MATRIX ANALYSIS

Through Coordinate Geometry

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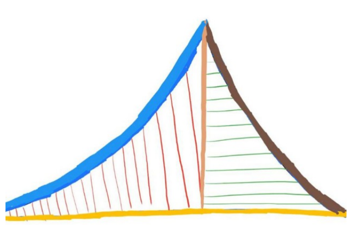

Contents

Introduction

This book links high school coordinate geometry to linear algebra and matrix analysis through solved problems.

Chapter 1

Quadrilaterals

1.1. Properties

1. The angles of quadrilateral are in the ratio 3:5:9:13. Find all the angles of the quadrilateral.

2. If diagonals of a parallelogram are equal then show that it is a rectangle.

   Solution: See Fig. 1.1. From (A.1.23),

$$\mathbf{B} - \mathbf{A} = \mathbf{C} - \mathbf{D} \tag{1.1}$$

$$\implies \mathbf{B} - \mathbf{C} = \mathbf{A} - \mathbf{D} \tag{1.2}$$

   Also, it is given that the diagonals of $ABCD$ are equal. Hence,

$$\|\mathbf{C} - \mathbf{A}\|^2 = \|\mathbf{D} - \mathbf{B}\|^2 \tag{1.3}$$

$$\implies \|(\mathbf{C} - \mathbf{B}) + (\mathbf{B} - \mathbf{A})\|^2 = \|(\mathbf{D} - \mathbf{C}) + (\mathbf{C} - \mathbf{B}\|^2 \tag{1.4}$$

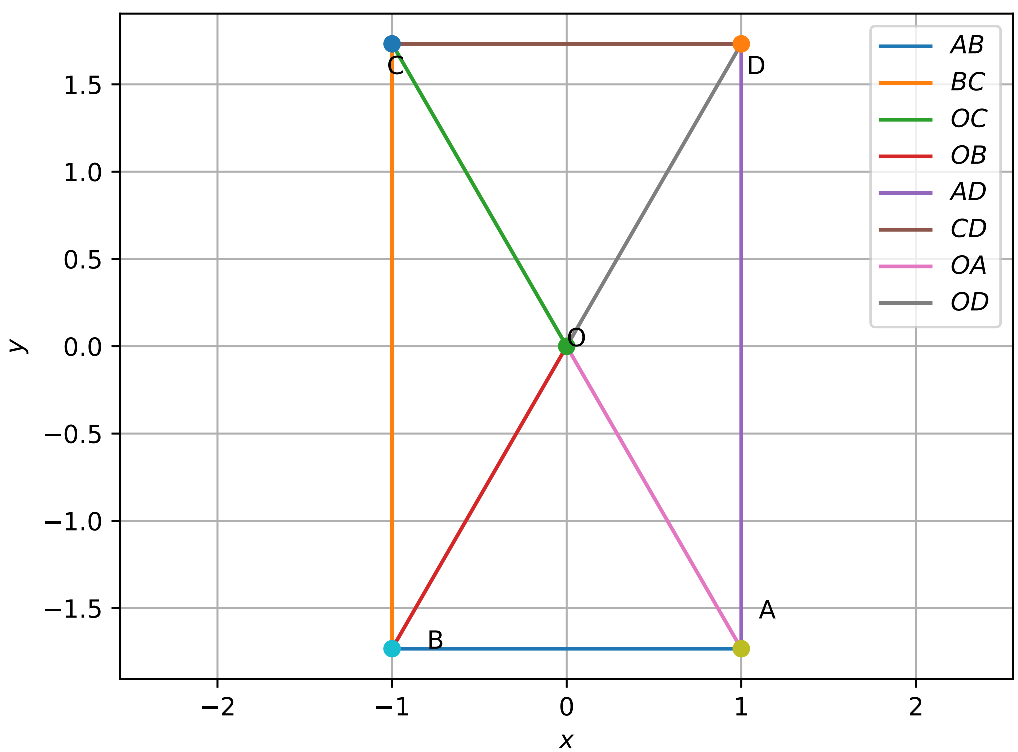

Figure 1.1:

which can be expressed as

$$\|\mathbf{C}-\mathbf{B}\|^2+\|\mathbf{B}-\mathbf{A}\|^2+2(\mathbf{C}-\mathbf{B})^\top(\mathbf{B}-\mathbf{A}) \\ =\|\mathbf{D}-\mathbf{C}\|^2+\|\mathbf{C}-\mathbf{B}\|^2+2(\mathbf{D}-\mathbf{C})^\top(\mathbf{C}-\mathbf{B}) \quad (1.5)$$

which, can be simplified to obtain

$$(\mathbf{C}-\mathbf{B})^\top(\mathbf{B}-\mathbf{A})=(\mathbf{D}-\mathbf{C})^\top(\mathbf{C}-\mathbf{B}) \quad (1.6)$$

since

$$\|\mathbf{D} - \mathbf{C}\| = \|\mathbf{B} - \mathbf{A}\| \tag{1.7}$$

yielding

$$(\mathbf{A} - \mathbf{B})^{\top}(\mathbf{B} - \mathbf{C}) = \mathbf{0} \tag{1.8}$$

from (1.1).

3. Show that if the diagonals of a quadrilateral bisect each other at right angles, then it is a rhombus.

   Solution: See Fig. 1.2. From the given information,

$$\frac{\mathbf{B} + \mathbf{D}}{2} = \frac{\mathbf{A} + \mathbf{C}}{2} \tag{1.9}$$

$$(\mathbf{B} - \mathbf{D})^{\top} (\mathbf{A} - \mathbf{C}) = 0 \tag{1.10}$$

From (1.9),

$$\mathbf{B} - \mathbf{A} = \mathbf{C} - \mathbf{D} \tag{1.11}$$

which, from (A.1.23), is the definition of a parallelogram. Further, substituting

$$\mathbf{B} - \mathbf{D} = (\mathbf{B} - \mathbf{A}) + (\mathbf{A} - \mathbf{D}) \tag{1.12}$$

$$\mathbf{A} - \mathbf{C} = (\mathbf{A} - \mathbf{B}) + (\mathbf{B} - \mathbf{C}) \tag{1.13}$$

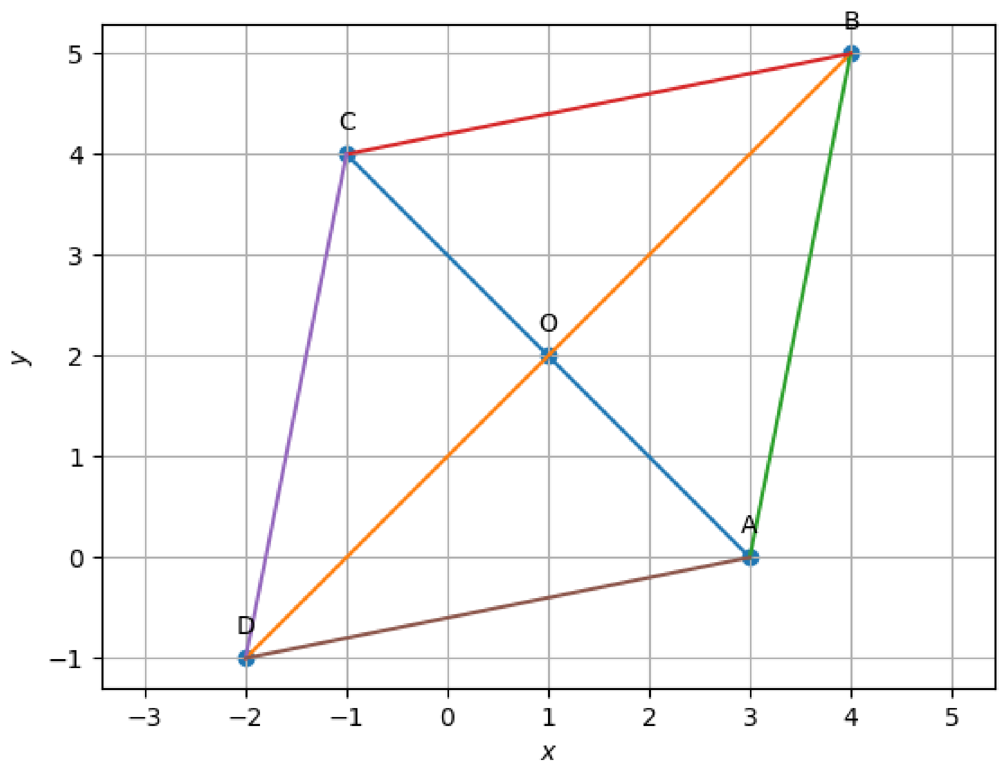

Figure 1.2: Rhombus

in (1.10),

$$
\begin{aligned}
&[(\mathbf{B}-\mathbf{A})+(\mathbf{A}-\mathbf{D})]^\top\,[(\mathbf{A}-\mathbf{B})+(\mathbf{B}-\mathbf{C})]=0\\
&\implies -\left\|\mathbf{B}-\mathbf{A}\right\|^2+(\mathbf{B}-\mathbf{A})^\top(\mathbf{B}-\mathbf{C})+\\
&\qquad(\mathbf{A}-\mathbf{D})^\top(\mathbf{A}-\mathbf{B})+(\mathbf{A}-\mathbf{D})^\top(\mathbf{B}-\mathbf{C})=0 \quad (1.14)
\end{aligned}
$$

From (1.11),

$$\mathbf{B} - \mathbf{C} = \mathbf{A} - \mathbf{D} \tag{1.15}$$

$$\implies (\mathbf{B} - \mathbf{A})^\top (\mathbf{B} - \mathbf{C}) + (\mathbf{A} - \mathbf{D})^\top (\mathbf{A} - \mathbf{B}) = \mathbf{0} \tag{1.16}$$

and

$$(\mathbf{A} - \mathbf{D})^\top (\mathbf{B} - \mathbf{C}) = \|\mathbf{B} - \mathbf{C}\|^2 \tag{1.17}$$

Substituting from

(1.16) and (1.17) in (1.14),

$$\|\mathbf{A} - \mathbf{B}\|^2 = \|\mathbf{B} - \mathbf{C}\|^2 \tag{1.18}$$

which means that the adjacent sides of the parallelogram are equal. Thus, the quadrilateral is a rhombus

4. Show that the diagonals of a square are equal and bisect each other at right angles.
   Solution: This is obvious from Problems (2) and (3).

5.

6. Diagonal AC of a parallelogram ABCD bisects $\angle A$ in Fig (1.3). Show that

   (a) it bisects $\angle C$ also

   (b) $ABCD$ is a rhombus

   Solution:

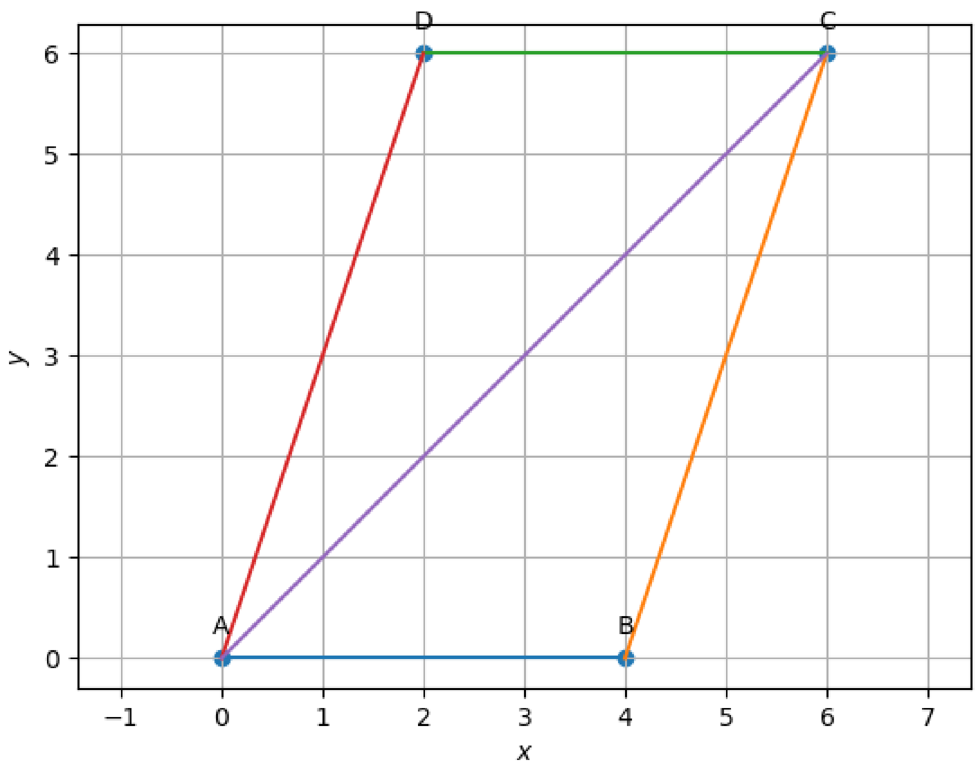

Figure 1.3:

(a) From (A.25),

$$\angle BAC = \angle DAC \tag{1.19}$$

$$\implies \frac{(\mathbf{A}-\mathbf{B})^T(\mathbf{A}-\mathbf{C})}{\|\mathbf{A}-\mathbf{B}\|\,\|\mathbf{A}-\mathbf{C}\|} = \frac{(\mathbf{A}-\mathbf{D})^T(\mathbf{A}-\mathbf{C})}{\|\mathbf{A}-\mathbf{D}\|\,\|\mathbf{A}-\mathbf{C}\|} \tag{1.20}$$

Also,

$$\cos\angle ACD = \frac{(\mathbf{C}-\mathbf{D})^T(\mathbf{C}-\mathbf{A})}{\|\mathbf{C}-\mathbf{D}\|\,\|\mathbf{C}-\mathbf{A}\|} \tag{1.21}$$

From Appendix A.1.23,

$$\mathbf{B} - \mathbf{A} = \mathbf{C} - \mathbf{D} \tag{1.22}$$

$$\implies \frac{(\mathbf{C} - \mathbf{D})^T(\mathbf{C} - \mathbf{A})}{\|\mathbf{C} - \mathbf{D}\| \, \|\mathbf{C} - \mathbf{A}\|} = \frac{(\mathbf{B} - \mathbf{A})^T(\mathbf{C} - \mathbf{A})}{\|\mathbf{B} - \mathbf{A}\| \, \|\mathbf{C} - \mathbf{A}\|} \tag{1.23}$$

upon substituting in (1.21). Thus, from (1.21) and (1.19),

$$\angle BAC = \angle DAC = \angle ACD \tag{1.24}$$

Similarly, it can be shown that

$$\angle ACD = \angle ACB \tag{1.25}$$

(b)

7. $ABCD$ is a rhombus. Show that the diagonal AC bisects angle A as well as angle C and diagonal BD bisects angle B as well as angle D.

Solution: For the rhombus in Fig. 1.4,

$$\begin{aligned} \|\mathbf{A} - \mathbf{B}\| &= \|\mathbf{A} - \mathbf{D}\| \\ \mathbf{A} - \mathbf{B} &= \mathbf{D} - \mathbf{C} \end{aligned} \tag{1.26}$$

From (A.25),

$$\begin{aligned} \cos\angle BAC &= \frac{(\mathbf{A} - \mathbf{B})^T(\mathbf{A} - \mathbf{C})}{\|\mathbf{A} - \mathbf{B}\| \, \|\mathbf{A} - \mathbf{C}\|} \\ \cos\angle DAC &= \frac{(\mathbf{C} - \mathbf{D})^T(\mathbf{C} - \mathbf{A})}{\|\mathbf{C} - \mathbf{D}\| \, \|\mathbf{C} - \mathbf{A}\|} \end{aligned} \tag{1.27}$$

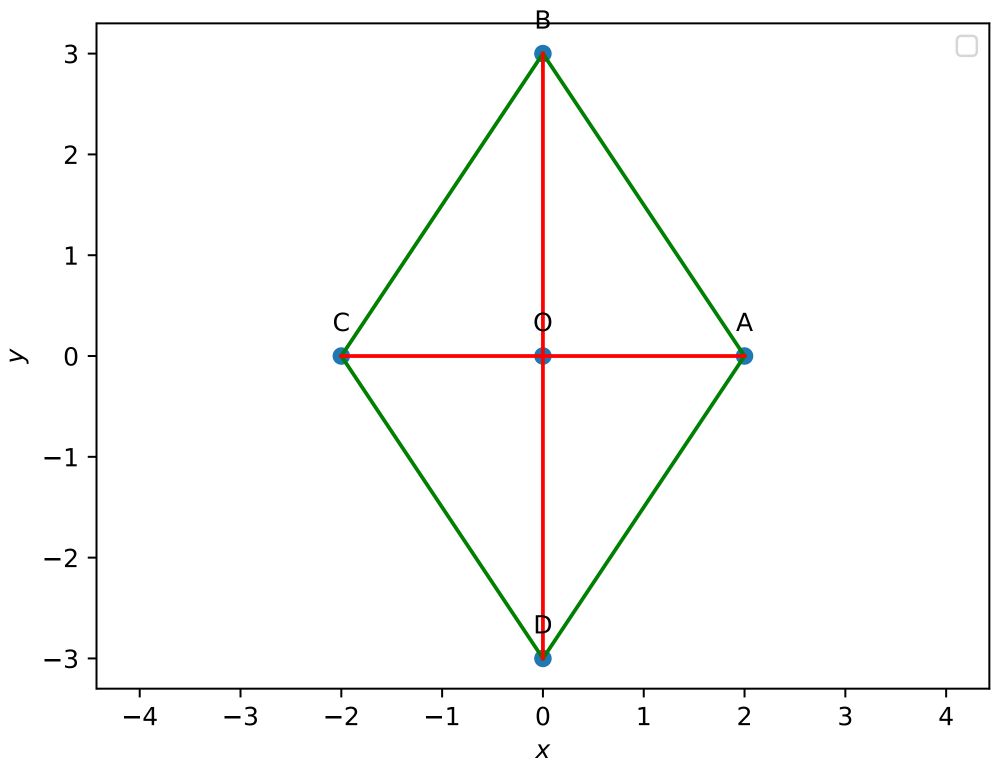

Figure 1.4:

From (1.26) and (1.27), we obtain

$$\cos\angle BAC = \cos\angle DAC \tag{1.28}$$

Thus, AC bisects $\angle A$. Similarly, the remaining results can be proved.

8.

9. In parallelogram $ABCD$, two points $\mathbf{P}$ and $\mathbf{Q}$ are taken on diagonal BD such that $DP = BQ$. Show that

   (a) $\triangle APD \cong \triangle CQB$

   (b) $AP = CQ$

   (c) $\triangle AQB \cong \triangle CPD$

   (d) $AQ = CP$

   (e) $APCQ$ is a parallelogram

   Solution: See Fig. 1.5.

   From (A.25) and the given information,

$$\mathbf{A} - \mathbf{B} = \mathbf{D} - \mathbf{C} \tag{1.29}$$

$$\implies \mathbf{A} - \mathbf{D} = \mathbf{B} - \mathbf{C} \tag{1.30}$$

$$\mathbf{B} - \mathbf{Q} = \mathbf{P} - \mathbf{D} \quad \text{(given)} \tag{1.31}$$

   From (1.29) and (1.31)

$$\mathbf{A} - \mathbf{P} = \mathbf{Q} - \mathbf{C} \tag{1.32}$$

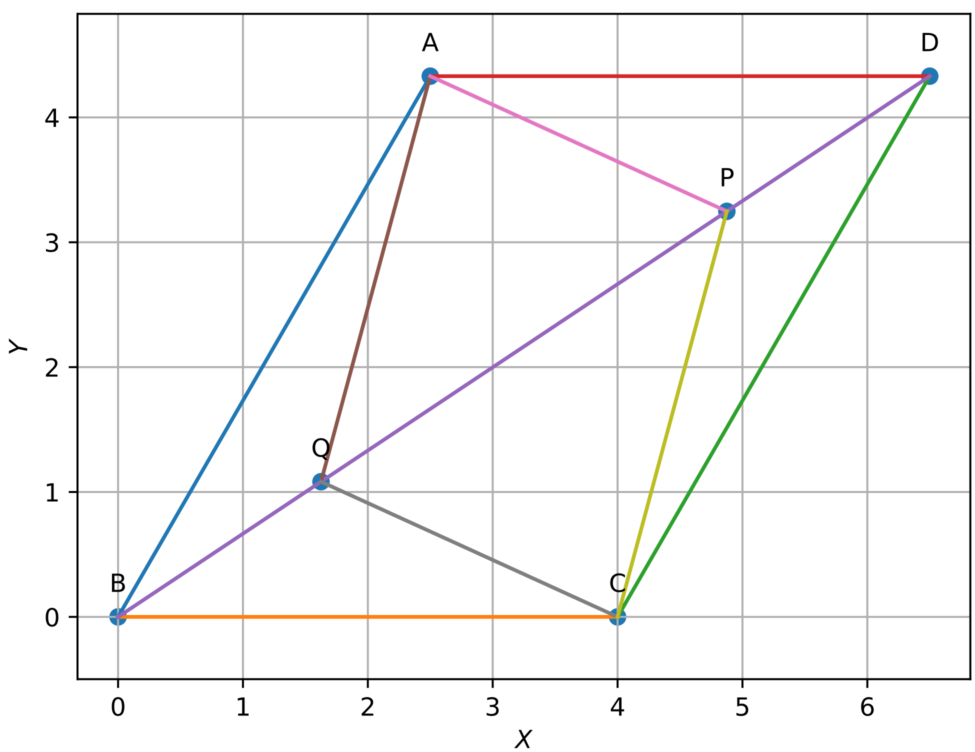

Figure 1.5:

(a) From (1.29), (1.31) and (1.32) taking the norms of the respective sides,

$$\triangle APD \cong \triangle CQB \tag{1.33}$$

(b) From (1.32), taking the norm,

$$AP = CQ \tag{1.34}$$

(c) From (1.29), (1.31) and (1.32) taking the norms of the respective sides,

$$\triangle AQB \cong \triangle CPD \tag{1.35}$$

(d) From (1.32),

$$AQ = CP \tag{1.36}$$

10. $ABCD$ is a parallelogram and AP and CQ are perpendiculars from vertices **A** and **C** on diagonal BD . Show that

(a) $\triangle APB \cong \triangle CQD$

(b) $AP = CQ$

Solution: From Fig. 1.6, and (A.25),

$$\begin{aligned}\cos\angle ABD &= \frac{(\mathbf{A}-\mathbf{B})^T(\mathbf{D}-\mathbf{B})}{\|\mathbf{A}-\mathbf{B}\|\,\|\mathbf{D}-\mathbf{B}\|}\\ \cos\angle CDB &= \frac{(\mathbf{C}-\mathbf{D})^T(\mathbf{B}-\mathbf{D})}{\|\mathbf{C}-\mathbf{D}\|\,\|\mathbf{B}-\mathbf{D}\|}\end{aligned} \tag{1.37}$$

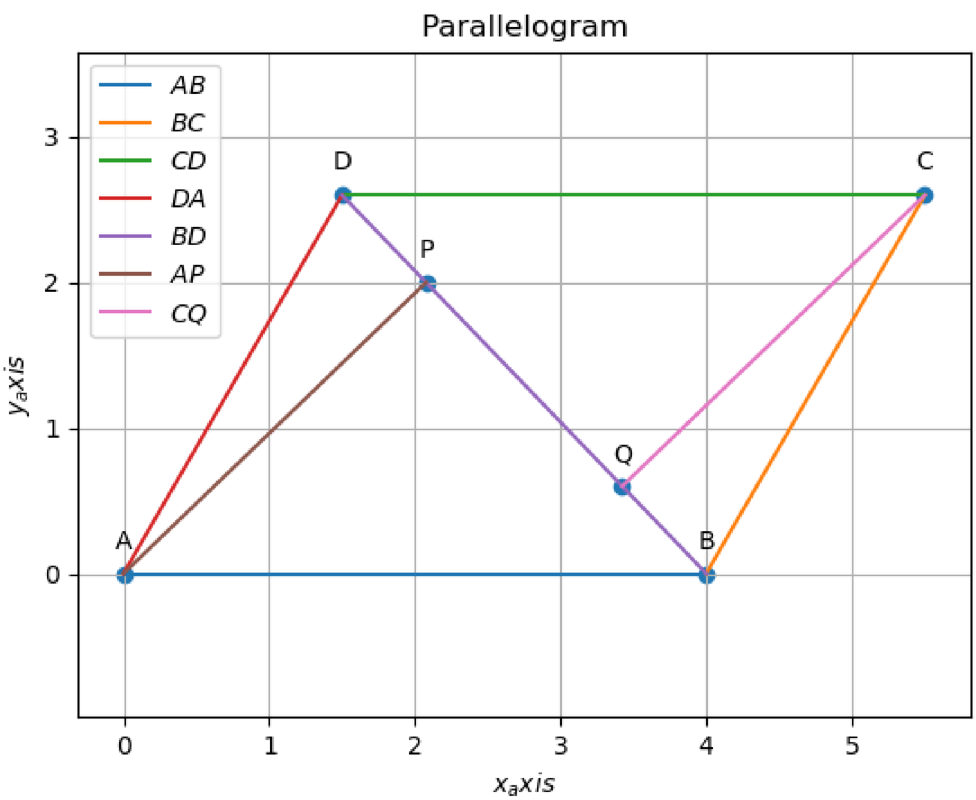
Parallelogram
AB
BC
CD
DA
BD
AP
CQ
D
C
P
Q
A
B
3
2
1
0
0
1
2
3
4
5
y_axis
x_axis

Figure 1.6:

From Appendix A.1.23,

$$\mathbf{A} - \mathbf{B} = \mathbf{D} - \mathbf{C} \tag{1.38}$$

Substituting in (1.37),

$$\cos \angle ABD = \cos \angle CDB \tag{1.39}$$

Using SAS congruence, 10a is proved. 10b follows from 10a.

11. In $\triangle ABC$ and $\triangle DEF, AB = DE, AB \parallel DE, BC = EF$ and $BC \parallel EF$. Vertices $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ are joined to vertices $\mathbf{D}, \mathbf{E}$ and $\mathbf{F}$ respectively (see Figure 1.7). Show that

    (a) quadrilateral $ABED$ is a parallelogram

    (b) quadrilateral $BEFC$ is a parallelogram

    (c) $AD \parallel CF$ and $AD = CF$

    (d) quadrilateral $ACFD$ is a parallelogram

    (e) $AC = DF$

    (f) $\triangle ABC \cong \triangle DEF$.

    Solution: From the given information

$$\mathbf{A} - \mathbf{B} = \mathbf{D} - \mathbf{E} \tag{1.40}$$

$$\mathbf{B} - \mathbf{E} = \mathbf{C} - \mathbf{F} \tag{1.41}$$

(a) From Appendix A.1.23, (1.40) defines the parallelogram $ABED$.

(b) Similarly, (1.41) defines the parallelogram $BEFC$.

