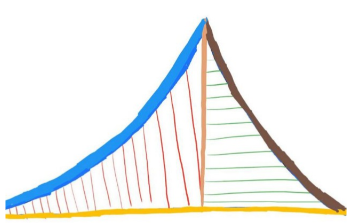
# MATRIX ANALYSIS

## Through Coordinate Geometry

**G. V. V. Sharma**



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# Introduction

This book links high school coordinate geometry to linear algebra and matrix analysis through solved problems.

# Chapter 1

# Vectors

## 1.1. Distance Formula

1.1.1 Find the distances between the following pairs of points:

(a) $(2,3),(4,1)$

(b) $(-5,7),(-1,3)$

(c) $(a,b),(-a,-b)$

1.1.2 Find the distance between the points $(0,0)$ and $(36,15)$.

1.1.3 Determine if the points $(1,5),(2,3)$ and $(-2,-11)$ are collinear.

**Solution:** We know that points $\mathbf{A},\mathbf{B}$ and $\mathbf{C}$ are collinear, if

$$\operatorname{rank}\begin{pmatrix}\mathbf{A}^\top \\ \mathbf{B}^\top \\ \mathbf{C}^\top\end{pmatrix} = 1 \tag{1.1.3.1}$$

Since

$$\begin{pmatrix} \mathbf{A}^\top \\ \mathbf{B}^\top \\ \mathbf{C}^\top \end{pmatrix} = \begin{pmatrix} 1 & 5 \\ 2 & 3 \\ -2 & -11 \end{pmatrix} \tag{1.1.3.2}$$

$$\xleftrightarrow[R_3 \to R_3 + 2R_1]{R_2 \to R_2 - 2R_1} \begin{pmatrix} 1 & 5 \\ 0 & -7 \\ 0 & -1 \end{pmatrix} \xleftrightarrow{R_3 \to R_3 - \frac{1}{7}R_2} \begin{pmatrix} 1 & 5 \\ 0 & -7 \\ 0 & 0 \end{pmatrix}, \tag{1.1.3.3}$$

the rank of the mareix is 2. From (1.1.3.1), the points are not collinear. This is verified by Fig. 1.1.3.1, where the given points constitute a triangle and not a line.

1.1.4 Check whether$(5, -2), (6, 4)$ and $(7, -2)$ are the vertices of an isosceles triangle.

1.1.5 Name the type of quadrilateral formed,if any,by the following points,and give reasons for your answer

(a) $(-1, -2), (1, 0), (-1, 2), (-3, 0)$

(b) $(-3, 5), (-3, 1), (0, 3), (-1, -4)$

(c) $(4, 5), (7, 6), (4, 3), (1, 2)$

1.1.6 Find the point on the x-axis which is equidistant from $(2, -5)$ and $(-2, 9)$.

1.1.7 Find the values of $y$ for which the distance between the points $\mathbf{P}(2, -3)$ and $\mathbf{Q}(10, y)$ is 10 units.

1.1.8 If $\mathbf{Q}(0, 1)$ is equidistant from $\mathbf{P}(5, -3)$ and $\mathbf{R}(x, 6)$, find the values of $x$. Also find the distances $QR$ and $PR$.

1.1.9 Find a relation between $x$ and $y$ such that the point $(x, y)$ is equidistant from the point $(3, 6)$ and $(-3, 4)$.

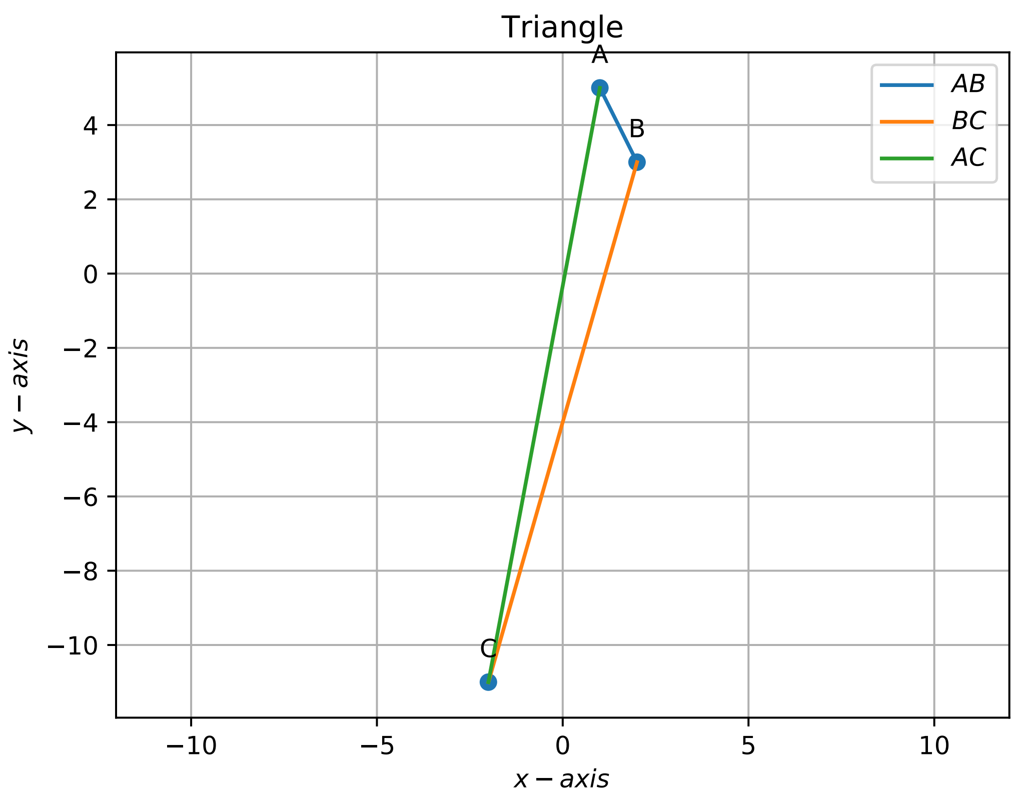


Figure 1.1.3.1:

# 1.2. Section Formula

# 1.3. Scalar Product

1.3.1 Find the angle between two vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ with magnitudes $\sqrt{3}$ and 2 respectively having $\overrightarrow{a}.\overrightarrow{b} = \sqrt{6}$.

1.3.2 Find the angle between the the vectors $\hat{i} - 2\hat{j} + 3\hat{k}$ and $3\hat{i} - 2\hat{j} + \hat{k}$.

1.3.3 Find the projection of the vector $\hat{i} - \hat{j}$ on the vector $\hat{i} + \hat{j}$.

1.3.4 Find the projection of the vector $\hat{i}+3\hat{j}+7\hat{k}$ on the vector $7\hat{i}-\hat{j}+8\hat{k}$.

1.3.5 Show that each of the given three vectors is a unit vector:

$\frac{1}{7}(2\hat{i}+3\hat{j}+6\hat{k}), \frac{1}{7}(3\hat{i}-6\hat{j}+2\hat{k}), \frac{1}{7}(6\hat{i}+2\hat{j}-3\hat{k})$

Also,show that they are mutually perpendicular to each other.

1.3.6 Find $|\overrightarrow{a}|$ and $\left|\overrightarrow{b}\right|$,if $(\overrightarrow{a}+\overrightarrow{b}).(\overrightarrow{a}-\overrightarrow{b})=8$ and $|\overrightarrow{a}|=8\left|\overrightarrow{b}\right|$.

1.3.7 Evaluate the product$(3\overrightarrow{a}-5\overrightarrow{b}).(2\overrightarrow{a}+7\overrightarrow{b})$.

1.3.8 Find the magnitude of two vectors $\overrightarrow{a}$ and $\overrightarrow{b}$, having the same magnitude and such that the angle between them is $60°$ and their scalar product is $\frac{1}{2}$

1.3.9 Find $|\overrightarrow{x}|$,if for a unit vector $\overrightarrow{a}$, $(\overrightarrow{x}-\overrightarrow{a}).(\overrightarrow{x}+\overrightarrow{a})$=12.

**Solution:** From the given information,

$$(\mathbf{x}-\mathbf{a})^\top(\mathbf{x}+\mathbf{a})=12 \tag{1.3.9.1}$$

$$\implies \mathbf{x}^\top\mathbf{x}-\mathbf{a}^\top\mathbf{x}+\mathbf{x}^\top\mathbf{a}-\mathbf{a}^\top\mathbf{a}=12 \tag{1.3.9.2}$$

$$\implies \|\mathbf{x}\|^2-\|\mathbf{a}\|^2=12 \tag{1.3.9.3}$$

$$\implies \|\mathbf{x}\|^2-1=12 \tag{1.3.9.4}$$

$$\text{or, } \|\mathbf{x}\|=\sqrt{13} \tag{1.3.9.5}$$

1.3.10 If $\overrightarrow{a}=2\hat{i}+2\hat{j}3\hat{k}, \overrightarrow{b}=\hat{-i}+2\hat{j}+\hat{k}$ and $\overrightarrow{c}=3\hat{i}+\hat{j}$ are such that $\overrightarrow{a}+\lambda\overrightarrow{b}$ is perpendicular to $\overrightarrow{c}$,then find the value of $\lambda$.

1.3.11 Show that $|\overrightarrow{a}|\overrightarrow{b}+\left|\overrightarrow{b}\right|\overrightarrow{a}$ is perpendicular to $|\overrightarrow{a}|\overrightarrow{b}-\left|\overrightarrow{b}\right|\overrightarrow{a}$, for any two nonzero vectors $\overrightarrow{a}$ and $\overrightarrow{b}$.

1.3.12 If $\overrightarrow{a}.\overrightarrow{a}$=0 and $\overrightarrow{a}.\overrightarrow{b}$=0, then what can be conculded about the vector $\overrightarrow{b}$?

1.3.13 If $\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}$ are unit vectors such that $\overrightarrow{a} + \overrightarrow{b} + \overrightarrow{c} = \overrightarrow{0}$, find the value of $\overrightarrow{a}.\overrightarrow{b} + \overrightarrow{b}.\overrightarrow{c} + \overrightarrow{c}.\overrightarrow{a}$.

1.3.14 If either vector $\overrightarrow{a} = 0$ or $\overrightarrow{b} = 0$, then $\overrightarrow{a}.\overrightarrow{b}$=0. But the converse need not be true .Justify your answer with an example.

1.3.15 If the vertices A,B,C of a triangle ABC are (1,2,3),(-1,0,0)(0,1,2), respectively , then find $\angle ABC$.[$\angle ABC$ is the angle between the vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$].

1.3.16 show that the points A(1,2,7),B(2,6,3)and C(3,10,-1) are collinear.

1.3.17 show that the vectors $2\hat{i} - \hat{j} + \hat{k}, \hat{i} - 3\hat{j} - 5\hat{k}$ and $3\hat{i} - 4\hat{j} - 4\hat{k}$ from the vertices of a right angled triangle.

1.3.18 If $\overrightarrow{a}$ is a nonzero vector of magnitude 'a' and $\lambda$ a nonzero scalar , then $\lambda\overrightarrow{a}$ is unit vector if

1. $\lambda = 1$
2. $\lambda = -1$
3. $a = |\lambda|$
4. $a = 1/|\lambda|$

# 1.4. Area of a Triangle

1.4.1 Find the area of the triangle whose vertices are

(a) $(2,3), (-1,0), (2,-4)$

(b) $(-5,-1), (3,-5), (5,2)$

**Solution:**

(a) In this case, the area is given by

$$\frac{1}{2}\left\|(\mathbf{A}-\mathbf{B})\times(\mathbf{A}-\mathbf{C})\right\| \tag{1.4.1.1}$$

$$\tag{1.4.1.2}$$

Since

$$\mathbf{A}-\mathbf{B} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} \tag{1.4.1.3}$$

$$\mathbf{A}-\mathbf{C} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ -4 \end{pmatrix} = \begin{pmatrix} 0 \\ 7 \end{pmatrix} \tag{1.4.1.4}$$

the desired area is given by

$$\frac{1}{2}\begin{vmatrix} 3 & 0 \\ 3 & 7 \end{vmatrix} = \frac{21}{2} \tag{1.4.1.5}$$

(b) In this case,

$$\mathbf{A}-\mathbf{B} = \begin{pmatrix} -5 \\ -1 \end{pmatrix} - \begin{pmatrix} 3 \\ -5 \end{pmatrix} = \begin{pmatrix} -8 \\ 4 \end{pmatrix} \tag{1.4.1.6}$$

$$\mathbf{A}-\mathbf{C} = \begin{pmatrix} -5 \\ -1 \end{pmatrix} - \begin{pmatrix} 5 \\ 2 \end{pmatrix} = \begin{pmatrix} -10 \\ -3 \end{pmatrix} \tag{1.4.1.7}$$

$$\implies \text{Area} = \frac{1}{2}\begin{vmatrix} -8 & -10 \\ 4 & -3 \end{vmatrix} = 32 \tag{1.4.1.8}$$

1.4.2 In each of the following, find the value of '$k$', for which the points are collinear.

(a) $(7,-2),(5,1),(3,k)$

(b) $(8,1),(k,-4),(2,-5)$

1.4.3 Find the area of the triangle formed by joining the mid-points of the sides of the triangle whose vertices are $(0,-1),(2,1)$ and $(0,3)$. Find the ratio of this area to the area of the given triangle.

1.4.4 Find the area of the quadrilateral whose vertices, taken in order, are $(-4,-2),(-3,-5),(3,-2)$ and $(2,3)$.

1.4.5 Verify that a median of a triangle divides it into two triangles of equal areas for $\triangle ABC$ whose vertices are $\mathbf{A}(4,-6),\mathbf{B}(3,2),$ and $\mathbf{C}(5,2)$.

1.4.6 Find the area of region bounded by the triangle whose vertices are $(1,0),(2,2)$ and $(3,1)$.

1.4.7 Find the area of region bounded by the triangle whose vertices are $(-1,0),(1,3)$ and $(3,2)$.

1.4.8 Find the area of the $\triangle ABC$, coordinates of whose vertices are $\mathbf{A}(2,0),\mathbf{B}(4,5),$ and $\mathbf{C}(6,3)$.

## 1.5. Miscellaneous Exercises

1.5.1 Determine the ratio in which the line $2x+y-4=0$ divides the line segment joining the points $\mathbf{A}(2,-2)$ and $\mathbf{B}(3,7)$.

1.5.2 Find a relation between $x$ and $y$ if the points $(x,y),(1,2)$ and $(7,0)$ are collinear.

1.5.3 Find the centre of a circle passing through the points $(6,-6),(3,-7)$ and $(3,3)$.

1.5.4 The two opposite vertices of a square are $(-1, 2)$ and $(3, 2)$. Find the coordinates of the other two vertices.

**Solution:** Let

$$\mathbf{A} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \tag{1.5.4.1}$$

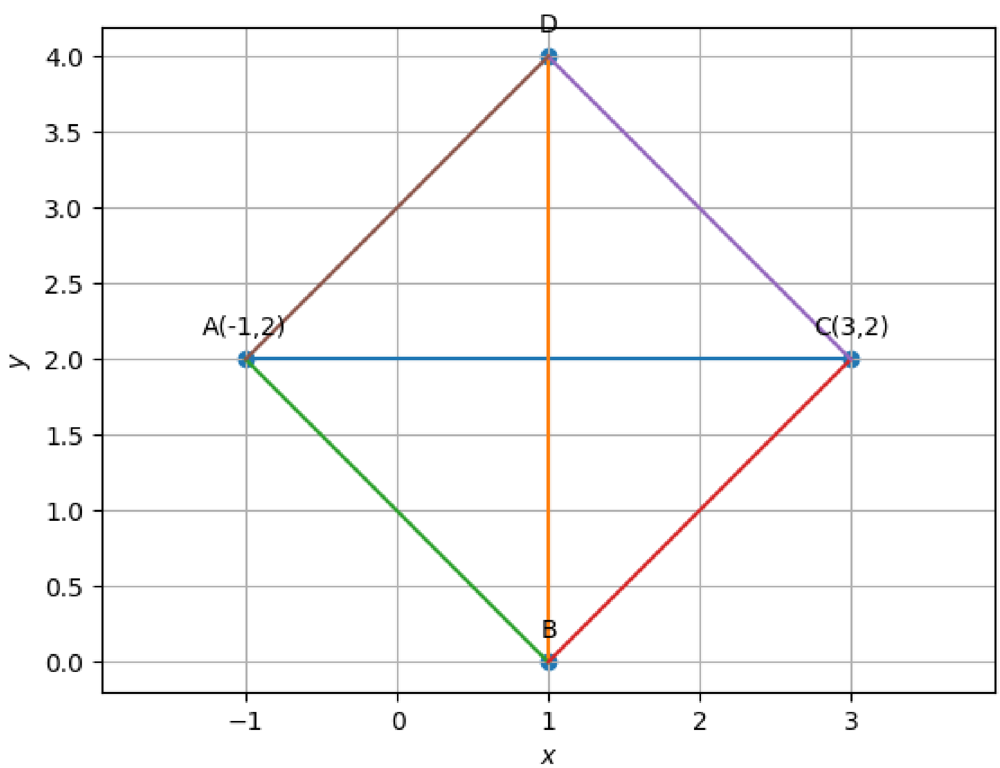


Figure 1.5.4.1:

Shifting $\mathbf{A}$ to origin with reference to Fig. 1.5.4.2,

$$\mathbf{A}' = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{C}' = \mathbf{C} - \mathbf{A} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \tag{1.5.4.2}$$

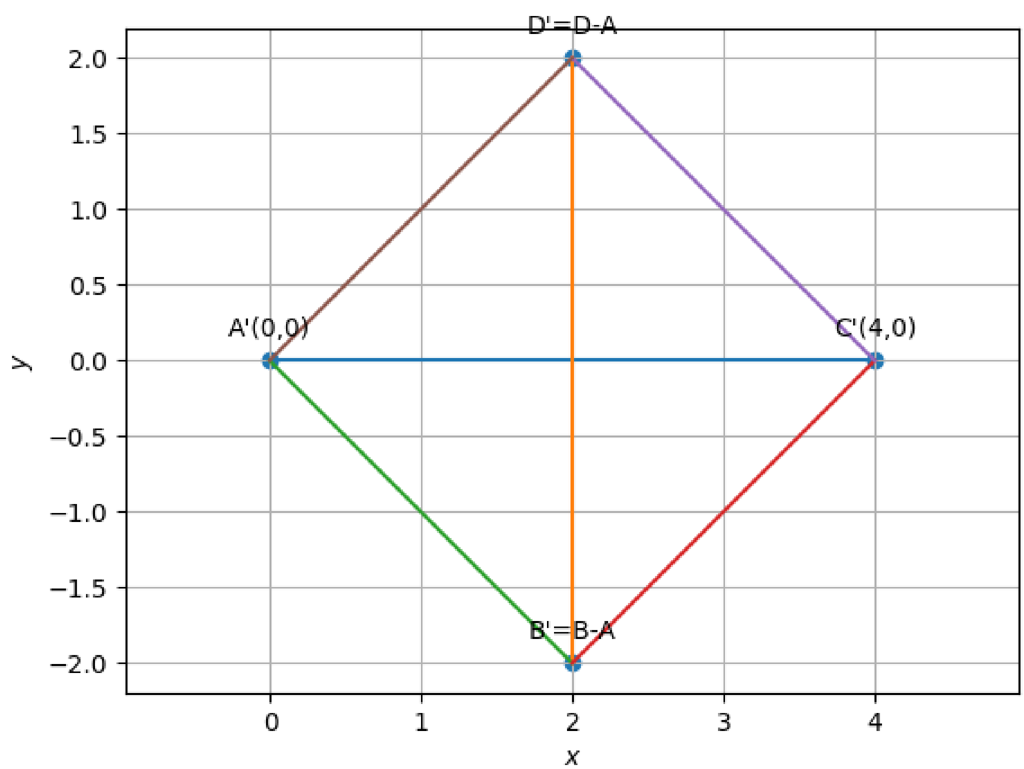


Figure 1.5.4.2:

Since

$$\mathbf{C} - \mathbf{A} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tan\theta = \frac{0}{4} \implies \theta = 0^\circ \tag{1.5.4.3}$$

where $\theta$ is the angle made by $AC$ with the x-axis. Considering the rotation matrix

$$\mathbf{P} = \begin{pmatrix} \cos\left(\frac{\pi}{4} - \theta\right) & -\sin\left(\frac{\pi}{4} - \theta\right) \\ \sin\left(\frac{\pi}{4} - \theta\right) & \cos\left(\frac{\pi}{4} - \theta\right) \end{pmatrix} \tag{1.5.4.4}$$

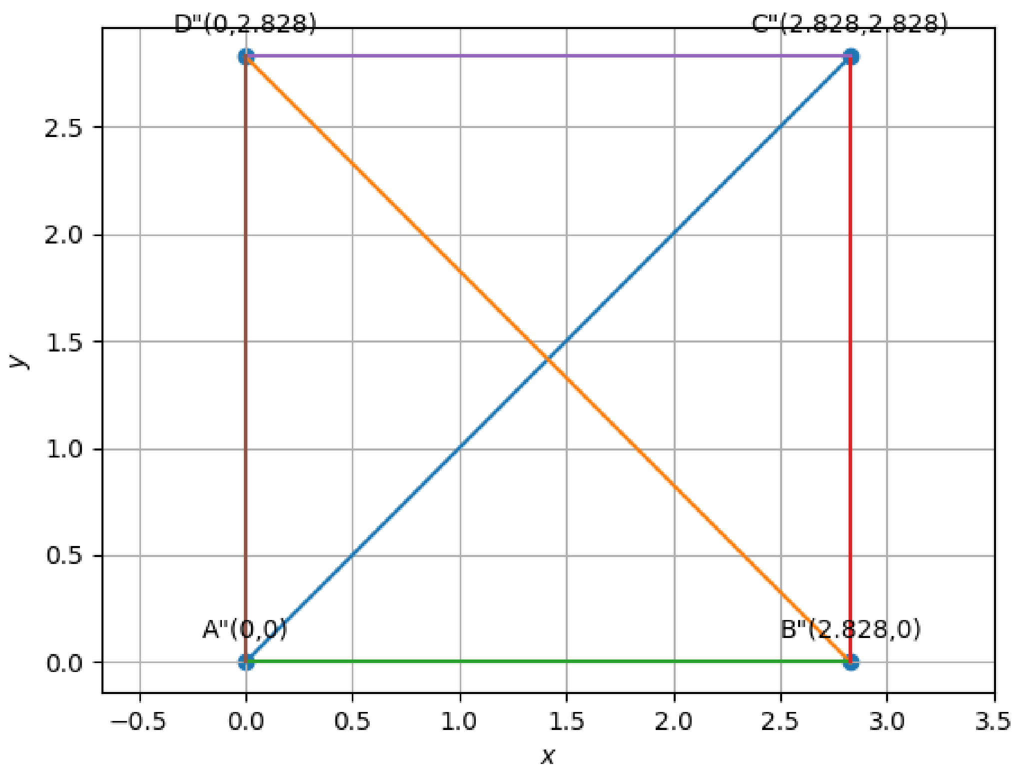


Figure 1.5.4.3:

from Figure 1.5.4.3,

$$\mathbf{C}'' = \mathbf{P}(\mathbf{C} - \mathbf{A}) \tag{1.5.4.5}$$

$$\mathbf{B}'' = \begin{pmatrix} \mathbf{e}_1 & \mathbf{0} \end{pmatrix} \mathbf{C}'' \tag{1.5.4.6}$$

$$\mathbf{D}'' = \begin{pmatrix} \mathbf{0} & \mathbf{e}_2 \end{pmatrix} \mathbf{C}'' \tag{1.5.4.7}$$

Now,

$$\mathbf{B} = \mathbf{P}^\top \mathbf{B}'' + \mathbf{A} \tag{1.5.4.8}$$

$$\mathbf{D} = \mathbf{P}^\top \mathbf{D}'' + \mathbf{A} \tag{1.5.4.9}$$

by reversing the process of translation and rotation. Thus, from (1.5.4.8) (1.5.4.6), (1.5.4.9) and (1.5.4.7)

$$\mathbf{B} = \mathbf{P}^\top \begin{pmatrix} \mathbf{e}_1 & \mathbf{0} \end{pmatrix} \mathbf{P}(\mathbf{C} - \mathbf{A}) + \mathbf{A} \tag{1.5.4.10}$$

$$\mathbf{D} = \mathbf{P}^\top \begin{pmatrix} \mathbf{0} & \mathbf{e}_2 \end{pmatrix} \mathbf{P}(\mathbf{C} - \mathbf{A}) + \mathbf{A} \tag{1.5.4.11}$$

yielding

$$\mathbf{B} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{D} \begin{pmatrix} 1 \\ 4 \end{pmatrix}. \tag{1.5.4.12}$$

1.5.5 The vertices of a $\triangle ABC$ are $\mathbf{A}(4,6), \mathbf{B}(1,5)$ and $\mathbf{C}(7,2)$. A line is drawn to intersect sides $AB$ and $AC$ at $\mathbf{D}$ and $\mathbf{E}$ respectively, such that $\frac{AD}{AB} = \frac{AE}{AC} = \frac{1}{4}$. Calculate the area of $\triangle ADE$ and compare it with the area of the $\triangle ABC$.

1.5.6 Let $\mathbf{A}(4,2), \mathbf{B}(6,5)$ and $\mathbf{C}(1,4)$ be the vertices of $\triangle ABC$.

(a) The median from $\mathbf{A}$ meets $BC$ at $\mathbf{D}$. Find the coordinates of the point $\mathbf{D}$.

(b) Find the coordinates of the point $\mathbf{P}$ on $AD$ such that $AP : PD = 2 : 1$.

(c) Find the coordinates of points $\mathbf{Q}$ and $\mathbf{R}$ on medians $BE$ and $CF$ respectively such that $BQ : QE = 2 : 1$ and $CR : RF = 2 : 1$.

(d) What do you observe?

(e) If $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ are the vertices of $\triangle ABC$, find the coordinates of the centroid of the triangle.

1.5.7 $ABCD$ is a rectangle formed by the points $\mathbf{A}(-1,-1), \mathbf{B}(-1,4), \mathbf{C}(5,4)$ and $\mathbf{D}(5,-1)$. $\mathbf{P}, \mathbf{Q}, \mathbf{R}$ and $\mathbf{S}$ are the mid-points of $AB, BC, CD$ and $DA$ respectively. Is the quadrilateral $PQRS$ a square? a rectangle? or a rhombus? Justify your answer.

**Solution:** See Fig. 1.5.7.1.

$$\mathbf{P} = \frac{1}{2}(\mathbf{A}+\mathbf{B}) = \frac{1}{2}\left(\begin{pmatrix}-1\\-1\end{pmatrix}+\begin{pmatrix}-1\\4\end{pmatrix}\right) = \begin{pmatrix}-1\\\frac{3}{2}\end{pmatrix} \tag{1.5.7.1}$$

$$\mathbf{Q} = \frac{1}{2}(\mathbf{B}+\mathbf{C}) = \frac{1}{2}\left(\begin{pmatrix}-1\\4\end{pmatrix}+\begin{pmatrix}5\\4\end{pmatrix}\right) = \begin{pmatrix}2\\4\end{pmatrix} \tag{1.5.7.2}$$

$$\mathbf{R} = \frac{1}{2}(\mathbf{C}+\mathbf{D}) = \frac{1}{2}\left(\begin{pmatrix}5\\4\end{pmatrix}+\begin{pmatrix}5\\-1\end{pmatrix}\right) = \begin{pmatrix}5\\\frac{3}{2}\end{pmatrix} \tag{1.5.7.3}$$

$$\mathbf{S} = \frac{1}{2}(\mathbf{D}+\mathbf{A}) = \frac{1}{2}\left(\begin{pmatrix}5\\-1\end{pmatrix}+\begin{pmatrix}-1\\-1\end{pmatrix}\right) = \begin{pmatrix}2\\-1\end{pmatrix} \tag{1.5.7.4}$$

We know that PQRS is a parallelogram. To know, if it is a rectangle, we need to ascertain whether any of the two adjacent sides are perpendicular. That means

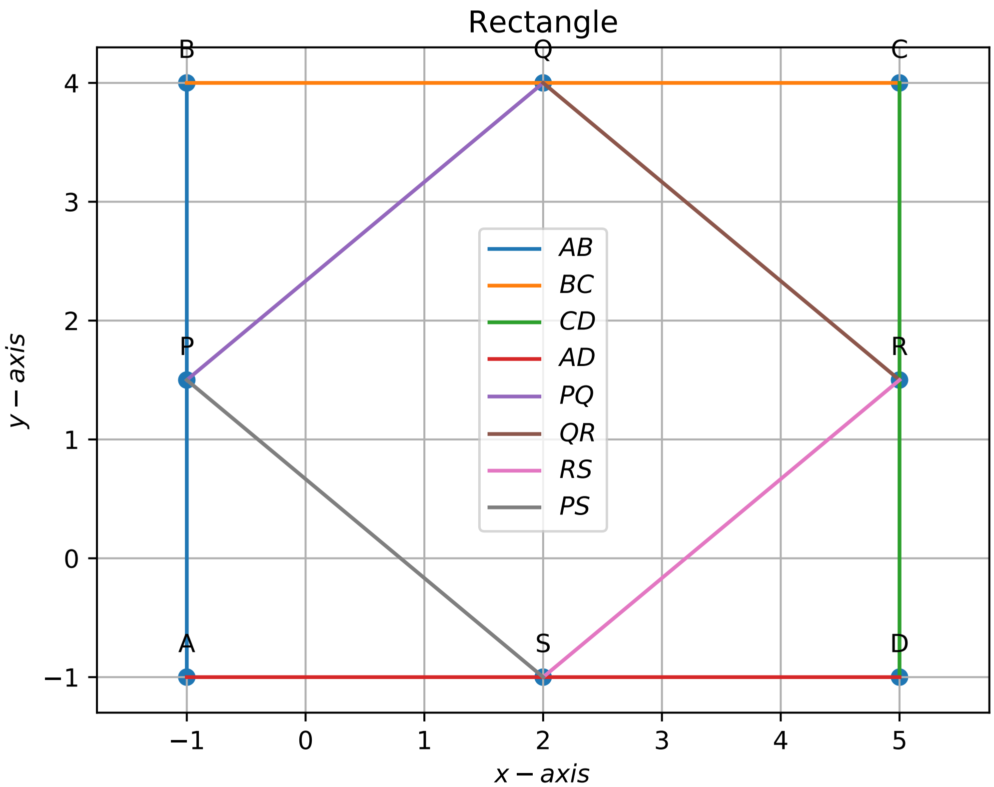


Figure 1.5.7.1:

$(\mathbf{Q}-\mathbf{P})^{\top}(\mathbf{R}-\mathbf{Q})$ should be equal to zero.

$$\mathbf{Q}-\mathbf{P} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} - \begin{pmatrix} -1 \\ \frac{3}{2} \end{pmatrix} = \begin{pmatrix} 3 \\ \frac{5}{2} \end{pmatrix} \tag{1.5.7.5}$$

$$\mathbf{R}-\mathbf{Q} = \begin{pmatrix} 5 \\ \frac{3}{2} \end{pmatrix} - \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ -\frac{5}{2} \end{pmatrix} \tag{1.5.7.6}$$

$$(\mathbf{Q}-\mathbf{P})^{\top}(\mathbf{R}-\mathbf{Q}) = \begin{pmatrix} 3 & \frac{5}{2} \end{pmatrix} \begin{pmatrix} 3 \\ -\frac{5}{2} \end{pmatrix} \neq 0 \tag{1.5.7.7}$$

Therefore PQRS is not a rectangle. Let us check if it is a rhombus. For a rhombus, the diagonals bisect perpendicularly. That means $(\mathbf{R}-\mathbf{P})^\top(\mathbf{S}-\mathbf{Q})$ should be equal to zero.

$$\mathbf{R}-\mathbf{P} = \begin{pmatrix} 5 \\ \frac{3}{2} \end{pmatrix} - \begin{pmatrix} -1 \\ \frac{3}{2} \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \end{pmatrix} \tag{1.5.7.8}$$

$$\mathbf{S}-\mathbf{Q} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} - \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ -5 \end{pmatrix} \tag{1.5.7.9}$$

$$(\mathbf{R}-\mathbf{P})^\top(\mathbf{S}-\mathbf{Q}) = \begin{pmatrix} 6 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ -5 \end{pmatrix} = 0 \tag{1.5.7.10}$$

Therefore $PQRS$ is a rhombus.

## 1.6. Line Preliminaries

1.6.1 Draw a quadrilateral in the Cartesian plane, whose vertices are (-4,5), (0,7), (5,-5), (-4,-2). Also, find its area.

**Solution:** See Fig. 1.6.1.1.

$$ar(\triangle ABC) = \frac{1}{2}\left\|(\mathbf{B}-\mathbf{A})\times(\mathbf{B}-\mathbf{C})\right\| \tag{1.6.1.1}$$

$$= \frac{1}{2}\begin{vmatrix} 4 & 2 \\ -5 & 12 \end{vmatrix} = 29 \tag{1.6.1.2}$$

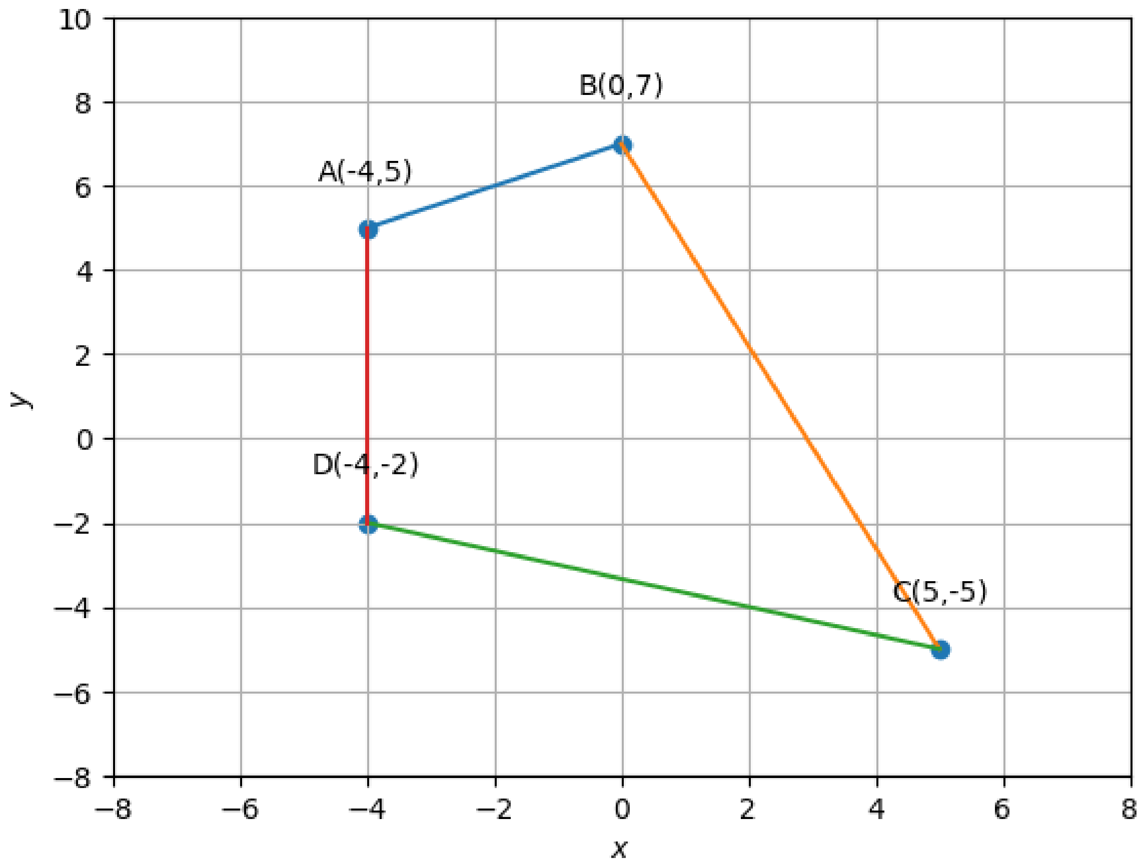


Figure 1.6.1.1:

Similarly,

$$ar(\triangle ADC) = \frac{1}{2}\left\|(\mathbf{D}-\mathbf{A})\times(\mathbf{D}-\mathbf{C})\right\| \tag{1.6.1.3}$$

$$= \frac{1}{2}\begin{vmatrix} 0 & -7 \\ -9 & 3 \end{vmatrix} = 31.5 \tag{1.6.1.4}$$

Thus,

$$ar(ABCD) = ar(\triangle ABC) + ar(\triangle ADC) = 60.5 \tag{1.6.1.5}$$

1.6.2 The base of an equilateral triangle with side $2a$ lies along the y-axis such that the mid-point of the base is at the origin. Find vertices of the triangle.

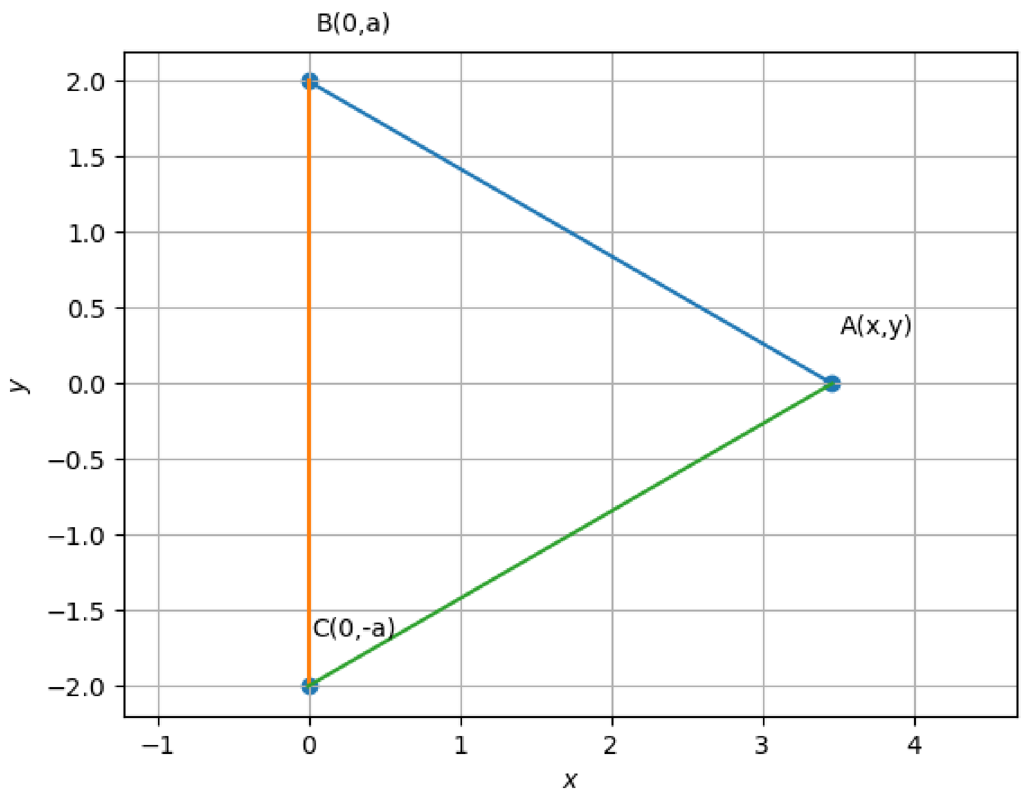


Figure 1.6.2.1:

**Solution:** Let the base be $BC$. From the given information,

$$\mathbf{B} = a\mathbf{e}_2, \mathbf{C} = -a\mathbf{e}_2 \tag{1.6.2.1}$$

Since $\mathbf{A}$ lies on the $x$-axis,

$$\mathbf{A} = k\mathbf{e}_1 \tag{1.6.2.2}$$

and

$$\|\mathbf{A} - \mathbf{C}\|^2 = (2a)^2 \tag{1.6.2.3}$$

$$\implies \|\mathbf{A}\|^2 + \|\mathbf{C}\|^2 - 2\mathbf{A}^\top\mathbf{C} = 4a^2 \tag{1.6.2.4}$$

$$\implies k^2 + a^2 = 4a^2 \tag{1.6.2.5}$$

$$\text{or, } k = \pm a\sqrt{3} \tag{1.6.2.6}$$

Thus,

$$\mathbf{A} = \pm\sqrt{3}a\mathbf{e}_1 \tag{1.6.2.7}$$

Fig. 1.6.2.1 is plotted for $a = 2$.

1.6.3 Find a point on the x-axis,which is equidistant from the points $\begin{pmatrix} 7 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

**Solution:** From the given information

$$\|\mathbf{x} - \mathbf{A}\|^2 = \|\mathbf{x} - \mathbf{B}\|^2 \tag{1.6.3.1}$$

$$\implies (\mathbf{x} - \mathbf{A})^\top (\mathbf{x} - \mathbf{A}) = (\mathbf{x} - \mathbf{B})^\top (\mathbf{x} - \mathbf{B}) \tag{1.6.3.2}$$

$$\implies \|\mathbf{x}\|^2 - 2\mathbf{A}^\top\mathbf{x} + \|\mathbf{A}\|^2 = \|\mathbf{x}\|^2 - 2\mathbf{B}^\top\mathbf{x} + \|\mathbf{B}\|^2 \tag{1.6.3.3}$$

$$\text{or, } (\mathbf{A} - \mathbf{B})^\top \mathbf{x} = \frac{\|\mathbf{A}\|^2 - \|\mathbf{B}\|^2}{2} \tag{1.6.3.4}$$

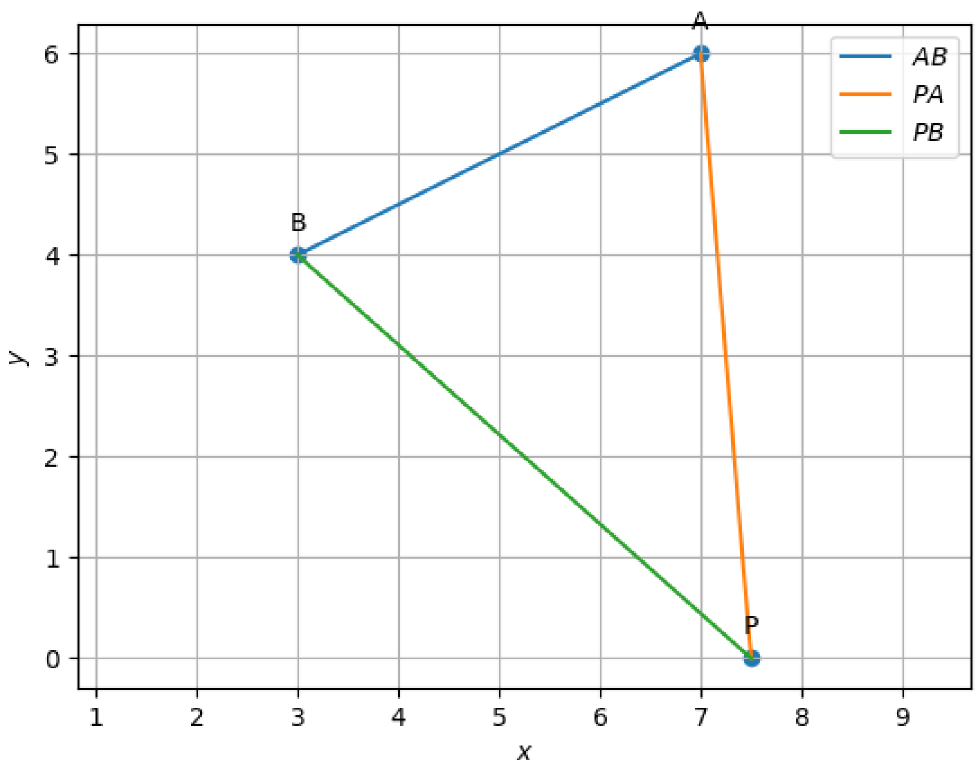


Figure 1.6.3.1:

Since $\mathbf{x}$ lies on the $x$-axis,

$$\mathbf{x} = k\mathbf{e}_1 \tag{1.6.3.5}$$

which, upon substituting in (1.6.3.4) yields

$$k = \frac{15}{2} \tag{1.6.3.6}$$

1.6.4

1.6.5 Find the slope of a line,which passes through the origin and the mid point of the line segment joining the points $\mathbf{P}$(0,-4) and $\mathbf{B}$(8,0).

