

# Matrix Analysis

G V V Sharma\*

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\*The author is with the Department of Electrical Engineering, Indian Institute of Technology, Hyderabad 502285 India e-mail: gadepall@iith.ac.in. All content in this manual is released under GNU GPL. Free and open source.

## Appendix E

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### Abstract

This manual provides an introduction to vectors and their properties, based on the question papers, year 2020, from Class 10 and 12, CBSE.

#### 1.2 × 1 VECTORS

1.1. Let

$$\mathbf{A} \equiv \vec{A} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \quad (1.1.1)$$

$$\equiv a_1 \vec{i} + a_2 \vec{j}, \quad (1.1.2)$$

$$\mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \quad (1.1.3)$$

be  $2 \times 1$  vectors. Then, the determinant of the  $2 \times 2$  matrix

$$\mathbf{M} = (\mathbf{A} \ \mathbf{B}) \quad (1.1.4)$$

is defined as

$$|\mathbf{M}| = |\mathbf{A} \ \mathbf{B}| \quad (1.1.5)$$

$$= \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1 \quad (1.1.6)$$

1.2. The value of the cross product of two vectors is given by (1.1.5).

1.3. The area of the triangle with vertices  $\mathbf{A}, \mathbf{B}, \mathbf{C}$  is given by the absolute value of

$$\frac{1}{2} |\mathbf{A} - \mathbf{B} \ \mathbf{A} - \mathbf{C}| \quad (1.3.1)$$

1.4. The transpose of  $\mathbf{A}$  is defined as

$$\mathbf{A}^\top = (a_1 \ a_2) \quad (1.4.1)$$

1.5. The *inner product* or *dot product* is defined as

$$\mathbf{A}^\top \mathbf{B} \equiv \mathbf{A} \cdot \mathbf{B} \quad (1.5.1)$$

$$= (a_1 \ a_2) \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = a_1 b_1 + a_2 b_2 \quad (1.5.2)$$

1.6. *norm* of  $\mathbf{A}$  is defined as

$$\|\mathbf{A}\| \equiv |\vec{A}| \quad (1.6.1)$$

$$= \sqrt{\mathbf{A}^\top \mathbf{A}} = \sqrt{a_1^2 + a_2^2} \quad (1.6.2)$$

Thus,

$$\|\lambda \mathbf{A}\| \equiv |\lambda \vec{A}| \quad (1.6.3)$$

$$= |\lambda| \|\mathbf{A}\| \quad (1.6.4)$$

1.7. The distance between the points  $\mathbf{A}$  and  $\mathbf{B}$  is given by

$$\|\mathbf{A} - \mathbf{B}\| \quad (1.7.1)$$

1.8. Let  $\mathbf{x}$  be equidistant from the points  $\mathbf{A}$  and  $\mathbf{B}$ . Then

$$(\mathbf{A} - \mathbf{B})^\top \mathbf{x} = \frac{\|\mathbf{A}\|^2 - \|\mathbf{B}\|^2}{2} \quad (1.8.1)$$

**Solution:**

$$\|\mathbf{x} - \mathbf{A}\| = \|\mathbf{A} - \mathbf{B}\| \quad (1.8.2)$$

$$\implies \|\mathbf{x} - \mathbf{A}\|^2 = \|\mathbf{x} - \mathbf{B}\|^2 \quad (1.8.3)$$

which can be expressed as

$$\begin{aligned} (\mathbf{x} - \mathbf{A})^\top (\mathbf{x} - \mathbf{A}) &= (\mathbf{x} - \mathbf{B})^\top (\mathbf{x} - \mathbf{B}) \\ \implies \|\mathbf{x}\|^2 - 2\mathbf{x}^\top \mathbf{A} + \|\mathbf{A}\|^2 &= \|\mathbf{x}\|^2 - 2\mathbf{x}^\top \mathbf{B} + \|\mathbf{B}\|^2 \end{aligned} \quad (1.8.4)$$

which can be simplified to obtain (1.8.1).

1.9. If  $\mathbf{x}$  lies on the  $x$ -axis and is equidistant from the points  $\mathbf{A}$  and  $\mathbf{B}$ ,

$$\mathbf{x} = x\mathbf{e}_1 \quad (1.9.1)$$

where

$$x = \frac{\|\mathbf{A}\|^2 - \|\mathbf{B}\|^2}{2(\mathbf{A} - \mathbf{B})^\top \mathbf{e}_1} \quad (1.9.2)$$

**Solution:** From (1.8.1).

$$x(\mathbf{A} - \mathbf{B})^\top \mathbf{e}_1 = \frac{\|\mathbf{A}\|^2 - \|\mathbf{B}\|^2}{2} \quad (1.9.3)$$

yielding (1.9.2).

1.10. The angle between two vectors is given by

$$\theta = \cos^{-1} \frac{\mathbf{A}^\top \mathbf{B}}{\|\mathbf{A}\| \|\mathbf{B}\|} \quad (1.10.1)$$

1.11. If two vectors are orthogonal (perpendicular),

$$\mathbf{A}^\top \mathbf{B} = 0 \quad (1.11.1)$$

1.12. The *direction vector* of the line joining two points  $\mathbf{A}, \mathbf{B}$  is given by

$$\mathbf{m} = \mathbf{A} - \mathbf{B} \quad (1.12.1)$$

1.13. The unit vector in the direction of  $\mathbf{m}$  is defined as

$$\frac{\mathbf{m}}{\|\mathbf{m}\|} \quad (1.13.1)$$

1.14. If the direction vector of a line is expressed as

$$\mathbf{m} = \begin{pmatrix} 1 \\ m \end{pmatrix}, \quad (1.14.1)$$

the  $m$  is defined to be the slope of the line.

1.15. The *normal vector* to  $\mathbf{m}$  is defined by

$$\mathbf{m}^\top \mathbf{n} = 0 \quad (1.15.1)$$

1.16. The point  $\mathbf{P}$  that divides the line segment  $AB$  in the ratio  $k : 1$  is given by

$$\mathbf{P} = \frac{k\mathbf{B} + \mathbf{A}}{k + 1} \quad (1.16.1)$$

1.17. The standard basis vectors are defined as

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad (1.17.1)$$

$$\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (1.17.2)$$

## 2 $3 \times 1$ VECTORS

2.1. Let

$$\mathbf{A} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \equiv a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{j}, \quad (2.1.1)$$

$$\mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \quad (2.1.2)$$

and

$$\mathbf{A}_{ij} = \begin{pmatrix} a_i \\ a_j \end{pmatrix}, \quad (2.1.3)$$

$$\mathbf{B}_{ij} = \begin{pmatrix} b_i \\ b_j \end{pmatrix}. \quad (2.1.4)$$

2.2. The *cross product* or *vector product* of  $\mathbf{A}, \mathbf{B}$  is defined as

$$\mathbf{A} \times \mathbf{B} = \begin{pmatrix} \mathbf{A}_{23} & \mathbf{B}_{23} \\ \mathbf{A}_{31} & \mathbf{B}_{31} \\ \mathbf{A}_{12} & \mathbf{B}_{12} \end{pmatrix} \quad (2.2.1)$$

2.3. Verify that

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} \quad (2.3.1)$$

2.4. The area of a triangle is given by

$$\frac{1}{2} \|\mathbf{A} \times \mathbf{B}\| \quad (2.4.1)$$

## 3 MATRICES

### 3.1 Eigenvalues and Eigenvectors

3.1.1. The eigenvalue  $\lambda$  and the eigenvector  $\mathbf{x}$  for a matrix  $\mathbf{A}$  are defined as,

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x} \quad (3.1.1.1)$$

3.1.2. The eigenvalues are calculated by solving the equation

$$f(\lambda) = |\lambda\mathbf{I} - \mathbf{A}| = 0 \quad (3.1.2.1)$$

The above equation is known as the characteristic equation.

3.1.3. According to the Cayley-Hamilton theorem,

$$f(\lambda) = 0 \implies f(\mathbf{A}) = 0 \quad (3.1.3.1)$$

3.1.4. The trace of a square matrix is defined to be the sum of the diagonal elements.

$$\text{tr}(\mathbf{A}) = \sum_{i=1}^N a_{ii}. \quad (3.1.4.1)$$

where  $a_{ii}$  is the  $i$ th diagonal element of the matrix  $\mathbf{A}$ .

3.1.5. The trace of a matrix is equal to the sum of the eigenvalues

$$\text{tr}(\mathbf{A}) = \sum_{i=1}^N \lambda_i \quad (3.1.5.1)$$

### 3.2 Determinants

3.2.1. Let

$$\mathbf{A} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}. \quad (3.2.1.1)$$

be a  $3 \times 3$  matrix. Then,

$$|\mathbf{A}| = a_1 \begin{pmatrix} b_2 & c_2 \\ b_3 & c_3 \end{pmatrix} - a_2 \begin{pmatrix} b_1 & c_1 \\ b_3 & c_3 \end{pmatrix} + a_3 \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}. \quad (3.2.1.2)$$

3.2.2. Let  $\lambda_1, \lambda_2, \dots, \lambda_n$  be the eigenvalues of a matrix  $\mathbf{A}$ . Then, the product of the eigenvalues is equal to the determinant of  $\mathbf{A}$ .

$$|\mathbf{A}| = \prod_{i=1}^n \lambda_i \quad (3.2.2.1)$$

3.2.3.

$$|\mathbf{AB}| = |\mathbf{A}| |\mathbf{B}| \quad (3.2.3.1)$$

3.2.4. If  $\mathbf{A}$  be an  $n \times n$  matrix,

$$|k\mathbf{A}| = k^n |\mathbf{A}| \quad (3.2.4.1)$$

### 3.3 Rank of a Matrix

3.3.1. The rank of a matrix is defined as the number of linearly independent rows. This is also known as the row rank.

3.3.2. Row rank = Column rank.

3.3.3. The rank of a matrix is obtained as the number of nonzero rows obtained after row reduction.

3.3.4. An  $n \times n$  matrix is invertible if and only if its rank is  $n$ .

### 3.4 Inverse of a Matrix

3.4.1. For a  $2 \times 2$  matrix

$$\mathbf{A} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}, \quad (3.4.1.1)$$

the inverse is given by

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|} \begin{pmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{pmatrix}, \quad (3.4.1.2)$$

3.4.2. For higher order matrices, the inverse should be calculated using row operations.

### 3.5 Orthogonality

3.5.1. The rotation matrix is defined as

$$\mathbf{R}_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad \theta \in [0, 2\pi] \quad (3.5.1.1)$$

3.5.2. The rotation matrix is *orthogonal*

$$\mathbf{R}_\theta^\top \mathbf{R}_\theta = \mathbf{R}_\theta \mathbf{R}_\theta^\top = \mathbf{I} \quad (3.5.2.1)$$

3.5.3.

$$\mathbf{m}^\top \mathbf{n} = 0 \implies \mathbf{n} = \mathbf{R}_{\frac{\pi}{2}} \mathbf{m} \quad (3.5.3.1)$$

3.5.4.

$$\mathbf{n}^\top \mathbf{h} = 1 \implies \mathbf{n} = \frac{\mathbf{e}_1}{\mathbf{e}_1^\top \mathbf{h}} + \mu \mathbf{R}_{\frac{\pi}{2}} \mathbf{h}, \quad \mu \in \mathbb{R}. \quad (3.5.4.1)$$

## 4 LINEAR FORMS

### 4.1 Two Dimensions

4.1.1. The equation of a line is given by

$$\mathbf{n}^\top \mathbf{x} = c \quad (4.1.1.1)$$

where  $\mathbf{n}$  is the normal vector of the line.