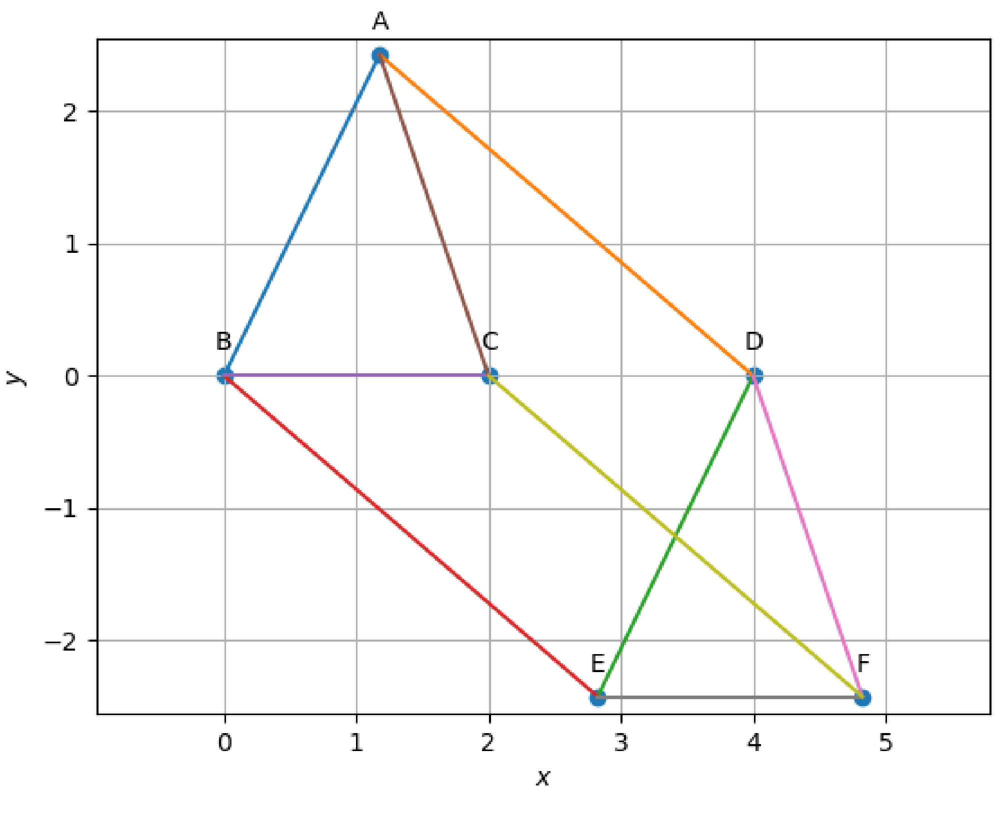

Figure 1.7:

(c) From (1.40) and (1.41),

$$\mathbf{A} - \mathbf{D} = \mathbf{C} - \mathbf{F} \tag{1.42}$$

which yields 11c.

(d) (1.42) implies that $ACFD$ is a parallelogram.

(e) (1.42) implies $AC = DF$.

(f) Obvious from the fact the $ABCD, BEFC$ and $ACFD$ are parallelograms.

12. $ABCD$ is trapezium in which $AB \parallel CD$ and $AD = BC$. Show that,

    (a) $\angle A = \angle B$

    (b) $\angle C = \angle D$

    (c) Diagonal AC = Diagonal BD

    (d) $\triangle ABC = \triangle BAD$

1.2. Mid Point Theorem

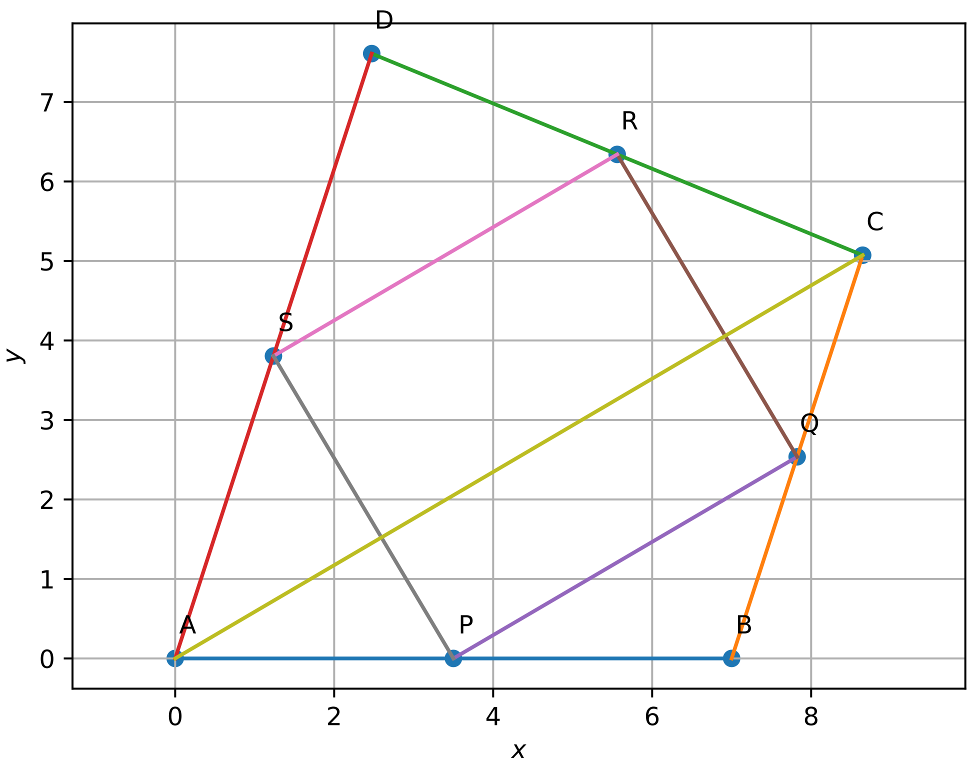

Figure 1.8:

1. ABCD is a quadrilateral in which $\mathbf{P}, \mathbf{Q}, \mathbf{R}$ and $\mathbf{S}$ are mid-points of the sides AB, BC, CD and DA (see Fig 1.8). AC is a diagonal.

   Show that

   (a) $SR \parallel AC$ and $SR = \frac{1}{2}AC$

   (b) $PQ = SR$

   (c) $PQRS$ is a parallelogram.

Solution: Using (A.35),

$$\begin{aligned}\mathbf{P} &= \frac{\mathbf{A}+\mathbf{B}}{2}\\ \mathbf{Q} &= \frac{\mathbf{C}+\mathbf{B}}{2}\\ \mathbf{R} &= \frac{\mathbf{C}+\mathbf{D}}{2}\\ \mathbf{S} &= \frac{\mathbf{D}+\mathbf{A}}{2}\end{aligned} \tag{1.43}$$

(a) Consequently,

$$\mathbf{R}-\mathbf{S} = \frac{\mathbf{C}-\mathbf{A}}{2} \tag{1.44}$$

$$\implies SR \parallel AC \tag{1.45}$$

Also,

$$\left\|\mathbf{R}-\mathbf{S}\right\| = \frac{\left\|\mathbf{C}-\mathbf{A}\right\|}{2} \tag{1.46}$$

$$\implies SR = \frac{1}{2}AC \tag{1.47}$$

(b) From (1.43),

$$\mathbf{R}-\mathbf{S} = \mathbf{Q}-\mathbf{P} \tag{1.48}$$

which means that $PQRS$ is a parallelogram and $PQ = SR$.

2.

3. $ABCD$ is a rectangle and $\mathbf{P}, \mathbf{Q}, \mathbf{R}$ and $\mathbf{S}$ are mid-points of the sides AB, BC, CD and DA respectively. Show that the quadrilateral $PQRS$ is a rhombus.

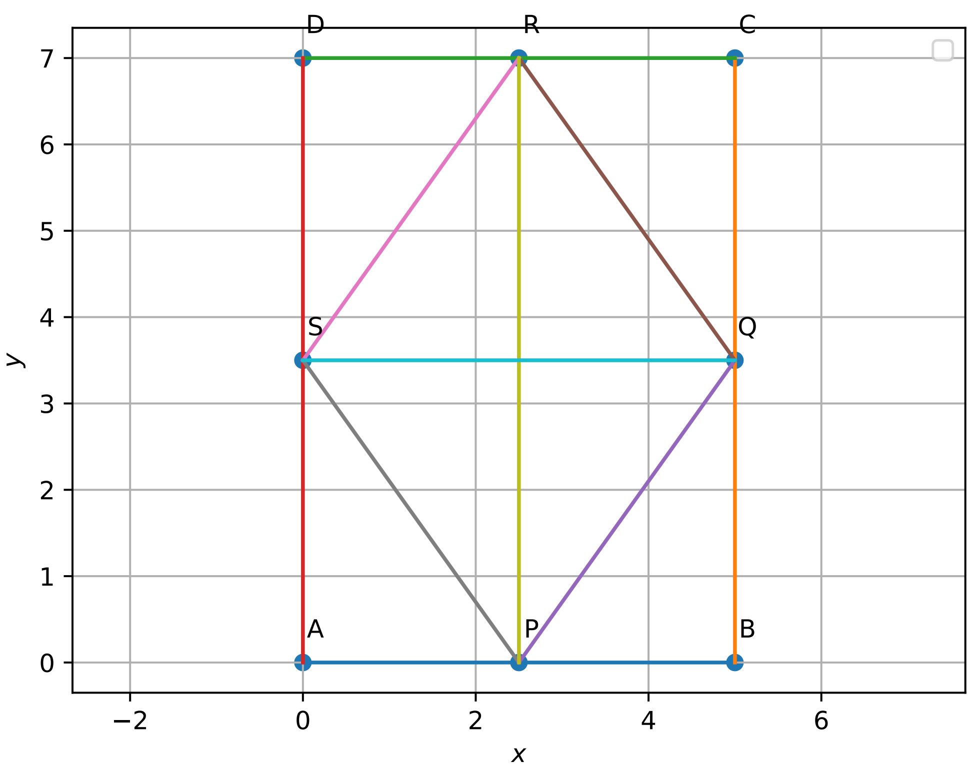

Figure 1.9:

Solution: From Problem 1, it is obvious that $PQRS$ is a parallelogram. Further, from (1.43),

$$\left(\mathbf{P}-\mathbf{R}\right)^{\top}\left(\mathbf{S}-\mathbf{Q}\right)=\left(\mathbf{A}+\mathbf{B}-\mathbf{C}-\mathbf{D}\right)^{\top}\left(\mathbf{A}+\mathbf{D}-\mathbf{B}-\mathbf{C}\right) \tag{1.49}$$

$$=\mathbf{0} \tag{1.50}$$

since

$$\left(\mathbf{A}-\mathbf{D}\right)^{\top}\left(\mathbf{A}-\mathbf{B}\right)=\mathbf{0} \tag{1.51}$$

$$\left\|\mathbf{A}-\mathbf{D}\right\|=\left\|\mathbf{A}-\mathbf{B}\right\| \tag{1.52}$$

as $ABCD$ is a rectangle. Thus, the diagonals PR and SQ bisect each other proving that $PQRS$ is a rhombus.

4.

5. In a parallelogram ABCD, **E** and **F** are the mid-points of sides AB and CD respectively (see Fig. 1.10) Show that the line segments AF and EC trisect the diagonal BD.

Proof. From the given information,

$$\mathbf{E}=\frac{\mathbf{A}+\mathbf{B}}{2} \tag{1.53}$$

$$\mathbf{F}=\frac{\mathbf{C}+\mathbf{D}}{2} \tag{1.54}$$

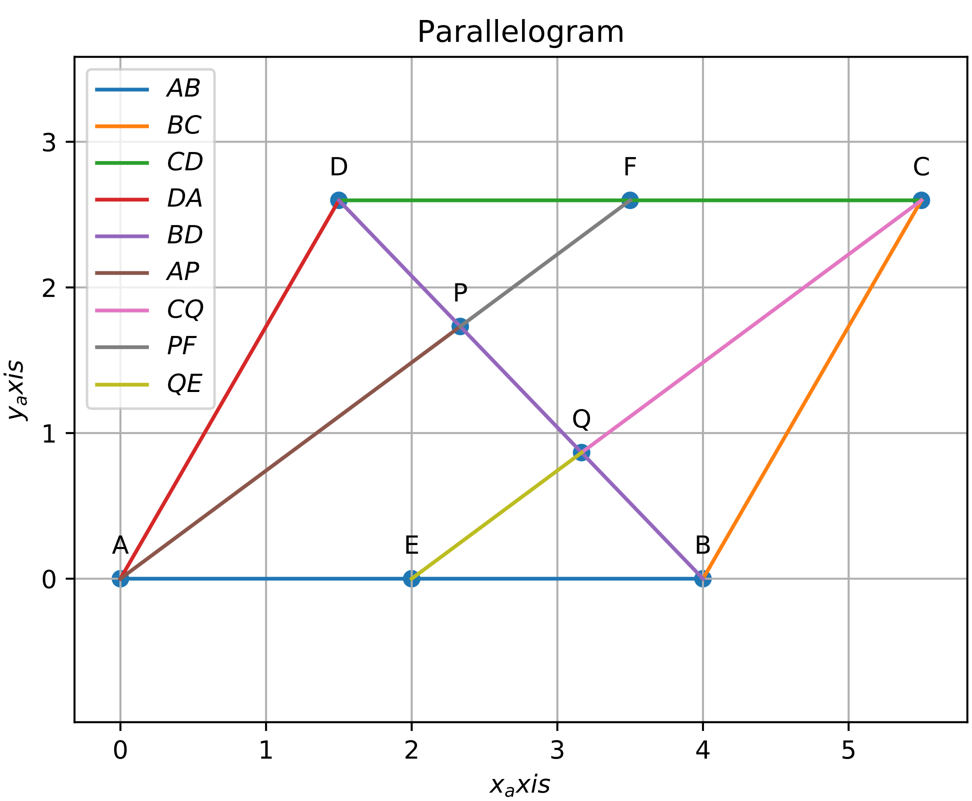
Parallelogram
AB
BC
CD
DA
BD
AP
CQ
PF
QE
D
F
C
P
Q
A
E
B
3
2
1
0
0
1
2
3
4
5
x_axis
y_axis

Figure 1.10:

Hence,

$$\mathbf{E} - \mathbf{C} = \frac{\mathbf{A} - \mathbf{C} + \mathbf{B} - \mathbf{C}}{2} \tag{1.55}$$

$$\mathbf{A} - \mathbf{F} = \frac{\mathbf{A} - \mathbf{C} + \mathbf{A} - \mathbf{D}}{2} \tag{1.56}$$

Since $ABCD$ is a parallelogram,

$$\mathbf{B} - \mathbf{C} = \mathbf{A} - \mathbf{D} \tag{1.57}$$

$$\implies \mathbf{E} - \mathbf{C} = \mathbf{A} - \mathbf{F} \tag{1.58}$$

Thus, $AF \parallel EC$. From Appendix A.1.28, using the fact that $\mathbf{F}$ is the mid point of CD, we conclude that $\mathbf{P}$ is the mid point of DQ. Similarly, it can be shown that $\mathbf{Q}$ is the mid point of BP. $\square$

6.

7. ABC is a triangle right angled at $\mathbf{C}$. A line through the mid-point $\mathbf{M}$ of hypotenuse AB and parallel to BC intersects AC at D (see Fig. 1.11). Show that

   (a) D is the mid-point of AC

   (b) $MD \perp AC$

   (c) $CM = MA = \frac{1}{2}AB$

   Solution:

   (a) Trivial from Appendix A.1.28.

   (b) Since ABC is right angled at C,

$$\left(\mathbf{C} - \mathbf{A}\right)^{\top}\left(\mathbf{C} - \mathbf{B}\right) = 0 \tag{1.59}$$

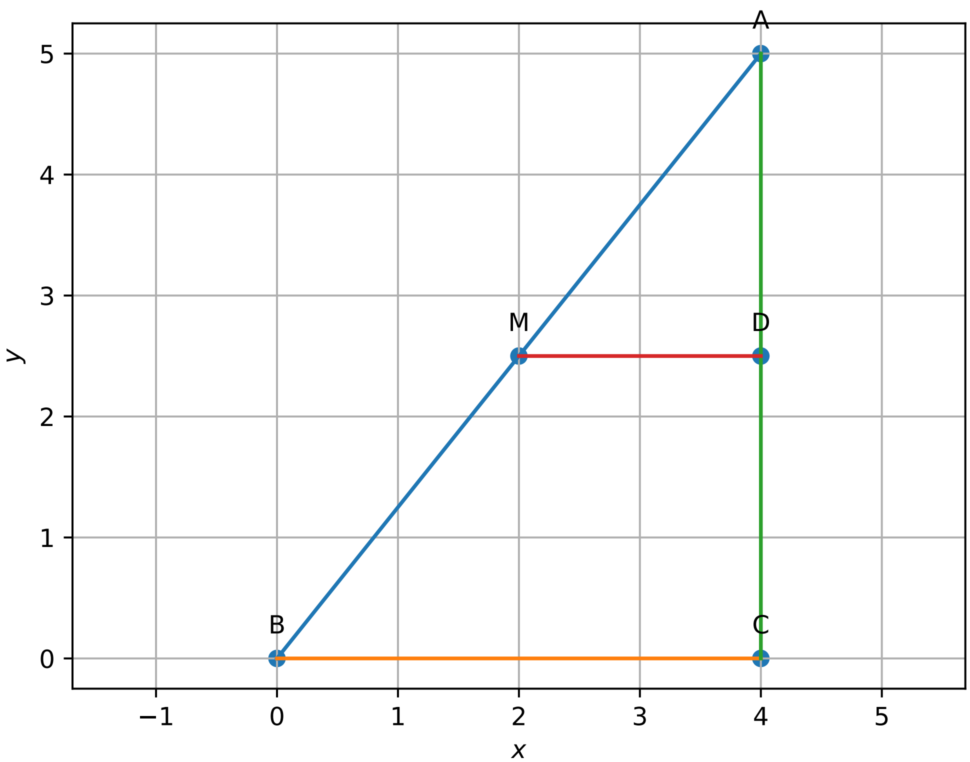

Figure 1.11:

Given that MD is parallel to BC, so

$$(\mathbf{C} - \mathbf{B}) = \lambda(\mathbf{M} - \mathbf{D}) \tag{1.60}$$

Substituting (1.60) in (1.59) and dividing by λ, we get

$$(\mathbf{C} - \mathbf{A})^\top (\mathbf{M} - \mathbf{D}) = 0 \tag{1.61}$$

From (1.61) it can be concluded that $MD \perp AC$.

(c) Since

$$\|\mathbf{C}-\mathbf{M}\|^2-\|\mathbf{A}-\mathbf{M}\|^2=\|\mathbf{C}\|^2-\|\mathbf{A}\|^2-2\left(\mathbf{C}-\mathbf{A}\right)^\top\mathbf{M} \tag{1.62}$$

$$=\left(\mathbf{C}-\mathbf{A}\right)^\top\left(\mathbf{C}+\mathbf{A}-2\mathbf{M}\right) \tag{1.63}$$

$$=\left(\mathbf{C}-\mathbf{A}\right)^\top\left(\mathbf{C}-\mathbf{B}\right)=\mathbf{0} \tag{1.64}$$

upon substituting from Property 7a and (1.59). Thus, $CM = AM$.

Chapter 2

Areas

2.1. Parallelograms

1. In the Figure 2.1, $ABCD$ is a parallelogram, $AE \perp DC$ and $CF \perp AD$. If $AB = 16cm$, $AE = 8cm$, and $CF = 10cm$, find AD.

2. If $\mathbf{E}, \mathbf{F}, \mathbf{G}$ and $\mathbf{H}$ are respectively the mid-points of the sides of a parallelogram $ABCD$, show that

$$ar(EFGH) = \frac{1}{2} ar(ABCD) \tag{2.1}$$

Proof. From Problem 1, $EFGH$ is also a parallelogram and

$$\mathbf{E} - \mathbf{F} = \frac{\mathbf{A} - \mathbf{C}}{2} \tag{2.2}$$

$$\mathbf{E} - \mathbf{H} = \frac{\mathbf{A} - \mathbf{D}}{2} \tag{2.3}$$

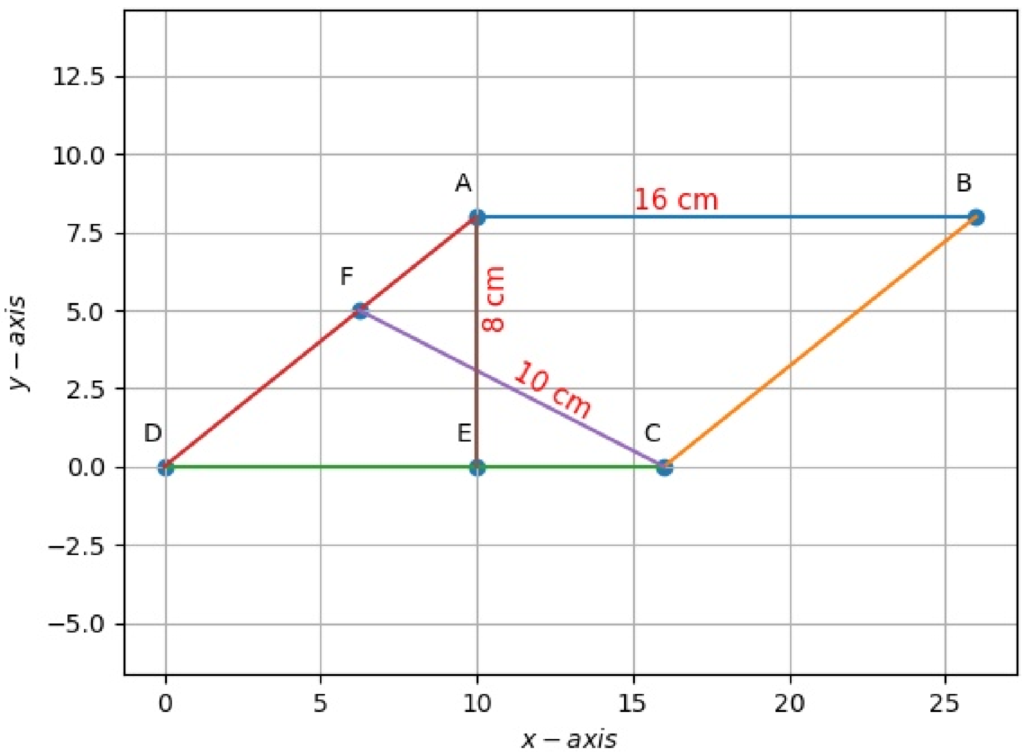

Figure 2.1:

Thus, the area off $EFGH$ is obtained from (A.39) as

$$\|(\mathbf{E}-\mathbf{F})\times(\mathbf{E}-\mathbf{H})\| = \frac{1}{4}\|(\mathbf{A}-\mathbf{C})\times(\mathbf{B}-\mathbf{D})\| \tag{2.4}$$

From Appendix A.1.23,

$$\mathbf{D} = \mathbf{C}-\mathbf{B}+\mathbf{A} \tag{2.5}$$

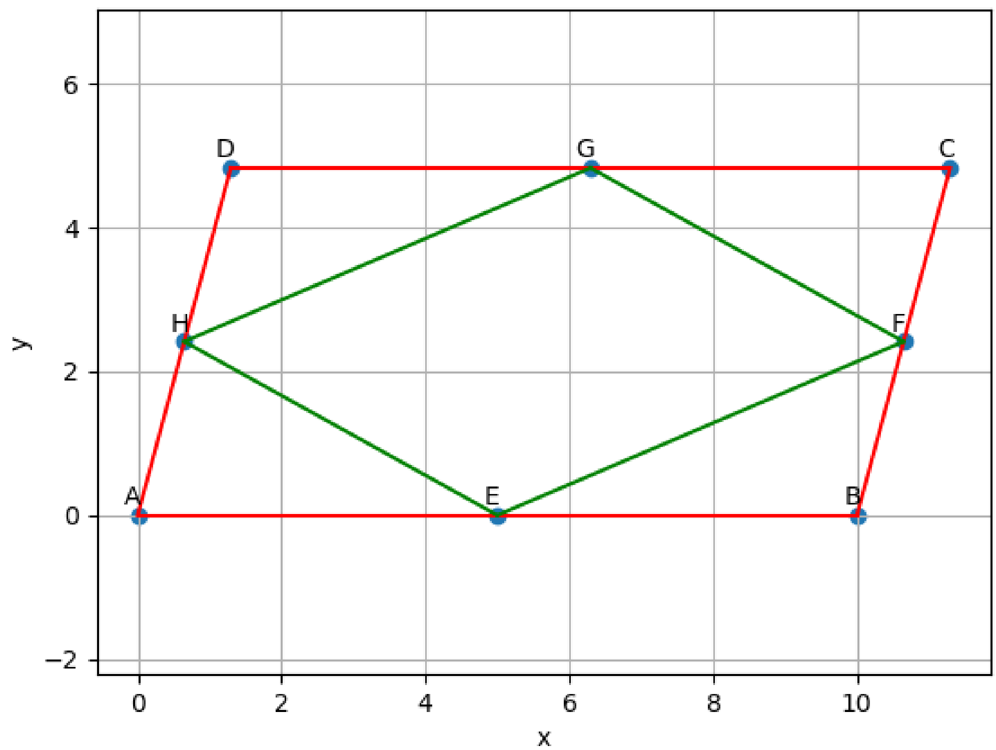

Figure 2.2:

which,

$$(\mathbf{A}-\mathbf{C})\times(\mathbf{B}-\mathbf{D}) = (\mathbf{A}-\mathbf{C})\times(2\mathbf{B}-\mathbf{C}-\mathbf{A}) \tag{2.6}$$

$$= 2\left(\mathbf{A}\times\mathbf{B}+\mathbf{B}\times\mathbf{C}+\mathbf{C}\times\mathbf{A}\right) \tag{2.7}$$

Substituting (2.7) in (2.4) yields

$$\left\|(\mathbf{E}-\mathbf{F})\times(\mathbf{E}-\mathbf{H})\right\| = \frac{1}{2}\left\|\mathbf{A}\times\mathbf{B}+\mathbf{B}\times\mathbf{C}+\mathbf{C}\times\mathbf{A}\right\| \tag{2.8}$$

The area of $ABCD$ is

$$\|(\mathbf{A}-\mathbf{B})\times(\mathbf{A}-\mathbf{D})\| = \|\mathbf{A}\times\mathbf{B}+\mathbf{B}\times\mathbf{C}+\mathbf{C}\times\mathbf{A}\| \tag{2.9}$$

upon substituting from Appendix A.1.23 and simplifying. From (2.8) and (2.9) we obtain (2.1). □

3.

4. For a given Parallelogram $ABCD$, show that for any point $\mathbf{P}$ inside the parallelogram,

   (a) $Ar(APD) + Ar(PBC) = \frac{1}{2}Ar(ABCD)$

   (b) $Ar(APD) + Ar(PBC) = Ar(APB) + Ar(PCD)$

5. In Fig.1, $PQRS$ and $ABRS$ are parallelograms and $\mathbf{X}$ is any point on side BR. Show that

   (a) $ar(PQRS) = ar(ABRS)$

   (b) $ar(AXS) = \frac{1}{2}ar(PQRS)$

Proof. (a) From Appendix A.1.23,

$$\mathbf{A}-\mathbf{B} = \mathbf{S}-\mathbf{R} = \mathbf{P}-\mathbf{Q} \tag{2.10}$$

and from Appendix A.1.25, using (2.10), we obtain Property 5a.

