V} = \left\|\mathbf{n}\right\|^2 \mathbf{I} - e^2 \mathbf{n}\mathbf{n}^\top, \tag{D.4}$$

$$\mathbf{u} = ce^2 \mathbf{n} - \left\|\mathbf{n}\right\|^2 \mathbf{F}, \tag{D.5}$$

$$f = \left\|\mathbf{n}\right\|^2 \left\|\mathbf{F}\right\|^2 - c^2 e^2 \tag{D.6}$$

Proof. Using Definition D.1.1 and Lemma C.11, for any point $\mathbf{x}$ on the conic,

$$\left\|\mathbf{x} - \mathbf{F}\right\|^2 = e^2 \frac{\left(\mathbf{n}^\top \mathbf{x} - c\right)^2}{\left\|\mathbf{n}\right\|^2} \tag{D.7}$$

$$\implies \left\|\mathbf{n}\right\|^2 \left(\mathbf{x} - \mathbf{F}\right)^\top \left(\mathbf{x} - \mathbf{F}\right) = e^2 \left(\mathbf{n}^\top \mathbf{x} - c\right)^2 \tag{D.8}$$

$$\implies \left\|\mathbf{n}\right\|^2 \left(\mathbf{x}^\top \mathbf{x} - 2\mathbf{F}^\top \mathbf{x} + \left\|\mathbf{F}\right\|^2\right) = e^2 \left(c^2 + \left(\mathbf{n}^\top \mathbf{x}\right)^2 - 2c\mathbf{n}^\top \mathbf{x}\right) \tag{D.9}$$

$$= e^2 \left(c^2 + \left(\mathbf{x}^\top \mathbf{n}\mathbf{n}^\top \mathbf{x}\right) - 2c\mathbf{n}^\top \mathbf{x}\right) \tag{D.10}$$

which can be expressed as (D.3) after simplification.

□

D.1.3. The eccentricity, directrices and foci of (D.3) are given by

$$e = \sqrt{1 - \frac{\lambda_1}{\lambda_2}} \tag{D.11}$$

$$\mathbf{n} = \sqrt{\lambda_2}\mathbf{p}_1,$$

$$c = \begin{cases} \frac{e\mathbf{u}^\top\mathbf{n} \pm \sqrt{e^2\left(\mathbf{u}^\top\mathbf{n}\right)^2 - \lambda_2(e^2-1)\left(\|\mathbf{u}\|^2 - \lambda_2 f\right)}}{\lambda_2 e(e^2-1)} & e \neq 1 \\ \frac{\|\mathbf{u}\|^2 - \lambda_2 f}{2\mathbf{u}^\top\mathbf{n}} & e = 1 \end{cases} \tag{D.12}$$

$$\mathbf{F} = \frac{ce^2\mathbf{n} - \mathbf{u}}{\lambda_2} \tag{D.13}$$

Proof. From (D.4), using the fact that $\mathbf{V}$ is symmetric with $\mathbf{V} = \mathbf{V}^\top$,

$$\mathbf{V}^\top\mathbf{V} = \left(\|\mathbf{n}\|^2\mathbf{I} - e^2\mathbf{n}\mathbf{n}^\top\right)^\top\left(\|\mathbf{n}\|^2\mathbf{I} - e^2\mathbf{n}\mathbf{n}^\top\right) \tag{D.14}$$

$$\implies \mathbf{V}^2 = \|\mathbf{n}\|^4\mathbf{I} + e^4\mathbf{n}\mathbf{n}^\top\mathbf{n}\mathbf{n}^\top - 2e^2\|\mathbf{n}\|^2\mathbf{n}\mathbf{n}^\top \tag{D.15}$$

$$= \|\mathbf{n}\|^4\mathbf{I} + e^4\|\mathbf{n}\|^2\mathbf{n}\mathbf{n}^\top - 2e^2\|\mathbf{n}\|^2\mathbf{n}\mathbf{n}^\top \tag{D.16}$$

$$= \|\mathbf{n}\|^4\mathbf{I} + e^2\left(e^2 - 2\right)\|\mathbf{n}\|^2\mathbf{n}\mathbf{n}^\top \tag{D.17}$$

$$= \|\mathbf{n}\|^4\mathbf{I} + \left(e^2 - 2\right)\|\mathbf{n}\|^2\left(\|\mathbf{n}\|^2\mathbf{I} - \mathbf{V}\right) \tag{D.18}$$

which can be expressed as

$$\mathbf{V}^2 + \left(e^2 - 2\right)\|\mathbf{n}\|^2\mathbf{V} - \left(e^2 - 1\right)\|\mathbf{n}\|^4\mathbf{I} = 0 \tag{D.19}$$

Using the Cayley-Hamilton theorem, (D.19) results in the characteristic equation,

$$\lambda^2 - \left(2 - e^2\right)\|\mathbf{n}\|^2\lambda + \left(1 - e^2\right)\|\mathbf{n}\|^4 = 0 \tag{D.20}$$

which can be expressed as

$$\left(\frac{\lambda}{\|\mathbf{n}\|^2}\right)^2 - \left(2-e^2\right)\left(\frac{\lambda}{\|\mathbf{n}\|^2}\right) + \left(1-e^2\right) = 0 \tag{D.21}$$

$$\implies \frac{\lambda}{\|\mathbf{n}\|^2} = 1-e^2, 1 \tag{D.22}$$

$$\text{or, } \lambda_2 = \|\mathbf{n}\|^2, \lambda_1 = \left(1-e^2\right)\lambda_2 \tag{D.23}$$