4.1.2. The equation of a line with normal vector  $\mathbf{n}$  and passing through a point  $\mathbf{A}$  is given by

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{A}) = 0 \quad (4.1.2.1)$$

4.1.3. The equation of a line  $L$  is also given by

$$\mathbf{n}^\top \mathbf{x} = \begin{cases} 0 & \mathbf{0} \in L \\ 1 & \text{otherwise} \end{cases} \quad (4.1.3.1)$$

4.1.4. The parametric equation of a line is given by

$$\mathbf{x} = \mathbf{A} + \lambda \mathbf{m} \quad (4.1.4.1)$$

where  $\mathbf{m}$  is the direction vector of the line and  $\mathbf{A}$  is any point on the line.

4.1.5. Let  $\mathbf{A}$  and  $\mathbf{B}$  be two points on a straight line and let  $\mathbf{P} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$  be any point on it. If  $p_2$  is known, then

$$\mathbf{P} = \mathbf{A} + \frac{p_2 - \mathbf{e}_2^\top \mathbf{A}}{\mathbf{e}_2^\top (\mathbf{B} - \mathbf{A})} (\mathbf{B} - \mathbf{A}) \quad (4.1.5.1)$$

**Solution:** The equation of the line can be expressed in parametric form as

$$\mathbf{x} = \mathbf{A} + \lambda (\mathbf{B} - \mathbf{A}) \quad (4.1.5.2)$$

$$\implies \mathbf{P} = \mathbf{A} + \lambda (\mathbf{B} - \mathbf{A}) \quad (4.1.5.3)$$

$$\implies \mathbf{e}_2^\top \mathbf{P} = \mathbf{e}_2^\top \mathbf{A} + \lambda \mathbf{e}_2^\top (\mathbf{B} - \mathbf{A}) \quad (4.1.5.4)$$

$$\implies p_2 = \mathbf{e}_2^\top \mathbf{A} + \lambda \mathbf{e}_2^\top (\mathbf{B} - \mathbf{A}) \quad (4.1.5.5)$$

$$\text{or, } \lambda = \frac{p_2 - \mathbf{e}_2^\top \mathbf{A}}{\mathbf{e}_2^\top (\mathbf{B} - \mathbf{A})} \quad (4.1.5.6)$$

yielding (4.1.5.1).

4.1.6. The distance from a point  $\mathbf{P}$  to the line in (4.1.1.1) is given by

$$d = \frac{|\mathbf{n}^\top \mathbf{P} - c|}{\|\mathbf{n}\|} \quad (4.1.6.1)$$

**Solution:** Without loss of generality, let  $\mathbf{A}$  be the foot of the perpendicular from  $\mathbf{P}$  to the line in (4.1.4.1). The equation of the normal to (4.1.1.1) can then be expressed as

$$\mathbf{x} = \mathbf{A} + \lambda \mathbf{n} \quad (4.1.6.2)$$

$$\implies \mathbf{P} - \mathbf{A} = \lambda \mathbf{n} \quad (4.1.6.3)$$

$\therefore \mathbf{P}$  lies on (4.1.6.2). From the above, the desired distance can be expressed as

$$d = \|\mathbf{P} - \mathbf{A}\| = |\lambda| \|\mathbf{n}\| \quad (4.1.6.4)$$

From (4.1.6.3),

$$\mathbf{n}^\top (\mathbf{P} - \mathbf{A}) = \lambda \mathbf{n}^\top \mathbf{n} = \lambda \|\mathbf{n}\|^2 \quad (4.1.6.5)$$

$$\implies |\lambda| = \frac{|\mathbf{n}^\top (\mathbf{P} - \mathbf{A})|}{\|\mathbf{n}\|^2} \quad (4.1.6.6)$$

Substituting the above in (4.1.6.4) and using the fact that

$$\mathbf{n}^\top \mathbf{A} = c \quad (4.1.6.7)$$

from (4.1.1.1), yields (4.1.6.1)

4.1.7. The distance from the origin to the line in (4.1.1.1) is given by

$$d = \frac{|c|}{\|\mathbf{n}\|} \quad (4.1.7.1)$$

4.1.8. The distance between the parallel lines

$$\begin{aligned} \mathbf{n}^\top \mathbf{x} &= c_1 \\ \mathbf{n}^\top \mathbf{x} &= c_2 \end{aligned} \quad (4.1.8.1)$$

is given by

$$d = \frac{|c_1 - c_2|}{\|\mathbf{n}\|} \quad (4.1.8.2)$$

4.1.9. The equation of the line perpendicular to (4.1.1.1) and passing through the point  $\mathbf{P}$  is given by

$$\mathbf{m}^\top (\mathbf{x} - \mathbf{P}) = 0 \quad (4.1.9.1)$$

4.1.10. The foot of the perpendicular from  $\mathbf{P}$  to the line in (4.1.1.1) is given by

$$(\mathbf{m} \ \mathbf{n})^\top \mathbf{x} = \begin{pmatrix} \mathbf{m}^\top \mathbf{P} \\ c \end{pmatrix} \quad (4.1.10.1)$$

**Solution:** From (4.1.1.1) and (4.1.2.1) the foot of the perpendicular satisfies the equations

$$\mathbf{n}^\top \mathbf{x} = c \quad (4.1.10.2)$$

$$\mathbf{m}^\top (\mathbf{x} - \mathbf{P}) = 0 \quad (4.1.10.3)$$

where  $\mathbf{m}$  is the direction vector of the given line. Combining the above into a matrix equation results in (4.1.10.1).

## 4.2 Three Dimensions

4.2.1. The area of a triangle with vertices  $\mathbf{A}, \mathbf{B}, \mathbf{C}$  is given by

$$\frac{1}{2} \|(\mathbf{A} - \mathbf{B}) \times (\mathbf{A} - \mathbf{C})\| \quad (4.2.1.1)$$

4.2.2. Points  $\mathbf{A}, \mathbf{B}, \mathbf{C}$  are on a line if

$$\text{rank} \begin{pmatrix} \mathbf{A} \\ \mathbf{B} \\ \mathbf{C} \end{pmatrix} = 1 \quad (4.2.2.1)$$

4.2.3. Points  $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$  form a parallelogram if

$$\text{rank} \begin{pmatrix} \mathbf{A} \\ \mathbf{B} \\ \mathbf{C} \\ \mathbf{D} \end{pmatrix} = 1, \text{rank} \begin{pmatrix} \mathbf{A} \\ \mathbf{B} \\ \mathbf{C} \end{pmatrix} = 2 \quad (4.2.3.1)$$

4.2.4. The equation of a line is given by (4.1.4.1)

4.2.5. The equation of a plane is given by (4.1.1.1)

4.2.6. The distance from the origin to the line in (4.1.1.1) is given by (4.1.7.1)

4.2.7. The distance from a point  $\mathbf{P}$  to the line in (4.1.4.1) is given by

$$d = \|\mathbf{A} - \mathbf{P}\|^2 - \frac{\{\mathbf{m}^\top (\mathbf{A} - \mathbf{P})\}^2}{\|\mathbf{m}\|^2} \quad (4.2.7.1)$$

**Solution:**

$$d(\lambda) = \|\mathbf{A} + \lambda \mathbf{m} - \mathbf{P}\| \quad (4.2.7.2)$$

$$\implies d^2(\lambda) = \|\mathbf{A} + \lambda \mathbf{m} - \mathbf{P}\|^2 \quad (4.2.7.3)$$

which can be simplified to obtain

$$d^2(\lambda) = \lambda^2 \|\mathbf{m}\|^2 + 2\lambda \mathbf{m}^\top (\mathbf{A} - \mathbf{P}) + \|\mathbf{A} - \mathbf{P}\|^2 \quad (4.2.7.4)$$

which is of the form

$$d^2(\lambda) = a\lambda^2 + 2b\lambda + c \quad (4.2.7.5)$$

$$= a \left\{ \left( \lambda + \frac{b}{a} \right)^2 + \left[ \frac{c}{a} - \left( \frac{b}{a} \right)^2 \right] \right\} \quad (4.2.7.6)$$



with

$$a = \|\mathbf{m}\|^2, b = \mathbf{m}^\top (\mathbf{A} - \mathbf{P}), c = \|\mathbf{A} - \mathbf{P}\|^2 \quad (4.2.7.7)$$

which can be expressed as From the above,  $d^2(\lambda)$  is smallest when upon substituting from (4.2.7.7)

$$\lambda + \frac{b}{2a} = 0 \implies \lambda = -\frac{b}{2a} \quad (4.2.7.8)$$

$$= -\frac{\mathbf{m}^\top (\mathbf{A} - \mathbf{P})}{\|\mathbf{m}\|^2} \quad (4.2.7.9)$$

and consequently,

$$d_{\min}(\lambda) = a \left( \frac{c}{a} - \left( \frac{b}{a} \right)^2 \right) \quad (4.2.7.10)$$

$$= c - \frac{b^2}{a} \quad (4.2.7.11)$$

yielding (4.2.7.1) after substituting from (4.2.7.7).

4.2.8. The distance between the parallel planes (4.1.8.1) is given by (4.1.8.2).

4.2.9. The plane

$$\mathbf{n}^\top \mathbf{x} = c \quad (4.2.9.1)$$

contains the line

$$\mathbf{x} = \mathbf{A} + \lambda \mathbf{m} \quad (4.2.9.2)$$

if

$$\mathbf{m}^\top \mathbf{n} = 0 \quad (4.2.9.3)$$

**Solution:** Any point on the line (4.2.9.2) should also satisfy (4.2.9.1). Hence,

$$\mathbf{n}^\top (\mathbf{A} + \lambda \mathbf{m}) = \mathbf{n}^\top \mathbf{A} = c \quad (4.2.9.4)$$

which can be simplified to obtain (4.2.9.3)

4.2.10. The foot of the perpendicular from a point  $\mathbf{P}$  to the plane

$$\mathbf{n}^\top \mathbf{x} = c \quad (4.2.10.1)$$

is given by

**Solution:** The equation of the line perpendicular to the given plane and passing through  $\mathbf{P}$  is

$$\mathbf{x} = \mathbf{P} + \lambda \mathbf{n} \quad (4.2.10.2)$$

From (4.2.13.1), the intersection of the above line with the given plane is

$$\mathbf{x} = \mathbf{P} + \frac{c - \mathbf{n}^\top \mathbf{P}}{\|\mathbf{n}\|^2} \mathbf{n} \quad (4.2.10.3)$$

4.2.11. The image of a point  $\mathbf{P}$  with respect to the plane

$$\mathbf{n}^\top \mathbf{x} = c \quad (4.2.11.1)$$

is given by

$$\mathbf{R} = \mathbf{P} + 2 \frac{c - \mathbf{n}^\top \mathbf{P}}{\|\mathbf{n}\|^2} \mathbf{n} \quad (4.2.11.2)$$

**Solution:** Let  $\mathbf{R}$  be the desired image. Then, substituting the expression for the foot of the perpendicular from  $\mathbf{P}$  to the given plane using (4.2.10.3),

$$\frac{\mathbf{P} + \mathbf{R}}{2} = \mathbf{P} + \frac{c - \mathbf{n}^\top \mathbf{P}}{\|\mathbf{n}\|^2} \quad (4.2.11.3)$$

4.2.12. Let a plane pass through the points  $\mathbf{A}, \mathbf{B}$  and be perpendicular to the plane

$$\mathbf{n}^\top \mathbf{x} = c \quad (4.2.12.1)$$

Then the equation of this plane is given by

$$\mathbf{p}^\top \mathbf{x} = 1 \quad (4.2.12.2)$$

where

$$\mathbf{p} = (\mathbf{A} \ \mathbf{B} \ \mathbf{n})^{-\top} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad (4.2.12.3)$$

**Solution:** From the given information,

$$\mathbf{p}^\top \mathbf{A} = d \quad (4.2.12.4)$$

$$\mathbf{p}^\top \mathbf{B} = d \quad (4.2.12.5)$$

$$\mathbf{p}^\top \mathbf{n} = 0 \quad (4.2.12.6)$$

$\therefore$  the normal vectors to the two planes will also be perpendicular. The system of equations in (4.2.12.6) can be expressed as the matrix equation

$$(\mathbf{A} \ \mathbf{B} \ \mathbf{n})^\top \mathbf{p} = d \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad (4.2.12.7)$$

which yields (4.2.12.3) upon normalising with  $d$ .