(b) Using section formula, let

$$\mathbf{X} = \frac{\mathbf{R}+k\mathbf{B}}{1+k}. \tag{2.11}$$

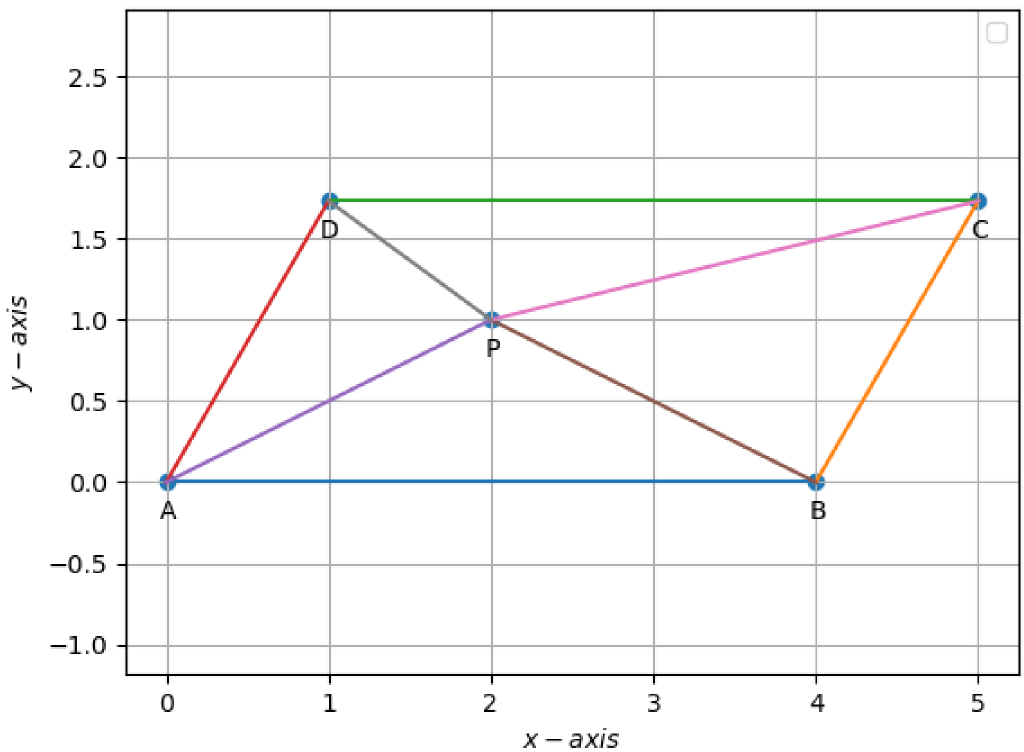

Figure 2.3:

Then,

$$ar\left(AXS\right) = \frac{1}{2}\left\|\mathbf{S}\times\mathbf{X}+\mathbf{X}\times\mathbf{A}+\mathbf{A}\times\mathbf{S}\right\| \tag{2.12}$$

$$= \frac{1}{2}\left\|\frac{\mathbf{S}\times\mathbf{R}+k\mathbf{S}\times\mathbf{B}+\mathbf{R}\times\mathbf{A}+k\mathbf{B}\times\mathbf{A}}{k+1}+\mathbf{A}\times\mathbf{S}\right\| \tag{2.13}$$

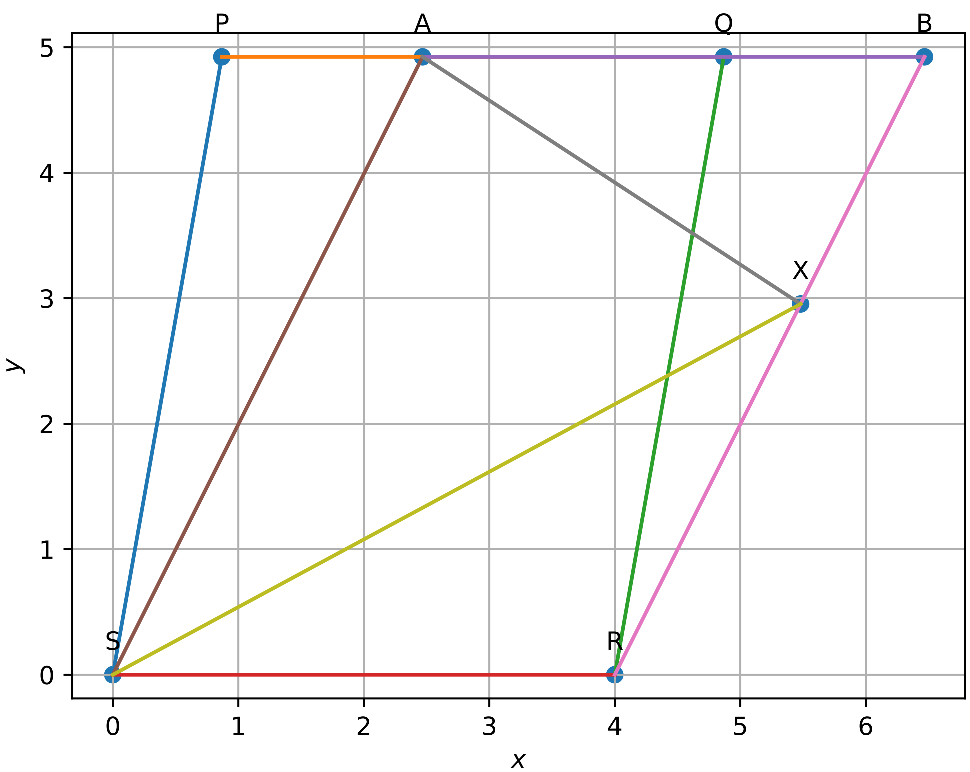

Figure 2.4:

Substituting for $\mathbf{B}$ from (2.10) in the above,

$$ar\left(AXS\right) = \frac{1}{2}\left\|\frac{\mathbf{S}\times\mathbf{R}+\mathbf{R}\times\mathbf{A}+k\left(\mathbf{S}-\mathbf{A}\right)\times\left(\mathbf{A}-\mathbf{S}+\mathbf{R}\right)}{k+1}+\mathbf{A}\times\mathbf{S}\right\| \tag{2.14}$$

$$= \frac{1}{2}\left\|\frac{\mathbf{S}\times\mathbf{R}+\mathbf{R}\times\mathbf{A}+k\left(\mathbf{S}-\mathbf{A}\right)\times\mathbf{R}}{k+1}+\mathbf{A}\times\mathbf{S}\right\| \tag{2.15}$$

$$= \frac{1}{2}\left\|\mathbf{S}\times\mathbf{R}+\mathbf{R}\times\mathbf{A}+\mathbf{A}\times\mathbf{S}\right\| \tag{2.16}$$

$$= \frac{1}{2}ar\left(ABRS\right) \tag{2.17}$$

$\square$

2.2. Triangles

1. In the Figure 2.5, **E** is any point on median AD of a $\triangle ABC$. Show that

$$ar(ABE) = ar(ACE). \tag{2.18}$$

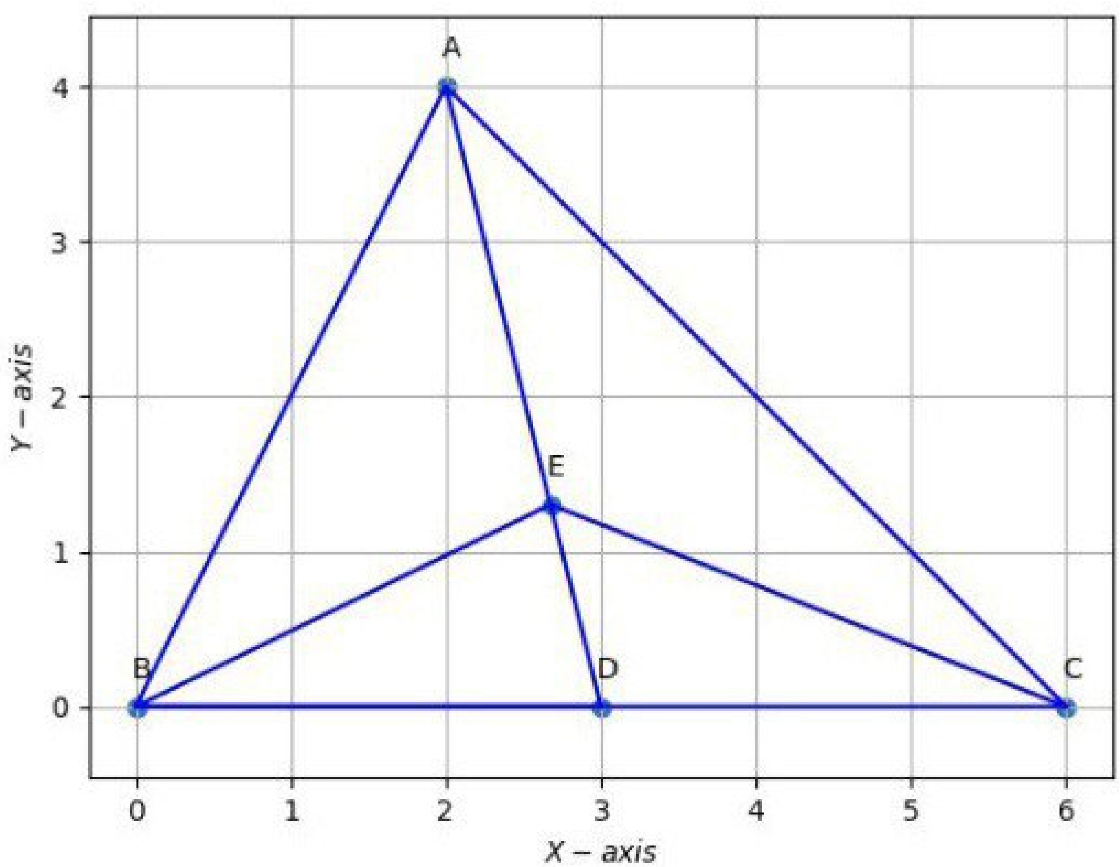

Figure 2.5:

Proof. From (A.7)

$$
\begin{aligned}
ar\,(BDE) &= \frac{1}{2}\left\|\mathbf{B}\times\mathbf{D}+\mathbf{D}\times\mathbf{E}+\mathbf{E}\times\mathbf{B}\right\| && (2.19)\\
&= \frac{1}{2}\left\|\mathbf{B}\times\left(\frac{\mathbf{B}+\mathbf{C}}{2}\right)+\left(\frac{\mathbf{B}+\mathbf{C}}{2}\right)\times\mathbf{E}+\mathbf{E}\times\mathbf{B}\right\| && (2.20)\\
&= \frac{1}{4}\left\|\mathbf{B}\times\mathbf{C}+\mathbf{C}\times\mathbf{E}+\mathbf{E}\times\mathbf{B}\right\| && (2.21)
\end{aligned}
$$

after simplification. Similarly, it can be shown that

$$
\begin{aligned}
ar\,(EDC) &= \frac{1}{4}\left\|\mathbf{B}\times\mathbf{C}+\mathbf{C}\times\mathbf{E}+\mathbf{E}\times\mathbf{B}\right\| && (2.22)\\
&= ar\,(BDE) && (2.23)
\end{aligned}
$$

The same approach can be used to show that

$$ar\,(ADB) = ar\,(ADC) \tag{2.24}$$

Subtracting (2.23) from (2.24) yields (2.18)

□

2. In $\triangle ABC, \mathbf{E}$ is the mid-point of median AD. Show that

$$ar(\triangle BED) = \frac{1}{4} ar(\triangle ABC) \tag{2.25}$$

Proof. From Problem 2,

$$ar(\triangle BED) = \frac{1}{4}\left\|\mathbf{B}\times\mathbf{C}+\mathbf{C}\times\mathbf{E}+\mathbf{E}\times\mathbf{B}\right\| \tag{2.26}$$

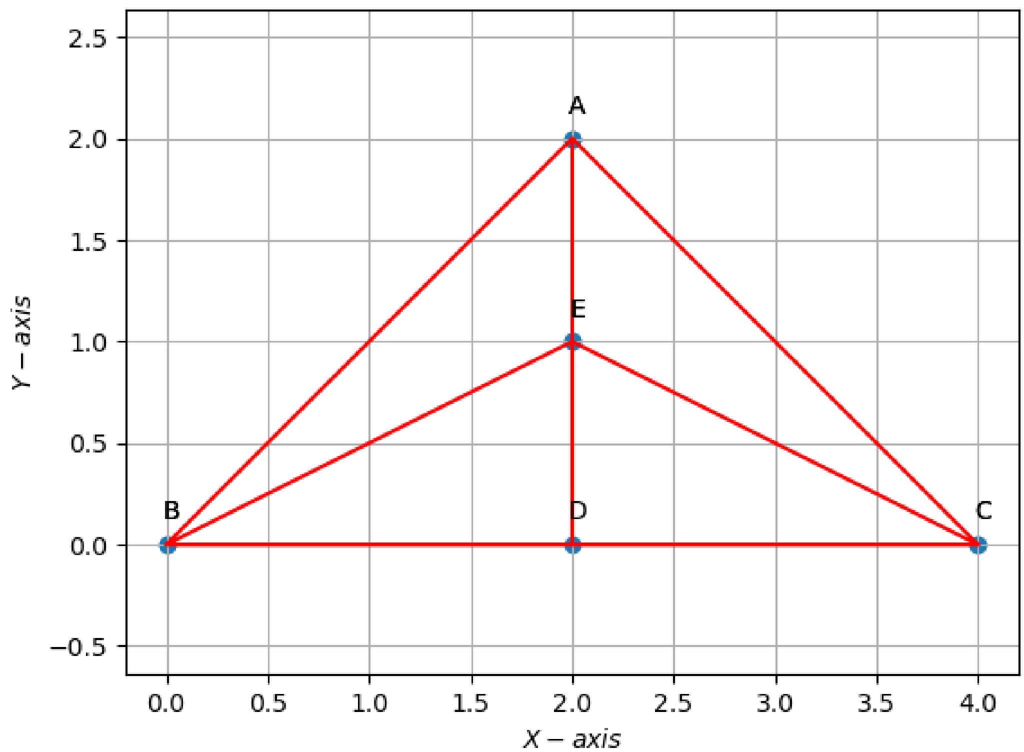

Figure 2.6:

Since

$$\mathbf{E} = \frac{\mathbf{A} + \mathbf{D}}{2} \tag{2.27}$$

$$= \frac{2\mathbf{A} + \mathbf{B} + \mathbf{C}}{4}, \tag{2.28}$$

substituting the above in (2.26) yields

$$ar(\triangle BED) = \frac{1}{4}\left\|\mathbf{B}\times\mathbf{C} + \mathbf{C}\times\frac{2\mathbf{A}+\mathbf{B}+\mathbf{C}}{4} + \frac{2\mathbf{A}+\mathbf{B}+\mathbf{C}}{4}\times\mathbf{B}\right\| \tag{2.29}$$

$$= \frac{1}{8}\left\|\mathbf{A}\times\mathbf{B} + \mathbf{B}\times\mathbf{C} + \mathbf{C}\times\mathbf{A}\right\| \tag{2.30}$$

resulting in (2.25). □

3. Show that the diagonals of a parallelogram divide it into four triangles of equal area.

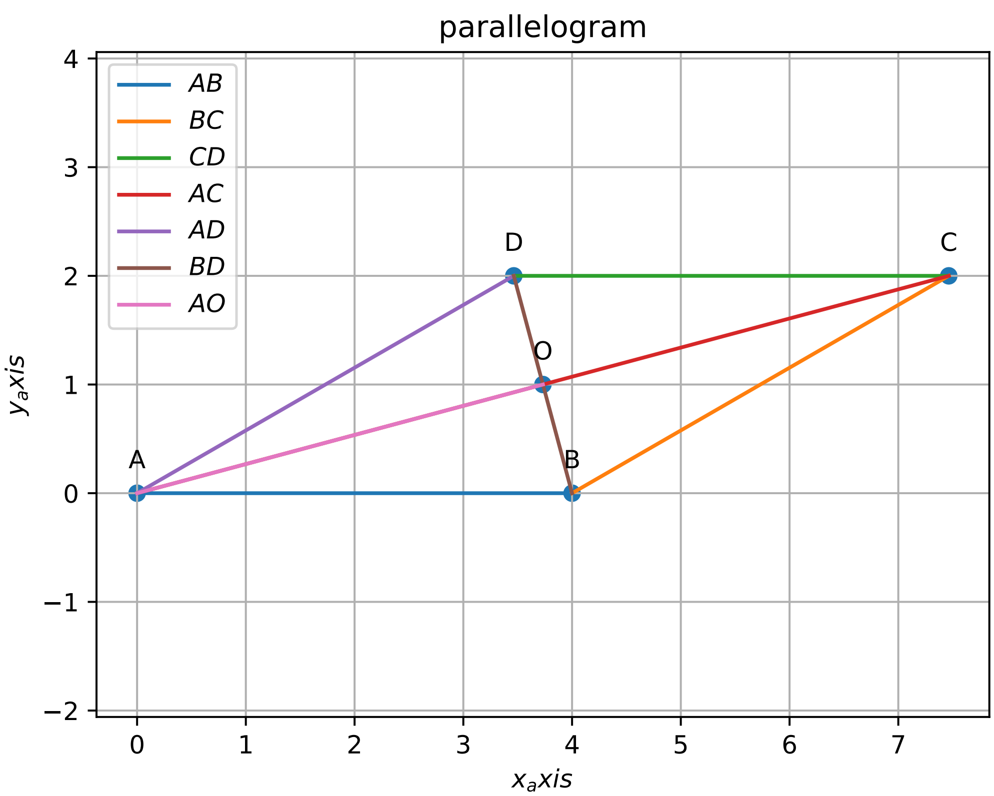

Figure 2.7:

Proof. See Fig. 2.7. From Appendix A.1.24 and A.1.3

$$ar(AOB) = \frac{1}{2}\left\|\mathbf{A}\times\mathbf{O}+\mathbf{O}\times\mathbf{B}+\mathbf{B}\times\mathbf{A}\right\| \tag{2.31}$$

$$= \frac{1}{2}\left\|\mathbf{A}\times\left(\frac{\mathbf{A}+\mathbf{C}}{2}\right)+\left(\frac{\mathbf{A}+\mathbf{C}}{2}\right)\times\mathbf{B}+\mathbf{B}\times\mathbf{A}\right\| \tag{2.32}$$

$$= \frac{1}{4}\left\|\mathbf{A}\times\mathbf{C}+\mathbf{C}\times\mathbf{B}+\mathbf{B}\times\mathbf{A}\right\| \tag{2.33}$$

yielding the desired result from Appendix A.1.25 □

4. ABC, ABD are 2 triangles on same base AB, if line segment CD is bisected by AB at $\mathbf{O}$, show that

$$ar(ABC) = ar(ABD) \tag{2.34}$$

Proof. See Fig. 2.8. AO and OB are medians of triangles ADC and BDC. From Appendix A.1.5, (2.34) is trivial. □

5.

6.

7.

8.

9. The side AB of a parallelogram $ABCD$ is produced to any point $\mathbf{P}$. A line through $\mathbf{A}$ and parallel to CP meets CB produced at $\mathbf{Q}$ and then parallelogram $PBQR$ is

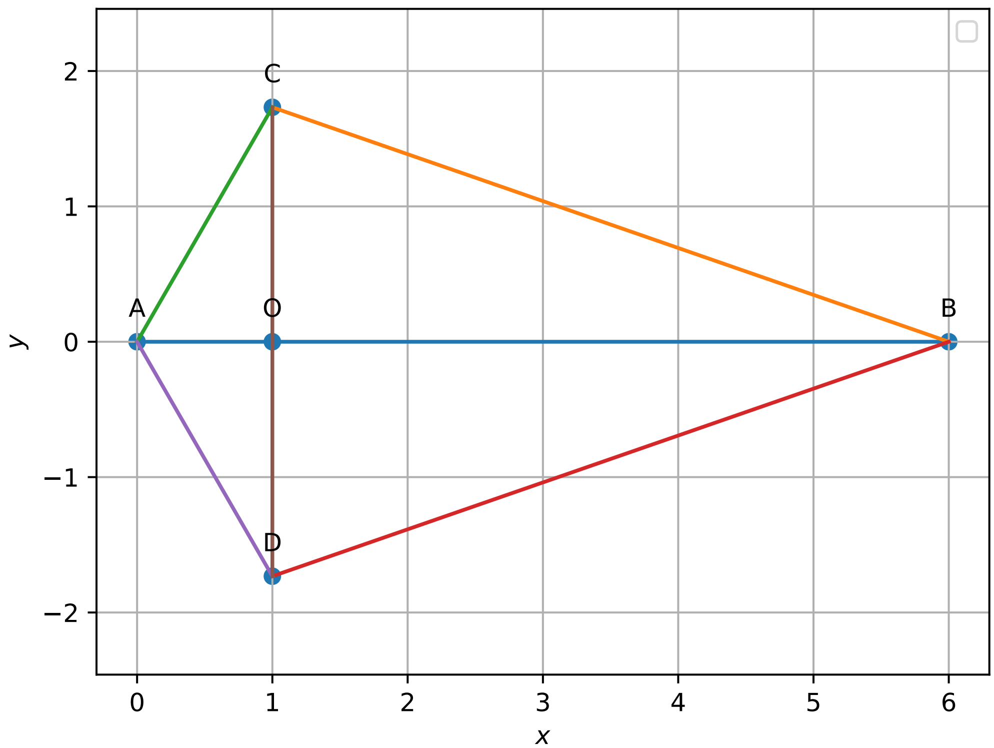

Figure 2.8:

completed. Show that

$$ar(ABCD) = ar(PBQR) \tag{2.35}$$

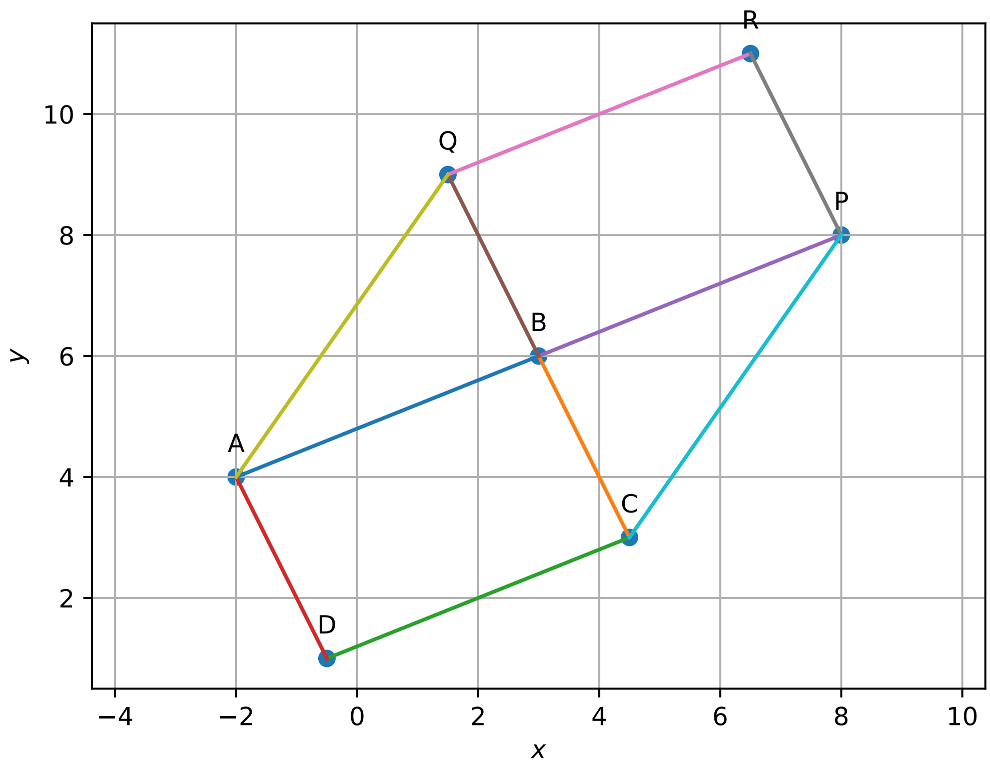

Figure 2.9:

Proof. From the given information, using section formula,

$$\mathbf{Q} = \frac{k_1\mathbf{C} + \mathbf{B}}{k_1 + 1} \tag{2.36}$$

$$\mathbf{P} = \frac{k_2\mathbf{A} + \mathbf{B}}{k_2 + 1} \tag{2.37}$$

Also, since $AQ \parallel CP$,

$$\mathbf{A} - \mathbf{Q} = k\left(\mathbf{C} - \mathbf{P}\right) \tag{2.38}$$

Substituting from (2.36) and (2.37) in the above,

$$\mathbf{A} - \frac{k_1\mathbf{C} + \mathbf{B}}{k_1 + 1} = k\left(\mathbf{C} - \frac{k_2\mathbf{A} + \mathbf{B}}{k_2 + 1}\right) \tag{2.39}$$

which, after some algebra, yields

$$\left(1 + \frac{kk_2}{k_2 + 1}\right)\mathbf{A} + \left(\frac{k}{k_2 + 1} - \frac{1}{k_1 + 1}\right)\mathbf{B} - \left(\frac{k_1}{k_1 + 1} + k\right)\mathbf{C} = \mathbf{0} \tag{2.40}$$

From Appendix A.1.26, (2.40) results in

$$\left(\frac{k}{k_2 + 1} - \frac{1}{k_1 + 1}\right) = \left(\frac{k_1}{k_1 + 1} + k\right) = 0 \tag{2.41}$$

$$\text{or, } k_1 + k_2 = -1 \tag{2.42}$$

From Appendix A.1.25

$$ar\left(PBQR\right) = \left\|\mathbf{P} \times \mathbf{B} + \mathbf{B} \times \mathbf{Q} + \mathbf{Q} \times \mathbf{P}\right\| \tag{2.43}$$

The R.H.S. in the above can be expressed as

$$\frac{k_2\mathbf{A} + \mathbf{B}}{k_2 + 1} \times \mathbf{B} + \mathbf{B} \times \frac{k_1\mathbf{C} + \mathbf{B}}{k_1 + 1} + \frac{k_1\mathbf{C} + \mathbf{B}}{k_1 + 1} \times \frac{k_2\mathbf{A} + \mathbf{B}}{k_2 + 1} \tag{2.44}$$

leading to

$$\begin{multline}\left(\frac{k_2}{k_2 + 1} - \frac{k_2}{\left(k_1 + 1\right)\left(k_2 + 1\right)}\right)\mathbf{A} \times \mathbf{B} \\ + \mathbf{B} \times \mathbf{C}\left(\frac{k_1}{k_1 + 1} - \frac{k_1}{\left(k_1 + 1\right)\left(k_2 + 1\right)}\right) \\ + \frac{k_1k_2}{\left(k_1 + 1\right)\left(k_2 + 1\right)}\mathbf{C} \times \mathbf{A} \end{multline} \tag{2.45}$$

that can be simplified to obtain

$$ar\,(PBQR) = \frac{k_1 k_2}{(k_1+1)\,(k_2+1)}\,\|(\mathbf{A}\times\mathbf{B}+\mathbf{B}\times\mathbf{C}+\mathbf{C}\times\mathbf{A})\| \tag{2.46}$$

$$= \|(\mathbf{A}\times\mathbf{B}+\mathbf{B}\times\mathbf{C}+\mathbf{C}\times\mathbf{A})\| \tag{2.47}$$

using the fact that

$$\frac{k_1 k_2}{(k_1+1)\,(k_2+1)} = 1 \tag{2.48}$$

from (2.42). Also, from Appendix A.1.25,

$$ar\,(ABCD) = \|(\mathbf{A}\times\mathbf{B}+\mathbf{B}\times\mathbf{C}+\mathbf{C}\times\mathbf{A})\| \tag{2.49}$$

yielding (2.35) from (2.47). □

10.