From (D.23), the eccentricity of (D.3) is given by (D.11). Multiplying both sides of (D.4) by $\mathbf{n}$,

$$\mathbf{V}\mathbf{n} = \|\mathbf{n}\|^2 \mathbf{n} - e^2 \mathbf{n}\mathbf{n}^\top \mathbf{n} \tag{D.24}$$

$$= \|\mathbf{n}\|^2 \left(1-e^2\right)\mathbf{n} \tag{D.25}$$

$$= \lambda_1 \mathbf{n} \tag{D.26}$$

$$\tag{D.27}$$

from (D.23). Thus, λ_1 is the corresponding eigenvalue for $\mathbf{n}$. From (B.22) and (D.27), this implies that

$$\mathbf{p}_1 = \frac{\mathbf{n}}{\|\mathbf{n}\|} \tag{D.28}$$

$$\text{or, } \mathbf{n} = \|\mathbf{n}\|\,\mathbf{p}_1 = \sqrt{\lambda_2}\mathbf{p}_1 \tag{D.29}$$

from (D.23) . From (D.5) and (D.23),

$$\mathbf{F} = \frac{ce^2\mathbf{n} - \mathbf{u}}{\lambda_2} \tag{D.30}$$

$$\implies \|\mathbf{F}\|^2 = \frac{\left(ce^2\mathbf{n} - \mathbf{u}\right)^\top \left(ce^2\mathbf{n} - \mathbf{u}\right)}{\lambda_2^2} \tag{D.31}$$

$$\implies \lambda_2^2 \|\mathbf{F}\|^2 = c^2e^4\lambda_2 - 2ce^2\mathbf{u}^\top\mathbf{n} + \|\mathbf{u}\|^2 \tag{D.32}$$

Also, (D.6) can be expressed as

$$\lambda_2 \|\mathbf{F}\|^2 = f + c^2e^2 \tag{D.33}$$

From (D.32) and (D.33),

$$c^2e^4\lambda_2 - 2ce^2\mathbf{u}^\top\mathbf{n} + \|\mathbf{u}\|^2 = \lambda_2 \left(f + c^2e^2\right) \tag{D.34}$$

$$\implies \lambda_2 e^2 \left(e^2 - 1\right) c^2 - 2ce^2\mathbf{u}^\top\mathbf{n} + \|\mathbf{u}\|^2 - \lambda_2 f = 0 \tag{D.35}$$

yielding (D.13). □

D.1.4. (D.3) represents

(a) a parabola for $\left|\mathbf{V}\right| = 0$,

(b) ellipse for $\left|\mathbf{V}\right| > 0$ and

(c) hyperbola for $\left|\mathbf{V}\right| < 0$.

Proof. From (D.11),

$$\frac{\lambda_1}{\lambda_2} = 1 - e^2 \tag{D.36}$$

Also,

$$\left|\mathbf{V}\right| = \lambda_1\lambda_2 \tag{D.37}$$

yielding Table D.2 $\square$

Eccentricity	**Conic**	**Eigenvalue**	**Determinant**
$e = 1$	Parabola	$\lambda_1 = 0$	$\left|\mathbf{V}\right| = 0$
$e < 1$	Ellipse	$\lambda_1 > 0, \lambda_2 > 0$	$\left|\mathbf{V}\right| > 0$
$e > 1$	Hyperbola	$\lambda_1 < 0, \lambda_2 > 0$	$\left|\mathbf{V}\right| < 0$

Table D.2:

D.2. Standard Form

D.2.1. Using the affine transformation in (B.19), the conic in (D.3) can be expressed in standard form as

$$\mathbf{y}^\top \left(\frac{\mathbf{D}}{f_0}\right)\mathbf{y} = 1 \qquad |\mathbf{V}| \neq 0 \tag{D.38}$$

$$\mathbf{y}^\top\mathbf{D}\mathbf{y} = -\eta\mathbf{e}_1^\top\mathbf{y} \qquad |\mathbf{V}| = 0 \tag{D.39}$$

where

$$f_0 = \mathbf{u}^\top \mathbf{V}^{-1} \mathbf{u} - f \neq 0 \tag{D.40}$$

$$\eta = 2\mathbf{u}^\top \mathbf{p}_1 \tag{D.41}$$

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{D.42}$$

Proof. Using (B.19) (D.3) can be expressed as

$$(\mathbf{P}\mathbf{y} + \mathbf{c})^\top \mathbf{V} (\mathbf{P}\mathbf{y} + \mathbf{c}) + 2\mathbf{u}^\top (\mathbf{P}\mathbf{y} + \mathbf{c}) + f = 0, \tag{D.43}$$

yielding

$$\mathbf{y}^\top \mathbf{P}^\top \mathbf{V}\mathbf{P}\mathbf{y} + 2 (\mathbf{V}\mathbf{c} + \mathbf{u})^\top \mathbf{P}\mathbf{y} + \mathbf{c}^\top \mathbf{V}\mathbf{c} + 2\mathbf{u}^\top \mathbf{c} + f = 0 \tag{D.44}$$

From (D.44) and (B.20),

$$\mathbf{y}^\top \mathbf{D}\mathbf{y} + 2 (\mathbf{V}\mathbf{c} + \mathbf{u})^\top \mathbf{P}\mathbf{y} + \mathbf{c}^\top (\mathbf{V}\mathbf{c} + \mathbf{u}) + \mathbf{u}^\top \mathbf{c} + f = 0 \tag{D.45}$$

When $\mathbf{V}^{-1}$ exists, choosing

$$\mathbf{V}\mathbf{c} + \mathbf{u} = \mathbf{0}, \quad \text{or}, \mathbf{c} = -\mathbf{V}^{-1}\mathbf{u}, \tag{D.46}$$

and substituting (D.46) in (D.45) yields (D.38). When $|\mathbf{V}| = 0, \lambda_1 = 0$ and

$$\mathbf{V}\mathbf{p}_1 = 0, \mathbf{V}\mathbf{p}_2 = \lambda_2 \mathbf{p}_2. \tag{D.47}$$

where $\mathbf{p}_1, \mathbf{p}_2$ are the eigenvectors of $\mathbf{V}$ such that (B.20)

$$\mathbf{P} = \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix}, \tag{D.48}$$