4.2.13. The intersection of the line represented by (4.1.4.1) with the plane represented by (4.1.1.1) is given by

$$\mathbf{x} = \mathbf{A} + \frac{c - \mathbf{n}^\top \mathbf{A}}{\mathbf{n}^\top \mathbf{m}} \mathbf{m} \quad (4.2.13.1)$$

**Solution:** From (4.1.4.1) and (4.1.1.1),

$$\mathbf{x} = \mathbf{A} + \lambda \mathbf{m} \quad (4.2.13.2)$$

$$\mathbf{n}^\top \mathbf{x} = c \quad (4.2.13.3)$$

$$\implies \mathbf{n}^\top (\mathbf{A} + \lambda \mathbf{m}) = c \quad (4.2.13.4)$$

which can be simplified to obtain

$$\mathbf{n}^\top \mathbf{A} + \lambda \mathbf{n}^\top \mathbf{m} = c \quad (4.2.13.5)$$

$$\implies \lambda = \frac{c - \mathbf{n}^\top \mathbf{A}}{\mathbf{n}^\top \mathbf{m}} \quad (4.2.13.6)$$

Substituting the above in (4.2.13.4) yields (4.2.13.1).

4.2.14. The foot of the perpendicular from the point  $\mathbf{P}$  to the line represented by (4.1.4.1) is given by

$$\mathbf{x} = \mathbf{A} + \frac{\mathbf{m}^\top (\mathbf{P} - \mathbf{A})}{\|\mathbf{m}\|^2} \mathbf{m} \quad (4.2.14.1)$$

**Solution:** Let the equation of the line be

$$\mathbf{x} = \mathbf{A} + \lambda \mathbf{m} \quad (4.2.14.2)$$

The equation of the plane perpendicular to the given line passing through  $\mathbf{P}$  is given by

$$\mathbf{m}^\top (\mathbf{x} - \mathbf{P}) = 0 \quad (4.2.14.3)$$

$$\implies \mathbf{m}^\top \mathbf{x} = \mathbf{m}^\top \mathbf{P} \quad (4.2.14.4)$$

The desired foot of the perpendicular is the intersection of (4.2.14.2) with (4.2.14.3) which can be obtained from (4.2.13.1) as (4.2.14.1)

4.2.15. The foot of the perpendicular from a point  $\mathbf{P}$  to a plane is  $\mathbf{Q}$ . The equation of the plane is given by

$$(\mathbf{P} - \mathbf{Q})^\top (\mathbf{x} - \mathbf{Q}) = 0 \quad (4.2.15.1)$$

**Solution:** The normal vector to the plane is given by

$$\mathbf{n} = \mathbf{P} - \mathbf{Q} \quad (4.2.15.2)$$

Hence, the equation of the plane is (4.2.15.1).

4.2.16. Let  $\mathbf{A}, \mathbf{B}, \mathbf{C}$  be points on a plane. The equation of the plane is then given by

$$(\mathbf{A} \ \mathbf{B} \ \mathbf{C})^\top \mathbf{n} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad (4.2.16.1)$$

**Solution:** Let the equation of the plane be

$$\mathbf{n}^\top \mathbf{x} = 1 \quad (4.2.16.2)$$

Then

$$\mathbf{n}^\top \mathbf{A} = 1 \quad (4.2.16.3)$$

$$\mathbf{n}^\top \mathbf{B} = 1 \quad (4.2.16.4)$$

$$\mathbf{n}^\top \mathbf{C} = 1 \quad (4.2.16.5)$$

which can be combined to obtain (4.2.16.1).

4.2.17. (Parallelogram Law) Let  $\mathbf{A}, \mathbf{B}, \mathbf{D}$  be three vertices of a parallelogram. Then the vertex  $\mathbf{C}$  is given by

$$\mathbf{C} = \mathbf{B} + \mathbf{D} - \mathbf{A} \quad (4.2.17.1)$$

**Solution:** Shifting  $\mathbf{A}$  to the origin, we obtain a parallelogram with corresponding vertices

$$\mathbf{0}, \mathbf{B} - \mathbf{A}, \mathbf{D} - \mathbf{A} \quad (4.2.17.2)$$

The fourth vertex of this parallelogram is then obtained as

$$(\mathbf{B} - \mathbf{A}) + (\mathbf{D} - \mathbf{A}) = \mathbf{D} + \mathbf{B} - 2\mathbf{A} \quad (4.2.17.3)$$

Shifting the origin to  $\mathbf{A}$ , the fourth vertex is obtained as

$$\mathbf{C} = \mathbf{D} + \mathbf{B} - 2\mathbf{A} + \mathbf{A} \quad (4.2.17.4)$$

$$= \mathbf{D} + \mathbf{B} - \mathbf{A} \quad (4.2.17.5)$$

4.2.18. (Affine Transformation) Let  $\mathbf{A}, \mathbf{C}$ , be opposite vertices of a square. The other two points can be obtained as

$$\mathbf{B} = \frac{\|\mathbf{A} - \mathbf{C}\|}{\sqrt{2}} \mathbf{P} \mathbf{e}_1 + \mathbf{A} \quad (4.2.18.1)$$

$$\mathbf{D} = \frac{\|\mathbf{A} - \mathbf{C}\|}{\sqrt{2}} \mathbf{P} \mathbf{e}_2 + \mathbf{A} \quad (4.2.18.2)$$

where

$$\mathbf{P} = \begin{pmatrix} \cos\left(\theta - \frac{\pi}{4}\right) & \sin\left(\theta - \frac{\pi}{4}\right) \\ \sin\left(\theta - \frac{\pi}{4}\right) & \cos\left(\theta - \frac{\pi}{4}\right) \end{pmatrix} \quad (4.2.18.3)$$

and

$$\cos \theta = \frac{(\mathbf{C} - \mathbf{A})^\top \mathbf{e}_1}{\|\mathbf{A} - \mathbf{C}\| \|\mathbf{e}_1\|} \quad (4.2.18.4)$$

## 5 QUADRATIC FORMS

### 5.1 Definitions

**Definition 1.** The affine transformation is given by

$$\mathbf{x} = \mathbf{P} \mathbf{y} + \mathbf{c} \quad (\text{Affine Transformation}) \quad (5.1.1)$$

where  $\mathbf{P}$  is invertible.

**Definition 2.** The eigenvalue decomposition of a symmetric matrix  $\mathbf{V}$  is given by

$$\mathbf{P}^\top \mathbf{V} \mathbf{P} = \mathbf{D}. \quad (\text{Eigenvalue Decomposition}) \quad (5.1.2)$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad (5.1.3)$$

$$\mathbf{P} = (\mathbf{p}_1 \quad \mathbf{p}_2), \quad \mathbf{P}^\top = \mathbf{P}^{-1}, \quad (5.1.4)$$

### 5.2 The Quadratic Form

**Definition 3.** Let  $\mathbf{q}$  be a point such that the ratio of its distance from a fixed point  $\mathbf{F}$  and the distance ( $d$ ) from a fixed line

$$L : \mathbf{n}^\top \mathbf{x} = c \quad (5.2.1)$$

is constant, given by

$$\frac{\|\mathbf{q} - \mathbf{F}\|}{d} = e \quad (5.2.2)$$

The locus of  $\mathbf{q}$  is known as a conic section. The line  $L$  is known as the directrix and the point  $\mathbf{F}$  is the focus.  $e$  is defined to be the eccentricity of the conic.

- 1) For  $e = 1$ , the conic is a parabola
- 2) For  $e < 1$ , the conic is an ellipse
- 3) For  $e > 1$ , the conic is a hyperbola

**Theorem 5.1.** The equation of a conic with directrix  $\mathbf{n}^\top \mathbf{x} = c$ , eccentricity  $e$  and focus  $\mathbf{F}$  is given by

$$\mathbf{x}^\top \mathbf{V} \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \quad (5.2.1)$$

where

$$\mathbf{V} = \|\mathbf{n}\|^2 \mathbf{I} - e^2 \mathbf{n} \mathbf{n}^\top, \quad (5.2.2)$$

$$\mathbf{u} = ce^2 \mathbf{n} - \|\mathbf{n}\|^2 \mathbf{F}, \quad (5.2.3)$$

$$f = \|\mathbf{n}\|^2 \|\mathbf{F}\|^2 - c^2 e^2 \quad (5.2.4)$$

*Proof.* Using Definition 3 and Lemma 4.1.6.1, for any point  $\mathbf{x}$  on the conic,

$$\|\mathbf{x} - \mathbf{F}\|^2 = e^2 \frac{(\mathbf{n}^\top \mathbf{x} - c)^2}{\|\mathbf{n}\|^2} \quad (5.2.5)$$

$$\implies \|\mathbf{n}\|^2 (\mathbf{x} - \mathbf{F})^\top (\mathbf{x} - \mathbf{F}) = e^2 (\mathbf{n}^\top \mathbf{x} - c)^2 \quad (5.2.6)$$

$$\implies \|\mathbf{n}\|^2 (\mathbf{x}^\top \mathbf{x} - 2\mathbf{F}^\top \mathbf{x} + \|\mathbf{F}\|^2) = e^2 (c^2 + (\mathbf{n}^\top \mathbf{x})^2 - 2c\mathbf{n}^\top \mathbf{x}) \quad (5.2.7)$$

$$= e^2 (c^2 + (\mathbf{x}^\top \mathbf{n} \mathbf{n}^\top \mathbf{x}) - 2c\mathbf{n}^\top \mathbf{x}) \quad (5.2.8)$$

which can be expressed as (5.2.1) after simplification. □

**Theorem 5.2.** *The eccentricity, directrices and foci of (5.2.1) are given by*

$$e = \sqrt{1 - \frac{\lambda_1}{\lambda_2}} \quad (5.2.9)$$

$$\mathbf{n} = \sqrt{\lambda_2} \mathbf{p}_1, \quad (5.2.10)$$

$$c = \begin{cases} \frac{e\mathbf{u}^\top \mathbf{n} \pm \sqrt{e^2 (\mathbf{u}^\top \mathbf{n})^2 - \lambda_2 (e^2 - 1) (\|\mathbf{u}\|^2 - \lambda_2 f)}}{\lambda_2 e (e^2 - 1)} & e \neq 1 \\ \frac{\|\mathbf{u}\|^2 - \lambda_2 f}{2\mathbf{u}^\top \mathbf{n}} & e = 1 \end{cases}$$

$$\mathbf{F} = \frac{ce^2 \mathbf{n} - \mathbf{u}}{\lambda_2} \quad (5.2.11)$$

*Proof.* See Appendix A □

**Theorem 5.3.** (5.2.1) represents

- 1) a parabola for  $|\mathbf{V}| = 0$ ,
- 2) ellipse for  $|\mathbf{V}| > 0$  and
- 3) hyperbola for  $|\mathbf{V}| < 0$ .