11. $ABCDE$ is a pentagon. A line through **B** parallel to AC meets DC produced at F. Show that

$$ar(ACB) = ar(ACF) \tag{2.50}$$

$$ar(AEDF) = ar(ABCDE) \tag{2.51}$$

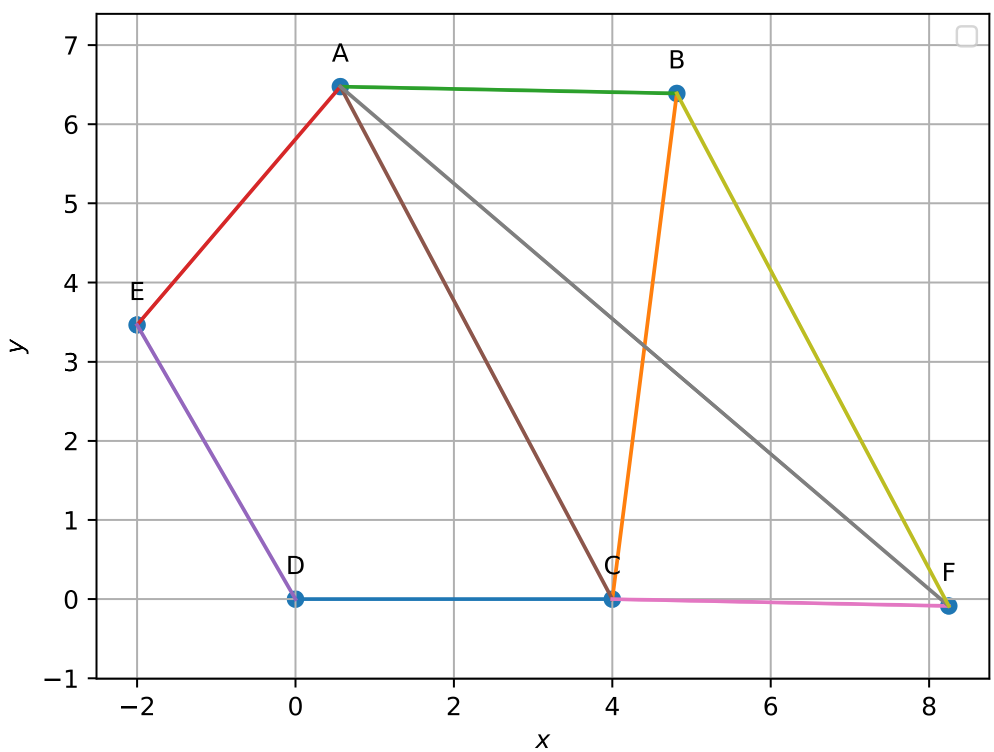

Figure 2.10:

Proof. Since $BF \parallel AC$,

$$\mathbf{F} - \mathbf{B} = k\left(\mathbf{C} - \mathbf{A}\right) \tag{2.52}$$

$$\implies \mathbf{F} = \mathbf{B} + k\left(\mathbf{C} - \mathbf{A}\right) \tag{2.53}$$

Thus, from Appendix A.1.3,

$$ar(ACF) = \frac{1}{2}\left\|\mathbf{F} \times \mathbf{A} + \mathbf{A} \times \mathbf{C} + \mathbf{C} \times \mathbf{F}\right\| \tag{2.54}$$

Substituting from (2.53) in (2.54),

$$ar(ACF) = \frac{1}{2}\left\|\{\mathbf{B} + k(\mathbf{C} - \mathbf{A})\} \times \mathbf{A} + \mathbf{A} \times \mathbf{C} + \mathbf{C} \times \{\mathbf{B} + k(\mathbf{C} - \mathbf{A})\}\right\| \tag{2.55}$$

$$= \frac{1}{2}\left\|\mathbf{B} \times \mathbf{A} + \mathbf{A} \times \mathbf{C} + \mathbf{C} \times \mathbf{B}\right\| \tag{2.56}$$

$$= ar(ACB) \tag{2.57}$$

upon substituting from from Appendix A.1.3. (2.51) follows from (2.50).

□

12.

13.

14.

15.

16. In the Figure 2.11,

$$ar(DRC) = ar(DPC) \tag{2.58}$$

$$ar(BDP) = ar(ARC). \tag{2.59}$$

Show that the quadrilaterals $ABCD$ and $DCPR$ are trapeziums.

Proof. From Appendix A.1.4 and (2.58),

$$\frac{1}{2}\left\|(\mathbf{D} - \mathbf{R}) \times (\mathbf{D} - \mathbf{C})\right\| = \frac{1}{2}\left\|(\mathbf{C} - \mathbf{D}) \times (\mathbf{C} - \mathbf{P})\right\| \tag{2.60}$$

$$\implies (\mathbf{D} - \mathbf{R}) \times (\mathbf{D} - \mathbf{C}) = (\mathbf{C} - \mathbf{D}) \times (\mathbf{C} - \mathbf{P}) \tag{2.61}$$

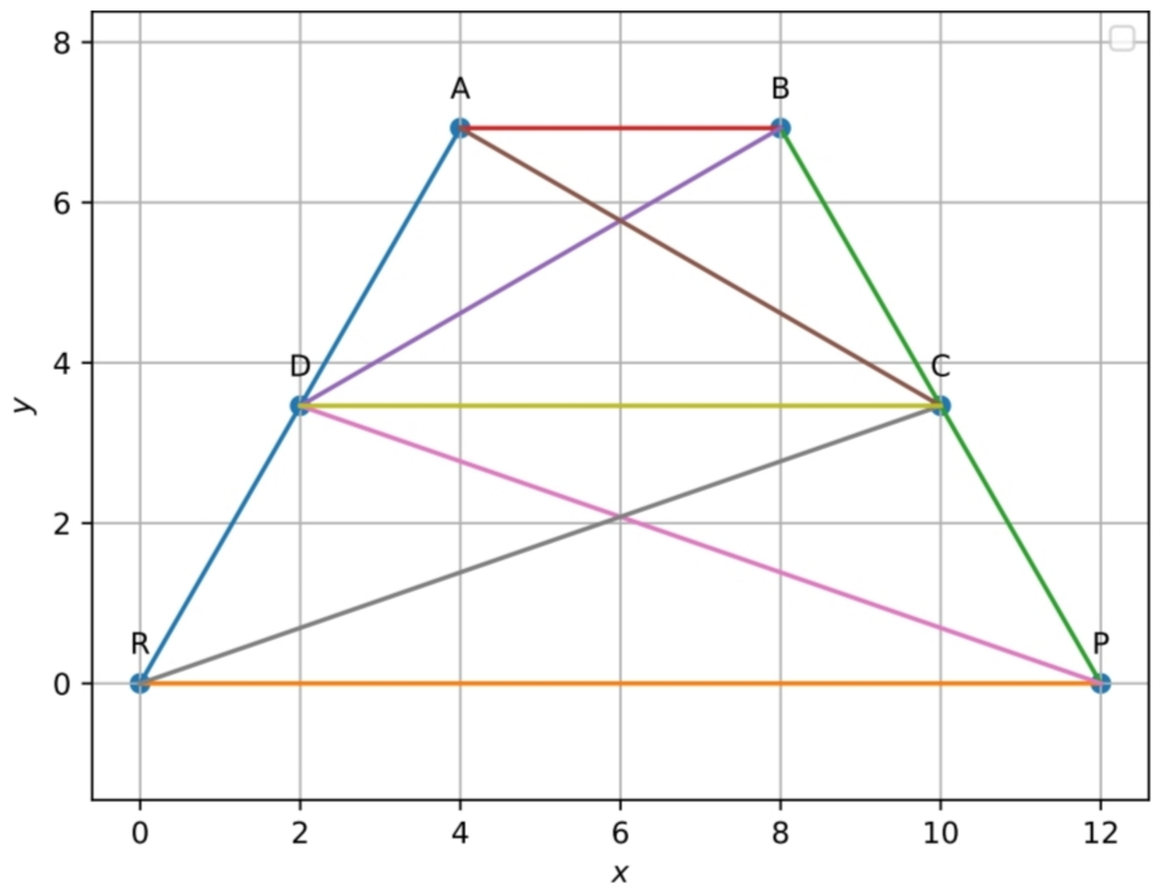

Figure 2.11:

which can be expressed as

$$(\mathbf{C}-\mathbf{D})\times(\mathbf{C}-\mathbf{D}+\mathbf{R}-\mathbf{P})=\mathbf{0} \tag{2.62}$$

$$\implies (\mathbf{C}-\mathbf{D})\times(\mathbf{R}-\mathbf{P})=\mathbf{0} \tag{2.63}$$

$$\text{or, } CD \parallel RP \tag{2.64}$$

Hence, $DCPR$ is a trapezium. Similarly, it can be shown that $ABCD$ is also a trapezium.

□

2.3. Triangles and Parallelograms

1.

2.

3. In Fig. 2.12 $ABCD, DCFE$ and $ABFE$ are parallelograms. Show that

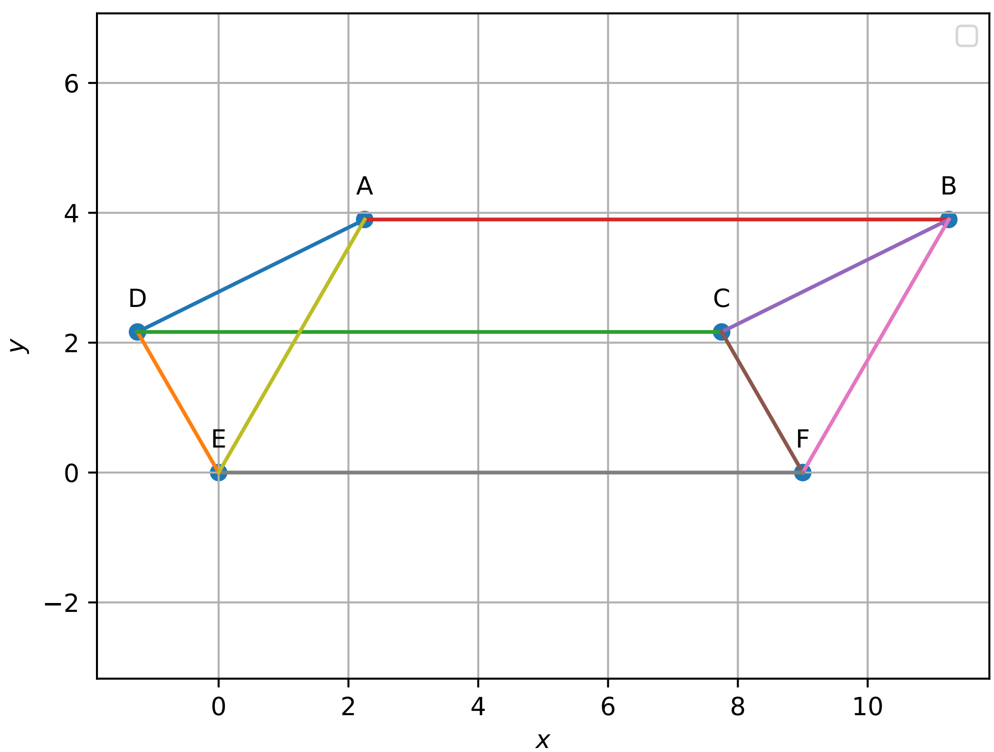

Figure 2.12:

$$ar(ADE) = ar(BCF) \tag{2.65}$$

Proof. From the given information and Appendix A.1.23,

$$\mathbf{B} - \mathbf{A} = \mathbf{C} - \mathbf{D} \tag{2.66}$$

$$\mathbf{C} - \mathbf{D} = \mathbf{F} - \mathbf{E} \tag{2.67}$$

$$\mathbf{B} - \mathbf{A} = \mathbf{F} - \mathbf{E} \tag{2.68}$$

Thus, from Appendix A.1.25,

$$ar(ADE) = \|(\mathbf{D} - \mathbf{E}) \times (\mathbf{D} - \mathbf{A})\| \tag{2.69}$$

$$= \|(\mathbf{C} - \mathbf{F}) \times (\mathbf{C} - \mathbf{B})\| \tag{2.70}$$

$$= ar(ADE) \tag{2.71}$$

upon substituting from (2.66) and (2.67). □

4. In figure below, $ABCD$ is a parallelogram and BC is produced to a point **Q** such that $AD = CQ$. If AQ intersect DC at **P**, show that

$$ar(BPC) = ar(DPQ). \tag{2.72}$$

5. In Fig. 2.14, ABC and BDE are two equilateral triangles such that **D** is the mid-point

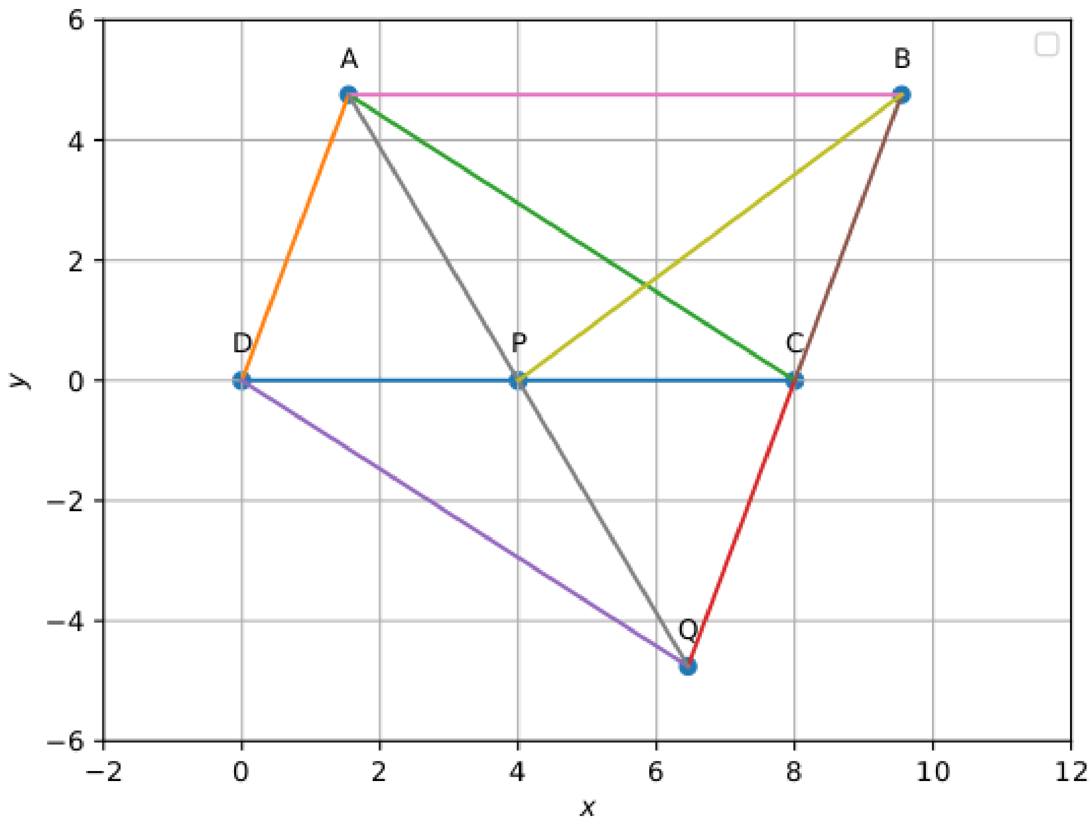

Figure 2.13:

of BC. If AE intersects BC at $\mathbf{F}$, show that

$$ar(BDE) = \frac{1}{4}ar(ABC) \tag{2.73}$$

$$ar(BDE) = \frac{1}{2}ar(BAE) \tag{2.74}$$

$$ar(ABC) = 2ar(BEC) \tag{2.75}$$

$$ar(BFE) = ar(AFD) \tag{2.76}$$

$$ar(BFE) = 2ar(FED) \tag{2.77}$$

$$ar(FED) = \frac{1}{8}ar(AFC) \tag{2.78}$$

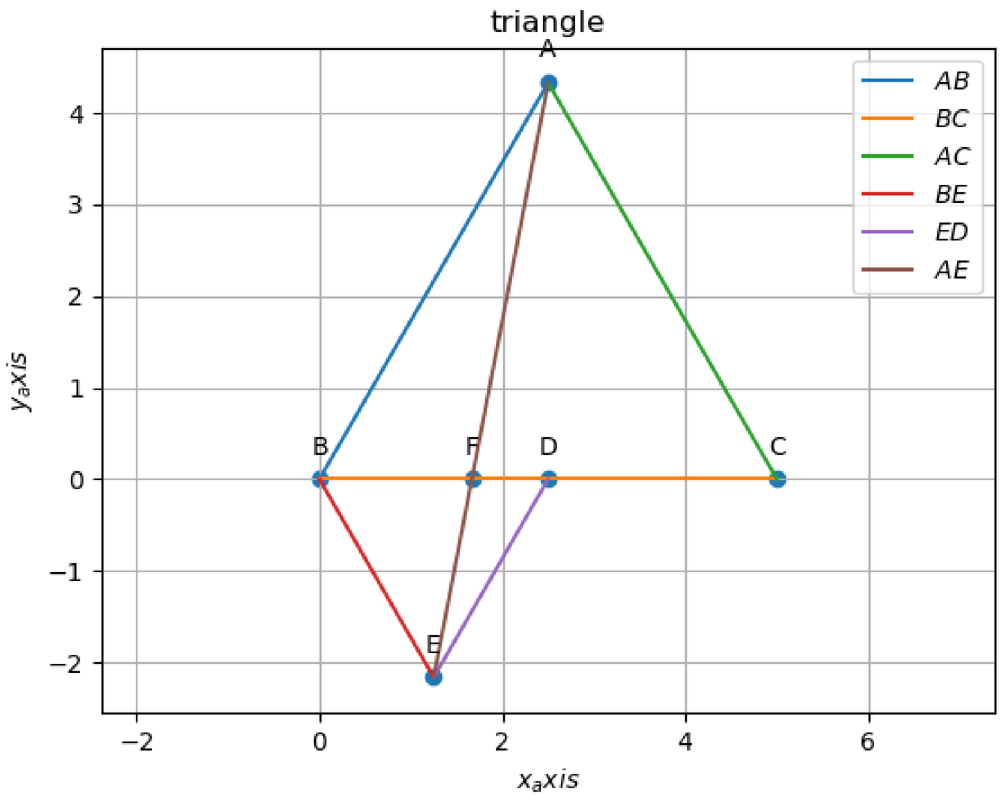

Figure 2.14:

6.

7.

8.

Chapter 3

Circles

3.1. Equal Chords

1. Two circles of radii 5cm and 3cm intersect at two points and the distance between their center is 4cm. Find the length of the common chord.

   Solution: See Fig. 3.1. and

Parameter	**Value**	**Description**
$\mathbf{c}_1$	$\mathbf{0}$	Center of Circle 1
$\mathbf{c}_2$	$4\mathbf{e}_1$	Center of Circle 2
r_1	5	Radius of Circle 1
r_2	3	Radius of Circle 2

Table 3.2:

   From Table 3.2, (D.38) and (D.39), the equations of the two circles are

$$\begin{aligned} \|\mathbf{x}\|^2 - 25 = 0 \\ \|\mathbf{x}\|^2 - 8\mathbf{e}_1^\top \mathbf{x} + 7 = 0 \end{aligned} \tag{3.1}$$

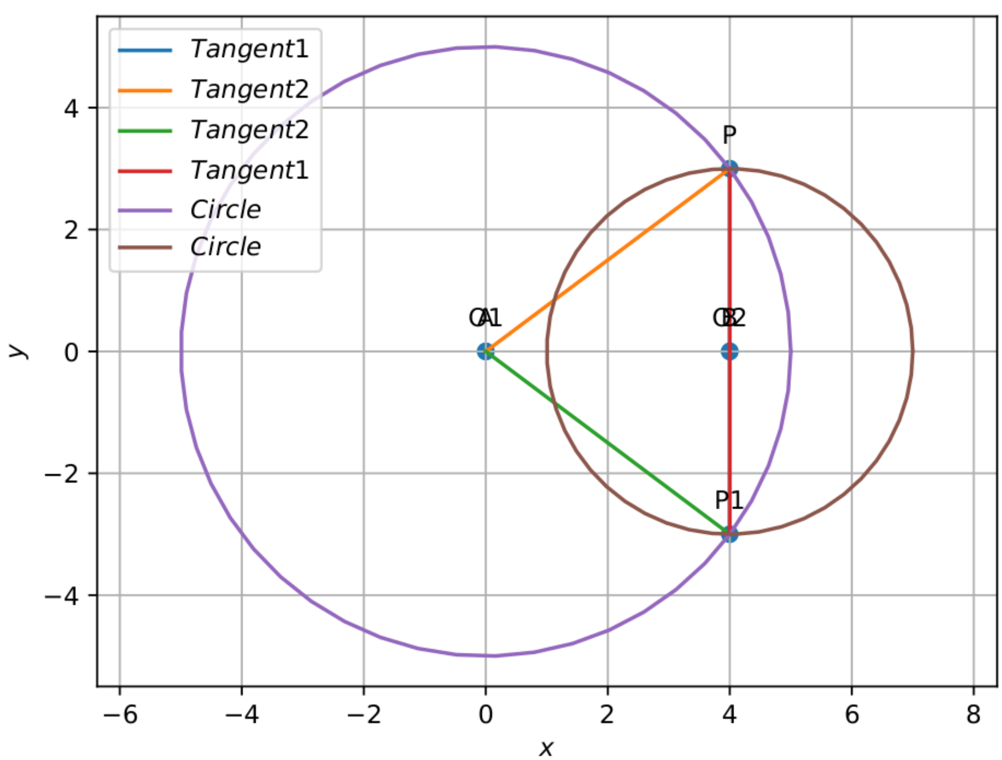

Figure 3.1:

From (3.1) and (D.41) the equation of the common chord is

$$\mathbf{e}_1^\top \mathbf{x} = 4 \tag{3.2}$$

It is easy to verify that

$$\mathbf{q} = 4\mathbf{e}_1 \tag{3.3}$$

is a point on (3.2). Substituting

$$\mathbf{m} = \mathbf{e}_2, \mathbf{q} = 4\mathbf{e}_1, \mathbf{V} = \mathbf{I}, \mathbf{u} = \mathbf{0}, f = -25 \tag{3.4}$$

in (F.32), the length of the chord in (F.24) is given by

$$\frac{2\sqrt{\left[\mathbf{e}_2^\top\left(4\mathbf{e}_1\right)\right]^2-\left(16\mathbf{e}_1^\top\mathbf{e}_1-25\right)\left(\mathbf{e}_2^\top\mathbf{e}_2\right)}}{\mathbf{e}_2^\top\mathbf{e}_2}\left\|\mathbf{e}_2\right\|=6 \tag{3.5}$$

2.

3.

4.

5.

6.

3.2. Inscribed Polygons

1. In Fig. 3.2, $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ are three points with centre $\mathbf{O}$ such that $\angle BOC = 30°$ and $\angle AOB = 60°$. If $\mathbf{D}$ is a point on the circle other than the arc ABC, find $\angle ADC$.

   Solution: See Fig. (3.2).

$$\mathbf{A}=\mathbf{e}_2, \mathbf{B}=\begin{pmatrix}\cos 30\\ \sin 30\end{pmatrix}, \mathbf{C}=\mathbf{e}_1 \text{ and } \mathbf{D}=\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}. \tag{3.6}$$

2.

3. Let $\angle PQR = 100°$ where $\mathbf{PQ}, \mathbf{R}$ are points on a circle with centre $\mathbf{O}$. Find $\angle OPR$.

   Solution: In Fig. 3.3,

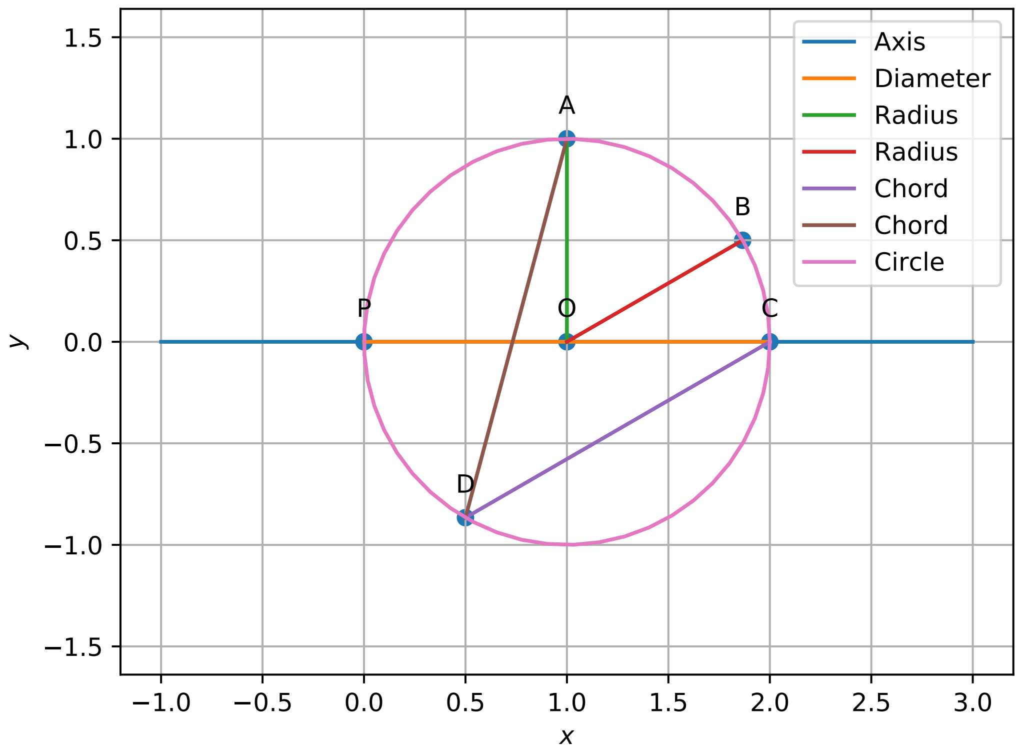

Figure 3.2:

$$\mathbf{P} = \begin{pmatrix} \cos(\theta + 160) \\ \sin(\theta + 160) \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}, \mathbf{R} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}. \tag{3.7}$$

3.3. Tangent to a Circle

1.

