

Matrix Analysis

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Abstract

This manual provides an introduction to vectors and their properties, based on the question papers, year 2020, from Class 10 and 12, CBSE.

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1 DEFINITIONS

1.1 2×1 vectors

1.1.1. Let

$$\mathbf{A} \equiv \vec{A} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \quad (1.1.1.1)$$

$$\equiv a_1 \vec{i} + a_2 \vec{j}, \quad (1.1.1.2)$$

$$\mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \quad (1.1.1.3)$$

be 2×1 vectors. Then, the determinant of the 2×2 matrix

$$\mathbf{M} = (\mathbf{A} \ \mathbf{B}) \quad (1.1.1.4)$$

is defined as

$$|\mathbf{M}| = |\mathbf{A} \ \mathbf{B}| \quad (1.1.1.5)$$

$$= \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1 \quad (1.1.1.6)$$

1.1.2. The value of the cross product of two vectors is given by (1.1.1.5).

1.1.3. The area of the triangle with vertices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ is given by the absolute value of

$$\frac{1}{2} |\mathbf{A} - \mathbf{B} \ \mathbf{A} - \mathbf{C}| \quad (1.1.3.1)$$

1.1.4. The transpose of \mathbf{A} is defined as

$$\mathbf{A}^\top = (a_1 \ a_2) \quad (1.1.4.1)$$

1.1.5. The *inner product* or *dot product* is defined as

$$\mathbf{A}^\top \mathbf{B} \equiv \mathbf{A} \cdot \mathbf{B} \quad (1.1.5.1)$$

$$= (a_1 \ a_2) \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = a_1 b_1 + a_2 b_2 \quad (1.1.5.2)$$

1.1.6. *norm* of \mathbf{A} is defined as

$$\|\mathbf{A}\| \equiv |\vec{A}| \quad (1.1.6.1)$$

$$= \sqrt{\mathbf{A}^\top \mathbf{A}} = \sqrt{a_1^2 + a_2^2} \quad (1.1.6.2)$$

Thus,

$$\|\lambda \mathbf{A}\| \equiv |\lambda \vec{A}| \quad (1.1.6.3)$$

$$= |\lambda| \|\mathbf{A}\| \quad (1.1.6.4)$$

1.1.7. The distance between the points \mathbf{A} and \mathbf{B} is given by

$$\|\mathbf{A} - \mathbf{B}\| \quad (1.1.7.1)$$

1.1.8. Let \mathbf{x} be equidistant from the points \mathbf{A} and \mathbf{B} . Then

$$(\mathbf{A} - \mathbf{B})^\top \mathbf{x} = \frac{\|\mathbf{A}\|^2 - \|\mathbf{B}\|^2}{2} \quad (1.1.8.1)$$

Solution:

$$\|\mathbf{x} - \mathbf{A}\| = \|\mathbf{A} - \mathbf{B}\| \quad (1.1.8.2)$$

$$\implies \|\mathbf{x} - \mathbf{A}\|^2 = \|\mathbf{x} - \mathbf{B}\|^2 \quad (1.1.8.3)$$

which can be expressed as

$$\begin{aligned} (\mathbf{x} - \mathbf{A})^\top (\mathbf{x} - \mathbf{A}) &= (\mathbf{x} - \mathbf{B})^\top (\mathbf{x} - \mathbf{B}) \\ \implies \|\mathbf{x}\|^2 - 2\mathbf{x}^\top \mathbf{A} + \|\mathbf{A}\|^2 &= \|\mathbf{x}\|^2 - 2\mathbf{x}^\top \mathbf{B} + \|\mathbf{B}\|^2 \end{aligned} \quad (1.1.8.4)$$

which can be simplified to obtain (1.1.8.1).

1.1.9. If \mathbf{x} lies on the x -axis and is equidistant from the points \mathbf{A} and \mathbf{B} ,

$$\mathbf{x} = x\mathbf{e}_1 \quad (1.1.9.1)$$

where

$$x = \frac{\|\mathbf{A}\|^2 - \|\mathbf{B}\|^2}{2(\mathbf{A} - \mathbf{B})^\top \mathbf{e}_1} \quad (1.1.9.2)$$

Solution: From (1.1.8.1).

$$x(\mathbf{A} - \mathbf{B})^\top \mathbf{e}_1 = \frac{\|\mathbf{A}\|^2 - \|\mathbf{B}\|^2}{2} \quad (1.1.9.3)$$

yielding (1.1.9.2).

1.1.10. The angle between two vectors is given by

$$\theta = \cos^{-1} \frac{\mathbf{A}^\top \mathbf{B}}{\|\mathbf{A}\| \|\mathbf{B}\|} \quad (1.1.10.1)$$

1.1.11. If two vectors are orthogonal (perpendicular),

$$\mathbf{A}^\top \mathbf{B} = 0 \quad (1.1.11.1)$$

1.1.12. The *direction vector* of the line joining two points \mathbf{A}, \mathbf{B} is given by

$$\mathbf{m} = \mathbf{A} - \mathbf{B} \quad (1.1.12.1)$$

1.1.13. The unit vector in the direction of \mathbf{m} is defined as

$$\frac{\mathbf{m}}{\|\mathbf{m}\|} \quad (1.1.13.1)$$

1.1.14. If the direction vector of a line is expressed as

$$\mathbf{m} = \begin{pmatrix} 1 \\ m \end{pmatrix}, \quad (1.1.14.1)$$

the m is defined to be the slope of the line.

1.1.15. The *normal vector* to \mathbf{m} is defined by

$$\mathbf{m}^\top \mathbf{n} = 0 \quad (1.1.15.1)$$

1.1.16. The point \mathbf{P} that divides the line segment AB in the ratio $k : 1$ is given by

$$\mathbf{P} = \frac{k\mathbf{B} + \mathbf{A}}{k + 1} \quad (1.1.16.1)$$

1.1.17. The standard basis vectors are defined as

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (1.1.17.1)$$

1.2 3×1 vectors

1.2.1. Let

$$\mathbf{A} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \equiv a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{j}, \quad (1.2.1.1)$$

$$\mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \quad (1.2.1.2)$$

and

$$\mathbf{A}_{ij} = \begin{pmatrix} a_i \\ a_j \end{pmatrix}, \mathbf{B}_{ij} = \begin{pmatrix} b_i \\ b_j \end{pmatrix}, \quad (1.2.1.3)$$

1.2.2. The *cross product* or *vector product* of \mathbf{A}, \mathbf{B} is defined as

$$\mathbf{A} \times \mathbf{B} = \begin{pmatrix} \begin{vmatrix} \mathbf{A}_{23} & \mathbf{B}_{23} \\ \mathbf{A}_{31} & \mathbf{B}_{31} \end{vmatrix} \\ \begin{vmatrix} \mathbf{A}_{31} & \mathbf{B}_{31} \\ \mathbf{A}_{12} & \mathbf{B}_{12} \end{vmatrix} \\ \begin{vmatrix} \mathbf{A}_{12} & \mathbf{B}_{12} \\ \mathbf{A}_{23} & \mathbf{B}_{23} \end{vmatrix} \end{pmatrix} \quad (1.2.2.1)$$

1.2.3. Verify that

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} \quad (1.2.3.1)$$

1.2.4. The area of a triangle is given by

$$\frac{1}{2} \|\mathbf{A} \times \mathbf{B}\| \quad (1.2.4.1)$$

1.3 Eigenvalues and Eigenvectors

1.3.1. The eigenvalue λ and the eigenvector \mathbf{x} for a matrix \mathbf{A} are defined as,

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x} \quad (1.3.1.1)$$

1.3.2. The eigenvalues are calculated by solving the equation

$$f(\lambda) = |\lambda\mathbf{I} - \mathbf{A}| = 0 \quad (1.3.2.1)$$

The above equation is known as the characteristic equation.

1.3.3. According to the Cayley-Hamilton theorem,

$$f(\lambda) = 0 \implies f(\mathbf{A}) = 0 \quad (1.3.3.1)$$

1.3.4. The trace of a square matrix is defined to be the sum of the diagonal elements.

$$\text{tr}(\mathbf{A}) = \sum_{i=1}^N a_{ii}. \quad (1.3.4.1)$$

where a_{ii} is the i th diagonal element of the matrix \mathbf{A} .

1.3.5. The trace of a matrix is equal to the sum of the eigenvalues

$$\text{tr}(\mathbf{A}) = \sum_{i=1}^N \lambda_i \quad (1.3.5.1)$$

1.4 Determinants

1.4.1. Let

$$\mathbf{A} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}. \quad (1.4.1.1)$$

be a 3×3 matrix. Then,

$$|\mathbf{A}| = a_1 \begin{pmatrix} b_2 & c_2 \\ b_3 & c_3 \end{pmatrix} - a_2 \begin{pmatrix} b_1 & c_1 \\ b_3 & c_3 \end{pmatrix} + a_3 \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}. \quad (1.4.1.2)$$

1.4.2. Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigenvalues of a matrix \mathbf{A} . Then, the product of the eigenvalues is equal to the determinant of \mathbf{A} .

$$|\mathbf{A}| = \prod_{i=1}^n \lambda_i \quad (1.4.2.1)$$

1.4.3.

$$|\mathbf{AB}| = |\mathbf{A}| |\mathbf{B}| \quad (1.4.3.1)$$

1.4.4. If \mathbf{A} be an $n \times n$ matrix,

$$|k\mathbf{A}| = k^n |\mathbf{A}| \quad (1.4.4.1)$$

1.5 Rank of a Matrix

1.5.1. The rank of a matrix is defined as the number of linearly independent rows. This is also known as the row rank.

1.5.2. Row rank = Column rank.

1.5.3. The rank of a matrix is obtained as the number of nonzero rows obtained after row reduction.

1.5.4. An $n \times n$ matrix is invertible if and only if its rank is n .

1.6 Inverse of a Matrix

1.6.1. For a 2×2 matrix

$$\mathbf{A} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}, \quad (1.6.1.1)$$

the inverse is given by

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|} \begin{pmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{pmatrix}, \quad (1.6.1.2)$$

1.6.2. For higher order matrices, the inverse should be calculated using row operations.

2 LINEAR FORMS

2.1 Two Dimensions

2.1.1. The equation of a line is given by

$$\mathbf{n}^\top \mathbf{x} = c \quad (2.1.1.1)$$

where \mathbf{n} is the normal vector of the line.

2.1.2. The equation of a line with normal vector \mathbf{n} and passing through a point \mathbf{A} is given by

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{A}) = 0 \quad (2.1.2.1)$$

2.1.3. The parametric equation of a line is given by

$$\mathbf{x} = \mathbf{A} + \lambda \mathbf{m} \quad (2.1.3.1)$$

where \mathbf{m} is the direction vector of the line and \mathbf{A} is any point on the line.