*Proof.* From (5.2.9),

$$\frac{\lambda_1}{\lambda_2} = 1 - e^2 \quad (5.2.1)$$

Also,

$$|\mathbf{V}| = \lambda_1 \lambda_2 \quad (5.2.2)$$

yielding Table 3 □

Eccentricity	Conic	Eigenvalue	Determinant
$e = 1$	Parabola	$\lambda_1 = 0$	$\mathbf{V} = 0$
$e < 1$	Ellipse	$\lambda_1 > 0, \lambda_2 > 0$	$\mathbf{V} > 0$
$e > 1$	Hyperbola	$\lambda_1 < 0, \lambda_2 > 0$	$\mathbf{V} < 0$

TABLE 3

### 5.3 Standard Form

**Theorem 5.4.** Using the affine transformation in (5.1.1), the conic in (5.2.1) can be expressed in standard form as

$$\mathbf{y}^\top \left( \frac{\mathbf{D}}{f_0} \right) \mathbf{y} = 1 \quad |\mathbf{V}| \neq 0 \quad (5.3.1)$$

$$\mathbf{y}^\top \mathbf{D} \mathbf{y} = -\eta \mathbf{e}_1^\top \mathbf{y} \quad |\mathbf{V}| = 0 \quad (5.3.2)$$

where

$$f_0 = \mathbf{u}^\top \mathbf{V}^{-1} \mathbf{u} - f \neq 0 \quad (5.3.3)$$

$$\eta = 2\mathbf{u}^\top \mathbf{p}_1 \quad (5.3.4)$$

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (5.3.5)$$

*Proof.* See Appendix B. □

**Corollary 5.5.** For the standard conic,

$$\mathbf{P} = \mathbf{I} \quad (5.3.6)$$

$$\mathbf{u} = \begin{cases} 0 & e \neq 1 \\ \frac{\eta}{2} \mathbf{e}_1 & e = 1 \end{cases} \quad (5.3.7)$$

$$\lambda_1 \begin{cases} = 0 & e = 1 \\ \neq 0 & e \neq 1 \end{cases} \quad (5.3.8)$$

where

$$\mathbf{I} = (\mathbf{e}_1 \quad \mathbf{e}_2) \quad (5.3.9)$$

is the identity matrix.

**Theorem 5.6.**

1) The directrices for the standard conic are given by

$$\mathbf{e}_1^\top \mathbf{y} = \pm \frac{1}{e} \sqrt{\frac{|f_0|}{\lambda_2(1-e^2)}} \quad e \neq 1 \quad (5.3.1)$$

$$\mathbf{e}_1^\top \mathbf{y} = \frac{\eta}{2\lambda_2} \quad e = 1 \quad (5.3.2)$$

2) The foci of the standard ellipse and hyperbola are given by

$$\mathbf{F} = \begin{cases} \pm e \sqrt{\frac{|f_0|}{\lambda_2(1-e^2)}} \mathbf{e}_1 & e \neq 1 \\ -\frac{\eta}{4\lambda_2} \mathbf{e}_1 & e = 1 \end{cases} \quad (5.3.1)$$

*Proof.* See Appendix C. □

#### 5.4 Corollaries-Standard Form

**Corollary 5.7.** *The center of the standard ellipse/hyperbola, defined to be the mid point of the line joining the foci, is the origin.*

**Corollary 5.8.** *The principal (major) axis of the standard ellipse/hyperbola, defined to be the line joining the two foci is the  $x$ -axis.*

*Proof.* From (5.3.1), it is obvious that the line joining the foci passes through the origin. Also, the direction vector of this line is  $\mathbf{e}_1$ . Thus, the principal axis is the  $x$ -axis.  $\square$

**Corollary 5.9.** *The minor axis of the standard ellipse/hyperbola, defined to be the line orthogonal to the  $x$ -axis is the  $y$ -axis.*

**Corollary 5.10.** *The axis of symmetry of the standard parabola, defined to be the line perpendicular to the directrix and passing through the focus, is the  $x$ -axis.*

*Proof.* From (3.0.2.4) and (5.3.1), the axis of the parabola can be expressed using (4.1.2.1) as

$$\mathbf{e}_2^\top \left( \mathbf{y} + \frac{\eta}{4\lambda_2} \mathbf{e}_1 \right) = 0 \quad (5.4.1)$$

$$\implies \mathbf{e}_2^\top \mathbf{y} = 0, \quad (5.4.2)$$

which is the equation of the  $x$ -axis.  $\square$

**Corollary 5.11.** *The point where the parabola intersects its axis of symmetry is called the vertex. For the standard parabola, the vertex is the origin.*

*Proof.* (5.4.2) can be expressed as

$$\mathbf{y} = \alpha \mathbf{e}_1, \quad (5.4.3)$$

using (4.1.2.1). Substituting (5.4.3) in (5.3.2),

$$\alpha^2 \mathbf{e}_1^\top \mathbf{D} \mathbf{e}_1 = -\eta \alpha \mathbf{e}_1^\top \mathbf{e}_1 \quad (5.4.4)$$

$$\implies \alpha = 0, \text{ or, } \mathbf{y} = \mathbf{0}. \quad (5.4.5)$$

$\square$

**Corollary 5.12.** *The focal length of the standard parabola, , defined to be the distance between the vertex and the focus, measured along the axis of symmetry, is  $\left| \frac{\eta}{4\lambda_2} \right|$*

#### 5.5 Corollaries-Quadratic Form

**Corollary 5.13.** *The center/vertex of a conic section are given by*

$$\mathbf{c} = -\mathbf{V}^{-1} \mathbf{u} \quad |\mathbf{V}| \neq 0 \quad (5.5.1)$$

$$\begin{pmatrix} \mathbf{u}^\top + \frac{\eta}{2} \mathbf{p}_1^\top \\ \mathbf{v} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -f \\ \frac{\eta}{2} \mathbf{p}_1 - \mathbf{u} \end{pmatrix} \quad |\mathbf{V}| = 0 \quad (5.5.2)$$

*Proof.* In (5.1.1), substituting  $\mathbf{y} = \mathbf{0}$ , the center/vertex for the quadratic form is obtained as

$$\mathbf{x} = \mathbf{c}, \quad (5.5.3)$$

where  $\mathbf{c}$  is derived as (5.5.1) and (5.5.2) in Appendix B.  $\square$

**Corollary 5.14.** *The equation of the minor and major axes for the ellipse/hyperbola are respectively given by*

$$\mathbf{p}_i^\top (\mathbf{x} - \mathbf{c}) = 0, i = 1, 2 \quad (5.5.4)$$

The axis of symmetry for the parabola is also given by (5.5.4).

*Proof.* From (5.8), the major/symmetry axis for the hyperbola/ellipse/parabola can be expressed using (5.1.1) as

$$\mathbf{e}_2^\top \mathbf{P}^\top (\mathbf{x} - \mathbf{c}) = 0 \quad (5.5.5)$$

$$\implies (\mathbf{P}\mathbf{e}_2)^\top (\mathbf{x} - \mathbf{c}) = 0 \quad (5.5.6)$$

yielding (5.5.4), and the proof for the minor axis is similar.  $\square$

### 5.6 Pair of Straight Lines

**Lemma 5.1** (Asymptotes). *The asymptotes of the hyperbola in (5.3.1), defined to be the lines that do not intersect the hyperbola, are given by*

$$(\sqrt{|\lambda_1|} \pm \sqrt{|\lambda_2|}) \mathbf{y} = 0 \quad (5.6.1)$$

*Proof.* From (5.3.1), it is obvious that the pair of lines represented by

$$\mathbf{y}^\top \mathbf{D} \mathbf{y} = 0 \quad (5.6.2)$$

do not intersect the conic

$$\mathbf{y}^\top \mathbf{D} \mathbf{y} = f_0 \quad (5.6.3)$$

Thus, (5.6.2) represents the asymptotes of the hyperbola in (5.3.1) and can be expressed as

$$\lambda_1 y_1^2 + \lambda_2 y_2^2 = 0, \quad (5.6.4)$$

which can then be simplified to obtain (5.6.1).  $\square$

**Corollary 5.15.** (5.2.1) represents a pair of straight lines if

$$\mathbf{u}^\top \mathbf{V}^{-1} \mathbf{u} - f = 0 \quad (5.6.5)$$

**Theorem 5.16.** (5.2.1) represents a pair of straight lines if the matrix

$$\begin{pmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^\top & f \end{pmatrix} \quad (5.6.6)$$

is singular.

*Proof.* Let

$$\begin{pmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^\top & f \end{pmatrix} \mathbf{x} = \mathbf{0} \quad (5.6.7)$$

Expressing

$$\mathbf{x} = \begin{pmatrix} \mathbf{y} \\ y_3 \end{pmatrix}, \quad (5.6.8)$$

$$\begin{pmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^\top & f \end{pmatrix} \begin{pmatrix} \mathbf{y} \\ y_3 \end{pmatrix} = \mathbf{0} \quad (5.6.9)$$

$$\implies \mathbf{V}\mathbf{y} + y_3 \mathbf{u} = \mathbf{0} \quad \text{and} \quad (5.6.10)$$

$$\mathbf{u}^\top \mathbf{y} + f y_3 = 0 \quad (5.6.11)$$



From (5.6.10) we obtain,

$$\mathbf{y}^\top \mathbf{V} \mathbf{y} + y_3 \mathbf{y}^\top \mathbf{u} = 0 \quad (5.6.12)$$

$$\implies \mathbf{y}^\top \mathbf{V} \mathbf{y} + y_3 \mathbf{u}^\top \mathbf{y} = 0 \quad (5.6.13)$$

yielding (5.6.5) upon substituting from (5.6.11).  $\square$

**Corollary 5.17.** *Using the affine transformation, (5.6.1) can be expressed as the lines*

$$(\sqrt{|\lambda_1|} \pm \sqrt{|\lambda_2|}) \mathbf{P}^\top (\mathbf{x} - \mathbf{c}) = 0 \quad (5.6.14)$$

**Corollary 5.18.** *The angle between the asymptotes can be expressed as*

$$\cos \theta = \frac{|\lambda_1| - |\lambda_2|}{|\lambda_1| + |\lambda_2|} \quad (5.6.15)$$

*Proof.* The normal vectors of the lines in (5.6.14) are

$$\begin{aligned} \mathbf{n}_1 &= \mathbf{P} \begin{pmatrix} \sqrt{|\lambda_1|} \\ \sqrt{|\lambda_2|} \end{pmatrix} \\ \mathbf{n}_2 &= \mathbf{P} \begin{pmatrix} \sqrt{|\lambda_1|} \\ -\sqrt{|\lambda_2|} \end{pmatrix} \end{aligned} \quad (5.6.16)$$

The angle between the asymptotes is given by

$$\cos \theta = \frac{\mathbf{n}_1^\top \mathbf{n}_2}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} \quad (5.6.17)$$

The orthogonal matrix  $\mathbf{P}$  preserves the norm, i.e.

$$\|\mathbf{n}_1\| = \left\| \mathbf{P} \begin{pmatrix} \sqrt{|\lambda_1|} \\ \sqrt{|\lambda_2|} \end{pmatrix} \right\| = \left\| \begin{pmatrix} \sqrt{|\lambda_1|} \\ \sqrt{|\lambda_2|} \end{pmatrix} \right\| \quad (5.6.18)$$

$$= \sqrt{|\lambda_1| + |\lambda_2|} = \|\mathbf{n}_2\| \quad (5.6.19)$$

It is easy to verify that

$$\mathbf{n}_1^\top \mathbf{n}_2 = |\lambda_1| - |\lambda_2| \quad (5.6.20)$$

Thus, the angle between the asymptotes is obtained from (5.6.17) as (5.6.15).  $\square$