2. Draw a circle and two lines parallel to a given line such that one is a tangent and the other is a secant to the circle

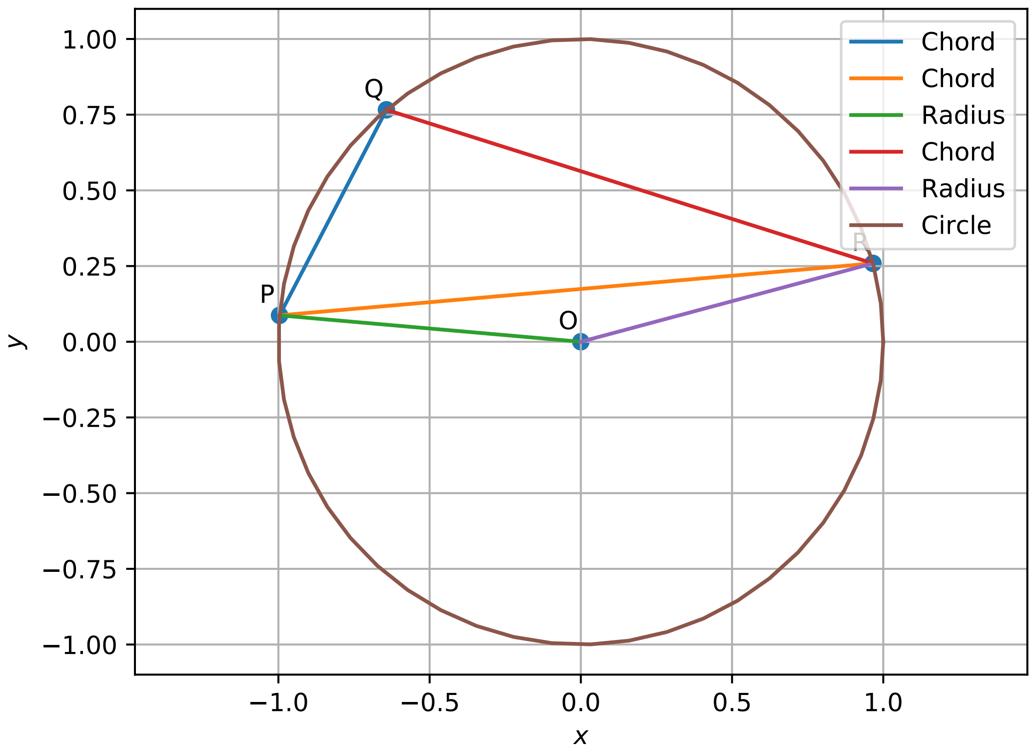

Figure 3.3:

Solution: The parameters of the circle in Fig. 3.4 are

$$\mathbf{u} = \mathbf{0}, f = -16 \tag{3.8}$$

Considering the given line to be

$$\mathbf{e}_1^\top \mathbf{x} = 5 \tag{3.9}$$

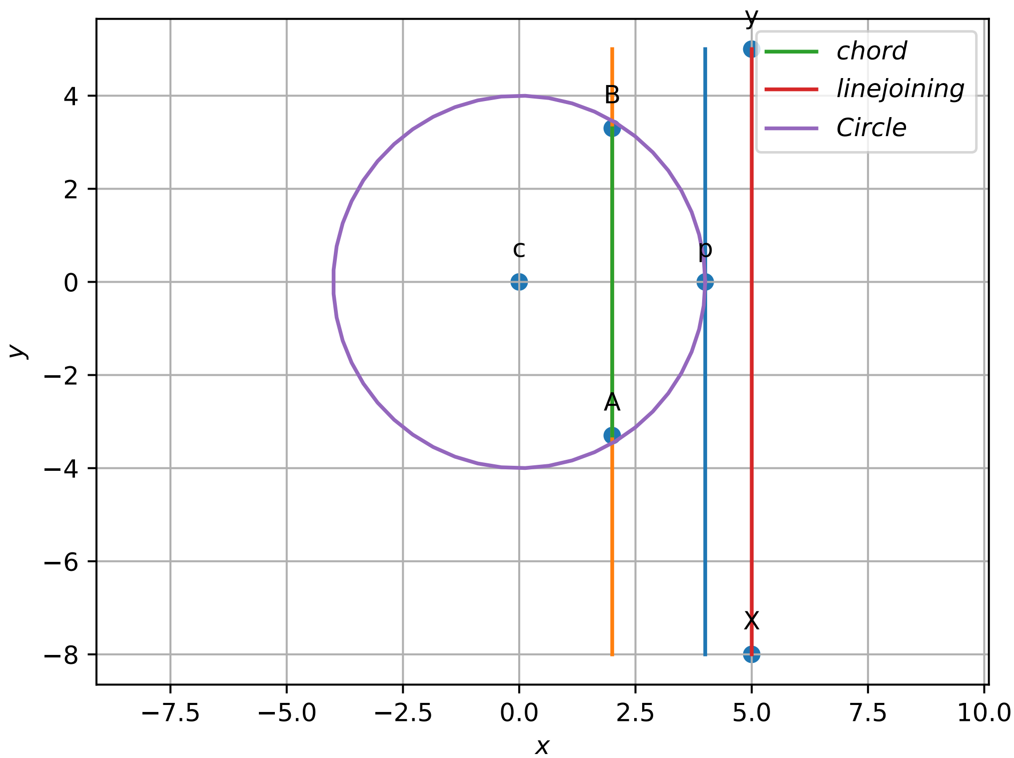

Figure 3.4:

the tangent to the circle will be

$$\mathbf{e}_1^\top \mathbf{x} = 4 \tag{3.10}$$

and the secant will be

$$\mathbf{e}_1^\top \mathbf{x} = c \tag{3.11}$$

where

$$|c| < 4 \tag{3.12}$$

3.4. Tangents from a Point

1.

2.

3.

4. Show that the tangents of circle drawn at the ends of diameter are parallel.

   Solution: See Fig. 3.5. Let $\mathbf{A}, \mathbf{B}$ be the end points of the diameter of the circle through which the tangents are drawn. From (D.39),

$$\frac{\mathbf{A}+\mathbf{B}}{2} = -\mathbf{u} \tag{3.13}$$

$$\implies \mathbf{A}+\mathbf{B} = -2\mathbf{u} \tag{3.14}$$

   From (F.29),

$$\mathbf{m}_1^\top (\mathbf{A}+\mathbf{u}) = 0 \tag{3.15}$$

$$\mathbf{m}_2^\top (\mathbf{B}+\mathbf{u}) = 0 \tag{3.16}$$

   where $\mathbf{m_1}, \mathbf{m_2}$ are the direction vectors of the tangents at $\mathbf{A}, \mathbf{B}$ respectively. Then,

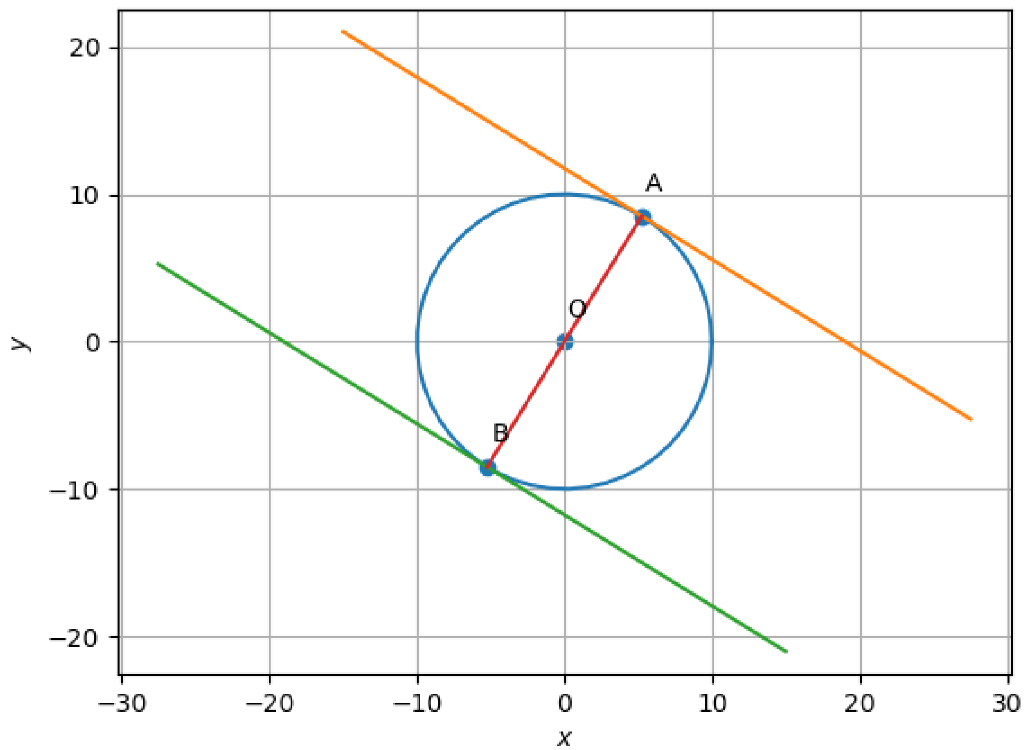

Figure 3.5:

the normal vectors at the point of contact of tangets are

$$\mathbf{A} + \mathbf{u} = k_1\mathbf{n_1} \tag{3.17}$$

$$\mathbf{B} + \mathbf{u} = k_2\mathbf{n_2} \tag{3.18}$$

Adding (3.17) and (3.18),

$$k_1\mathbf{n_1} + k_2\mathbf{n_2} = \mathbf{A} + \mathbf{B} + 2\mathbf{u} \tag{3.19}$$

$$= \mathbf{0} \tag{3.20}$$

from (3.14), (3.20) can be expressed as

$$k_1\mathbf{n_1} + k_2\mathbf{n_2} = 0 \tag{3.21}$$

$$k_1\mathbf{n_1} = -k_2\mathbf{n_2} \tag{3.22}$$

Since

$$\mathbf{n_1} \times \mathbf{n_2} = \mathbf{0}, \tag{3.23}$$

$$\implies n_1 \parallel n_2 \tag{3.24}$$

$$\text{or, } m_1 \parallel m_2 \tag{3.25}$$

5.

6. The length of a tangent from a point **A** at distance 5 cm from the centre of the circle is 4 cm. Find the radius of the circle.

   Solution:

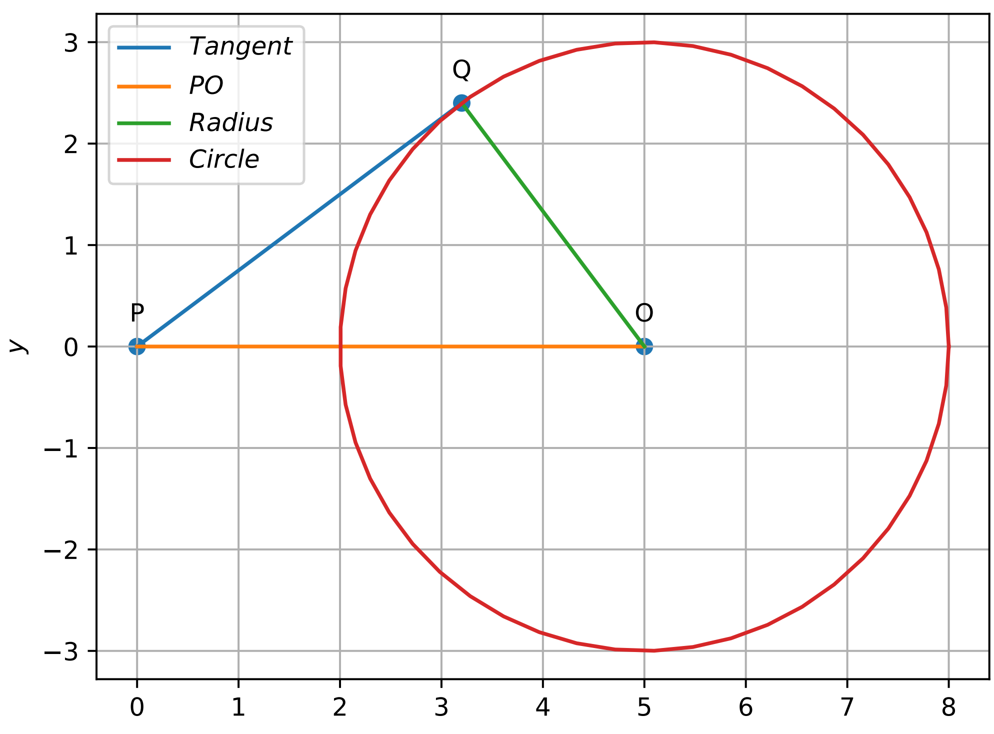

Figure 3.6:

Chapter 4

Triangle

4.1. Triangle Constructions

1. Construct a triangle ABC in which $BC = 7cm, \angle B = 75^\circ$ and $AB + AC = 13cm$.

   Solution: See Fig. 4.1.

   Using the cosine formula in $\triangle ABC$,

$$b^2 = a^2 + c^2 - 2ac\cos B \tag{4.1}$$

$$\implies (b+c)(b-c) = a^2 - 2ac\cos B \tag{4.2}$$

$$\text{or, } K(b-c) = a^2 - 2ac\cos B \tag{4.3}$$

   where

$$K = b + c \tag{4.4}$$

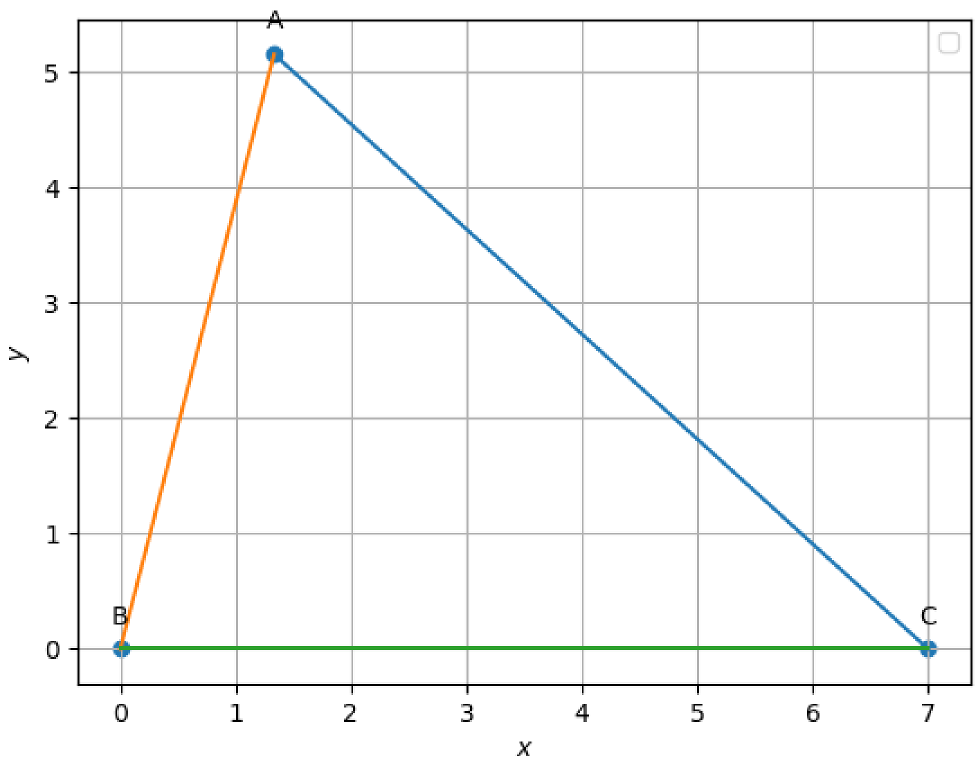

Figure 4.1:

From (4.3) and (4.4),

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} b \\ c \end{pmatrix} = \begin{pmatrix} \frac{a^2 - 2ac\cos B}{K} \\ K \end{pmatrix} \tag{4.5}$$

$$\implies \begin{pmatrix} b \\ c \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \frac{a^2 - 2ac\cos B}{K} \\ K \end{pmatrix} \tag{4.6}$$

$$\because \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = 2\mathbf{I} \tag{4.7}$$

From (4.6)

$$c = \frac{1}{2}\mathbf{e}_2^\top \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \frac{a^2}{K} \\ K \end{pmatrix} - \frac{2ac\cos B}{K} \tag{4.8}$$

$$\implies c = \frac{1}{2\left(1 + \frac{2a\cos B}{K}\right)}\mathbf{e}_2^\top \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \frac{a^2}{K} \\ K \end{pmatrix} \tag{4.9}$$

The coordinates of $\triangle ABC$ can then be expressed as

$$\mathbf{A} = c \begin{pmatrix} \cos B \\ \sin B \end{pmatrix}, \mathbf{B} = \mathbf{0}, \mathbf{C} = \begin{pmatrix} a \\ 0 \end{pmatrix}. \tag{4.10}$$

2. Construct a triangle ABC in which $BC = 8cm, \angle B = 45°$ and $AB - AC = 3.5cm$.

   Solution: See Fig. 4.2. Using the cosine formula in $\triangle ABC$,

$$b^2 = a^2 + c^2 - 2ac\cos B \tag{4.11}$$

$$\implies (b+c)(b-c) = a^2 - 2ac\cos B \tag{4.12}$$

$$\text{or, } K(b+c) = a^2 - 2ac\cos B \tag{4.13}$$

where

$$-K = b - c \tag{4.14}$$

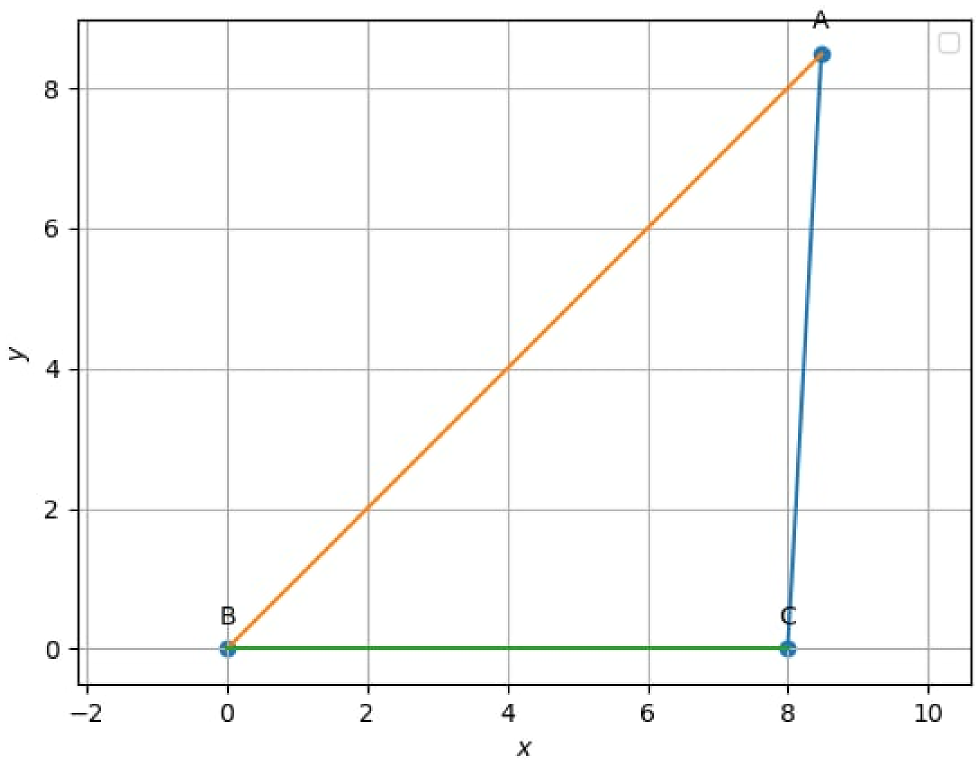

Figure 4.2:

From (4.13) and (4.14),

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} b \\ c \end{pmatrix} = \begin{pmatrix} \frac{a^2 - 2ac\cos B}{K} \\ -K \end{pmatrix} \tag{4.15}$$

$$\implies \begin{pmatrix} b \\ c \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \frac{a^2 - 2ac\cos B}{K} \\ -K \end{pmatrix} \tag{4.16}$$

$$\because \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = 2\mathbf{I} \tag{4.17}$$

From (4.16)

$$c = \frac{1}{2}\mathbf{e}_2^\top \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \frac{a^2}{K} \\ -K \end{pmatrix} - \frac{2ac\cos B}{K} \tag{4.18}$$

$$\implies c = \frac{1}{2\left(1 + \frac{2a\cos B}{K}\right)}\mathbf{e}_2^\top \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \frac{a^2}{K} \\ -K \end{pmatrix} \tag{4.19}$$

The coordinates of $\triangle ABC$ can then be expressed as

$$\mathbf{A} = c \begin{pmatrix} \cos B \\ \sin B \end{pmatrix}, \mathbf{B} = \mathbf{0}, \mathbf{C} = \begin{pmatrix} a \\ 0 \end{pmatrix}. \tag{4.20}$$

3. Construct a triangle PQR in which $QR = 6cm, \angle Q = 60^\circ$ and $PR - PQ = 2cm$.

   Solution: Same as Problem 1 with

$$\angle Q = \angle B, QR = a, PR = b, PQ = c \tag{4.21}$$

4. Construct a triangle XYZ in which $\angle Y = 30^\circ, \angle Z = 90^\circ$ and $XY+YZ+ZX = 11cm$.

   Solution: From the given information,

$$x + y + z = K \tag{4.22}$$

$$y\cos Z + z\cos Y - x = 0 \tag{4.23}$$

$$y\sin Z - z\sin Y = 0 \tag{4.24}$$

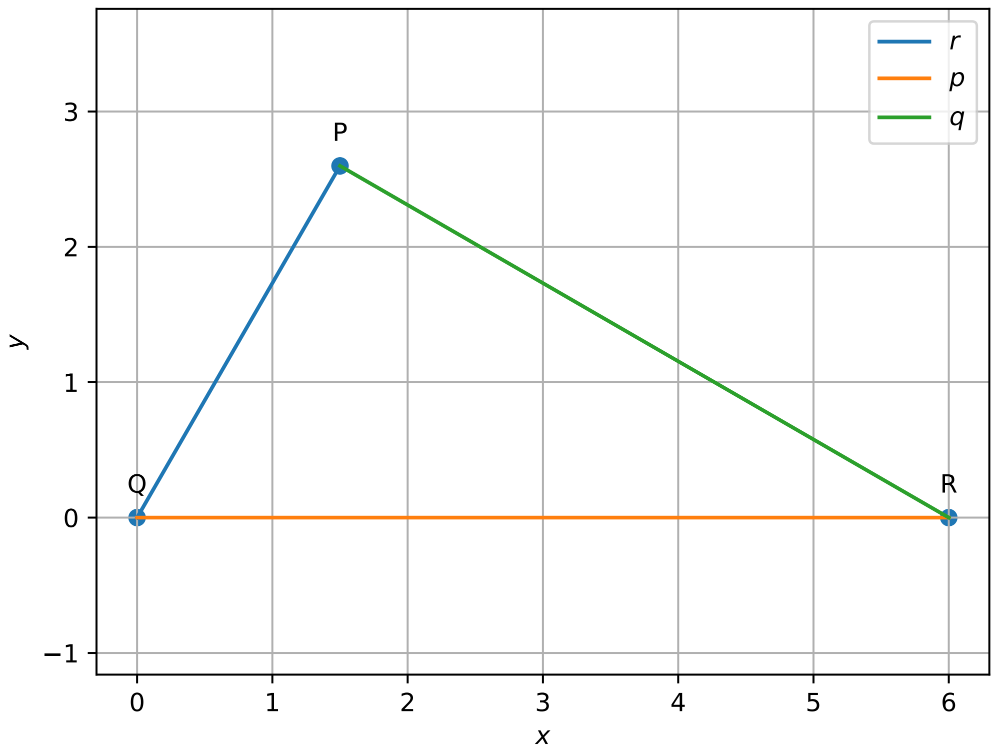

Figure 4.3:

resulting in the matrix equation

$$\begin{pmatrix} 1 & 1 & 1 \\ \cos Z & \cos Y & -1 \\ \sin Z & -\sin Y & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = K\mathbf{e}_1 \tag{4.25}$$

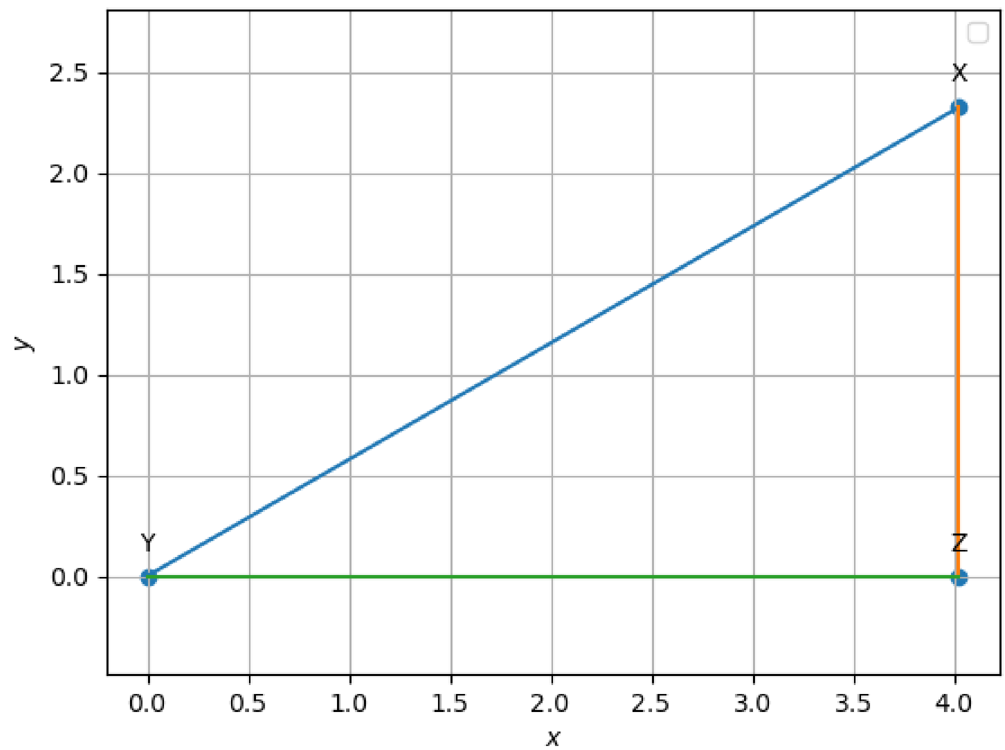

Figure 4.4:

which can be solved to obtain all the sides. $\triangle XYZ$ can then be plotted using

$$\mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix}, \mathbf{Y} = \mathbf{0}, \mathbf{Z} = \begin{pmatrix} x \\ 0 \end{pmatrix} \tag{4.26}$$

5. Construct a right triangle whose base is 12cm and sum of its hypotenuse and other side is 18cm.

Solution: From the given information, let

$$a = 12, \angle B = 90^\circ, b + c = 18 \tag{4.27}$$

We need to find b. This is similar to Problem 1.

Appendix A

Vectors

A.1. 2×1 vectors

A.1.1. Let

$$\mathbf{A} \equiv \overrightarrow{A} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \tag{A.1}$$

$$\equiv a_1 \overrightarrow{i} + a_2 \overrightarrow{j}, \tag{A.2}$$

$$\mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \tag{A.3}$$

be 2×1 vectors. Then, the determinant of the 2×2 matrix

$$\mathbf{M} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \end{pmatrix} \tag{A.4}$$

is defined as

$$\begin{vmatrix}\mathbf{M}\end{vmatrix} = \begin{vmatrix}\mathbf{A} & \mathbf{B}\end{vmatrix} \tag{A.5}$$

$$= \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1 \tag{A.6}$$

A.1.2. The value of the cross product of two vectors is given by (A.5).

A.1.3. The area of the triangle with vertices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ is given by

$$\frac{1}{2}\left\|(\mathbf{A}-\mathbf{B}) \times (\mathbf{A}-\mathbf{C})\right\| = \frac{1}{2}\left\|\mathbf{A}\times\mathbf{B} + \mathbf{B}\times\mathbf{C} + \mathbf{C}\times\mathbf{A}\right\| \tag{A.7}$$

A.1.4. If

$$\left\|\mathbf{A}\times\mathbf{B}\right\| = \left\|\mathbf{C}\times\mathbf{D}\right\|, \quad \text{then} \tag{A.8}$$

$$\mathbf{A}\times\mathbf{B} = \pm\left(\mathbf{C}\times\mathbf{D}\right) \tag{A.9}$$

where the sign depends on the orientation of the vectors.

A.1.5. The median divides the triangle into two triangles of equal area.