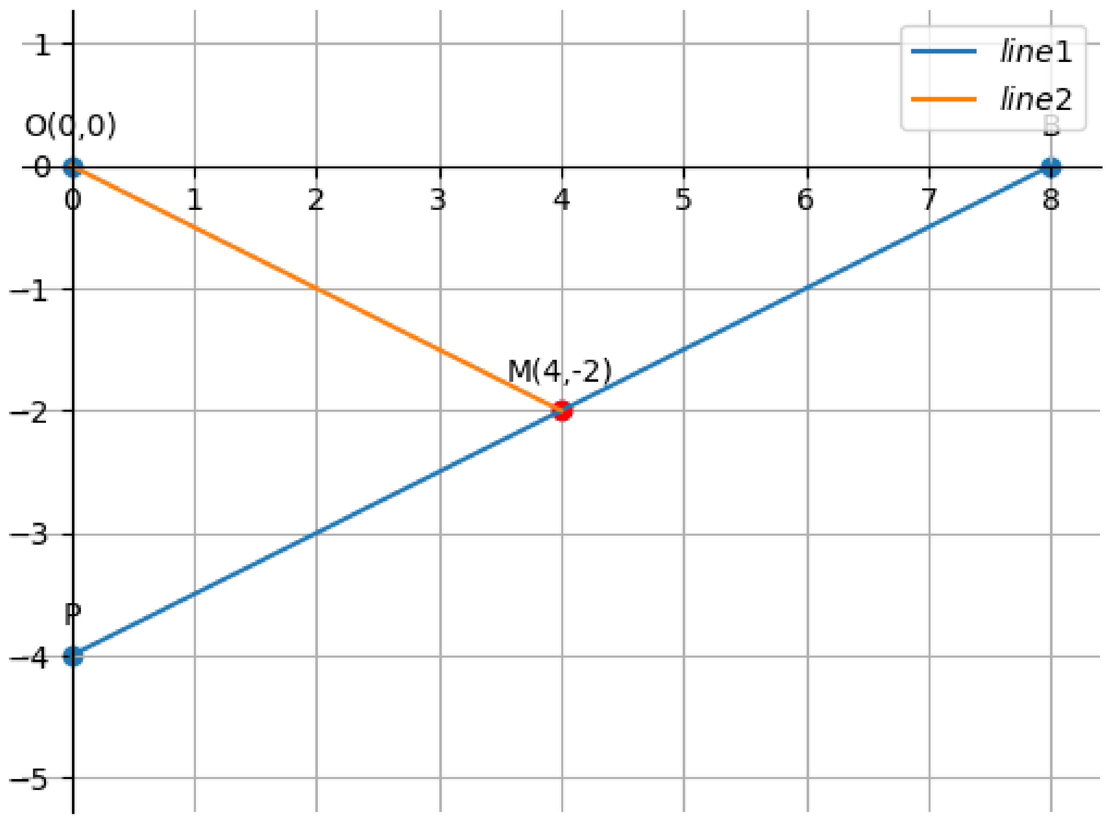


Figure 1.6.5.1:

**Solution:** The mid point of $PB$ is

$$\mathbf{M} = \frac{1}{2}(\mathbf{P} + \mathbf{B}) = \begin{pmatrix} 4 \\ -2 \end{pmatrix} \tag{1.6.5.1}$$

The direction vector of line joining $\mathbf{O}, \mathbf{M}$ is

$$\mathbf{m} = \mathbf{O} - \mathbf{M} = -\mathbf{M} \tag{1.6.5.2}$$

which can be expressed as

$$\begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix} \tag{1.6.5.3}$$

Thus the slope is

$$m = -\frac{1}{2} \tag{1.6.5.4}$$

1.6.6 Without using the Baudhayana theorem, show that the points $(4, 4), (3, 5)$ and $(-1, -1)$ are the vertices of a right angled triangle.

$$\mathbf{C} - \mathbf{A} = \begin{pmatrix} -5 \\ -5 \end{pmatrix}, \tag{1.6.6.1}$$

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{1.6.6.2}$$

$$\implies (\mathbf{C} - \mathbf{A})^\top (\mathbf{A} - \mathbf{B}) = 0 \tag{1.6.6.3}$$

Thus, $AB \perp AC$.

1.6.7 If three points $(x, -1), (2, 1)$ and $(4, 5)$ are collinear, find the value of $x$.

**Solution:** Let

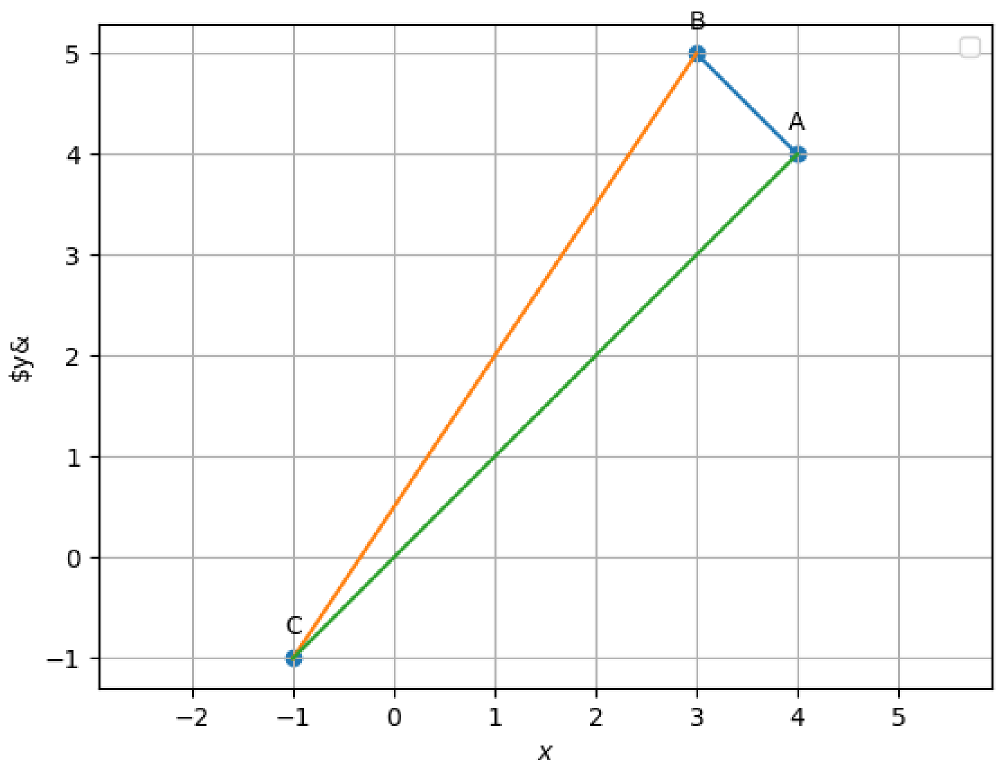


Figure 1.6.6.1:

$$\mathbf{A} = \begin{pmatrix} x \\ -1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}. \tag{1.6.7.1}$$

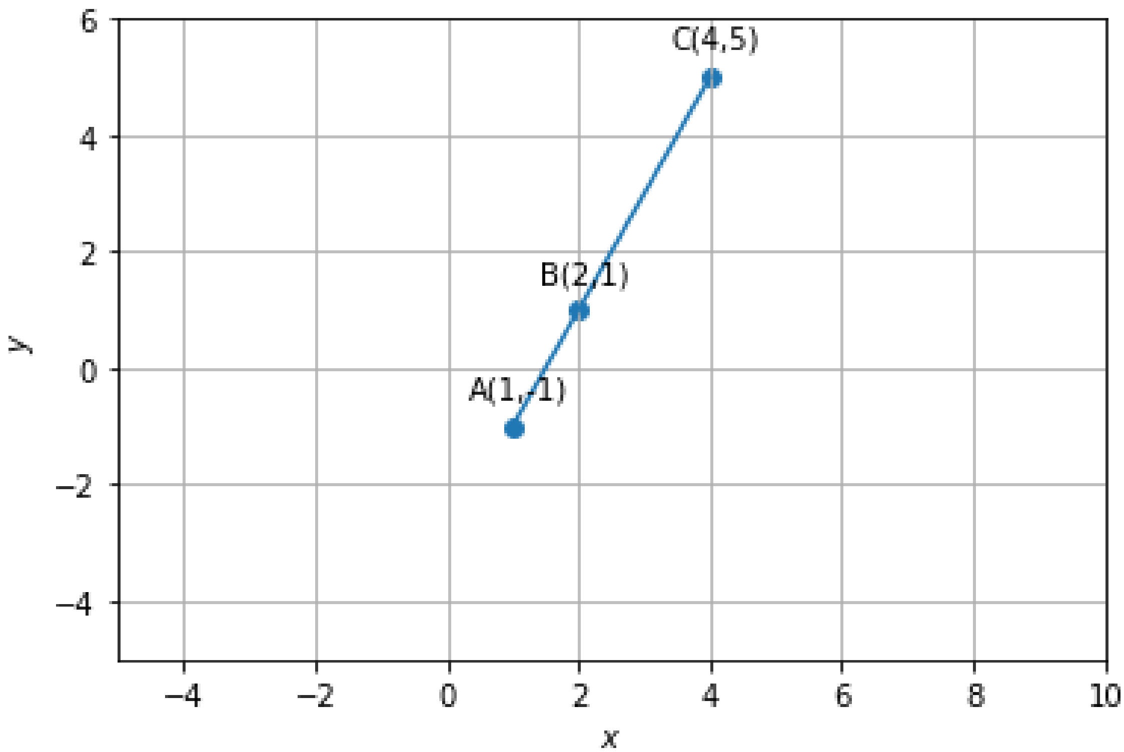


Figure 1.6.7.1:

Then

$$\mathbf{A}-\mathbf{B}=\begin{pmatrix}x-2\\-2\end{pmatrix} \tag{1.6.7.2}$$

$$\mathbf{A}-\mathbf{C}=\begin{pmatrix}4-x\\6\end{pmatrix} \tag{1.6.7.3}$$

Forming the collinearity matrix using (C.1.4.1),

$$\begin{pmatrix}x-2 & -2\\4-x & 6\end{pmatrix}\xleftrightarrow{R_1=3R_1+R-2}=\begin{pmatrix}2x-2 & 0\\4-x & 6\end{pmatrix} \tag{1.6.7.4}$$

If the rank of the matrix is 1, any one of the rows must be zero. So, making the first element in the above matrix 0,

$$x = 1 \tag{1.6.7.5}$$

1.6.8 Without using distance formula, show that points (– 2, – 1), (4, 0), (3, 3) and (–3, 2) are the vertices of a parallelogram.

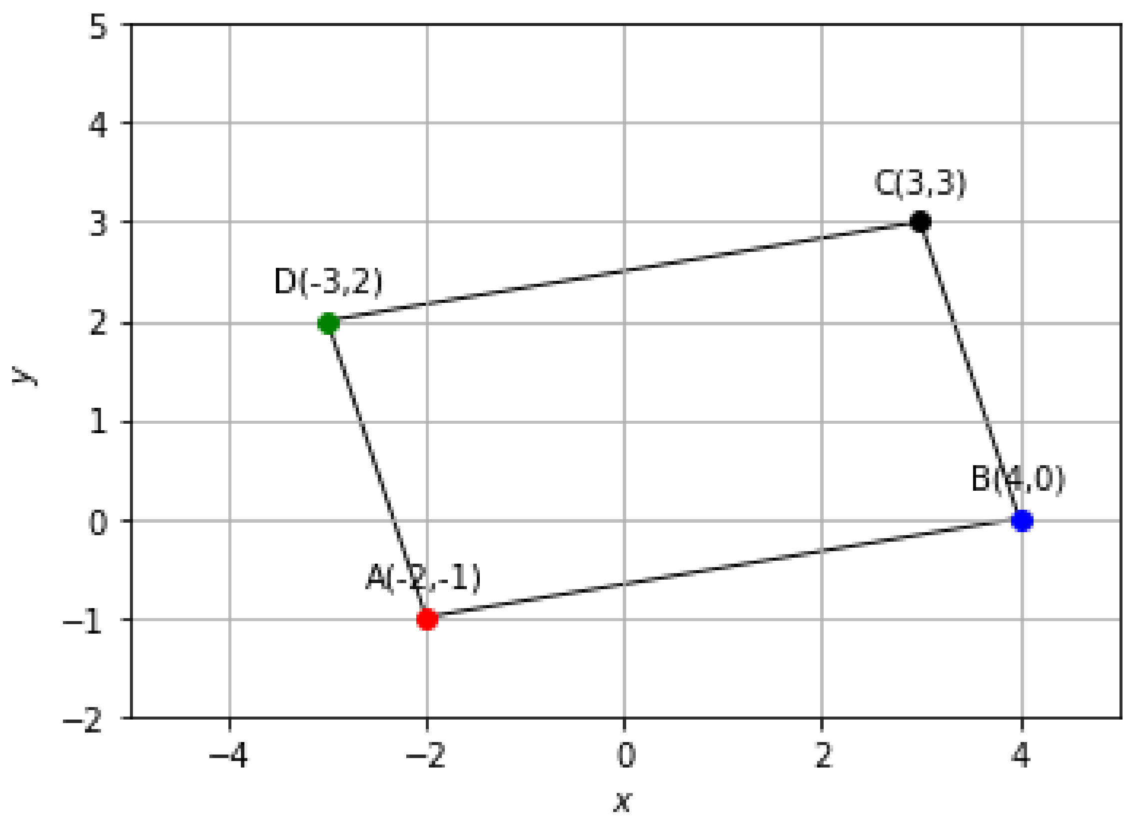


Figure 1.6.8.1:

**Solution:** See Fig. 1.6.8.1.

$$\mathbf{A} = \begin{pmatrix} -2 \\ -1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} -3 \\ 2 \end{pmatrix} \tag{1.6.8.1}$$

and

$$\mathbf{P} = \mathbf{B} - \mathbf{A} = \begin{pmatrix} 6 \\ 1 \end{pmatrix} \tag{1.6.8.2}$$

$$\mathbf{Q} = \mathbf{C} - \mathbf{D} = \begin{pmatrix} 6 \\ 1 \end{pmatrix} \tag{1.6.8.3}$$

$$\mathbf{R} = \mathbf{A} - \mathbf{C} = \begin{pmatrix} 1 \\ -3 \end{pmatrix} \tag{1.6.8.4}$$

$$\mathbf{S} = \mathbf{A} - \mathbf{D} = \begin{pmatrix} 1 \\ -3 \end{pmatrix} \tag{1.6.8.5}$$

Since $\mathbf{P} = \mathbf{Q}$ and $\mathbf{R} = \mathbf{S}$, from (A.1.24.1), $ABCD$ is a parallelogram

1.6.9 Find the angle between x-axis and the line joining points (3,-1) and (4,-2)

**Solution:** See Fig. 1.6.9.1.

Let

$$\mathbf{P} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} 4 \\ -2 \end{pmatrix} \tag{1.6.9.1}$$

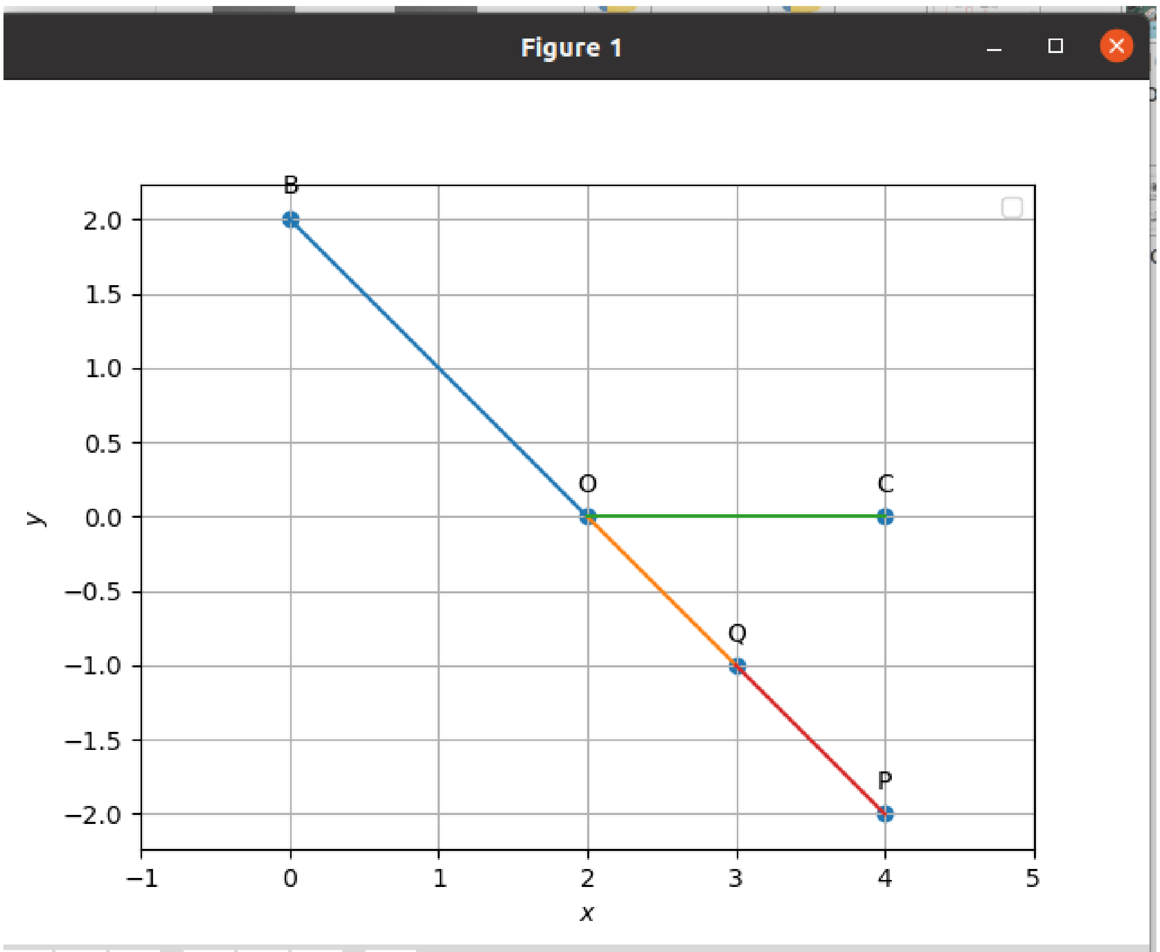


Figure 1.6.9.1:

Then

$$\mathbf{C} = \mathbf{P} - \mathbf{Q} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{1.6.9.2}$$

The desired angle is given by

$$\cos\theta = \frac{\mathbf{C}^T\mathbf{e}_1}{\|\mathbf{C}\|\,\|\mathbf{e}_1\|} \tag{1.6.9.3}$$

$$= -\frac{1}{\sqrt{2}} \tag{1.6.9.4}$$

$$\implies \theta = 135^\circ \tag{1.6.9.5}$$

1.6.10 The slope of a line is double of the slope of another line. If tangent of the angle between them is $1/3$, find the slopes of the lines.

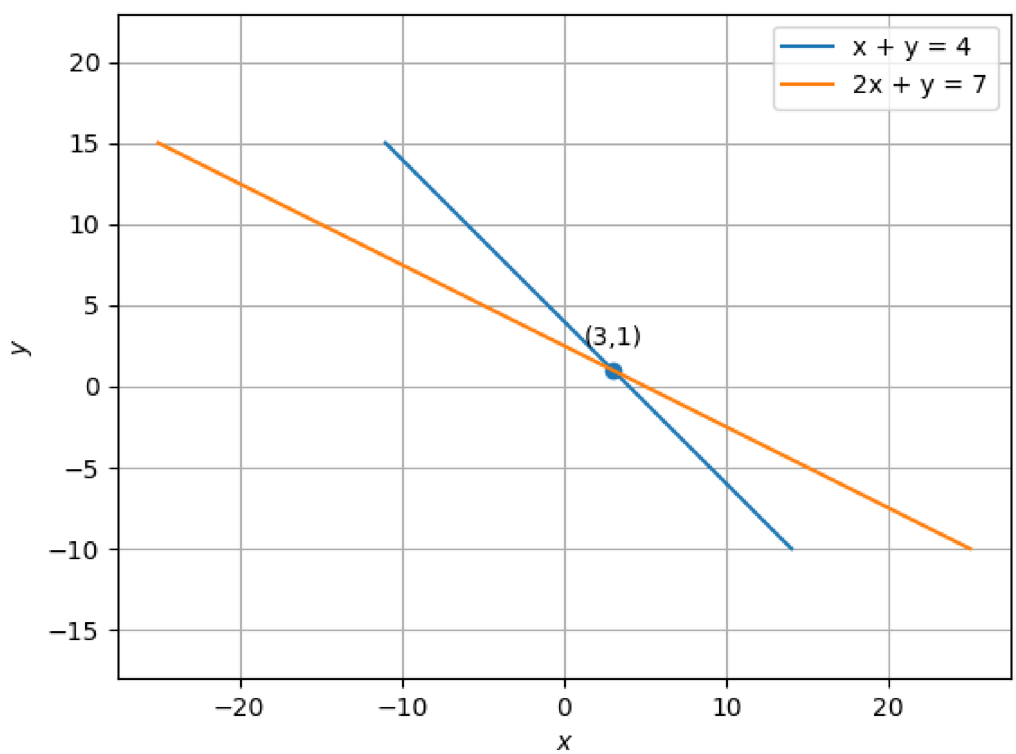


Figure 1.6.10.1:

**Solution:** The direction vector of a line is expressed as

$$\mathbf{m} = \begin{pmatrix} 1 \\ m \end{pmatrix} \tag{1.6.10.1}$$

where $m$ is defined to be the slope of the line. If the angle between the lines be $\theta$,

$$\tan\theta = \frac{1}{3} \implies \cos\theta = \frac{3}{\sqrt{10}} \tag{1.6.10.2}$$

The angle between two vectors is then expressed as

$$\frac{3}{\sqrt{10}} = \frac{\mathbf{m}_1^\top \mathbf{m}_2}{\|\mathbf{m}_1\|\,\|\mathbf{m}_2\|} \tag{1.6.10.3}$$

$$= \frac{\begin{pmatrix} 1 & m \end{pmatrix}\begin{pmatrix} 1 \\ 2m \end{pmatrix}}{\left\|\begin{pmatrix} 1 \\ m \end{pmatrix}\right\|\left\|\begin{pmatrix} 1 \\ 2m \end{pmatrix}\right\|} \tag{1.6.10.4}$$

$$= \frac{2m^2+1}{\sqrt{m^2+1}\sqrt{4m^2+1}} \tag{1.6.10.5}$$

$$\implies \frac{9}{10} = \frac{4m^4+4m^2+1}{4m^4+5m^2+1} \tag{1.6.10.6}$$

$$\text{or, } 4m^4 - 5m^2 + 1 = 0 \tag{1.6.10.7}$$

yielding

$$m = \pm\frac{1}{2}, \pm 1 \tag{1.6.10.8}$$

1.6.11 A line passes through $(x_1, y_1)$ and $(h, k)$. If slope of the line is m show that

$(k - y_1) = m(h - x_1)$.

**Solution:** Given

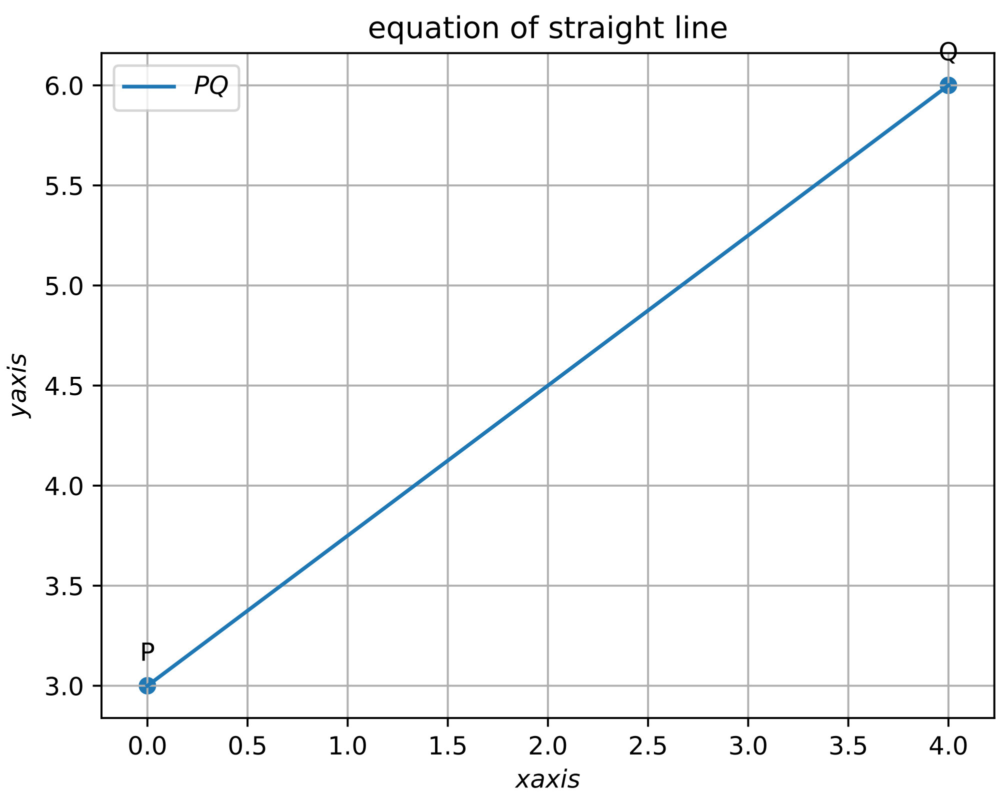


Figure 1.6.11.1:

$$\mathbf{A} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} h \\ k \end{pmatrix} \tag{1.6.11.1}$$

The direction vector

$$\mathbf{m} = \mathbf{B} - \mathbf{A} \tag{1.6.11.2}$$

$$= \begin{pmatrix} h - x_1 \\ k - y_1 \end{pmatrix} \equiv \begin{pmatrix} 1 \\ \frac{k-y_1}{h-x_1} \end{pmatrix} \tag{1.6.11.3}$$

which yields the desired relation from (A.1.18.1).

1.6.12 If three points $(h, 0), (a, b)$ and $(0, k)$ lie on a line, show that

$$\frac{a}{h} + \frac{b}{k} = 1 \tag{1.6.12.1}$$

**Solution:** Let

$$\mathbf{A} = \begin{pmatrix} h \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} a \\ b \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 0 \\ k \end{pmatrix} \tag{1.6.12.2}$$

Forming the matrix in (C.1.4.1), we obtain, upon row reduction

$$\begin{pmatrix} h - a & -b \\ h & -k \end{pmatrix} \xleftrightarrow{\frac{R_1}{h-a}} \begin{pmatrix} 1 & \frac{-b}{h-a} \\ h & -k \end{pmatrix} \tag{1.6.12.3}$$

$$\xleftrightarrow{R_2 \to R_2 - hR_1} \begin{pmatrix} 1 & \frac{-b}{h-a} \\ 0 & -k + \frac{bh}{h-a} \end{pmatrix} \tag{1.6.12.4}$$

For obtaining a rank 1 matrix,

$$-k + \frac{bh}{h-a} = 0 \tag{1.6.12.5}$$

$$\implies \frac{a}{h} + \frac{b}{k} = 1 \tag{1.6.12.6}$$

upon simplification.

# Chapter 2

# Line

## 2.1. Equation of a Line

2.1.1

2.1.2

2.1.3

2.1.4

2.1.5

2.1.6

2.1.7

2.1.8

2.1.9 The Vertices of Triangle $PQR$ is $\mathbf{P}(2,1), \mathbf{Q}(-2,3), \mathbf{R}(4,5)$. Find the equation of the Median Through $\mathbf{R}$.

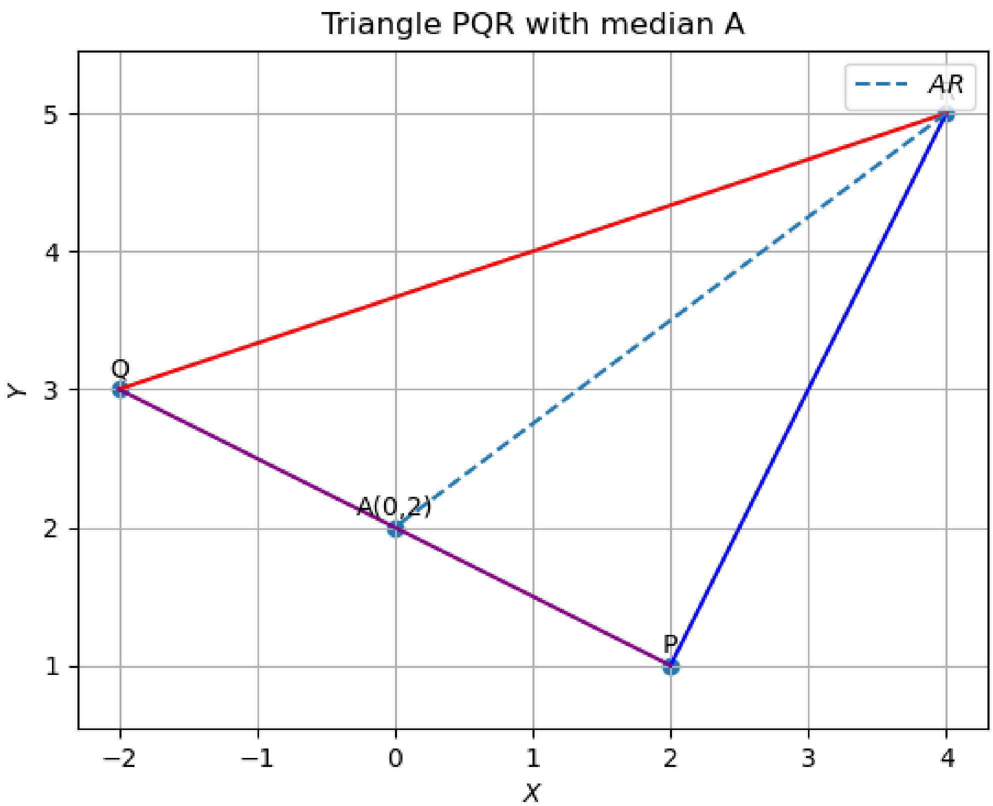


Figure 2.1.9.1:

**Solution:** See Fig. 2.1.9.1. Using Section Formula,

$$\mathbf{A} = \frac{\mathbf{P}+\mathbf{Q}}{2} \tag{2.1.9.1}$$

$$= \begin{pmatrix} 0 \\ 2 \end{pmatrix} \tag{2.1.9.2}$$

So , the Direction Vector of $AR$ is

$$\mathbf{m} = \mathbf{R} - \mathbf{A} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \tag{2.1.9.3}$$

$$\implies \mathbf{n} = \begin{pmatrix} 3 \\ -4 \end{pmatrix} \tag{2.1.9.4}$$

which is the normal vector. Thus, from (C.1.2.1), the equation of the line is

$$\begin{pmatrix} 3 & -4 \end{pmatrix} (\mathbf{x} - \mathbf{R}) = 0 \tag{2.1.9.5}$$

$$\implies \begin{pmatrix} 3 & -4 \end{pmatrix} \mathbf{x} = 8 \tag{2.1.9.6}$$

2.1.10 Find the equation of the line passing through (-3,5) and perpendicular to the line through the points (2,5) and (-3,6).

**Solution:** Let

$$\mathbf{A} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -3 \\ 6 \end{pmatrix}, \mathbf{P} = \begin{pmatrix} -3 \\ 5 \end{pmatrix} \tag{2.1.10.1}$$

The normal vector of the desired line is then given by

$$\mathbf{n} = \mathbf{B} - \mathbf{A} = \begin{pmatrix} 5 \\ -1 \end{pmatrix} \tag{2.1.10.2}$$

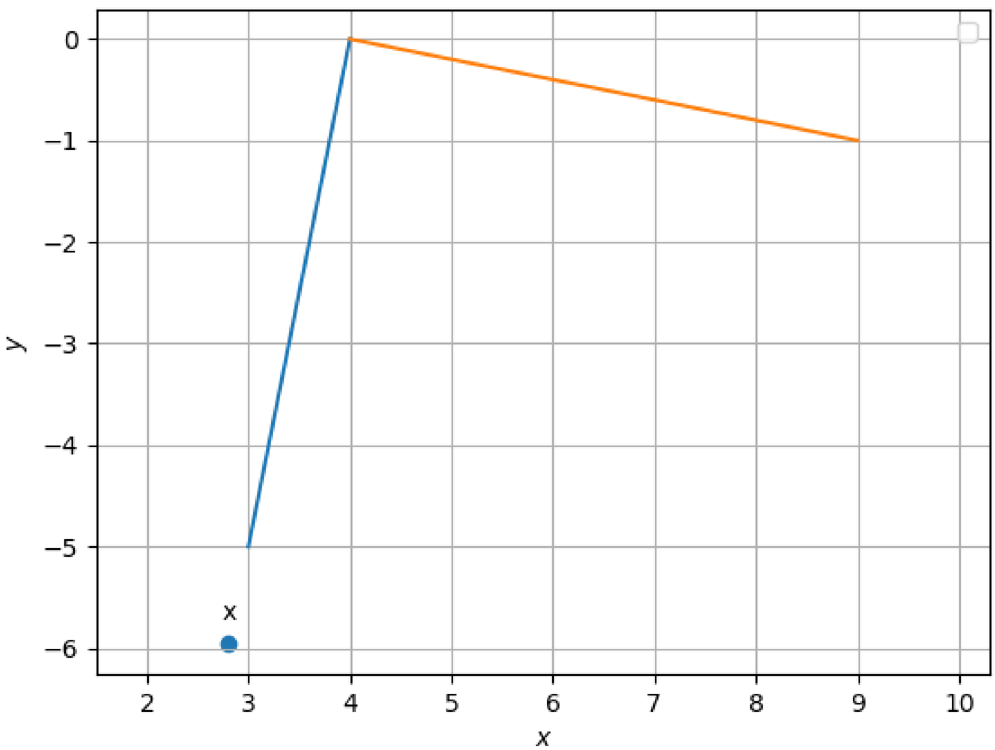


Figure 2.1.10.1:

Thus, the equation of the line is

$$\begin{pmatrix} 5 & -1 \end{pmatrix} \left( \mathbf{x} - \begin{pmatrix} -3 \\ 5 \end{pmatrix} \right) = 0 \tag{2.1.10.3}$$

$$\implies \begin{pmatrix} 5 & -1 \end{pmatrix} \mathbf{x} = -20 \tag{2.1.10.4}$$

2.1.11 A line perpendicular to the line segement joining the points (1,0) and (2,3) divides it in the ratio $1 : n$. Find the equation of the line.

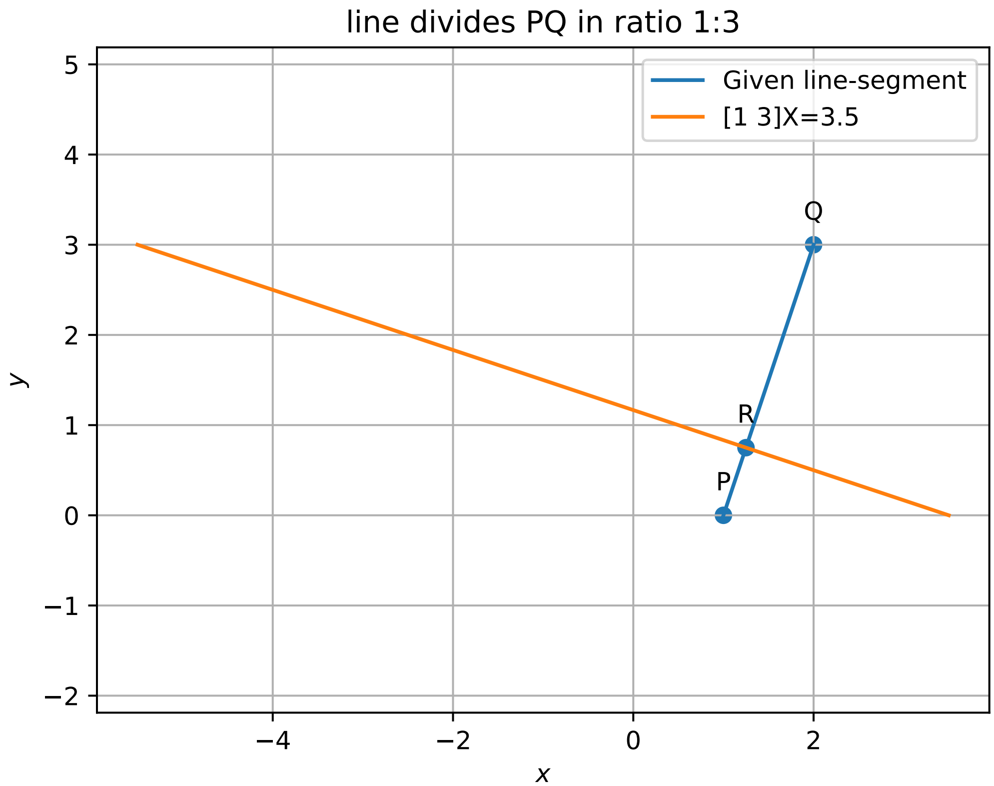


Figure 2.1.11.1:

**Solution:** Let

$$\mathbf{P} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \tag{2.1.11.1}$$

The direction vector of $PQ$ is

$$\mathbf{m} = \mathbf{Q} - \mathbf{P} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \tag{2.1.11.2}$$

Also, using section formula,

$$\mathbf{R} = \frac{\mathbf{Q} + n\mathbf{P}}{1+n} \tag{2.1.11.3}$$

and the equation of line passing through $\mathbf{R}$ is

$$\mathbf{m}^\top (\mathbf{x} - \mathbf{R}) = 0 \tag{2.1.11.4}$$

$$\implies \begin{pmatrix} 1 & 3 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 \; 3 \end{pmatrix} \begin{pmatrix} \frac{2+n}{1+n} \\ \frac{3}{1+n} \end{pmatrix} \tag{2.1.11.5}$$

$$= \frac{11+n}{1+n} \tag{2.1.11.6}$$

2.1.12

2.1.13 Find equation of a line passing trough a point (2,2) and cutting off intercepts on the axes whose sum is 9.

**Solution:** Let the $x$ intercept be $a$ and the $y$ intercept be $b$. Then

$$a + b = 9 \tag{2.1.13.1}$$

Let

$$\mathbf{P} = \begin{pmatrix} a \\ 0 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} 0 \\ b \end{pmatrix}, \mathbf{R} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \tag{2.1.13.2}$$

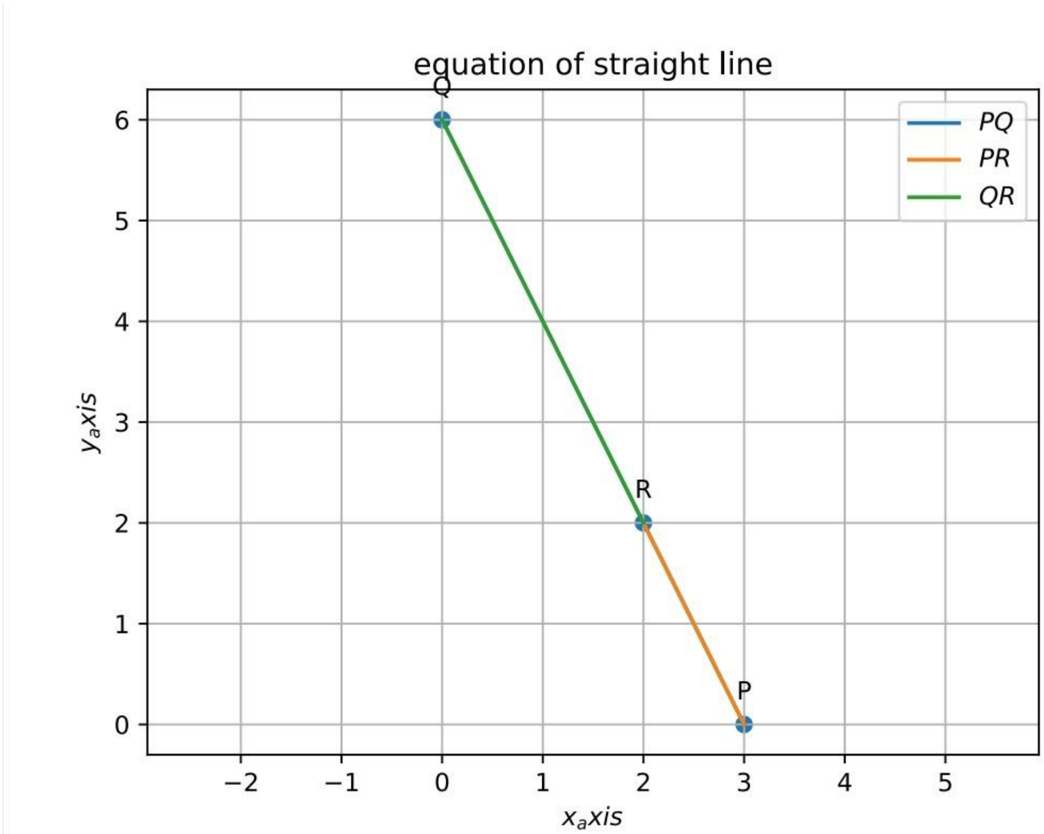


Figure 2.1.13.1:

Since the points are collinear, from (C.1.4.1), we obtain the matrix

$$\begin{pmatrix} \mathbf{P}-\mathbf{Q} & \mathbf{P}-\mathbf{R} \end{pmatrix} = \begin{pmatrix} a & a-2 \\ -b & -2 \end{pmatrix} \tag{2.1.13.3}$$

which is singular if the determinant

$$-2a + b(a-2) = ab - 2\,(a+b) = 0 \tag{2.1.13.4}$$

yielding

$$ab = 18 \tag{2.1.13.5}$$

upon substituting from (2.1.13.1). (2.1.13.5) and (2.1.13.1) form

$$x^2 - 9x + 18 = 0 \tag{2.1.13.6}$$

with roots

$$x = 6, 3 \tag{2.1.13.7}$$

$$\text{or, } \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \end{pmatrix} \tag{2.1.13.8}$$

Since the direction vector of the line is

$$\mathbf{P} - \mathbf{Q} = \begin{pmatrix} a \\ -b \end{pmatrix}, \tag{2.1.13.9}$$

the normal vector is

$$\mathbf{n} = \begin{pmatrix} b \\ a \end{pmatrix} \equiv \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{2.1.13.10}$$

Thus, the possible equations of the line are

$$\begin{pmatrix} 1 & 2 \end{pmatrix} \mathbf{x} = 6 \tag{2.1.13.11}$$

$$\begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} = 6 \tag{2.1.13.12}$$

2.1.14 Find the equation of the line through the point (0,2) making an angle

$$2\pi/3 \tag{2.1.14.1}$$

with the positive X-axis. Also find the equation of the line parallel to it and crossing the Y-axis at a distance of 2 units below the origin

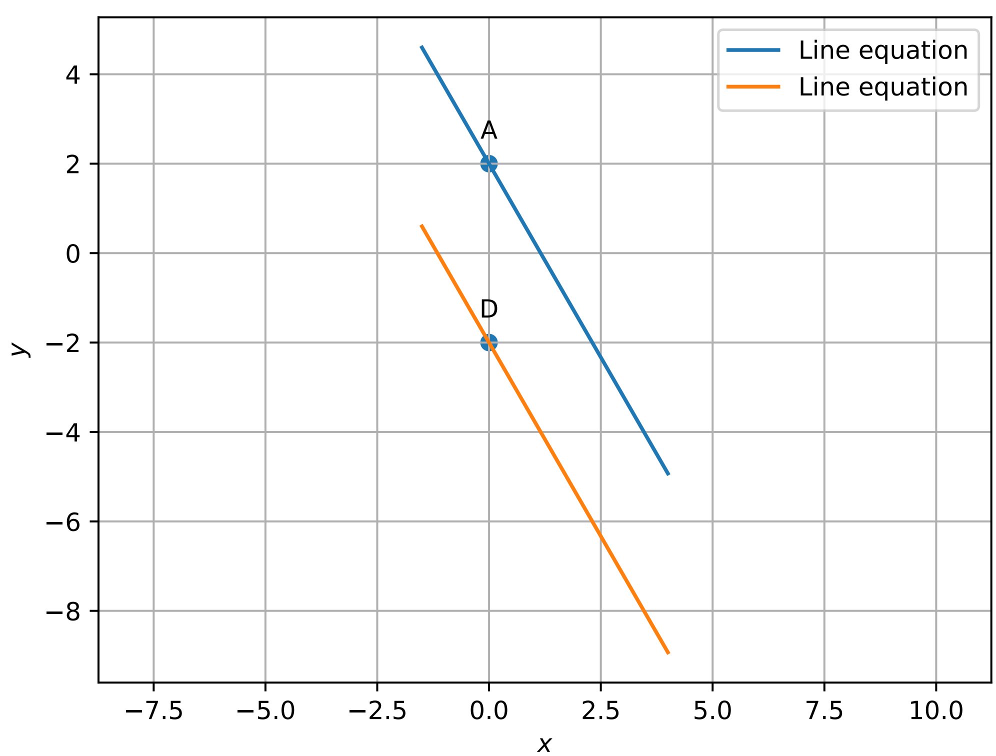


Figure 2.1.14.1:

**Solution:** From the given information, the direction vector is

$$\mathbf{m} = \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} \tag{2.1.14.2}$$

Thus, the normal vector is

$$\mathbf{n} = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix} \tag{2.1.14.3}$$

and the equation of the line is

$$\begin{pmatrix} \sqrt{3} & 1 \end{pmatrix} \left( \mathbf{x} - \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right) = 0 \tag{2.1.14.4}$$

$$\implies \begin{pmatrix} \sqrt{3} & 1 \end{pmatrix} \mathbf{x} = 2 \tag{2.1.14.5}$$

The equation of the parallel crossing the Y-axis at a distance of 2 units below the origin is given by

$$\begin{pmatrix} \sqrt{3} & 1 \end{pmatrix} \left( \mathbf{x} - \begin{pmatrix} 0 \\ -2 \end{pmatrix} \right) = 0 \tag{2.1.14.6}$$

$$\implies \begin{pmatrix} \sqrt{3} & 1 \end{pmatrix} \mathbf{x} = -2 \tag{2.1.14.7}$$

2.1.15 The perpendicular from the origin to a line meets it at the point (-2,9). Find the equation of the line.

**Solution:**

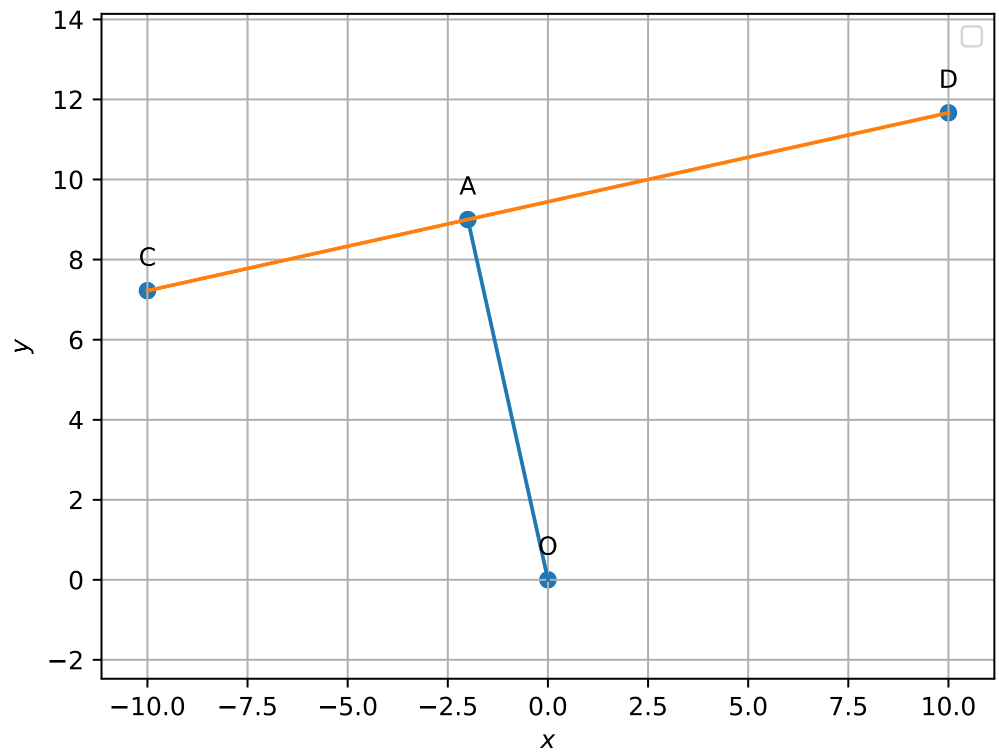


Figure 2.1.15.1:

Given

$$\mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{A} = \begin{pmatrix} -2 \\ 9 \end{pmatrix} \tag{2.1.15.1}$$

The normal vector is

$$\mathbf{n} = \mathbf{O} - \mathbf{A} \tag{2.1.15.2}$$

$$= \begin{pmatrix} 2 \\ -9 \end{pmatrix} \tag{2.1.15.3}$$

yielding the equation of the line as

$$\begin{pmatrix} 2 & -9 \end{pmatrix} \left( \mathbf{x} - \begin{pmatrix} 2 \\ -9 \end{pmatrix} \right) = 0 \tag{2.1.15.4}$$

$$\implies \begin{pmatrix} 2 & -9 \end{pmatrix} \mathbf{x} = 85 \tag{2.1.15.5}$$

2.1.16

2.1.17

2.1.18

2.1.19

2.1.20 By using the concept of equation of a line, prove that the three points (3, 0), (– 2, – 2) and (8, 2) are collinear.