2.1.4. Let \mathbf{A} and \mathbf{B} be two points on a straight line and let $\mathbf{P} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$ be any point on it. If p_2 is known, then

$$\mathbf{P} = \mathbf{A} + \frac{p_2 - \mathbf{e}_2^\top \mathbf{A}}{\mathbf{e}_2^\top (\mathbf{B} - \mathbf{A})} (\mathbf{B} - \mathbf{A}) \quad (2.1.4.1)$$

Solution: The equation of the line can be expressed in parametric form as

$$\mathbf{x} = \mathbf{A} + \lambda (\mathbf{B} - \mathbf{A}) \quad (2.1.4.2)$$

$$\implies \mathbf{P} = \mathbf{A} + \lambda (\mathbf{B} - \mathbf{A}) \quad (2.1.4.3)$$

$$\implies \mathbf{e}_2^\top \mathbf{P} = \mathbf{e}_2^\top \mathbf{A} + \lambda \mathbf{e}_2^\top (\mathbf{B} - \mathbf{A}) \quad (2.1.4.4)$$

$$\implies p_2 = \mathbf{e}_2^\top \mathbf{A} + \lambda \mathbf{e}_2^\top (\mathbf{B} - \mathbf{A}) \quad (2.1.4.5)$$

$$\text{or, } \lambda = \frac{p_2 - \mathbf{e}_2^\top \mathbf{A}}{\mathbf{e}_2^\top (\mathbf{B} - \mathbf{A})} \quad (2.1.4.6)$$

yielding (2.1.4.1).

2.1.5. The distance from a point \mathbf{P} to the line in (2.1.1.1) is given by

$$d = \frac{|\mathbf{n}^\top \mathbf{P} - c|}{\|\mathbf{n}\|} \quad (2.1.5.1)$$

Solution: Without loss of generality, let \mathbf{A} be the foot of the perpendicular from \mathbf{P} to the line in (2.1.3.1). The equation of the normal to (2.1.1.1) can then be expressed as

$$\mathbf{x} = \mathbf{A} + \lambda \mathbf{n} \quad (2.1.5.2)$$

$$\implies \mathbf{P} - \mathbf{A} = \lambda \mathbf{n} \quad (2.1.5.3)$$

$\therefore \mathbf{P}$ lies on (2.1.5.2). From the above, the desired distance can be expressed as

$$d = \|\mathbf{P} - \mathbf{A}\| = |\lambda| \|\mathbf{n}\| \quad (2.1.5.4)$$

From (2.1.5.3),

$$\mathbf{n}^\top (\mathbf{P} - \mathbf{A}) = \lambda \mathbf{n}^\top \mathbf{n} = \lambda \|\mathbf{n}\|^2 \quad (2.1.5.5)$$

$$\implies |\lambda| = \frac{|\mathbf{n}^\top (\mathbf{P} - \mathbf{A})|}{\|\mathbf{n}\|^2} \quad (2.1.5.6)$$

Substituting the above in (2.1.5.4) and using the fact that

$$\mathbf{n}^\top \mathbf{A} = c \quad (2.1.5.7)$$

from (2.1.1.1), yields (2.1.5.1)

2.1.6. The distance from the origin to the line in (2.1.1.1) is given by

$$d = \frac{|c|}{\|\mathbf{n}\|} \quad (2.1.6.1)$$

2.1.7. The distance between the parallel lines

$$\begin{aligned} \mathbf{n}^\top \mathbf{x} &= c_1 \\ \mathbf{n}^\top \mathbf{x} &= c_2 \end{aligned} \quad (2.1.7.1)$$

is given by

$$d = \frac{|c_1 - c_2|}{\|\mathbf{n}\|} \quad (2.1.7.2)$$

2.1.8. The equation of the line perpendicular to (2.1.1.1) and passing through the point \mathbf{P} is given by

$$\mathbf{m}^\top (\mathbf{x} - \mathbf{P}) = 0 \quad (2.1.8.1)$$

2.1.9. The foot of the perpendicular from \mathbf{P} to the line in (2.1.1.1) is given by

$$(\mathbf{m} \quad \mathbf{n})^\top \mathbf{x} = \begin{pmatrix} \mathbf{m}^\top \mathbf{P} \\ c \end{pmatrix} \quad (2.1.9.1)$$

Solution: From (2.1.1.1) and (2.1.2.1) the foot of the perpendicular satisfies the equations

$$\mathbf{n}^\top \mathbf{x} = c \quad (2.1.9.2)$$

$$\mathbf{m}^\top (\mathbf{x} - \mathbf{P}) = 0 \quad (2.1.9.3)$$

where \mathbf{m} is the direction vector of the given line. Combining the above into a matrix equation results in (2.1.9.1).

2.2 Three Dimensions

2.2.1. The area of a triangle with vertices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ is given by

$$\frac{1}{2} \|(\mathbf{A} - \mathbf{B}) \times (\mathbf{A} - \mathbf{C})\| \quad (2.2.1.1)$$

2.2.2. Points $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are on a line if

$$\text{rank} \begin{pmatrix} \mathbf{A} \\ \mathbf{B} \\ \mathbf{C} \end{pmatrix} = 1 \quad (2.2.2.1)$$

2.2.3. Points $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ form a parallelogram if

$$\text{rank} \begin{pmatrix} \mathbf{A} \\ \mathbf{B} \\ \mathbf{C} \\ \mathbf{D} \end{pmatrix} = 1, \text{rank} \begin{pmatrix} \mathbf{A} \\ \mathbf{B} \\ \mathbf{C} \end{pmatrix} = 2 \quad (2.2.3.1)$$

2.2.4. The equation of a line is given by (2.1.3.1)

2.2.5. The equation of a plane is given by (2.1.1.1)

2.2.6. The distance from the origin to the line in (2.1.1.1) is given by (2.1.6.1)

2.2.7. The distance from a point \mathbf{P} to the line in (2.1.3.1) is given by

$$d = \|\mathbf{A} - \mathbf{P}\|^2 - \frac{\{\mathbf{m}^\top (\mathbf{A} - \mathbf{P})\}^2}{\|\mathbf{m}\|^2} \quad (2.2.7.1)$$

Solution:

$$d(\lambda) = \|\mathbf{A} + \lambda\mathbf{m} - \mathbf{P}\| \quad (2.2.7.2)$$

$$\implies d^2(\lambda) = \|\mathbf{A} + \lambda\mathbf{m} - \mathbf{P}\|^2 \quad (2.2.7.3)$$

which can be simplified to obtain

$$d^2(\lambda) = \lambda^2 \|\mathbf{m}\|^2 + 2\lambda\mathbf{m}^\top (\mathbf{A} - \mathbf{P}) + \|\mathbf{A} - \mathbf{P}\|^2 \quad (2.2.7.4)$$

which is of the form

$$d^2(\lambda) = a\lambda^2 + 2b\lambda + c \quad (2.2.7.5)$$

$$= a \left\{ \left(\lambda + \frac{b}{a} \right)^2 + \left[\frac{c}{a} - \left(\frac{b}{a} \right)^2 \right] \right\} \quad (2.2.7.6)$$

with

$$a = \|\mathbf{m}\|^2, b = \mathbf{m}^\top (\mathbf{A} - \mathbf{P}), c = \|\mathbf{A} - \mathbf{P}\|^2 \quad (2.2.7.7)$$

which can be expressed as From the above, $d^2(\lambda)$ is smallest when upon substituting from (2.2.7.7)

$$\lambda + \frac{b}{2a} = 0 \implies \lambda = -\frac{b}{2a} \quad (2.2.7.8)$$

$$= -\frac{\mathbf{m}^\top (\mathbf{A} - \mathbf{P})}{\|\mathbf{m}\|^2} \quad (2.2.7.9)$$

and consequently,

$$d_{\min}(\lambda) = a \left(\frac{c}{a} - \left(\frac{b}{a} \right)^2 \right) \quad (2.2.7.10)$$

$$= c - \frac{b^2}{a} \quad (2.2.7.11)$$

yielding (2.2.7.1) after substituting from (2.2.7.7).

2.2.8. The distance between the parallel planes (2.1.7.1) is given by (2.1.7.2).

2.2.9. The plane

$$\mathbf{n}^\top \mathbf{x} = c \quad (2.2.9.1)$$

contains the line

$$\mathbf{x} = \mathbf{A} + \lambda\mathbf{m} \quad (2.2.9.2)$$

if

$$\mathbf{m}^\top \mathbf{n} = 0 \quad (2.2.9.3)$$

Solution: Any point on the line (2.2.9.2) should also satisfy (2.2.9.1). Hence,

$$\mathbf{n}^\top (\mathbf{A} + \lambda\mathbf{m}) = \mathbf{n}^\top \mathbf{A} = c \quad (2.2.9.4)$$

which can be simplified to obtain (2.2.9.3)

2.2.10. The foot of the perpendicular from a point \mathbf{P} to the plane

$$\mathbf{n}^\top \mathbf{x} = c \quad (2.2.10.1)$$

is given by

Solution: The equation of the line perpendicular to the given plane and passing through \mathbf{P} is

$$\mathbf{x} = \mathbf{P} + \lambda \mathbf{n} \quad (2.2.10.2)$$

From (2.2.13.1), the intersection of the above line with the given plane is

$$\mathbf{x} = \mathbf{P} + \frac{c - \mathbf{n}^\top \mathbf{P}}{\|\mathbf{n}\|^2} \mathbf{n} \quad (2.2.10.3)$$

2.2.11. The image of a point \mathbf{P} with respect to the plane

$$\mathbf{n}^\top \mathbf{x} = c \quad (2.2.11.1)$$

is given by

$$\mathbf{R} = \mathbf{P} + 2 \frac{c - \mathbf{n}^\top \mathbf{P}}{\|\mathbf{n}\|^2} \mathbf{n} \quad (2.2.11.2)$$

Solution: Let \mathbf{R} be the desired image. Then, substituting the expression for the foot of the perpendicular from \mathbf{P} to the given plane using (2.2.10.3),

$$\frac{\mathbf{P} + \mathbf{R}}{2} = \mathbf{P} + \frac{c - \mathbf{n}^\top \mathbf{P}}{\|\mathbf{n}\|^2} \mathbf{n} \quad (2.2.11.3)$$

2.2.12. Let a plane pass through the points \mathbf{A}, \mathbf{B} and be perpendicular to the plane

$$\mathbf{n}^\top \mathbf{x} = c \quad (2.2.12.1)$$

Then the equation of this plane is given by

$$\mathbf{p}^\top \mathbf{x} = 1 \quad (2.2.12.2)$$

where

$$\mathbf{p} = (\mathbf{A} \ \mathbf{B} \ \mathbf{n})^{-\top} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad (2.2.12.3)$$

Solution: From the given information,

$$\mathbf{p}^\top \mathbf{A} = d \quad (2.2.12.4)$$

$$\mathbf{p}^\top \mathbf{B} = d \quad (2.2.12.5)$$

$$\mathbf{p}^\top \mathbf{n} = 0 \quad (2.2.12.6)$$

\therefore the normal vectors to the two planes will also be perpendicular. The system of equations in (2.2.12.6) can be expressed as the matrix equation

$$(\mathbf{A} \ \mathbf{B} \ \mathbf{n})^\top \mathbf{p} = d \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad (2.2.12.7)$$

which yields (2.2.12.3) upon normalising with d .