A.1.6. The transpose of $\mathbf{A}$ is defined as

$$\mathbf{A}^{\top} = \begin{pmatrix} a_1 & a_2 \end{pmatrix} \tag{A.10}$$

A.1.7. The inner product or dot product is defined as

$$\mathbf{A}^\top \mathbf{B} \equiv \mathbf{A} \cdot \mathbf{B} \tag{A.11}$$

$$= \begin{pmatrix} a_1 & a_2 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = a_1 b_1 + a_2 b_2 \tag{A.12}$$

A.1.8. norm of **A** is defined as

$$\|A\| \equiv \left| \overrightarrow{A} \right| \tag{A.13}$$

$$= \sqrt{\mathbf{A}^\top \mathbf{A}} = \sqrt{a_1^2 + a_2^2} \tag{A.14}$$

Thus,

$$\|\lambda \mathbf{A}\| \equiv \left| \lambda \overrightarrow{A} \right| \tag{A.15}$$

$$= |\lambda| \, \|\mathbf{A}\| \tag{A.16}$$

A.1.9. The distance betwen the points **A** and **B** is given by

$$\|\mathbf{A} - \mathbf{B}\| \tag{A.17}$$

A.1.10. Let **x** be equidistant from the points **A** and **B**. Then

$$(\mathbf{A} - \mathbf{B})^\top \mathbf{x} = \frac{\|\mathbf{A}\|^2 - \|\mathbf{B}\|^2}{2} \tag{A.18}$$

Solution:

$$\|\mathbf{x}-\mathbf{A}\| = \|\mathbf{A}-\mathbf{B}\| \tag{A.19}$$

$$\implies \|\mathbf{x}-\mathbf{A}\|^2 = \|\mathbf{x}-\mathbf{B}\|^2 \tag{A.20}$$

which can be expressed as

$$\begin{aligned}(\mathbf{x}-\mathbf{A})^\top(\mathbf{x}-\mathbf{A}) = (\mathbf{x}-\mathbf{B})^\top(\mathbf{x}-\mathbf{B})\\ \implies \|\mathbf{x}\|^2 - 2\mathbf{x}^\top\mathbf{A} + \|\mathbf{A}\|^2\\ = \|\mathbf{x}\|^2 - 2\mathbf{x}^\top\mathbf{B} + \|\mathbf{B}\|^2\end{aligned} \tag{A.21}$$

which can be simplified to obtain (A.18).

A.1.11. If $\mathbf{x}$ lies on the x-axis and is equidistant from the points $\mathbf{A}$ and $\mathbf{B}$,

$$\mathbf{x} = x\mathbf{e}_1 \tag{A.22}$$

where

$$x = \frac{\|\mathbf{A}\|^2 - \|\mathbf{B}\|^2}{2(\mathbf{A}-\mathbf{B})^\top\mathbf{e}_1} \tag{A.23}$$

Solution: From (A.18).

$$x(\mathbf{A}-\mathbf{B})^\top\mathbf{e}_1 = \frac{\|\mathbf{A}\|^2 - \|\mathbf{B}\|^2}{2} \tag{A.24}$$

yielding (A.23).

A.1.12. The angle between two vectors is given by

$$\theta = \cos^{-1} \frac{\mathbf{A}^\top \mathbf{B}}{\|A\| \|B\|} \tag{A.25}$$

A.1.13. If two vectors are orthogonal (perpendicular),

$$\mathbf{A}^\top \mathbf{B} = 0 \tag{A.26}$$

A.1.14. For an isoceles triangle ABC ith $AB = AC$, the median $AD \perp BC$.

A.1.15. The direction vector of the line joining two points $\mathbf{A}, \mathbf{B}$ is given by

$$\mathbf{m} = \mathbf{A} - \mathbf{B} \tag{A.27}$$

A.1.16. The unit vector in the direction of $\mathbf{m}$ is defined as

$$\frac{\mathbf{m}}{\|\mathbf{m}\|} \tag{A.28}$$

A.1.17. If the direction vector of a line is expressed as

$$\mathbf{m} = \begin{pmatrix} 1 \\ m \end{pmatrix}, \tag{A.29}$$

the m is defined to be the slope of the line.

A.1.18. $AB \parallel CD$ if

$$\mathbf{A} - \mathbf{B} = k \left(\mathbf{C} - \mathbf{D} \right) \tag{A.30}$$

A.1.19. The normal vector to $\mathbf{m}$ is defined by

$$\mathbf{m}^{\top}\mathbf{n} = 0 \tag{A.31}$$

A.1.20. If

$$\mathbf{m}^{\top}\mathbf{n}_1 = 0 \tag{A.32}$$

$$\mathbf{m}^{\top}\mathbf{n}_2 = 0, \tag{A.33}$$

$$\mathbf{n}_1 \parallel \mathbf{n}_2 \tag{A.34}$$

A.1.21. The point $\mathbf{P}$ that divides the line segment AB in the ratio $k : 1$ is given by

$$\mathbf{P} = \frac{k\mathbf{B} + \mathbf{A}}{k+1} \tag{A.35}$$

A.1.22. The standard basis vectors are defined as

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{A.36}$$

$$\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{A.37}$$

A.1.23. If $ABCD$ be a parallelogram,

$$\mathbf{B} - \mathbf{A} = \mathbf{C} - \mathbf{D} \tag{A.38}$$

A.1.24. Diagonals of a parallelogram bisect each other.

A.1.25. The area of the parallelogram with vertices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and $\mathbf{D}$ is given by

$$\left\|(\mathbf{A}-\mathbf{B}) \times (\mathbf{A}-\mathbf{D})\right\| = \left\|\mathbf{A} \times \mathbf{B} + \mathbf{B} \times \mathbf{C} + \mathbf{C} \times \mathbf{A}\right\| \tag{A.39}$$

A.1.26. Points $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ form a triangle if

$$p\left(\mathbf{A}-\mathbf{B}\right) + q\left(\mathbf{A}-\mathbf{C}\right) = 0 \tag{A.40}$$

$$\text{or, } \left(p+q\right)\mathbf{A} - p\mathbf{B} - q\mathbf{C} = 0 \tag{A.41}$$

$$\implies p = 0, q = 0 \tag{A.42}$$

are linearly independent.

A.1.27. In $\triangle ABC$, if $\mathbf{D}, \mathbf{E}$ divide the lines AB, AC in the ratio $k:1$ respectively, then $DE \parallel BC$.

Proof. From (A.35),

$$\mathbf{D} = \frac{k\mathbf{B}+\mathbf{A}}{k+1} \tag{A.43}$$

$$\mathbf{E} = \frac{k\mathbf{C}+\mathbf{A}}{k+1} \tag{A.44}$$

$$\implies \mathbf{D}-\mathbf{E} = \frac{k}{k+1}\left(\mathbf{B}-\mathbf{C}\right) \tag{A.45}$$

Thus, from Appendix A.1.17, $DE \parallel BC$.

□

A.1.28. In $\triangle ABC$, if $DE \parallel BC$, $\mathbf{D}$ and $\mathbf{E}$ divide the lines AB, AC in the same ratio.

Proof. If $DE \parallel BC$, from (A.30)

$$(\mathbf{B}-\mathbf{C}) = k\,(\mathbf{D}-\mathbf{E}) \tag{A.46}$$

Using (A.35), let

$$\mathbf{D} = \frac{k_1\mathbf{B}+\mathbf{A}}{k_1+1} \tag{A.47}$$

$$\mathbf{E} = \frac{k_2\mathbf{C}+\mathbf{A}}{k_2+1} \tag{A.48}$$

Subtituting the above in (A.46), after some algebra, we obtain

$$(p+q)\,\mathbf{A} - p\mathbf{B} - q\mathbf{C} = 0 \tag{A.49}$$

where

$$p = \frac{1}{k} - \frac{k_1}{k_1+1}, q = \frac{1}{k} - \frac{k_1}{k_1+1} \tag{A.50}$$

From (A.41),

$$p = q = 0 \tag{A.51}$$

$$\implies k_1 = k_2 = \frac{1}{k-1} \tag{A.52}$$

□

A.2. 3×1 vectors

A.2.1. Let

$$\mathbf{A} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \equiv a_1 \overrightarrow{i} + a_2 \overrightarrow{j} + a_3 \overrightarrow{j}, \tag{A.53}$$

$$\mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \tag{A.54}$$

and

$$\mathbf{A}_{ij} = \begin{pmatrix} a_i \\ a_j \end{pmatrix}, \tag{A.55}$$

$$\mathbf{B}_{ij} = \begin{pmatrix} b_i \\ b_j \end{pmatrix}. \tag{A.56}$$

A.2.2. The <u>cross product</u> or <u>vector product</u> of $\mathbf{A}, \mathbf{B}$ is defined as

$$\mathbf{A} \times \mathbf{B} = \left(\begin{vmatrix} \mathbf{A}_{23} & \mathbf{B}_{23} \\ \mathbf{A}_{31} & \mathbf{B}_{31} \\ \mathbf{A}_{12} & \mathbf{B}_{12} \end{vmatrix} \right) \tag{A.57}$$

A.2.3. Verify that

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} \tag{A.58}$$

A.2.4. The area of a triangle is given by

$$\frac{1}{2}\left\|\mathbf{A} \times \mathbf{B}\right\| \tag{A.59}$$

Appendix B

Matrices

B.1. Eigenvalues and Eigenvectors

B.1.1. The eigenvalue λ and the eigenvector $\mathbf{x}$ for a matrix $\mathbf{A}$ are defined as,

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x} \tag{B.1}$$

B.1.2. The eigenvalues are calculated by solving the equation

$$f(\lambda) = \left|\lambda\mathbf{I} - \mathbf{A}\right| = 0 \tag{B.2}$$

The above equation is known as the characteristic equation.

B.1.3. According to the Cayley-Hamilton theorem,

$$f(\lambda) = 0 \implies f(\mathbf{A}) = 0 \tag{B.3}$$

B.1.4. The trace of a square matrix is defined to be the sum of the diagonal elements.

$$\mathrm{tr}\left(\mathbf{A}\right) = \sum_{i=1}^{N} a_{ii}. \tag{B.4}$$

where a_{ii} is the ith diagonal element of the matrix $\mathbf{A}$.

B.1.5. The trace of a matrix is equal to the sum of the eigenvalues

$$\mathrm{tr}\,(\mathbf{A}) = \sum_{i=1}^{N} \lambda_i \tag{B.5}$$

B.2. Determinants

B.2.1. Let

$$\mathbf{A} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}. \tag{B.6}$$

be a 3×3 matrix. Then,

$$\left|\mathbf{A}\right| = a_1 \begin{pmatrix} b_2 & c_2 \\ b_3 & c_3 \end{pmatrix} - a_2 \begin{pmatrix} b_1 & c_1 \\ b_3 & c_3 \end{pmatrix} + a_3 \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}. \tag{B.7}$$

B.2.2. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of a matrix $\mathbf{A}$. Then, the product of the eigenvalues is equal to the determinant of $\mathbf{A}$.

$$\left|\mathbf{A}\right| = \prod_{i=1}^{n} \lambda_i \tag{B.8}$$

B.2.3.

$$\left|\mathbf{AB}\right| = \left|\mathbf{A}\right| \left|\mathbf{B}\right| \tag{B.9}$$

B.2.4. If $\mathbf{A}$ be an $n \times n$ matrix,

$$\left|k\mathbf{A}\right| = k^n \left|\mathbf{A}\right| \tag{B.10}$$

B.3. Rank of a Matrix

B.3.1. The rank of a matrix is defined as the number of linearly independent rows. This is also known as the row rank.

B.3.2. Row rank = Column rank.

B.3.3. The rank of a matrix is obtained as the number of nonzero rows obtained after row reduction.

B.3.4. An $n \times n$ matrix is invertible if and only if its rank is n.

B.3.5. Points $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are on a line if

$$\operatorname{rank} \begin{pmatrix} \mathbf{A} \\ \mathbf{B} \\ \mathbf{C} \end{pmatrix} = 1 \tag{B.11}$$

B.3.6. Points $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ form a paralelogram if

$$\operatorname{rank}\begin{pmatrix}\mathbf{A}\\ \mathbf{B}\\ \mathbf{C}\\ \mathbf{D}\end{pmatrix} = 1, \operatorname{rank}\begin{pmatrix}\mathbf{A}\\ \mathbf{B}\\ \mathbf{C}\end{pmatrix} = 2 \tag{B.12}$$

B.4. Inverse of a Matrix

B.4.1. For a 2×2 matrix

$$\mathbf{A} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}, \tag{B.13}$$

the inverse is given by

$$\mathbf{A}^{-1} = \frac{1}{\left|\mathbf{A}\right|}\begin{pmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{pmatrix}, \tag{B.14}$$

B.4.2. For higher order matrices, the inverse should be calculated using row operations.

B.5. Orthogonality

B.5.1. The rotation matrix is defined as

$$\mathbf{R}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad \theta \in [0, 2\pi] \tag{B.15}$$

B.5.2. The rotation matrix is orthogonal

$$\mathbf{R}_\theta^\top \mathbf{R}_\theta = \mathbf{R}_\theta \mathbf{R}_\theta^\top = \mathbf{I} \tag{B.16}$$

B.5.3. If the angle of rotation is $\frac{\pi}{2}$,

$$\mathbf{m}^\top \mathbf{n} = 0 \implies \mathbf{n} = \mathbf{R}_{\frac{\pi}{2}} \mathbf{m} \tag{B.17}$$

B.5.4.

$$\mathbf{n}^\top \mathbf{h} = 1 \implies \mathbf{n} = \frac{\mathbf{e}_1}{\mathbf{e}_1^\top \mathbf{h}} + \mu \mathbf{R}_{\frac{\pi}{2}} \mathbf{h}, \quad \mu \in \mathbb{R}. \tag{B.18}$$

B.5.5. The affine transformation is given by

$$\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c} \quad \text{(Affine Transformation)} \tag{B.19}$$

where $\mathbf{P}$ is invertible.

B.5.6. The eigenvalue decomposition of a symmetric matrix $\mathbf{V}$ is given by

$$\mathbf{P}^\top \mathbf{V} \mathbf{P} = \mathbf{D}. \quad \text{(Eigenvalue Decomposition)} \tag{B.20}$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \tag{B.21}$$

$$\mathbf{P} = \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix}, \quad \mathbf{P}^\top = \mathbf{P}^{-1}, \tag{B.22}$$

Appendix C

Linear Forms

C.1. Two Dimensions

C.1.1. The equation of a line is given by

$$\mathbf{n}^\top \mathbf{x} = c \tag{C.1}$$

where $\mathbf{n}$ is the normal vector of the line.

C.1.2. The equation of a line with normal vector $\mathbf{n}$ and passing through a point $\mathbf{A}$ is given by

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{A}) = 0 \tag{C.2}$$

C.1.3. The equation of a line L is also given by

$$\mathbf{n}^\top \mathbf{x} = \begin{cases} 0 & \mathbf{0} \in L \\ 1 & \text{otherwise} \end{cases} \tag{C.3}$$

C.1.4. The parametric equation of a line is given by

$$\mathbf{x} = \mathbf{A} + \lambda\mathbf{m} \tag{C.4}$$

where $\mathbf{m}$ is the direction vector of the line and $\mathbf{A}$ is any point on the line.

C.1.5. Let $\mathbf{A}$ and $\mathbf{B}$ be two points on a straight line and let $\mathbf{P} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$ be any point on it. If p_2 is known, then

$$\mathbf{P} = \mathbf{A} + \frac{p_2 - \mathbf{e}_2^\top \mathbf{A}}{\mathbf{e}_2^\top (\mathbf{B} - \mathbf{A})} (\mathbf{B} - \mathbf{A}) \tag{C.5}$$

Solution: The equation of the line can be expressed in parametric from as

$$\mathbf{x} = \mathbf{A} + \lambda (\mathbf{B} - \mathbf{A}) \tag{C.6}$$

$$\implies \mathbf{P} = \mathbf{A} + \lambda (\mathbf{B} - \mathbf{A}) \tag{C.7}$$

$$\implies \mathbf{e}_2^\top \mathbf{P} = \mathbf{e}_2^\top \mathbf{A} + \lambda \mathbf{e}_2^\top (\mathbf{B} - \mathbf{A}) \tag{C.8}$$

$$\implies p_2 = \mathbf{e}_2^\top \mathbf{A} + \lambda \mathbf{e}_2^\top (\mathbf{B} - \mathbf{A}) \tag{C.9}$$

$$\text{or, } \lambda = \frac{p_2 - \mathbf{e}_2^\top \mathbf{A}}{\mathbf{e}_2^\top (\mathbf{B} - \mathbf{A})} \tag{C.10}$$

yielding (C.5).

C.1.6. The distance from a point $\mathbf{P}$ to the line in (C.1) is given by

$$d = \frac{\left|\mathbf{n}^\top \mathbf{P} - c\right|}{\left\|\mathbf{n}\right\|} \tag{C.11}$$

Solution: Without loss of generality, let $\mathbf{A}$ be the foot of the perpendicular from $\mathbf{P}$

to the line in (C.4). The equation of the normal to (C.1) can then be expressed as

$$\mathbf{x} = \mathbf{A} + \lambda\mathbf{n} \tag{C.12}$$

$$\implies \mathbf{P} - \mathbf{A} = \lambda\mathbf{n} \tag{C.13}$$

$\because$ **P** lies on (C.12). From the above, the desired distance can be expressed as

$$d = \left\|\mathbf{P} - \mathbf{A}\right\| = \left|\lambda\right| \left\|\mathbf{n}\right\| \tag{C.14}$$

From (C.13),

$$\mathbf{n}^\top \left(\mathbf{P} - \mathbf{A}\right) = \lambda\mathbf{n}^\top\mathbf{n} = \lambda \left\|\mathbf{n}\right\|^2 \tag{C.15}$$

$$\implies \left|\lambda\right| = \frac{\left|\mathbf{n}^\top \left(\mathbf{P} - \mathbf{A}\right)\right|}{\left\|\mathbf{n}\right\|^2} \tag{C.16}$$

Substituting the above in (C.14) and using the fact that

$$\mathbf{n}^\top\mathbf{A} = c \tag{C.17}$$

from (C.1), yields (C.11)

C.1.7. The distance from the origin to the line in (C.1) is given by

$$d = \frac{\left|c\right|}{\left\|\mathbf{n}\right\|} \tag{C.18}$$

C.1.8. The distance between the parallel lines

$$\begin{aligned}\mathbf{n}^\top \mathbf{x} &= c_1 \\ \mathbf{n}^\top \mathbf{x} &= c_2\end{aligned} \tag{C.19}$$

is given by

$$d = \frac{|c_1 - c_2|}{\|\mathbf{n}\|} \tag{C.20}$$

C.1.9. The equation of the line perpendicular to (C.1) and passing through the point $\mathbf{P}$ is given by

$$\mathbf{m}^\top (\mathbf{x} - \mathbf{P}) = 0 \tag{C.21}$$

C.1.10. The foot of the perpendicular from $\mathbf{P}$ to the line in (C.1) is given by

$$\begin{pmatrix} \mathbf{m} & \mathbf{n} \end{pmatrix}^\top \mathbf{x} = \begin{pmatrix} \mathbf{m}^\top \mathbf{P} \\ c \end{pmatrix} \tag{C.22}$$

Solution: From (C.1) and (C.2) the foot of the perpendicular satisfies the equations

$$\mathbf{n}^\top \mathbf{x} = c \tag{C.23}$$

$$\mathbf{m}^\top (\mathbf{x} - \mathbf{P}) = 0 \tag{C.24}$$

where $\mathbf{m}$ is the direction vector of the given line. Combining the above into a matrix equation results in (C.22).

C.1.11. The equations of the angle bisectors of the lines

$$\mathbf{n}_1^\top \mathbf{x} = c_1 \tag{C.25}$$

$$\mathbf{n}_2^\top \mathbf{x} = c_2 \tag{C.26}$$

are given by

$$\frac{\mathbf{n}_1^\top \mathbf{x} - c_1}{\|\mathbf{n}_1\|} = \pm \frac{\mathbf{n}_2^\top \mathbf{x} - c_2}{\|\mathbf{n}_2\|} \tag{C.27}$$

Proof. Any point on the angle bisector is equidistant from the lines. □

C.2. Three Dimensions

C.2.1. Points $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are on a line if

$$\operatorname{rank} \begin{pmatrix} \mathbf{A} \\ \mathbf{B} \\ \mathbf{C} \end{pmatrix} = 1 \tag{C.28}$$

C.2.2. Points $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ form a paralelogram if

$$\operatorname{rank} \begin{pmatrix} \mathbf{A} \\ \mathbf{B} \\ \mathbf{C} \\ \mathbf{D} \end{pmatrix} = 1, \operatorname{rank} \begin{pmatrix} \mathbf{A} \\ \mathbf{B} \\ \mathbf{C} \end{pmatrix} = 2 \tag{C.29}$$

C.2.3. The equation of a line is given by (C.4)

C.2.4. The equation of a plane is given by (C.1)

C.2.5. The distance from the origin to the line in (C.1) is given by (C.18)

C.2.6. The distance from a point $\mathbf{P}$ to the line in (C.4) is given by

$$d = \left\|\mathbf{A} - \mathbf{P}\right\|^2 - \frac{\left\{\mathbf{m}^\top\left(\mathbf{A} - \mathbf{P}\right)\right\}^2}{\left\|\mathbf{m}\right\|^2} \tag{C.30}$$

Solution:

$$d\left(\lambda\right) = \left\|\mathbf{A} + \lambda\mathbf{m} - \mathbf{P}\right\| \tag{C.31}$$

$$\implies d^2\left(\lambda\right) = \left\|\mathbf{A} + \lambda\mathbf{m} - \mathbf{P}\right\|^2 \tag{C.32}$$

which can be simplified to obtain

$$d^2\left(\lambda\right) = \lambda^2\left\|\mathbf{m}\right\|^2 + 2\lambda\mathbf{m}^\top\left(\mathbf{A} - \mathbf{P}\right) + \left\|\mathbf{A} - \mathbf{P}\right\|^2 \tag{C.33}$$

which is of the form

$$d^2\left(\lambda\right) = a\lambda^2 + 2b\lambda + c \tag{C.34}$$

$$= a\left\{\left(\lambda + \frac{b}{a}\right)^2 + \left[\frac{c}{a} - \left(\frac{b}{a}\right)^2\right]\right\} \tag{C.35}$$

with

$$a = \left\|\mathbf{m}\right\|^2, b = \mathbf{m}^\top\left(\mathbf{A} - \mathbf{P}\right), c = \left\|\mathbf{A} - \mathbf{P}\right\|^2 \tag{C.36}$$

which can be expressed as From the above, $d^2\left(\lambda\right)$ is smallest when upon substituting from (C.36)

$$\lambda + \frac{b}{2a} = 0 \implies \lambda = -\frac{b}{2a} \tag{C.37}$$

$$= -\frac{\mathbf{m}^\top (\mathbf{A} - \mathbf{P})}{\|\mathbf{m}\|^2} \tag{C.38}$$

and consequently,

$$d_{\min}(\lambda) = a\left(\frac{c}{a} - \left(\frac{b}{a}\right)^2\right) \tag{C.39}$$

$$= c - \frac{b^2}{a} \tag{C.40}$$

yielding (C.30) after substituting from (C.36).

C.2.7. The distance between the parallel planes (C.19) is given by (C.20).

C.2.8. The plane

$$\mathbf{n}^\top \mathbf{x} = c \tag{C.41}$$

contains the line

$$\mathbf{x} = \mathbf{A} + \lambda \mathbf{m} \tag{C.42}$$

if

$$\mathbf{m}^\top \mathbf{n} = 0 \tag{C.43}$$

Solution: Any point on the line (C.42) should also satisfy (C.41). Hence,

$$\mathbf{n}^\top(\mathbf{A} + \lambda\mathbf{m}) = \mathbf{n}^\top\mathbf{A} = c \tag{C.44}$$

which can be simplified to obtain (C.43)

C.2.9. The foot of the perpendicular from a point $\mathbf{P}$ to the plane

$$\mathbf{n}^\top\mathbf{x} = c \tag{C.45}$$

is given by

$$\mathbf{x} = \mathbf{P} + \frac{c - \mathbf{n}^\top\mathbf{P}}{\|\mathbf{n}\|^2}\mathbf{n} \tag{C.46}$$

Solution: The equation of the line perpendicular to the given plane and passing through $\mathbf{P}$ is

$$\mathbf{x} = \mathbf{P} + \lambda\mathbf{n} \tag{C.47}$$

From (C.58), the intersection of the above line with the given plane is (C.46).

C.2.10. The image of a point $\mathbf{P}$ with respect to the plane

$$\mathbf{n}^\top\mathbf{x} = c \tag{C.48}$$

is given by

$$\mathbf{R} = \mathbf{P} + 2\frac{c - \mathbf{n}^\top \mathbf{P}}{\|\mathbf{n}\|^2} \tag{C.49}$$

Solution: Let $\mathbf{R}$ be the desired image. Then, subtituting the expression for the foot of the perpendicular from $\mathbf{P}$ to the given plane using (C.46),

$$\frac{\mathbf{P} + \mathbf{R}}{2} = \mathbf{P} + \frac{c - \mathbf{n}^\top \mathbf{P}}{\|\mathbf{n}\|^2} \tag{C.50}$$

C.2.11. Let a plane pass through the points $\mathbf{A}, \mathbf{B}$ and be perpendicular to the plane

$$\mathbf{n}^\top \mathbf{x} = c \tag{C.51}$$

Then the equation of this plane is given by

$$\mathbf{p}^\top \mathbf{x} = 1 \tag{C.52}$$

where

$$\mathbf{p} = \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{n} \end{pmatrix}^{-\top} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \tag{C.53}$$

Solution: From the given information,

$$\mathbf{p}^\top \mathbf{A} = d \tag{C.54}$$

$$\mathbf{p}^\top \mathbf{B} = d \tag{C.55}$$

$$\mathbf{p}^\top \mathbf{n} = 0 \tag{C.56}$$

$\because$ the normal vectors to the two planes will also be perpendicular. The system of equations in (C.56) can be expressed as the matrix equation

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{n} \end{pmatrix}^\top \mathbf{p} = d \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \tag{C.57}$$

which yields (C.53) upon normalising with d.

C.2.12. The intersection of the line represented by (C.4) with the plane represented by (C.1) is given by

$$\mathbf{x} = \mathbf{A} + \frac{c - \mathbf{n}^\top \mathbf{A}}{\mathbf{n}^\top \mathbf{m}} \mathbf{m} \tag{C.58}$$

Solution: From (C.4) and (C.1),

$$\mathbf{x} = \mathbf{A} + \lambda \mathbf{m} \tag{C.59}$$

$$\mathbf{n}^\top \mathbf{x} = c \tag{C.60}$$

$$\implies \mathbf{n}^\top (\mathbf{A} + \lambda \mathbf{m}) = c \tag{C.61}$$

which can be simplified to obtain

$$\mathbf{n}^\top \mathbf{A} + \lambda \mathbf{n}^\top \mathbf{m} = c \tag{C.62}$$

$$\implies \lambda = \frac{c - \mathbf{n}^\top \mathbf{A}}{\mathbf{n}^\top \mathbf{m}} \tag{C.63}$$

Substituting the above in (C.61) yields (C.58).