**Solution:** The collinearity matrix can be expressed as

$$\begin{pmatrix} -5 & -2 \\ 5 & 2 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_1 + R_2} = \begin{pmatrix} -5 & -2 \\ 0 & 0 \end{pmatrix} \tag{2.1.20.1}$$

which is a rank 1 matrix.

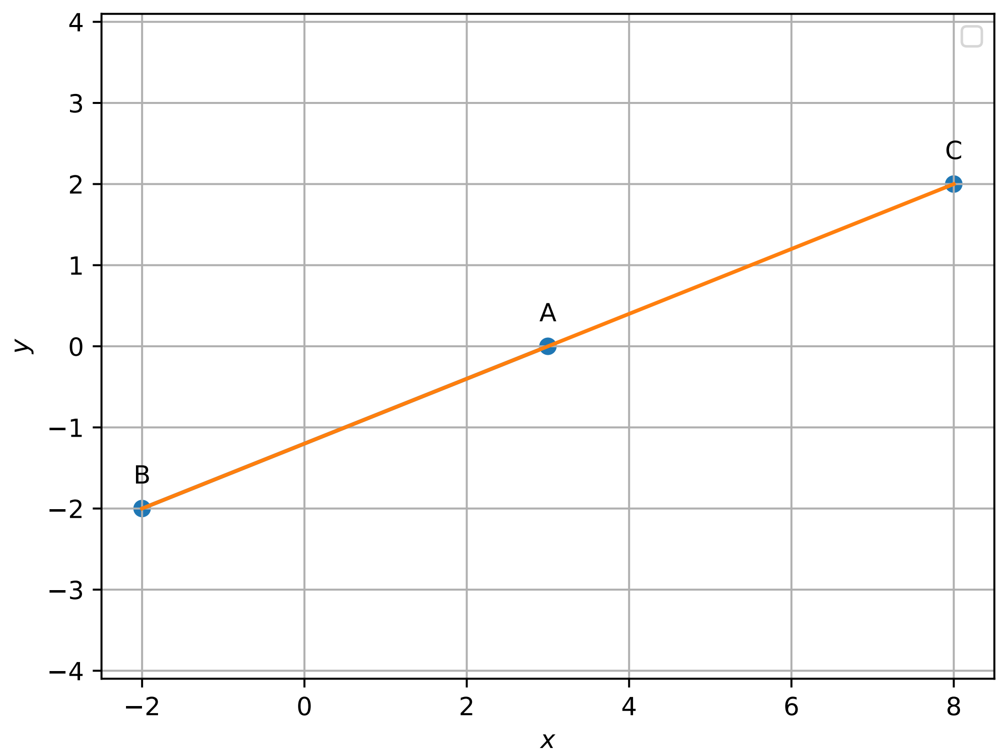


Figure 2.1.20.1:

## 2.2. General Equation of a Line

2.2.1

2.2.2

2.2.3

2.2.4

2.2.5

2.2.6

2.2.7 Find the equation of the line parallel to the line 3x-4y+2=0 and passing through the point (-2,3).

**Solution:** From the given information,

$$\mathbf{n} = \begin{pmatrix} 3 \\ -4 \end{pmatrix} \tag{2.2.7.1}$$

$$\implies \begin{pmatrix} 3 & -4 \end{pmatrix} \left\{ \mathbf{x} - \begin{pmatrix} -2 \\ 3 \end{pmatrix} \right\} = 0 \tag{2.2.7.2}$$

$$= -18 \tag{2.2.7.3}$$

which is the required equation of the line.

2.2.8

2.2.9 Find angle between the lines,$\sqrt{3}x + y = 1$ and $x + \sqrt{3}y$=1.

**Solution:** From the given equations, the normal vectors can be expressed as

$$\mathbf{n}_1 = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}, \mathbf{n}_2 = \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} \tag{2.2.9.1}$$

The angle between the lines can then be expressed as

$$cos\theta = \frac{\mathbf{n}_1^T \mathbf{n}_2}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} \tag{2.2.9.2}$$

$$= \frac{\sqrt{3}}{2} \tag{2.2.9.3}$$

$$\text{or, } \theta = 30^\circ \tag{2.2.9.4}$$

2.2.10

2.2.11

2.2.12

2.2.13

2.2.14

2.2.15

2.2.16

2.2.17

2.2.18

## 2.3. Miscellaneous Exercises

# Chapter 3

# Circles

## 3.1. Equation

3.1.1

3.1.2

3.1.3

3.1.4

3.1.5

3.1.6

3.1.7

3.1.8

3.1.9

3.1.10

3.1.11 Find the equation of the circle passing through the points (2,3) and (-1,1) and whose centre is on the line $x - 3y - 11 = 0$

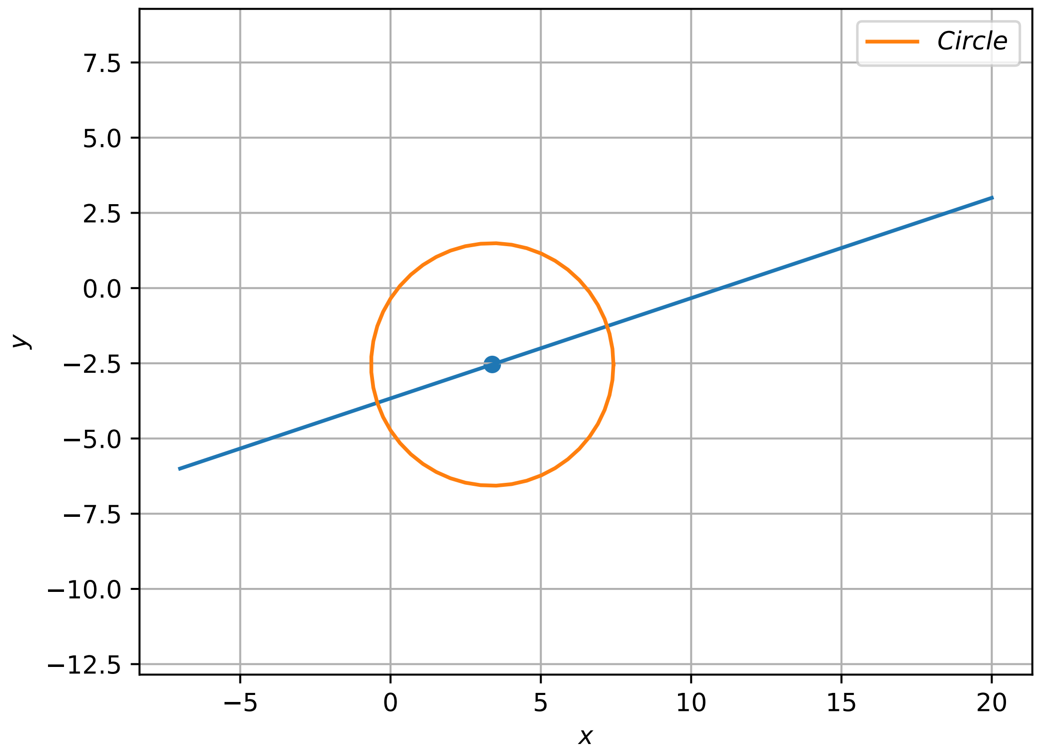


Figure 3.1.11.1:

**Solution:** See Fig. From (D.2.1.1), and the given information,

$$\|\mathbf{P}\|^2 + 2\mathbf{u}^\top \mathbf{P} + f = 0 \tag{3.1.11.1}$$

$$\|\mathbf{Q}\|^2 + 2\mathbf{u}^\top \mathbf{Q} + f = 0 \tag{3.1.11.2}$$

$$-\mathbf{n}^\top \mathbf{u} = c \tag{3.1.11.3}$$

by noting that the centre of the circle is $-\mathbf{u}$. Substituting numerical values, we obtain the matrix equation

$$\begin{pmatrix} 4 & 6 & 1 \\ -2 & 2 & 1 \\ -1 & 3 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ f \end{pmatrix} = \begin{pmatrix} -13 \\ -2 \\ 11 \end{pmatrix} \tag{3.1.11.4}$$

$$\tag{3.1.11.5}$$

The augmented matrix for (3.1.11.4) can be expressed as

$$\xleftrightarrow{1/4R_1 \leftrightarrow R_1} \left(\begin{array}{ccc|c} 1 & 3/2 & .1/4 & -13/4 \\ -2 & 2 & 1 & -2 \\ -1 & 3 & 0 & 11 \end{array}\right) \tag{3.1.11.6}$$

which can be reduced to echelon form using row operations to obtain

$$\mathbf{u} = \begin{pmatrix} -7/2 \\ 5/2 \end{pmatrix}, f = -14 \tag{3.1.11.7}$$

3.1.12 Find the equation of circle with radius 5 whose center lies on x-axis and passes through point $(2, 3)$.

**Solution:** See Fig. 3.1.12.1. From the given information, the following equations can be formulated using (D.2.1.1).

$$\|\mathbf{P}\|^2 + 2\mathbf{u}^\top \mathbf{P} + f = 0 \tag{3.1.12.1}$$

$$\mathbf{u} = k\mathbf{e}_1 \tag{3.1.12.2}$$

$$\|\mathbf{u}\|^2 - f = r^2 \tag{3.1.12.3}$$

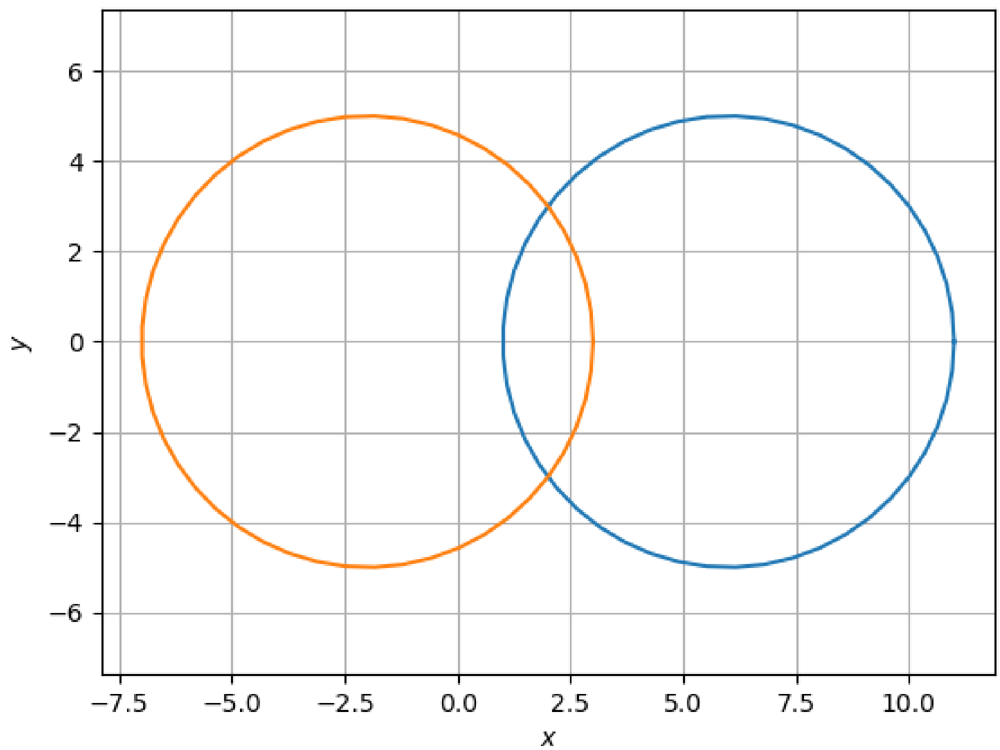


Figure 3.1.12.1:

where

$$\mathbf{P} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \text{ and } r = 5 \tag{3.1.12.4}$$

From (3.1.12.1) and (3.1.12.3),

$$\|\mathbf{P}\|^2 + 2\mathbf{u}^\top \mathbf{P} + \|\mathbf{u}\|^2 = r^2 \tag{3.1.12.5}$$

Substituting from (3.1.12.2) in the above,

$$k^2 + 2k\mathbf{e}_1^\top \mathbf{P} + \|\mathbf{P}\|^2 - r^2 = 0 \tag{3.1.12.6}$$

resulting in

$$k = -\mathbf{e}_1^\top \mathbf{P} \pm \sqrt{\left(\mathbf{e}_1^\top \mathbf{P}\right)^2 + r^2 - \|\mathbf{P}\|^2} \tag{3.1.12.7}$$

Substituting numerical values,

$$k = 2, -6 \tag{3.1.12.8}$$

resulting in circles with centre

$$-\mathbf{u} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} 6 \\ 0 \end{pmatrix}. \tag{3.1.12.9}$$

This is verified in Fig. (3.1.12.1).

## 3.2. Construction of Tangents to a Circle

3.2.1 Draw a circle of radius 6 cm. From a point 10 cm away from its centre, construct the pair of tangents to the circle and measure their lengths.

**Solution:** Follow the approach in Problem 6.4.6.

3.2.2 Construct a tangent to a circle of radius 4cm from a point on the concentric circle of radius 6cm and measure its length. Also verify the measurement by actual calculation.

**Solution:** See Fig. 3.2.2.1.

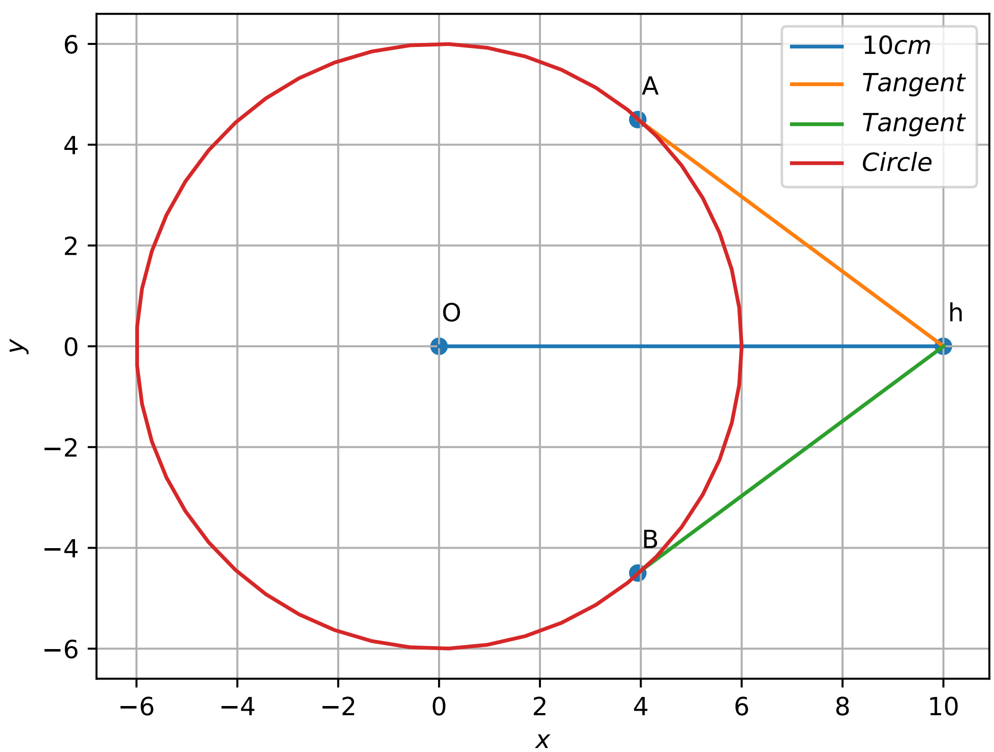


Figure 3.2.1.1:

3.2.3 Draw a circle of radius 3 cm. Take two points **P** and **Q** on one of its extended diameter each at a distance of 7 cm from its centre. Draw tangents to the circle from these two points **P** and **Q**.

**Solution:** See Fig. 3.2.3.1.

3.2.4 Draw a pair of tangents to a circle of radius 5 cm which are inclined to each other at an angle of 60°.

**Solution:** See Fig. 3.2.4.1.

3.2.5 Draw a line segment $AB$ of length 8cm. Taking **A** as centre, draw a circle of radius

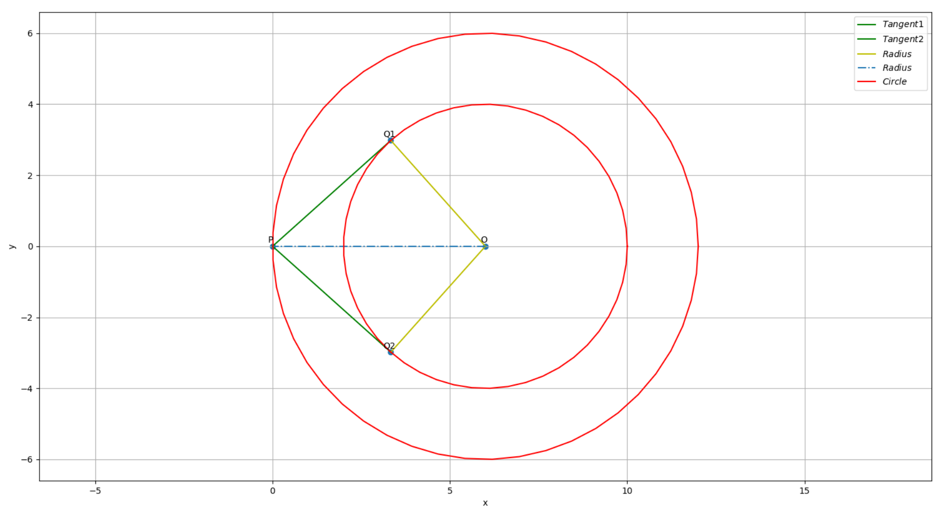


Figure 3.2.2.1:

4cm and taking **B** as centre, draw another circle of radius 3cm. Construct tangents to each circle from the centre of the circle.

**Solution:** See Fig. 3.2.5.1.

3.2.6 Let $ABC$ be a right triangle in which $AB = 6cm, BC = 8cm$ and $\angle B = 90°$. $BD$ is the perpendicular from **B** on $AC$. The circle through **B**, **C**, **D** is drawn. Construct the tangents from **A** to this circle.

**Solution:** See Fig. 3.2.6.1.

$$BD \perp AC \implies \mathbf{O} = \frac{\mathbf{B}+\mathbf{C}}{2} \tag{3.2.6.1}$$

From (C.1.11.1), the coordinates of **D** can be obtained.

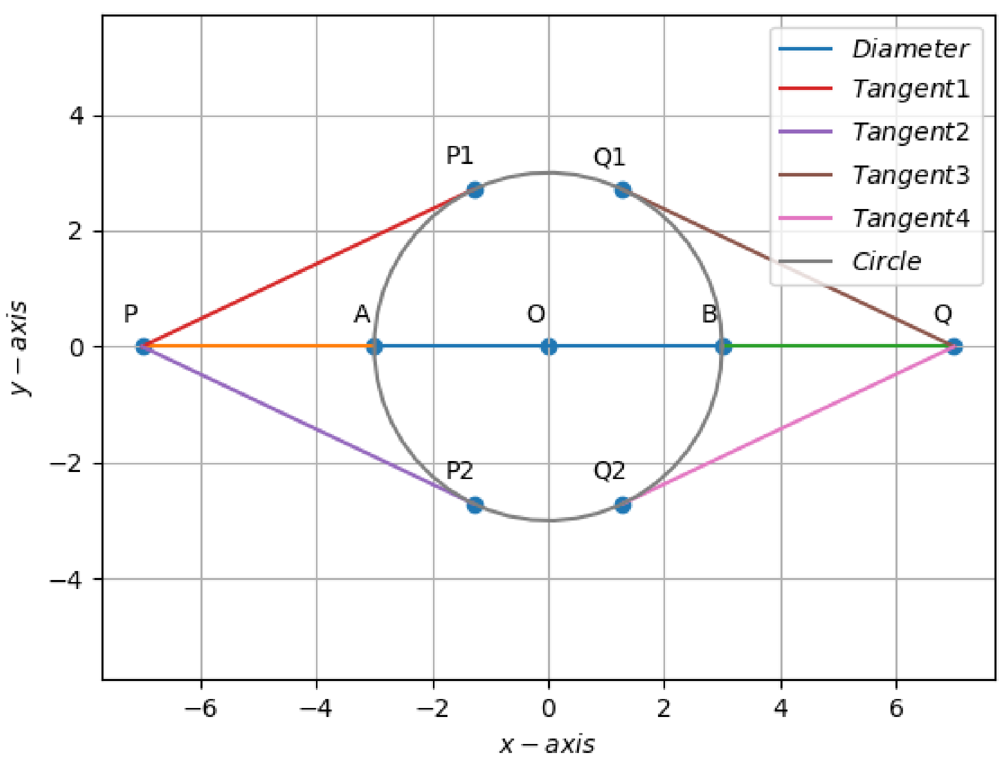
Diameter
Tangent1
Tangent2
Tangent3
Tangent4
Circle
P1
Q1
P
A
O
B
Q
P2
Q2
y − axis
x − axis
4
2
0
−2
−4
−6
6

Figure 3.2.3.1:

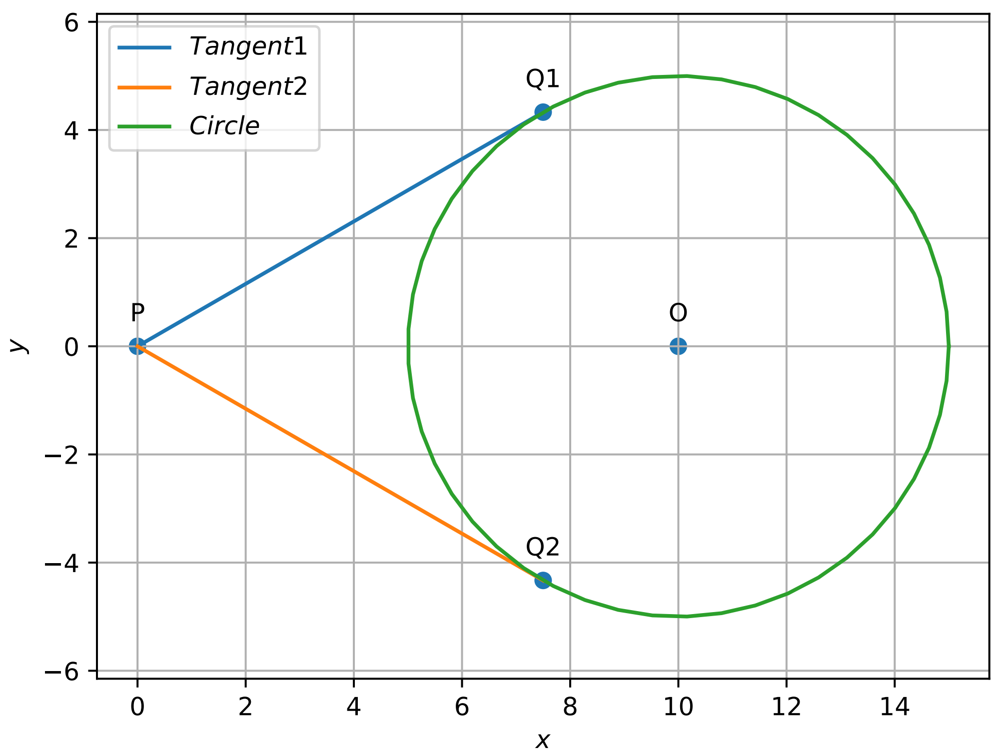


Figure 3.2.4.1:

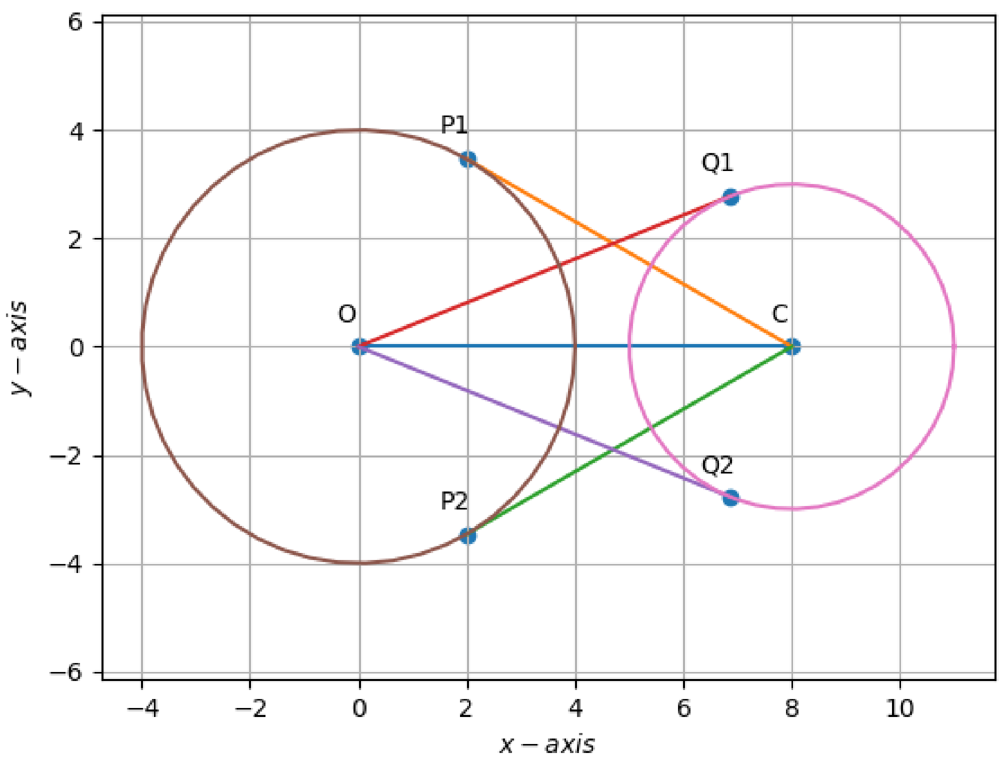


Figure 3.2.5.1:

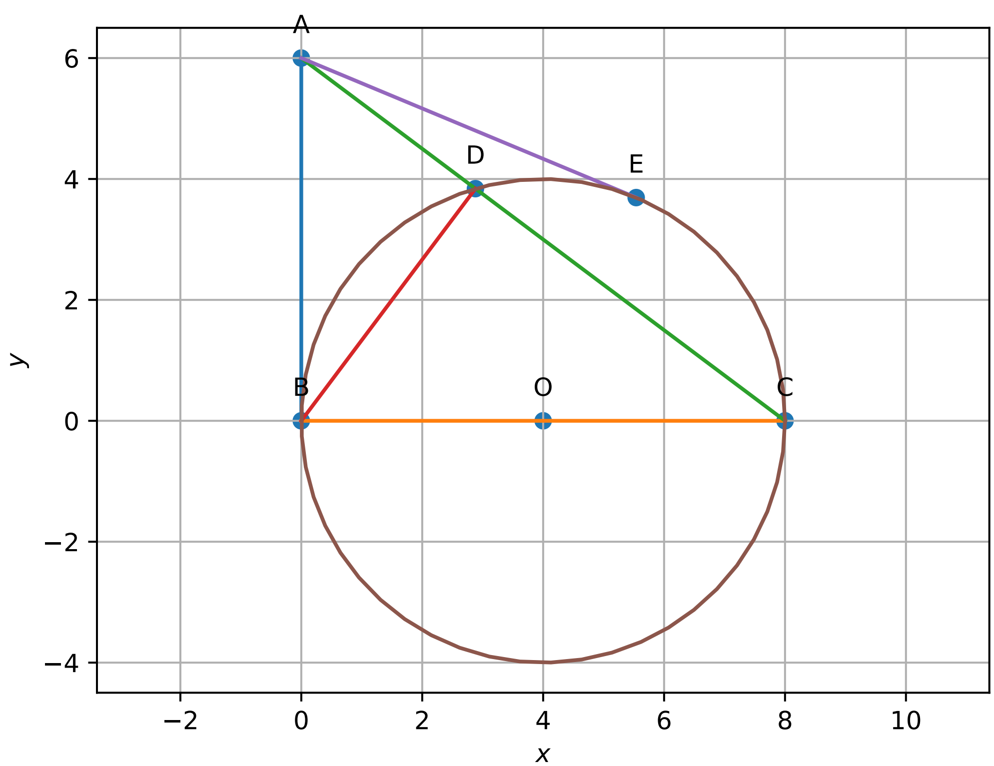


Figure 3.2.6.1:

# Chapter 4

# Triangle Constructions

## 4.1. Introduction

4.1.1 Construct a triangle $ABC$ in which $BC = 7cm, \angle B = 75°$ and $AB + AC = 13cm$.

**Solution:** See Fig. 4.1.1.1.

Using the cosine formula in $\triangle ABC$,

$$b^2 = a^2 + c^2 - 2ac\cos B \tag{4.1.1.1}$$

$$\implies (b+c)(b-c) = a^2 - 2ac\cos B \tag{4.1.1.2}$$

$$\text{or, } K(b-c) = a^2 - 2ac\cos B \tag{4.1.1.3}$$

where

$$K = b + c \tag{4.1.1.4}$$

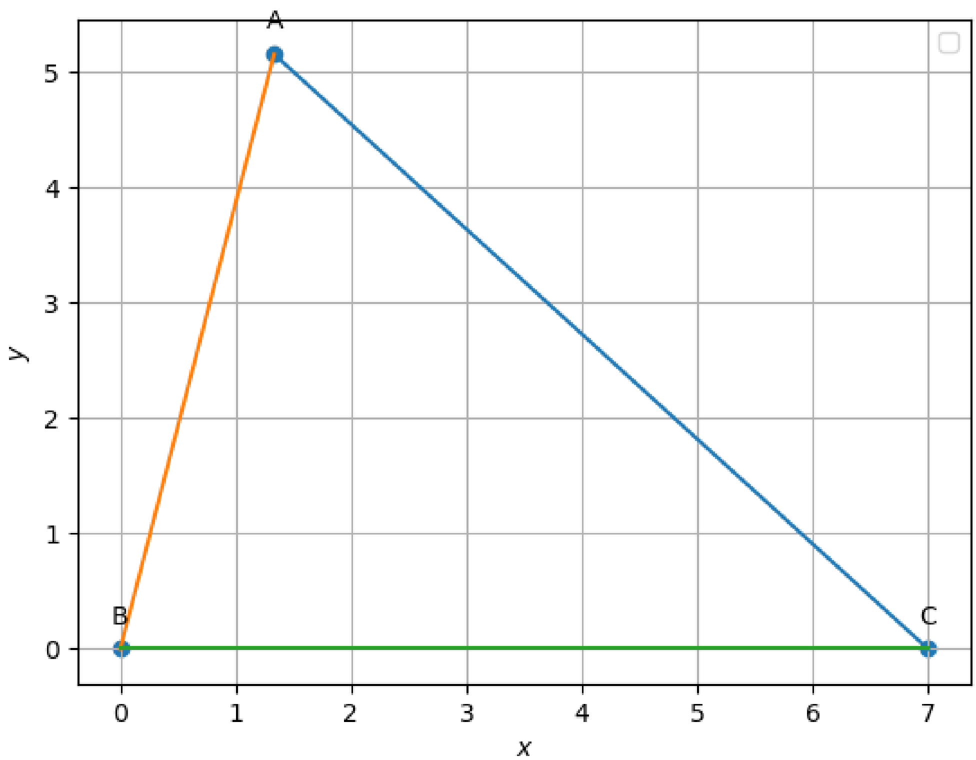


Figure 4.1.1.1:

From (4.1.1.3) and (4.1.1.4),

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} b \\ c \end{pmatrix} = \begin{pmatrix} \frac{a^2-2ac\cos B}{K} \\ K \end{pmatrix} \tag{4.1.1.5}$$

$$\implies \begin{pmatrix} b \\ c \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \frac{a^2-2ac\cos B}{K} \\ K \end{pmatrix} \tag{4.1.1.6}$$

$$\because \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = 2\mathbf{I} \tag{4.1.1.7}$$

From (4.1.1.6)

$$c = \frac{1}{2}\mathbf{e}_2^\top \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \frac{a^2}{K} \\ K \end{pmatrix} - \frac{2ac\cos B}{K} \tag{4.1.1.8}$$

$$\implies c = \frac{1}{2\left(1 + \frac{2a\cos B}{K}\right)}\mathbf{e}_2^\top \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \frac{a^2}{K} \\ K \end{pmatrix} \tag{4.1.1.9}$$

The coordinates of $\triangle ABC$ can then be expressed as

$$\mathbf{A} = c \begin{pmatrix} \cos B \\ \sin B \end{pmatrix}, \mathbf{B} = \mathbf{0}, \mathbf{C} = \begin{pmatrix} a \\ 0 \end{pmatrix}. \tag{4.1.1.10}$$

4.1.2 Construct a triangle $ABC$ in which $BC = 8cm, \angle B = 45°$ and $AB - AC = 3.5cm$.

**Solution:** See Fig. 4.1.2.1. Using the cosine formula in $\triangle ABC$,

$$b^2 = a^2 + c^2 - 2ac\cos B \tag{4.1.2.1}$$

$$\implies (b+c)(b-c) = a^2 - 2ac\cos B \tag{4.1.2.2}$$

$$\text{or, } K(b+c) = a^2 - 2ac\cos B \tag{4.1.2.3}$$

where

$$-K = b - c \tag{4.1.2.4}$$

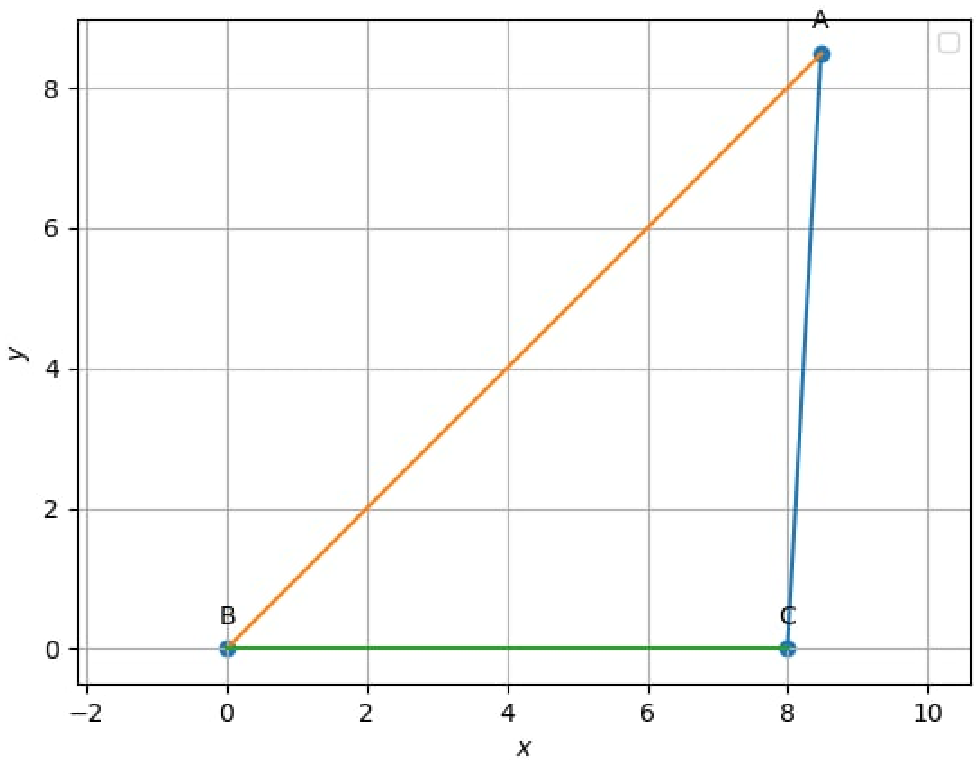


Figure 4.1.2.1:

From (4.1.2.3) and (4.1.2.4),

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} b \\ c \end{pmatrix} = \begin{pmatrix} \frac{a^2-2ac\cos B}{K} \\ -K \end{pmatrix} \tag{4.1.2.5}$$

$$\implies \begin{pmatrix} b \\ c \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \frac{a^2-2ac\cos B}{K} \\ -K \end{pmatrix} \tag{4.1.2.6}$$

$$\because \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = 2\mathbf{I} \tag{4.1.2.7}$$

From (4.1.2.6)

$$c = \frac{1}{2}\mathbf{e}_2^\top \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \frac{a^2}{K} \\ -K \end{pmatrix} - \frac{2ac\cos B}{K} \tag{4.1.2.8}$$

$$\implies c = \frac{1}{2\left(1 + \frac{2a\cos B}{K}\right)}\mathbf{e}_2^\top \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \frac{a^2}{K} \\ -K \end{pmatrix} \tag{4.1.2.9}$$

The coordinates of $\triangle ABC$ can then be expressed as

$$\mathbf{A} = c \begin{pmatrix} \cos B \\ \sin B \end{pmatrix}, \mathbf{B} = \mathbf{0}, \mathbf{C} = \begin{pmatrix} a \\ 0 \end{pmatrix}. \tag{4.1.2.10}$$

4.1.3 Construct a triangle $PQR$ in which $QR = 6cm, \angle Q = 60°$ and $PR - PQ = 2cm$.

**Solution:** Same as Problem 4.1.1 with

$$\angle Q = \angle B, QR = a, PR = b, PQ = c \tag{4.1.3.1}$$

4.1.4 Construct a triangle $XYZ$ in which $\angle Y = 30°, \angle Z = 90°$ and $XY + YZ + ZX = 11cm$.

**Solution:** From the given information,

$$x + y + z = K \tag{4.1.4.1}$$

$$y\cos Z + z\cos Y - x = 0 \tag{4.1.4.2}$$

$$y\sin Z - z\sin Y = 0 \tag{4.1.4.3}$$

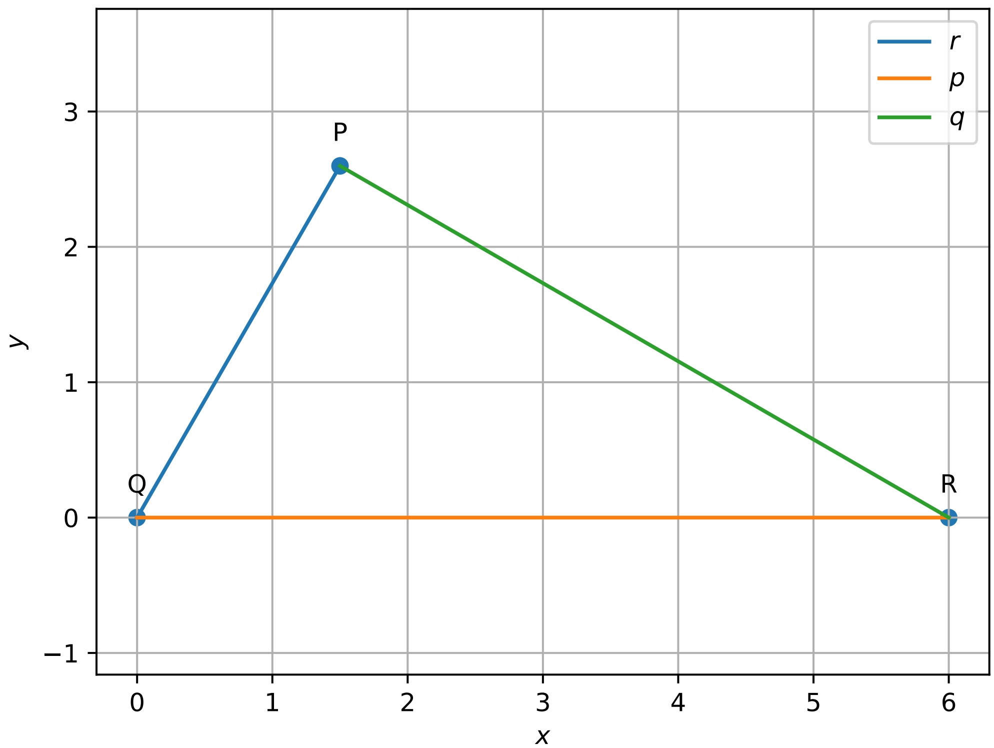


Figure 4.1.3.1:

resulting in the matrix equation

$$\begin{pmatrix} 1 & 1 & 1 \\ \cos Z & \cos Y & -1 \\ \sin Z & -\sin Y & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = K\mathbf{e}_1 \tag{4.1.4.4}$$

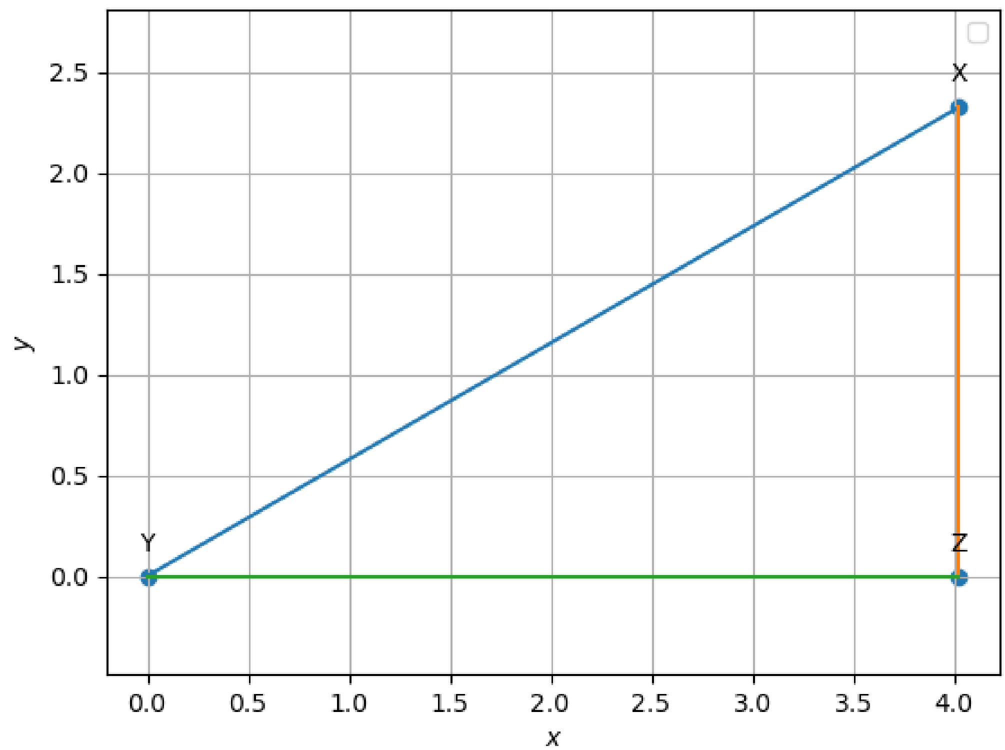


Figure 4.1.4.1:

which can be solved to obtain all the sides. $\triangle XYZ$ can then be plotted using

$$\mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix}, \mathbf{Y} = \mathbf{0}, \mathbf{Z} = \begin{pmatrix} x \\ 0 \end{pmatrix} \tag{4.1.4.5}$$

4.1.5 Construct a right triangle whose base is 12cm and sum of its hypotenuse and other side is 18cm.

**Solution:** From the given information, let

$$a = 12, \angle B = 90^\circ, b + c = 18 \tag{4.1.5.1}$$

We need to find $b$. This is similar to Problem 4.1.1.