Substituting (D.48) in (D.45),

$$\mathbf{y}^\top \mathbf{D} \mathbf{y} + 2 \left(\mathbf{c}^\top \mathbf{V} + \mathbf{u}^\top \right) \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix} \mathbf{y} + \mathbf{c}^\top \left(\mathbf{V}\mathbf{c} + \mathbf{u} \right) + \mathbf{u}^\top \mathbf{c} + f = 0 \tag{D.49}$$

$$\implies \mathbf{y}^\top \mathbf{D} \mathbf{y} + 2 \begin{pmatrix} \left(\mathbf{c}^\top \mathbf{V} + \mathbf{u}^\top \right) \mathbf{p}_1 & \left(\mathbf{c}^\top \mathbf{V} + \mathbf{u}^\top \right) \mathbf{p}_2 \end{pmatrix} \mathbf{y} + \mathbf{c}^\top \left(\mathbf{V}\mathbf{c} + \mathbf{u} \right) + \mathbf{u}^\top \mathbf{c} + f = 0 \tag{D.50}$$

$$\implies \mathbf{y}^\top \mathbf{D} \mathbf{y} + 2 \begin{pmatrix} \mathbf{u}^\top \mathbf{p}_1 & \left(\lambda_2 \mathbf{c}^\top + \mathbf{u}^\top \right) \mathbf{p}_2 \end{pmatrix} \mathbf{y} + \mathbf{c}^\top \left(\mathbf{V}\mathbf{c} + \mathbf{u} \right) + \mathbf{u}^\top \mathbf{c} + f = 0 \tag{D.51}$$

upon substituting from (D.47) yielding

$$\lambda_2 y_2^2 + 2 \left(\mathbf{u}^\top \mathbf{p}_1 \right) y_1 + 2 y_2 \left(\lambda_2 \mathbf{c} + \mathbf{u} \right)^\top \mathbf{p}_2 + \mathbf{c}^\top \left(\mathbf{V}\mathbf{c} + \mathbf{u} \right) + \mathbf{u}^\top \mathbf{c} + f = 0 \tag{D.52}$$

Thus, (D.52) can be expressed as (D.39) by choosing

$$\eta = 2 \mathbf{u}^\top \mathbf{p}_1 \tag{D.53}$$

and $\mathbf{c}$ in (D.45) such that

$$2 \mathbf{P}^\top \left(\mathbf{V}\mathbf{c} + \mathbf{u} \right) = \eta \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{D.54}$$

$$\mathbf{c}^\top \left(\mathbf{V}\mathbf{c} + \mathbf{u} \right) + \mathbf{u}^\top \mathbf{c} + f = 0 \tag{D.55}$$

$\because \mathbf{P}^\top\mathbf{P} = \mathbf{I}$, multiplying (D.54) by $\mathbf{P}$ yields

$$(\mathbf{V}\mathbf{c} + \mathbf{u}) = \frac{\eta}{2}\mathbf{p}_1, \tag{D.56}$$

which, upon substituting in (D.55) results in

$$\frac{\eta}{2}\mathbf{c}^\top\mathbf{p}_1 + \mathbf{u}^\top\mathbf{c} + f = 0 \tag{D.57}$$

(D.56) and (D.57) can be clubbed together to obtain (E.7). $\square$

D.2.2. For the standard conic,

$$\mathbf{P} = \mathbf{I} \tag{D.58}$$

$$\mathbf{u} = \begin{cases} 0 & e \neq 1 \\ \frac{\eta}{2}\mathbf{e}_1 & e = 1 \end{cases} \tag{D.59}$$

$$\lambda_1 \begin{cases} = 0 & e = 1 \\ \neq 0 & e \neq 1 \end{cases} \tag{D.60}$$

where

$$\mathbf{I} = \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 \end{pmatrix} \tag{D.61}$$

is the identity matrix.

D.2.3.

(a) The directrices for the standard conic are given by

$$\mathbf{e}_1^\top \mathbf{y} = \pm\frac{1}{e}\sqrt{\frac{|f_0|}{\lambda_2\left(1-e^2\right)}} \qquad e \neq 1 \tag{D.62}$$

$$\mathbf{e}_1^\top \mathbf{y} = \frac{\eta}{2\lambda_2} \qquad e = 1 \tag{D.63}$$

(b) The foci of the standard ellipse and hyperbola are given by

$$\mathbf{F} = \begin{cases} \pm e\sqrt{\frac{|f_0|}{\lambda_2(1-e^2)}}\mathbf{e}_1 & e \neq 1 \\ -\frac{\eta}{4\lambda_2}\mathbf{e}_1 & e = 1 \end{cases} \tag{D.64}$$

Proof. (a) For the standard hyperbola/ellipse in (D.38), from (D.58), (D.12) and (D.59),

$$\mathbf{n} = \sqrt{\frac{\lambda_2}{f_0}}\mathbf{e}_1 \tag{D.65}$$

$$c = \pm\frac{\sqrt{-\frac{\lambda_2}{f_0}\left(e^2-1\right)\left(\frac{\lambda_2}{f_0}\right)}}{\frac{\lambda_2}{f_0}e\left(e^2-1\right)} \tag{D.66}$$

$$= \pm\frac{1}{e\sqrt{1-e^2}} \tag{D.67}$$

yielding (D.62) upon substituting from (D.11) and simplifying. For the standard parabola in (D.39), from (D.58), (D.12) and (D.59), noting that $f = 0$,

$$\mathbf{n} = \sqrt{\lambda_2}\mathbf{e}_1 \tag{D.68}$$

$$c = \frac{\left\|\frac{\eta}{2}\mathbf{e}_1\right\|^2}{2\left(\frac{\eta}{\mathbf{2}}\right)(\mathbf{e_1})^\top \mathbf{n}} \tag{D.69}$$

$$\tag{D.70}$$

$$= \frac{\eta}{4\sqrt{\lambda_2}} \tag{D.71}$$

yielding (D.63).