C.2.13. The foot of the perpendicular from the point $\mathbf{P}$ to the line represented by (C.4) is given by

$$\mathbf{x} = \mathbf{A} + \frac{\mathbf{m}^\top (\mathbf{P} - \mathbf{A})}{\|\mathbf{m}\|^2}\mathbf{m} \tag{C.64}$$

Solution: Let the equation of the line be

$$\mathbf{x} = \mathbf{A} + \lambda \mathbf{m} \tag{C.65}$$

The equation of the plane perpendicular to the given line passing through $\mathbf{P}$ is given by

$$\mathbf{m}^\top (\mathbf{x} - \mathbf{P}) = 0 \tag{C.66}$$

$$\implies \mathbf{m}^\top \mathbf{x} = \mathbf{m}^\top \mathbf{P} \tag{C.67}$$

The desired foot of the perpendicular is the intersection of (C.65) with (C.66) which can be obtained from (C.58) as (C.64)

C.2.14. The foot of the perpendicular from a point $\mathbf{P}$ to a plane is $\mathbf{Q}$. The equation of the plane is given by

$$
(\mathbf{P}-\mathbf{Q})^\top (\mathbf{x}-\mathbf{Q}) = 0 \tag{C.68}
$$

Solution: The normal vector to the plane is given by

$$
\mathbf{n} = \mathbf{P}-\mathbf{Q} \tag{C.69}
$$

Hence, the equation of the plane is (C.68).

C.2.15. Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be points on a plane. The equation of the plane is then given by

$$
\begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix}^\top \mathbf{n} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{C.70}
$$

Solution: Let the equation of the plane be

$$
\mathbf{n}^\top \mathbf{x} = 1 \tag{C.71}
$$

Then

$$
\mathbf{n}^\top \mathbf{A} = 1 \tag{C.72}
$$

$$
\mathbf{n}^\top \mathbf{B} = 1 \tag{C.73}
$$

$$
\mathbf{n}^\top \mathbf{C} = 1 \tag{C.74}
$$

which can be combined to obtain (C.70).

C.2.16. (Parallelogram Law) Let $\mathbf{A}, \mathbf{B}, \mathbf{D}$ be three vertices of a parallelogram. Then the vertex

$\mathbf{C}$ is given by

$$\mathbf{C} = \mathbf{B} + \mathbf{C} - \mathbf{A} \tag{C.75}$$

Solution: Shifting $\mathbf{A}$ to the origin, we obtain a parallelogram with corresponding vertices

$$\mathbf{0}, \mathbf{B} - \mathbf{A}, \mathbf{D} - \mathbf{A} \tag{C.76}$$

The fourth vertex of this parallelogram is then obtained as

$$(\mathbf{B} - \mathbf{A}) + (\mathbf{D} - \mathbf{A}) = \mathbf{D} + \mathbf{B} - 2\mathbf{A} \tag{C.77}$$

Shifting the origin to $\mathbf{A}$, the fourth vertex is obtained as

$$\mathbf{C} = \mathbf{D} + \mathbf{B} - 2\mathbf{A} + \mathbf{A} \tag{C.78}$$

$$= \mathbf{D} + \mathbf{B} - \mathbf{A} \tag{C.79}$$

C.2.17. (Affine Transformation) Let $\mathbf{A}, \mathbf{C}$, be opposite vertices of a square. The other two points can be obtained as

$$\mathbf{B} = \frac{\|\mathbf{A} - \mathbf{C}\|}{\sqrt{2}}\mathbf{P}\mathbf{e}_1 + \mathbf{A} \tag{C.80}$$

$$\mathbf{D} = \frac{\|\mathbf{A} - \mathbf{C}\|}{\sqrt{2}}\mathbf{P}\mathbf{e}_2 + \mathbf{A} \tag{C.81}$$

where

$$\mathbf{P} = \begin{pmatrix} \cos\left(\theta - \frac{\pi}{4}\right) & \sin\left(\theta - \frac{\pi}{4}\right) \\ \sin\left(\theta - \frac{\pi}{4}\right) & \cos\left(\theta - \frac{\pi}{4}\right) \end{pmatrix} \tag{C.82}$$

and

$$\cos\theta = \frac{(\mathbf{C} - \mathbf{A})^{\top} \mathbf{e}_1}{\|\mathbf{A} - \mathbf{C}\| \, \|\mathbf{e}_1\|} \tag{C.83}$$

Appendix D

Quadratic Forms

D.1. Conic equation

D.1.1. Let $\mathbf{q}$ be a point such that the ratio of its distance from a fixed point $\mathbf{F}$ and the distance (d) from a fixed line

$$L : \mathbf{n}^\top \mathbf{x} = c \tag{D.1}$$

is constant, given by

$$\frac{\|\mathbf{q} - \mathbf{F}\|}{d} = e \tag{D.2}$$

The locus of $\mathbf{q}$ is known as a conic section. The line L is known as the directrix and the point $\mathbf{F}$ is the focus. e is defined to be the eccentricity of the conic.

(a) For $e = 1$, the conic is a parabola

(b) For $e < 1$, the conic is an ellipse

(c) For $e > 1$, the conic is a hyperbola

D.1.2. The equation of a conic with directrix $\mathbf{n}^\top \mathbf{x} = c$, eccentricity e and focus $\mathbf{F}$ is given by

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{D.3}$$

where

$$\mathbf{V} = \|\mathbf{n}\|^2 \mathbf{I} - e^2 \mathbf{n}\mathbf{n}^\top, \tag{D.4}$$

$$\mathbf{u} = ce^2\mathbf{n} - \|\mathbf{n}\|^2 \mathbf{F}, \tag{D.5}$$

$$f = \|\mathbf{n}\|^2 \|\mathbf{F}\|^2 - c^2 e^2 \tag{D.6}$$

Proof. Using Definition D.1.1 and Lemma C.11, for any point $\mathbf{x}$ on the conic,

$$\|\mathbf{x} - \mathbf{F}\|^2 = e^2 \frac{\left(\mathbf{n}^\top \mathbf{x} - c\right)^2}{\|\mathbf{n}\|^2} \tag{D.7}$$

$$\implies \|\mathbf{n}\|^2 (\mathbf{x} - \mathbf{F})^\top (\mathbf{x} - \mathbf{F}) = e^2 \left(\mathbf{n}^\top \mathbf{x} - c\right)^2 \tag{D.8}$$

$$\implies \|\mathbf{n}\|^2 \left(\mathbf{x}^\top \mathbf{x} - 2\mathbf{F}^\top \mathbf{x} + \|\mathbf{F}\|^2\right) = e^2 \left(c^2 + \left(\mathbf{n}^\top \mathbf{x}\right)^2 - 2c\mathbf{n}^\top \mathbf{x}\right) \tag{D.9}$$

$$= e^2 \left(c^2 + \left(\mathbf{x}^\top \mathbf{n}\mathbf{n}^\top \mathbf{x}\right) - 2c\mathbf{n}^\top \mathbf{x}\right) \tag{D.10}$$

which can be expressed as (D.3) after simplification.

□

D.1.3. The eccentricity, directrices and foci of (D.3) are given by

$$e = \sqrt{1 - \frac{\lambda_1}{\lambda_2}} \tag{D.11}$$

$$\mathbf{n} = \sqrt{\lambda_2}\mathbf{p}_1,$$

$$c = \begin{cases} \frac{e\mathbf{u}^\top \mathbf{n} \pm \sqrt{e^2\left(\mathbf{u}^\top \mathbf{n}\right)^2 - \lambda_2(e^2-1)\left(\|\mathbf{u}\|^2 - \lambda_2 f\right)}}{\lambda_2 e(e^2-1)} & e \neq 1 \\ \frac{\|\mathbf{u}\|^2 - \lambda_2 f}{2\mathbf{u}^\top \mathbf{n}} & e = 1 \end{cases} \tag{D.12}$$

$$\mathbf{F} = \frac{ce^2\mathbf{n} - \mathbf{u}}{\lambda_2} \tag{D.13}$$

Proof. From (D.4), using the fact that $\mathbf{V}$ is symmetric with $\mathbf{V} = \mathbf{V}^\top$,

$$\mathbf{V}^\top \mathbf{V} = \left(\|\mathbf{n}\|^2 \mathbf{I} - e^2\mathbf{n}\mathbf{n}^\top\right)^\top \left(\|\mathbf{n}\|^2 \mathbf{I} - e^2\mathbf{n}\mathbf{n}^\top\right) \tag{D.14}$$

$$\implies \mathbf{V}^2 = \|\mathbf{n}\|^4 \mathbf{I} + e^4\mathbf{n}\mathbf{n}^\top\mathbf{n}\mathbf{n}^\top - 2e^2 \|\mathbf{n}\|^2 \mathbf{n}\mathbf{n}^\top \tag{D.15}$$

$$= \|\mathbf{n}\|^4 \mathbf{I} + e^4 \|\mathbf{n}\|^2 \mathbf{n}\mathbf{n}^\top - 2e^2 \|\mathbf{n}\|^2 \mathbf{n}\mathbf{n}^\top \tag{D.16}$$

$$= \|\mathbf{n}\|^4 \mathbf{I} + e^2\left(e^2 - 2\right) \|\mathbf{n}\|^2 \mathbf{n}\mathbf{n}^\top \tag{D.17}$$

$$= \|\mathbf{n}\|^4 \mathbf{I} + \left(e^2 - 2\right) \|\mathbf{n}\|^2 \left(\|\mathbf{n}\|^2 \mathbf{I} - \mathbf{V}\right) \tag{D.18}$$

which can be expressed as

$$\mathbf{V}^2 + \left(e^2 - 2\right) \|\mathbf{n}\|^2 \mathbf{V} - \left(e^2 - 1\right) \|\mathbf{n}\|^4 \mathbf{I} = 0 \tag{D.19}$$

Using the Cayley-Hamilton theorem, (D.19) results in the characteristic equation,

$$\lambda^2 - \left(2 - e^2\right) \|\mathbf{n}\|^2 \lambda + \left(1 - e^2\right) \|\mathbf{n}\|^4 = 0 \tag{D.20}$$

which can be expressed as

$$\left(\frac{\lambda}{\|\mathbf{n}\|^2}\right)^2 - \left(2 - e^2\right)\left(\frac{\lambda}{\|\mathbf{n}\|^2}\right) + \left(1 - e^2\right) = 0 \tag{D.21}$$

$$\implies \frac{\lambda}{\|\mathbf{n}\|^2} = 1 - e^2, 1 \tag{D.22}$$

$$\text{or, } \lambda_2 = \|\mathbf{n}\|^2, \lambda_1 = \left(1 - e^2\right)\lambda_2 \tag{D.23}$$

From (D.23), the eccentricity of (D.3) is given by (D.11). Multiplying both sides of (D.4) by $\mathbf{n}$,

$$\mathbf{V}\mathbf{n} = \|\mathbf{n}\|^2 \mathbf{n} - e^2 \mathbf{n}\mathbf{n}^\top \mathbf{n} \tag{D.24}$$

$$= \|\mathbf{n}\|^2 \left(1 - e^2\right) \mathbf{n} \tag{D.25}$$

$$= \lambda_1 \mathbf{n} \tag{D.26}$$

$$\tag{D.27}$$

from (D.23). Thus, λ_1 is the corresponding eigenvalue for $\mathbf{n}$. From (B.22) and (D.27), this implies that

$$\mathbf{p}_1 = \frac{\mathbf{n}}{\|\mathbf{n}\|} \tag{D.28}$$

$$\text{or, } \mathbf{n} = \|\mathbf{n}\| \mathbf{p}_1 = \sqrt{\lambda_2}\mathbf{p}_1 \tag{D.29}$$

from (D.23) . From (D.5) and (D.23),

$$\mathbf{F} = \frac{ce^2\mathbf{n} - \mathbf{u}}{\lambda_2} \tag{D.30}$$

$$\implies \|\mathbf{F}\|^2 = \frac{\left(ce^2\mathbf{n} - \mathbf{u}\right)^\top \left(ce^2\mathbf{n} - \mathbf{u}\right)}{\lambda_2^2} \tag{D.31}$$

$$\implies \lambda_2^2 \|\mathbf{F}\|^2 = c^2e^4\lambda_2 - 2ce^2\mathbf{u}^\top\mathbf{n} + \|\mathbf{u}\|^2 \tag{D.32}$$

Also, (D.6) can be expressed as

$$\lambda_2 \|\mathbf{F}\|^2 = f + c^2e^2 \tag{D.33}$$

From (D.32) and (D.33),

$$c^2e^4\lambda_2 - 2ce^2\mathbf{u}^\top\mathbf{n} + \|\mathbf{u}\|^2 = \lambda_2\left(f + c^2e^2\right) \tag{D.34}$$

$$\implies \lambda_2e^2\left(e^2 - 1\right)c^2 - 2ce^2\mathbf{u}^\top\mathbf{n} + \|\mathbf{u}\|^2 - \lambda_2 f = 0 \tag{D.35}$$

yielding (D.13). □

D.1.4. (D.3) represents

(a) a parabola for $\left|\mathbf{V}\right| = 0$,

(b) ellipse for $\left|\mathbf{V}\right| > 0$ and

(c) hyperbola for $\left|\mathbf{V}\right| < 0$.

Proof. From (D.11),

$$\frac{\lambda_1}{\lambda_2} = 1 - e^2 \tag{D.36}$$

Also,

$$\left|\mathbf{V}\right| = \lambda_1\lambda_2 \tag{D.37}$$

yielding Table D.2 $\square$

Eccentricity	**Conic**	**Eigenvalue**	**Determinant**
$e = 1$	Parabola	$\lambda_1 = 0$	$\left\|\mathbf{V}\right\| = 0$
$e < 1$	Ellipse	$\lambda_1 > 0, \lambda_2 > 0$	$\left\|\mathbf{V}\right\| > 0$
$e > 1$	Hyperbola	$\lambda_1 < 0, \lambda_2 > 0$	$\left\|\mathbf{V}\right\| < 0$

Table D.2:

D.2. Circles

D.2.1. The equation of a circle is given by

$$\left\|\mathbf{x}\right\|^2 + 2\mathbf{u}^\top\mathbf{x} + f = 0 \tag{D.38}$$

D.2.2. For a circle with centre $\mathbf{c}$ and radius r,

$$\mathbf{u} = -\mathbf{c}, f = \left\|\mathbf{u}\right\|^2 - r^2 \tag{D.39}$$

D.2.3. Any point $\mathbf{x}$ on a circle can be expressed as

$$\mathbf{x} = \mathbf{c} + r \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}. \tag{D.40}$$

D.2.4. The equation of the common chord of intersection of two circles is given by

$$\mathbf{u}_1^\top \mathbf{x} - \mathbf{u}_2^\top \mathbf{x} + f_1 - f_2 = 0 \tag{D.41}$$

D.2.5. Let

$$\mathbf{A} = \begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} \cos\theta_2 \\ \sin\theta_2 \end{pmatrix}, \tag{D.42}$$

be points on a unit circle with centre $\mathbf{O}$ at the origin. Then

$$\cos AOB = \mathbf{A}^\top \mathbf{B} \tag{D.43}$$

D.2.6. Let

$$\mathbf{A} = \begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} \cos\theta_2 \\ \sin\theta_2 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \tag{D.44}$$

be points on a unit circle. Then

$$\cos ACB = \frac{(\mathbf{C} - \mathbf{A})^\top (\mathbf{C} - \mathbf{B})}{\|\mathbf{C} - \mathbf{A}\| \, \|\mathbf{C} - \mathbf{B}\|} \tag{D.45}$$

$$= \cos\left(\frac{\theta_1 - \theta_2}{2}\right) \tag{D.46}$$

Proof. Since

$$
\begin{aligned}
(\mathbf{C}-\mathbf{A})^\top(\mathbf{C}-\mathbf{B}) &= \|\mathbf{C}\|^2 - \mathbf{C}^\top(\mathbf{A}+\mathbf{B}) + \mathbf{A}^\top\mathbf{B} && \text{(D.47)}\\
&= 1-\cos(\theta-\theta_1)-\cos(\theta-\theta_2)+\cos(\theta_1-\theta_2) && \text{(D.48)}\\
&= 2\cos^2\left(\frac{\theta_1-\theta_2}{2}\right) - 2\cos\left(\frac{\theta_1-\theta_2}{2}\right)\cos\left(\theta-\frac{\theta_1+\theta_2}{2}\right) && \text{(D.49)}\\
&= 4\cos\left(\frac{\theta_1-\theta_2}{2}\right)\sin\left(\frac{\theta-\theta_1}{2}\right)\sin\left(\frac{\theta-\theta_2}{2}\right), && \text{(D.50)}
\end{aligned}
$$

and

$$
\begin{aligned}
\|\mathbf{C}-\mathbf{A}\|^2 &= \|\mathbf{C}\|^2 + \|\mathbf{A}\|^2 - 2\mathbf{C}^\top\mathbf{A}, && \text{(D.51)}\\
&= 4\sin^2\left(\frac{\theta-\theta_1}{2}\right), && \text{(D.52)}\\
\|\mathbf{C}-\mathbf{B}\|^2 &= \|\mathbf{C}\|^2 + \|\mathbf{B}\|^2 - 2\mathbf{C}^\top\mathbf{B}, && \text{(D.53)}\\
&= 4\sin^2\left(\frac{\theta-\theta_2}{2}\right), && \text{(D.54)}
\end{aligned}
$$

(D.45) can be expressed as

$$
\frac{\cos\left(\frac{\theta_1-\theta_2}{2}\right)\sin\left(\frac{\theta-\theta_1}{2}\right)\sin\left(\frac{\theta-\theta_2}{2}\right)}{\sin\left(\frac{\theta-\theta_1}{2}\right)\sin\left(\frac{\theta-\theta_1}{2}\right)} \tag{D.55}
$$

yielding (D.46) $\square$

D.2.7. From (D.43) and (D.46),

$$
\angle AOB = 2\angle AOC \tag{D.56}
$$

D.3. Standard Form

D.3.1. Using the affine transformation in (B.19), the conic in (D.3) can be expressed in standard form as

$$\mathbf{y}^\top \left(\frac{\mathbf{D}}{f_0}\right) \mathbf{y} = 1 \qquad |\mathbf{V}| \neq 0 \tag{D.57}$$

$$\mathbf{y}^\top \mathbf{D} \mathbf{y} = -\eta \mathbf{e}_1^\top \mathbf{y} \qquad |\mathbf{V}| = 0 \tag{D.58}$$

where

$$f_0 = \mathbf{u}^\top \mathbf{V}^{-1} \mathbf{u} - f \neq 0 \tag{D.59}$$

$$\eta = 2\mathbf{u}^\top \mathbf{p}_1 \tag{D.60}$$

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{D.61}$$

Proof. Using (B.19) (D.3) can be expressed as

$$(\mathbf{P}\mathbf{y} + \mathbf{c})^\top \mathbf{V} (\mathbf{P}\mathbf{y} + \mathbf{c}) + 2\mathbf{u}^\top (\mathbf{P}\mathbf{y} + \mathbf{c}) + f = 0, \tag{D.62}$$

yielding

$$\mathbf{y}^\top \mathbf{P}^\top \mathbf{V} \mathbf{P} \mathbf{y} + 2(\mathbf{V}\mathbf{c} + \mathbf{u})^\top \mathbf{P}\mathbf{y} + \mathbf{c}^\top \mathbf{V} \mathbf{c} + 2\mathbf{u}^\top \mathbf{c} + f = 0 \tag{D.63}$$

From (D.63) and (B.20),

$$\mathbf{y}^\top \mathbf{D} \mathbf{y} + 2(\mathbf{V}\mathbf{c} + \mathbf{u})^\top \mathbf{P}\mathbf{y} + \mathbf{c}^\top (\mathbf{V}\mathbf{c} + \mathbf{u}) + \mathbf{u}^\top \mathbf{c} + f = 0 \tag{D.64}$$

When $\mathbf{V}^{-1}$ exists, choosing

$$\mathbf{V}\mathbf{c} + \mathbf{u} = \mathbf{0}, \quad \text{or}, \mathbf{c} = -\mathbf{V}^{-1}\mathbf{u}, \tag{D.65}$$

and substituting (D.65) in (D.64) yields (D.57). When $|\mathbf{V}| = 0, \lambda_1 = 0$ and

$$\mathbf{V}\mathbf{p}_1 = 0, \mathbf{V}\mathbf{p}_2 = \lambda_2 \mathbf{p}_2. \tag{D.66}$$

where $\mathbf{p}_1, \mathbf{p}_2$ are the eigenvectors of $\mathbf{V}$ such that (B.20)

$$\mathbf{P} = \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix}, \tag{D.67}$$

Substituting (D.67) in (D.64),

$$\mathbf{y}^\top \mathbf{D}\mathbf{y} + 2\left(\mathbf{c}^\top \mathbf{V} + \mathbf{u}^\top\right)\begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix}\mathbf{y} + \mathbf{c}^\top\left(\mathbf{V}\mathbf{c} + \mathbf{u}\right) + \mathbf{u}^\top \mathbf{c} + f = 0 \tag{D.68}$$

$$\implies \mathbf{y}^\top \mathbf{D}\mathbf{y} + 2\begin{pmatrix} \left(\mathbf{c}^\top \mathbf{V} + \mathbf{u}^\top\right)\mathbf{p}_1 & \left(\mathbf{c}^\top \mathbf{V} + \mathbf{u}^\top\right)\mathbf{p}_2 \end{pmatrix}\mathbf{y} + \mathbf{c}^\top\left(\mathbf{V}\mathbf{c} + \mathbf{u}\right) + \mathbf{u}^\top \mathbf{c} + f = 0 \tag{D.69}$$

$$\implies \mathbf{y}^\top \mathbf{D}\mathbf{y} + 2\begin{pmatrix} \mathbf{u}^\top \mathbf{p}_1 & \left(\lambda_2 \mathbf{c}^\top + \mathbf{u}^\top\right)\mathbf{p}_2 \end{pmatrix}\mathbf{y} + \mathbf{c}^\top\left(\mathbf{V}\mathbf{c} + \mathbf{u}\right) + \mathbf{u}^\top \mathbf{c} + f = 0 \tag{D.70}$$

upon substituting from (D.66) yielding

$$\lambda_2 y_2^2 + 2\left(\mathbf{u}^\top \mathbf{p}_1\right) y_1 + 2y_2\left(\lambda_2 \mathbf{c} + \mathbf{u}\right)^\top \mathbf{p}_2 + \mathbf{c}^\top\left(\mathbf{V}\mathbf{c} + \mathbf{u}\right) + \mathbf{u}^\top \mathbf{c} + f = 0 \tag{D.71}$$

Thus, (D.71) can be expressed as (D.58) by choosing

$$\eta = 2\mathbf{u}^\top \mathbf{p}_1 \tag{D.72}$$

and $\mathbf{c}$ in (D.64) such that

$$2\mathbf{P}^\top \left(\mathbf{V}\mathbf{c} + \mathbf{u}\right) = \eta \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{D.73}$$

$$\mathbf{c}^\top \left(\mathbf{V}\mathbf{c} + \mathbf{u}\right) + \mathbf{u}^\top \mathbf{c} + f = 0 \tag{D.74}$$

$\because$ $\mathbf{P}^\top \mathbf{P} = \mathbf{I}$, multiplying (D.73) by $\mathbf{P}$ yields

$$\left(\mathbf{V}\mathbf{c} + \mathbf{u}\right) = \frac{\eta}{2}\mathbf{p}_1, \tag{D.75}$$

which, upon substituting in (D.74) results in

$$\frac{\eta}{2}\mathbf{c}^\top \mathbf{p}_1 + \mathbf{u}^\top \mathbf{c} + f = 0 \tag{D.76}$$

(D.75) and (D.76) can be clubbed together to obtain (E.7). $\square$

D.3.2. For the standard conic,

$$\mathbf{P} = \mathbf{I} \tag{D.77}$$

$$\mathbf{u} = \begin{cases} 0 & e \neq 1 \\ \frac{\eta}{2}\mathbf{e}_1 & e = 1 \end{cases} \tag{D.78}$$

$$\lambda_1 \begin{cases} = 0 & e = 1 \\ \neq 0 & e \neq 1 \end{cases} \tag{D.79}$$

where

$$\mathbf{I} = \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 \end{pmatrix} \tag{D.80}$$

is the identity matrix.

D.3.3.

(a) The directrices for the standard conic are given by

$$\mathbf{e}_1^\top \mathbf{y} = \pm \frac{1}{e} \sqrt{\frac{|f_0|}{\lambda_2 (1 - e^2)}} \qquad e \neq 1 \tag{D.81}$$

$$\mathbf{e}_1^\top \mathbf{y} = \frac{\eta}{2\lambda_2} \qquad e = 1 \tag{D.82}$$

(b) The foci of the standard ellipse and hyperbola are given by

$$\mathbf{F} = \begin{cases} \pm e \sqrt{\frac{|f_0|}{\lambda_2(1-e^2)}}\mathbf{e}_1 & e \neq 1 \\ -\frac{\eta}{4\lambda_2}\mathbf{e}_1 & e = 1 \end{cases} \tag{D.83}$$

Proof. (a) For the standard hyperbola/ellipse in (D.57), from (D.77), (D.12) and (D.78),

$$\mathbf{n} = \sqrt{\frac{\lambda_2}{f_0}}\mathbf{e}_1 \tag{D.84}$$

$$c = \pm\frac{\sqrt{-\frac{\lambda_2}{f_0}\left(e^2-1\right)\left(\frac{\lambda_2}{f_0}\right)}}{\frac{\lambda_2}{f_0}e\left(e^2-1\right)} \tag{D.85}$$

$$= \pm\frac{1}{e\sqrt{1-e^2}} \tag{D.86}$$

yielding (D.81) upon substituting from (D.11) and simplifying. For the standard parabola in (D.58), from (D.77), (D.12) and (D.78), noting that $f = 0$,

$$\mathbf{n} = \sqrt{\lambda_2}\mathbf{e}_1 \tag{D.87}$$

$$c = \frac{\left\|\frac{\eta}{2}\mathbf{e}_1\right\|^2}{2\left(\frac{\eta}{\mathbf{2}}\right)\left(\mathbf{e_1}\right)^\top\mathbf{n}} \tag{D.88}$$

$$\tag{D.89}$$

$$= \frac{\eta}{4\sqrt{\lambda_2}} \tag{D.90}$$

yielding (D.82).

(b) For the standard ellipse/hyperbola, substituting from (D.86), (D.84), (D.78) and (D.11) in (D.13),

$$\mathbf{F} = \pm\frac{\left(\frac{1}{e\sqrt{1-e^2}}\right)\left(e^2\right)\sqrt{\frac{\lambda_2}{f_0}}\mathbf{e}_1}{\frac{\lambda_2}{f_0}} \tag{D.91}$$

yielding (D.83) after simplification. For the standard parabola, substituting from

(D.90), (D.87), (D.78) and (D.11) in (D.13),

$$\mathbf{F} = \frac{\left(\frac{\eta}{4\sqrt{\lambda_2}}\right)\sqrt{\lambda_2}\mathbf{e}_1 - \frac{\eta}{\mathbf{2}}\mathbf{e_1}}{\lambda_2} \tag{D.92}$$

$$\tag{D.93}$$

yielding (D.83) after simplification.

$\square$

Appendix E

Conic Parameters

E.1. Standard Form

E.1.1. The center of the standard ellipse/hyperbola, defined to be the mid point of the line joining the foci, is the origin.

E.1.2. The principal (major) axis of the standard ellipse/hyperbola, defined to be the line joining the two foci is the x-axis.

Proof. From (D.83), it is obvious that the line joining the foci passes through the origin. Also, the direction vector of this line is $\mathbf{e}_1$. Thus, the principal axis is the x-axis. □

E.1.3. The minor axis of the standard ellipse/hyperbola, defined to be the line orthogonal to the x-axis is the y-axis.

E.1.4. The axis of symmetry of the standard parabola, defined to be the line perpendicular to the directrix and passing through the focus, is the x- axis.