2.2.13. The intersection of the line represented by (2.1.3.1) with the plane represented by (2.1.1.1) is given by

$$\mathbf{x} = \mathbf{A} + \frac{c - \mathbf{n}^\top \mathbf{A}}{\mathbf{n}^\top \mathbf{m}} \mathbf{m} \quad (2.2.13.1)$$

Solution: From (2.1.3.1) and (2.1.1.1),

$$\mathbf{x} = \mathbf{A} + \lambda \mathbf{m} \quad (2.2.13.2)$$

$$\mathbf{n}^\top \mathbf{x} = c \quad (2.2.13.3)$$

$$\implies \mathbf{n}^\top (\mathbf{A} + \lambda \mathbf{m}) = c \quad (2.2.13.4)$$

which can be simplified to obtain

$$\mathbf{n}^\top \mathbf{A} + \lambda \mathbf{n}^\top \mathbf{m} = c \quad (2.2.13.5)$$

$$\implies \lambda = \frac{c - \mathbf{n}^\top \mathbf{A}}{\mathbf{n}^\top \mathbf{m}} \quad (2.2.13.6)$$

Substituting the above in (2.2.13.4) yields (2.2.13.1).

2.2.14. The foot of the perpendicular from the point \mathbf{P} to the line represented by (2.1.3.1) is given by

$$\mathbf{x} = \mathbf{A} + \frac{\mathbf{m}^\top (\mathbf{P} - \mathbf{A})}{\|\mathbf{m}\|^2} \mathbf{m} \quad (2.2.14.1)$$

Solution: Let the equation of the line be

$$\mathbf{x} = \mathbf{A} + \lambda \mathbf{m} \quad (2.2.14.2)$$

The equation of the plane perpendicular to the given line passing through \mathbf{P} is given by

$$\mathbf{m}^\top (\mathbf{x} - \mathbf{P}) = 0 \quad (2.2.14.3)$$

$$\implies \mathbf{m}^\top \mathbf{x} = \mathbf{m}^\top \mathbf{P} \quad (2.2.14.4)$$

The desired foot of the perpendicular is the intersection of (2.2.14.2) with (2.2.14.3) which can be obtained from (2.2.13.1) as (2.2.14.1)

2.2.15. The foot of the perpendicular from a point \mathbf{P} to a plane is \mathbf{Q} . The equation of the plane is given by

$$(\mathbf{P} - \mathbf{Q})^\top (\mathbf{x} - \mathbf{Q}) = 0 \quad (2.2.15.1)$$

Solution: The normal vector to the plane is given by

$$\mathbf{n} = \mathbf{P} - \mathbf{Q} \quad (2.2.15.2)$$

Hence, the equation of the plane is (2.2.15.1).

2.2.16. Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be points on a plane. The equation of the plane is then given by

$$(\mathbf{A} \ \mathbf{B} \ \mathbf{C})^\top \mathbf{n} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad (2.2.16.1)$$

Solution: Let the equation of the plane be

$$\mathbf{n}^\top \mathbf{x} = 1 \quad (2.2.16.2)$$

Then

$$\mathbf{n}^\top \mathbf{A} = 1 \quad (2.2.16.3)$$

$$\mathbf{n}^\top \mathbf{B} = 1 \quad (2.2.16.4)$$

$$\mathbf{n}^\top \mathbf{C} = 1 \quad (2.2.16.5)$$

which can be combined to obtain (2.2.16.1).

2.2.17. (Parallelogram Law) Let $\mathbf{A}, \mathbf{B}, \mathbf{D}$ be three vertices of a parallelogram. Then the vertex \mathbf{C} is given by

$$\mathbf{C} = \mathbf{B} + \mathbf{D} - \mathbf{A} \quad (2.2.17.1)$$

Solution: Shifting \mathbf{A} to the origin, we obtain a parallelogram with corresponding vertices

$$\mathbf{0}, \mathbf{B} - \mathbf{A}, \mathbf{D} - \mathbf{A} \quad (2.2.17.2)$$

The fourth vertex of this parallelogram is then obtained as

$$(\mathbf{B} - \mathbf{A}) + (\mathbf{D} - \mathbf{A}) = \mathbf{D} + \mathbf{B} - 2\mathbf{A} \quad (2.2.17.3)$$

Shifting the origin to \mathbf{A} , the fourth vertex is obtained as

$$\mathbf{C} = \mathbf{D} + \mathbf{B} - 2\mathbf{A} + \mathbf{A} \quad (2.2.17.4)$$

$$= \mathbf{D} + \mathbf{B} - \mathbf{A} \quad (2.2.17.5)$$

2.2.18. (Affine Transformation) Let \mathbf{A}, \mathbf{C} , be opposite vertices of a square. The other two points can be obtained as

$$\mathbf{B} = \frac{\|\mathbf{A} - \mathbf{C}\|}{\sqrt{2}} \mathbf{P} \mathbf{e}_1 + \mathbf{A} \quad (2.2.18.1)$$

$$\mathbf{D} = \frac{\|\mathbf{A} - \mathbf{C}\|}{\sqrt{2}} \mathbf{P} \mathbf{e}_2 + \mathbf{A} \quad (2.2.18.2)$$

where

$$\mathbf{P} = \begin{pmatrix} \cos\left(\theta - \frac{\pi}{4}\right) & \sin\left(\theta - \frac{\pi}{4}\right) \\ \sin\left(\theta - \frac{\pi}{4}\right) & \cos\left(\theta - \frac{\pi}{4}\right) \end{pmatrix} \quad (2.2.18.3)$$

and

$$\cos \theta = \frac{(\mathbf{C} - \mathbf{A})^\top \mathbf{e}_1}{\|\mathbf{A} - \mathbf{C}\| \|\mathbf{e}_1\|} \quad (2.2.18.4)$$

3 QUADRATIC FORMS

3.1 Definitions

Definition 1. The affine transformation is given by

$$\mathbf{x} = \mathbf{P} \mathbf{y} + \mathbf{c} \quad (\text{Affine Transformation}) \quad (3.1.1)$$

where \mathbf{P} is invertible.

Definition 2. The eigenvalue decomposition of a symmetric matrix \mathbf{V} is given by

$$\mathbf{P}^\top \mathbf{V} \mathbf{P} = \mathbf{D}. \quad (\text{Eigenvalue Decomposition}) \quad (3.1.2)$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad (3.1.3)$$

$$\mathbf{P} = (\mathbf{p}_1 \quad \mathbf{p}_2), \quad \mathbf{P}^\top = \mathbf{P}^{-1}, \quad (3.1.4)$$

3.2 The Quadratic Form

Definition 3. Let \mathbf{q} be a point such that the ratio of its distance from a fixed point \mathbf{F} and the distance (d) from a fixed line

$$L : \mathbf{n}^\top \mathbf{x} = c \quad (3.2.1)$$

is constant, given by

$$\frac{\|\mathbf{q} - \mathbf{F}\|}{d} = e \quad (3.2.2)$$

The locus of \mathbf{q} is known as a conic section. The line L is known as the directrix and the point \mathbf{F} is the focus. e is defined to be the eccentricity of the conic.

- 1) For $e = 1$, the conic is a parabola
- 2) For $e < 1$, the conic is an ellipse
- 3) For $e > 1$, the conic is a hyperbola

Theorem 3.1. The equation of a conic with directrix $\mathbf{n}^\top \mathbf{x} = c$, eccentricity e and focus \mathbf{F} is given by

$$\mathbf{x}^\top \mathbf{V} \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \quad (3.2.1)$$

where

$$\mathbf{V} = \|\mathbf{n}\|^2 \mathbf{I} - e^2 \mathbf{n} \mathbf{n}^\top, \quad (3.2.2)$$

$$\mathbf{u} = ce^2 \mathbf{n} - \|\mathbf{n}\|^2 \mathbf{F}, \quad (3.2.3)$$

$$f = \|\mathbf{n}\|^2 \|\mathbf{F}\|^2 - c^2 e^2 \quad (3.2.4)$$

Proof. Using Definition 3 and Lemma 2.1.5.1, for any point \mathbf{x} on the conic,

$$\|\mathbf{x} - \mathbf{F}\|^2 = e^2 \frac{(\mathbf{n}^\top \mathbf{x} - c)^2}{\|\mathbf{n}\|^2} \quad (3.2.5)$$

$$\implies \|\mathbf{n}\|^2 (\mathbf{x} - \mathbf{F})^\top (\mathbf{x} - \mathbf{F}) = e^2 (\mathbf{n}^\top \mathbf{x} - c)^2 \quad (3.2.6)$$

$$\implies \|\mathbf{n}\|^2 (\mathbf{x}^\top \mathbf{x} - 2\mathbf{F}^\top \mathbf{x} + \|\mathbf{F}\|^2) = e^2 (c^2 + (\mathbf{n}^\top \mathbf{x})^2 - 2c\mathbf{n}^\top \mathbf{x}) \quad (3.2.7)$$

$$= e^2 (c^2 + (\mathbf{x}^\top \mathbf{n} \mathbf{n}^\top \mathbf{x}) - 2c\mathbf{n}^\top \mathbf{x}) \quad (3.2.8)$$

which can be expressed as (3.2.1) after simplification. □

Theorem 3.2. The eccentricity, directrices and foci of (3.2.1) are given by

$$e = \sqrt{1 - \frac{\lambda_1}{\lambda_2}} \quad (3.2.9)$$

$$\mathbf{n} = \sqrt{\lambda_2} \mathbf{p}_1,$$

$$c = \begin{cases} \frac{e\mathbf{u}^\top \mathbf{n} \pm \sqrt{e^2 (\mathbf{u}^\top \mathbf{n})^2 - \lambda_2 (e^2 - 1) (\|\mathbf{u}\|^2 - \lambda_2 f)}}{\lambda_2 e (e^2 - 1)} & e \neq 1 \\ \frac{\|\mathbf{u}\|^2 - \lambda_2 f}{2\mathbf{u}^\top \mathbf{n}} & e = 1 \end{cases} \quad (3.2.10)$$

$$\mathbf{F} = \frac{ce^2 \mathbf{n} - \mathbf{u}}{\lambda_2} \quad (3.2.11)$$

Proof. See Appendix A □

Theorem 3.3. (3.2.1) represents

- 1) a parabola for $|\mathbf{V}| = 0$,

- 2) ellipse for $|\mathbf{V}| > 0$ and
 3) hyperbola for $|\mathbf{V}| < 0$.