(b) For the standard ellipse/hyperbola, substituting from (D.67), (D.65), (D.59) and (D.11) in (D.13),

$$\mathbf{F} = \pm\frac{\left(\frac{1}{e\sqrt{1-e^2}}\right)(e^2)\sqrt{\frac{\lambda_2}{f_0}}\mathbf{e}_1}{\frac{\lambda_2}{f_0}} \tag{D.72}$$

yielding (D.64) after simplification. For the standard parabola, substituting from (D.71), (D.68), (D.59) and (D.11) in (D.13),

$$\mathbf{F} = \frac{\left(\frac{\eta}{4\sqrt{\lambda_2}}\right)\sqrt{\lambda_2}\mathbf{e}_1 - \frac{\eta}{\mathbf{2}}\mathbf{e_1}}{\lambda_2} \tag{D.73}$$

$$\tag{D.74}$$

yielding (D.64) after simplification.

□

Appendix E

Conic Parameters

E.1. Standard Form

E.1.1. The center of the standard ellipse/hyperbola, defined to be the mid point of the line joining the foci, is the origin.

E.1.2. The principal (major) axis of the standard ellipse/hyperbola, defined to be the line joining the two foci is the x-axis.

Proof. From (D.64), it is obvious that the line joining the foci passes through the origin. Also, the direction vector of this line is $\mathbf{e}_1$. Thus, the principal axis is the x-axis. □

E.1.3. The minor axis of the standard ellipse/hyperbola, defined to be the line orthogonal to the x-axis is the y-axis.

E.1.4. The axis of symmetry of the standard parabola, defined to be the line perpendicular to the directrix and passing through the focus, is the x- axis.

Proof. From (D.68) and (D.64), the axis of the parabola can be expressed using (C.2) as

$$\mathbf{e}_2^\top \left(\mathbf{y} + \frac{\eta}{4\lambda_2} \mathbf{e}_1 \right) = 0 \tag{E.1}$$

$$\implies \mathbf{e}_2^\top \mathbf{y} = 0, \tag{E.2}$$

which is the equation of the x-axis. $\square$

E.1.5. The point where the parabola intersects its axis of symmetry is called the vertex. For the standard parabola, the vertex is the origin.

Proof. (E.2) can be expressed as

$$\mathbf{y} = \alpha \mathbf{e}_1, \tag{E.3}$$

using (C.2). Substituting (E.3) in (D.39),

$$\alpha^2 \mathbf{e}_1^\top \mathbf{D} \mathbf{e}_1 = -\eta \alpha \mathbf{e}_1^\top \mathbf{e}_1 \tag{E.4}$$

$$\implies \alpha = 0, \text{ or, } \mathbf{y} = \mathbf{0}. \tag{E.5}$$

$\square$

E.1.6. The <u>focal length</u> of the standard parabola, , defined to be the distance between the vertex and the focus, measured along the axis of symmetry, is $\left| \frac{\eta}{4\lambda_2} \right|$

E.2. Quadratic Form

E.2.1. The center/vertex of a conic section are given by

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \qquad \left|\mathbf{V}\right| \neq 0 \tag{E.6}$$

$$\begin{pmatrix} \mathbf{u}^\top + \frac{\eta}{2}\mathbf{p}_1^\top \\ \mathbf{V} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -f \\ \frac{\eta}{2}\mathbf{p}_1 - \mathbf{u} \end{pmatrix} \qquad \left|\mathbf{V}\right| = 0 \tag{E.7}$$

Proof. In (B.19), substituting $\mathbf{y} = \mathbf{0}$, the center/vertex for the quadratic form is obtained as

$$\mathbf{x} = \mathbf{c}, \tag{E.8}$$

where $\mathbf{c}$ is derived as (E.6) and (E.7) in Appendix D.2.1. □

E.2.2. The equation of the minor and major axes for the ellipse/hyperbola are respectively given by

$$\mathbf{p}_i^\top (\mathbf{x} - \mathbf{c}) = 0, i = 1, 2 \tag{E.9}$$

The axis of symmetry for the parabola is also given by (E.9).

Proof. From (E.1.2), the major/symmetry axis for the hyperbola/ellipse/parabola can be expressed using (B.19) as

$$\mathbf{e}_2^\top \mathbf{P}^\top (\mathbf{x} - \mathbf{c}) = 0 \tag{E.10}$$

$$\implies (\mathbf{P}\mathbf{e}_2)^\top (\mathbf{x} - \mathbf{c}) = 0 \tag{E.11}$$

yielding (E.9), and the proof for the minor axis is similar. $\square$

Appendix F

Conic Lines

F.1. Pair of Straight Lines

F.1.1. The asymptotes of the hyperbola in (D.38), defined to be the lines that do not intersect the hyperbola, are given by

$$\left(\sqrt{|\lambda_1|} \quad \pm\sqrt{|\lambda_2|}\right)\mathbf{y} = 0 \tag{F.1}$$

Proof. From (D.38), it is obvious that the pair of lines represented by

$$\mathbf{y}^\top \mathbf{D}\mathbf{y} = 0 \tag{F.2}$$

do not intersect the conic

$$\mathbf{y}^\top \mathbf{D}\mathbf{y} = f_0 \tag{F.3}$$

Thus, (F.2) represents the asysmptotes of the hyperbola in (D.38) and can be expressed as

$$\lambda_1 y_1^2 + \lambda_2 y_1^2 = 0, \tag{F.4}$$

which can then be simplified to obtain (F.1).

$\square$

F.1.2. (D.3) represents a pair of straight lines if

$$\mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} - f = 0 \tag{F.5}$$

F.1.3. (D.3) represents a pair of straight lines if the matrix

$$\begin{pmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u} & f \end{pmatrix} \tag{F.6}$$

is singular.