Proof. From (D.87) and (D.83), the axis of the parabola can be expressed using (C.2) as

$$\mathbf{e}_2^\top \left(\mathbf{y} + \frac{\eta}{4\lambda_2} \mathbf{e}_1 \right) = 0 \tag{E.1}$$

$$\implies \mathbf{e}_2^\top \mathbf{y} = 0, \tag{E.2}$$

which is the equation of the x-axis. □

E.1.5. The point where the parabola intersects its axis of symmetry is called the vertex. For the standard parabola, the vertex is the origin.

Proof. (E.2) can be expressed as

$$\mathbf{y} = \alpha \mathbf{e}_1, \tag{E.3}$$

using (C.2). Substituting (E.3) in (D.58),

$$\alpha^2 \mathbf{e}_1^\top \mathbf{D} \mathbf{e}_1 = -\eta \alpha \mathbf{e}_1^\top \mathbf{e}_1 \tag{E.4}$$

$$\implies \alpha = 0, \text{ or, } \mathbf{y} = \mathbf{0}. \tag{E.5}$$

□

E.1.6. The focal length of the standard parabola, , defined to be the distance between the vertex and the focus, measured along the axis of symmetry, is $\left| \frac{\eta}{4\lambda_2} \right|$

E.2. Quadratic Form

E.2.1. The center/vertex of a conic section are given by

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \qquad \left|\mathbf{V}\right| \neq 0 \tag{E.6}$$

$$\begin{pmatrix} \mathbf{u}^\top + \frac{\eta}{2}\mathbf{p}_1^\top \\ \mathbf{v} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -f \\ \frac{\eta}{2}\mathbf{p}_1 - \mathbf{u} \end{pmatrix} \qquad \left|\mathbf{V}\right| = 0 \tag{E.7}$$

Proof. In (B.19), substituting $\mathbf{y} = \mathbf{0}$, the center/vertex for the quadratic form is obtained as

$$\mathbf{x} = \mathbf{c}, \tag{E.8}$$

where $\mathbf{c}$ is derived as (E.6) and (E.7) in Appendix D.3.1. □

E.2.2. The equation of the minor and major axes for the ellipse/hyperbola are respectively given by

$$\mathbf{p}_i^\top (\mathbf{x} - \mathbf{c}) = 0, i = 1, 2 \tag{E.9}$$

The axis of symmetry for the parabola is also given by (E.9).

Proof. From (E.1.2), the major/symmetry axis for the hyperbola/ellipse/parabola can be expressed using (B.19) as

$$\mathbf{e}_2^\top \mathbf{P}^\top (\mathbf{x} - \mathbf{c}) = 0 \tag{E.10}$$

$$\implies (\mathbf{P}\mathbf{e}_2)^\top (\mathbf{x} - \mathbf{c}) = 0 \tag{E.11}$$

yielding (E.9), and the proof for the minor axis is similar. $\square$

Appendix F

Conic Lines

F.1. Pair of Straight Lines

F.1.1. The asymptotes of the hyperbola in (D.57), defined to be the lines that do not intersect the hyperbola, are given by

$$\left(\sqrt{|\lambda_1|} \quad \pm\sqrt{|\lambda_2|}\right)\mathbf{y} = 0 \tag{F.1}$$

Proof. From (D.57), it is obvious that the pair of lines represented by

$$\mathbf{y}^\top \mathbf{D}\mathbf{y} = 0 \tag{F.2}$$

do not intersect the conic

$$\mathbf{y}^\top \mathbf{D}\mathbf{y} = f_0 \tag{F.3}$$

Thus, (F.2) represents the asysmptotes of the hyperbola in (D.57) and can be expressed as

$$\lambda_1 y_1^2 + \lambda_2 y_1^2 = 0, \tag{F.4}$$

which can then be simplified to obtain (F.1).

$\square$

F.1.2. (D.3) represents a pair of straight lines if

$$\mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} - f = 0 \tag{F.5}$$

F.1.3. (D.3) represents a pair of straight lines if the matrix

$$\begin{pmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u} & f \end{pmatrix} \tag{F.6}$$

is singular.

Proof. Let

$$\begin{pmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u} & f \end{pmatrix} \mathbf{x} = \mathbf{0} \tag{F.7}$$

Expressing

$$\mathbf{x} = \begin{pmatrix} \mathbf{y} \\ y_3 \end{pmatrix}, \tag{F.8}$$

$$\begin{pmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^\top & f \end{pmatrix} \begin{pmatrix} \mathbf{y} \\ y_3 \end{pmatrix} = \mathbf{0} \tag{F.9}$$

$$\implies \mathbf{V}\mathbf{y} + y_3\mathbf{u} = \mathbf{0} \quad \text{and} \tag{F.10}$$

$$\mathbf{u}^\top\mathbf{y} + f y_3 = 0 \tag{F.11}$$

From (F.10) we obtain,

$$\mathbf{y}^\top\mathbf{V}\mathbf{y} + y_3\mathbf{y}^\top\mathbf{u} = \mathbf{0} \tag{F.12}$$

$$\implies \mathbf{y}^\top\mathbf{V}\mathbf{y} + y_3\mathbf{u}^\top\mathbf{y} = \mathbf{0} \tag{F.13}$$

yielding (F.5) upon substituting from (F.11). □

F.1.4. Using the affine transformation, (F.1) can be expressed as the lines

$$\begin{pmatrix} \sqrt{|\lambda_1|} & \pm\sqrt{|\lambda_2|} \end{pmatrix} \mathbf{P}^\top (\mathbf{x} - \mathbf{c}) = 0 \tag{F.14}$$

F.1.5. The angle between the asymptotes can be expressed as

$$\cos\theta = \frac{|\lambda_1| - |\lambda_2|}{|\lambda_1| + |\lambda_2|} \tag{F.15}$$

Proof. The normal vectors of the lines in (F.14) are

$$\begin{aligned} \mathbf{n}_1 &= \mathbf{P} \begin{pmatrix} \sqrt{|\lambda_1|} \\ \sqrt{|\lambda_2|} \end{pmatrix} \\ \mathbf{n}_2 &= \mathbf{P} \begin{pmatrix} \sqrt{|\lambda_1|} \\ -\sqrt{|\lambda_2|} \end{pmatrix} \end{aligned} \tag{F.16}$$

The angle between the asymptotes is given by

$$\cos\theta = \frac{\mathbf{n_1}^\top \mathbf{n_2}}{\|\mathbf{n_1}\|\,\|\mathbf{n_2}\|} \tag{F.17}$$

The orthogonal matrix $\mathbf{P}$ preserves the norm, i.e.

$$\|\mathbf{n_1}\| = \left\| \mathbf{P} \begin{pmatrix} \sqrt{|\lambda_1|} \\ \sqrt{|\lambda_2|} \end{pmatrix} \right\| = \left\| \begin{pmatrix} \sqrt{|\lambda_1|} \\ \sqrt{|\lambda_2|} \end{pmatrix} \right\| \tag{F.18}$$

$$= \sqrt{|\lambda_1| + |\lambda_2|} = \|\mathbf{n_2}\| \tag{F.19}$$

It is easy to verify that

$$\mathbf{n_1}^\top \mathbf{n_2} = |\lambda_1| - |\lambda_2| \tag{F.20}$$

Thus, the angle between the asymptotes is obtained from (F.17) as (F.15). □

F.2. Intersection of Conics

F.2.1. Let

$$\mathbf{x}^\top \mathbf{V}_i \mathbf{x} + 2\mathbf{u}_i^\top \mathbf{x} + f_i = 0, \quad i = 1, 2 \tag{F.21}$$

be the equation of two conics. The locus of their intersection is a pair of straight lines if

$$\begin{vmatrix} \mathbf{V}_1 + \mu \mathbf{V}_2 & \mathbf{u}_1 + \mu \mathbf{u}_2 \\ (\mathbf{u}_1 + \mu \mathbf{u}_2)^\top & f \end{vmatrix} = 0, \left| \mathbf{V}_1 + \mu \mathbf{V}_2 \right| < 0 \tag{F.22}$$

Proof. The intersection of the conics in (F.21) is given by the curve

$$\mathbf{x}^\top (\mathbf{V}_1 + \mu \mathbf{V}_2) \mathbf{x} + 2 (\mathbf{u}_1 + \mu \mathbf{u}_2)^\top \mathbf{x} + f_1 + \mu f_2 = 0, \tag{F.23}$$

which, from Theorem F.1.3 represents a pair of straight lines if (F.22) is satisfied. □

F.2.2. The points of intersection of the conics in (F.21) are the points of the intersection of the lines in (F.23).

F.3. Chords of a Conic

F.3.1. The points of intersection of the line

$$L: \quad \mathbf{x} = \mathbf{q} + \mu \mathbf{m} \quad \mu \in \mathbb{R} \tag{F.24}$$

with the conic section in (D.3) are given by

$$\mathbf{x}_i = \mathbf{q} + \mu_i \mathbf{m} \tag{F.25}$$

where

$$\mu_i = \frac{1}{\mathbf{m}^\top \mathbf{V}\mathbf{m}} \Big(-\mathbf{m}^\top \left(\mathbf{V}\mathbf{q} + \mathbf{u}\right) \\ \pm \sqrt{\left[\mathbf{m}^\top \left(\mathbf{V}\mathbf{q} + \mathbf{u}\right)\right]^2 - \left(\mathbf{q}^\top \mathbf{V}\mathbf{q} + 2\mathbf{u}^\top \mathbf{q} + f\right)\left(\mathbf{m}^\top \mathbf{V}\mathbf{m}\right)} \Big) \tag{F.26}$$

Proof. Substituting (F.24) in (D.3),

$$\left(\mathbf{q} + \mu\mathbf{m}\right)^\top \mathbf{V}\left(\mathbf{q} + \mu\mathbf{m}\right) + 2\mathbf{u}^\top \left(\mathbf{q} + \mu\mathbf{m}\right) + f = 0 \tag{F.27}$$

$$\implies \mu^2 \mathbf{m}^\top \mathbf{V}\mathbf{m} + 2\mu\mathbf{m}^\top \left(\mathbf{V}\mathbf{q} + \mathbf{u}\right) + \mathbf{q}^\top \mathbf{V}\mathbf{q} + 2\mathbf{u}^\top \mathbf{q} + f = 0 \tag{F.28}$$

Solving the above quadratic in (F.28) yields (F.26). □

F.3.2. If L in (F.24) touches (D.3) at exactly one point $\mathbf{q}$,

$$\mathbf{m}^\top \left(\mathbf{V}\mathbf{q} + \mathbf{u}\right) = 0 \tag{F.29}$$

Proof. In this case, (F.28) has exactly one root. Hence, in (F.26)

$$\left[\mathbf{m}^\top \left(\mathbf{V}\mathbf{q} + \mathbf{u}\right)\right]^2 - \left(\mathbf{m}^\top \mathbf{V}\mathbf{m}\right)\left(\mathbf{q}^\top \mathbf{V}\mathbf{q} + 2\mathbf{u}^\top \mathbf{q} + f\right) = 0 \tag{F.30}$$

$\because$ $\mathbf{q}$ is the point of contact, $\mathbf{q}$ satisfies (D.3) and

$$\mathbf{q}^\top \mathbf{V}\mathbf{q} + 2\mathbf{u}^\top \mathbf{q} + f = 0 \tag{F.31}$$

Substituting (F.31) in (F.30) and simplifying, we obtain (F.29). □

F.3.3. The length of the chord in (F.24) is given by

$$\frac{2\sqrt{\left[\mathbf{m}^\top\left(\mathbf{V}\mathbf{q}+\mathbf{u}\right)\right]^2-\left(\mathbf{q}^\top\mathbf{V}\mathbf{q}+2\mathbf{u}^\top\mathbf{q}+f\right)\left(\mathbf{m}^\top\mathbf{V}\mathbf{m}\right)}}{\mathbf{m}^\top\mathbf{V}\mathbf{m}}\left\|\mathbf{m}\right\| \tag{F.32}$$

Proof. The distance between the points in (F.25) is given by

$$\left\|\mathbf{x}_1-\mathbf{x}_2\right\| = \left|\mu_1-\mu_2\right|\left\|\mathbf{m}\right\| \tag{F.33}$$

Substituing μ_i from (F.26) in (F.33) yields (F.32). □

F.3.4. The affine transform for the conic section, preserves the norm. This implies that the length of any chord of a conic is invariant to translation and/or rotation.

Proof. Let

$$\mathbf{x}_i = \mathbf{P}\mathbf{y}_i + \mathbf{c} \tag{F.34}$$

be any two points on the conic. Then the distance between the points is given by

$$\left\|\mathbf{x}_1-\mathbf{x}_2\right\| = \left\|\mathbf{P}\left(\mathbf{y}_1-\mathbf{y}_2\right)\right\| \tag{F.35}$$

which can be expressed as

$$\left\|\mathbf{x}_1-\mathbf{x}_2\right\|^2 = \left(\mathbf{y}_1-\mathbf{y}_2\right)^\top\mathbf{P}^\top\mathbf{P}\left(\mathbf{y}_1-\mathbf{y}_2\right) \tag{F.36}$$

$$= \left\|\mathbf{y}_1-\mathbf{y}_2\right\|^2 \tag{F.37}$$

since

$$\mathbf{P}^\top \mathbf{P} = \mathbf{I} \tag{F.38}$$

□

F.3.5. For the standard hyperbola/ellipse, the length of the major axis is

$$2\sqrt{\left|\frac{f_0}{\lambda_1}\right|} \tag{F.39}$$

and the minor axis is

$$2\sqrt{\left|\frac{f_0}{\lambda_2}\right|} \tag{F.40}$$

Proof. Since the major axis passes through the origin,

$$\mathbf{q} = \mathbf{0} \tag{F.41}$$

Further, from Corollary (E.1.2),

$$\mathbf{m} = \mathbf{e}_2, \tag{F.42}$$

and from (D.57),

$$\mathbf{V} = \frac{\mathbf{D}}{f_0}, \mathbf{u} = 0, f = -1 \tag{F.43}$$

Substituting the above in (F.32),

$$\frac{2\sqrt{\mathbf{e}_1^\top \frac{\mathbf{D}}{f_0}\mathbf{e}_1}}{\mathbf{e}_1^\top \frac{\mathbf{D}}{f_0}\mathbf{e}_1}\|\mathbf{e}_1\| \tag{F.44}$$

yielding (F.39). Similarly, for the minor axis, the only different parameter is

$$\mathbf{m} = \mathbf{e}_2, \tag{F.45}$$

Substituting the above in (F.32),

$$\frac{2\sqrt{\mathbf{e}_2^\top \frac{\mathbf{D}}{f_0}\mathbf{e}_2}}{\mathbf{e}_2^\top \frac{\mathbf{D}}{f_0}\mathbf{e}_2}\|\mathbf{e}_2\| \tag{F.46}$$

yielding (F.40).

□

F.3.6. The latus rectum of a conic section is the chord that passes through the focus and is perpendicular to the major axis. The length of the latus rectum for a conic is given by

$$l = \begin{cases} 2\frac{\sqrt{|f_0\lambda_1|}}{\lambda_2} & e \neq 1 \\ \frac{\eta}{\lambda_2} & e = 1 \end{cases} \tag{F.47}$$

Proof. The latus rectum is perpendicular to the major axis for the standard conic. Hence, from Corollary (E.1.2),

$$\mathbf{m} = \mathbf{e}_2, \tag{F.48}$$

Since it passes through the focus, from (D.83)

$$\mathbf{q} = \mathbf{F} = \pm e\sqrt{\frac{f_0}{\lambda_2\left(1-e^2\right)}}\mathbf{e}_1 \tag{F.49}$$

for the standard hyperbola/ellipse. Also, from (D.57),

$$\mathbf{V} = \frac{\mathbf{D}}{f_0}, \mathbf{u} = 0, f = -1 \tag{F.50}$$

Substituting the above in (F.32),

$$\frac{2\sqrt{\left[\mathbf{e}_2^\top\left(\frac{\mathbf{D}}{f_0}e\sqrt{\frac{f_0}{\lambda_2(1-e^2)}}\mathbf{e}_1\right)\right]^2 - \left(e\sqrt{\frac{f_0}{\lambda_2(1-e^2)}}\mathbf{e}_1^\top\frac{\mathbf{D}}{f_0}e\sqrt{\frac{f_0}{\lambda_2(1-e^2)}}\mathbf{e}_1 - 1\right)\left(\mathbf{e}_2^\top\frac{\mathbf{D}}{f_0}\mathbf{e}_2\right)}}{\mathbf{e}_2^\top\frac{\mathbf{D}}{f_0}\mathbf{e}_2}\|\mathbf{e}_2\| \tag{F.51}$$

Since

$$\mathbf{e}_2^\top\mathbf{D}\mathbf{e}_1 = 0, \mathbf{e}_1^\top\mathbf{D}\mathbf{e}_1 = \lambda_1, \mathbf{e}_1^\top\mathbf{e}_1 = 1, \|\mathbf{e}_2\| = 1, \mathbf{e}_2^\top\mathbf{D}\mathbf{e}_2 = \lambda_2, \tag{F.52}$$

(F.51) can be expressed as

$$\frac{2\sqrt{\left(1-\frac{\lambda_1 e^2}{\lambda_2(1-e^2)}\right)\left(\frac{\lambda_2}{f_0}\right)}}{\frac{\lambda_2}{f_0}} \tag{F.53}$$

$$= 2\frac{\sqrt{f_0\lambda_1}}{\lambda_2} \qquad \left(\because e^2 = 1 - \frac{\lambda_1}{\lambda_2}\right) \tag{F.54}$$

For the standard parabola, the parameters in (F.32) are

$$\mathbf{q} = \mathbf{F} = -\frac{\eta}{4\lambda_2}\mathbf{e}_1, \mathbf{m} = \mathbf{e}_1, \mathbf{V} = \mathbf{D}, \mathbf{u} = \frac{\eta}{2}\mathbf{e}_1^\top, f = 0 \tag{F.55}$$

Substituting the above in (F.32), the length of the latus rectum can be expressed as

$$2\frac{\sqrt{\left[\mathbf{e}_2^\top\left(\mathbf{D}\left(-\frac{\eta}{4\lambda_2}\mathbf{e}_1\right)+\frac{\eta}{2}\mathbf{e}_1\right)\right]^2-\left(\left(-\frac{\eta}{4\lambda_2}\mathbf{e}_1\right)^\top\mathbf{D}\left(-\frac{\eta}{4\lambda_2}\mathbf{e}_1\right)+2\frac{\eta}{2}\mathbf{e}_1^\top\left(-\frac{\eta}{4\lambda_2}\mathbf{e}_1\right)\right)\left(\mathbf{e}_2^\top\mathbf{D}\mathbf{e}_2\right)}}{\mathbf{e}_2^\top\mathbf{D}\mathbf{e}_2}\left\|\mathbf{e}_2\right\| \tag{F.56}$$

Since

$$\mathbf{e}_2^\top\mathbf{D}\mathbf{e}_1=0,\mathbf{e}_2^\top\mathbf{e}_2=0,\mathbf{e}_1^\top\mathbf{D}\mathbf{e}_1=0,\mathbf{e}_1^\top\mathbf{e}_1=1,\left\|\mathbf{e}_1\right\|=1,\mathbf{e}_2^\top\mathbf{D}\mathbf{e}_2=\lambda_2, \tag{F.57}$$

(F.56) can be expressed as

$$2\frac{\sqrt{\frac{\eta^2}{4\lambda_2}\lambda_2}}{\lambda_2}=\frac{\eta}{\lambda_2} \tag{F.58}$$

□

F.4. Tangent and Normal

F.4.1. Given the point of contact $\mathbf{q}$, the equation of a tangent to (D.3) is

$$\left(\mathbf{V}\mathbf{q}+\mathbf{u}\right)^\top\mathbf{x}+\mathbf{u}^\top\mathbf{q}+f=0 \tag{F.59}$$

Proof. The normal vector is obtained from (F.29) and (A.31) as

$$\kappa\mathbf{n}=\mathbf{V}\mathbf{q}+\mathbf{u},\kappa\in\mathbb{R} \tag{F.60}$$

From (F.60) and (C.2), the equation of the tangent is

$$(\mathbf{V}\mathbf{q}+\mathbf{u})^{\top}(\mathbf{x}-\mathbf{q})=0 \tag{F.61}$$

$$\implies (\mathbf{V}\mathbf{q}+\mathbf{u})^{\top}\mathbf{x}-\mathbf{q}^{\top}\mathbf{V}\mathbf{q}-\mathbf{u}^{\top}\mathbf{q}=0 \tag{F.62}$$

which, upon substituting from (F.31) and simplifying yields (F.59) □

F.4.2. If $\mathbf{V}^{-1}$ exists, given the normal vector $\mathbf{n}$, the tangent points of contact to (D.3) are given by

$$\begin{gathered}\mathbf{q}_i=\mathbf{V}^{-1}(\kappa_i\mathbf{n}-\mathbf{u}),\, i=1,2\\ \text{where } \kappa_i=\pm\sqrt{\frac{f_0}{\mathbf{n}^{\top}\mathbf{V}^{-1}\mathbf{n}}}\end{gathered} \tag{F.63}$$

Proof. From (F.60),

$$\mathbf{q}=\mathbf{V}^{-1}(\kappa\mathbf{n}-\mathbf{u}), \quad \kappa\in\mathbb{R} \tag{F.64}$$

Substituting (F.64) in (F.31),

$$(\kappa\mathbf{n}-\mathbf{u})^{\top}\mathbf{V}^{-1}(\kappa\mathbf{n}-\mathbf{u})+2\mathbf{u}^{\top}\mathbf{V}^{-1}(\kappa\mathbf{n}-\mathbf{u})+f=0 \tag{F.65}$$

$$\implies \kappa^2\mathbf{n}^{\top}\mathbf{V}^{-1}\mathbf{n}-\mathbf{u}^{\top}\mathbf{V}^{-1}\mathbf{u}+f=0 \tag{F.66}$$

$$\text{or, } \kappa=\pm\sqrt{\frac{f_0}{\mathbf{n}^{\top}\mathbf{V}^{-1}\mathbf{n}}} \tag{F.67}$$

Substituting (F.67) in (F.64) yields (F.63). □

F.4.3. If $\mathbf{V}$ is not invertible, given the normal vector $\mathbf{n}$, the point of contact to (D.3) is given by the matrix equation

$$\begin{pmatrix} (\mathbf{u}+\kappa\mathbf{n})^\top \\ \mathbf{V} \end{pmatrix} \mathbf{q} = \begin{pmatrix} -f \\ \kappa\mathbf{n}-\mathbf{u} \end{pmatrix} \tag{F.68}$$

$$\text{where } \kappa = \frac{\mathbf{p}_1^\top \mathbf{u}}{\mathbf{p}_1^\top \mathbf{n}}, \quad \mathbf{V}\mathbf{p}_1 = 0 \tag{F.69}$$

Proof. If $\mathbf{V}$ is non-invertible, it has a zero eigenvalue. If the corresponding eigenvector is $\mathbf{p}_1$, then,

$$\mathbf{V}\mathbf{p}_1 = 0 \tag{F.70}$$

From (F.60),

$$\kappa\mathbf{n} = \mathbf{V}\mathbf{q} + \mathbf{u}, \quad \kappa \in \mathbb{R} \tag{F.71}$$

$$\Longrightarrow \kappa\mathbf{p}_1^\top \mathbf{n} = \mathbf{p}_1^\top \mathbf{V}\mathbf{q} + \mathbf{p}_1^\top \mathbf{u} \tag{F.72}$$

$$\text{or, } \kappa\mathbf{p}_1^\top \mathbf{n} = \mathbf{p}_1^\top \mathbf{u}, \quad \because \mathbf{p}_1^\top \mathbf{V} = 0, \quad (\text{ from (F.70)}) \tag{F.73}$$

yielding κ in (F.69). From (F.71),

$$\kappa\mathbf{q}^\top \mathbf{n} = \mathbf{q}^\top \mathbf{V}\mathbf{q} + \mathbf{q}^\top \mathbf{u} \tag{F.74}$$

$$\Longrightarrow \kappa\mathbf{q}^\top \mathbf{n} = -f - \mathbf{q}^\top \mathbf{u} \quad \text{from (F.31)}, \tag{F.75}$$

$$\text{or, } (\kappa\mathbf{n}+\mathbf{u})^\top \mathbf{q} = -f \tag{F.76}$$

(F.71) can be expressed as

$$\mathbf{V}\mathbf{q} = \kappa\mathbf{n} - \mathbf{u}. \tag{F.77}$$

(F.76) and (F.77) clubbed together result in (F.68). $\square$

F.4.4. The normal vectors of the tangents to the conic in (D.3) satisfy

$$\mathbf{n}^\top \mathbf{V}^{-1}\mathbf{n} - f_0 = 0 \tag{F.78}$$

Proof. From (F.29), the normal vector to the tangent at $\mathbf{q}$ can be expressed as

$$\mathbf{n} = \mathbf{V}\mathbf{q} + \mathbf{u} \tag{F.79}$$

$$\implies \mathbf{q} = \mathbf{V}^{-1}\left(\mathbf{n} - \mathbf{u}\right) \tag{F.80}$$

which upon substituting in (D.3) yields

$$\left(\mathbf{n} - \mathbf{u}\right)^\top \mathbf{V}^{-1}\mathbf{V}\mathbf{V}^{-1}\left(\mathbf{n} - \mathbf{u}\right) + 2\mathbf{u}^\top \mathbf{V}^{-1}\left(\mathbf{n} - \mathbf{u}\right) + f = 0 \tag{F.81}$$

which can be simplified to obtain (F.78). $\square$

F.4.5. The normal vectors of the tangents to the conic in (D.3) from a point $\mathbf{h}$ are given by

$$\begin{aligned} \mathbf{n}_1 &= \mathbf{P}\begin{pmatrix} \sqrt{|\lambda_1|} \\ \sqrt{|\lambda_2|} \end{pmatrix} \\ \mathbf{n}_2 &= \mathbf{P}\begin{pmatrix} \sqrt{|\lambda_1|} \\ -\sqrt{|\lambda_2|} \end{pmatrix} \end{aligned} \tag{F.82}$$

where $\lambda_i, \mathbf{P}$ are the eigenparameters of

$$\mathbf{\Sigma} = \left(\mathbf{V}\mathbf{h} + \mathbf{u}\right)\left(\mathbf{V}\mathbf{h} + \mathbf{u}\right)^\top - \left(\mathbf{h}^\top \mathbf{V}\mathbf{h} + 2\mathbf{u}^\top \mathbf{h} + f\right)\mathbf{V}. \tag{F.83}$$

Proof. From (F.26), and (F.30)

$$\left[\mathbf{m}^\top (\mathbf{V}\mathbf{h}+\mathbf{u})\right]^2 - \left(\mathbf{m}^\top \mathbf{V}\mathbf{m}\right)\left(\mathbf{h}^\top \mathbf{V}\mathbf{h} + 2\mathbf{u}^\top \mathbf{h} + f\right) = 0 \tag{F.84}$$

$$\implies \mathbf{m}^\top \left[(\mathbf{V}\mathbf{h}+\mathbf{u})(\mathbf{V}\mathbf{h}+\mathbf{u})^\top - \mathbf{V}\left(\mathbf{h}^\top \mathbf{V}\mathbf{h} + 2\mathbf{u}^\top \mathbf{h} + f\right)\right]\mathbf{m} = 0 \tag{F.85}$$

yielding (F.83). Consequently, from (F.16), (F.82) can be obtained. □