Proof. From (3.2.9),

$$\frac{\lambda_1}{\lambda_2} = 1 - e^2 \quad (3.2.1)$$

Also,

$$|\mathbf{V}| = \lambda_1 \lambda_2 \quad (3.2.2)$$

yielding Table 3 □

Eccentricity	Conic	Eigenvalue	Determinant
$e = 1$	Parabola	$\lambda_1 = 0$	$ \mathbf{V} = 0$
$e < 1$	Ellipse	$\lambda_1 > 0, \lambda_2 > 0$	$ \mathbf{V} > 0$
$e > 1$	Hyperbola	$\lambda_1 < 0, \lambda_2 > 0$	$ \mathbf{V} < 0$

TABLE 3

3.3 Standard Form

Theorem 3.4. Using the affine transformation in (3.1.1), the conic in (3.2.1) can be expressed in standard form as

$$\mathbf{y}^\top \left(\frac{\mathbf{D}}{f_0} \right) \mathbf{y} = 1 \quad |\mathbf{V}| \neq 0 \quad (3.3.1)$$

$$\mathbf{y}^\top \mathbf{D} \mathbf{y} = -\eta \mathbf{e}_1^\top \mathbf{y} \quad |\mathbf{V}| = 0 \quad (3.3.2)$$

where

$$f_0 = \mathbf{u}^\top \mathbf{V}^{-1} \mathbf{u} - f \neq 0 \quad (3.3.3)$$

$$\eta = 2\mathbf{u}^\top \mathbf{p}_1 \quad (3.3.4)$$

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (3.3.5)$$

Proof. See Appendix B. □

Corollary 3.5. For the standard conic,

$$\mathbf{P} = \mathbf{I} \quad (3.3.6)$$

$$\mathbf{u} = \begin{cases} 0 & e \neq 1 \\ \frac{\eta}{2} \mathbf{e}_1 & e = 1 \end{cases} \quad (3.3.7)$$

$$\lambda_1 \begin{cases} = 0 & e = 1 \\ \neq 0 & e \neq 1 \end{cases} \quad (3.3.8)$$

where

$$\mathbf{I} = (\mathbf{e}_1 \quad \mathbf{e}_2) \quad (3.3.9)$$

is the identity matrix.

Theorem 3.6.

1) The directrices for the standard conic are given by

$$\mathbf{e}_1^\top \mathbf{y} = \pm \frac{1}{e} \sqrt{\frac{|f_0|}{\lambda_2(1-e^2)}} \quad e \neq 1 \quad (3.3.1)$$

$$\mathbf{e}_1^\top \mathbf{y} = \frac{\eta}{2\lambda_2} \quad e = 1 \quad (3.3.2)$$

2) The foci of the standard ellipse and hyperbola are given by

$$\mathbf{F} = \begin{cases} \pm e \sqrt{\frac{|f_0|}{\lambda_2(1-e^2)}} \mathbf{e}_1 & e \neq 1 \\ -\frac{\eta}{4\lambda_2} \mathbf{e}_1 & e = 1 \end{cases} \quad (3.3.1)$$

Proof. See Appendix C. □

3.4 Corollaries-Standard Form

Corollary 3.7. The center of the standard ellipse/hyperbola, defined to be the mid point of the line joining the foci, is the origin.

Corollary 3.8. The principal (major) axis of the standard ellipse/hyperbola, defined to be the line joining the two foci is the x -axis.

Proof. From (3.3.1), it is obvious that the line joining the foci passes through the origin. Also, the direction vector of this line is \mathbf{e}_1 . Thus, the principal axis is the x -axis. □

Corollary 3.9. The minor axis of the standard ellipse/hyperbola, defined to be the line orthogonal to the x -axis is the y -axis.

Corollary 3.10. The axis of symmetry of the standard parabola, defined to be the line perpendicular to the directrix and passing through the focus, is the x -axis.

Proof. From (3.0.2.4) and (3.3.1), the axis of the parabola can be expressed using (2.1.2.1) as

$$\mathbf{e}_2^\top \left(\mathbf{y} + \frac{\eta}{4\lambda_2} \mathbf{e}_1 \right) = 0 \quad (3.4.1)$$

$$\implies \mathbf{e}_2^\top \mathbf{y} = 0, \quad (3.4.2)$$

which is the equation of the x -axis. □

Corollary 3.11. The point where the parabola intersects its axis of symmetry is called the vertex. For the standard parabola, the vertex is the origin.

Proof. (3.4.2) can be expressed as

$$\mathbf{y} = \alpha \mathbf{e}_1, \quad (3.4.3)$$

using (2.1.2.1). Substituting (3.4.3) in (3.3.2),

$$\alpha^2 \mathbf{e}_1^\top \mathbf{D} \mathbf{e}_1 = -\eta \alpha \mathbf{e}_1^\top \mathbf{e}_1 \quad (3.4.4)$$

$$\implies \alpha = 0, \text{ or, } \mathbf{y} = \mathbf{0}. \quad (3.4.5)$$

□

Corollary 3.12. The focal length of the standard parabola, defined to be the distance between the vertex and the focus, measured along the axis of symmetry, is $\left| \frac{\eta}{4\lambda_2} \right|$

3.5 Corollaries-Quadratic Form

Corollary 3.13. *The center/vertex of a conic section are given by*

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \quad |\mathbf{V}| \neq 0 \quad (3.5.1)$$

$$\begin{pmatrix} \mathbf{u}^\top + \frac{\eta}{2}\mathbf{p}_1^\top \\ \mathbf{v} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -f \\ \frac{\eta}{2}\mathbf{p}_1 - \mathbf{u} \end{pmatrix} \quad |\mathbf{V}| = 0 \quad (3.5.2)$$

Proof. In (3.1.1), substituting $\mathbf{y} = 0$, the center/vertex for the quadratic form is obtained as

$$\mathbf{x} = \mathbf{c}, \quad (3.5.3)$$

where \mathbf{c} is derived as (3.5.1) and (3.5.2) in Appendix B. \square

Corollary 3.14. *The equation of the minor and major axes for the ellipse/hyperbola are respectively given by*

$$\mathbf{p}_i^\top (\mathbf{x} - \mathbf{c}) = 0, i = 1, 2 \quad (3.5.4)$$

The axis of symmetry for the parabola is also given by (3.5.4).

Proof. From (3.8), the major/symmetry axis for the hyperbola/ellipse/parabola can be expressed using (3.1.1) as

$$\mathbf{e}_2^\top \mathbf{P}^\top (\mathbf{x} - \mathbf{c}) = 0 \quad (3.5.5)$$

$$\implies (\mathbf{P}\mathbf{e}_2)^\top (\mathbf{x} - \mathbf{c}) = 0 \quad (3.5.6)$$

yielding (3.5.4), and the proof for the minor axis is similar. \square

3.6 Pair of Straight Lines

Lemma 3.1 (Asymptotes). *The asymptotes of the hyperbola in (3.3.1), defined to be the lines that do not intersect the hyperbola, are given by*

$$(\sqrt{|\lambda_1|} \pm \sqrt{|\lambda_2|}) \mathbf{y} = 0 \quad (3.6.1)$$

Proof. From (3.3.1), it is obvious that the pair of lines represented by

$$\mathbf{y}^\top \mathbf{D} \mathbf{y} = 0 \quad (3.6.2)$$

do not intersect the conic

$$\mathbf{y}^\top \mathbf{D} \mathbf{y} = f_0 \quad (3.6.3)$$

Thus, (3.6.2) represents the asymptotes of the hyperbola in (3.3.1) and can be expressed as

$$\lambda_1 y_1^2 + \lambda_2 y_2^2 = 0, \quad (3.6.4)$$

which can then be simplified to obtain (3.6.1). \square

Corollary 3.15. (3.2.1) represents a pair of straight lines if

$$\mathbf{u}^\top \mathbf{V}^{-1} \mathbf{u} - f = 0 \quad (3.6.5)$$

Theorem 3.16. (3.2.1) represents a pair of straight lines if the matrix

$$\begin{pmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u} & f \end{pmatrix} \quad (3.6.6)$$

is singular.

Proof. Let

$$\begin{pmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u} & f \end{pmatrix} \mathbf{x} = \mathbf{0} \quad (3.6.7)$$

Expressing

$$\mathbf{x} = \begin{pmatrix} \mathbf{y} \\ y_3 \end{pmatrix}, \quad (3.6.8)$$

$$\begin{pmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^\top & f \end{pmatrix} \begin{pmatrix} \mathbf{y} \\ y_3 \end{pmatrix} = \mathbf{0} \quad (3.6.9)$$

$$\implies \mathbf{V}\mathbf{y} + y_3\mathbf{u} = \mathbf{0} \quad \text{and} \quad (3.6.10)$$

$$\mathbf{u}^\top \mathbf{y} + f y_3 = 0 \quad (3.6.11)$$

From (3.6.10) we obtain,

$$\mathbf{y}^\top \mathbf{V} \mathbf{y} + y_3 \mathbf{y}^\top \mathbf{u} = 0 \quad (3.6.12)$$

$$\implies \mathbf{y}^\top \mathbf{V} \mathbf{y} + y_3 \mathbf{u}^\top \mathbf{y} = 0 \quad (3.6.13)$$

yielding (3.6.5) upon substituting from (3.6.11). \square

Corollary 3.17. *Using the affine transformation, (3.6.1) can be expressed as the lines*

$$(\sqrt{|\lambda_1|} \pm \sqrt{|\lambda_2|}) \mathbf{P}^\top (\mathbf{x} - \mathbf{c}) = 0 \quad (3.6.14)$$

Corollary 3.18. *The angle between the asymptotes can be expressed as*

$$\cos \theta = \frac{|\lambda_1| - |\lambda_2|}{|\lambda_1| + |\lambda_2|} \quad (3.6.15)$$

Proof. The normal vectors of the lines in (3.6.14) are

$$\begin{aligned} \mathbf{n}_1 &= \mathbf{P} \begin{pmatrix} \sqrt{|\lambda_1|} \\ \sqrt{|\lambda_2|} \end{pmatrix} \\ \mathbf{n}_2 &= \mathbf{P} \begin{pmatrix} \sqrt{|\lambda_1|} \\ -\sqrt{|\lambda_2|} \end{pmatrix} \end{aligned} \quad (3.6.16)$$

The angle between the asymptotes is given by

$$\cos \theta = \frac{\mathbf{n}_1^\top \mathbf{n}_2}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} \quad (3.6.17)$$

The orthogonal matrix \mathbf{P} preserves the norm, i.e.

$$\|\mathbf{n}_1\| = \left\| \mathbf{P} \begin{pmatrix} \sqrt{|\lambda_1|} \\ \sqrt{|\lambda_2|} \end{pmatrix} \right\| = \left\| \begin{pmatrix} \sqrt{|\lambda_1|} \\ \sqrt{|\lambda_2|} \end{pmatrix} \right\| \quad (3.6.18)$$

$$= \sqrt{|\lambda_1| + |\lambda_2|} = \|\mathbf{n}_2\| \quad (3.6.19)$$

It is easy to verify that

$$\mathbf{n}_1^\top \mathbf{n}_2 = |\lambda_1| - |\lambda_2| \quad (3.6.20)$$

Thus, the angle between the asymptotes is obtained from (3.6.17) as (3.6.15). \square