## 5.7 Intersection of Conics

**Lemma 5.2.** *Let*

$$\mathbf{x}^\top \mathbf{V}_i \mathbf{x} + 2\mathbf{u}_i^\top \mathbf{x} + f_i = 0, \quad i = 1, 2 \quad (5.7.1)$$

*be the equation of two conics. The locus of their intersection is a pair of straight lines if*

$$\begin{vmatrix} \mathbf{V}_1 + \mu \mathbf{V}_2 & \mathbf{u}_1 + \mu \mathbf{u}_2 \\ (\mathbf{u}_1 + \mu \mathbf{u}_2)^\top & f \end{vmatrix} = 0, \quad |\mathbf{V}_1 + \mu \mathbf{V}_2| < 0 \quad (5.7.2)$$

*Proof.* The intersection of the conics in (5.7.1) is given by the curve

$$\mathbf{x}^\top (\mathbf{V}_1 + \mu \mathbf{V}_2) \mathbf{x} + 2(\mathbf{u}_1 + \mu \mathbf{u}_2)^\top \mathbf{x} + f_1 + \mu f_2 = 0, \quad (5.7.3)$$

which, from Theorem 5.16 represents a pair of straight lines if (5.7.2) is satisfied.  $\square$

**Corollary 5.19.** *The points of intersection of the conics in (5.7.1) are the points of the intersection of the lines in (5.7.3).*

## 6 CONIC LINES

### 6.1 Chords of a Conic

**Theorem 6.1** (Chord). *The points of intersection of the line*

$$L : \mathbf{x} = \mathbf{q} + \mu \mathbf{m} \quad \mu \in \mathbb{R} \quad (6.1.1)$$

*with the conic section in (5.2.1) are given by*

$$\mathbf{x}_i = \mathbf{q} + \mu_i \mathbf{m} \quad (6.1.2)$$

where

$$\mu_i = \frac{1}{\mathbf{m}^\top \mathbf{V} \mathbf{m}} \left( -\mathbf{m}^\top (\mathbf{V} \mathbf{q} + \mathbf{u}) \pm \sqrt{[\mathbf{m}^\top (\mathbf{V} \mathbf{q} + \mathbf{u})]^2 - (\mathbf{q}^\top \mathbf{V} \mathbf{q} + 2\mathbf{u}^\top \mathbf{q} + f)(\mathbf{m}^\top \mathbf{V} \mathbf{m})} \right) \quad (6.1.3)$$

*Proof.* Substituting (6.1.1) in (5.2.1),

$$(\mathbf{q} + \mu \mathbf{m})^\top \mathbf{V} (\mathbf{q} + \mu \mathbf{m}) + 2\mathbf{u}^\top (\mathbf{q} + \mu \mathbf{m}) + f = 0 \quad (6.1.4)$$

$$\implies \mu^2 \mathbf{m}^\top \mathbf{V} \mathbf{m} + 2\mu \mathbf{m}^\top (\mathbf{V} \mathbf{q} + \mathbf{u}) + \mathbf{q}^\top \mathbf{V} \mathbf{q} + 2\mathbf{u}^\top \mathbf{q} + f = 0 \quad (6.1.5)$$

Solving the above quadratic in (6.1.5) yields (6.1.3).  $\square$

**Corollary 6.2.** *If  $L$  in (6.1.1) touches (5.2.1) at exactly one point  $\mathbf{q}$ ,*

$$\mathbf{m}^\top (\mathbf{V} \mathbf{q} + \mathbf{u}) = 0 \quad (6.1.6)$$

*Proof.* In this case, (6.1.5) has exactly one root. Hence, in (6.1.3)

$$[\mathbf{m}^\top (\mathbf{V} \mathbf{q} + \mathbf{u})]^2 - (\mathbf{m}^\top \mathbf{V} \mathbf{m})(\mathbf{q}^\top \mathbf{V} \mathbf{q} + 2\mathbf{u}^\top \mathbf{q} + f) = 0 \quad (6.1.7)$$

$\therefore \mathbf{q}$  is the point of contact,  $\mathbf{q}$  satisfies (5.2.1) and

$$\mathbf{q}^\top \mathbf{V} \mathbf{q} + 2\mathbf{u}^\top \mathbf{q} + f = 0 \quad (6.1.8)$$

Substituting (6.1.8) in (6.1.7) and simplifying, we obtain (6.1.6).  $\square$

**Theorem 6.3.** *The length of the chord in (6.1.1) is given by*

$$\frac{2\sqrt{[\mathbf{m}^\top (\mathbf{V} \mathbf{q} + \mathbf{u})]^2 - (\mathbf{q}^\top \mathbf{V} \mathbf{q} + 2\mathbf{u}^\top \mathbf{q} + f)(\mathbf{m}^\top \mathbf{V} \mathbf{m})}}{\mathbf{m}^\top \mathbf{V} \mathbf{m}} \|\mathbf{m}\| \quad (6.1.9)$$

*Proof.* The distance between the points in (6.1.2) is given by

$$\|\mathbf{x}_1 - \mathbf{x}_2\| = |\mu_1 - \mu_2| \|\mathbf{m}\| \quad (6.1.10)$$

Substituting  $\mu_i$  from (6.1.3) in (6.1.10) yields (6.1.9).  $\square$

**Theorem 6.4.** *The affine transform for the conic section, preserves the norm. This implies that the length of any chord of a conic is invariant to translation and/or rotation.*

*Proof.* Let

$$\mathbf{x}_i = \mathbf{P} \mathbf{y}_i + \mathbf{c} \quad (6.1.11)$$

be any two points on the conic. Then the distance between the points is given by

$$\|\mathbf{x}_1 - \mathbf{x}_2\| = \|\mathbf{P}(\mathbf{y}_1 - \mathbf{y}_2)\| \quad (6.1.12)$$

which can be expressed as

$$\|\mathbf{x}_1 - \mathbf{x}_2\|^2 = (\mathbf{y}_1 - \mathbf{y}_2)^\top \mathbf{P}^\top \mathbf{P} (\mathbf{y}_1 - \mathbf{y}_2) \quad (6.1.13)$$

$$= \|\mathbf{y}_1 - \mathbf{y}_2\|^2 \quad (6.1.14)$$

since

$$\mathbf{P}^\top \mathbf{P} = \mathbf{I} \quad (6.1.15)$$

□

**Corollary 6.5.** *For the standard hyperbola/ellipse, the length of the major axis is*

$$2\sqrt{\left|\frac{f_0}{\lambda_1}\right|} \quad (6.1.16)$$

and the minor axis is

$$2\sqrt{\left|\frac{f_0}{\lambda_2}\right|} \quad (6.1.17)$$

*Proof.* See Appendix D

□

**Theorem 6.6** (latus rectum). *The latus rectum of a conic section is the chord that passes through the focus and is perpendicular to the major axis. The length of the latus rectum for a conic is given by*

$$l = \begin{cases} 2\frac{\sqrt{|f_0\lambda_1|}}{\lambda_2} & e \neq 1 \\ \frac{\eta}{\lambda_2} & e = 1 \end{cases} \quad (6.1.18)$$

*Proof.* See Appendix E.

□

## 6.2 Tangent and Normal

**Theorem 6.7** (Tangent). *Given the point of contact  $\mathbf{q}$ , the equation of a tangent to (5.2.1) is*

$$(\mathbf{V}\mathbf{q} + \mathbf{u})^\top \mathbf{x} + \mathbf{u}^\top \mathbf{q} + f = 0 \quad (6.2.1)$$

*Proof.* The normal vector is obtained from (6.1.6) and (1.15.1) as

$$\kappa \mathbf{n} = \mathbf{V}\mathbf{q} + \mathbf{u}, \kappa \in \mathbb{R} \quad (6.2.2)$$

From (6.2.2) and (4.1.2.1), the equation of the tangent is

$$(\mathbf{V}\mathbf{q} + \mathbf{u})^\top (\mathbf{x} - \mathbf{q}) = 0 \quad (6.2.3)$$

$$\implies (\mathbf{V}\mathbf{q} + \mathbf{u})^\top \mathbf{x} - \mathbf{q}^\top \mathbf{V}\mathbf{q} - \mathbf{u}^\top \mathbf{q} = 0 \quad (6.2.4)$$

which, upon substituting from (6.1.8) and simplifying yields (6.2.1)

□

**Theorem 6.8.** *If  $\mathbf{V}^{-1}$  exists, given the normal vector  $\mathbf{n}$ , the tangent points of contact to (5.2.1) are given by*

$$\mathbf{q}_i = \mathbf{V}^{-1}(\kappa_i \mathbf{n} - \mathbf{u}), i = 1, 2$$

$$\text{where } \kappa_i = \pm \sqrt{\frac{f_0}{\mathbf{n}^\top \mathbf{V}^{-1} \mathbf{n}}} \quad (6.2.5)$$

*Proof.* From (6.2.2),

$$\mathbf{q} = \mathbf{V}^{-1}(\kappa \mathbf{n} - \mathbf{u}), \quad \kappa \in \mathbb{R} \quad (6.2.6)$$

Substituting (6.2.6) in (6.1.8),

$$(\kappa \mathbf{n} - \mathbf{u})^\top \mathbf{V}^{-1} (\kappa \mathbf{n} - \mathbf{u}) + 2\mathbf{u}^\top \mathbf{V}^{-1} (\kappa \mathbf{n} - \mathbf{u}) + f = 0 \quad (6.2.7)$$

$$\implies \kappa^2 \mathbf{n}^\top \mathbf{V}^{-1} \mathbf{n} - \mathbf{u}^\top \mathbf{V}^{-1} \mathbf{u} + f = 0 \quad (6.2.8)$$

$$\text{or, } \kappa = \pm \sqrt{\frac{f_0}{\mathbf{n}^\top \mathbf{V}^{-1} \mathbf{n}}} \quad (6.2.9)$$

Substituting (6.2.9) in (6.2.6) yields (6.2.5).  $\square$

**Theorem 6.9.** *If  $\mathbf{V}$  is not invertible, given the normal vector  $\mathbf{n}$ , the point of contact to (5.2.1) is given by the matrix equation*

$$\begin{pmatrix} (\mathbf{u} + \kappa \mathbf{n})^\top \\ \mathbf{V} \end{pmatrix} \mathbf{q} = \begin{pmatrix} -f \\ \kappa \mathbf{n} - \mathbf{u} \end{pmatrix} \quad (6.2.10)$$

$$\text{where } \kappa = \frac{\mathbf{p}_1^\top \mathbf{u}}{\mathbf{p}_1^\top \mathbf{n}}, \quad \mathbf{V} \mathbf{p}_1 = 0 \quad (6.2.11)$$

*Proof.* If  $\mathbf{V}$  is non-invertible, it has a zero eigenvalue. If the corresponding eigenvector is  $\mathbf{p}_1$ , then,

$$\mathbf{V} \mathbf{p}_1 = 0 \quad (6.2.12)$$

From (6.2.2),

$$\kappa \mathbf{n} = \mathbf{V} \mathbf{q} + \mathbf{u}, \quad \kappa \in \mathbb{R} \quad (6.2.13)$$

$$\implies \kappa \mathbf{p}_1^\top \mathbf{n} = \mathbf{p}_1^\top \mathbf{V} \mathbf{q} + \mathbf{p}_1^\top \mathbf{u} \quad (6.2.14)$$

$$\text{or, } \kappa \mathbf{p}_1^\top \mathbf{n} = \mathbf{p}_1^\top \mathbf{u}, \quad \because \mathbf{p}_1^\top \mathbf{V} = 0, \quad (\text{from (6.2.12)}) \quad (6.2.15)$$

yielding  $\kappa$  in (6.2.11). From (6.2.13),

$$\kappa \mathbf{q}^\top \mathbf{n} = \mathbf{q}^\top \mathbf{V} \mathbf{q} + \mathbf{q}^\top \mathbf{u} \quad (6.2.16)$$

$$\implies \kappa \mathbf{q}^\top \mathbf{n} = -f - \mathbf{q}^\top \mathbf{u} \quad \text{from (6.1.8),} \quad (6.2.17)$$

$$\text{or, } (\kappa \mathbf{n} + \mathbf{u})^\top \mathbf{q} = -f \quad (6.2.18)$$

(6.2.13) can be expressed as

$$\mathbf{V} \mathbf{q} = \kappa \mathbf{n} - \mathbf{u}. \quad (6.2.19)$$

(6.2.18) and (6.2.19) clubbed together result in (6.2.10).  $\square$

**Theorem 6.10.** *The normal vectors of the tangents from a point  $\mathbf{h}$  to the conic in (5.2.1) are given by*

$$\mathbf{n} = \frac{\mathbf{e}_1}{\mathbf{e}_1^\top \mathbf{h}} + \mu_i \mathbf{R} \mathbf{h}, \quad (6.2.20)$$

where

$$\mathbf{R} = \mathbf{R}_{\frac{\pi}{2}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad (6.2.21)$$

and  $\mu_i$  are obtained from (6.1.3) by substituting

$$\mathbf{m} = \mathbf{R} \mathbf{h}, \mathbf{u} = 0, \mathbf{q} = \frac{\mathbf{e}_1}{\mathbf{e}_1^\top \mathbf{h}} \quad (6.2.22)$$

and replacing  $\mathbf{V}$  with  $\mathbf{V}^{-1}$ .