## 4.2. Properties

4.2.1 In the Figure 4.2.1.1, $\mathbf{E}$ is any point on median $AD$ of a $\triangle ABC$. Show that

$$ar(ABE) = ar(ACE). \tag{4.2.1.1}$$

*Proof.* From (A.1.3.1)

$$ar\left(BDE\right) = \frac{1}{2}\left\|\mathbf{B}\times\mathbf{D} + \mathbf{D}\times\mathbf{E} + \mathbf{E}\times\mathbf{B}\right\| \tag{4.2.1.2}$$

$$= \frac{1}{2}\left\|\mathbf{B}\times\left(\frac{\mathbf{B}+\mathbf{C}}{2}\right) + \left(\frac{\mathbf{B}+\mathbf{C}}{2}\right)\times\mathbf{E} + \mathbf{E}\times\mathbf{B}\right\| \tag{4.2.1.3}$$

$$= \frac{1}{4}\left\|\mathbf{B}\times\mathbf{C} + \mathbf{C}\times\mathbf{E} + \mathbf{E}\times\mathbf{B}\right\| \tag{4.2.1.4}$$

after simplification. Similarly, it can be shown that

$$ar\left(EDC\right) = \frac{1}{4}\left\|\mathbf{B}\times\mathbf{C} + \mathbf{C}\times\mathbf{E} + \mathbf{E}\times\mathbf{B}\right\| \tag{4.2.1.5}$$

$$= ar\left(BDE\right) \tag{4.2.1.6}$$

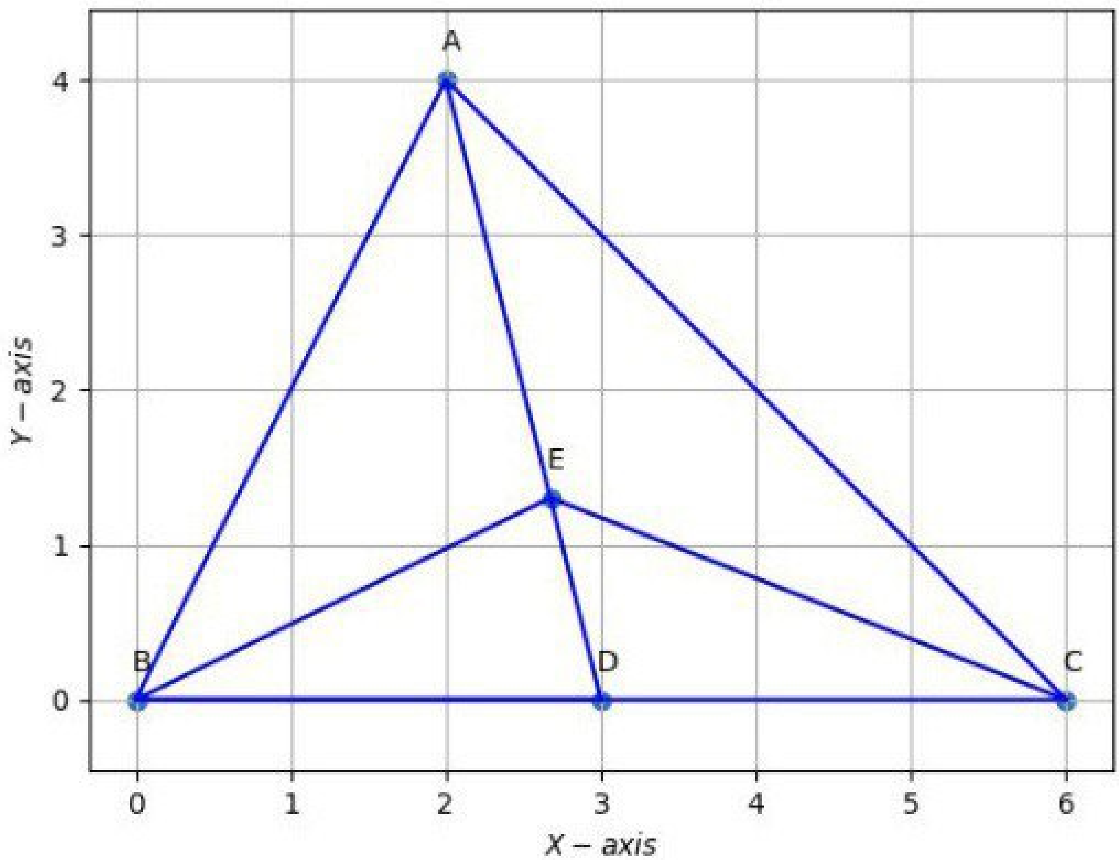


Figure 4.2.1.1:

The same approach can be used to show that

$$ar\,(ADB) = ar\,(ADC) \tag{4.2.1.7}$$

Subtracting (4.2.1.6) from (4.2.1.7) yields (4.2.1.1)

$\square$

4.2.2 In $\triangle ABC$, $\mathbf{E}$ is the mid-point of median $AD$. Show that

$$ar(\triangle BED) = \frac{1}{4} ar(\triangle ABC) \tag{4.2.2.1}$$

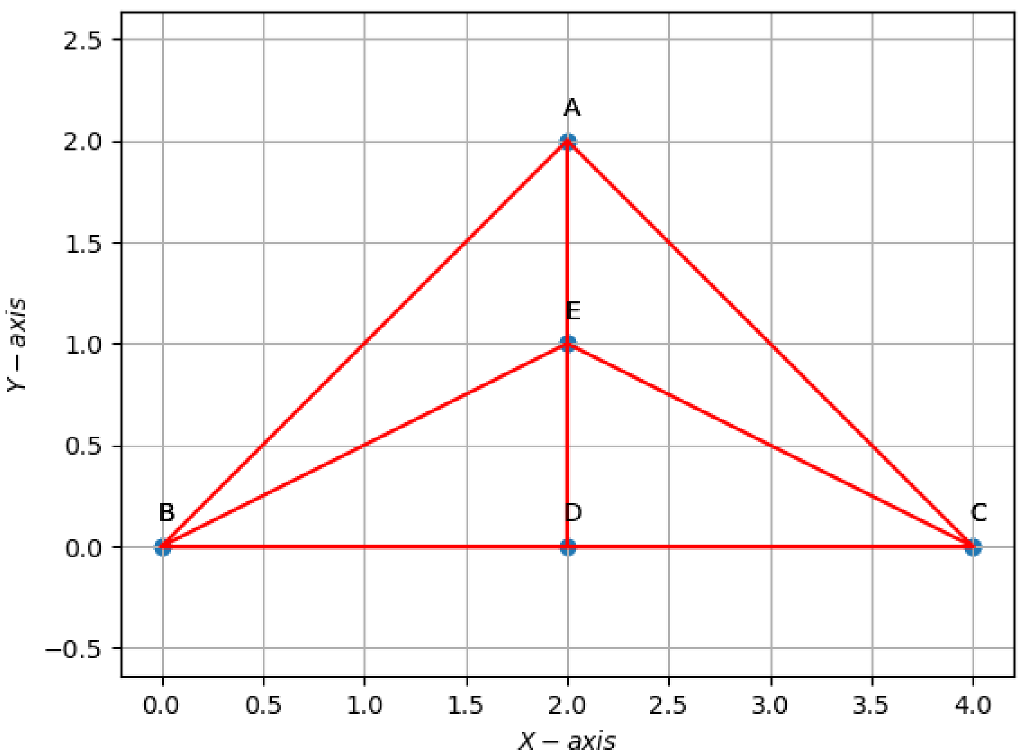


Figure 4.2.2.1:

*Proof.* From Problem 4.2.2,

$$ar(\triangle BED) = \frac{1}{4} \left\| \mathbf{B} \times \mathbf{C} + \mathbf{C} \times \mathbf{E} + \mathbf{E} \times \mathbf{B} \right\| \tag{4.2.2.2}$$

Since

$$\mathbf{E} = \frac{\mathbf{A}+\mathbf{D}}{2} \tag{4.2.2.3}$$
$$= \frac{2\mathbf{A}+\mathbf{B}+\mathbf{C}}{4}, \tag{4.2.2.4}$$

substituting the above in (4.2.2.2) yields

$$ar(\triangle BED) = \frac{1}{4}\left\|\mathbf{B}\times\mathbf{C}+\mathbf{C}\times\frac{2\mathbf{A}+\mathbf{B}+\mathbf{C}}{4}+\frac{2\mathbf{A}+\mathbf{B}+\mathbf{C}}{4}\times\mathbf{B}\right\| \tag{4.2.2.5}$$
$$= \frac{1}{8}\left\|\mathbf{A}\times\mathbf{B}+\mathbf{B}\times\mathbf{C}+\mathbf{C}\times\mathbf{A}\right\| \tag{4.2.2.6}$$

resulting in (4.2.2.1). □

4.2.3 Show that the diagonals of a parallelogram divide it into four triangles of equal area.

*Proof.* See Fig. 4.2.3.1. From Appendix A.1.25 and A.1.3

$$ar\,(AOB) = \frac{1}{2}\left\|\mathbf{A}\times\mathbf{O}+\mathbf{O}\times\mathbf{B}+\mathbf{B}\times\mathbf{A}\right\| \tag{4.2.3.1}$$
$$= \frac{1}{2}\left\|\mathbf{A}\times\left(\frac{\mathbf{A}+\mathbf{C}}{2}\right)+\left(\frac{\mathbf{A}+\mathbf{C}}{2}\right)\times\mathbf{B}+\mathbf{B}\times\mathbf{A}\right\| \tag{4.2.3.2}$$
$$= \frac{1}{4}\left\|\mathbf{A}\times\mathbf{C}+\mathbf{C}\times\mathbf{B}+\mathbf{B}\times\mathbf{A}\right\| \tag{4.2.3.3}$$

yielding the desired result from Appendix A.1.26 □

4.2.4 $ABC, ABD$ are 2 triangles on same base $AB$, if line segment $CD$ is bisected by $AB$ at $\mathbf{O}$, show that

$$ar(ABC) = ar(ABD) \tag{4.2.4.1}$$

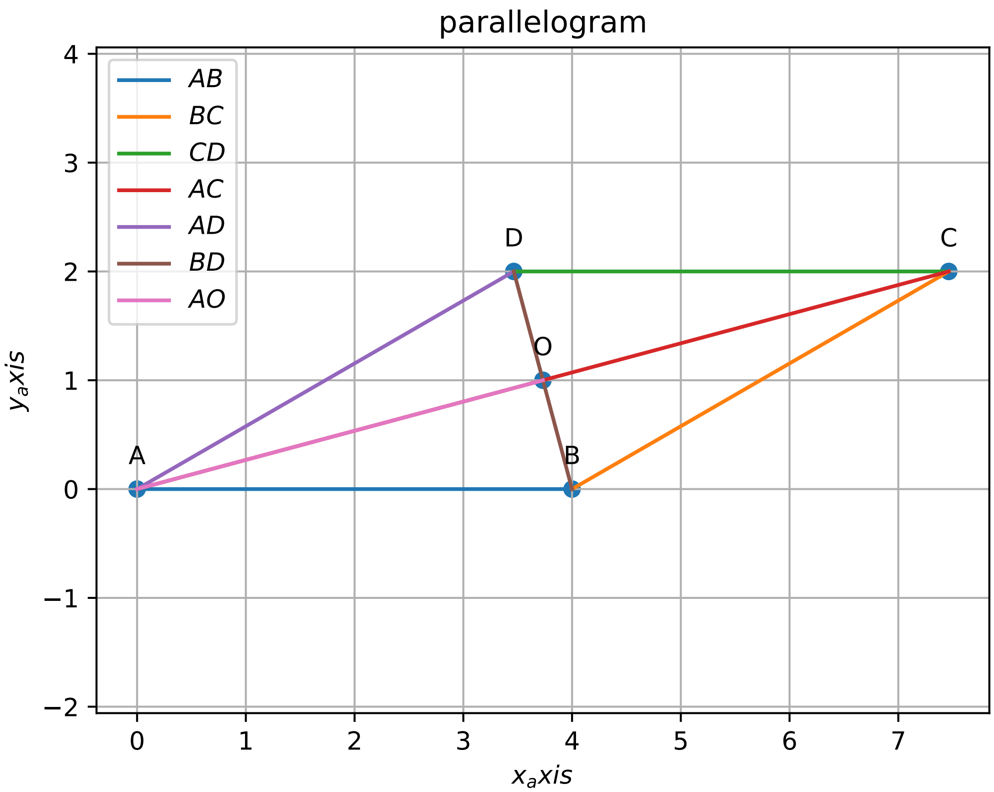


Figure 4.2.3.1:

*Proof.* See Fig. 4.2.4.1. $AO$ and $OB$ are medians of triangles $ADC$ and $BDC$. From Appendix A.1.5, (4.2.4.1) is trivial. □

4.2.5

4.2.6

4.2.7

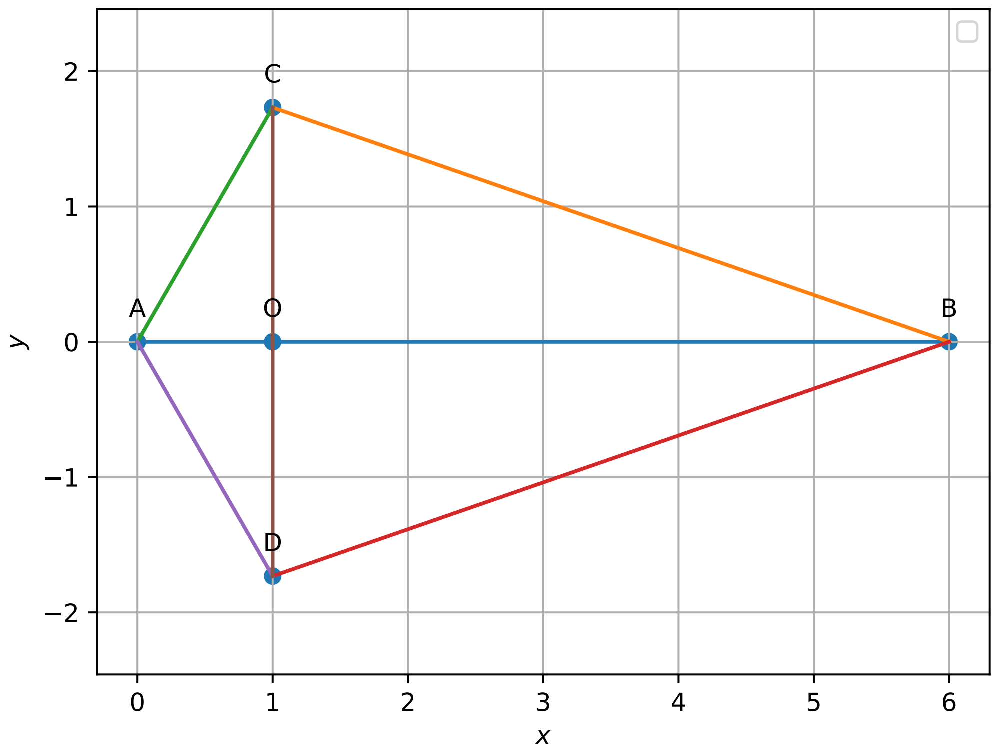


Figure 4.2.4.1:

4.2.8

4.2.9 The side $AB$ of a parallelogram $ABCD$ is produced to any point **P**. A line through **A** and parallel to $CP$ meets $CB$ produced at **Q** and then parallelogram $PBQR$ is completed. Show that

$$ar(ABCD) = ar(PBQR) \tag{4.2.9.1}$$

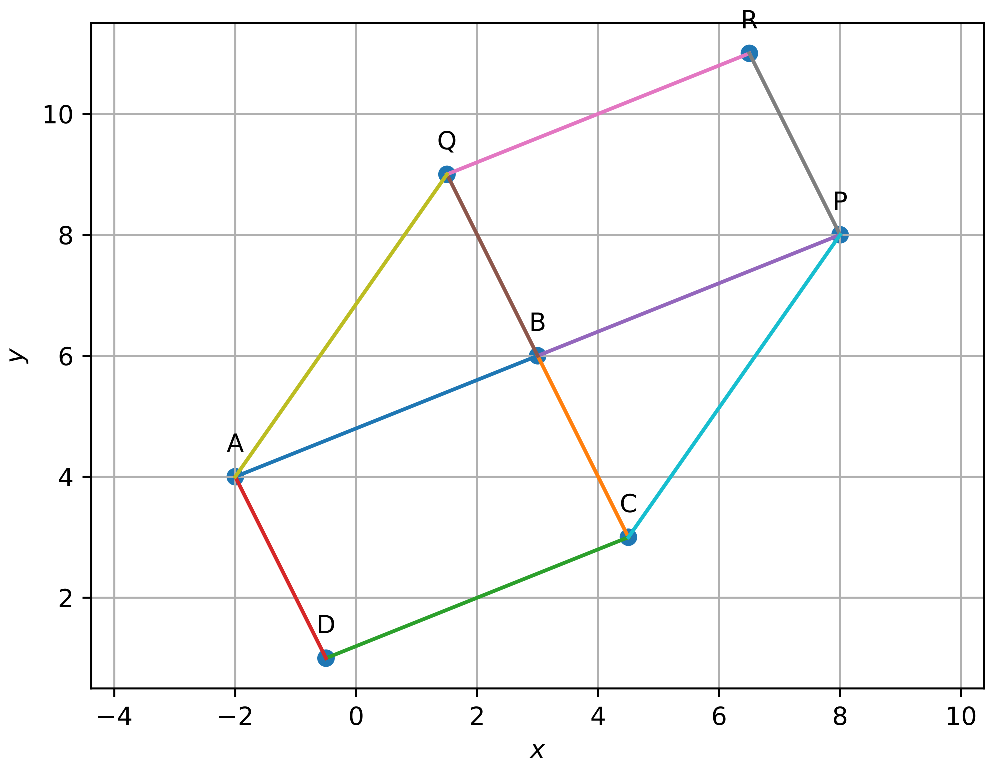


Figure 4.2.9.1:

*Proof.* From the given information, using section formula,

$$\mathbf{Q} = \frac{k_1\mathbf{C} + \mathbf{B}}{k_1 + 1} \tag{4.2.9.2}$$

$$\mathbf{P} = \frac{k_2\mathbf{A} + \mathbf{B}}{k_2 + 1} \tag{4.2.9.3}$$

Also, since $AQ \parallel CP$,

$$\mathbf{A} - \mathbf{Q} = k\left(\mathbf{C} - \mathbf{P}\right) \tag{4.2.9.4}$$

Substituting from (4.2.9.2) and (4.2.9.3) in the above,

$$\mathbf{A} - \frac{k_1\mathbf{C} + \mathbf{B}}{k_1 + 1} = k\left(\mathbf{C} - \frac{k_2\mathbf{A} + \mathbf{B}}{k_2 + 1}\right) \tag{4.2.9.5}$$

which, after some algebra, yields

$$\left(1 + \frac{kk_2}{k_2 + 1}\right)\mathbf{A} + \left(\frac{k}{k_2 + 1} - \frac{1}{k_1 + 1}\right)\mathbf{B} - \left(\frac{k_1}{k_1 + 1} + k\right)\mathbf{C} = \mathbf{0} \tag{4.2.9.6}$$

From Appendix A.1.27, (4.2.9.6) results in

$$\left(\frac{k}{k_2 + 1} - \frac{1}{k_1 + 1}\right) = \left(\frac{k_1}{k_1 + 1} + k\right) = 0 \tag{4.2.9.7}$$

$$\text{or, } k_1 + k_2 = -1 \tag{4.2.9.8}$$

From Appendix A.1.26

$$ar\left(PBQR\right) = \left\|\mathbf{P} \times \mathbf{B} + \mathbf{B} \times \mathbf{Q} + \mathbf{Q} \times \mathbf{P}\right\| \tag{4.2.9.9}$$

The R.H.S. in the above can be expressed as

$$\frac{k_2\mathbf{A} + \mathbf{B}}{k_2 + 1} \times \mathbf{B} + \mathbf{B} \times \frac{k_1\mathbf{C} + \mathbf{B}}{k_1 + 1} + \frac{k_1\mathbf{C} + \mathbf{B}}{k_1 + 1} \times \frac{k_2\mathbf{A} + \mathbf{B}}{k_2 + 1} \tag{4.2.9.10}$$

leading to

$$\begin{aligned}\left(\frac{k_2}{k_2 + 1} - \frac{k_2}{(k_1 + 1)(k_2 + 1)}\right)\mathbf{A} \times \mathbf{B}\\ + \mathbf{B} \times \mathbf{C}\left(\frac{k_1}{k_1 + 1} - \frac{k_1}{(k_1 + 1)(k_2 + 1)}\right)\\ + \frac{k_1k_2}{(k_1 + 1)(k_2 + 1)}\mathbf{C} \times \mathbf{A}\end{aligned} \tag{4.2.9.11}$$

that can be simplified to obtain

$$ar\,(PBQR) = \frac{k_1 k_2}{(k_1+1)\,(k_2+1)}\,\|(\mathbf{A}\times\mathbf{B}+\mathbf{B}\times\mathbf{C}+\mathbf{C}\times\mathbf{A})\| \tag{4.2.9.12}$$

$$= \|(\mathbf{A}\times\mathbf{B}+\mathbf{B}\times\mathbf{C}+\mathbf{C}\times\mathbf{A})\| \tag{4.2.9.13}$$

using the fact that

$$\frac{k_1 k_2}{(k_1+1)\,(k_2+1)} = 1 \tag{4.2.9.14}$$

from (4.2.9.8). Also, from Appendix A.1.26,

$$ar\,(ABCD) = \|(\mathbf{A}\times\mathbf{B}+\mathbf{B}\times\mathbf{C}+\mathbf{C}\times\mathbf{A})\| \tag{4.2.9.15}$$

yielding (4.2.9.1) from (4.2.9.13). □

4.2.10

4.2.11 $ABCDE$ is a pentagon. A line through $\mathbf{B}$ parallel to $AC$ meets $DC$ produced at $F$. Show that

$$ar(ACB) = ar(ACF) \tag{4.2.11.1}$$

$$ar(AEDF) = ar(ABCDE) \tag{4.2.11.2}$$

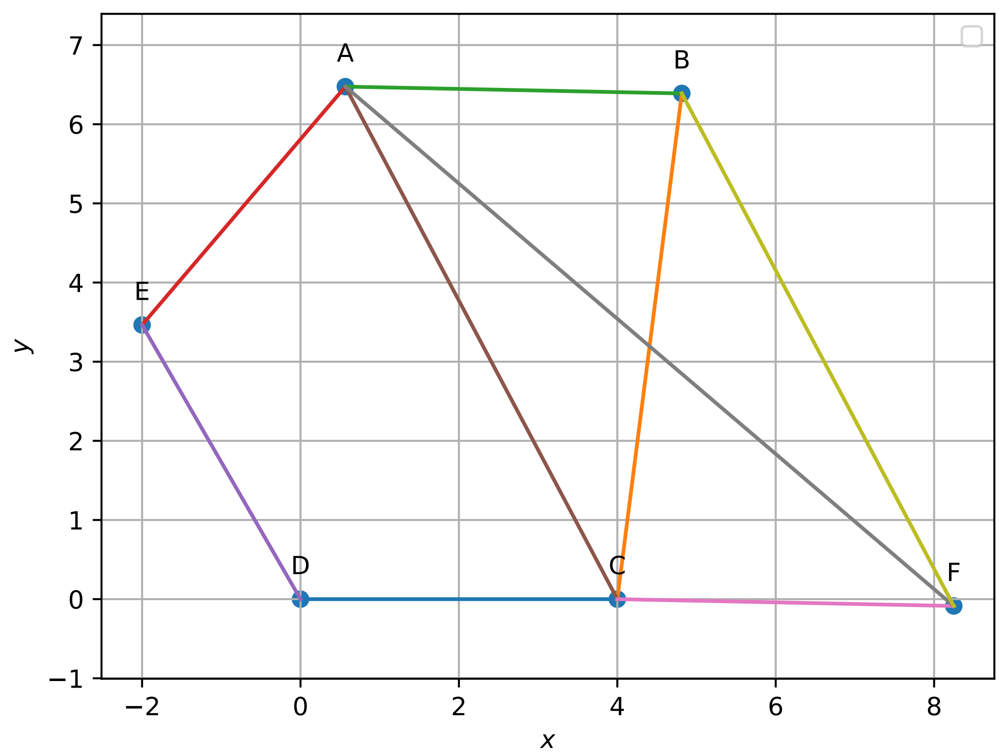


Figure 4.2.11.1:

*Proof.* Since $BF \parallel AC$,

$$\mathbf{F} - \mathbf{B} = k\left(\mathbf{C} - \mathbf{A}\right) \tag{4.2.11.3}$$

$$\implies \mathbf{F} = \mathbf{B} + k\left(\mathbf{C} - \mathbf{A}\right) \tag{4.2.11.4}$$

Thus, from Appendix A.1.3,

$$ar(ACF) = \frac{1}{2}\left\|\mathbf{F} \times \mathbf{A} + \mathbf{A} \times \mathbf{C} + \mathbf{C} \times \mathbf{F}\right\| \tag{4.2.11.5}$$

Substituting from (4.2.11.4) in (4.2.11.5),

$$ar(ACF) = \frac{1}{2}\left\|\{\mathbf{B} + k(\mathbf{C} - \mathbf{A})\} \times \mathbf{A} + \mathbf{A} \times \mathbf{C} + \mathbf{C} \times \{\mathbf{B} + k(\mathbf{C} - \mathbf{A})\}\right\| \tag{4.2.11.6}$$

$$= \frac{1}{2}\left\|\mathbf{B} \times \mathbf{A} + \mathbf{A} \times \mathbf{C} + \mathbf{C} \times \mathbf{B}\right\| \tag{4.2.11.7}$$

$$= ar(ACB) \tag{4.2.11.8}$$

upon substituting from from Appendix A.1.3. (4.2.11.2) follows from (4.2.11.1).

□

4.2.12

4.2.13

4.2.14

4.2.15

4.2.16 In the Figure 4.2.16.1,

$$ar(DRC) = ar(DPC) \tag{4.2.16.1}$$

$$ar(BDP) = ar(ARC). \tag{4.2.16.2}$$

Show that the quadrilaterals $ABCD$ and $DCPR$ are trapeziums.

*Proof.* From Appendix A.1.4 and (4.2.16.1),

$$\frac{1}{2}\left\|(\mathbf{D} - \mathbf{R}) \times (\mathbf{D} - \mathbf{C})\right\| = \frac{1}{2}\left\|(\mathbf{C} - \mathbf{D}) \times (\mathbf{C} - \mathbf{P})\right\| \tag{4.2.16.3}$$

$$\implies (\mathbf{D} - \mathbf{R}) \times (\mathbf{D} - \mathbf{C}) = (\mathbf{C} - \mathbf{D}) \times (\mathbf{C} - \mathbf{P}) \tag{4.2.16.4}$$

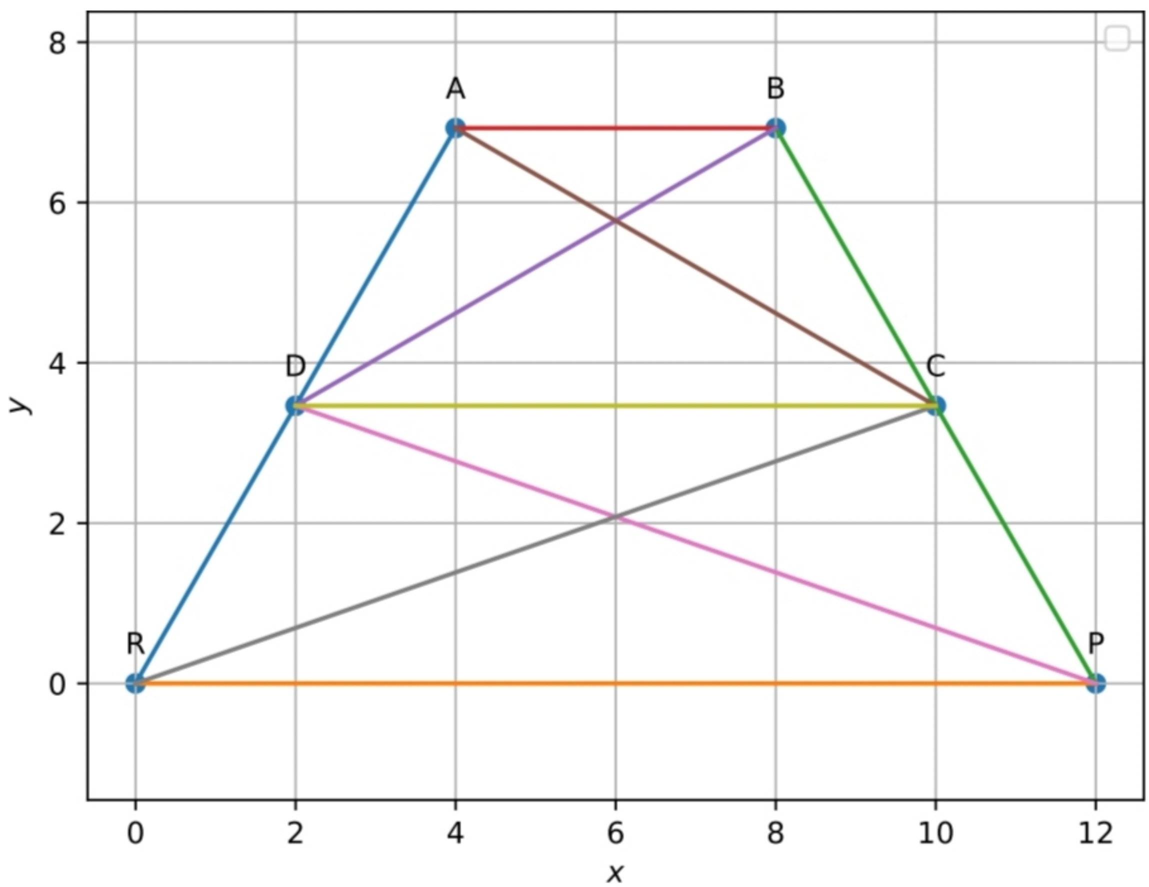


Figure 4.2.16.1:

which can be expressed as

$$(\mathbf{C}-\mathbf{D})\times(\mathbf{C}-\mathbf{D}+\mathbf{R}-\mathbf{P})=\mathbf{0} \tag{4.2.16.5}$$

$$\implies (\mathbf{C}-\mathbf{D})\times(\mathbf{R}-\mathbf{P})=\mathbf{0} \tag{4.2.16.6}$$

$$\text{or, } CD \parallel RP \tag{4.2.16.7}$$

Hence, $DCPR$ is a trapezium. Similarly, it can be shown that $ABCD$ is also a trapezium.

$\square$

# Chapter 5

# Quadrilateral Construction

## 5.1. Properties

5.1.1 The angles of quadrilateral are in the ratio 3:5:9:13. Find all the angles of the quadrilateral.

5.1.2 If diagonals of a parallelogram are equal then show that it is a rectangle.

**Solution:** See Fig. 5.1.2.1. From (A.1.24.1),

$$\mathbf{B} - \mathbf{A} = \mathbf{C} - \mathbf{D} \tag{5.1.2.1}$$

$$\implies \mathbf{B} - \mathbf{C} = \mathbf{A} - \mathbf{D} \tag{5.1.2.2}$$

Also, it is given that the diagonals of $ABCD$ are equal. Hence,

$$\|\mathbf{C} - \mathbf{A}\|^2 = \|\mathbf{D} - \mathbf{B}\|^2 \tag{5.1.2.3}$$

$$\implies \|(\mathbf{C} - \mathbf{B}) + (\mathbf{B} - \mathbf{A})\|^2 = \|(\mathbf{D} - \mathbf{C}) + (\mathbf{C} - \mathbf{B}\|^2 \tag{5.1.2.4}$$

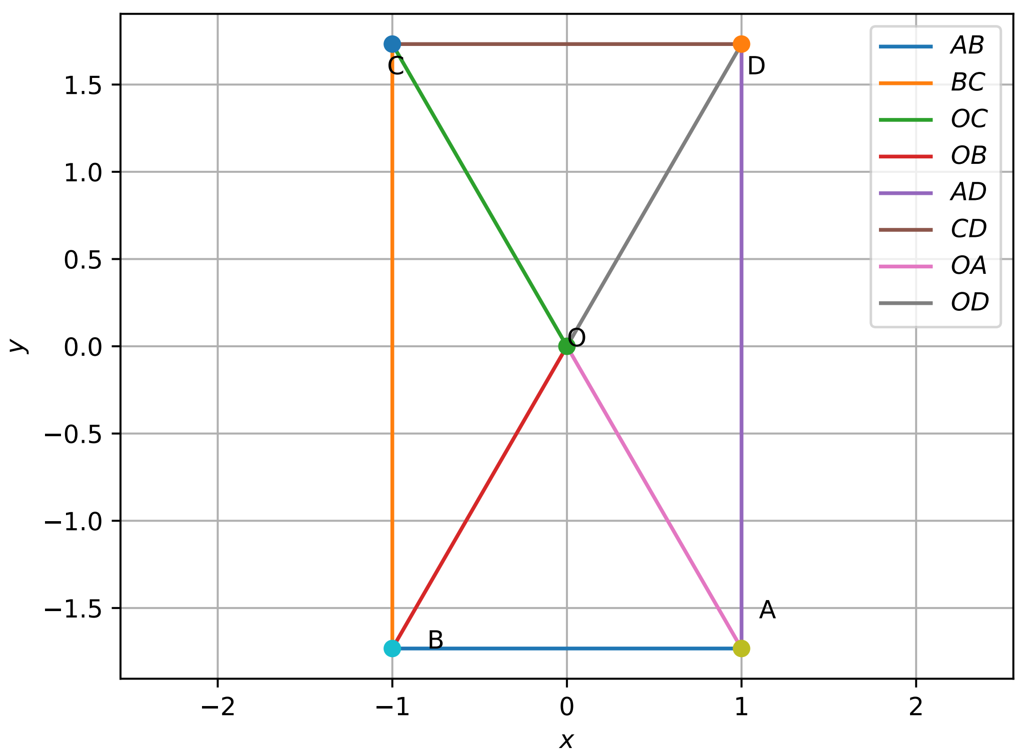


Figure 5.1.2.1:

which can be expressed as

$$\|\mathbf{C}-\mathbf{B}\|^2+\|\mathbf{B}-\mathbf{A}\|^2+2(\mathbf{C}-\mathbf{B})^\top(\mathbf{B}-\mathbf{A}) \\ =\|\mathbf{D}-\mathbf{C}\|^2+\|\mathbf{C}-\mathbf{B}\|^2+2(\mathbf{D}-\mathbf{C})^\top(\mathbf{C}-\mathbf{B}) \quad (5.1.2.5)$$

which, can be simplified to obtain

$$(\mathbf{C}-\mathbf{B})^\top(\mathbf{B}-\mathbf{A})=(\mathbf{D}-\mathbf{C})^\top(\mathbf{C}-\mathbf{B}) \quad (5.1.2.6)$$

since

$$\|\mathbf{D} - \mathbf{C}\| = \|\mathbf{B} - \mathbf{A}\| \tag{5.1.2.7}$$

yielding

$$(\mathbf{A} - \mathbf{B})^\top (\mathbf{B} - \mathbf{C}) = \mathbf{0} \tag{5.1.2.8}$$

from (5.1.2.1).

5.1.3 Show that if the diagonals of a quadrilateral bisect each other at right angles, then it is a rhombus.

**Solution:** See Fig. 5.1.3.1. From the given information,

$$\frac{\mathbf{B} + \mathbf{D}}{2} = \frac{\mathbf{A} + \mathbf{C}}{2} \tag{5.1.3.1}$$

$$(\mathbf{B} - \mathbf{D})^\top (\mathbf{A} - \mathbf{C}) = 0 \tag{5.1.3.2}$$

From (5.1.3.1),

$$\mathbf{B} - \mathbf{A} = \mathbf{C} - \mathbf{D} \tag{5.1.3.3}$$

which, from (A.1.24.1), is the definition of a parallelogram. Further, substituting

$$\mathbf{B} - \mathbf{D} = (\mathbf{B} - \mathbf{A}) + (\mathbf{A} - \mathbf{D}) \tag{5.1.3.4}$$

$$\mathbf{A} - \mathbf{C} = (\mathbf{A} - \mathbf{B}) + (\mathbf{B} - \mathbf{C}) \tag{5.1.3.5}$$

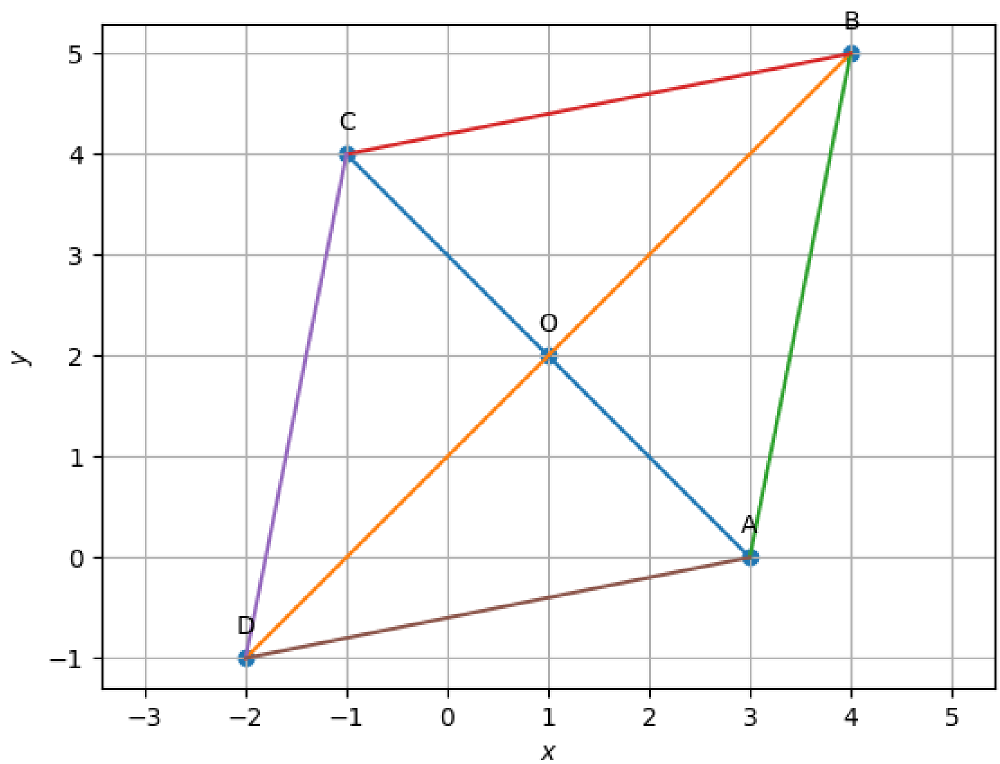


Figure 5.1.3.1: Rhombus

in (5.1.3.2),

$$[(\mathbf{B}-\mathbf{A})+(\mathbf{A}-\mathbf{D})]^\top [(\mathbf{A}-\mathbf{B})+(\mathbf{B}-\mathbf{C})]=0$$

$$\implies -\|\mathbf{B}-\mathbf{A}\|^2+(\mathbf{B}-\mathbf{A})^\top(\mathbf{B}-\mathbf{C})+$$

$$(\mathbf{A}-\mathbf{D})^\top(\mathbf{A}-\mathbf{B})+(\mathbf{A}-\mathbf{D})^\top(\mathbf{B}-\mathbf{C})=0 \quad (5.1.3.6)$$

From (5.1.3.3),

$$\mathbf{B}-\mathbf{C}=\mathbf{A}-\mathbf{D} \tag{5.1.3.7}$$

$$\implies (\mathbf{B}-\mathbf{A})^{\top}(\mathbf{B}-\mathbf{C})+(\mathbf{A}-\mathbf{D})^{\top}(\mathbf{A}-\mathbf{B})=\mathbf{0} \tag{5.1.3.8}$$

and

$$(\mathbf{A}-\mathbf{D})^{\top}(\mathbf{B}-\mathbf{C})=\left\|\mathbf{B}-\mathbf{C}\right\|^2 \tag{5.1.3.9}$$

Substituting from

(5.1.3.8) and (5.1.3.9) in (5.1.3.6),

$$\left\|\mathbf{A}-\mathbf{B}\right\|^2=\left\|\mathbf{B}-\mathbf{C}\right\|^2 \tag{5.1.3.10}$$

which means that the adjacent sides of the parallelogram are equal. Thus, the quadrilateral is a rhombus

5.1.4 Show that the diagonals of a square are equal and bisect each other at right angles.
**Solution:** This is obvious from Problems (5.1.2) and (5.1.3).

5.1.5

5.1.6 Diagonal AC of a parallelogram ABCD bisects $\angle A$ in Fig (5.1.6.1). Show that

(a) it bisects $\angle C$ also

(b) $ABCD$ is a rhombus

**Solution:**

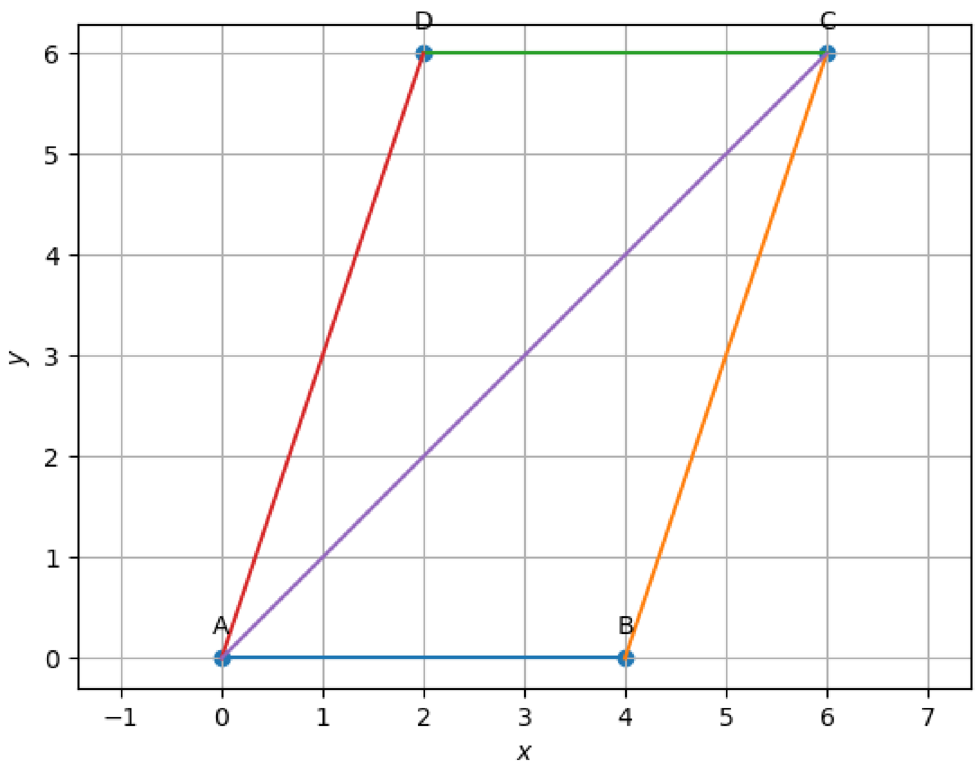


Figure 5.1.6.1:

(a) From (A.1.12.1),

$$\angle BAC = \angle DAC \tag{5.1.6.1}$$

$$\implies \frac{(\mathbf{A}-\mathbf{B})^T(\mathbf{A}-\mathbf{C})}{\|\mathbf{A}-\mathbf{B}\|\,\|\mathbf{A}-\mathbf{C}\|} = \frac{(\mathbf{A}-\mathbf{D})^T(\mathbf{A}-\mathbf{C})}{\|\mathbf{A}-\mathbf{D}\|\,\|\mathbf{A}-\mathbf{C}\|} \tag{5.1.6.2}$$

Also,

$$\cos\angle ACD = \frac{(\mathbf{C}-\mathbf{D})^T(\mathbf{C}-\mathbf{A})}{\|\mathbf{C}-\mathbf{D}\|\,\|\mathbf{C}-\mathbf{A}\|} \tag{5.1.6.3}$$

From Appendix A.1.24.1,

$$\mathbf{B} - \mathbf{A} = \mathbf{C} - \mathbf{D} \tag{5.1.6.4}$$

$$\implies \frac{(\mathbf{C} - \mathbf{D})^T (\mathbf{C} - \mathbf{A})}{\|\mathbf{C} - \mathbf{D}\| \, \|\mathbf{C} - \mathbf{A}\|} = \frac{(\mathbf{B} - \mathbf{A})^T (\mathbf{C} - \mathbf{A})}{\|\mathbf{B} - \mathbf{A}\| \, \|\mathbf{C} - \mathbf{A}\|} \tag{5.1.6.5}$$

upon substituting in (5.1.6.3). Thus, from (5.1.6.3) and (5.1.6.1),

$$\angle BAC = \angle DAC = \angle ACD \tag{5.1.6.6}$$

Similarly, it can be shown that

$$\angle ACD = \angle ACB \tag{5.1.6.7}$$

(b)

5.1.7 $ABCD$ is a rhombus. Show that the diagonal $AC$ bisects angle $A$ as well as angle $C$ and diagonal $BD$ bisects angle $B$ as well as angle $D$.