Proof. Let

$$\begin{pmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u} & f \end{pmatrix} \mathbf{x} = \mathbf{0} \tag{F.7}$$

Expressing

$$\mathbf{x} = \begin{pmatrix} \mathbf{y} \\ y_3 \end{pmatrix}, \tag{F.8}$$

$$\begin{pmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^\top & f \end{pmatrix} \begin{pmatrix} \mathbf{y} \\ y_3 \end{pmatrix} = \mathbf{0} \tag{F.9}$$

$$\implies \mathbf{V}\mathbf{y} + y_3\mathbf{u} = \mathbf{0} \quad \text{and} \tag{F.10}$$

$$\mathbf{u}^\top\mathbf{y} + f y_3 = 0 \tag{F.11}$$

From (F.10) we obtain,

$$\mathbf{y}^\top\mathbf{V}\mathbf{y} + y_3\mathbf{y}^\top\mathbf{u} = \mathbf{0} \tag{F.12}$$

$$\implies \mathbf{y}^\top\mathbf{V}\mathbf{y} + y_3\mathbf{u}^\top\mathbf{y} = \mathbf{0} \tag{F.13}$$

yielding (F.5) upon substituting from (F.11). □

F.1.4. Using the affine transformation, (F.1) can be expressed as the lines

$$\begin{pmatrix} \sqrt{|\lambda_1|} & \pm\sqrt{|\lambda_2|} \end{pmatrix} \mathbf{P}^\top (\mathbf{x} - \mathbf{c}) = 0 \tag{F.14}$$

F.1.5. The angle between the asymptotes can be expressed as

$$\cos\theta = \frac{|\lambda_1| - |\lambda_2|}{|\lambda_1| + |\lambda_2|} \tag{F.15}$$

Proof. The normal vectors of the lines in (F.14) are

$$\begin{aligned} \mathbf{n}_1 &= \mathbf{P} \begin{pmatrix} \sqrt{|\lambda_1|} \\ \sqrt{|\lambda_2|} \end{pmatrix} \\ \mathbf{n}_2 &= \mathbf{P} \begin{pmatrix} \sqrt{|\lambda_1|} \\ -\sqrt{|\lambda_2|} \end{pmatrix} \end{aligned} \tag{F.16}$$

The angle between the asymptotes is given by

$$\cos\theta = \frac{\mathbf{n_1}^\top \mathbf{n_2}}{\|\mathbf{n_1}\|\,\|\mathbf{n_2}\|} \tag{F.17}$$

The orthogonal matrix $\mathbf{P}$ preserves the norm, i.e.

$$\|\mathbf{n_1}\| = \left\| \mathbf{P} \begin{pmatrix} \sqrt{|\lambda_1|} \\ \sqrt{|\lambda_2|} \end{pmatrix} \right\| = \left\| \begin{pmatrix} \sqrt{|\lambda_1|} \\ \sqrt{|\lambda_2|} \end{pmatrix} \right\| \tag{F.18}$$

$$= \sqrt{|\lambda_1| + |\lambda_2|} = \|\mathbf{n_2}\| \tag{F.19}$$

It is easy to verify that

$$\mathbf{n_1}^\top \mathbf{n_2} = |\lambda_1| - |\lambda_2| \tag{F.20}$$

Thus, the angle between the asymptotes is obtained from (F.17) as (F.15). □

F.2. Intersection of Conics

F.2.1. Let

$$\mathbf{x}^\top \mathbf{V}_i \mathbf{x} + 2\mathbf{u}_i^\top \mathbf{x} + f_i = 0, \quad i = 1, 2 \tag{F.21}$$

be the equation of two conics. The locus of their intersection is a pair of straight lines if

$$\begin{vmatrix} \mathbf{V}_1 + \mu\mathbf{V}_2 & \mathbf{u}_1 + \mu\mathbf{u}_2 \\ (\mathbf{u}_1 + \mu\mathbf{u}_2)^\top & f \end{vmatrix} = 0, \begin{vmatrix} \mathbf{V}_1 + \mu\mathbf{V}_2 \end{vmatrix} < 0 \tag{F.22}$$

Proof. The intersection of the conics in (F.21) is given by the curve

$$\mathbf{x}^\top (\mathbf{V}_1 + \mu\mathbf{V}_2)\,\mathbf{x} + 2\,(\mathbf{u}_1 + \mu\mathbf{u}_2)^\top \mathbf{x} + f_1 + \mu f_2 = 0, \tag{F.23}$$

which, from Theorem F.1.3 represents a pair of straight lines if (F.22) is satisfied. □

F.2.2. The points of intersection of the conics in (F.21) are the points of the intersection of the lines in (F.23).

F.3. Chords of a Conic

F.3.1. The points of intersection of the line

$$L: \quad \mathbf{x} = \mathbf{q} + \mu\mathbf{m} \quad \mu \in \mathbb{R} \tag{F.24}$$

with the conic section in (D.3) are given by

$$\mathbf{x}_i = \mathbf{q} + \mu_i\mathbf{m} \tag{F.25}$$

where

$$\mu_i = \frac{1}{\mathbf{m}^\top \mathbf{V}\mathbf{m}} \Big(-\mathbf{m}^\top (\mathbf{V}\mathbf{q} + \mathbf{u}) \\ \pm \sqrt{\left[\mathbf{m}^\top (\mathbf{V}\mathbf{q} + \mathbf{u})\right]^2 - \left(\mathbf{q}^\top \mathbf{V}\mathbf{q} + 2\mathbf{u}^\top \mathbf{q} + f\right)\left(\mathbf{m}^\top \mathbf{V}\mathbf{m}\right)} \Big) \tag{F.26}$$

Proof. Substituting (F.24) in (D.3),

$$(\mathbf{q} + \mu\mathbf{m})^\top \mathbf{V} (\mathbf{q} + \mu\mathbf{m}) + 2\mathbf{u}^\top (\mathbf{q} + \mu\mathbf{m}) + f = 0 \tag{F.27}$$

$$\implies \mu^2 \mathbf{m}^\top \mathbf{V}\mathbf{m} + 2\mu\mathbf{m}^\top (\mathbf{V}\mathbf{q} + \mathbf{u}) + \mathbf{q}^\top \mathbf{V}\mathbf{q} + 2\mathbf{u}^\top \mathbf{q} + f = 0 \tag{F.28}$$