*Proof.* From (6.1.6), the normal vector to the tangent at  $\mathbf{q}$  can be expressed as

$$\mathbf{n} = \mathbf{V}\mathbf{q} + \mathbf{u} \quad (6.2.23)$$

$$\implies \mathbf{q} = \mathbf{V}^{-1}(\mathbf{n} - \mathbf{u}) \quad (6.2.24)$$

which upon substituting in (5.2.1) yields

$$(\mathbf{n} - \mathbf{u})^\top \mathbf{V}^{-1} \mathbf{V} \mathbf{V}^{-1} (\mathbf{n} - \mathbf{u}) + 2\mathbf{u}^\top \mathbf{V}^{-1} (\mathbf{n} - \mathbf{u}) + f = 0 \quad (6.2.25)$$

$$\implies \mathbf{n}^\top \mathbf{V}^{-1} \mathbf{n} - f_0 = 0 \quad (6.2.26)$$

Since the tangents pass through the point  $\mathbf{h}$ ,

$$\mathbf{n}^\top \mathbf{h} = 1 \quad (6.2.27)$$

and from (3.5.4.1), we obtain (6.2.20).  $\square$

**Theorem 6.11.** *The normal vectors of the tangents from a point  $\mathbf{h}$  to the conic in (5.2.1) are given by*

$$\begin{aligned} \mathbf{n}_1 &= \mathbf{P} \begin{pmatrix} \sqrt{|\lambda_1|} \\ \sqrt{|\lambda_2|} \end{pmatrix} \\ \mathbf{n}_2 &= \mathbf{P} \begin{pmatrix} \sqrt{|\lambda_1|} \\ -\sqrt{|\lambda_2|} \end{pmatrix} \end{aligned} \quad (6.2.28)$$

where  $\lambda_i, \mathbf{P}$  are the eigenparameters of

$$\Sigma = (\mathbf{V}\mathbf{h} + \mathbf{u})(\mathbf{V}\mathbf{h} + \mathbf{u})^\top - \mathbf{V}(\mathbf{h}^\top \mathbf{V}\mathbf{h} + 2\mathbf{u}^\top \mathbf{h} + f). \quad (6.2.29)$$

*Proof.* From (6.1.3), and (6.1.7)

$$[\mathbf{m}^\top (\mathbf{V}\mathbf{h} + \mathbf{u})]^2 - (\mathbf{m}^\top \mathbf{V}\mathbf{m})(\mathbf{h}^\top \mathbf{V}\mathbf{h} + 2\mathbf{u}^\top \mathbf{h} + f) = 0 \quad (6.2.30)$$

$$\implies \mathbf{m}^\top \left[ (\mathbf{V}\mathbf{h} + \mathbf{u})(\mathbf{V}\mathbf{h} + \mathbf{u})^\top - \mathbf{V}(\mathbf{h}^\top \mathbf{V}\mathbf{h} + 2\mathbf{u}^\top \mathbf{h} + f) \right] \mathbf{m} = 0 \quad (6.2.31)$$

yielding (6.2.29). Consequently, from (5.6.16), (6.2.28) can be obtained.  $\square$

## 7 EXAMPLES

### 7.1 Loney

**Example 7.1** (parabola). *To show that*

$$9x^2 - 24xy + 16y^2 - 18x - 101y + 19 = 0 \quad (7.1.1)$$

*is the equation of a parabola with latus rectum of length 3, vertex*

$$\frac{1}{25} \begin{pmatrix} -29 \\ 25 \end{pmatrix} \quad (7.1.2)$$

*and axis*

$$3x - 4y + 7 = 0 \quad (7.1.3)$$

**Solution:** Comparing (7.1.1) with (5.2.1),

$$\mathbf{V} = \begin{pmatrix} 9 & -12 \\ -12 & 16 \end{pmatrix} \quad (7.1.4)$$

$$\mathbf{u} = -\frac{1}{2} \begin{pmatrix} 18 \\ 101 \end{pmatrix} \quad (7.1.5)$$

$$f = 19 \quad (7.1.6)$$

The eigenvalues of  $\mathbf{V}$  are obtained as

$$|\lambda \mathbf{I} - \mathbf{V}| = 0 \quad (7.1.7)$$

$$\implies \lambda_1 = 0, \lambda_2 = 25 \quad (7.1.8)$$

Since the  $\mathbf{V}$  matrix has a 0 eigenvalue, (7.1.1) is a parabola. The eigenvector corresponding to the 0 eigenvalue is given by

$$\begin{pmatrix} 9 & -12 \\ -12 & 16 \end{pmatrix} \mathbf{p}_1 = \mathbf{0} \quad (7.1.9)$$

yielding

$$\mathbf{p}_1 = \frac{1}{5} \begin{pmatrix} 4 \\ 3 \end{pmatrix} \quad (7.1.10)$$

Substituting from (7.1.4), (7.1.5), (7.1.6), (7.1.10) and (7.1.8) in (5.3.3), (5.3.4) and (6.1.18) the latus rectum is obtained as

$$\eta = \frac{|2\mathbf{u}^\top \mathbf{p}_1|}{\lambda_2} = 3 \quad (7.1.11)$$

The vertex of the parabola is obtained from (5.5.2) as

$$\begin{pmatrix} -39 & -73 \\ 9 & -12 \\ -12 & 16 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -19 \\ -21 \\ 28 \end{pmatrix} \quad (7.1.12)$$

yielding (7.1.2). The second eigenvector of  $\mathbf{V}$  is orthogonal to  $\mathbf{p}_1$  and obtained as

$$\mathbf{p}_2 = \frac{1}{5} \begin{pmatrix} 3 \\ -4 \end{pmatrix} \quad (7.1.13)$$

Substituting from (7.1.2) and (7.1.12) in (5.5.4), the equation of the axis of symmetry for the parabola can be expressed as (7.1.3).

**Example 7.2** (ellipse). *To show that the equation*

$$14x^2 - 4xy + 11y^2 - 44x - 58y + 71 = 0 \quad (7.1.14)$$

*represents an ellipse with centre*

$$\mathbf{c} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad (7.1.15)$$

*and lengths of semi-axes*

$$\sqrt{6} \text{ and } 2 \quad (7.1.16)$$

**Solution:** The parameters for (7.1.14), are

$$\mathbf{V} = \begin{pmatrix} 14 & -2 \\ -2 & 11 \end{pmatrix} \quad (7.1.17)$$

$$\mathbf{u} = \begin{pmatrix} 22 \\ 29 \end{pmatrix} \quad (7.1.18)$$

$$f = 71 \quad (7.1.19)$$

Since

$$|\mathbf{V}| = 150 > 0, \quad (7.1.20)$$

(7.1.14) is an ellipse. Substituting from (7.1.17) and (7.1.18) in (5.5.1), the center of the ellipse is obtained as (7.1.15). Also, the eigenvalues of  $\mathbf{V}$  are

$$\lambda_1 = 10, \lambda_2 = 15 \quad (7.1.21)$$

Substituting from (7.1.17), (7.1.18) and (7.1.19) in (5.5.4),

$$f_0 = 60 \quad (7.1.22)$$

Substituting from (7.1.21) and (7.1.22) in (6.1.16) and (6.1.17), the lengths of the semi-axes are obtained as (7.1.16).

**Example 7.3** (hyperbola). *To show that the equation*

$$x^2 - 3xy + y^2 + 10x - 10y + 21 = 0 \quad (7.1.23)$$

represents a hyperbola with centre

$$\mathbf{c} = \begin{pmatrix} -2 \\ 2 \end{pmatrix} \quad (7.1.24)$$

and length of semi-axes

$$\sqrt{2} \text{ and } \sqrt{\frac{2}{5}} \quad (7.1.25)$$

**Solution:** The conic parameters for (7.1.23), are

$$\mathbf{V} = \frac{1}{2} \begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix} \quad (7.1.26)$$

$$\mathbf{u} = \begin{pmatrix} 5 \\ -5 \end{pmatrix} \quad (7.1.27)$$

$$f = 21 \quad (7.1.28)$$

Since

$$|\mathbf{V}| = -\frac{5}{4} < 0, \quad (7.1.29)$$

(7.1.23) is a hyperbola. Substituting from (7.1.26) and (7.1.26) in (5.5.1), the center of the hyperbola is obtained as (7.1.15). Also, the eigenvalues of  $\mathbf{V}$  are

$$\lambda_1 = -\frac{1}{2}, \lambda_2 = \frac{5}{2} \quad (7.1.30)$$

From (5.5.4),

$$f_0 = -1 \quad (7.1.31)$$

Substituting from (7.1.30) and (7.1.31) in (6.1.16) and (6.1.17), the lengths of the semi-axes are then obtained as (7.1.25).

**Example 7.4** (tangents). *To show that the tangents to the curve*

$$x^2 + 4xy + 3y^2 - 5x - 6y + 3 = 0 \quad (7.1.32)$$

*parallel to the line*

$$x + 4y + c = 0 \quad (7.1.33)$$

are

$$\begin{aligned} x + 4y - 5 &= 0 \\ x + 4y - 8 &= 0 \end{aligned} \quad (7.1.34)$$

**Solution:** The conic parameters for (7.1.32) are

$$\mathbf{V} = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \quad (7.1.35)$$

$$\mathbf{u} = \begin{pmatrix} -\frac{5}{2} \\ -3 \end{pmatrix} \quad (7.1.36)$$

$$f = 3 \quad (7.1.37)$$

Since

$$|\mathbf{V}| = -1 < 0, \quad (7.1.38)$$

(7.1.32) is a hyperbola. From (7.1.33), the normal vector to the tangent is

$$\mathbf{n} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \quad (7.1.39)$$

The equation of the tangent can be expressed as

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{q}_i) = 0 \quad (7.1.40)$$

where  $\mathbf{q}_i$  are the points of contact. Comparing the above with (7.1.33),

$$c = -\mathbf{n}^\top \mathbf{q}_i \quad (7.1.41)$$

which, upon substituting from (6.2.5) can be expressed as

$$c = -\mathbf{n}^\top \{ \mathbf{V}^{-1} (\kappa_i \mathbf{n} - \mathbf{u}) \} \quad (7.1.42)$$

$$= -\mathbf{n}^\top \left\{ \mathbf{V}^{-1} \left( \pm \sqrt{\frac{f_0}{\mathbf{n}^\top \mathbf{V}^{-1} \mathbf{n}}} \mathbf{n} - \mathbf{u} \right) \right\} \quad (7.1.43)$$

$$= \pm \sqrt{f_0 \mathbf{n}^\top \mathbf{V}^{-1} \mathbf{n}} + \mathbf{n}^\top \mathbf{V}^{-1} \mathbf{u}. \quad (7.1.44)$$

Substituting from (7.1.35), (7.1.36) and (7.1.37) in (5.3.3),

$$\mathbf{f}_0 = -\frac{3}{4} \quad (7.1.45)$$

Substituting from (7.1.35), (7.1.36), (7.1.39) and (7.1.45) in (6.2.9),

$$c = -5 \text{ or } c = -8 \quad (7.1.46)$$

yielding (7.1.34).