**Solution:** For the rhombus in Fig. 5.1.7.1,

$$\begin{aligned} \|\mathbf{A} - \mathbf{B}\| &= \|\mathbf{A} - \mathbf{D}\| \\ \mathbf{A} - \mathbf{B} &= \mathbf{D} - \mathbf{C} \end{aligned} \tag{5.1.7.1}$$

From (A.1.12.1),

$$\begin{aligned} \cos\angle BAC &= \frac{(\mathbf{A} - \mathbf{B})^T (\mathbf{A} - \mathbf{C})}{\|\mathbf{A} - \mathbf{B}\| \, \|\mathbf{A} - \mathbf{C}\|} \\ \cos\angle DAC &= \frac{(\mathbf{C} - \mathbf{D})^T (\mathbf{C} - \mathbf{A})}{\|\mathbf{C} - \mathbf{D}\| \, \|\mathbf{C} - \mathbf{A}\|} \end{aligned} \tag{5.1.7.2}$$

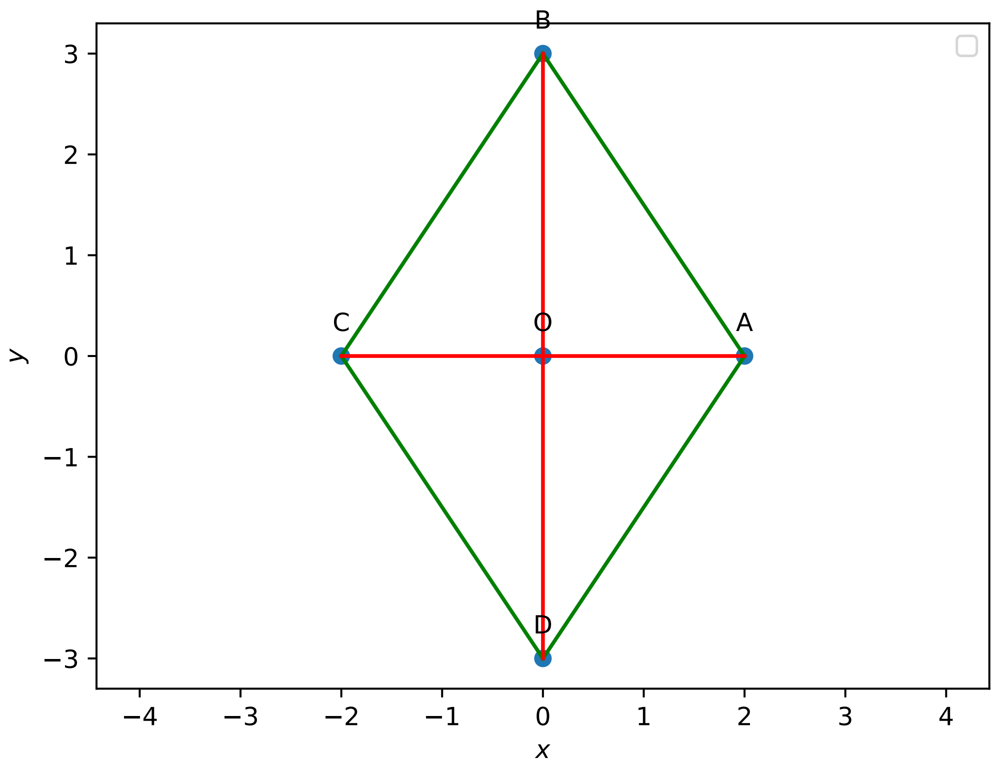


Figure 5.1.7.1:

From (5.1.7.1) and (5.1.7.2), we obtain

$$\cos \angle BAC = \cos \angle DAC \tag{5.1.7.3}$$

Thus, $AC$ bisects $\angle A$. Similarly, the remaining results can be proved.

5.1.8

5.1.9 In parallelogram $ABCD$, two points **P** and **Q** are taken on diagonal $BD$ such that $DP = BQ$. Show that

(a) $\triangle APD \cong \triangle CQB$

(b) $AP = CQ$

(c) $\triangle AQB \cong \triangle CPD$

(d) $AQ = CP$

(e) $APCQ$ is a parallelogram

**Solution:** See Fig. 5.1.9.1.

From (A.1.12.1) and the given information,

$$\mathbf{A} - \mathbf{B} = \mathbf{D} - \mathbf{C} \tag{5.1.9.1}$$

$$\implies \mathbf{A} - \mathbf{D} = \mathbf{B} - \mathbf{C} \tag{5.1.9.2}$$

$$\mathbf{B} - \mathbf{Q} = \mathbf{P} - \mathbf{D} \quad \text{(given)} \tag{5.1.9.3}$$

From (5.1.9.1) and (5.1.9.3)

$$\mathbf{A} - \mathbf{P} = \mathbf{Q} - \mathbf{C} \tag{5.1.9.4}$$

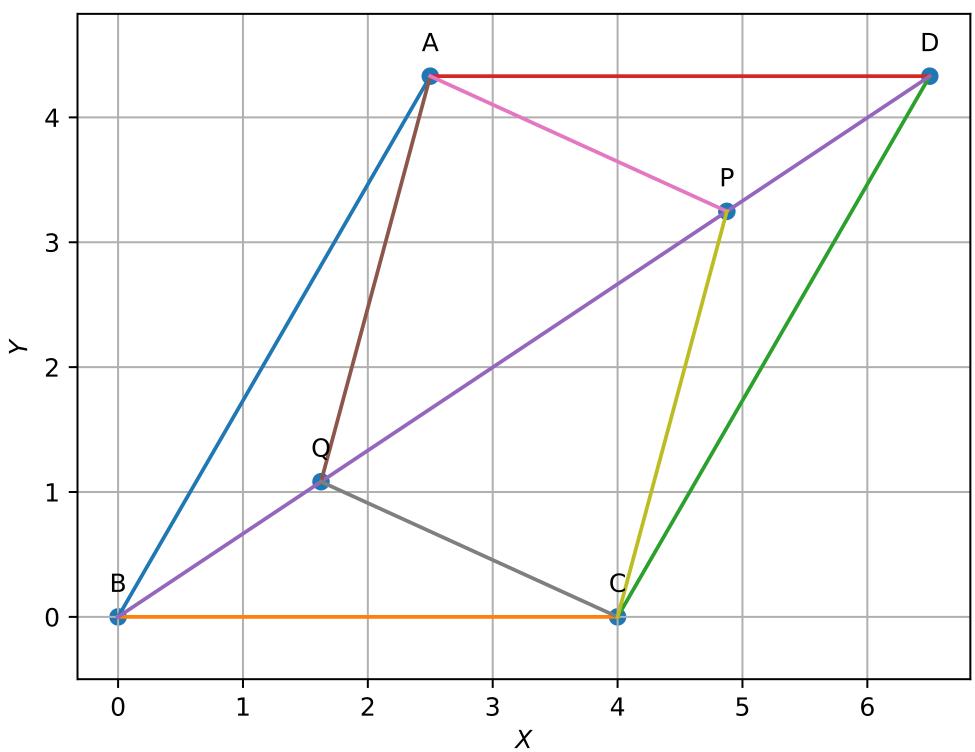


Figure 5.1.9.1:

(a) From (5.1.9.1), (5.1.9.3) and (5.1.9.4) taking the norms of the respective sides,

$$\triangle APD \cong \triangle CQB \tag{5.1.9.5}$$

(b) From (5.1.9.4), taking the norm,

$$AP = CQ \tag{5.1.9.6}$$

(c) From (5.1.9.1), (5.1.9.3) and (5.1.9.4) taking the norms of the respective sides,

$$\triangle AQB \cong \triangle CPD \tag{5.1.9.7}$$

(d) From (5.1.9.4),

$$AQ = CP \tag{5.1.9.8}$$

5.1.10 $ABCD$ is a parallelogram and $AP$ and $CQ$ are perpendiculars from vertices **A** and **C** on diagonal $BD$ . Show that

(a) $\triangle APB \cong \triangle CQD$

(b) $AP = CQ$

**Solution:** From Fig. 5.1.10.1, and (A.1.12.1),

$$\begin{aligned} \cos\angle ABD &= \frac{(\mathbf{A}-\mathbf{B})^T(\mathbf{D}-\mathbf{B})}{\|\mathbf{A}-\mathbf{B}\|\,\|\mathbf{D}-\mathbf{B}\|} \\ \cos\angle CDB &= \frac{(\mathbf{C}-\mathbf{D})^T(\mathbf{B}-\mathbf{D})}{\|\mathbf{C}-\mathbf{D}\|\,\|\mathbf{B}-\mathbf{D}\|} \end{aligned} \tag{5.1.10.1}$$

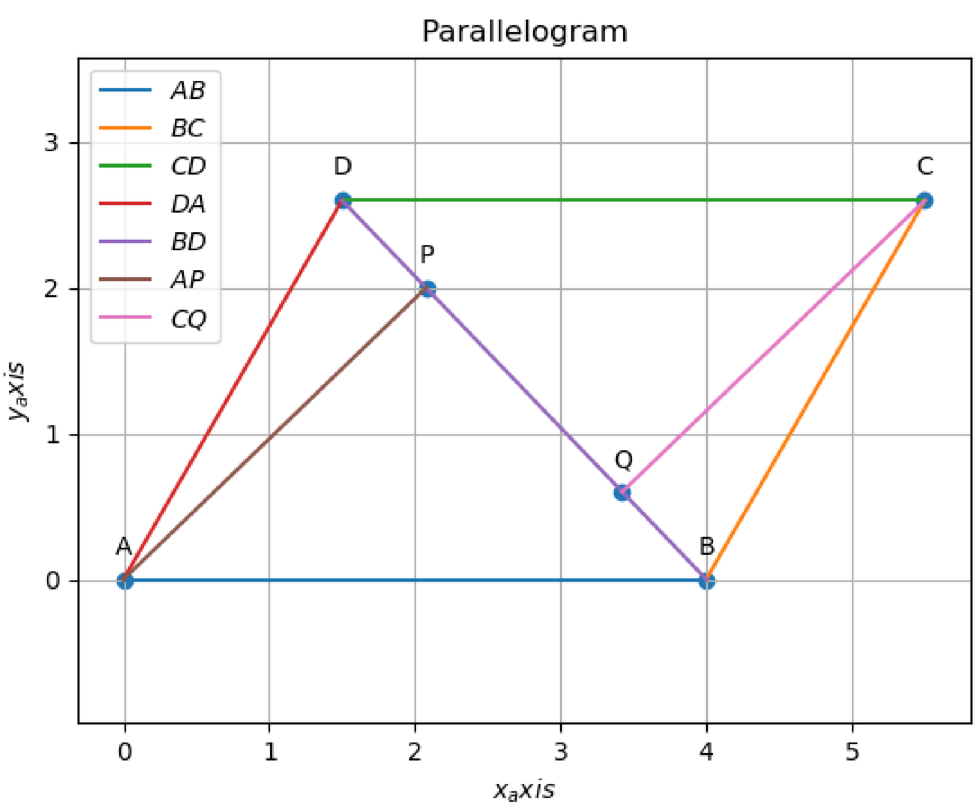


Figure 5.1.10.1:

From Appendix A.1.24.1,

$$\mathbf{A} - \mathbf{B} = \mathbf{D} - \mathbf{C} \tag{5.1.10.2}$$

Substituting in (5.1.10.1),

$$\cos \angle ABD = \cos \angle CDB \tag{5.1.10.3}$$

Using SAS congruence, 5.1.10a is proved. 5.1.10b follows from 5.1.10a.

5.1.11 In $\triangle ABC$ and $\triangle DEF$, $AB = DE$, $AB \parallel DE$, $BC = EF$ and $BC \parallel EF$. Vertices $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ are joined to vertices $\mathbf{D}, \mathbf{E}$ and $\mathbf{F}$ respectively (see Figure 5.1.11.1 ). Show that

(a) quadrilateral $ABED$ is a parallelogram

(b) quadrilateral $BEFC$ is a parallelogram

(c) $AD \parallel CF$ and $AD = CF$

(d) quadrilateral $ACFD$ is a parallelogram

(e) $AC = DF$

(f) $\triangle ABC \cong \triangle DEF$.

**Solution:** From the given information

$$\mathbf{A} - \mathbf{B} = \mathbf{D} - \mathbf{E} \tag{5.1.11.1}$$

$$\mathbf{B} - \mathbf{E} = \mathbf{C} - \mathbf{F} \tag{5.1.11.2}$$

(a) From Appendix A.1.24.1, (5.1.11.1) defines the parallelogram $ABED$.

(b) Similarly, (5.1.11.2) defines the parallelogram $BEFC$.

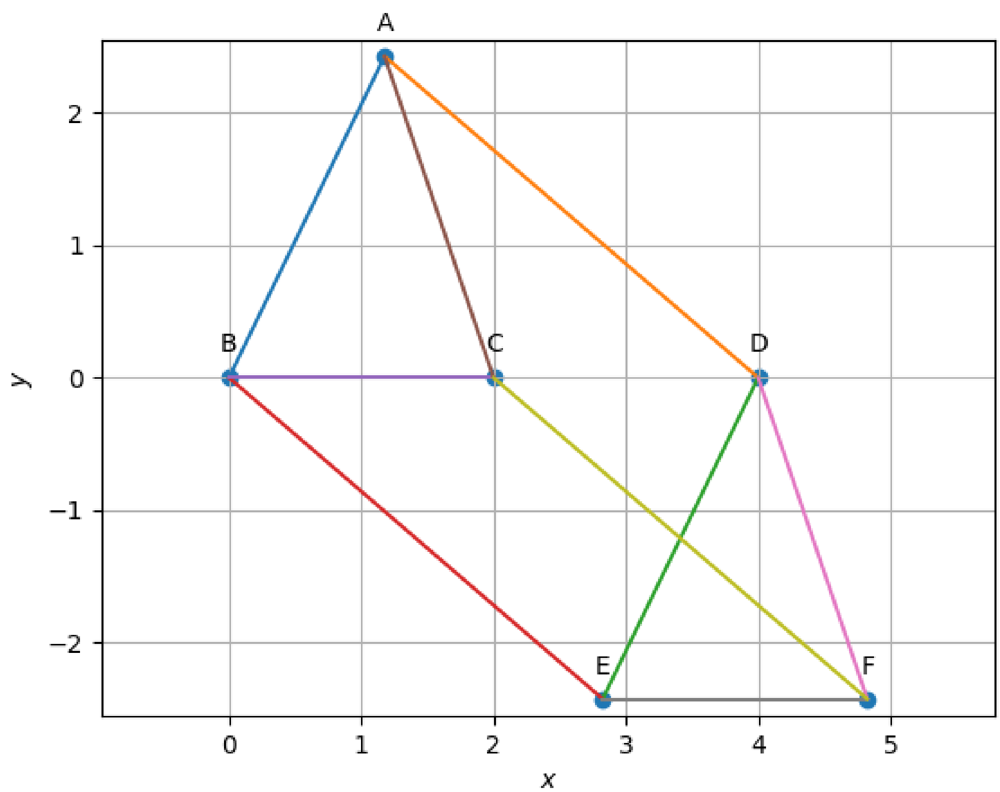


Figure 5.1.11.1:

(c) From (5.1.11.1) and (5.1.11.2),

$$\mathbf{A} - \mathbf{D} = \mathbf{C} - \mathbf{F} \tag{5.1.11.3}$$

which yields 5.1.11c.

(d) (5.1.11.3) implies that $ACFD$ is a parallelogram.

(e) (5.1.11.3) implies $AC = DF$.

(f) Obvious from the fact the $ABCD, BEFC$ and $ACFD$ are parallelograms.

5.1.12 $ABCD$ is trapezium in which $AB \parallel CD$ and $AD = BC$. Show that,

(a) $\angle A = \angle B$

(b) $\angle C = \angle D$

(c) Diagonal $AC$ = Diagonal $BD$

(d) $\triangle ABC = \triangle BAD$

# 5.2. Mid Point Theorem

5.2.1 ABCD is a quadrilateral in which $\mathbf{P}, \mathbf{Q}, \mathbf{R}$ and $\mathbf{S}$ are mid-points of the sides AB, BC, CD and DA (see Fig 5.2.1.1). AC is a diagonal.

Show that

(a) $SR \parallel AC$ and $SR = \frac{1}{2}AC$

(b) $PQ = SR$

(c) $PQRS$ is a parallelogram.

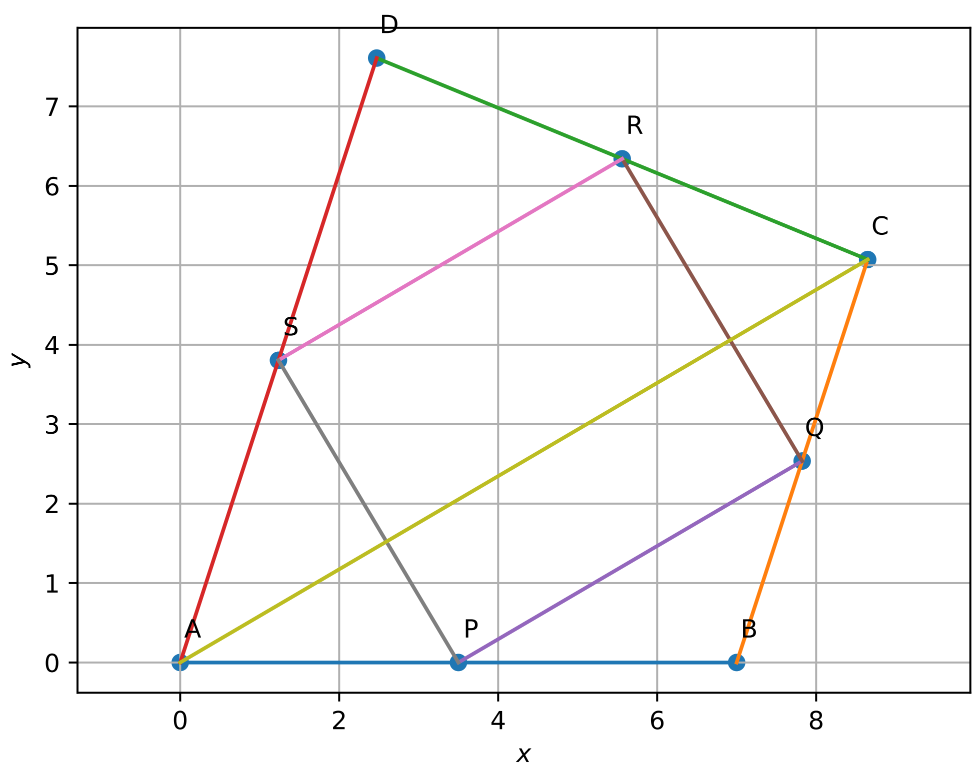


Figure 5.2.1.1:

**Solution:** Using (A.1.22.1),

$$\begin{aligned}\mathbf{P} &= \frac{\mathbf{A}+\mathbf{B}}{2}\\ \mathbf{Q} &= \frac{\mathbf{C}+\mathbf{B}}{2}\\ \mathbf{R} &= \frac{\mathbf{C}+\mathbf{D}}{2}\\ \mathbf{S} &= \frac{\mathbf{D}+\mathbf{A}}{2}\end{aligned} \tag{5.2.1.1}$$

(a) Consequently,

$$\mathbf{R}-\mathbf{S} = \frac{\mathbf{C}-\mathbf{A}}{2} \tag{5.2.1.2}$$

$$\implies SR \parallel AC \tag{5.2.1.3}$$

Also,

$$\left\|\mathbf{R}-\mathbf{S}\right\| = \frac{\left\|\mathbf{C}-\mathbf{A}\right\|}{2} \tag{5.2.1.4}$$

$$\implies SR = \frac{1}{2}AC \tag{5.2.1.5}$$

(b) From (5.2.1.1),

$$\mathbf{R}-\mathbf{S} = \mathbf{Q}-\mathbf{P} \tag{5.2.1.6}$$

which means that $PQRS$ is a parallelogram and $PQ = SR$.

5.2.2

5.2.3 $ABCD$ is a rectangle and $\mathbf{P}, \mathbf{Q}, \mathbf{R}$ and $\mathbf{S}$ are mid-points of the sides $AB, BC, CD$ and $DA$ respectively. Show that the quadrilateral $PQRS$ is a rhombus.

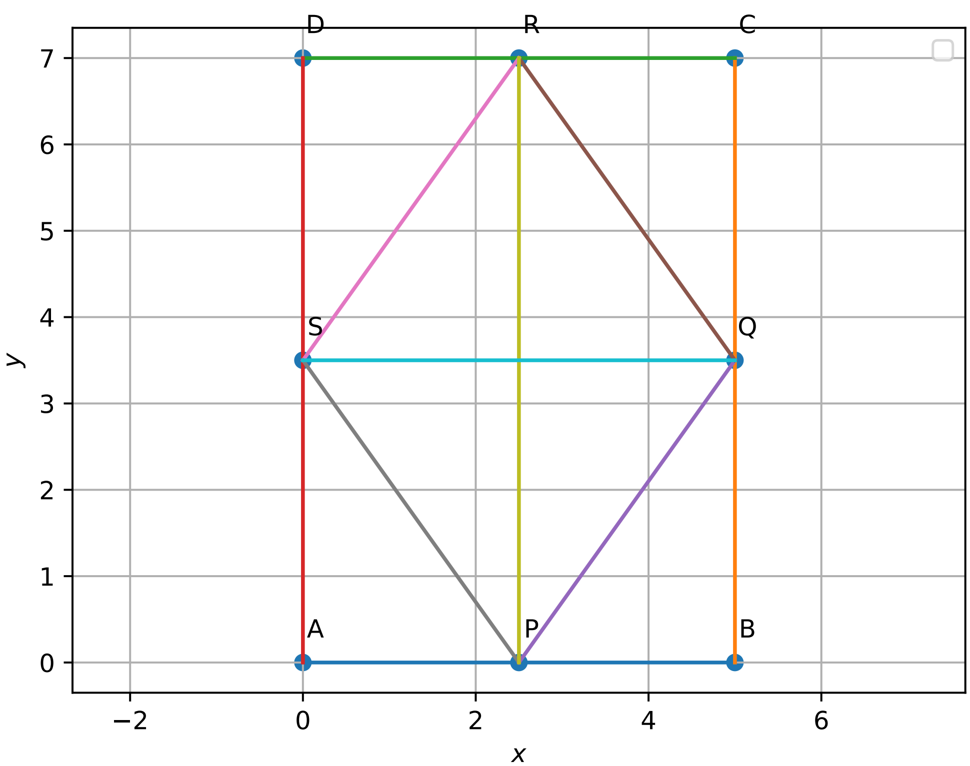


Figure 5.2.3.1:

**Solution:** From Problem 5.2.1, it is obvious that $PQRS$ is a parallelogram. Further, from (5.2.1.1),

$$\left(\mathbf{P}-\mathbf{R}\right)^{\top}\left(\mathbf{S}-\mathbf{Q}\right) = \left(\mathbf{A}+\mathbf{B}-\mathbf{C}-\mathbf{D}\right)^{\top}\left(\mathbf{A}+\mathbf{D}-\mathbf{B}-\mathbf{C}\right) \tag{5.2.3.1}$$

$$= \mathbf{0} \tag{5.2.3.2}$$

since

$$\left(\mathbf{A}-\mathbf{D}\right)^{\top}\left(\mathbf{A}-\mathbf{B}\right) = \mathbf{0} \tag{5.2.3.3}$$

$$\left\|\mathbf{A}-\mathbf{D}\right\| = \left\|\mathbf{A}-\mathbf{B}\right\| \tag{5.2.3.4}$$

as $ABCD$ is a rectangle. Thus, the diagonals $PR$ and $SQ$ bisect each other proving that $PQRS$ is a rhombus.

5.2.4

5.2.5 In a parallelogram ABCD, **E** and **F** are the mid-points of sides AB and CD respectively (see Fig. 5.2.5.1) Show that the line segments AF and EC trisect the diagonal BD.

*Proof.* From the given information,

$$\mathbf{E} = \frac{\mathbf{A}+\mathbf{B}}{2} \tag{5.2.5.1}$$

$$\mathbf{F} = \frac{\mathbf{C}+\mathbf{D}}{2} \tag{5.2.5.2}$$

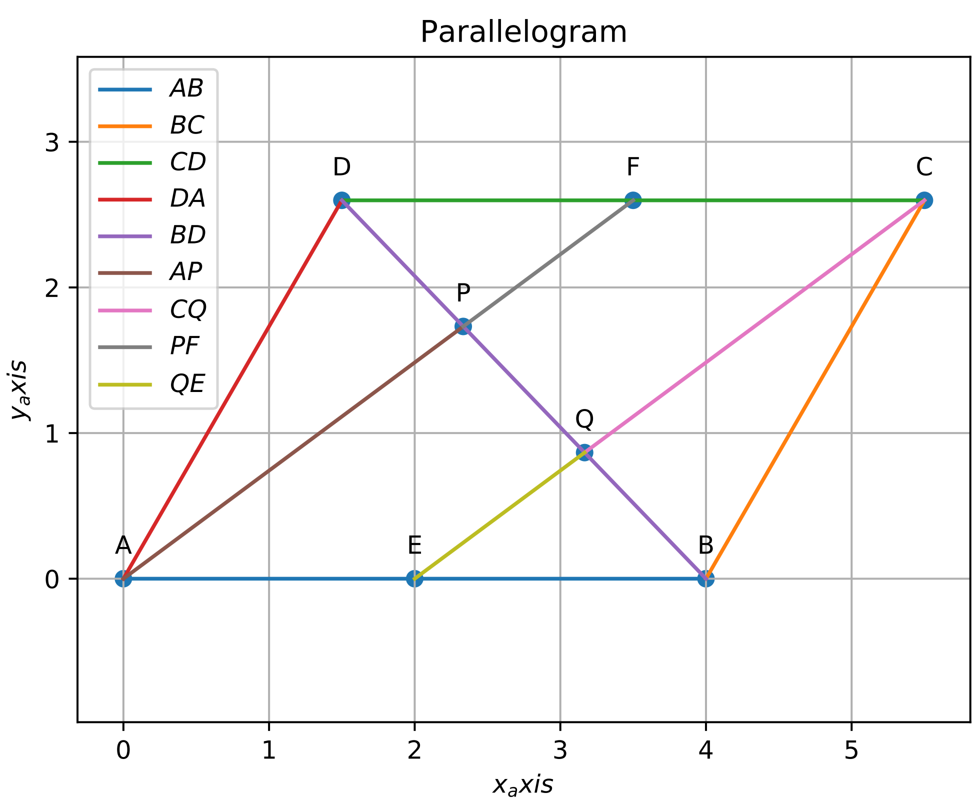
Parallelogram
AB
BC
CD
DA
BD
AP
CQ
PF
QE
D
F
C
P
Q
A
E
B
3
2
1
0
0
1
2
3
4
5
$x_axis$
$y_axis$

Figure 5.2.5.1:

Hence,

$$\mathbf{E} - \mathbf{C} = \frac{\mathbf{A} - \mathbf{C} + \mathbf{B} - \mathbf{C}}{2} \tag{5.2.5.3}$$
$$\mathbf{A} - \mathbf{F} = \frac{\mathbf{A} - \mathbf{C} + \mathbf{A} - \mathbf{D}}{2} \tag{5.2.5.4}$$

Since $ABCD$ is a parallelogram,

$$\mathbf{B} - \mathbf{C} = \mathbf{A} - \mathbf{D} \tag{5.2.5.5}$$

$$\implies \mathbf{E} - \mathbf{C} = \mathbf{A} - \mathbf{F} \tag{5.2.5.6}$$

Thus, $AF \parallel EC$. From Appendix A.1.29, using the fact that $\mathbf{F}$ is the mid point of $CD$, we conclude that $\mathbf{P}$ is the mid point of $DQ$. Similarly, it can be shown that $\mathbf{Q}$ is the mid point of $BP$. □

5.2.6

5.2.7 $ABC$ is a triangle right angled at $\mathbf{C}$. A line through the mid-point $\mathbf{M}$ of hypotenuse $AB$ and parallel to $BC$ intersects $AC$ at $D$ (see Fig. 5.2.7.1). Show that

(a) $D$ is the mid-point of $AC$

(b) $MD \perp AC$

(c) $CM = MA = \frac{1}{2}AB$

**Solution:**

(a) Trivial from Appendix A.1.29.

(b) Since $ABC$ is right angled at $C$,

$$(\mathbf{C} - \mathbf{A})^\top (\mathbf{C} - \mathbf{B}) = 0 \tag{5.2.7.1}$$

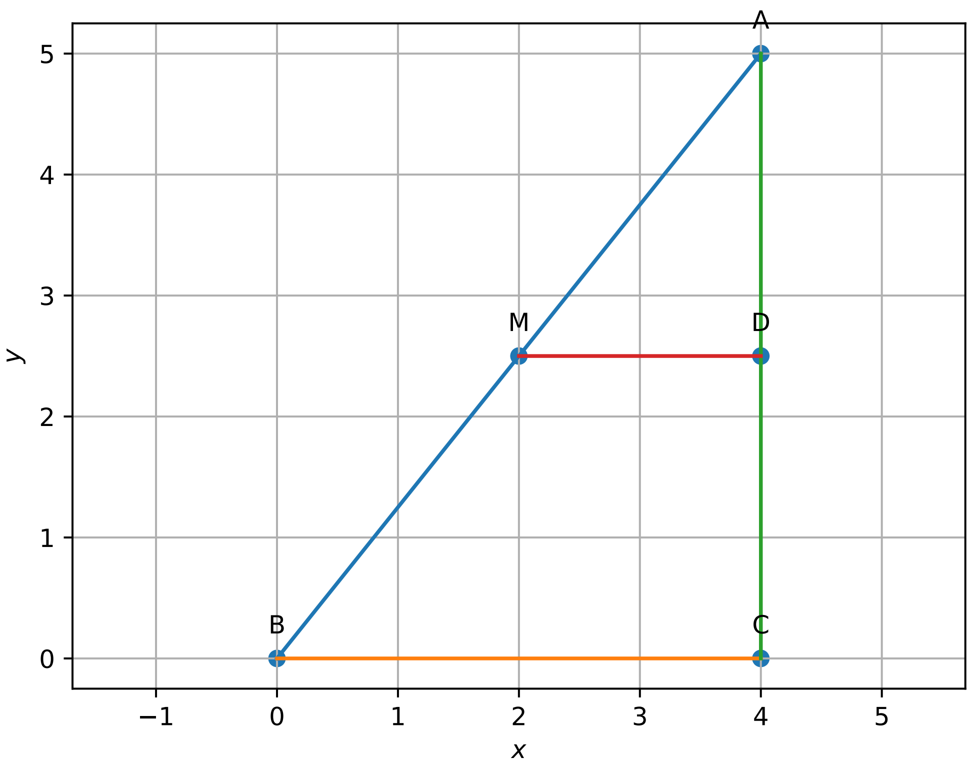


Figure 5.2.7.1:

Given that $MD$ is parallel to $BC$, so

$$(\mathbf{C}-\mathbf{B}) = \lambda(\mathbf{M}-\mathbf{D}) \tag{5.2.7.2}$$

Substituting (5.2.7.2) in (5.2.7.1) and dividing by $\lambda$, we get

$$(\mathbf{C}-\mathbf{A})^{\top}(\mathbf{M}-\mathbf{D}) = 0 \tag{5.2.7.3}$$

From (5.2.7.3) it can be concluded that $MD \perp AC$.

(c) Since

$$\|\mathbf{C}-\mathbf{M}\|^2 - \|\mathbf{A}-\mathbf{M}\|^2 = \|\mathbf{C}\|^2 - \|\mathbf{A}\|^2 - 2\left(\mathbf{C}-\mathbf{A}\right)^\top \mathbf{M} \tag{5.2.7.4}$$

$$= \left(\mathbf{C}-\mathbf{A}\right)^\top \left(\mathbf{C}+\mathbf{A}-2\mathbf{M}\right) \tag{5.2.7.5}$$

$$= \left(\mathbf{C}-\mathbf{A}\right)^\top \left(\mathbf{C}-\mathbf{B}\right) = \mathbf{0} \tag{5.2.7.6}$$

upon substituting from Property 5.2.7a and (5.2.7.1). Thus, $CM = AM$.

## 5.3. Parallelograms

5.3.1 In the Figure 5.3.1.1, $ABCD$ is a parallelogram, $AE \perp DC$ and $CF \perp AD$. If $AB = 16cm$, $AE = 8cm$, and $CF = 10cm$, find $AD$.

5.3.2 If $\mathbf{E}, \mathbf{F}, \mathbf{G}$ and $\mathbf{H}$ are respectively the mid-points of the sides of a parallelogram $ABCD$, show that

$$ar(EFGH) = \frac{1}{2} ar(ABCD) \tag{5.3.2.1}$$

*Proof.* From Problem 5.2.1, $EFGH$ is also a parallelogram and

$$\mathbf{E}-\mathbf{F} = \frac{\mathbf{A}-\mathbf{C}}{2} \tag{5.3.2.2}$$

$$\mathbf{E}-\mathbf{H} = \frac{\mathbf{A}-\mathbf{D}}{2} \tag{5.3.2.3}$$

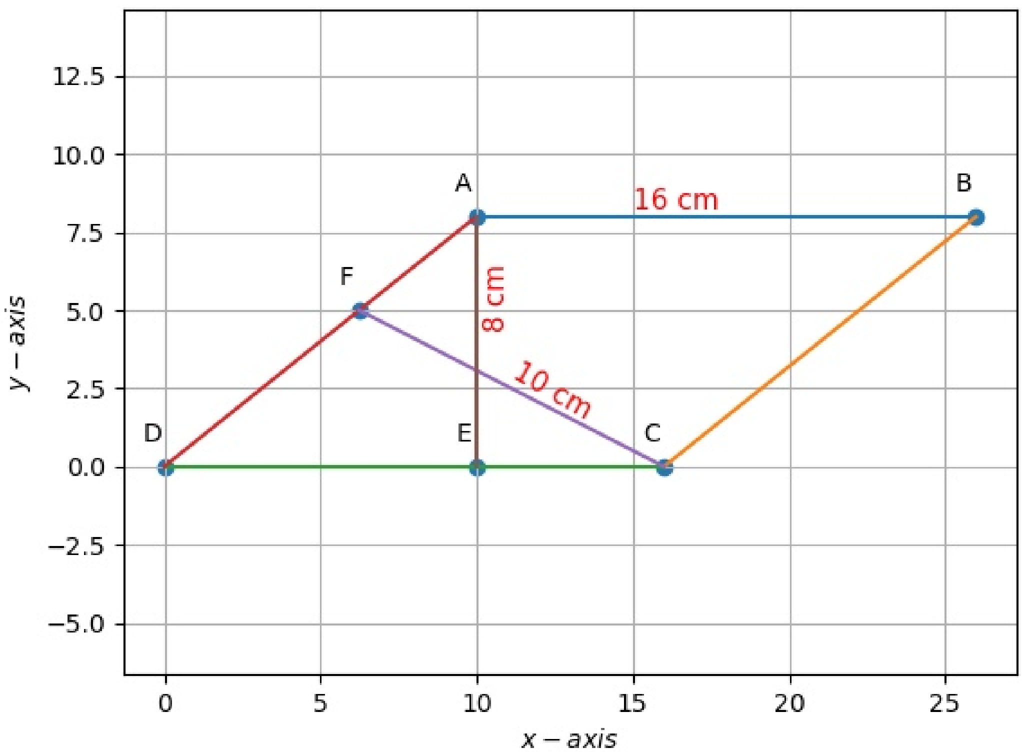


Figure 5.3.1.1:

Thus, the area off $EFGH$ is obtained from (A.1.26.1) as

$$\|(\mathbf{E}-\mathbf{F})\times(\mathbf{E}-\mathbf{H})\| = \frac{1}{4}\|(\mathbf{A}-\mathbf{C})\times(\mathbf{B}-\mathbf{D})\| \tag{5.3.2.4}$$

From Appendix A.1.24.1,

$$\mathbf{D} = \mathbf{C}-\mathbf{B}+\mathbf{A} \tag{5.3.2.5}$$

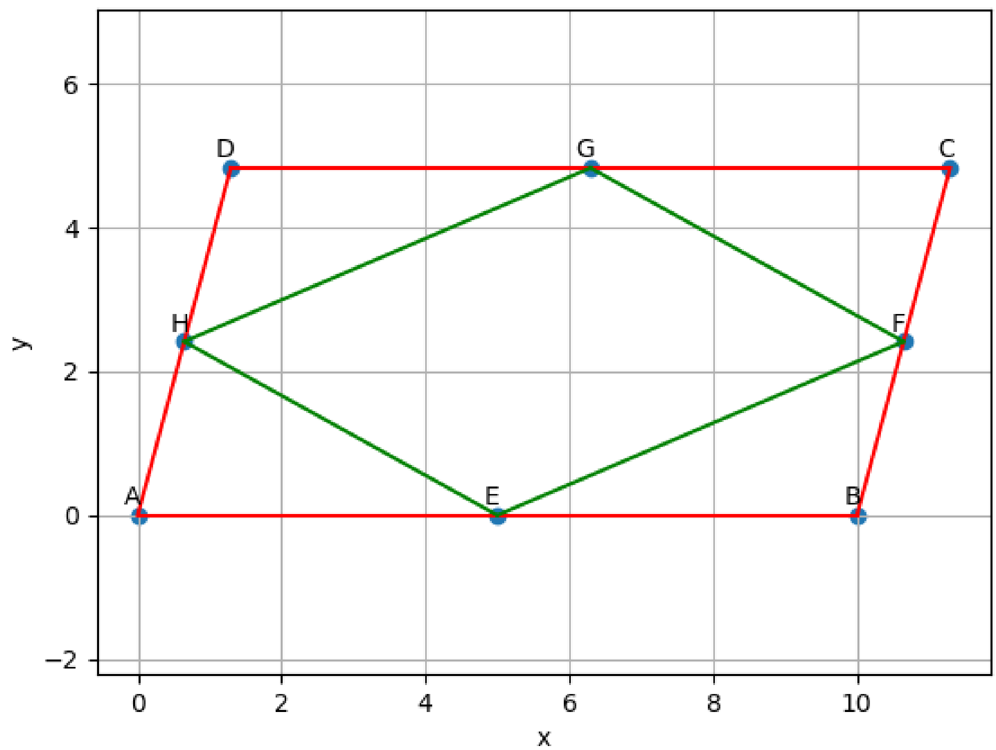


Figure 5.3.2.1:

which,

$$(\mathbf{A}-\mathbf{C})\times(\mathbf{B}-\mathbf{D}) = (\mathbf{A}-\mathbf{C})\times(2\mathbf{B}-\mathbf{C}-\mathbf{A}) \tag{5.3.2.6}$$

$$= 2\left(\mathbf{A}\times\mathbf{B}+\mathbf{B}\times\mathbf{C}+\mathbf{C}\times\mathbf{A}\right) \tag{5.3.2.7}$$

Substituting (5.3.2.7) in (5.3.2.4) yields

$$\left\|(\mathbf{E}-\mathbf{F})\times(\mathbf{E}-\mathbf{H})\right\| = \frac{1}{2}\left\|\mathbf{A}\times\mathbf{B}+\mathbf{B}\times\mathbf{C}+\mathbf{C}\times\mathbf{A}\right\| \tag{5.3.2.8}$$

The area of $ABCD$ is

$$\|(\mathbf{A}-\mathbf{B})\times(\mathbf{A}-\mathbf{D})\| = \|\mathbf{A}\times\mathbf{B}+\mathbf{B}\times\mathbf{C}+\mathbf{C}\times\mathbf{A}\| \tag{5.3.2.9}$$

upon substituting from Appendix A.1.24.1 and simplifying. From (5.3.2.8) and (5.3.2.9) we obtain (5.3.2.1). $\square$

5.3.3

5.3.4 For a given Parallelogram $ABCD$, show that for any point $\mathbf{P}$ inside the parallelogram,

(a) $Ar(APD) + Ar(PBC) = \frac{1}{2}Ar(ABCD)$

(b) $Ar(APD) + Ar(PBC) = Ar(APB) + Ar(PCD)$

5.3.5 In Fig.1, $PQRS$ and $ABRS$ are parallelograms and $\mathbf{X}$ is any point on side $BR$. Show that

(a) $ar(PQRS) = ar(ABRS)$

(b) $ar(AXS) = \frac{1}{2}ar(PQRS)$

*Proof.* (a) From Appendix A.1.24.1,

$$\mathbf{A}-\mathbf{B} = \mathbf{S}-\mathbf{R} = \mathbf{P}-\mathbf{Q} \tag{5.3.5.1}$$

and from Appendix A.1.26, using (5.3.5.1), we obtain Property 5.3.5a.