Solving the above quadratic in (F.28) yields (F.26). □

F.3.2. If L in (F.24) touches (D.3) at exactly one point $\mathbf{q}$,

$$\mathbf{m}^\top (\mathbf{V}\mathbf{q} + \mathbf{u}) = 0 \tag{F.29}$$

Proof. In this case, (F.28) has exactly one root. Hence, in (F.26)

$$\left[\mathbf{m}^\top (\mathbf{V}\mathbf{q} + \mathbf{u})\right]^2 - \left(\mathbf{m}^\top \mathbf{V}\mathbf{m}\right)\left(\mathbf{q}^\top \mathbf{V}\mathbf{q} + 2\mathbf{u}^\top \mathbf{q} + f\right) = 0 \tag{F.30}$$

$\because$ $\mathbf{q}$ is the point of contact, $\mathbf{q}$ satisfies (D.3) and

$$\mathbf{q}^\top \mathbf{V}\mathbf{q} + 2\mathbf{u}^\top \mathbf{q} + f = 0 \tag{F.31}$$

Substituting (F.31) in (F.30) and simplifying, we obtain (F.29). □

F.3.3. The length of the chord in (F.24) is given by

$$\frac{2\sqrt{\left[\mathbf{m}^\top\left(\mathbf{V}\mathbf{q}+\mathbf{u}\right)\right]^2-\left(\mathbf{q}^\top\mathbf{V}\mathbf{q}+2\mathbf{u}^\top\mathbf{q}+f\right)\left(\mathbf{m}^\top\mathbf{V}\mathbf{m}\right)}}{\mathbf{m}^\top\mathbf{V}\mathbf{m}}\|\mathbf{m}\| \tag{F.32}$$

Proof. The distance between the points in (F.25) is given by

$$\|\mathbf{x}_1-\mathbf{x}_2\|=|\mu_1-\mu_2|\,\|\mathbf{m}\| \tag{F.33}$$

Substituing μ_i from (F.26) in (F.33) yields (F.32). □

F.3.4. The affine transform for the conic section, preserves the norm. This implies that the length of any chord of a conic is invariant to translation and/or rotation.

Proof. Let

$$\mathbf{x}_i=\mathbf{P}\mathbf{y}_i+\mathbf{c} \tag{F.34}$$

be any two points on the conic. Then the distance between the points is given by

$$\|\mathbf{x}_1-\mathbf{x}_2\|=\|\mathbf{P}\left(\mathbf{y}_1-\mathbf{y}_2\right)\| \tag{F.35}$$

which can be expressed as

$$\|\mathbf{x}_1-\mathbf{x}_2\|^2=\left(\mathbf{y}_1-\mathbf{y}_2\right)^\top\mathbf{P}^\top\mathbf{P}\left(\mathbf{y}_1-\mathbf{y}_2\right) \tag{F.36}$$

$$=\|\mathbf{y}_1-\mathbf{y}_2\|^2 \tag{F.37}$$

since

$$\mathbf{P}^\top \mathbf{P} = \mathbf{I} \tag{F.38}$$

$\square$

F.3.5. For the standard hyperbola/ellipse, the length of the major axis is

$$2\sqrt{\left|\frac{f_0}{\lambda_1}\right|} \tag{F.39}$$

and the minor axis is

$$2\sqrt{\left|\frac{f_0}{\lambda_2}\right|} \tag{F.40}$$

Proof. Since the major axis passes through the origin,

$$\mathbf{q} = \mathbf{0} \tag{F.41}$$

Further, from Corollary (E.1.2),

$$\mathbf{m} = \mathbf{e}_2, \tag{F.42}$$

and from (D.38),

$$\mathbf{V} = \frac{\mathbf{D}}{f_0}, \mathbf{u} = 0, f = -1 \tag{F.43}$$

Substituting the above in (F.32),

$$\frac{2\sqrt{\mathbf{e}_1^\top \frac{\mathbf{D}}{f_0}\mathbf{e}_1}}{\mathbf{e}_1^\top \frac{\mathbf{D}}{f_0}\mathbf{e}_1}\|\mathbf{e}_1\| \tag{F.44}$$

yielding (F.39). Similarly, for the minor axis, the only different parameter is

$$\mathbf{m} = \mathbf{e}_2, \tag{F.45}$$

Substituting the above in (F.32),

$$\frac{2\sqrt{\mathbf{e}_2^\top \frac{\mathbf{D}}{f_0}\mathbf{e}_2}}{\mathbf{e}_2^\top \frac{\mathbf{D}}{f_0}\mathbf{e}_2}\|\mathbf{e}_2\| \tag{F.46}$$

yielding (F.40).

□

F.3.6. The latus rectum of a conic section is the chord that passes through the focus and is perpendicular to the major axis. The length of the latus rectum for a conic is given by

$$l = \begin{cases} 2\frac{\sqrt{|f_0\lambda_1|}}{\lambda_2} & e \neq 1 \\ \frac{\eta}{\lambda_2} & e = 1 \end{cases} \tag{F.47}$$

Proof. The latus rectum is perpendicular to the major axis for the standard conic. Hence, from Corollary (E.1.2),

$$\mathbf{m} = \mathbf{e}_2, \tag{F.48}$$

Since it passes through the focus, from (D.64)

$$\mathbf{q} = \mathbf{F} = \pm e\sqrt{\frac{f_0}{\lambda_2\left(1-e^2\right)}}\mathbf{e}_1 \tag{F.49}$$

for the standard hyperbola/ellipse. Also, from (D.38),

$$\mathbf{V} = \frac{\mathbf{D}}{f_0}, \mathbf{u} = 0, f = -1 \tag{F.50}$$

Substituting the above in (F.32),

$$\frac{2\sqrt{\left[\mathbf{e}_2^\top\left(\frac{\mathbf{D}}{f_0}e\sqrt{\frac{f_0}{\lambda_2(1-e^2)}}\mathbf{e}_1\right)\right]^2-\left(e\sqrt{\frac{f_0}{\lambda_2(1-e^2)}}\mathbf{e}_1^\top\frac{\mathbf{D}}{f_0}e\sqrt{\frac{f_0}{\lambda_2(1-e^2)}}\mathbf{e}_1-1\right)\left(\mathbf{e}_2^\top\frac{\mathbf{D}}{f_0}\mathbf{e}_2\right)}}{\mathbf{e}_2^\top\frac{\mathbf{D}}{f_0}\mathbf{e}_2}\left\|\mathbf{e}_2\right\| \tag{F.51}$$