3.7 Chords of a Conic

Theorem 3.19 (Chord). *The points of intersection of the line*

$$L : \mathbf{x} = \mathbf{q} + \mu \mathbf{m} \quad \mu \in \mathbb{R} \quad (3.7.1)$$

with the conic section in (3.2.1) are given by

$$\mathbf{x}_i = \mathbf{q} + \mu_i \mathbf{m} \quad (3.7.2)$$

where

$$\mu_i = \frac{1}{\mathbf{m}^\top \mathbf{V} \mathbf{m}} \left(-\mathbf{m}^\top (\mathbf{V} \mathbf{q} + \mathbf{u}) \pm \sqrt{[\mathbf{m}^\top (\mathbf{V} \mathbf{q} + \mathbf{u})]^2 - (\mathbf{q}^\top \mathbf{V} \mathbf{q} + 2\mathbf{u}^\top \mathbf{q} + f)(\mathbf{m}^\top \mathbf{V} \mathbf{m})} \right) \quad (3.7.3)$$

Proof. Substituting (3.7.1) in (3.2.1),

$$(\mathbf{q} + \mu \mathbf{m})^\top \mathbf{V} (\mathbf{q} + \mu \mathbf{m}) + 2\mathbf{u}^\top (\mathbf{q} + \mu \mathbf{m}) + f = 0 \quad (3.7.4)$$

$$\implies \mu^2 \mathbf{m}^\top \mathbf{V} \mathbf{m} + 2\mu \mathbf{m}^\top (\mathbf{V} \mathbf{q} + \mathbf{u}) + \mathbf{q}^\top \mathbf{V} \mathbf{q} + 2\mathbf{u}^\top \mathbf{q} + f = 0 \quad (3.7.5)$$

Solving the above quadratic in (3.7.5) yields (3.7.3). \square

Corollary 3.20. *If L in (3.7.1) touches (3.2.1) at exactly one point \mathbf{q} ,*

$$\mathbf{m}^\top (\mathbf{V} \mathbf{q} + \mathbf{u}) = 0 \quad (3.7.6)$$

Proof. In this case, (3.7.5) has exactly one root. Hence, in (3.7.3)

$$[\mathbf{m}^\top (\mathbf{V} \mathbf{q} + \mathbf{u})]^2 - (\mathbf{m}^\top \mathbf{V} \mathbf{m}) (\mathbf{q}^\top \mathbf{V} \mathbf{q} + 2\mathbf{u}^\top \mathbf{q} + f) = 0 \quad (3.7.7)$$

$\therefore \mathbf{q}$ is the point of contact, \mathbf{q} satisfies (3.2.1) and

$$\mathbf{q}^\top \mathbf{V} \mathbf{q} + 2\mathbf{u}^\top \mathbf{q} + f = 0 \quad (3.7.8)$$

Substituting (3.7.8) in (3.7.7) and simplifying, we obtain (3.7.6). \square

Theorem 3.21. *The length of the chord in (3.7.1) is given by*

$$\frac{2\sqrt{[\mathbf{m}^\top (\mathbf{V} \mathbf{q} + \mathbf{u})]^2 - (\mathbf{q}^\top \mathbf{V} \mathbf{q} + 2\mathbf{u}^\top \mathbf{q} + f)(\mathbf{m}^\top \mathbf{V} \mathbf{m})}}{\mathbf{m}^\top \mathbf{V} \mathbf{m}} \|\mathbf{m}\| \quad (3.7.9)$$

Proof. The distance between the points in (3.7.2) is given by

$$\|\mathbf{x}_1 - \mathbf{x}_2\| = |\mu_1 - \mu_2| \|\mathbf{m}\| \quad (3.7.10)$$

Substituting μ_i from (3.7.3) in (3.7.10) yields (3.7.9). \square

Theorem 3.22. *The affine transform for the conic section, preserves the norm. This implies that the length of any chord of a conic is invariant to translation and/or rotation.*

Proof. Let

$$\mathbf{x}_i = \mathbf{P} \mathbf{y}_i + \mathbf{c} \quad (3.7.11)$$

be any two points on the conic. Then the distance between the points is given by

$$\|\mathbf{x}_1 - \mathbf{x}_2\| = \|\mathbf{P} (\mathbf{y}_1 - \mathbf{y}_2)\| \quad (3.7.12)$$

which can be expressed as

$$\|\mathbf{x}_1 - \mathbf{x}_2\|^2 = (\mathbf{y}_1 - \mathbf{y}_2)^\top \mathbf{P}^\top \mathbf{P} (\mathbf{y}_1 - \mathbf{y}_2) \quad (3.7.13)$$

$$= \|\mathbf{y}_1 - \mathbf{y}_2\|^2 \quad (3.7.14)$$

since

$$\mathbf{P}^\top \mathbf{P} = \mathbf{I} \quad (3.7.15)$$

□

Corollary 3.23. *For the standard hyperbola/ellipse, the length of the major axis is*

$$2\sqrt{\left|\frac{f_0}{\lambda_1}\right|} \quad (3.7.16)$$

and the minor axis is

$$2\sqrt{\left|\frac{f_0}{\lambda_2}\right|} \quad (3.7.17)$$

Proof. See Appendix D

□

Theorem 3.24 (latus rectum). *The latus rectum of a conic section is the chord that passes through the focus and is perpendicular to the major axis. The length of the latus rectum for a conic is given by*

$$l = \begin{cases} 2\frac{\sqrt{|f_0\lambda_1|}}{\lambda_2} & e \neq 1 \\ \frac{\eta}{\lambda_2} & e = 1 \end{cases} \quad (3.7.18)$$

Proof. See Appendix E.

□

3.8 Tangent and Normal

Theorem 3.25 (Tangent). *Given the point of contact \mathbf{q} , the equation of a tangent to (3.2.1) is*

$$(\mathbf{V}\mathbf{q} + \mathbf{u})^\top \mathbf{x} + \mathbf{u}^\top \mathbf{q} + f = 0 \quad (3.8.1)$$

Proof. The normal vector is obtained from (3.7.6) and (1.1.15.1) as

$$\kappa \mathbf{n} = \mathbf{V}\mathbf{q} + \mathbf{u}, \kappa \in \mathbb{R} \quad (3.8.2)$$

From (3.8.2) and (2.1.2.1), the equation of the tangent is

$$(\mathbf{V}\mathbf{q} + \mathbf{u})^\top (\mathbf{x} - \mathbf{q}) = 0 \quad (3.8.3)$$

$$\implies (\mathbf{V}\mathbf{q} + \mathbf{u})^\top \mathbf{x} - \mathbf{q}^\top \mathbf{V}\mathbf{q} - \mathbf{u}^\top \mathbf{q} = 0 \quad (3.8.4)$$

which, upon substituting from (3.7.8) and simplifying yields (3.8.1)

□

Theorem 3.26. *If \mathbf{V}^{-1} exists, given the normal vector \mathbf{n} , the tangent points of contact to (3.2.1) are given by*

$$\mathbf{q}_i = \mathbf{V}^{-1}(\kappa_i \mathbf{n} - \mathbf{u}), i = 1, 2$$

$$\text{where } \kappa_i = \pm \sqrt{\frac{f_0}{\mathbf{n}^\top \mathbf{V}^{-1} \mathbf{n}}} \quad (3.8.5)$$

Proof. From (3.8.2),

$$\mathbf{q} = \mathbf{V}^{-1}(\kappa \mathbf{n} - \mathbf{u}), \quad \kappa \in \mathbb{R} \quad (3.8.6)$$

Substituting (3.8.6) in (3.7.8),

$$(\kappa \mathbf{n} - \mathbf{u})^\top \mathbf{V}^{-1} (\kappa \mathbf{n} - \mathbf{u}) + 2\mathbf{u}^\top \mathbf{V}^{-1} (\kappa \mathbf{n} - \mathbf{u}) + f = 0 \quad (3.8.7)$$

$$\implies \kappa^2 \mathbf{n}^\top \mathbf{V}^{-1} \mathbf{n} - \mathbf{u}^\top \mathbf{V}^{-1} \mathbf{u} + f = 0 \quad (3.8.8)$$

$$\text{or, } \kappa = \pm \sqrt{\frac{f_0}{\mathbf{n}^\top \mathbf{V}^{-1} \mathbf{n}}} \quad (3.8.9)$$

Substituting (3.8.9) in (3.8.6) yields (3.8.5). \square

Theorem 3.27. *If \mathbf{V} is not invertible, given the normal vector \mathbf{n} , the point of contact to (3.2.1) is given by the matrix equation*

$$\begin{pmatrix} (\mathbf{u} + \kappa \mathbf{n})^\top \\ \mathbf{V} \end{pmatrix} \mathbf{q} = \begin{pmatrix} -f \\ \kappa \mathbf{n} - \mathbf{u} \end{pmatrix} \quad (3.8.10)$$

$$\text{where } \kappa = \frac{\mathbf{p}_1^\top \mathbf{u}}{\mathbf{p}_1^\top \mathbf{n}}, \quad \mathbf{V} \mathbf{p}_1 = 0 \quad (3.8.11)$$

Proof. If \mathbf{V} is non-invertible, it has a zero eigenvalue. If the corresponding eigenvector is \mathbf{p}_1 , then,

$$\mathbf{V} \mathbf{p}_1 = 0 \quad (3.8.12)$$

From (3.8.2),

$$\kappa \mathbf{n} = \mathbf{V} \mathbf{q} + \mathbf{u}, \quad \kappa \in \mathbb{R} \quad (3.8.13)$$

$$\implies \kappa \mathbf{p}_1^\top \mathbf{n} = \mathbf{p}_1^\top \mathbf{V} \mathbf{q} + \mathbf{p}_1^\top \mathbf{u} \quad (3.8.14)$$

$$\text{or, } \kappa \mathbf{p}_1^\top \mathbf{n} = \mathbf{p}_1^\top \mathbf{u}, \quad \because \mathbf{p}_1^\top \mathbf{V} = 0, \quad (\text{from (3.8.12)}) \quad (3.8.15)$$

yielding κ in (3.8.11). From (3.8.13),

$$\kappa \mathbf{q}^\top \mathbf{n} = \mathbf{q}^\top \mathbf{V} \mathbf{q} + \mathbf{q}^\top \mathbf{u} \quad (3.8.16)$$

$$\implies \kappa \mathbf{q}^\top \mathbf{n} = -f - \mathbf{q}^\top \mathbf{u} \quad \text{from (3.7.8),} \quad (3.8.17)$$

$$\text{or, } (\kappa \mathbf{n} + \mathbf{u})^\top \mathbf{q} = -f \quad (3.8.18)$$