(b) Using section formula, let

$$\mathbf{X} = \frac{\mathbf{R}+k\mathbf{B}}{1+k}. \tag{5.3.5.2}$$

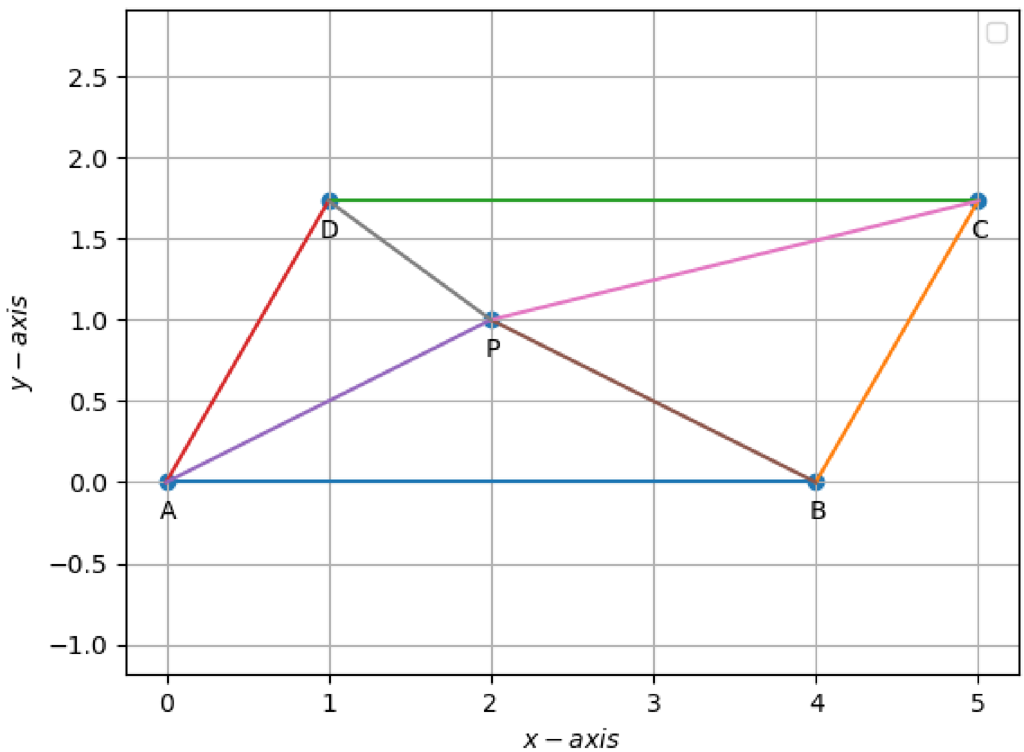


Figure 5.3.4.1:

Then,

$$ar\,(AXS) = \frac{1}{2}\left\|\mathbf{S}\times\mathbf{X}+\mathbf{X}\times\mathbf{A}+\mathbf{A}\times\mathbf{S}\right\| \tag{5.3.5.3}$$

$$= \frac{1}{2}\left\|\frac{\mathbf{S}\times\mathbf{R}+k\mathbf{S}\times\mathbf{B}+\mathbf{R}\times\mathbf{A}+k\mathbf{B}\times\mathbf{A}}{k+1}+\mathbf{A}\times\mathbf{S}\right\| \tag{5.3.5.4}$$

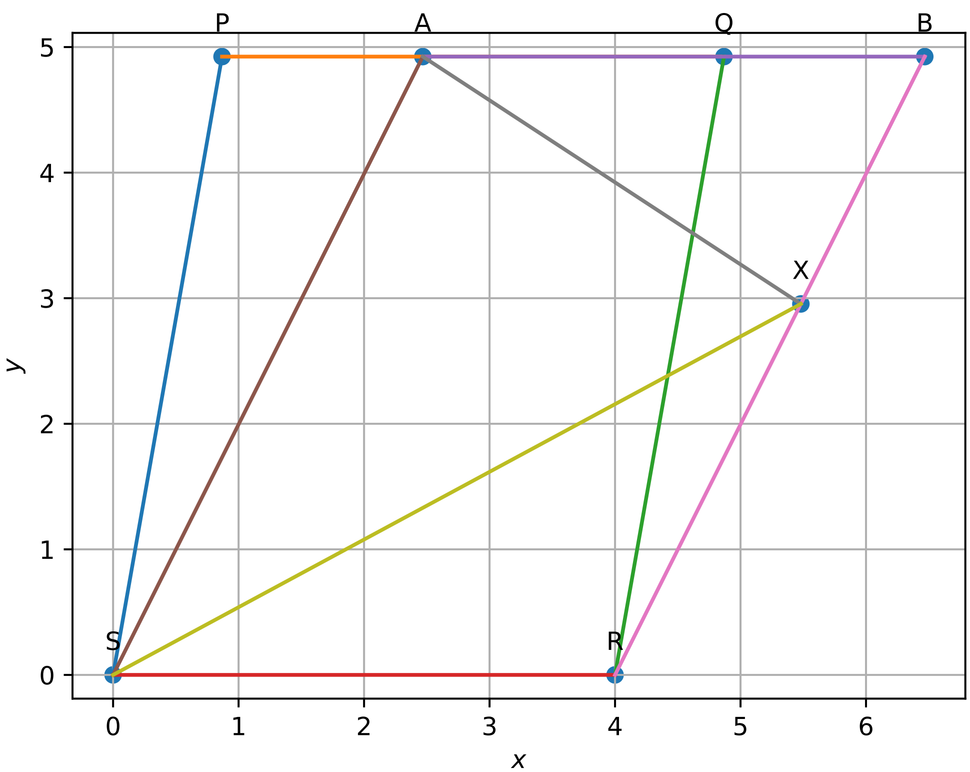


Figure 5.3.5.1:

Substituting for **B** from (5.3.5.1) in the above,

$$ar\left(AXS\right) = \frac{1}{2}\left\|\frac{\mathbf{S}\times\mathbf{R}+\mathbf{R}\times\mathbf{A}+k\left(\mathbf{S}-\mathbf{A}\right)\times\left(\mathbf{A}-\mathbf{S}+\mathbf{R}\right)}{k+1}+\mathbf{A}\times\mathbf{S}\right\| \tag{5.3.5.5}$$

$$= \frac{1}{2}\left\|\frac{\mathbf{S}\times\mathbf{R}+\mathbf{R}\times\mathbf{A}+k\left(\mathbf{S}-\mathbf{A}\right)\times\mathbf{R}}{k+1}+\mathbf{A}\times\mathbf{S}\right\| \tag{5.3.5.6}$$

$$= \frac{1}{2}\left\|\mathbf{S}\times\mathbf{R}+\mathbf{R}\times\mathbf{A}+\mathbf{A}\times\mathbf{S}\right\| \tag{5.3.5.7}$$

$$= \frac{1}{2}ar\left(ABRS\right) \tag{5.3.5.8}$$

□

## 5.4. Triangles and Parallelograms

5.4.1

5.4.2

5.4.3 In Fig. 5.4.3.1 $ABCD, DCFE$ and $ABFE$ are parallelograms. Show that

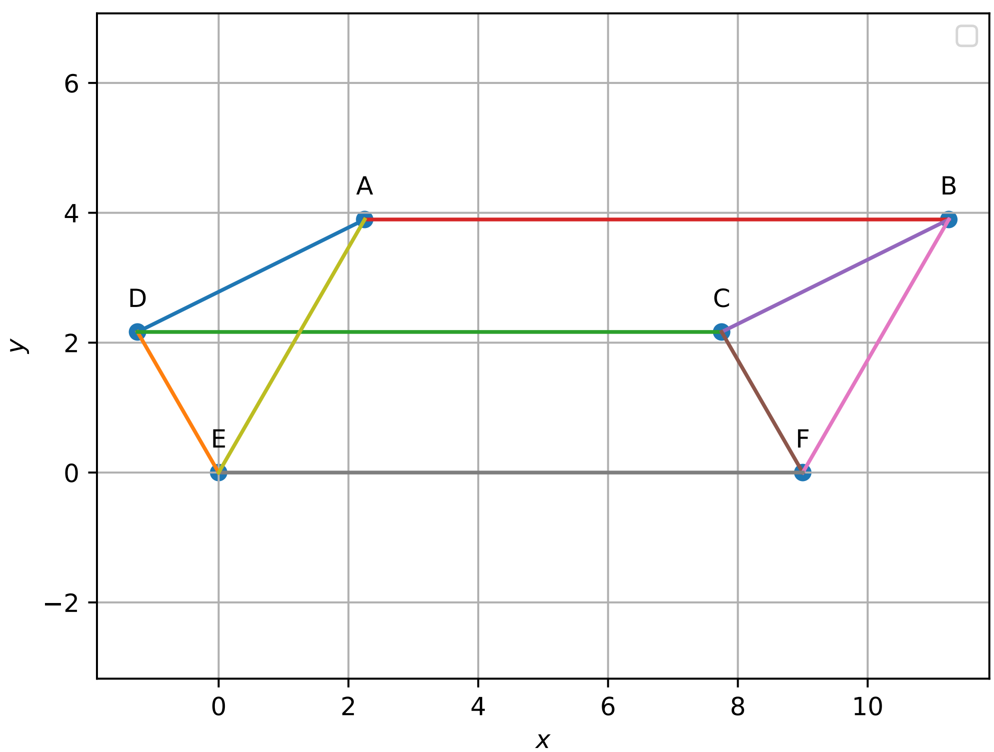


Figure 5.4.3.1:

$$ar(ADE) = ar(BCF) \tag{5.4.3.1}$$

*Proof.* From the given information and Appendix A.1.24.1,

$$\mathbf{B} - \mathbf{A} = \mathbf{C} - \mathbf{D} \tag{5.4.3.2}$$

$$\mathbf{C} - \mathbf{D} = \mathbf{F} - \mathbf{E} \tag{5.4.3.3}$$

$$\mathbf{B} - \mathbf{A} = \mathbf{F} - \mathbf{E} \tag{5.4.3.4}$$

Thus, from Appendix A.1.26,

$$ar(ADE) = \|(\mathbf{D} - \mathbf{E}) \times (\mathbf{D} - \mathbf{A})\| \tag{5.4.3.5}$$

$$= \|(\mathbf{C} - \mathbf{F}) \times (\mathbf{C} - \mathbf{B})\| \tag{5.4.3.6}$$

$$= ar(ADE) \tag{5.4.3.7}$$

upon substituting from (5.4.3.2) and (5.4.3.3). □

5.4.4 In figure below, $ABCD$ is a parallelogram and $BC$ is produced to a point $\mathbf{Q}$ such that $AD = CQ$. If $AQ$ intersect $DC$ at $\mathbf{P}$, show that

$$ar(BPC) = ar(DPQ). \tag{5.4.4.1}$$

5.4.5 In Fig. 5.4.5.1, $ABC$ and $BDE$ are two equilateral triangles such that $\mathbf{D}$ is the mid-

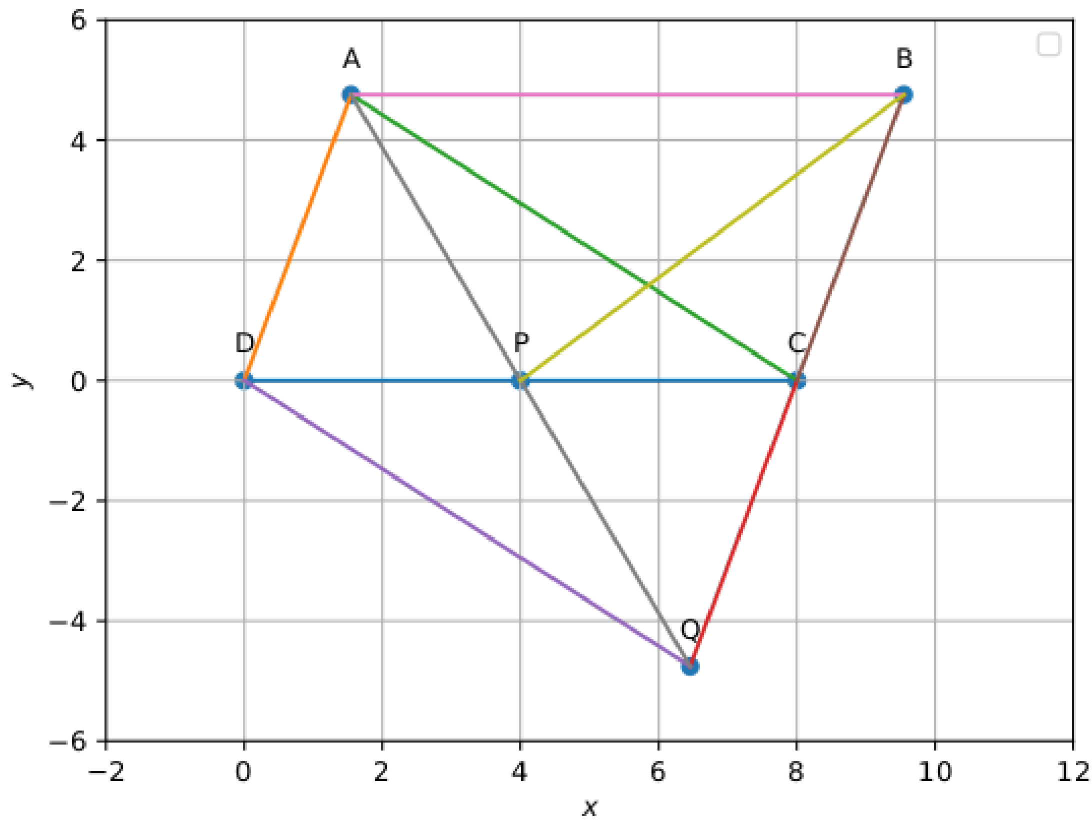


Figure 5.4.4.1:

point of $BC$. If $AE$ intersects $BC$ at $\mathbf{F}$, show that

$$ar(BDE) = \frac{1}{4}ar(ABC) \tag{5.4.5.1}$$

$$ar(BDE) = \frac{1}{2}ar(BAE) \tag{5.4.5.2}$$

$$ar(ABC) = 2ar(BEC) \tag{5.4.5.3}$$

$$ar(BFE) = ar(AFD) \tag{5.4.5.4}$$

$$ar(BFE) = 2ar(FED) \tag{5.4.5.5}$$

$$ar(FED) = \frac{1}{8}ar(AFC) \tag{5.4.5.6}$$

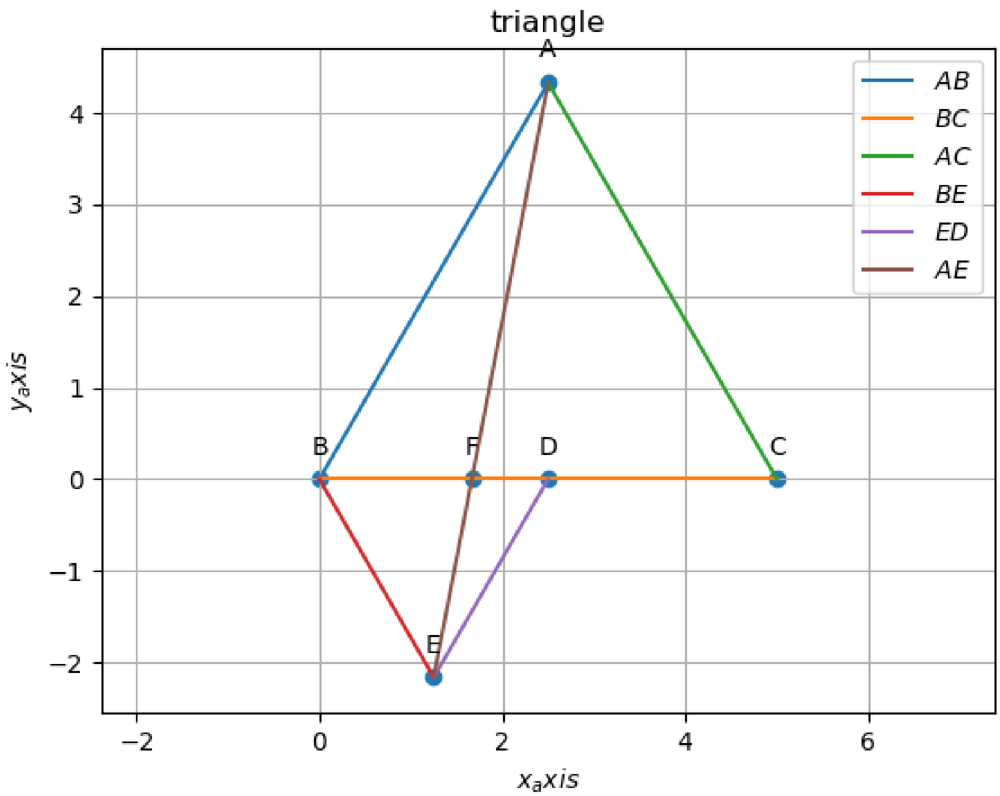


Figure 5.4.5.1:

5.4.6

5.4.7

5.4.8

# Chapter 6

# Circle Construction

## 6.1. Equal Chords

6.1.1 Two circles of radii 5cm and 3cm intersect at two points and the distance between their center is 4cm. Find the length of the common chord.

**Solution:** See Fig. 6.1.1.1. and

| Parameter | Value | Description |
|---|---|---|
| $\mathbf{c}_1$ | $\mathbf{0}$ | Center of Circle 1 |
| $\mathbf{c}_2$ | $4\mathbf{e}_1$ | Center of Circle 2 |
| $r_1$ | 5 | Radius of Circle 1 |
| $r_2$ | 3 | Radius of Circle 2 |

Table 6.1.1.2:

From Table 6.1.1.2, (D.2.1.1) and (D.2.2.1), the equations of the two circles are

$$\|\mathbf{x}\|^2 - 25 = 0$$
$$\|\mathbf{x}\|^2 - 8\mathbf{e}_1^\top\mathbf{x} + 7 = 0 \tag{6.1.1.1}$$

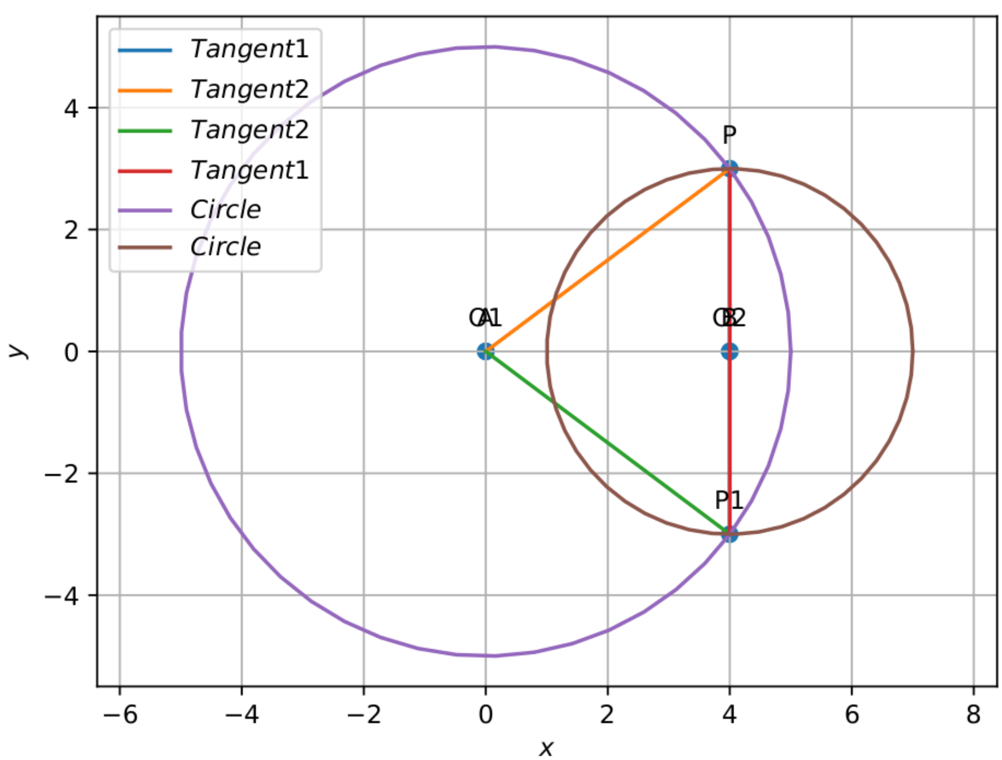


Figure 6.1.1.1:

From (6.1.1.1) and (D.2.4.1) the equation of the common chord is

$$\mathbf{e}_1^\top \mathbf{x} = 4 \tag{6.1.1.2}$$

It is easy to verify that

$$\mathbf{q} = 4\mathbf{e}_1 \tag{6.1.1.3}$$

is a point on (6.1.1.2). Substituting

$$\mathbf{m} = \mathbf{e}_2, \mathbf{q} = 4\mathbf{e}_1, \mathbf{V} = \mathbf{I}, \mathbf{u} = \mathbf{0}, f = -25 \tag{6.1.1.4}$$

in (F.3.3.1), the length of the chord in (F.3.1.1) is given by

$$\frac{2\sqrt{\left[\mathbf{e}_2^\top\left(4\mathbf{e}_1\right)\right]^2-\left(16\mathbf{e}_1^\top\mathbf{e}_1-25\right)\left(\mathbf{e}_2^\top\mathbf{e}_2\right)}}{\mathbf{e}_2^\top\mathbf{e}_2}\left\|\mathbf{e}_2\right\|=6 \tag{6.1.1.5}$$

6.1.2

6.1.3

6.1.4

6.1.5

6.1.6

## 6.2. Inscribed Polygons

6.2.1 In Fig. 6.2.1.1, $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ are three points with centre $\mathbf{O}$ such that $\angle BOC = 30°$ and $\angle AOB = 60°$. If $\mathbf{D}$ is a point on the circle other than the arc ABC, find $\angle ADC$.

**Solution:** See Fig. (6.2.1.1).

$$\mathbf{A}=\mathbf{e}_2, \mathbf{B}=\begin{pmatrix}\cos 30\\ \sin 30\end{pmatrix}, \mathbf{C}=\mathbf{e}_1 \text{ and } \mathbf{D}=\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}. \tag{6.2.1.1}$$

6.2.2

6.2.3 Let $\angle PQR = 100°$ where $\mathbf{PQ}, \mathbf{R}$ are points on a circle with centre $\mathbf{O}$. Find $\angle OPR$.

**Solution:** In Fig. 6.2.3.1,

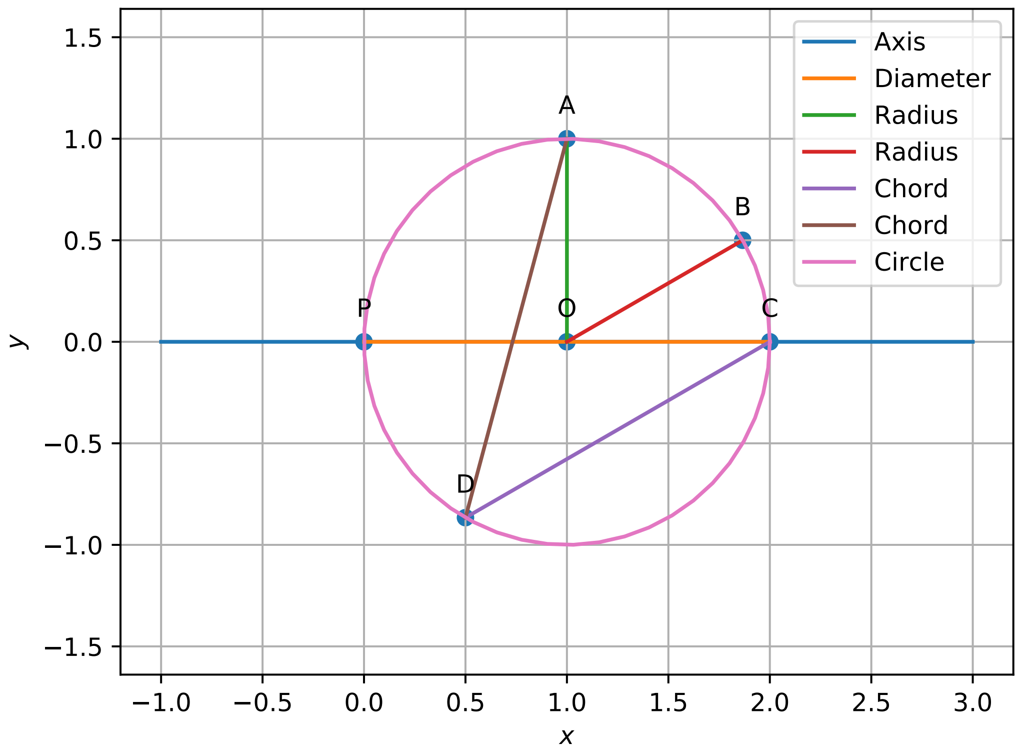


Figure 6.2.1.1:

$$\mathbf{P} = \begin{pmatrix} \cos(\theta + 160) \\ \sin(\theta + 160) \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}, \mathbf{R} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}. \tag{6.2.3.1}$$

# 6.3. Tangent to a Circle

6.3.1

6.3.2 Draw a circle and two lines parallel to a given line such that one is a tangent and the other is a secant to the circle

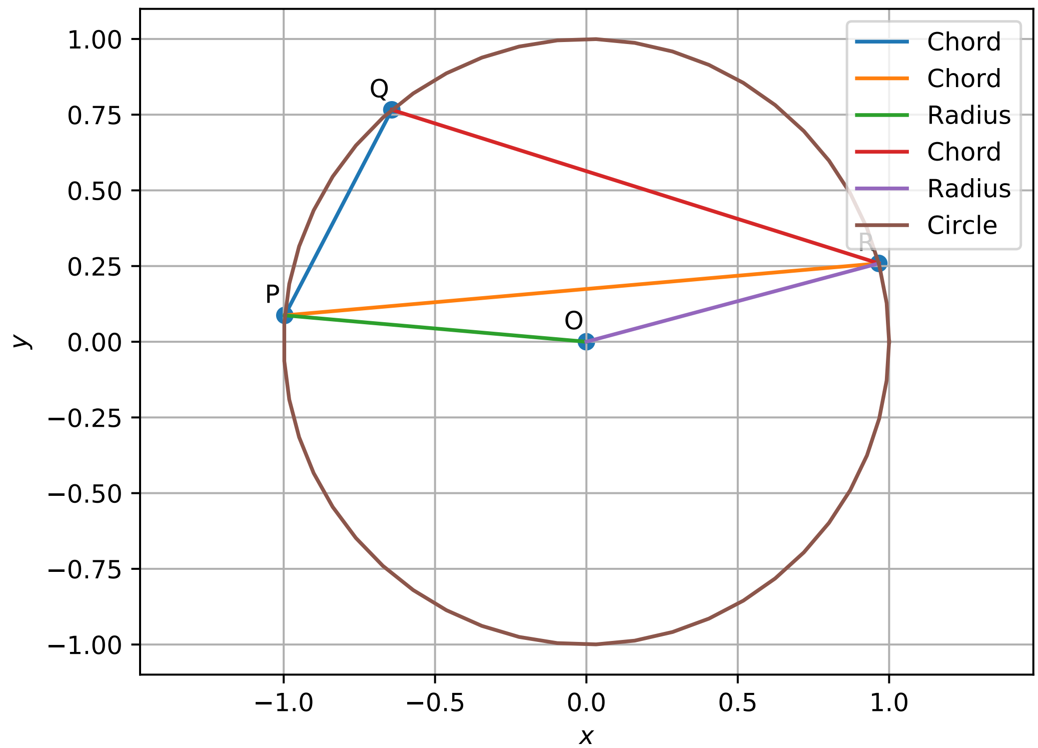


Figure 6.2.3.1:

**Solution:** The parameters of the circle in Fig. 6.3.2.1 are

$$\mathbf{u} = \mathbf{0}, f = -16 \tag{6.3.2.1}$$

Considering the given line to be

$$\mathbf{e}_1^\top \mathbf{x} = 5 \tag{6.3.2.2}$$

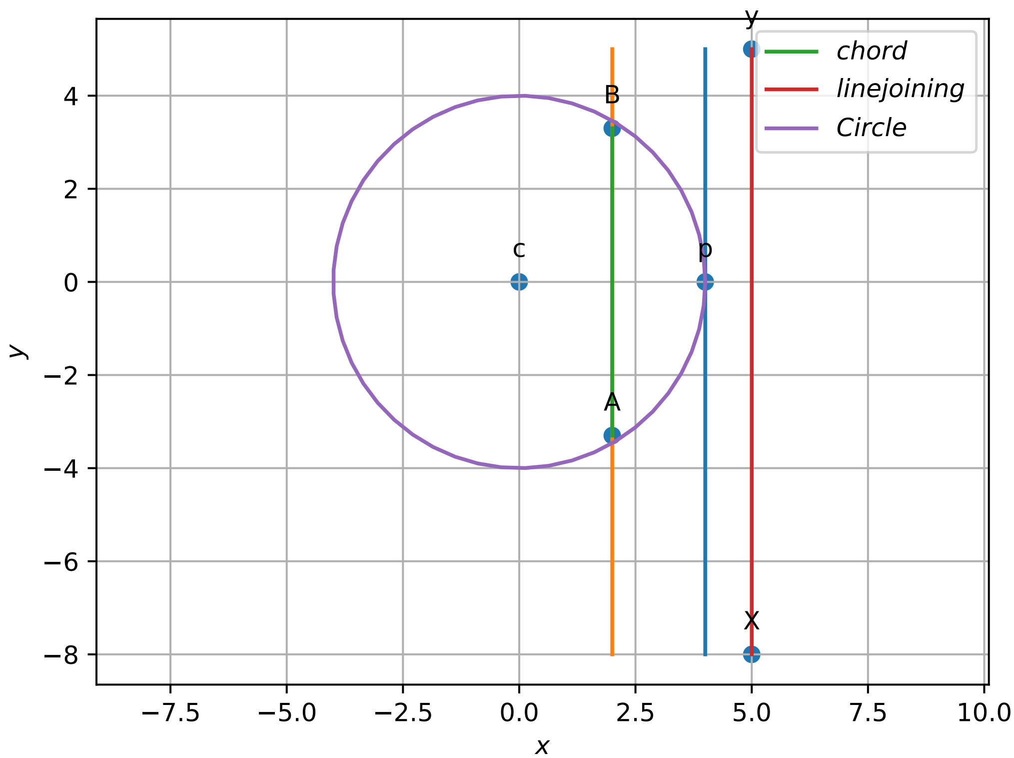


Figure 6.3.2.1:

the tangent to the circle will be

$$\mathbf{e}_1^\top \mathbf{x} = 4 \tag{6.3.2.3}$$

and the secant will be

$$\mathbf{e}_1^\top \mathbf{x} = c \tag{6.3.2.4}$$

where

$$|c| < 4 \tag{6.3.2.5}$$

## 6.4. Tangents from a Point

6.4.1

6.4.2

6.4.3

6.4.4 Show that the tangents of circle drawn at the ends of diameter are parallel.

**Solution:** See Fig. 6.4.4.1. Let $\mathbf{A}, \mathbf{B}$ be the end points of the diameter of the circle through which the tangents are drawn. From (D.2.2.1),

$$\frac{\mathbf{A} + \mathbf{B}}{2} = -\mathbf{u} \tag{6.4.4.1}$$

$$\implies \mathbf{A} + \mathbf{B} = -2\mathbf{u} \tag{6.4.4.2}$$

From (F.3.2.1),

$$\mathbf{m_1}^\top (\mathbf{A} + \mathbf{u}) = 0 \tag{6.4.4.3}$$

$$\mathbf{m_2}^\top (\mathbf{B} + \mathbf{u}) = 0 \tag{6.4.4.4}$$

where $\mathbf{m_1}, \mathbf{m_2}$ are the direction vectors of the tangents at $\mathbf{A}, \mathbf{B}$ respectively. Then,

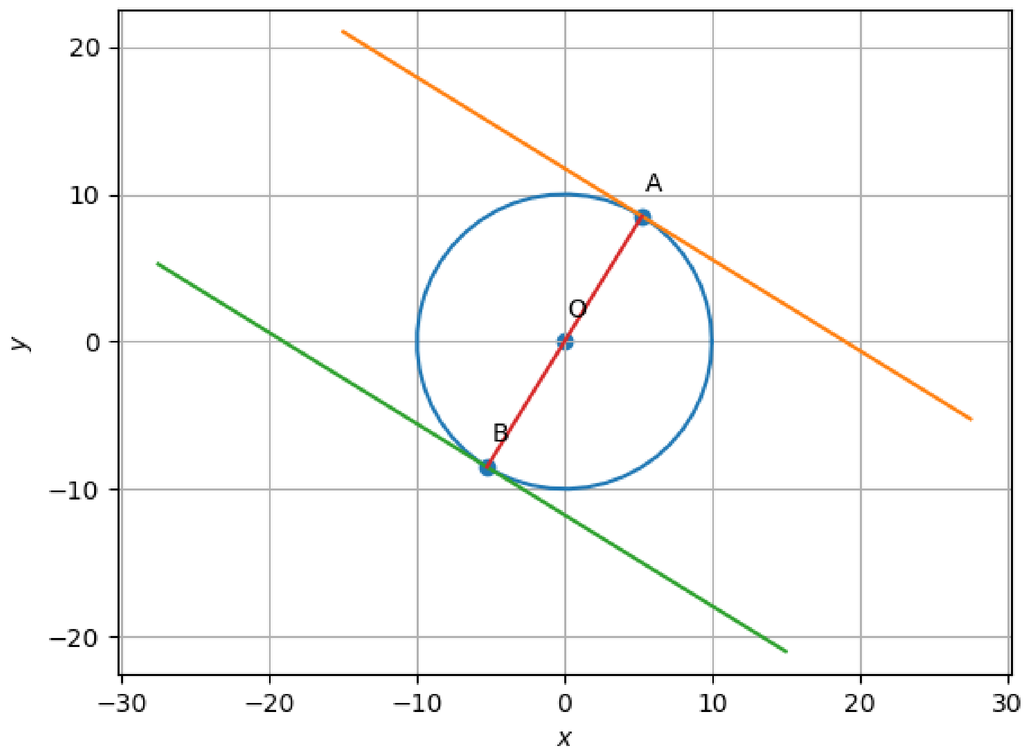


Figure 6.4.4.1:

the normal vectors at the point of contact of tangets are

$$\mathbf{A} + \mathbf{u} = k_1 \mathbf{n_1} \tag{6.4.4.5}$$

$$\mathbf{B} + \mathbf{u} = k_2 \mathbf{n_2} \tag{6.4.4.6}$$

Adding (6.4.4.5) and (6.4.4.6),

$$k_1\mathbf{n_1} + k_2\mathbf{n_2} = \mathbf{A} + \mathbf{B} + 2\mathbf{u} \tag{6.4.4.7}$$

$$= \mathbf{0} \tag{6.4.4.8}$$

from (6.4.4.2), (6.4.4.8) can be expressed as

$$k_1\mathbf{n_1} + k_2\mathbf{n_2} = 0 \tag{6.4.4.9}$$

$$k_1\mathbf{n_1} = -k_2\mathbf{n_2} \tag{6.4.4.10}$$

Since

$$\mathbf{n_1} \times \mathbf{n_2} = \mathbf{0}, \tag{6.4.4.11}$$

$$\mathbf{n_1} \parallel \mathbf{n_2} \implies \mathbf{m_1} \parallel \mathbf{m_2} \tag{6.4.4.12}$$

6.4.5

6.4.6 The length of a tangent from a point **A** at distance 5 cm from the centre of the circle is 4 cm. Find the radius of the circle.

**Solution:** From the Baudhayana theorem, the radius

$$r = 3 \tag{6.4.6.1}$$

Let

$$\mathbf{A} = \mathbf{O} \text{ and } \mathbf{O} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} \tag{6.4.6.2}$$

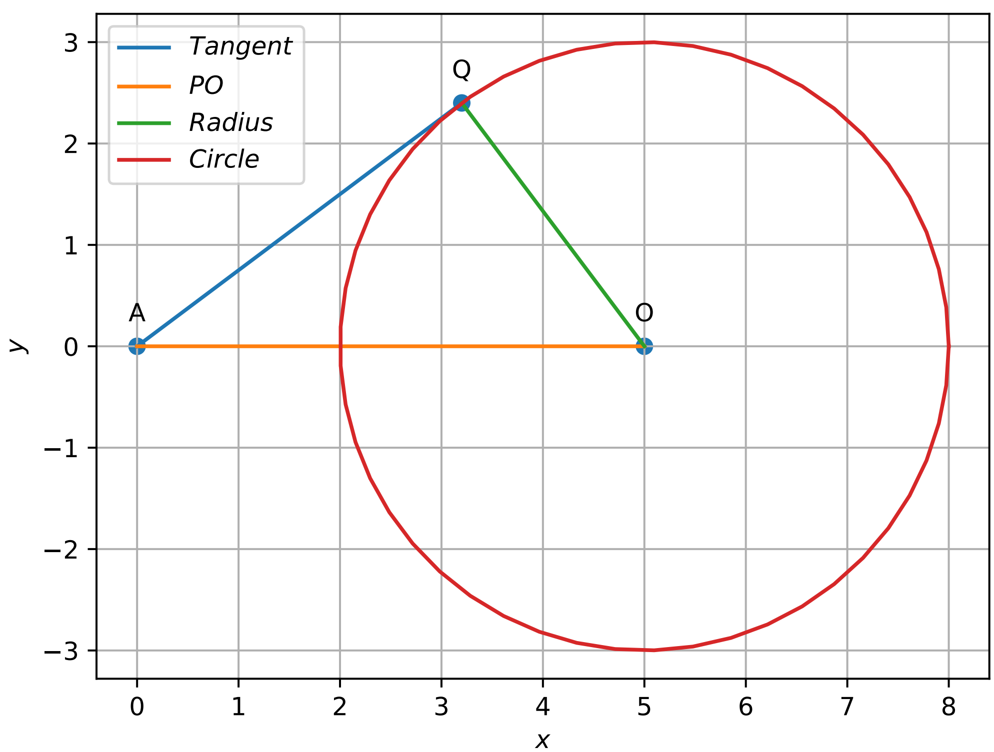


Figure 6.4.6.1:

The equation of the circle can then be expressed as

$$\|\mathbf{x}\|^2 + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{6.4.6.3}$$

where

$$\mathbf{u} = -\mathbf{O} = -\begin{pmatrix} 5 \\ 0 \end{pmatrix} \tag{6.4.6.4}$$

$$f = \|\mathbf{u}\|^2 - r^2 = 16 \tag{6.4.6.5}$$

From (F.4.9.2),

$$\boldsymbol{\Sigma} = (\mathbf{A} + \mathbf{u})(\mathbf{A} + \mathbf{u})^\top - \left(\mathbf{A}^\top \mathbf{A} + 2\mathbf{u}^\top \mathbf{A} + f\right)\mathbf{I} \tag{6.4.6.6}$$

$$= \begin{pmatrix} 9 & 0 \\ 0 & -16 \end{pmatrix} \tag{6.4.6.7}$$

Thus, from (F.4.9.1),

$$\mathbf{P} = \mathbf{I}, \lambda_1 = 9, \lambda_2 = -16 \tag{6.4.6.8}$$

$$\implies \mathbf{n}_1 = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \quad \text{and } \mathbf{n}_2 = \begin{pmatrix} 3 \\ -4 \end{pmatrix} \tag{6.4.6.9}$$

Substituting from the above in (F.4.6.1),

$$\mathbf{q}_{22} = \frac{1}{5}\begin{pmatrix} 16 \\ 12 \end{pmatrix} = \mathbf{Q} \tag{6.4.6.10}$$

in Fig. 6.4.6.1.

6.4.7 Two concentric circles are of radii 5cm and 3cm. Find the length of the chord of the larger circle which touches the smaller circle.

**Solution:** See Fig. 6.4.7.1. Let

$$\mathbf{O} = \mathbf{0} \tag{6.4.7.1}$$

$$r_1 = 5, r_2 = 3. \tag{6.4.7.2}$$

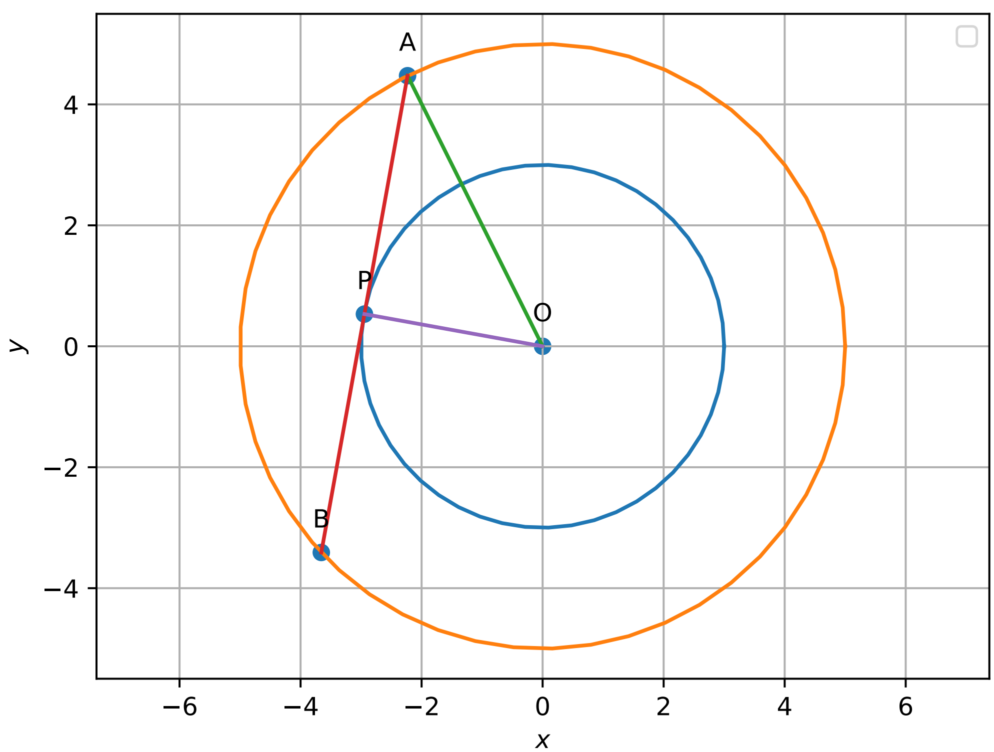


Figure 6.4.7.1:

Choosing

$$\mathbf{A} = r_1 \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \tag{6.4.7.3}$$

$\mathbf{P}$ can be obtained following the approach in Problem 6.4.7. From Appendix D.2.5, $\mathbf{P}$ is the mid point of $AB$. This can be used to obtain $\mathbf{B}$.

6.4.8 A quadrilateral $ABCD$ is drawn to circumscribe a circle. Show that $AB + CD$ is equal to $BC + AD$

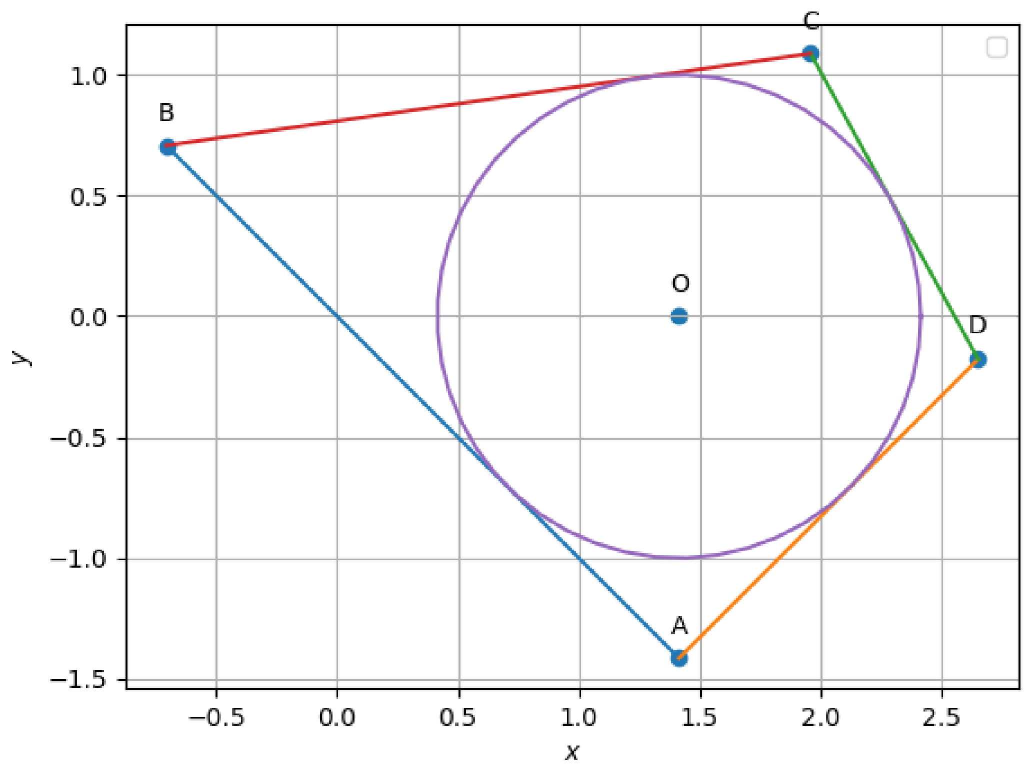


Figure 6.4.8.1:

**Solution:**

(a) Draw the circle.

(b) Choose the point **A**.

(c) Draw the tangents from **A** to the circle.

(d) Choose points $\mathbf{B}, \mathbf{D}$ on the tangents.

(e) From $\mathbf{B}, \mathbf{D}$, draw tangents to the circle intersecting at **C**.

6.4.9 In Fig. 6.4.9.1, $XY$ and $EF$ are two parallel tangents to a circle with centre **O** and another tangent $AB$ with point of contact **C** intersecting $XY$ at **A** and $EF$ at **B**. Prove that $\angle AOB = 90°$.

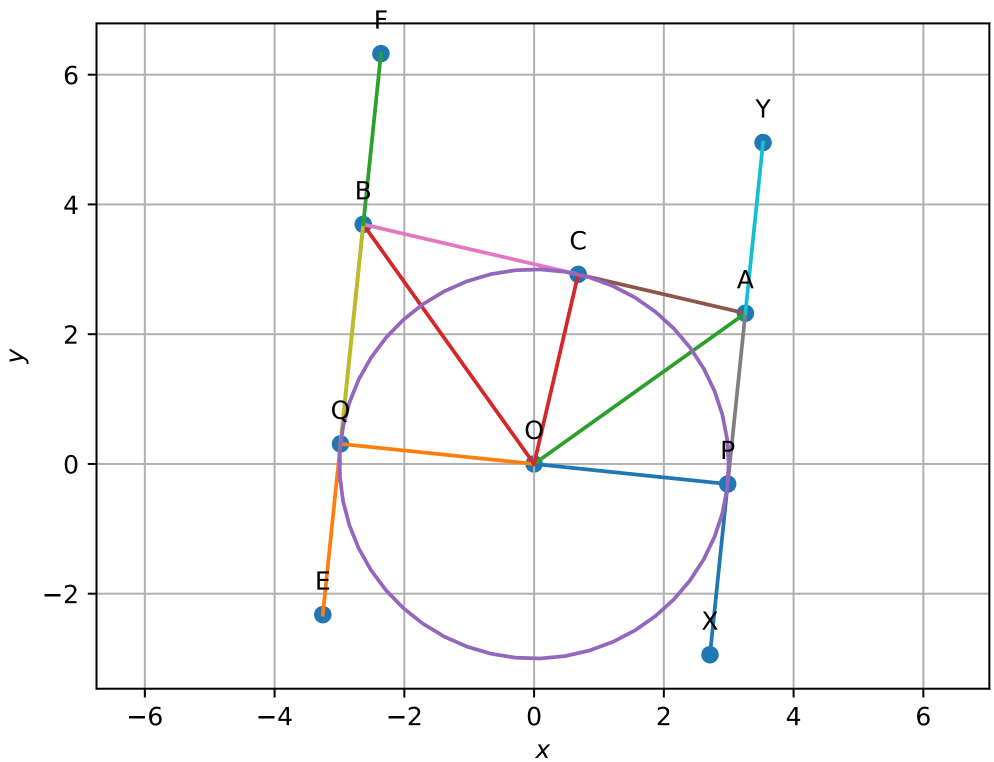


Figure 6.4.9.1:

**Solution:**

6.4.10 Prove that the angle between the two tangents drawn from an external point to a circle is supplementary to the angle subtended by the line-segment joining the points of contact at the centre.