## 7.2 Miscellaneous

7.2.1. Given unit basis vectors  $\mathbf{a}, \mathbf{b}$ , with angle  $\theta$  between them, the locus of the coordinates of a unit vector  $\mathbf{c}$  in the space spanned by  $\mathbf{a}, \mathbf{b}$  is given by

$$\mathbf{x}^\top \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \mathbf{x} = 1 \quad (7.2.1.1)$$

with  $\rho = \cos \theta$ .

**Solution:** Let

$$\mathbf{c} = x_1 \mathbf{a} + x_2 \mathbf{b} = \begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix} \mathbf{x} \quad (7.2.1.2)$$

Then,

$$\|\mathbf{c}\|^2 = \mathbf{x}^\top \begin{pmatrix} \mathbf{a}^\top \\ \mathbf{b}^\top \end{pmatrix} \begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix} \mathbf{x} \quad (7.2.1.3)$$

$$= \mathbf{x}^\top \begin{pmatrix} 1 & \mathbf{a}^\top \mathbf{b} \\ \mathbf{a}^\top \mathbf{b} & 1 \end{pmatrix} \mathbf{x} \quad (7.2.1.4)$$

which can be expressed as (7.2.1.1).

7.2.2. Given the coordinates of  $\mathbf{c}$ , the angle  $\theta$  between the basis vectors is given by

$$\rho = \frac{1 - \|\mathbf{x}\|^2}{\mathbf{x}^\top \mathbf{R} \mathbf{x}} \quad (7.2.2.1)$$

where

$$\mathbf{R} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (7.2.2.2)$$

**Solution:** Let

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad (7.2.2.3)$$

For

$$\mathbf{V} = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}, \mathbf{u} = 0, f = -1 \quad (7.2.2.4)$$

in (5.2.1),

Since

$$|\mathbf{V}| = 1 - \rho^2, 0 < |\mathbf{V}| < 1, \quad (7.2.2.5)$$

(7.2.1.1) represents the equation of an ellipse. Using eigenvalue decomposition,

$$\mathbf{V} = \mathbf{P}^\top \mathbf{D} \mathbf{P} \quad (7.2.2.6)$$

where

$$\mathbf{P} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 1+\rho & 0 \\ 0 & 1-\rho \end{pmatrix} \quad (7.2.2.7)$$

Using the affine transformation,

$$\mathbf{x} = \mathbf{P} \mathbf{y} \quad (7.2.2.8)$$

(7.2.1.1) can be expressed as

$$\mathbf{y}^\top \mathbf{D} \mathbf{y} = 1 \implies y_1^2 (1 + \rho) + y_2^2 (1 - \rho) = 1 \quad (7.2.2.9)$$

which can be simplified to obtain

$$\rho = \frac{1 - y_1^2 - y_2^2}{y_1^2 - y_2^2} \quad (7.2.2.10)$$

$$= \frac{1 - \|\mathbf{y}\|^2}{\mathbf{y}^\top \mathbf{Q} \mathbf{y}} \quad (7.2.2.11)$$

where

$$\mathbf{Q} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (7.2.2.12)$$

From (7.2.2.8), (7.2.2.11) can be expressed as (7.2.2.1)

$$\because \|\mathbf{a}\| = \|\mathbf{b}\| = 1. \quad (7.2.2.13)$$

#### APPENDIX A

From (5.2.2), using the fact that  $\mathbf{V}$  is symmetric with  $\mathbf{V} = \mathbf{V}^\top$ ,

$$\mathbf{V}^\top \mathbf{V} = (\|\mathbf{n}\|^2 \mathbf{I} - e^2 \mathbf{n} \mathbf{n}^\top)^\top (\|\mathbf{n}\|^2 \mathbf{I} - e^2 \mathbf{n} \mathbf{n}^\top) \quad (1.0.2.1)$$

$$\implies \mathbf{V}^2 = \|\mathbf{n}\|^4 \mathbf{I} + e^4 \mathbf{n} \mathbf{n}^\top \mathbf{n} \mathbf{n}^\top - 2e^2 \|\mathbf{n}\|^2 \mathbf{n} \mathbf{n}^\top \quad (1.0.2.2)$$

$$= \|\mathbf{n}\|^4 \mathbf{I} + e^4 \|\mathbf{n}\|^2 \mathbf{n} \mathbf{n}^\top - 2e^2 \|\mathbf{n}\|^2 \mathbf{n} \mathbf{n}^\top \quad (1.0.2.3)$$

$$= \|\mathbf{n}\|^4 \mathbf{I} + e^2 (e^2 - 2) \|\mathbf{n}\|^2 \mathbf{n} \mathbf{n}^\top \quad (1.0.2.4)$$

$$= \|\mathbf{n}\|^4 \mathbf{I} + (e^2 - 2) \|\mathbf{n}\|^2 (\|\mathbf{n}\|^2 \mathbf{I} - \mathbf{V}) \quad (1.0.2.5)$$

which can be expressed as

$$\mathbf{V}^2 + (e^2 - 2) \|\mathbf{n}\|^2 \mathbf{V} - (e^2 - 1) \|\mathbf{n}\|^4 \mathbf{I} = 0 \quad (1.0.2.6)$$

Using the Cayley-Hamilton theorem, (1.0.2.6) results in the characteristic equation,

$$\lambda^2 - (2 - e^2) \|\mathbf{n}\|^2 \lambda + (1 - e^2) \|\mathbf{n}\|^4 = 0 \quad (1.0.2.7)$$

which can be expressed as

$$\left( \frac{\lambda}{\|\mathbf{n}\|^2} \right)^2 - (2 - e^2) \left( \frac{\lambda}{\|\mathbf{n}\|^2} \right) + (1 - e^2) = 0 \quad (1.0.2.8)$$

$$\implies \frac{\lambda}{\|\mathbf{n}\|^2} = 1 - e^2, 1 \quad (1.0.2.9)$$

$$\text{or, } \lambda_2 = \|\mathbf{n}\|^2, \lambda_1 = (1 - e^2) \|\mathbf{n}\|^2 \quad (1.0.2.10)$$

From (1.0.2.10), the eccentricity of (5.2.1) is given by (5.2.9). Multiplying both sides of (5.2.2) by  $\mathbf{n}$ ,

$$\mathbf{V} \mathbf{n} = \|\mathbf{n}\|^2 \mathbf{n} - e^2 \mathbf{n} \mathbf{n}^\top \mathbf{n} \quad (1.0.2.11)$$

$$= \|\mathbf{n}\|^2 (1 - e^2) \mathbf{n} \quad (1.0.2.12)$$

$$= \lambda_1 \mathbf{n} \quad (1.0.2.13)$$

$$(1.0.2.14)$$

from (1.0.2.10). Thus,  $\lambda_1$  is the corresponding eigenvalue for  $\mathbf{n}$ . From (5.1.4) and (1.0.2.14), this implies that

$$\mathbf{p}_1 = \frac{\mathbf{n}}{\|\mathbf{n}\|} \quad (1.0.2.15)$$

$$\text{or, } \mathbf{n} = \|\mathbf{n}\| \mathbf{p}_1 = \sqrt{\lambda_2} \mathbf{p}_1 \quad (1.0.2.16)$$

from (1.0.2.10) . From (5.2.3) and (1.0.2.10),

$$\mathbf{F} = \frac{ce^2 \mathbf{n} - \mathbf{u}}{\lambda_2} \quad (1.0.2.17)$$

$$\implies \|\mathbf{F}\|^2 = \frac{(ce^2 \mathbf{n} - \mathbf{u})^\top (ce^2 \mathbf{n} - \mathbf{u})}{\lambda_2^2} \quad (1.0.2.18)$$

$$\implies \lambda_2^2 \|\mathbf{F}\|^2 = c^2 e^4 \lambda_2 - 2ce^2 \mathbf{u}^\top \mathbf{n} + \|\mathbf{u}\|^2 \quad (1.0.2.19)$$

Also, (5.2.4) can be expressed as

$$\lambda_2 \|\mathbf{F}\|^2 = f + c^2 e^2 \quad (1.0.2.20)$$

From (1.0.2.19) and (1.0.2.20),

$$c^2 e^4 \lambda_2 - 2ce^2 \mathbf{u}^\top \mathbf{n} + \|\mathbf{u}\|^2 = \lambda_2 (f + c^2 e^2) \quad (1.0.2.21)$$

$$\implies \lambda_2 e^2 (e^2 - 1) c^2 - 2ce^2 \mathbf{u}^\top \mathbf{n} + \|\mathbf{u}\|^2 - \lambda_2 f = 0 \quad (1.0.2.22)$$

yielding (5.2.11).

## APPENDIX B

Using (5.1.1) (5.2.1) can be expressed as

$$(\mathbf{P}\mathbf{y} + \mathbf{c})^\top \mathbf{V} (\mathbf{P}\mathbf{y} + \mathbf{c}) + 2\mathbf{u}^\top (\mathbf{P}\mathbf{y} + \mathbf{c}) + f = 0, \quad (2.0.2.1)$$

yielding

$$\mathbf{y}^\top \mathbf{P}^\top \mathbf{V} \mathbf{P} \mathbf{y} + 2(\mathbf{V}\mathbf{c} + \mathbf{u})^\top \mathbf{P} \mathbf{y} + \mathbf{c}^\top \mathbf{V} \mathbf{c} + 2\mathbf{u}^\top \mathbf{c} + f = 0 \quad (2.0.2.2)$$

From (2.0.2.2) and (5.1.2),

$$\mathbf{y}^\top \mathbf{D} \mathbf{y} + 2(\mathbf{V}\mathbf{c} + \mathbf{u})^\top \mathbf{P} \mathbf{y} + \mathbf{c}^\top (\mathbf{V}\mathbf{c} + \mathbf{u}) + \mathbf{u}^\top \mathbf{c} + f = 0 \quad (2.0.2.3)$$

When  $\mathbf{V}^{-1}$  exists, choosing

$$\mathbf{V}\mathbf{c} + \mathbf{u} = \mathbf{0}, \quad \text{or, } \mathbf{c} = -\mathbf{V}^{-1}\mathbf{u}, \quad (2.0.2.4)$$

and substituting (2.0.2.4) in (2.0.2.3) yields (5.3.1).