(3.8.13) can be expressed as

$$\mathbf{V} \mathbf{q} = \kappa \mathbf{n} - \mathbf{u}. \quad (3.8.19)$$

(3.8.18) and (3.8.19) clubbed together result in (3.8.10). \square

Theorem 3.28. *The normal vectors of the tangents from a point \mathbf{h} to the conic in (3.2.1) are given by*

$$\begin{aligned} \mathbf{n}_1 &= \mathbf{P} \begin{pmatrix} \sqrt{|\lambda_1|} \\ \sqrt{|\lambda_2|} \end{pmatrix} \\ \mathbf{n}_2 &= \mathbf{P} \begin{pmatrix} \sqrt{|\lambda_1|} \\ -\sqrt{|\lambda_2|} \end{pmatrix} \end{aligned} \quad (3.8.20)$$

where λ_i, \mathbf{P} are the eigenparameters of

$$\Sigma = (\mathbf{V} \mathbf{h} + \mathbf{u}) (\mathbf{V} \mathbf{h} + \mathbf{u})^\top - \mathbf{V} (\mathbf{h}^\top \mathbf{V} \mathbf{h} + 2\mathbf{u}^\top \mathbf{h} + f). \quad (3.8.21)$$

Proof. From (3.7.3), and (3.7.7)

$$[\mathbf{m}^\top (\mathbf{V} \mathbf{h} + \mathbf{u})]^2 - (\mathbf{m}^\top \mathbf{V} \mathbf{m}) (\mathbf{h}^\top \mathbf{V} \mathbf{h} + 2\mathbf{u}^\top \mathbf{h} + f) = 0 \quad (3.8.22)$$

$$\implies \mathbf{m}^\top \left[(\mathbf{V} \mathbf{h} + \mathbf{u}) (\mathbf{V} \mathbf{h} + \mathbf{u})^\top - \mathbf{V} (\mathbf{h}^\top \mathbf{V} \mathbf{h} + 2\mathbf{u}^\top \mathbf{h} + f) \right] \mathbf{m} = 0 \quad (3.8.23)$$

yielding (3.8.21). Consequently, from (3.6.16), (3.8.20) can be obtained. \square

3.9 Intersection of Conics

Lemma 3.2. *Let*

$$\mathbf{x}^\top \mathbf{V}_i \mathbf{x} + 2\mathbf{u}_i^\top \mathbf{x} + f_i = 0, \quad i = 1, 2 \quad (3.9.1)$$

be the equation of two conics. The locus of their intersection is a pair of straight lines if

$$\left| \begin{array}{cc} \mathbf{V}_1 + \mu \mathbf{V}_2 & \mathbf{u}_1 + \mu \mathbf{u}_2 \\ (\mathbf{u}_1 + \mu \mathbf{u}_2)^\top & f \end{array} \right| = 0, \quad |\mathbf{V}_1 + \mu \mathbf{V}_2| < 0 \quad (3.9.2)$$

Proof. The intersection of the conics in (3.9.1) is given by the curve

$$\mathbf{x}^\top (\mathbf{V}_1 + \mu \mathbf{V}_2) \mathbf{x} + 2(\mathbf{u}_1 + \mu \mathbf{u}_2)^\top \mathbf{x} + f_1 + \mu f_2 = 0, \quad (3.9.3)$$

which, from Theorem 3.16 represents a pair of straight lines if (3.9.2) is satisfied. \square

Corollary 3.29. *The points of intersection of the conics in (3.9.1) are the points of the intersection of the lines in (3.9.3).*

4 EXAMPLES

4.1 Loney

Example 4.1 (parabola). *To show that*

$$9x^2 - 24xy + 16y^2 - 18x - 101y + 19 = 0 \quad (4.1.1)$$

is the equation of a parabola with latus rectum of length 3, vertex

$$\frac{1}{25} \begin{pmatrix} -29 \\ 25 \end{pmatrix} \quad (4.1.2)$$

and axis

$$3x - 4y + 7 = 0 \quad (4.1.3)$$

Solution: Comparing (4.1.1) with (3.2.1),

$$\mathbf{V} = \begin{pmatrix} 9 & -12 \\ -12 & 16 \end{pmatrix} \quad (4.1.4)$$

$$\mathbf{u} = -\frac{1}{2} \begin{pmatrix} 18 \\ 101 \end{pmatrix} \quad (4.1.5)$$

$$f = 19 \quad (4.1.6)$$

The eigenvalues of \mathbf{V} are obtained as

$$|\lambda \mathbf{I} - \mathbf{V}| = 0 \quad (4.1.7)$$

$$\implies \lambda_1 = 0, \lambda_2 = 25 \quad (4.1.8)$$

Since the \mathbf{V} matrix has a 0 eigenvalue, (4.1.1) is a parabola. The eigenvector corresponding to the 0 eigenvalue is given by

$$\begin{pmatrix} 9 & -12 \\ -12 & 16 \end{pmatrix} \mathbf{p}_1 = \mathbf{0} \quad (4.1.9)$$

yielding

$$\mathbf{p}_1 = \frac{1}{5} \begin{pmatrix} 4 \\ 3 \end{pmatrix} \quad (4.1.10)$$

Substituting from (4.1.4), (4.1.5), (4.1.6), (4.1.10) and (4.1.8) in (3.3.3), (3.3.4) and (3.7.18) the latus rectum is obtained as

$$\eta = \frac{|2\mathbf{u}^\top \mathbf{p}_1|}{\lambda_2} = 3 \quad (4.1.11)$$

The vertex of the parabola is obtained from (3.5.2) as

$$\begin{pmatrix} -39 & -73 \\ 9 & -12 \\ -12 & 16 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -19 \\ -21 \\ 28 \end{pmatrix} \quad (4.1.12)$$

yielding (4.1.2). The second eigenvector of \mathbf{V} is orthogonal to \mathbf{p}_1 and obtained as

$$\mathbf{p}_2 = \frac{1}{5} \begin{pmatrix} 3 \\ -4 \end{pmatrix} \quad (4.1.13)$$

Substituting from (4.1.2) and (4.1.12) in (3.5.4), the equation of the axis of symmetry for the parabola can be expressed as (4.1.3).

Example 4.2 (ellipse). *To show that the equation*

$$14x^2 - 4xy + 11y^2 - 44x - 58y + 71 = 0 \quad (4.1.14)$$

represents an ellipse with centre

$$\mathbf{c} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad (4.1.15)$$

and lengths of semi-axes

$$\sqrt{6} \text{ and } 2 \quad (4.1.16)$$

Solution: The parameters for (4.1.14), are

$$\mathbf{V} = \begin{pmatrix} 14 & -2 \\ -2 & 11 \end{pmatrix} \quad (4.1.17)$$

$$\mathbf{u} = \begin{pmatrix} 22 \\ 29 \end{pmatrix} \quad (4.1.18)$$

$$f = 71 \quad (4.1.19)$$

Since

$$|\mathbf{V}| = 150 > 0, \quad (4.1.20)$$

(4.1.14) is an ellipse. Substituting from (4.1.17) and (4.1.18) in (3.5.1), the center of the ellipse is obtained as (4.1.15). Also, the eigenvalues of \mathbf{V} are

$$\lambda_1 = 10, \lambda_2 = 15 \quad (4.1.21)$$

Substituting from (4.1.17), (4.1.18) and (4.1.19) in (3.5.4),

$$f_0 = 60 \quad (4.1.22)$$

Substituting from (4.1.21) and (4.1.22) in (3.7.16) and (3.7.17), the lengths of the semi-axes are obtained as (4.1.16).

Example 4.3 (hyperbola). *To show that the equation*

$$x^2 - 3xy + y^2 + 10x - 10y + 21 = 0 \quad (4.1.23)$$

represents a hyperbola with centre

$$\mathbf{c} = \begin{pmatrix} -2 \\ 2 \end{pmatrix} \quad (4.1.24)$$

and length of semi-axes

$$\sqrt{2} \text{ and } \sqrt{\frac{2}{5}} \quad (4.1.25)$$

Solution: The conic parameters for (4.1.23), are

$$\mathbf{V} = \frac{1}{2} \begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix} \quad (4.1.26)$$

$$\mathbf{u} = \begin{pmatrix} 5 \\ -5 \end{pmatrix} \quad (4.1.27)$$

$$f = 21 \quad (4.1.28)$$

Since

$$|\mathbf{V}| = -\frac{5}{4} < 0, \quad (4.1.29)$$

(4.1.23) is a hyperbola. Substituting from (4.1.26) and (4.1.26) in (3.5.1), the center of the hyperbola is obtained as (4.1.15). Also, the eigenvalues of \mathbf{V} are

$$\lambda_1 = -\frac{1}{2}, \lambda_2 = \frac{5}{2} \quad (4.1.30)$$

From (3.5.4),

$$f_0 = -1 \quad (4.1.31)$$

Substituting from (4.1.30) and (4.1.31) in (3.7.16) and (3.7.17), the lengths of the semi-axes are then obtained as (4.1.25).

Example 4.4 (tangents). *To show that the tangents to the curve*

$$x^2 + 4xy + 3y^2 - 5x - 6y + 3 = 0 \quad (4.1.32)$$

parallel to the line

$$x + 4y + c = 0 \quad (4.1.33)$$

are

$$\begin{aligned} x + 4y - 5 &= 0 \\ x + 4y - 8 &= 0 \end{aligned} \quad (4.1.34)$$

Solution: The conic parameters for (4.1.32) are

$$\mathbf{V} = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \quad (4.1.35)$$

$$\mathbf{u} = \begin{pmatrix} -\frac{5}{2} \\ -3 \end{pmatrix} \quad (4.1.36)$$

$$f = 3 \quad (4.1.37)$$

Since

$$|\mathbf{V}| = -1 < 0, \quad (4.1.38)$$

(4.1.32) is a hyperbola. From (4.1.33), the normal vector to the tangent is

$$\mathbf{n} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \quad (4.1.39)$$

The equation of the tangent can be expressed as

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{q}_i) = 0 \quad (4.1.40)$$

where \mathbf{q}_i are the points of contact. Comparing the above with (4.1.33),

$$c = -\mathbf{n}^\top \mathbf{q}_i \quad (4.1.41)$$

which, upon substituting from (3.8.5) can be expressed as

$$c = -\mathbf{n}^\top \{ \mathbf{V}^{-1} (\kappa_i \mathbf{n} - \mathbf{u}) \} \quad (4.1.42)$$

$$= -\mathbf{n}^\top \left\{ \mathbf{V}^{-1} \left(\pm \sqrt{\frac{f_0}{\mathbf{n}^\top \mathbf{V}^{-1} \mathbf{n}}} \mathbf{n} - \mathbf{u} \right) \right\} \quad (4.1.43)$$

$$= \pm \sqrt{f_0 \mathbf{n}^\top \mathbf{V}^{-1} \mathbf{n}} + \mathbf{n}^\top \mathbf{V}^{-1} \mathbf{u}. \quad (4.1.44)$$

Substituting from (4.1.35), (4.1.36) and (4.1.37) in (3.3.3),

$$\mathbf{f}_0 = -\frac{3}{4} \quad (4.1.45)$$

Substituting from (4.1.35), (4.1.36), (4.1.39) and (4.1.45) in (3.8.9),

$$c = -5 \text{ or } c = -8 \quad (4.1.46)$$

yielding (4.1.34).