Since

$$\mathbf{e}_2^\top\mathbf{D}\mathbf{e}_1 = 0, \mathbf{e}_1^\top\mathbf{D}\mathbf{e}_1 = \lambda_1, \mathbf{e}_1^\top\mathbf{e}_1 = 1, \left\|\mathbf{e}_2\right\| = 1, \mathbf{e}_2^\top\mathbf{D}\mathbf{e}_2 = \lambda_2, \tag{F.52}$$

(F.51) can be expressed as

$$\frac{2\sqrt{\left(1-\frac{\lambda_1e^2}{\lambda_2(1-e^2)}\right)\left(\frac{\lambda_2}{f_0}\right)}}{\frac{\lambda_2}{f_0}} \tag{F.53}$$

$$= 2\frac{\sqrt{f_0\lambda_1}}{\lambda_2} \qquad \left(\because e^2 = 1-\frac{\lambda_1}{\lambda_2}\right) \tag{F.54}$$

For the standard parabola, the parameters in (F.32) are

$$\mathbf{q} = \mathbf{F} = -\frac{\eta}{4\lambda_2}\mathbf{e}_1, \mathbf{m} = \mathbf{e}_1, \mathbf{V} = \mathbf{D}, \mathbf{u} = \frac{\eta}{2}\mathbf{e}_1^\top, f = 0 \tag{F.55}$$

Substituting the above in (F.32), the length of the latus rectum can be expressed as

$$2\frac{\sqrt{\left[\mathbf{e}_2^\top\left(\mathbf{D}\left(-\frac{\eta}{4\lambda_2}\mathbf{e}_1\right)+\frac{\eta}{2}\mathbf{e}_1\right)\right]^2-\left(\left(-\frac{\eta}{4\lambda_2}\mathbf{e}_1\right)^\top\mathbf{D}\left(-\frac{\eta}{4\lambda_2}\mathbf{e}_1\right)+2\frac{\eta}{2}\mathbf{e}_1^\top\left(-\frac{\eta}{4\lambda_2}\mathbf{e}_1\right)\right)\left(\mathbf{e}_2^\top\mathbf{D}\mathbf{e}_2\right)}}{\mathbf{e}_2^\top\mathbf{D}\mathbf{e}_2}\left\|\mathbf{e}_2\right\| \tag{F.56}$$

Since

$$\mathbf{e}_2^\top\mathbf{D}\mathbf{e}_1=0,\mathbf{e}_2^\top\mathbf{e}_2=0,\mathbf{e}_1^\top\mathbf{D}\mathbf{e}_1=0,\mathbf{e}_1^\top\mathbf{e}_1=1,\left\|\mathbf{e}_1\right\|=1,\mathbf{e}_2^\top\mathbf{D}\mathbf{e}_2=\lambda_2, \tag{F.57}$$

(F.56) can be expressed as

$$2\frac{\sqrt{\frac{\eta^2}{4\lambda_2}\lambda_2}}{\lambda_2}=\frac{\eta}{\lambda_2} \tag{F.58}$$

□

F.4. Tangent and Normal

F.4.1. Given the point of contact $\mathbf{q}$, the equation of a tangent to (D.3) is

$$\left(\mathbf{V}\mathbf{q}+\mathbf{u}\right)^\top\mathbf{x}+\mathbf{u}^\top\mathbf{q}+f=0 \tag{F.59}$$

Proof. The normal vector is obtained from (F.29) and (A.28) as

$$\kappa\mathbf{n}=\mathbf{V}\mathbf{q}+\mathbf{u},\kappa\in\mathbb{R} \tag{F.60}$$

From (F.60) and (C.2), the equation of the tangent is

$$
\begin{aligned}
& (\mathbf{V}\mathbf{q}+\mathbf{u})^{\top}(\mathbf{x}-\mathbf{q}) = 0 && \text{(F.61)}\\
\implies & (\mathbf{V}\mathbf{q}+\mathbf{u})^{\top}\mathbf{x} - \mathbf{q}^{\top}\mathbf{V}\mathbf{q} - \mathbf{u}^{\top}\mathbf{q} = 0 && \text{(F.62)}
\end{aligned}
$$

which, upon substituting from (F.31) and simplifying yields (F.59) □

F.4.2. If $\mathbf{V}^{-1}$ exists, given the normal vector $\mathbf{n}$, the tangent points of contact to (D.3) are given by

$$
\begin{aligned}
\mathbf{q}_i &= \mathbf{V}^{-1}(\kappa_i\mathbf{n}-\mathbf{u}), i = 1, 2\\
\text{where } \kappa_i &= \pm\sqrt{\frac{f_0}{\mathbf{n}^{\top}\mathbf{V}^{-1}\mathbf{n}}}
\end{aligned}
\tag{F.63}
$$

Proof. From (F.60),

$$
\mathbf{q} = \mathbf{V}^{-1}(\kappa\mathbf{n}-\mathbf{u}), \quad \kappa \in \mathbb{R} \tag{F.64}
$$

Substituting (F.64) in (F.31),

$$
\begin{aligned}
& (\kappa\mathbf{n}-\mathbf{u})^{\top}\mathbf{V}^{-1}(\kappa\mathbf{n}-\mathbf{u}) + 2\mathbf{u}^{\top}\mathbf{V}^{-1}(\kappa\mathbf{n}-\mathbf{u}) + f = 0 && \text{(F.65)}\\
\implies & \kappa^2\mathbf{n}^{\top}\mathbf{V}^{-1}\mathbf{n} - \mathbf{u}^{\top}\mathbf{V}^{-1}\mathbf{u} + f = 0 && \text{(F.66)}\\
& \text{or, } \kappa = \pm\sqrt{\frac{f_0}{\mathbf{n}^{\top}\mathbf{V}^{-1}\mathbf{n}}} && \text{(F.67)}
\end{aligned}
$$