### B.1

When  $|\mathbf{V}| = 0, \lambda_1 = 0$  and

$$\mathbf{V}\mathbf{p}_1 = 0, \mathbf{V}\mathbf{p}_2 = \lambda_2 \mathbf{p}_2. \quad (2.1.2.1)$$

where  $\mathbf{p}_1, \mathbf{p}_2$  are the eigenvectors of  $\mathbf{V}$  such that (5.1.2)

$$\mathbf{P} = (\mathbf{p}_1 \quad \mathbf{p}_2), \quad (2.1.2.2)$$

Substituting (2.1.2.2) in (2.0.2.3),

$$\mathbf{y}^\top \mathbf{D} \mathbf{y} + 2(\mathbf{c}^\top \mathbf{V} + \mathbf{u}^\top) (\mathbf{p}_1 \quad \mathbf{p}_2) \mathbf{y} + \mathbf{c}^\top (\mathbf{V}\mathbf{c} + \mathbf{u}) + \mathbf{u}^\top \mathbf{c} + f = 0 \quad (2.1.2.3)$$

$$\implies \mathbf{y}^\top \mathbf{D} \mathbf{y} + 2((\mathbf{c}^\top \mathbf{V} + \mathbf{u}^\top) \mathbf{p}_1 (\mathbf{c}^\top \mathbf{V} + \mathbf{u}^\top) \mathbf{p}_2) \mathbf{y} + \mathbf{c}^\top (\mathbf{V}\mathbf{c} + \mathbf{u}) + \mathbf{u}^\top \mathbf{c} + f = 0 \quad (2.1.2.4)$$

$$\implies \mathbf{y}^\top \mathbf{D} \mathbf{y} + 2(\mathbf{u}^\top \mathbf{p}_1 \quad (\lambda_2 \mathbf{c}^\top + \mathbf{u}^\top) \mathbf{p}_2) \mathbf{y} + \mathbf{c}^\top (\mathbf{V}\mathbf{c} + \mathbf{u}) + \mathbf{u}^\top \mathbf{c} + f = 0 \quad (2.1.2.5)$$

upon substituting from (2.1.2.1) yielding

$$\lambda_2 y_2^2 + 2 (\mathbf{u}^\top \mathbf{p}_1) y_1 + 2 y_2 (\lambda_2 \mathbf{c} + \mathbf{u})^\top \mathbf{p}_2 + \mathbf{c}^\top (\mathbf{V} \mathbf{c} + \mathbf{u}) + \mathbf{u}^\top \mathbf{c} + f = 0 \quad (2.1.2.6)$$

Thus, (2.1.2.6) can be expressed as (5.3.2) by choosing

$$\eta = 2 \mathbf{u}^\top \mathbf{p}_1 \quad (2.1.2.7)$$

and  $\mathbf{c}$  in (2.0.2.3) such that

$$2 \mathbf{P}^\top (\mathbf{V} \mathbf{c} + \mathbf{u}) = \eta \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (2.1.2.8)$$

$$\mathbf{c}^\top (\mathbf{V} \mathbf{c} + \mathbf{u}) + \mathbf{u}^\top \mathbf{c} + f = 0 \quad (2.1.2.9)$$

$\therefore \mathbf{P}^\top \mathbf{P} = \mathbf{I}$ , multiplying (2.1.2.8) by  $\mathbf{P}$  yields

$$(\mathbf{V} \mathbf{c} + \mathbf{u}) = \frac{\eta}{2} \mathbf{p}_1, \quad (2.1.2.10)$$

which, upon substituting in (2.1.2.9) results in

$$\frac{\eta}{2} \mathbf{c}^\top \mathbf{p}_1 + \mathbf{u}^\top \mathbf{c} + f = 0 \quad (2.1.2.11)$$

(2.1.2.10) and (2.1.2.11) can be clubbed together to obtain (5.5.2).

## APPENDIX C

a) For the standard hyperbola/ellipse in (5.3.1), from (5.3.6), (5.2.10) and (5.3.7),

$$\mathbf{n} = \sqrt{\frac{\lambda_2}{f_0}} \mathbf{e}_1 \quad (3.0.2.1)$$

$$c = \pm \frac{\sqrt{-\frac{\lambda_2}{f_0} (e^2 - 1) \left( \frac{\lambda_2}{f_0} \right)}}{\frac{\lambda_2}{f_0} e (e^2 - 1)} \quad (3.0.2.2)$$

$$= \pm \frac{1}{e \sqrt{1 - e^2}} \quad (3.0.2.3)$$

yielding (5.3.1) upon substituting from (5.2.9) and simplifying. For the standard parabola in (5.3.2), from (5.3.6), (5.2.10) and (5.3.7), noting that  $f = 0$ ,

$$\mathbf{n} = \sqrt{\lambda_2} \mathbf{e}_1 \quad (3.0.2.4)$$

$$c = \frac{\left\| \frac{\eta}{2} \mathbf{e}_1 \right\|^2}{2 \left( \frac{\eta}{2} \right) (\mathbf{e}_1)^\top \mathbf{n}} \quad (3.0.2.5)$$

$$(3.0.2.6)$$

$$= \frac{\eta}{4 \sqrt{\lambda_2}} \quad (3.0.2.7)$$

yielding (5.3.2).

b) For the standard ellipse/hyperbola, substituting from (3.0.2.3), (3.0.2.1), (5.3.7) and (5.2.9) in (5.2.11),

$$\mathbf{F} = \pm \frac{\left( \frac{1}{e \sqrt{1 - e^2}} \right) (e^2) \sqrt{\frac{\lambda_2}{f_0}} \mathbf{e}_1}{\frac{\lambda_2}{f_0}} \quad (3.0.2.8)$$

yielding (5.3.1) after simplification. For the standard parabola, substituting from (3.0.2.7), (3.0.2.4), (5.3.7) and (5.2.9) in (5.2.11),

$$\mathbf{F} = \frac{\left(\frac{\eta}{4\sqrt{\lambda_2}}\right) \sqrt{\lambda_2} \mathbf{e}_1 - \frac{\eta}{2} \mathbf{e}_1}{\lambda_2} \quad (3.0.2.9)$$

$$(3.0.2.10)$$

yielding (5.3.1) after simplification.

#### APPENDIX D

Since the major axis passes through the origin,

$$\mathbf{q} = \mathbf{0} \quad (4.0.2.1)$$

Further, from Corollary (5.8),

$$\mathbf{m} = \mathbf{e}_2, \quad (4.0.2.2)$$

and from (5.3.1),

$$\mathbf{V} = \frac{\mathbf{D}}{f_0}, \mathbf{u} = 0, f = -1 \quad (4.0.2.3)$$

Substituting the above in (6.1.9),

$$\frac{2\sqrt{\mathbf{e}_1^\top \frac{\mathbf{D}}{f_0} \mathbf{e}_1}}{\mathbf{e}_1^\top \frac{\mathbf{D}}{f_0} \mathbf{e}_1} \|\mathbf{e}_1\| \quad (4.0.2.4)$$

yielding (6.1.16). Similarly, for the minor axis, the only different parameter is

$$\mathbf{m} = \mathbf{e}_2, \quad (4.0.2.5)$$

Substituting the above in (6.1.9),

$$\frac{2\sqrt{\mathbf{e}_2^\top \frac{\mathbf{D}}{f_0} \mathbf{e}_2}}{\mathbf{e}_2^\top \frac{\mathbf{D}}{f_0} \mathbf{e}_2} \|\mathbf{e}_2\| \quad (4.0.2.6)$$

yielding (6.1.17).

#### APPENDIX E

The latus rectum is perpendicular to the major axis for the standard conic. Hence, from Corollary (5.8),

$$\mathbf{m} = \mathbf{e}_2, \quad (5.0.2.1)$$

Since it passes through the focus, from (5.3.1)

$$\mathbf{q} = \mathbf{F} = \pm e \sqrt{\frac{f_0}{\lambda_2(1-e^2)}} \mathbf{e}_1 \quad (5.0.2.2)$$

for the standard hyperbola/ellipse. Also, from (5.3.1),

$$\mathbf{V} = \frac{\mathbf{D}}{f_0}, \mathbf{u} = 0, f = -1 \quad (5.0.2.3)$$

Substituting the above in (6.1.9),

$$\frac{2\sqrt{\left[\mathbf{e}_2^\top \left(\frac{\mathbf{D}}{f_0} e \sqrt{\frac{f_0}{\lambda_2(1-e^2)}} \mathbf{e}_1\right)\right]^2 - \left(e \sqrt{\frac{f_0}{\lambda_2(1-e^2)}} \mathbf{e}_1^\top \frac{\mathbf{D}}{f_0} e \sqrt{\frac{f_0}{\lambda_2(1-e^2)}} \mathbf{e}_1 - 1\right) \left(\mathbf{e}_2^\top \frac{\mathbf{D}}{f_0} \mathbf{e}_2\right)}}{\mathbf{e}_2^\top \frac{\mathbf{D}}{f_0} \mathbf{e}_2} \|\mathbf{e}_2\| \quad (5.0.2.4)$$

Since

$$\mathbf{e}_2^\top \mathbf{D} \mathbf{e}_1 = 0, \mathbf{e}_1^\top \mathbf{D} \mathbf{e}_1 = \lambda_1, \mathbf{e}_1^\top \mathbf{e}_1 = 1, \|\mathbf{e}_2\| = 1, \mathbf{e}_2^\top \mathbf{D} \mathbf{e}_2 = \lambda_2, \quad (5.0.2.5)$$

(5.0.2.4) can be expressed as

$$\frac{2\sqrt{\left(1 - \frac{\lambda_1 e^2}{\lambda_2(1-e^2)}\right) \left(\frac{\lambda_2}{f_0}\right)}}{\frac{\lambda_2}{f_0}} \quad (5.0.2.6)$$

$$= 2 \frac{\sqrt{f_0 \lambda_1}}{\lambda_2} \quad \left(\because e^2 = 1 - \frac{\lambda_1}{\lambda_2}\right) \quad (5.0.2.7)$$

For the standard parabola, the parameters in (6.1.9) are

$$\mathbf{q} = \mathbf{F} = -\frac{\eta}{4\lambda_2} \mathbf{e}_1, \mathbf{m} = \mathbf{e}_1, \mathbf{V} = \mathbf{D}, \mathbf{u} = \frac{\eta}{2} \mathbf{e}_1^\top, f = 0 \quad (5.0.2.8)$$

Substituting the above in (6.1.9), the length of the latus rectum can be expressed as

$$\frac{2\sqrt{\left[\mathbf{e}_2^\top \left(\mathbf{D} \left(-\frac{\eta}{4\lambda_2} \mathbf{e}_1\right) + \frac{\eta}{2} \mathbf{e}_1\right)\right]^2 - \left(\left(-\frac{\eta}{4\lambda_2} \mathbf{e}_1\right)^\top \mathbf{D} \left(-\frac{\eta}{4\lambda_2} \mathbf{e}_1\right) + 2\frac{\eta}{2} \mathbf{e}_1^\top \left(-\frac{\eta}{4\lambda_2} \mathbf{e}_1\right)\right) \left(\mathbf{e}_2^\top \mathbf{D} \mathbf{e}_2\right)}}{\mathbf{e}_2^\top \mathbf{D} \mathbf{e}_2} \|\mathbf{e}_2\| \quad (5.0.2.9)$$

Since

$$\mathbf{e}_2^\top \mathbf{D} \mathbf{e}_1 = 0, \mathbf{e}_2^\top \mathbf{e}_2 = 0, \mathbf{e}_1^\top \mathbf{D} \mathbf{e}_1 = 0, \mathbf{e}_1^\top \mathbf{e}_1 = 1, \|\mathbf{e}_1\| = 1, \mathbf{e}_2^\top \mathbf{D} \mathbf{e}_2 = \lambda_2, \quad (5.0.2.10)$$

(5.0.2.9) can be expressed as

$$2 \frac{\sqrt{\frac{\eta^2}{4\lambda_2} \lambda_2}}{\lambda_2} = \frac{\eta}{\lambda_2} \quad (5.0.2.11)$$