4.2 Miscellaneous

4.2.1. Given unit basis vectors \mathbf{a}, \mathbf{b} , with angle θ between them, the locus of the coordinates of a unit vector \mathbf{c} in the space spanned by \mathbf{a}, \mathbf{b} is given by

$$\mathbf{x}^\top \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \mathbf{x} = 1 \quad (4.2.1.1)$$

with $\rho = \cos \theta$.

Solution: Let

$$\mathbf{c} = x_1 \mathbf{a} + x_2 \mathbf{b} = \begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix} \mathbf{x} \quad (4.2.1.2)$$

Then,

$$\|\mathbf{c}\|^2 = \mathbf{x}^\top \begin{pmatrix} \mathbf{a}^\top \\ \mathbf{b}^\top \end{pmatrix} \begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix} \mathbf{x} \quad (4.2.1.3)$$

$$= \mathbf{x}^\top \begin{pmatrix} 1 & \mathbf{a}^\top \mathbf{b} \\ \mathbf{a}^\top \mathbf{b} & 1 \end{pmatrix} \mathbf{x} \quad (4.2.1.4)$$

which can be expressed as (4.2.1.1).

4.2.2. Given the coordinates of \mathbf{c} , the angle θ between the basis vectors is given by

$$\rho = \frac{1 - \|\mathbf{x}\|^2}{\mathbf{x}^\top \mathbf{R} \mathbf{x}} \quad (4.2.2.1)$$

where

$$\mathbf{R} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (4.2.2.2)$$

Solution: Let

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad (4.2.2.3)$$

For

$$\mathbf{V} = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}, \mathbf{u} = 0, f = -1 \quad (4.2.2.4)$$

in (3.2.1),

Since

$$|\mathbf{V}| = 1 - \rho^2, 0 < |\mathbf{V}| < 1, \quad (4.2.2.5)$$

(4.2.1.1) represents the equation of an ellipse. Using eigenvalue decomposition,

$$\mathbf{V} = \mathbf{P}^\top \mathbf{D} \mathbf{P} \quad (4.2.2.6)$$

where

$$\mathbf{P} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 1 + \rho & 0 \\ 0 & 1 - \rho \end{pmatrix} \quad (4.2.2.7)$$

Using the affine transformation,

$$\mathbf{x} = \mathbf{P} \mathbf{y} \quad (4.2.2.8)$$

(4.2.1.1) can be expressed as

$$\mathbf{y}^\top \mathbf{D} \mathbf{y} = 1 \implies y_1^2 (1 + \rho) + y_2^2 (1 - \rho) = 1 \quad (4.2.2.9)$$

which can be simplified to obtain

$$\rho = \frac{1 - y_1^2 - y_2^2}{y_1^2 - y_2^2} \quad (4.2.2.10)$$

$$= \frac{1 - \|\mathbf{y}\|^2}{\mathbf{y}^\top \mathbf{Q} \mathbf{y}} \quad (4.2.2.11)$$

where

$$\mathbf{Q} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (4.2.2.12)$$

From (4.2.2.8), (4.2.2.11) can be expressed as (4.2.2.1)

$$\therefore \|\mathbf{a}\| = \|\mathbf{b}\| = 1. \quad (4.2.2.13)$$

APPENDIX A

From (3.2.2), using the fact that \mathbf{V} is symmetric with $\mathbf{V} = \mathbf{V}^\top$,

$$\mathbf{V}^\top \mathbf{V} = (\|\mathbf{n}\|^2 \mathbf{I} - e^2 \mathbf{n} \mathbf{n}^\top)^\top (\|\mathbf{n}\|^2 \mathbf{I} - e^2 \mathbf{n} \mathbf{n}^\top) \quad (1.0.2.1)$$

$$\implies \mathbf{V}^2 = \|\mathbf{n}\|^4 \mathbf{I} + e^4 \mathbf{n} \mathbf{n}^\top \mathbf{n} \mathbf{n}^\top - 2e^2 \|\mathbf{n}\|^2 \mathbf{n} \mathbf{n}^\top \quad (1.0.2.2)$$

$$= \|\mathbf{n}\|^4 \mathbf{I} + e^4 \|\mathbf{n}\|^2 \mathbf{n} \mathbf{n}^\top - 2e^2 \|\mathbf{n}\|^2 \mathbf{n} \mathbf{n}^\top \quad (1.0.2.3)$$

$$= \|\mathbf{n}\|^4 \mathbf{I} + e^2 (e^2 - 2) \|\mathbf{n}\|^2 \mathbf{n} \mathbf{n}^\top \quad (1.0.2.4)$$

$$= \|\mathbf{n}\|^4 \mathbf{I} + (e^2 - 2) \|\mathbf{n}\|^2 (\|\mathbf{n}\|^2 \mathbf{I} - \mathbf{V}) \quad (1.0.2.5)$$

which can be expressed as

$$\mathbf{V}^2 + (e^2 - 2) \|\mathbf{n}\|^2 \mathbf{V} - (e^2 - 1) \|\mathbf{n}\|^4 \mathbf{I} = 0 \quad (1.0.2.6)$$

Using the Cayley-Hamilton theorem, (1.0.2.6) results in the characteristic equation,

$$\lambda^2 - (2 - e^2) \|\mathbf{n}\|^2 \lambda + (1 - e^2) \|\mathbf{n}\|^4 = 0 \quad (1.0.2.7)$$

which can be expressed as

$$\left(\frac{\lambda}{\|\mathbf{n}\|^2} \right)^2 - (2 - e^2) \left(\frac{\lambda}{\|\mathbf{n}\|^2} \right) + (1 - e^2) = 0 \quad (1.0.2.8)$$

$$\implies \frac{\lambda}{\|\mathbf{n}\|^2} = 1 - e^2, 1 \quad (1.0.2.9)$$

$$\text{or, } \lambda_2 = \|\mathbf{n}\|^2, \lambda_1 = (1 - e^2) \lambda_2 \quad (1.0.2.10)$$

From (1.0.2.10), the eccentricity of (3.2.1) is given by (3.2.9). Multiplying both sides of (3.2.2) by \mathbf{n} ,

$$\mathbf{V} \mathbf{n} = \|\mathbf{n}\|^2 \mathbf{n} - e^2 \mathbf{n} \mathbf{n}^\top \mathbf{n} \quad (1.0.2.11)$$

$$= \|\mathbf{n}\|^2 (1 - e^2) \mathbf{n} \quad (1.0.2.12)$$

$$= \lambda_1 \mathbf{n} \quad (1.0.2.13)$$

$$(1.0.2.14)$$

from (1.0.2.10). Thus, λ_1 is the corresponding eigenvalue for \mathbf{n} . From (3.1.4) and (1.0.2.14), this implies that

$$\mathbf{p}_1 = \frac{\mathbf{n}}{\|\mathbf{n}\|} \quad (1.0.2.15)$$

$$\text{or, } \mathbf{n} = \|\mathbf{n}\| \mathbf{p}_1 = \sqrt{\lambda_2} \mathbf{p}_1 \quad (1.0.2.16)$$

from (1.0.2.10). From (3.2.3) and (1.0.2.10),

$$\mathbf{F} = \frac{ce^2 \mathbf{n} - \mathbf{u}}{\lambda_2} \quad (1.0.2.17)$$

$$\implies \|\mathbf{F}\|^2 = \frac{(ce^2 \mathbf{n} - \mathbf{u})^\top (ce^2 \mathbf{n} - \mathbf{u})}{\lambda_2^2} \quad (1.0.2.18)$$

$$\implies \lambda_2^2 \|\mathbf{F}\|^2 = c^2 e^4 \lambda_2 - 2ce^2 \mathbf{u}^\top \mathbf{n} + \|\mathbf{u}\|^2 \quad (1.0.2.19)$$

Also, (3.2.4) can be expressed as

$$\lambda_2 \|\mathbf{F}\|^2 = f + c^2 e^2 \quad (1.0.2.20)$$

From (1.0.2.19) and (1.0.2.20),

$$c^2 e^4 \lambda_2 - 2ce^2 \mathbf{u}^\top \mathbf{n} + \|\mathbf{u}\|^2 = \lambda_2 (f + c^2 e^2) \quad (1.0.2.21)$$

$$\implies \lambda_2 e^2 (e^2 - 1) c^2 - 2ce^2 \mathbf{u}^\top \mathbf{n} + \|\mathbf{u}\|^2 - \lambda_2 f = 0 \quad (1.0.2.22)$$

yielding (3.2.11).

APPENDIX B

Using (3.1.1) (3.2.1) can be expressed as

$$(\mathbf{P}\mathbf{y} + \mathbf{c})^\top \mathbf{V} (\mathbf{P}\mathbf{y} + \mathbf{c}) + 2\mathbf{u}^\top (\mathbf{P}\mathbf{y} + \mathbf{c}) + f = 0, \quad (2.0.2.1)$$

yielding

$$\mathbf{y}^\top \mathbf{P}^\top \mathbf{V} \mathbf{P} \mathbf{y} + 2(\mathbf{V}\mathbf{c} + \mathbf{u})^\top \mathbf{P} \mathbf{y} + \mathbf{c}^\top \mathbf{V} \mathbf{c} + 2\mathbf{u}^\top \mathbf{c} + f = 0 \quad (2.0.2.2)$$

From (2.0.2.2) and (3.1.2),

$$\mathbf{y}^\top \mathbf{D} \mathbf{y} + 2(\mathbf{V}\mathbf{c} + \mathbf{u})^\top \mathbf{P} \mathbf{y} + \mathbf{c}^\top (\mathbf{V}\mathbf{c} + \mathbf{u}) + \mathbf{u}^\top \mathbf{c} + f = 0 \quad (2.0.2.3)$$

When \mathbf{V}^{-1} exists, choosing

$$\mathbf{V}\mathbf{c} + \mathbf{u} = \mathbf{0}, \quad \text{or, } \mathbf{c} = -\mathbf{V}^{-1}\mathbf{u}, \quad (2.0.2.4)$$

and substituting (2.0.2.4) in (2.0.2.3) yields (3.3.1).