Substituting (F.67) in (F.64) yields (F.63). □

F.4.3. If $\mathbf{V}$ is not invertible, given the normal vector $\mathbf{n}$, the point of contact to (D.3) is given by the matrix equation

$$\begin{pmatrix} (\mathbf{u} + \kappa\mathbf{n})^\top \\ \mathbf{V} \end{pmatrix} \mathbf{q} = \begin{pmatrix} -f \\ \kappa\mathbf{n} - \mathbf{u} \end{pmatrix} \tag{F.68}$$

$$\text{where } \kappa = \frac{\mathbf{p}_1^\top \mathbf{u}}{\mathbf{p}_1^\top \mathbf{n}}, \quad \mathbf{V}\mathbf{p}_1 = 0 \tag{F.69}$$

Proof. If $\mathbf{V}$ is non-invertible, it has a zero eigenvalue. If the corresponding eigenvector is $\mathbf{p}_1$, then,

$$\mathbf{V}\mathbf{p}_1 = 0 \tag{F.70}$$

From (F.60),

$$\kappa\mathbf{n} = \mathbf{V}\mathbf{q} + \mathbf{u}, \quad \kappa \in \mathbb{R} \tag{F.71}$$

$$\implies \kappa\mathbf{p}_1^\top\mathbf{n} = \mathbf{p}_1^\top\mathbf{V}\mathbf{q} + \mathbf{p}_1^\top\mathbf{u} \tag{F.72}$$

$$\text{or, } \kappa\mathbf{p}_1^\top\mathbf{n} = \mathbf{p}_1^\top\mathbf{u}, \quad \because \mathbf{p}_1^\top\mathbf{V} = 0, \quad (\text{ from (F.70)}) \tag{F.73}$$

yielding κ in (F.69). From (F.71),

$$\kappa\mathbf{q}^\top\mathbf{n} = \mathbf{q}^\top\mathbf{V}\mathbf{q} + \mathbf{q}^\top\mathbf{u} \tag{F.74}$$

$$\implies \kappa\mathbf{q}^\top\mathbf{n} = -f - \mathbf{q}^\top\mathbf{u} \quad \text{from (F.31)}, \tag{F.75}$$

$$\text{or, } (\kappa\mathbf{n} + \mathbf{u})^\top \mathbf{q} = -f \tag{F.76}$$

(F.71) can be expressed as

$$\mathbf{V}\mathbf{q} = \kappa\mathbf{n} - \mathbf{u}. \tag{F.77}$$

(F.76) and (F.77) clubbed together result in (F.68). □

F.4.4. The normal vectors of the tangents to the conic in (D.3) satisfy

$$\mathbf{n}^\top \mathbf{V}^{-1}\mathbf{n} - f_0 = 0 \tag{F.78}$$

Proof. From (F.29), the normal vector to the tangent at $\mathbf{q}$ can be expressed as

$$\mathbf{n} = \mathbf{V}\mathbf{q} + \mathbf{u} \tag{F.79}$$

$$\implies \mathbf{q} = \mathbf{V}^{-1}\left(\mathbf{n} - \mathbf{u}\right) \tag{F.80}$$

which upon substituting in (D.3) yields

$$\left(\mathbf{n} - \mathbf{u}\right)^\top \mathbf{V}^{-1}\mathbf{V}\mathbf{V}^{-1}\left(\mathbf{n} - \mathbf{u}\right) + 2\mathbf{u}^\top \mathbf{V}^{-1}\left(\mathbf{n} - \mathbf{u}\right) + f = 0 \tag{F.81}$$

which can be simplified to obtain (F.78). □

F.4.5. The normal vectors of the tangents to the conic in (D.3) from a point $\mathbf{h}$ are given by

Proof. Let the equation of the tangent be

$$\mathbf{n}^\top \mathbf{x} = c \tag{F.82}$$

If $\mathbf{q}$ be the point of contact, since $\mathbf{h}, \mathbf{q}$ lie on (F.82),

$$\mathbf{n}^\top \mathbf{q} = \mathbf{n}^\top \mathbf{h} = c \tag{F.83}$$

From (F.79), □

F.4.6. The normal vectors of the tangents to the conic in (D.3) from a point $\mathbf{h}$ are given by

$$
\begin{aligned}
\mathbf{n}_1 &= \mathbf{P}\begin{pmatrix}\sqrt{|\lambda_1|}\\ \sqrt{|\lambda_2|}\end{pmatrix}\\
\mathbf{n}_2 &= \mathbf{P}\begin{pmatrix}\sqrt{|\lambda_1|}\\ -\sqrt{|\lambda_2|}\end{pmatrix}
\end{aligned}
\tag{F.84}
$$

where $\lambda_i, \mathbf{P}$ are the eigenparameters of

$$
\mathbf{\Sigma} = (\mathbf{Vh}+\mathbf{u})(\mathbf{Vh}+\mathbf{u})^\top - \mathbf{V}\left(\mathbf{h}^\top\mathbf{Vh}+2\mathbf{u}^\top\mathbf{h}+f\right). \tag{F.85}
$$

Proof. From (F.26), and (F.30)

$$
\left[\mathbf{m}^\top(\mathbf{Vh}+\mathbf{u})\right]^2 - \left(\mathbf{m}^\top\mathbf{Vm}\right)\left(\mathbf{h}^\top\mathbf{Vh}+2\mathbf{u}^\top\mathbf{h}+f\right) = 0 \tag{F.86}
$$

$$
\implies \mathbf{m}^\top\left[(\mathbf{Vh}+\mathbf{u})(\mathbf{Vh}+\mathbf{u})^\top - \mathbf{V}\left(\mathbf{h}^\top\mathbf{Vh}+2\mathbf{u}^\top\mathbf{h}+f\right)\right]\mathbf{m} = 0 \tag{F.87}
$$

yielding (F.85). Consequently, from (F.16), (F.84) can be obtained. □