B.1

When $|\mathbf{V}| = 0, \lambda_1 = 0$ and

$$\mathbf{V}\mathbf{p}_1 = \mathbf{0}, \mathbf{V}\mathbf{p}_2 = \lambda_2 \mathbf{p}_2. \quad (2.1.2.1)$$

where $\mathbf{p}_1, \mathbf{p}_2$ are the eigenvectors of \mathbf{V} such that (3.1.2)

$$\mathbf{P} = (\mathbf{p}_1 \quad \mathbf{p}_2), \quad (2.1.2.2)$$

Substituting (2.1.2.2) in (2.0.2.3),

$$\mathbf{y}^\top \mathbf{D} \mathbf{y} + 2(\mathbf{c}^\top \mathbf{V} + \mathbf{u}^\top) (\mathbf{p}_1 \quad \mathbf{p}_2) \mathbf{y} + \mathbf{c}^\top (\mathbf{V}\mathbf{c} + \mathbf{u}) + \mathbf{u}^\top \mathbf{c} + f = 0 \quad (2.1.2.3)$$

$$\implies \mathbf{y}^\top \mathbf{D} \mathbf{y} + 2((\mathbf{c}^\top \mathbf{V} + \mathbf{u}^\top) \mathbf{p}_1 (\mathbf{c}^\top \mathbf{V} + \mathbf{u}^\top) \mathbf{p}_2) \mathbf{y} + \mathbf{c}^\top (\mathbf{V}\mathbf{c} + \mathbf{u}) + \mathbf{u}^\top \mathbf{c} + f = 0 \quad (2.1.2.4)$$

$$\implies \mathbf{y}^\top \mathbf{D} \mathbf{y} + 2(\mathbf{u}^\top \mathbf{p}_1 \quad (\lambda_2 \mathbf{c}^\top + \mathbf{u}^\top) \mathbf{p}_2) \mathbf{y} + \mathbf{c}^\top (\mathbf{V}\mathbf{c} + \mathbf{u}) + \mathbf{u}^\top \mathbf{c} + f = 0 \quad (2.1.2.5)$$

upon substituting from (2.1.2.1) yielding

$$\lambda_2 y_2^2 + 2(\mathbf{u}^\top \mathbf{p}_1) y_1 + 2y_2 (\lambda_2 \mathbf{c} + \mathbf{u})^\top \mathbf{p}_2 + \mathbf{c}^\top (\mathbf{V}\mathbf{c} + \mathbf{u}) + \mathbf{u}^\top \mathbf{c} + f = 0 \quad (2.1.2.6)$$

Thus, (2.1.2.6) can be expressed as (3.3.2) by choosing

$$\eta = 2\mathbf{u}^\top \mathbf{p}_1 \quad (2.1.2.7)$$

and \mathbf{c} in (2.0.2.3) such that

$$2\mathbf{P}^\top (\mathbf{V}\mathbf{c} + \mathbf{u}) = \eta \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (2.1.2.8)$$

$$\mathbf{c}^\top (\mathbf{V}\mathbf{c} + \mathbf{u}) + \mathbf{u}^\top \mathbf{c} + f = 0 \quad (2.1.2.9)$$

$\therefore \mathbf{P}^\top \mathbf{P} = \mathbf{I}$, multiplying (2.1.2.8) by \mathbf{P} yields

$$(\mathbf{V}\mathbf{c} + \mathbf{u}) = \frac{\eta}{2} \mathbf{p}_1, \quad (2.1.2.10)$$

which, upon substituting in (2.1.2.9) results in

$$\frac{\eta}{2} \mathbf{c}^\top \mathbf{p}_1 + \mathbf{u}^\top \mathbf{c} + f = 0 \quad (2.1.2.11)$$

(2.1.2.10) and (2.1.2.11) can be clubbed together to obtain (3.5.2).

APPENDIX C

a) For the standard hyperbola/ellipse in (3.3.1), from (3.3.6), (3.2.10) and (3.3.7),

$$\mathbf{n} = \sqrt{\frac{\lambda_2}{f_0}} \mathbf{e}_1 \quad (3.0.2.1)$$

$$c = \pm \frac{\sqrt{-\frac{\lambda_2}{f_0} (e^2 - 1) \left(\frac{\lambda_2}{f_0}\right)}}{\frac{\lambda_2}{f_0} e (e^2 - 1)} \quad (3.0.2.2)$$

$$= \pm \frac{1}{e\sqrt{1 - e^2}} \quad (3.0.2.3)$$

yielding (3.3.1) upon substituting from (3.2.9) and simplifying. For the standard parabola in (3.3.2), from (3.3.6), (3.2.10) and (3.3.7), noting that $f = 0$,

$$\mathbf{n} = \sqrt{\lambda_2} \mathbf{e}_1 \quad (3.0.2.4)$$

$$c = \frac{\left\| \frac{\eta}{2} \mathbf{e}_1 \right\|^2}{2 \left(\frac{\eta}{2} \right) (\mathbf{e}_1)^\top \mathbf{n}} \quad (3.0.2.5)$$

$$(3.0.2.6)$$

$$= \frac{\eta}{4\sqrt{\lambda_2}} \quad (3.0.2.7)$$

yielding (3.3.2).

b) For the standard ellipse/hyperbola, substituting from (3.0.2.3), (3.0.2.1), (3.3.7) and (3.2.9) in (3.2.11),

$$\mathbf{F} = \pm \frac{\left(\frac{1}{e\sqrt{1 - e^2}} \right) (e^2) \sqrt{\frac{\lambda_2}{f_0}} \mathbf{e}_1}{\frac{\lambda_2}{f_0}} \quad (3.0.2.8)$$

yielding (3.3.1) after simplification. For the standard parabola, substituting from (3.0.2.7), (3.0.2.4), (3.3.7) and (3.2.9) in (3.2.11),

$$\mathbf{F} = \frac{\left(\frac{\eta}{4\sqrt{\lambda_2}} \right) \sqrt{\lambda_2} \mathbf{e}_1 - \frac{\eta}{2} \mathbf{e}_1}{\lambda_2} \quad (3.0.2.9)$$

$$(3.0.2.10)$$

yielding (3.3.1) after simplification.

APPENDIX D

Since the major axis passes through the origin,

$$\mathbf{q} = \mathbf{0} \quad (4.0.2.1)$$

Further, from Corollary (3.8),

$$\mathbf{m} = \mathbf{e}_2, \quad (4.0.2.2)$$

and from (3.3.1),

$$\mathbf{V} = \frac{\mathbf{D}}{f_0}, \mathbf{u} = \mathbf{0}, f = -1 \quad (4.0.2.3)$$

Substituting the above in (3.7.9),

$$\frac{2\sqrt{\mathbf{e}_1^\top \frac{\mathbf{D}}{f_0} \mathbf{e}_1}}{\mathbf{e}_1^\top \frac{\mathbf{D}}{f_0} \mathbf{e}_1} \|\mathbf{e}_1\| \quad (4.0.2.4)$$

yielding (3.7.16). Similarly, for the minor axis, the only different parameter is

$$\mathbf{m} = \mathbf{e}_2, \quad (4.0.2.5)$$

Substituting the above in (3.7.9),

$$\frac{2\sqrt{\mathbf{e}_2^\top \frac{\mathbf{D}}{f_0} \mathbf{e}_2}}{\mathbf{e}_2^\top \frac{\mathbf{D}}{f_0} \mathbf{e}_2} \|\mathbf{e}_2\| \quad (4.0.2.6)$$

yielding (3.7.17).

APPENDIX E

The latus rectum is perpendicular to the major axis for the standard conic. Hence, from Corollary (3.8),

$$\mathbf{m} = \mathbf{e}_2, \quad (5.0.2.1)$$

Since it passes through the focus, from (3.3.1)

$$\mathbf{q} = \mathbf{F} = \pm e \sqrt{\frac{f_0}{\lambda_2(1-e^2)}} \mathbf{e}_1 \quad (5.0.2.2)$$

for the standard hyperbola/ellipse. Also, from (3.3.1),

$$\mathbf{V} = \frac{\mathbf{D}}{f_0}, \mathbf{u} = 0, f = -1 \quad (5.0.2.3)$$

Substituting the above in (3.7.9),

$$\frac{2\sqrt{\left[\mathbf{e}_2^\top \left(\frac{\mathbf{D}}{f_0} e \sqrt{\frac{f_0}{\lambda_2(1-e^2)}} \mathbf{e}_1\right)\right]^2 - \left(e \sqrt{\frac{f_0}{\lambda_2(1-e^2)}} \mathbf{e}_1^\top \frac{\mathbf{D}}{f_0} e \sqrt{\frac{f_0}{\lambda_2(1-e^2)}} \mathbf{e}_1 - 1\right) \left(\mathbf{e}_2^\top \frac{\mathbf{D}}{f_0} \mathbf{e}_2\right)}}{\mathbf{e}_2^\top \frac{\mathbf{D}}{f_0} \mathbf{e}_2} \|\mathbf{e}_2\| \quad (5.0.2.4)$$

Since

$$\mathbf{e}_2^\top \mathbf{D} \mathbf{e}_1 = 0, \mathbf{e}_1^\top \mathbf{D} \mathbf{e}_1 = \lambda_1, \mathbf{e}_1^\top \mathbf{e}_1 = 1, \|\mathbf{e}_2\| = 1, \mathbf{e}_2^\top \mathbf{D} \mathbf{e}_2 = \lambda_2, \quad (5.0.2.5)$$

(5.0.2.4) can be expressed as

$$\frac{2\sqrt{\left(1 - \frac{\lambda_1 e^2}{\lambda_2(1-e^2)}\right) \left(\frac{\lambda_2}{f_0}\right)}}{\frac{\lambda_2}{f_0}} \quad (5.0.2.6)$$

$$= 2 \frac{\sqrt{f_0 \lambda_1}}{\lambda_2} \quad \left(\because e^2 = 1 - \frac{\lambda_1}{\lambda_2}\right) \quad (5.0.2.7)$$

For the standard parabola, the parameters in (3.7.9) are

$$\mathbf{q} = \mathbf{F} = -\frac{\eta}{4\lambda_2} \mathbf{e}_1, \mathbf{m} = \mathbf{e}_1, \mathbf{V} = \mathbf{D}, \mathbf{u} = \frac{\eta}{2} \mathbf{e}_1^\top, f = 0 \quad (5.0.2.8)$$

Substituting the above in (3.7.9), the length of the latus rectum can be expressed as

$$2\sqrt{\frac{\left[\mathbf{e}_2^\top \left(\mathbf{D} \left(-\frac{\eta}{4\lambda_2}\mathbf{e}_1\right) + \frac{\eta}{2}\mathbf{e}_1\right)\right]^2 - \left(\left(-\frac{\eta}{4\lambda_2}\mathbf{e}_1\right)^\top \mathbf{D} \left(-\frac{\eta}{4\lambda_2}\mathbf{e}_1\right) + 2\frac{\eta}{2}\mathbf{e}_1^\top \left(-\frac{\eta}{4\lambda_2}\mathbf{e}_1\right)\right) (\mathbf{e}_2^\top \mathbf{D}\mathbf{e}_2)}{\mathbf{e}_2^\top \mathbf{D}\mathbf{e}_2}} \|\mathbf{e}_2\|} \quad (5.0.2.9)$$

Since

$$\mathbf{e}_2^\top \mathbf{D}\mathbf{e}_1 = 0, \mathbf{e}_2^\top \mathbf{e}_2 = 0, \mathbf{e}_1^\top \mathbf{D}\mathbf{e}_1 = 0, \mathbf{e}_1^\top \mathbf{e}_1 = 1, \|\mathbf{e}_1\| = 1, \mathbf{e}_2^\top \mathbf{D}\mathbf{e}_2 = \lambda_2, \quad (5.0.2.10)$$

(5.0.2.9) can be expressed as

$$2\frac{\sqrt{\frac{\eta^2}{4\lambda_2}\lambda_2}}{\lambda_2} = \frac{\eta}{\lambda_2} \quad (5.0.2.11)$$