**Solution:** Follow the approach in Problem 6.4.6 for constructing the tangents to the circle.

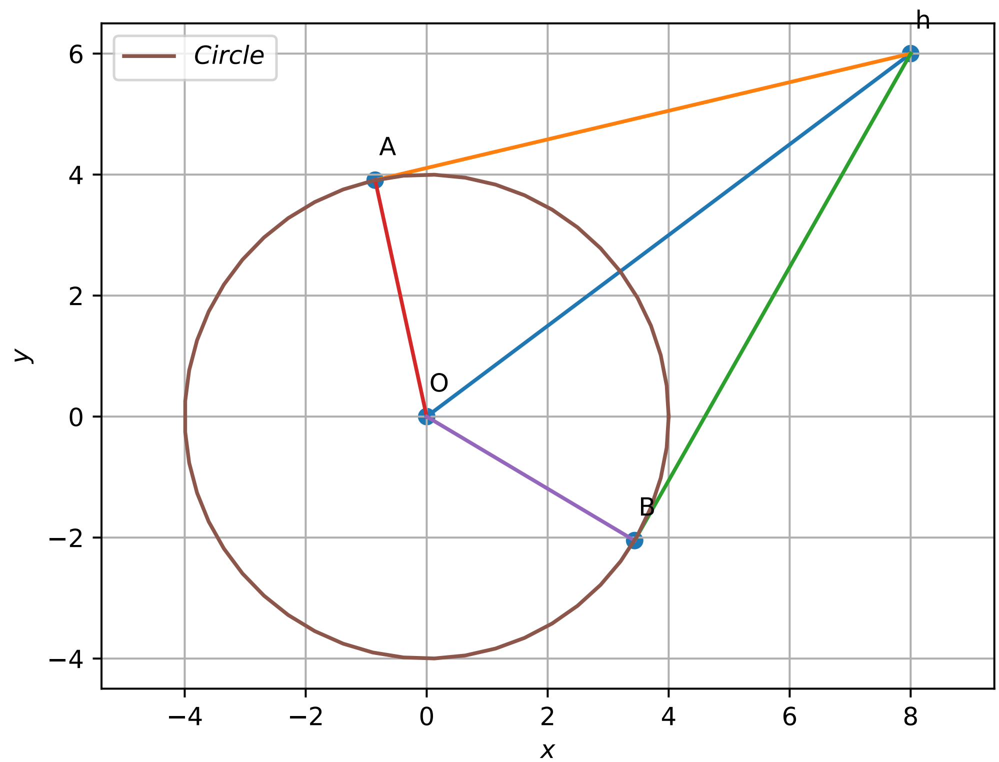


Figure 6.4.10.1:

6.4.11

6.4.12 A triangle $ABC$ is drawn to circumscribe a circle of radius 4cm such that the segments $BD$ and $DC$ into which $BC$ is divided by the point of contact $D$ are of lengths 8cm and 6cm respectively. Find the sides $AB$ and $AC$.

6.4.13

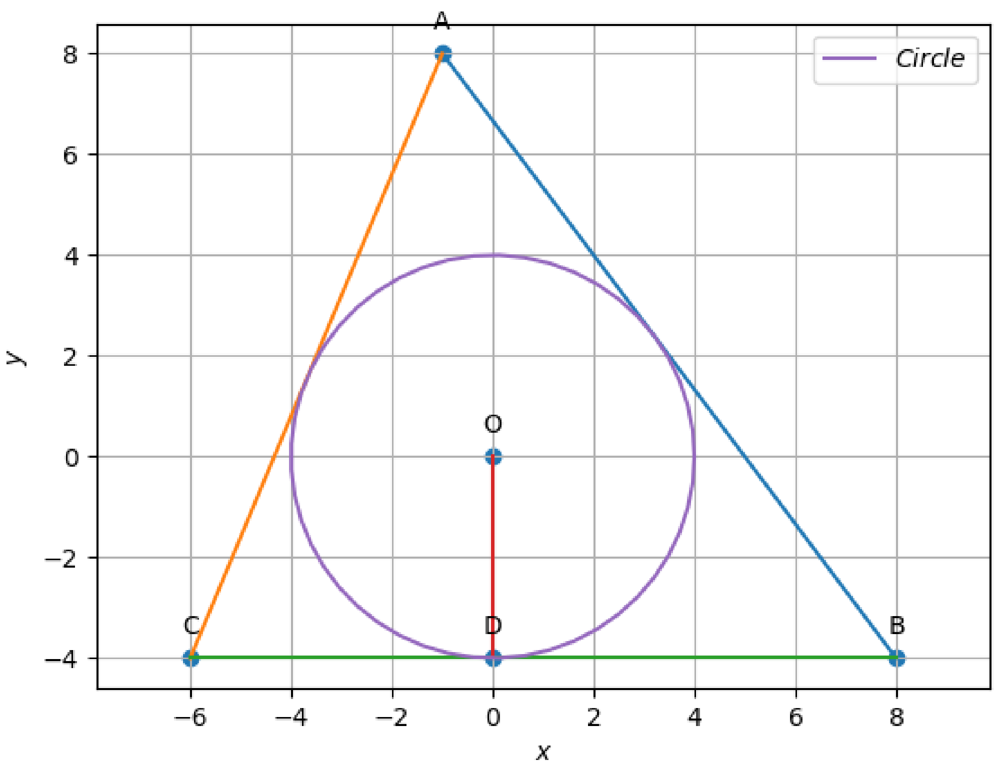


Figure 6.4.12.1:

# Chapter 7

# Conics

## 7.1. Parabola

## 7.2. Ellipse

## 7.3. Hyperbola

## 7.4. Miscellaneous

7.4.1

7.4.2 An arch is in the form of a parabola with its axis vertical. The arch is 10m high and 5m wide at the base. How wide is it 2m from the vertex of the parabola?

**Solution:**

7.4.3

7.4.4 An arch is in the form of a semi-ellipse. It is 8 m wide and 2 m high at the centre. Find the height of the arch at a point 1.5 m from one end.

**Solution:**

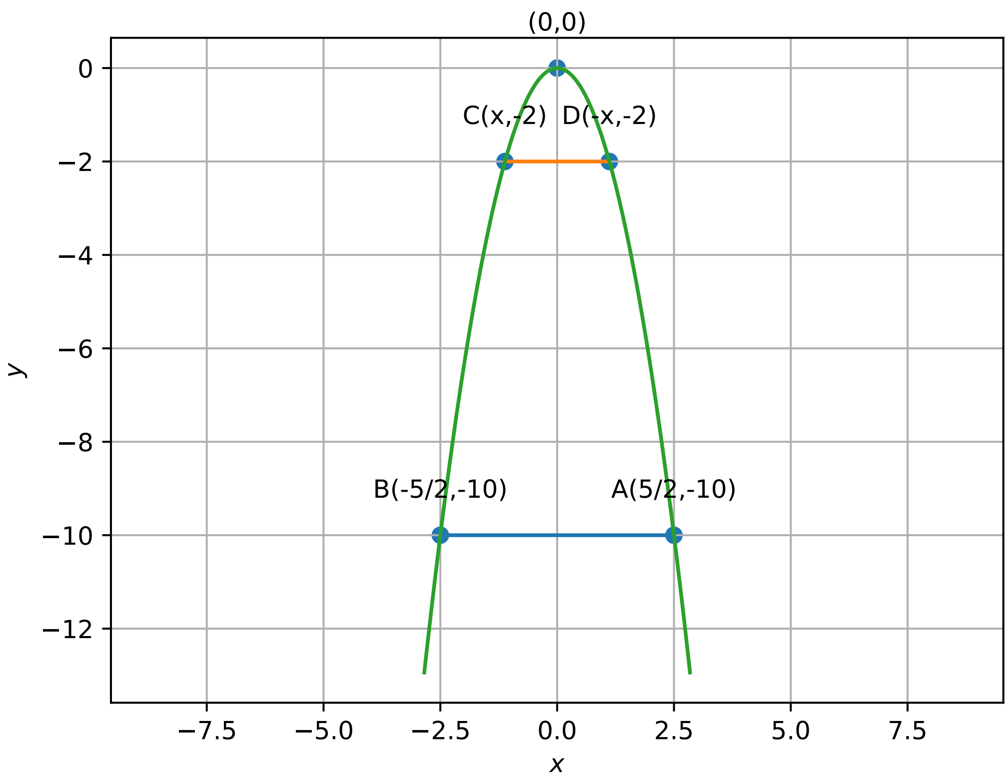


Figure 7.4.2.1:

7.4.5

7.4.6 Find the area of the triangle formed by the lines joining the vertex of the parabola $x^2 = 12y$ to the ends of its latus rectum.

7.4.7

7.4.8 An equilateral triangle is inscribed in the parabola $y^2 = 4ax$,where one vertex is at the vertex of the parabola. Find the length of the side of the triangle.

**Solution:**

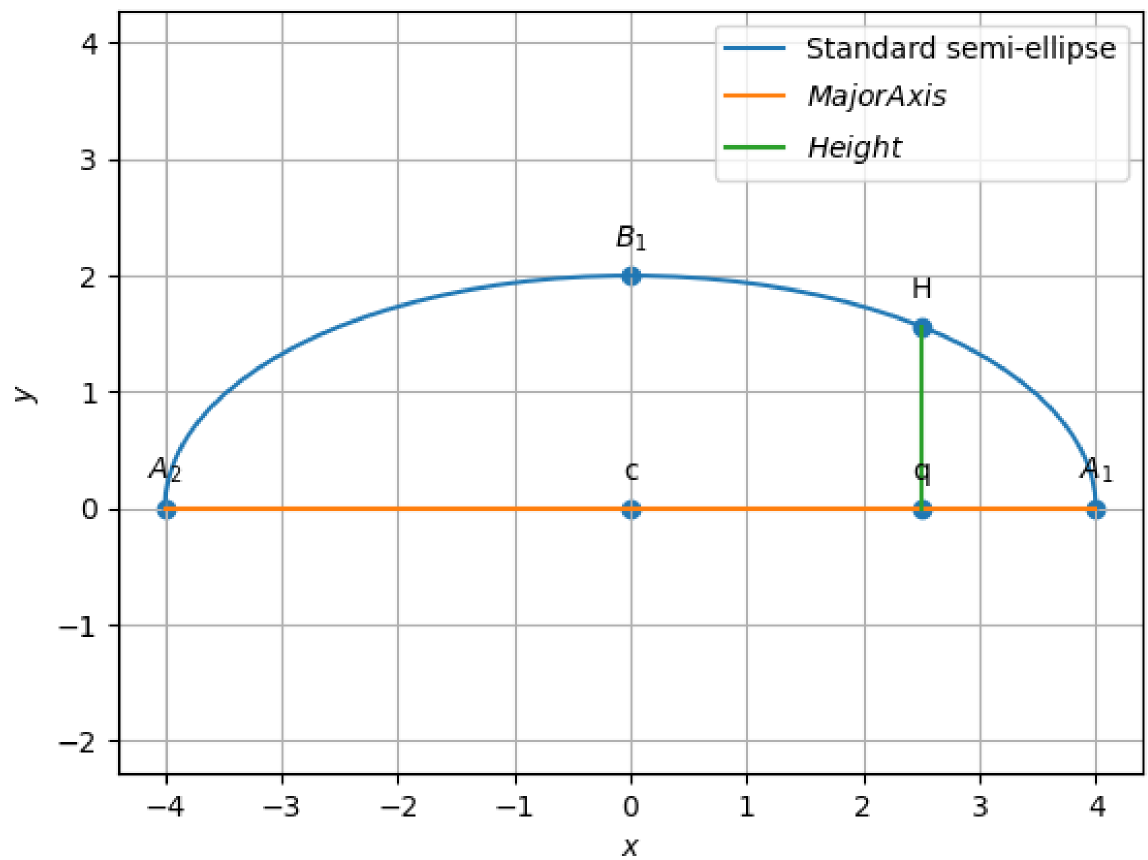
Standard semi-ellipse
MajorAxis
Height
$B_1$
H
$A_2$
c
q
$A_1$
y
x
4
3
2
1
0
−1
−2
−4
−3
−2
−1
0
1
2
3
4

Figure 7.4.4.1:

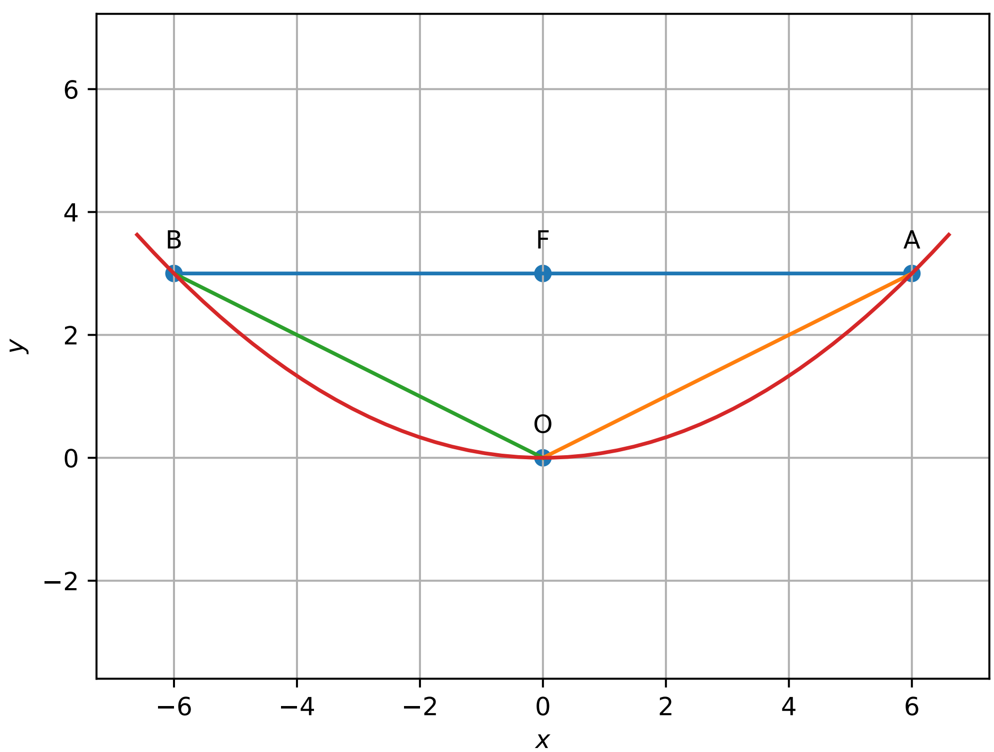


Figure 7.4.6.1:

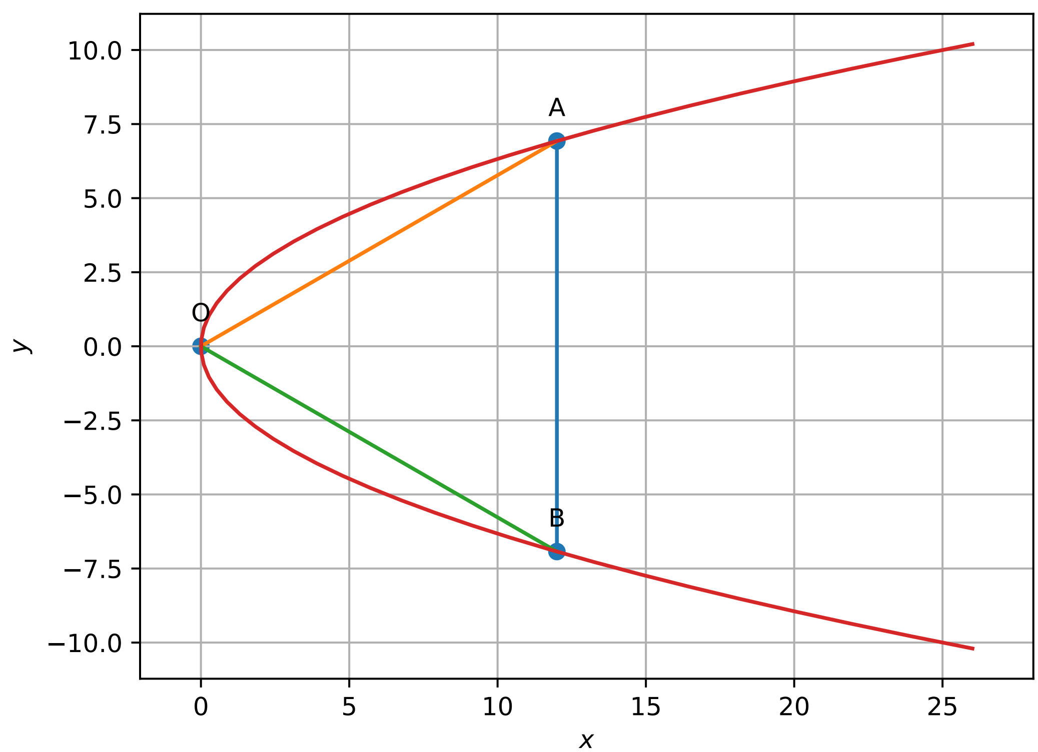


Figure 7.4.8.1:

# Chapter 8

# Intersection of Conics

## 8.1. Chords

8.1.1 Find the area of the region bounded by the curve $y^2 = x$ and the lines $x = 1$ and $x = 4$ and the axis in the first quadrant.

**Solution:**

The parameters of the conic are

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{u} = -\frac{1}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, f = 0 \tag{8.1.1.1}$$

For the line $x - 1 = 0$, the parameters are

$$\mathbf{q}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{m}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{8.1.1.2}$$

Substituting from the above in (F.3.1.3),

$$\mu_i = 1, -1 \tag{8.1.1.3}$$

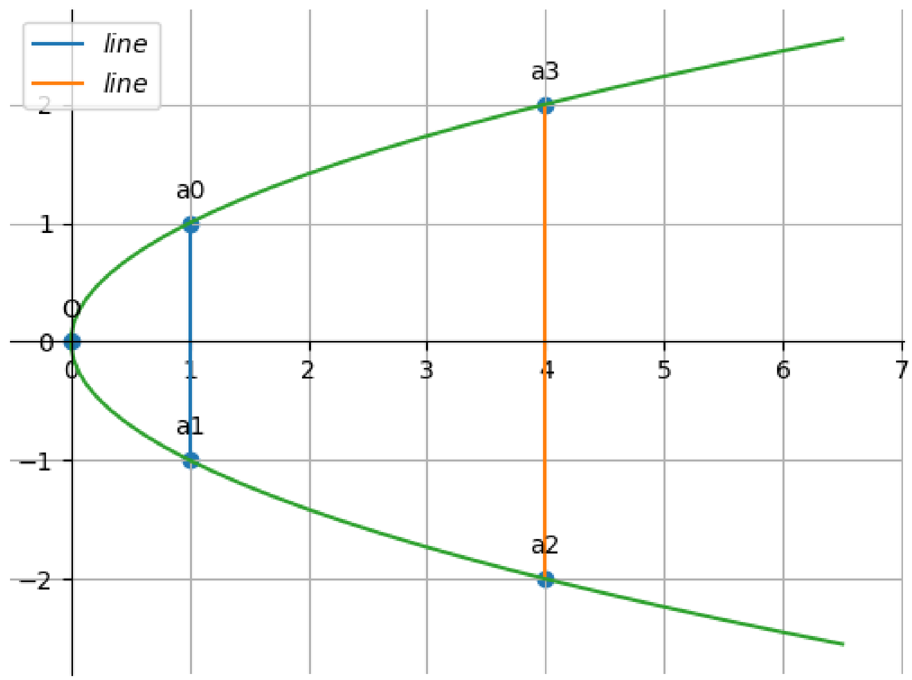


Figure 8.1.1.1:

yilelding the points of intersection

$$\mathbf{a}_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \mathbf{a}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{8.1.1.4}$$

Similarly, for the line $x - 4 = 0$

$$\mathbf{q_1} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}, \mathbf{m_1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{8.1.1.5}$$

yielding

$$\mu_i = 2, -2 \tag{8.1.1.6}$$

from which, the points of intersection are

$$\mathbf{a_3} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \mathbf{a_2} = \begin{pmatrix} 4 \\ -2 \end{pmatrix} \tag{8.1.1.7}$$

Thus, the area of the parabola in between the lines $x = 1$ and $x = 4$ is given by

$$\int_0^4 \sqrt{x}\, dx - \int_0^1 \sqrt{x}\, dx = 14/3 \tag{8.1.1.8}$$

8.1.2 Find the area of the region bounded by the curve $y^2 = 9x$ and the lines $x = 2$ and $x = 4$ and the axis in the first quadrant.

**Solution:** The parameters of the conic are

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{u} = \frac{9}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, f = 0. \tag{8.1.2.1}$$

The parameters of the line $x - 2 = 0$ are

$$\mathbf{q_2} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \mathbf{m_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{8.1.2.2}$$

Substituting in (F.3.1.3),

$$\mu_i = \pm 3\sqrt{2} \tag{8.1.2.3}$$

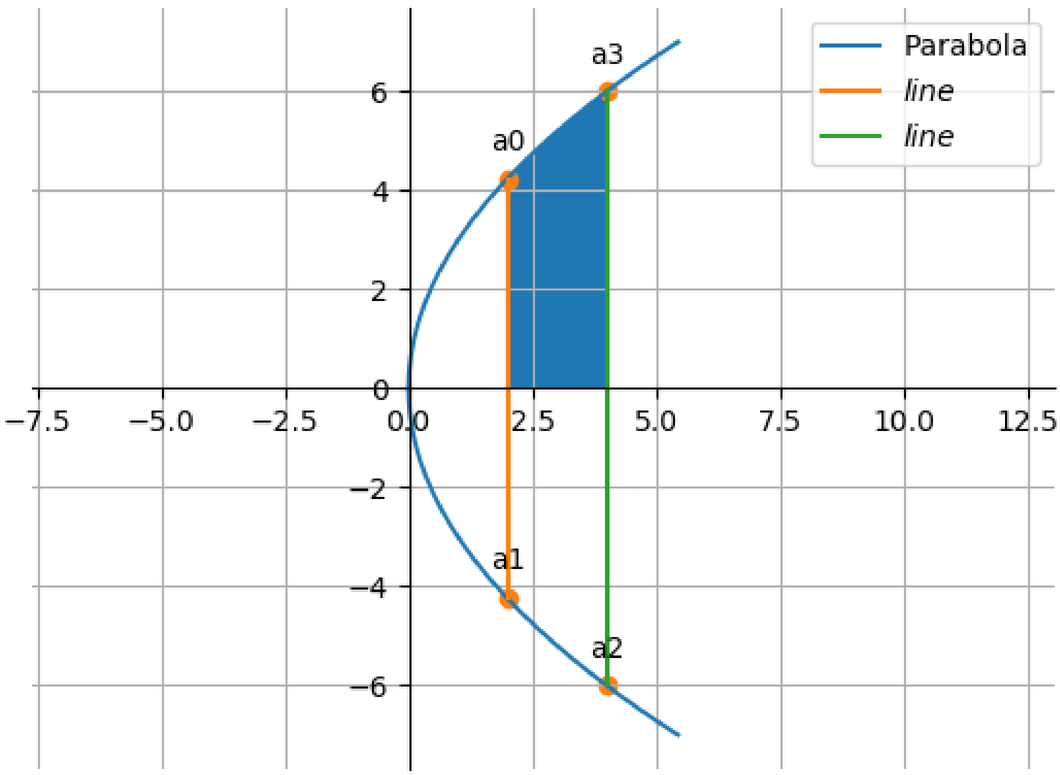


Figure 8.1.2.1:

yielding

$$\mathbf{a_0} = \begin{pmatrix} 2 \\ 3\sqrt{2} \end{pmatrix}, \mathbf{a_1} = \begin{pmatrix} 2 \\ -3\sqrt{2} \end{pmatrix}. \tag{8.1.2.4}$$

Similarly, for the line $x - 4 = 0$,

$$\mathbf{q_1} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}, \mathbf{m_1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{8.1.2.5}$$

yielding

$$\mu_i = \pm 6. \tag{8.1.2.6}$$

Thus,

$$\mathbf{a_3} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}, \mathbf{a_2} = \begin{pmatrix} 4 \\ -6 \end{pmatrix} \tag{8.1.2.7}$$

and the desired area of the parabola is

$$\int_0^4 3\sqrt{x}\,dx - \int_0^2 3\sqrt{x}\,dx = 16 - 4\sqrt{2} \tag{8.1.2.8}$$

8.1.3

8.1.4 Find the area of the region in the first quadrant enclosed by the x-axis, line $x = \sqrt{3}y$ and circle $x^2 + y^2 = 4$.

**Solution:** From the given information, the parameters of the circle and line are

$$f = -4, \mathbf{u} = \mathbf{0}, \mathbf{V} = \mathbf{I}, \mathbf{m} = \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}, \mathbf{h} = \mathbf{0} \tag{8.1.4.1}$$

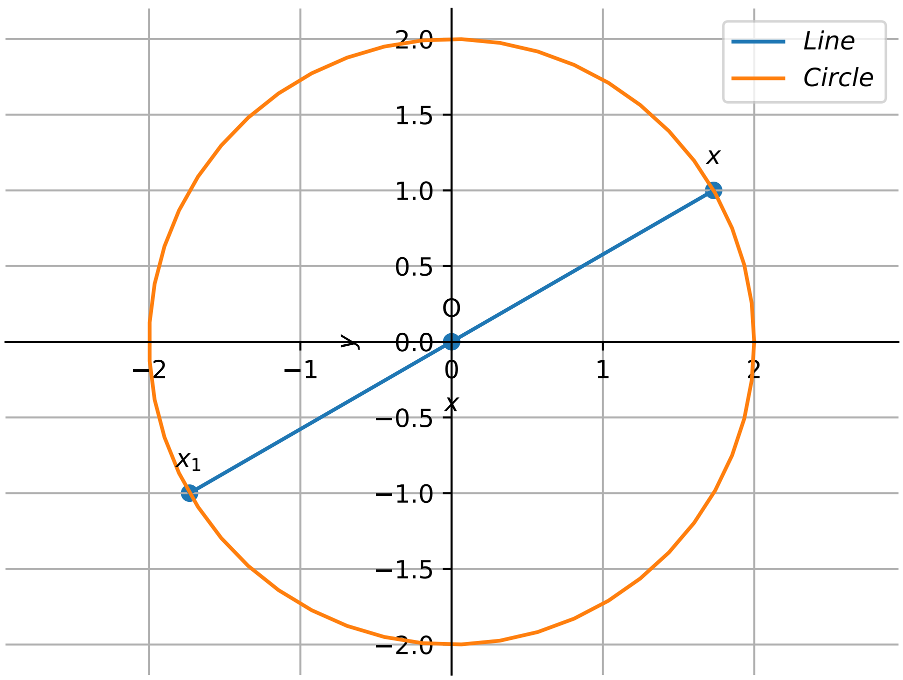


Figure 8.1.4.1:

Substituting the above parameters in (F.3.1.3),

$$\mu = \sqrt{3} \tag{8.1.4.2}$$

yielding the desired point of intersection as

$$\mathbf{x} = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix} \tag{8.1.4.3}$$

From (8.1.4.1), the angle between the given line and the x axis is

$$\theta = 30^\circ \tag{8.1.4.4}$$

and the area of the sector is

$$\frac{\theta}{360}\pi r^2 = \frac{\pi}{3} \tag{8.1.4.5}$$

8.1.5 Find the area of the smaller part of the circle $x^2 + y^2 = a^2$ cut off by the line x=$\frac{a}{\sqrt{2}}$.

**Solution:** The given circle can be expressed as a conic with parameters

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{u} = 0, f = -a^2 \tag{8.1.5.1}$$

The given line parameters are

$$\mathbf{h} = \begin{pmatrix} \frac{a}{\sqrt{2}} \\ 0 \end{pmatrix}, \mathbf{m} = \mathbf{e}_2. \tag{8.1.5.2}$$

Substituting the above in (F.3.1.3),

$$\mu = \pm\frac{a}{\sqrt{2}} \tag{8.1.5.3}$$

yielding the points of intersection of the line with circle as

$$\mathbf{A} = \begin{pmatrix} \frac{a}{\sqrt{2}} \\ -\frac{a}{\sqrt{2}} \end{pmatrix}, \mathbf{B} = \begin{pmatrix} \frac{a}{\sqrt{2}} \\ \frac{a}{\sqrt{2}} \end{pmatrix} \tag{8.1.5.4}$$

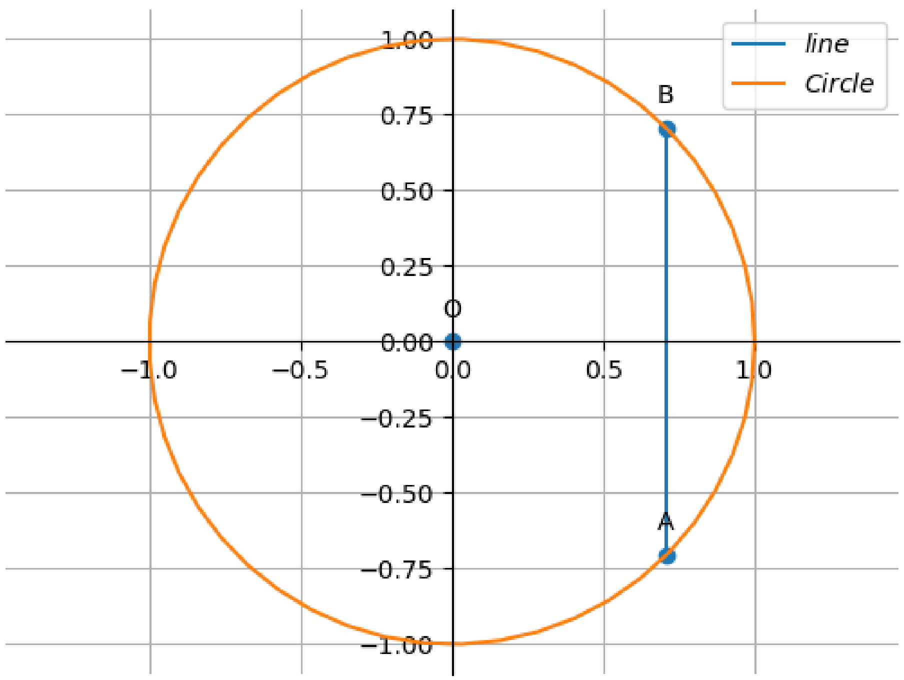


Figure 8.1.5.1:

From Fig. 8.1.5.1, the total area of the portion is given by

$$ar(APQ) = 2ar(APR) \tag{8.1.5.5}$$

$$= 2\int_0^{\frac{a}{\sqrt{2}}} \sqrt{a^2 - x^2}\, dx \tag{8.1.5.6}$$

$$= \frac{a^2}{2}\left(1 + \frac{\pi}{2}\right) \tag{8.1.5.7}$$

8.1.6 The area between $x = y^2$ and $x = 4$ is divided into two equal parts by the line $x = a$, find the value of a.

**Solution:** The given conic parameters are

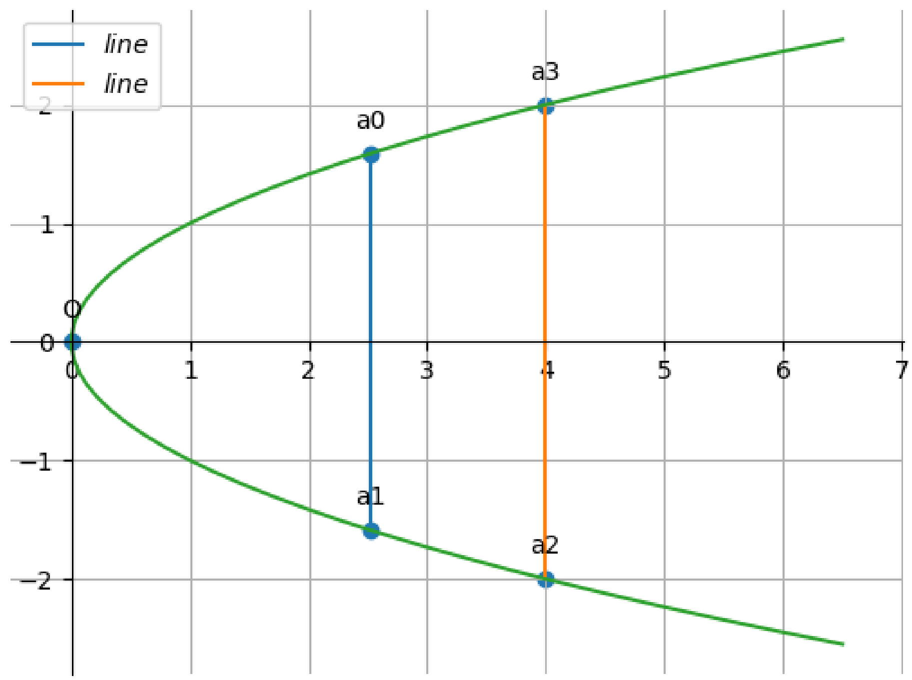


Figure 8.1.6.1:

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{u} = -\frac{1}{2}\mathbf{e}_1 f = 0 \tag{8.1.6.1}$$

The parameters of the lines are

$$\mathbf{q}_2 = \begin{pmatrix} a \\ 0 \end{pmatrix}, \mathbf{m}_2 = \mathbf{e}_2 \tag{8.1.6.2}$$

Substituting the above values in (F.3.1.3),

$$\mu_i = a, -a \tag{8.1.6.3}$$

yielding the points of intersection as

$$\mathbf{a_0} = \begin{pmatrix} a \\ a \end{pmatrix}, \mathbf{a_1} = \begin{pmatrix} a \\ -a \end{pmatrix} \tag{8.1.6.4}$$

Similarly, for the line $x - 4 = 0$,

$$\mathbf{q_1} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}, \mathbf{m_1} = \mathbf{e}_2 \tag{8.1.6.5}$$

yielding

$$\mu_i = 2, -2 \tag{8.1.6.6}$$

and

$$\mathbf{a}_3 = \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \mathbf{a}_2 = \begin{pmatrix} 4 \\ -2 \end{pmatrix}. \tag{8.1.6.7}$$

Area between parabola and the line $x = 4$ is divided equally by the line $x = a$. Thus,

$$A_1 = \int_0^a \sqrt{x}\, dx \tag{8.1.6.8}$$

$$A_2 = \int_a^4 \sqrt{x}\, dx \tag{8.1.6.9}$$

$$\text{and } A_1 = A_2 \tag{8.1.6.10}$$

$$\implies a = 4^{\frac{2}{3}} \tag{8.1.6.11}$$

8.1.7 Find the area of the region bounded by the parabola $y = x^2$ and $y = |x|$.

**Solution:**

8.1.8 Find the area bounded by the curve $x^2 = 4y$ and the line $x = 4y - 2$.

**Solution:** The given curve can be expressed as a conic with parameters

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}, f = 0 \tag{8.1.8.1}$$

The parameters of the given line are

$$\mathbf{q} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \tag{8.1.8.2}$$

The points of intersection can then be obtained from (F.3.1.3) as

$$\therefore \mathbf{x}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \mathbf{x}_2 = \begin{pmatrix} -1 \\ \frac{1}{4} \end{pmatrix} \tag{8.1.8.3}$$

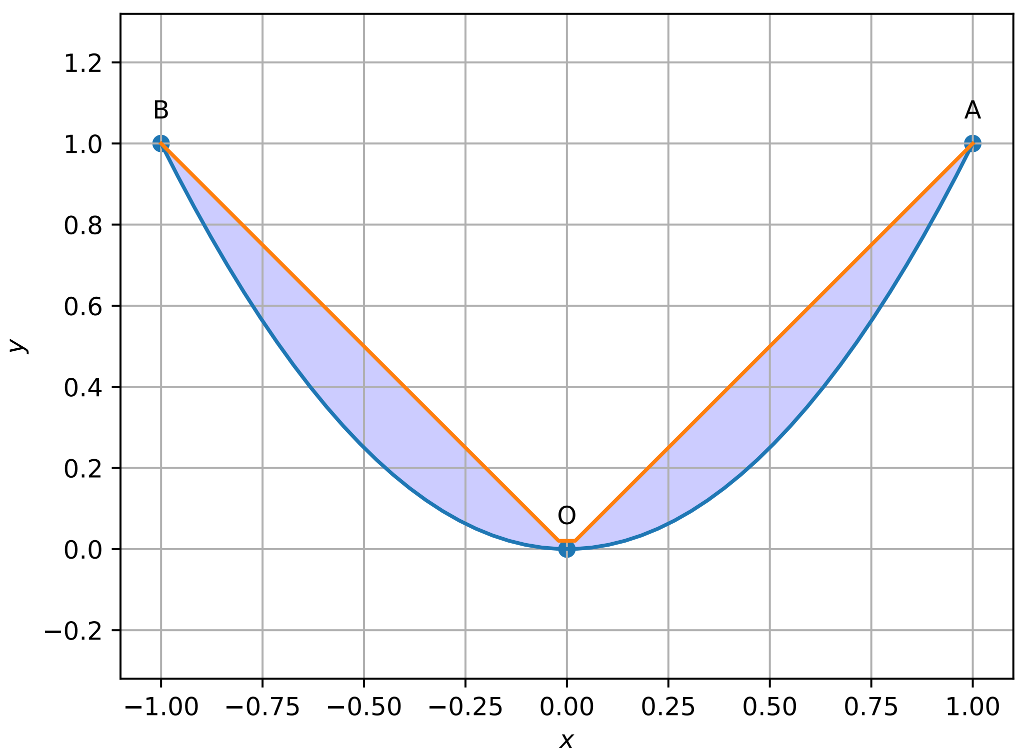


Figure 8.1.7.1:

The desired area is then obtained as

$$A = \int_{x_2}^{x_1} [f(x) - g(x)]\, dx \tag{8.1.8.4}$$

$$= \int_{-1}^{2} \left( \frac{x+2}{4} - \frac{x^2}{4} \right) dx \tag{8.1.8.5}$$

$$= \frac{9}{8} \tag{8.1.8.6}$$

8.1.9

8.1.10

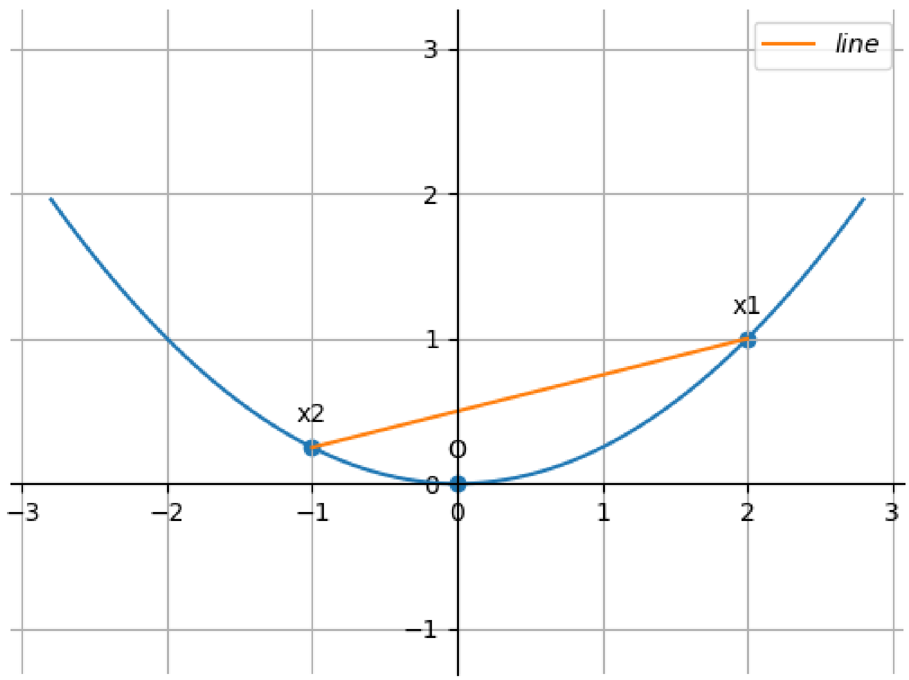


Figure 8.1.8.1:

8.1.11

## 8.2. Curves

8.2.1 Find the area of the circle $4x^2 + 4y^2 = 9$ which is interior to the parabola $x^2 = 4y$.

**Solution:** The given circle and parabola can be expressed as conics with parameters

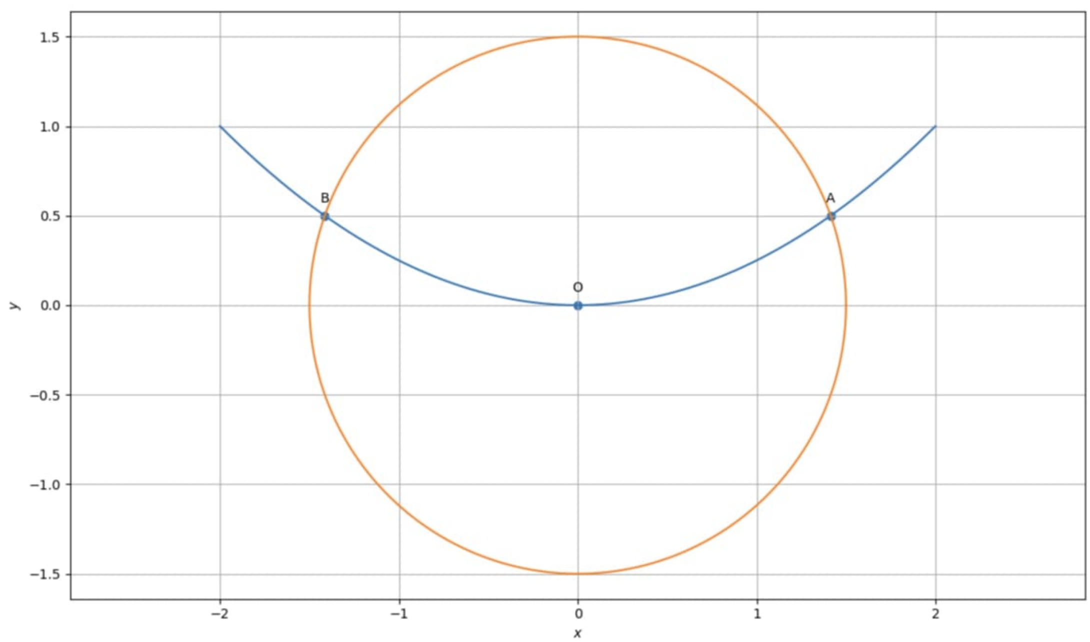


Figure 8.2.1.1:

$$\mathbf{V}_1 = 4\mathbf{I}, \mathbf{u_1} = \mathbf{0}, f_1 = -9 \tag{8.2.1.1}$$

$$\mathbf{V}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u_2} = -\begin{pmatrix} 0 \\ 2 \end{pmatrix}, f_2 = 0 \tag{8.2.1.2}$$

The intersection of the given conics is obtained as

$$\mathbf{x}^\top \left(\mathbf{V}_1 + \mu \mathbf{V}_2\right) \mathbf{x} + 2\left(\mathbf{u}_1 + \mu \mathbf{u}_2\right)^\top \mathbf{x} + \left(f_1 + \mu f_2\right) = 0 \tag{8.2.1.3}$$

This conic represents a pair of straight lines if

$$\begin{vmatrix} \mathbf{V}_1 + \mu \mathbf{V}_2 & \mathbf{u}_1 + \mu \mathbf{u}_2 \\ \left(\mathbf{u}_1 + \mu \mathbf{u}_2\right)^\top & f_1 + \mu f_2 \end{vmatrix} = 0 \tag{8.2.1.4}$$

which can be expressed as

$$\implies \begin{vmatrix} \mu+4 & 0 & 0 \\ 0 & 4 & -2\mu \\ 0 & -2\mu & -9 \end{vmatrix} = 0 \tag{8.2.1.5}$$

Solving the above equation we get,

$$\mu^3 + 4\mu^2 + 9\mu + 36 = 0 \tag{8.2.1.6}$$

yielding

$$\mu = -4. \tag{8.2.1.7}$$

Thus, the parameters for the pair of straight lines can be expressed as

$$\mathbf{V} = \mathbf{V}_1 + \mu\mathbf{V}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix}, \tag{8.2.1.8}$$

$$\mathbf{u} = \mathbf{u}_1 + \mu\mathbf{u}_2 = \begin{pmatrix} 0 \\ 8 \end{pmatrix} \tag{8.2.1.9}$$

$$f = -9, \tag{8.2.1.10}$$

$$\implies \mathbf{D} = \mathbf{V}, \mathbf{P} = \mathbf{I} \tag{8.2.1.11}$$

8.2.2

8.2.3 Find the area of the region bounded by the curves $y = x^2 + 2$, $y = x$, $x = 0$ and $x = 3$.

**Solution:** The conic parameters are

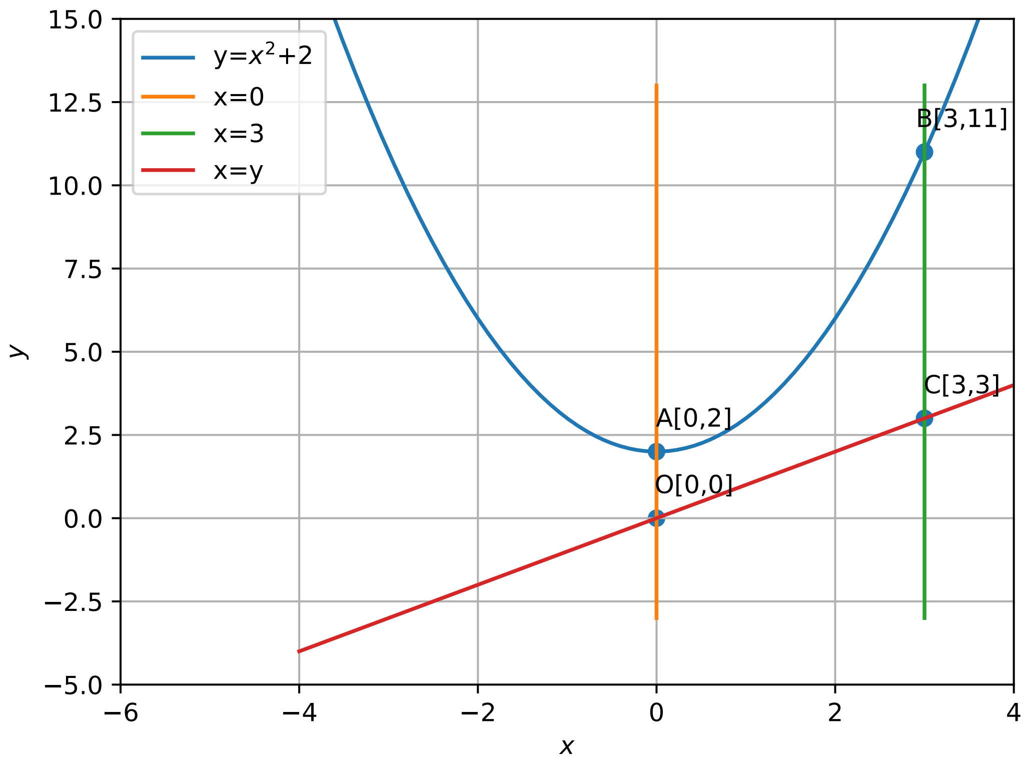


Figure 8.2.3.1:

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 0 \\ -1/2 \end{pmatrix}, f = 2. \tag{8.2.3.1}$$

8.2.4

8.2.5

8.2.6 Find the smaller area enclosed by the circle $x^2 + y^2 = 4$ and the line $x + y = 2$.

**Solution:** The given circle can be expressed as conics with parameters,

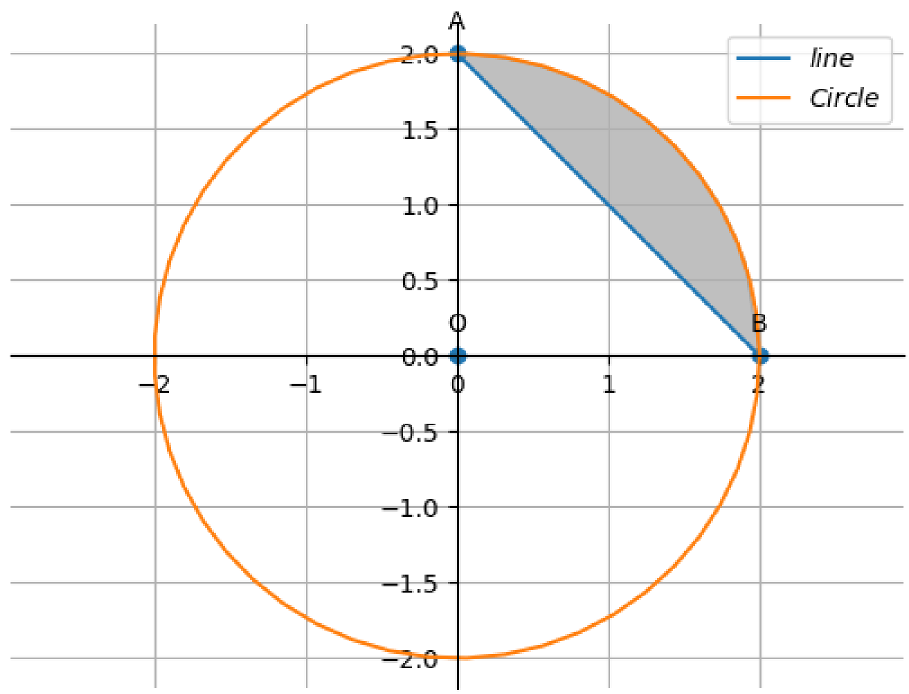


Figure 8.2.6.1:

$$\mathbf{V} = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}, \mathbf{u} = 0, f = -16 \tag{8.2.6.1}$$

The line parameters are

$$\mathbf{h} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} \tag{8.2.6.2}$$

Substituting the parameters in (F.3.1.3),

$$\mu = 0, -4 \tag{8.2.6.3}$$

yielding the points of intersection as

$$\mathbf{A} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \tag{8.2.6.4}$$

From Fig. 8.2.6.1, the desired area is

$$\int_0^2 \sqrt{4 - x^2}\, dx - \int_0^2 (2 - x)\, dx = \pi - 2 \tag{8.2.6.5}$$

8.2.7

## 8.3. Miscellaneous

8.3.1

8.3.2 Find the area between the curves $y = x$ and $y = x^2$.

**Solution:** The given curve can be expressed as a conic with parameters

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix}, f = 0 \tag{8.3.2.1}$$

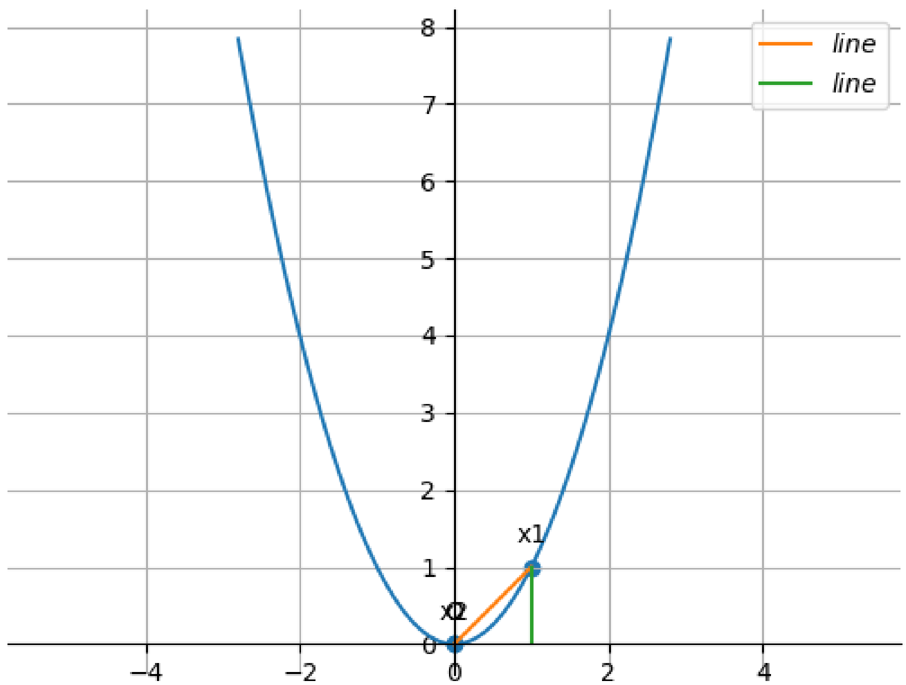


Figure 8.3.2.1:

The given line parameters are

$$\mathbf{h} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{8.3.2.2}$$

Substituting the given parameters in (F.3.1.3),

$$\mathbf{x_1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{x_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{8.3.2.3}$$

From Fig. 8.3.2.1, the area bounded by the curve $y = x^2$ and line $y = x$ is given by

$$\int_0^1 \left(x - \frac{x^2}{2}\right) dx = \frac{1}{6} \tag{8.3.2.4}$$

8.3.3 Find the area of the region bounded by the curve $x^2 = 4y$ and the lines y=2 and y=4 and the y-axis in the first quadrant.

**Solution:** The conic parameters are

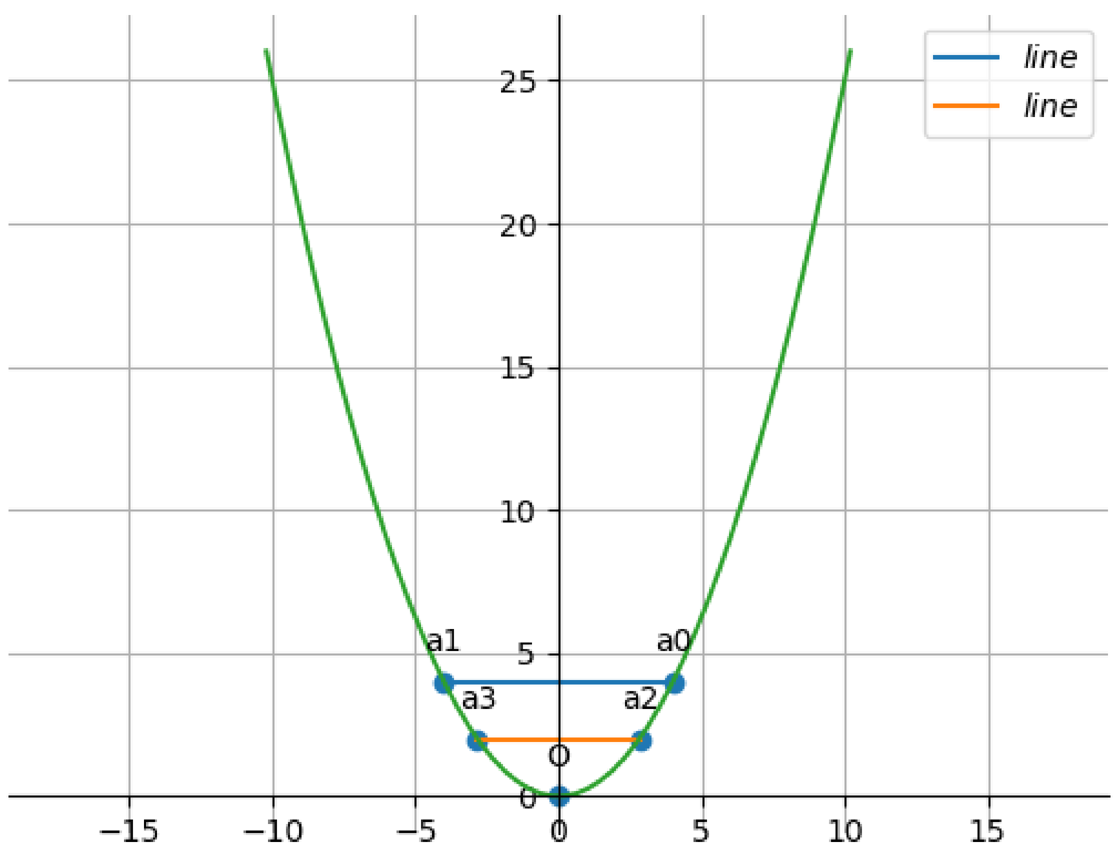


Figure 8.3.3.1:

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}, f = 0 \tag{8.3.3.1}$$

The vector parameters of $y - 4 = 0$ are

$$\mathbf{h}_1 = \begin{pmatrix} 0 \\ 4 \end{pmatrix}, \mathbf{m}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{8.3.3.2}$$

Substituting the above in (F.3.1.3),

$$\mu_i = 4, -4 \tag{8.3.3.3}$$

yielding the points of intersection with the parabola as

$$\mathbf{a}_0 = \begin{pmatrix} 4 \\ 4 \end{pmatrix}, \mathbf{a}_1 = \begin{pmatrix} -4 \\ 4 \end{pmatrix} \tag{8.3.3.4}$$

Similarly, for the line $y - 2 = 0$, the vector parameters are

$$\mathbf{h}_2 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \mathbf{m}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{8.3.3.5}$$

yielding

$$\mu_i = 2.8, -2.8 \tag{8.3.3.6}$$

and the points of intersection

$$\mathbf{a}_2 = \begin{pmatrix} 2.8 \\ 2 \end{pmatrix}, \mathbf{a}_3 = \begin{pmatrix} -2.8 \\ 2 \end{pmatrix} \tag{8.3.3.7}$$

From Fig. 8.3.3.1, the area of the parabola between the lines $y = 2$ and $y = 4$ is given by

$$\int_0^4 2\sqrt{y}\,dy - \int_0^2 2\sqrt{y}\,dy = 6.895 \tag{8.3.3.8}$$

8.3.4

8.3.5

8.3.6

8.3.7 Find the area enclosed by the parabola $4y = 3x^2$ and the line $2y = 3x + 12$.

**Solution:** The parameters of the given conic are

$$\mathbf{V} = \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}, f = 0. \tag{8.3.7.1}$$

For the line, the parameters are

$$\mathbf{h} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \tag{8.3.7.2}$$

yielding

$$\mu = -2.5, 2.7 \tag{8.3.7.3}$$

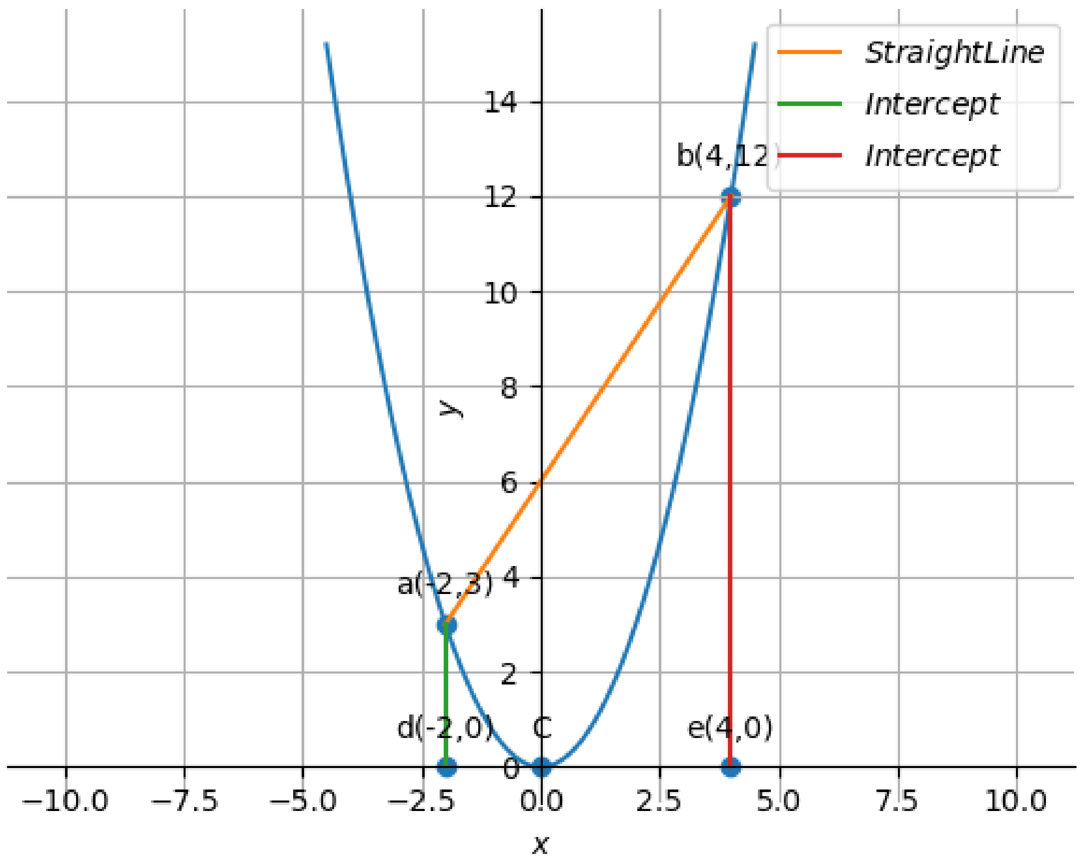


Figure 8.3.7.1:

upon substitution in (F.3.1.3) resulting in the points of intersection

$$\mathbf{A} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 4 \\ 12 \end{pmatrix}. \tag{8.3.7.4}$$

From Fig. 8.3.7.1, the desired area is

$$\int_{-2}^{4} \frac{3x+12}{2}\,dx - \int_{-2}^{4} \frac{3x^2}{4}\,dx = 27 \tag{8.3.7.5}$$

8.3.8 Find the area of the smaller region bounded by the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ and the line $\frac{x}{3} + \frac{y}{2} = 1$.

**Solution:** The given ellipse can be expressed as conics with parameters

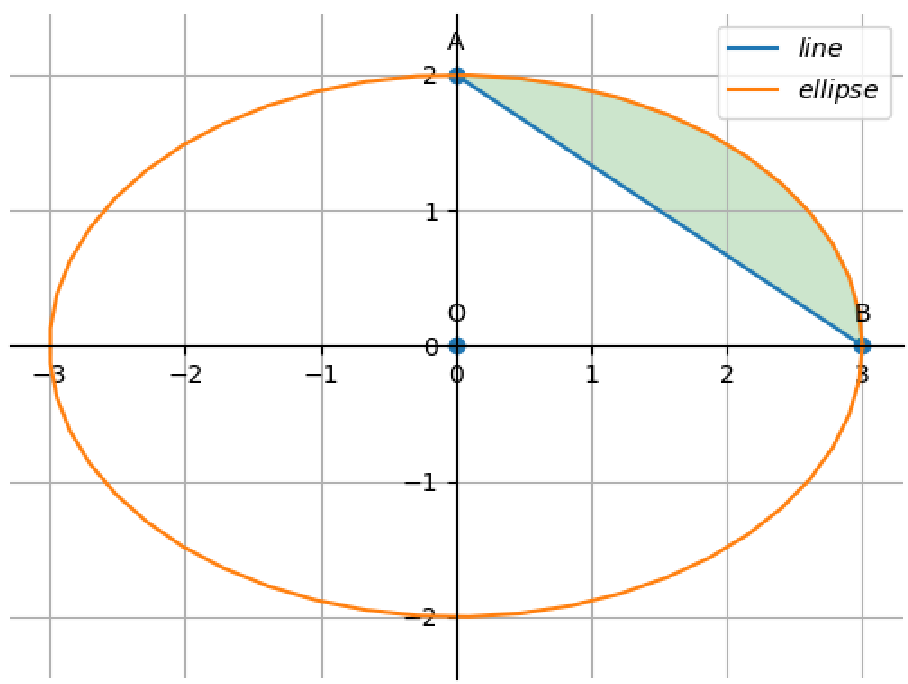


Figure 8.3.8.1:

$$\mathbf{V} = \begin{pmatrix} b^2 & 0 \\ 0 & a^2 \end{pmatrix}, \mathbf{u} = 0, f = -(a^2b^2). \tag{8.3.8.1}$$

The line parameters are

$$\mathbf{h} = \begin{pmatrix} a \\ 0 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} \frac{1}{b} \\ -\frac{1}{a} \end{pmatrix}. \tag{8.3.8.2}$$

Substituting the given parameters in (F.3.1.3),

$$\mu = 0, -6 \tag{8.3.8.3}$$

yielding the points of intersection

$$\mathbf{A} = \begin{pmatrix} a \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ b \end{pmatrix}. \tag{8.3.8.4}$$

From Fig. 8.3.8.1, the desired area is

$$\int_0^3 \frac{2}{3}\sqrt{9-x^2}\, dx - \int_0^3 \frac{2}{3}(3-x)\, dx = 3\left(\frac{\pi}{2} - 1\right) \tag{8.3.8.5}$$

8.3.9 Find the area of the smaller region bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the line $\frac{x}{a} + \frac{y}{b} = 1$.

**Solution:** The given ellipse can be expressed as a conic with parameters

$$\mathbf{V} = \begin{pmatrix} b^2 & 0 \\ 0 & a^2 \end{pmatrix}, \mathbf{u} = 0, f = -(a^2b^2). \tag{8.3.9.1}$$

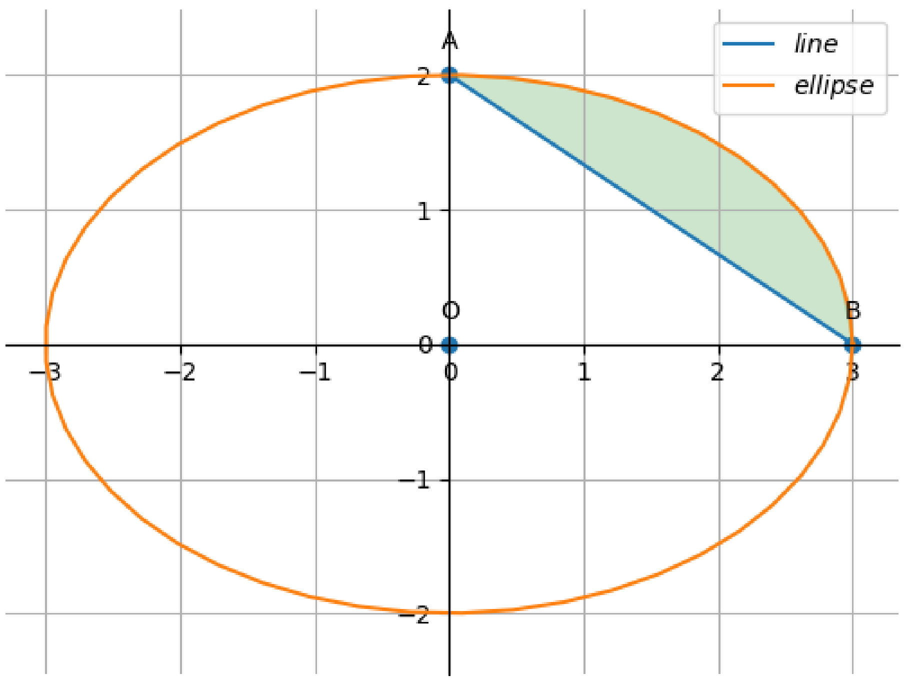


Figure 8.3.9.1:

The given line parameters are

$$\mathbf{h} = \begin{pmatrix} a \\ 0 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} \frac{1}{b} \\ -\frac{1}{a} \end{pmatrix}. \tag{8.3.9.2}$$

Substituting the given parameters in (F.3.1.3)

$$\mu = 0, -6 \tag{8.3.9.3}$$

yielding the points of intersection

$$\mathbf{A} = \begin{pmatrix} a \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ b \end{pmatrix} \tag{8.3.9.4}$$

From Fig. 8.3.9.1, the desired area is

$$\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\, dx - \int_0^a \frac{b}{a}(a - x)\, dx = \frac{ab}{2}\left(\frac{\pi}{2} - 1\right) \tag{8.3.9.5}$$

8.3.10 Find the area of the region bounded by the curve $x^2 = y$ and the lines $y = x + 2$ and the $x$ axis.

**Solution:**

8.3.11 Find the area bounded by the curve $y = x|x|$, $x$-axis and the ordinates $x$=-1 and $x$=1.

**Solution:** The parameters of the given conics are

$$\mathbf{V}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u_1} = \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix}, f_1 = 0 \tag{8.3.11.1}$$

$$\mathbf{V}_2 = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u_2} = \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix}, f_2 = 0 \tag{8.3.11.2}$$

The determinant equation for the intersection of two conics is

$$\begin{vmatrix} 1 - \mu & 0 & 0 \\ 0 & 0 & -\frac{1}{2} - \frac{\mu}{2} \\ 0 & -\frac{1}{2} - \frac{\mu}{2} & 0 \end{vmatrix} = 0 \tag{8.3.11.3}$$

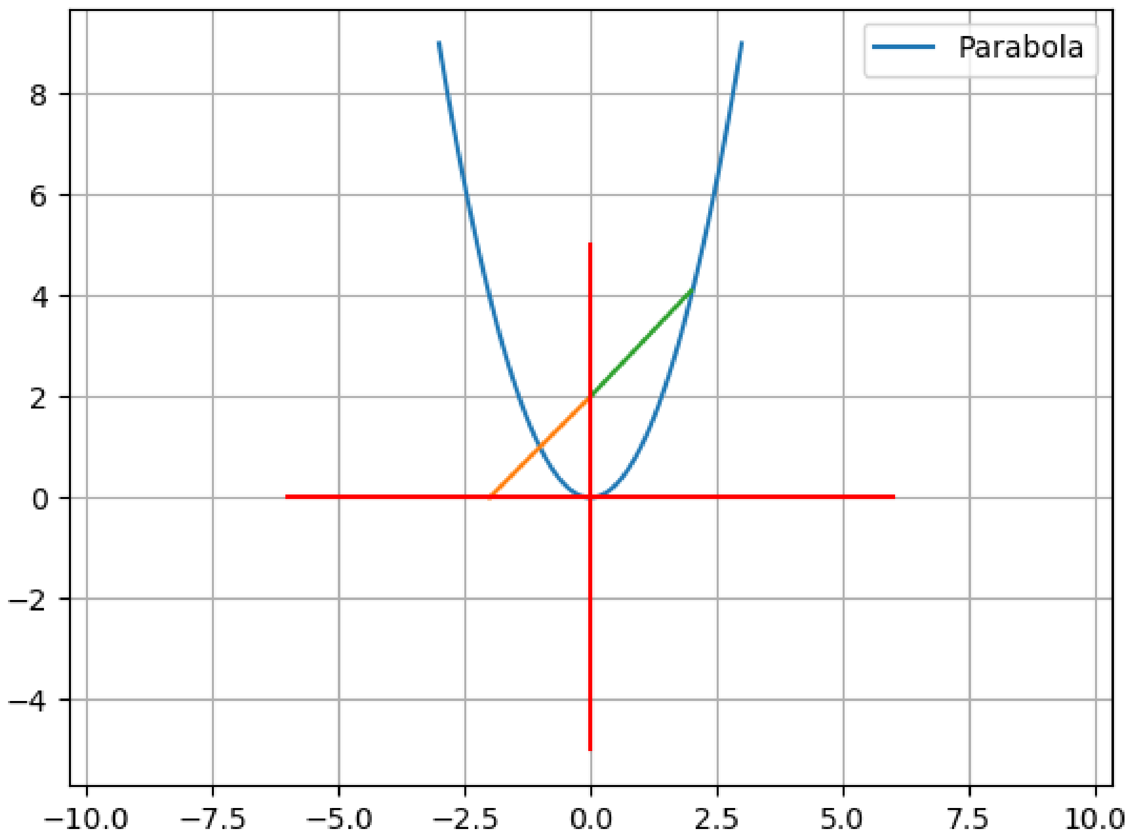


Figure 8.3.10.1:

yielding,

$$\mu^3 + \mu^2 - \mu - 1 = 0 \tag{8.3.11.4}$$

$$\implies \mu = -1, 1, 1 \tag{8.3.11.5}$$

8.3.12 Find the area of the circle $x^2 + y^2 = 16$ exterior to the parabola $y^2 = 6x$.

**Solution:** The given circle and parabola can be expressed as conics with respectivepa-

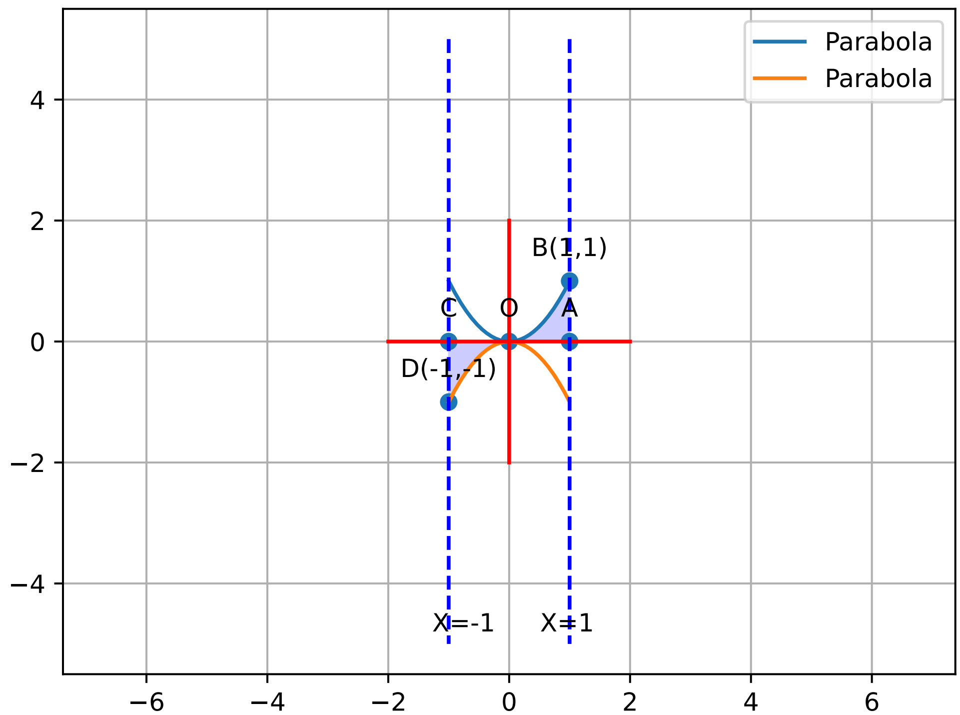


Figure 8.3.11.1:

rameters

$$\mathbf{V}_1 = \mathbf{I}, \mathbf{u_1} = 0, f_1 = -16, \tag{8.3.12.1}$$

$$\mathbf{V}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{u_2} = -\begin{pmatrix} 3 \\ 0 \end{pmatrix}, f_2 = 0 \tag{8.3.12.2}$$

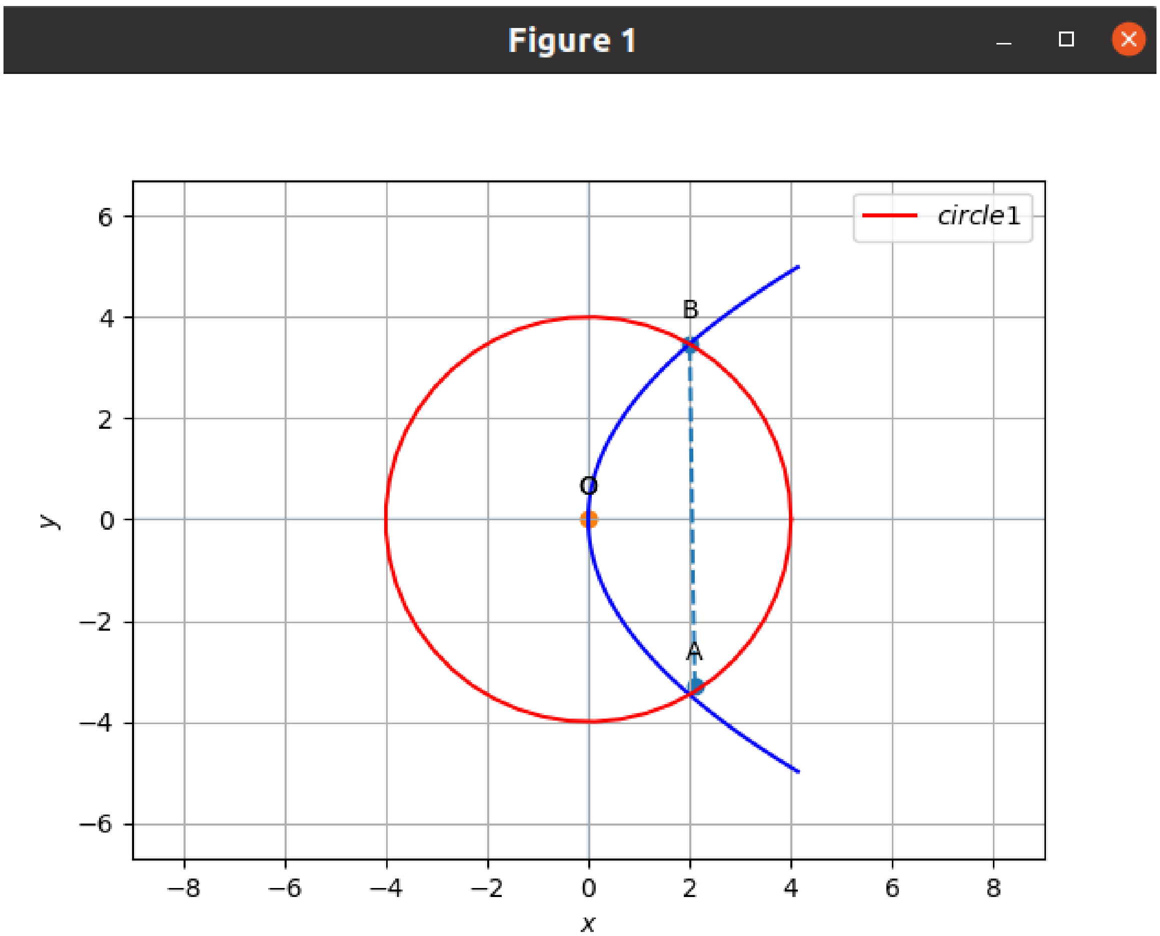


Figure 8.3.12.1:

The determinant of the intersection of the given conics is

$$\implies \begin{vmatrix} 1 & 0 & -3\mu \\ 0 & 1+\mu & 0 \\ -3\mu & 0 & -16 \end{vmatrix} = 0 \tag{8.3.12.3}$$

yielding

$$9\mu^3 + 9\mu^2 + 16\mu + 16 = 0 \tag{8.3.12.4}$$

$$\text{or, } \mu = -1 \tag{8.3.12.5}$$

# Chapter 9

# Tangent And Normal

## 9.1. Properties

9.1.1 Find the slope of the tangent to the curve

$$y = \frac{x-1}{x-2}, x \neq 2 \text{ at } x = 10. \tag{9.1.1.1}$$

9.1.2 Find a point on the curve

$$y = (x-2)^2 \tag{9.1.2.1}$$

at which a tangent is parallel to the chord joining the points (2,0) and (4,4).

**Solution:** The equation of the conic can be represented as

$$\mathbf{x}^\top \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} -2 & \frac{-1}{2} \end{pmatrix} \mathbf{x} + 4 = 0 \tag{9.1.2.2}$$

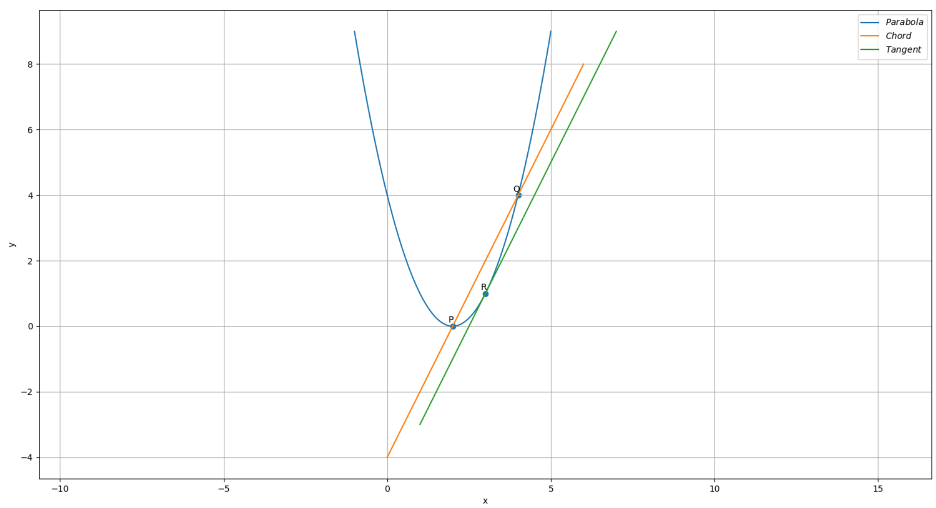


Figure 9.1.2.1:

So,

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u}^\top = \begin{pmatrix} -2 & \frac{-1}{2} \end{pmatrix}, f = 4 \tag{9.1.2.3}$$

The direction vector of the line passing through (2,0) and (4,4) is

$$\mathbf{m} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \implies \mathbf{n} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}. \tag{9.1.2.4}$$

From (F.4.7.1), the point of contact to parabola is given by

$$\begin{pmatrix} (\mathbf{u}+\kappa\mathbf{n})^\top \\ \mathbf{V} \end{pmatrix} \mathbf{q} = \begin{pmatrix} -f \\ \kappa\mathbf{n}-\mathbf{u} \end{pmatrix} \tag{9.1.2.5}$$

$$\text{where } \kappa = \frac{\mathbf{p}_1^\top \mathbf{u}}{\mathbf{p}_1^\top \mathbf{n}}, \quad \mathbf{V}\mathbf{p}_1 = 0 \tag{9.1.2.6}$$

The eigenvector corresponding to the zero eigenvalue is

$$\mathbf{p}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{9.1.2.7}$$

from which,

$$\kappa = \frac{\begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} -2 \\ \frac{-1}{2} \end{pmatrix}}{\begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix}} \tag{9.1.2.8}$$

$$= \frac{1}{2} \tag{9.1.2.9}$$

Substituting $\kappa$ in (9.1.2.5),

$$\begin{pmatrix} \left[\begin{pmatrix} -2 \\ \frac{-1}{2} \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 2 \\ -1 \end{pmatrix}\right]^{\top} \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \mathbf{q} = \begin{pmatrix} -4 \\ \frac{1}{2}\begin{pmatrix} 2 \\ -1 \end{pmatrix} - \begin{pmatrix} -2 \\ \frac{-1}{2} \end{pmatrix} \end{pmatrix} \tag{9.1.2.10}$$

$$\implies \begin{pmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{q} = \begin{pmatrix} -4 \\ 3 \\ 0 \end{pmatrix} \tag{9.1.2.11}$$

As the last row elements are all zero, we can eliminate that row

$$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{q} = \begin{pmatrix} -4 \\ 3 \end{pmatrix} \tag{9.1.2.12}$$

For applying row reduction method the augmented matrix is written as

$$\left(\begin{array}{cc|c} -1 & -1 & -4 \\ 1 & 0 & 3 \end{array}\right) \tag{9.1.2.13}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 + 2R_2} \left(\begin{array}{cc|c} 1 & -1 & 2 \\ 1 & 0 & 3 \end{array}\right) \tag{9.1.2.14}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - R_1} \left(\begin{array}{cc|c} 1 & -1 & 2 \\ 0 & 1 & 1 \end{array}\right) \tag{9.1.2.15}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 + R_2} \left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 1 \end{array}\right) \tag{9.1.2.16}$$

$$\implies \mathbf{q} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \tag{9.1.2.17}$$

which is the desired point of contact. See Fig. 9.1.2.1.

9.1.3 Find the equation of all lines having slope -1 that are tangents to the curve

$$y = \frac{1}{x-1}, x \neq 1 \tag{9.1.3.1}$$

**Solution:** From the given information,

$$\mathbf{V} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix}, f = -1, m = -1 \tag{9.1.3.2}$$

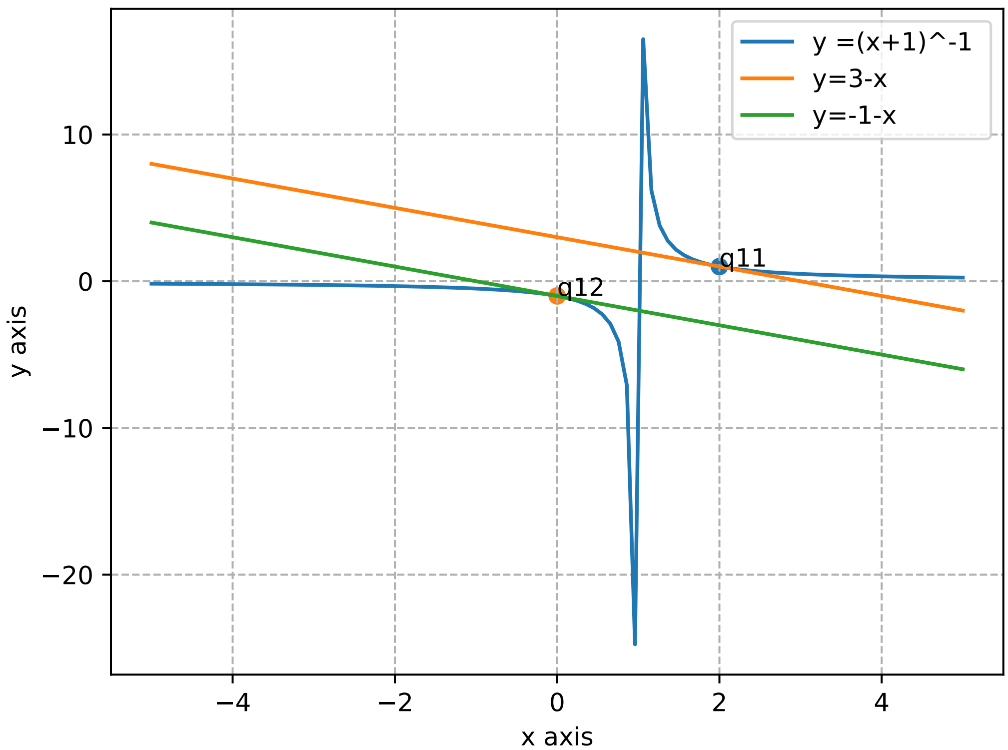


Figure 9.1.3.1:

From the above, the normal vector is

$$\mathbf{n} = \begin{pmatrix} -m \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{9.1.3.3}$$

From (F.4.4.1), the point(s) of contact are given by

$$\mathbf{q} = \mathbf{V}^{-1}(k_i\mathbf{n} - \mathbf{u}) \text{ where,} \tag{9.1.3.4}$$

$$k_i = \pm\sqrt{\frac{f_0}{\mathbf{n}^\top \mathbf{V}^{-1}\mathbf{n}}} \tag{9.1.3.5}$$

$$f_0 = f + \mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} \tag{9.1.3.6}$$

Substituting from (9.1.3.3) and (9.1.3.2) in the above,

$$\mathbf{q} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}. \tag{9.1.3.7}$$

From (F.4.1.1), the equations of tangents are given by

$$(\mathbf{V}\mathbf{q} + \mathbf{u})^\top \mathbf{x} + \mathbf{u}^\top \mathbf{q} + f = 0 \tag{9.1.3.8}$$

yielding

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} + 1 = 0 \tag{9.1.3.9}$$

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} - 3 = 0 \tag{9.1.3.10}$$

$$\tag{9.1.3.11}$$

See Fig. 9.1.3.1.

9.1.4 Find the equation of all lines having slope 2 which are tangents to the curve

$$y = \frac{1}{x-3}, x \neq 3 \tag{9.1.4.1}$$

**Solution:** From the given information

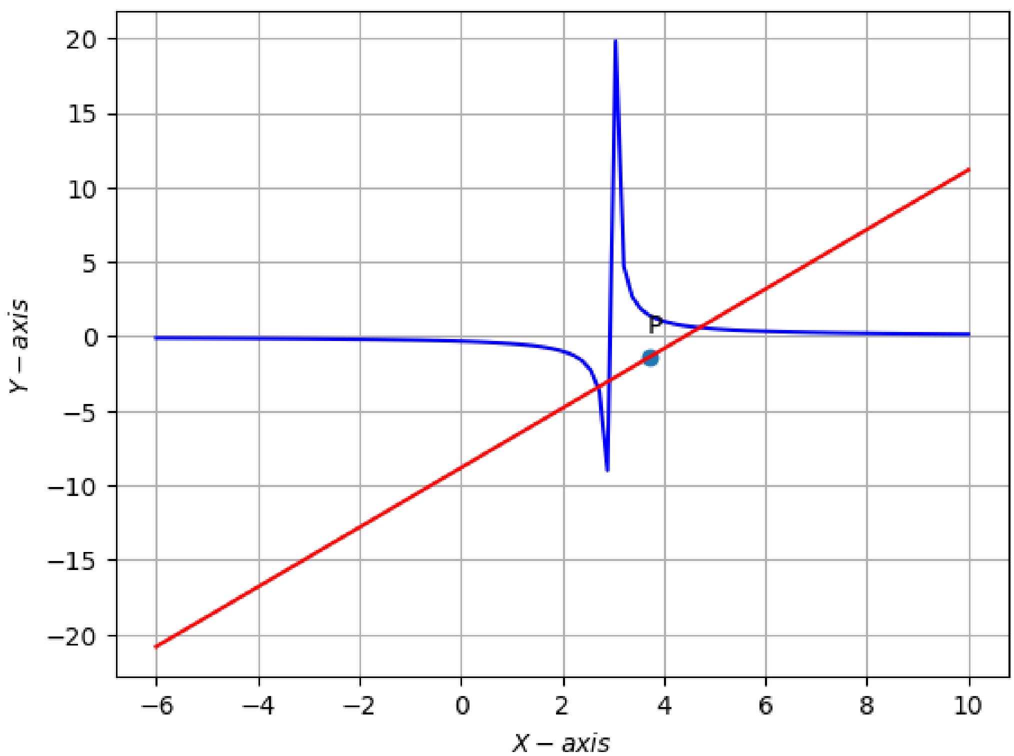


Figure 9.1.4.1:

$$\mathbf{V} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 0 \\ -\frac{3}{2} \end{pmatrix}, f = -1, m = 2 \tag{9.1.4.2}$$

$$\implies \mathbf{n} = \begin{pmatrix} -m \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \tag{9.1.4.3}$$

$$\tag{9.1.4.4}$$

Hence, the given curve is a hyperbola. Substituting numerical values, we obtain the condition in (F.4.5), which implies that the line with slope 2 is not a tangent. This can be verified from Fig. 9.1.4.1.

9.1.5 Find points on the curve $\frac{x^2}{9} + \frac{y^2}{16} = 1$ at which the tangents are

(a) parallel to x-axis

(b) parallel to y-axis

**Solution:** The parameters of the given conic are

$$\lambda_1 = 16, \lambda_2 = 9 \tag{9.1.5.1}$$

$$\mathbf{V} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, f = -144 \tag{9.1.5.2}$$

(a) The normal vector in this case is

$$\mathbf{n_1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{9.1.5.3}$$

which can be used along with the parameters in (9.1.5.2) to obtain

$$\mathbf{q_1} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}, \mathbf{q_2} = \begin{pmatrix} 0 \\ -4 \end{pmatrix} \tag{9.1.5.4}$$

using (F.4.4.1).

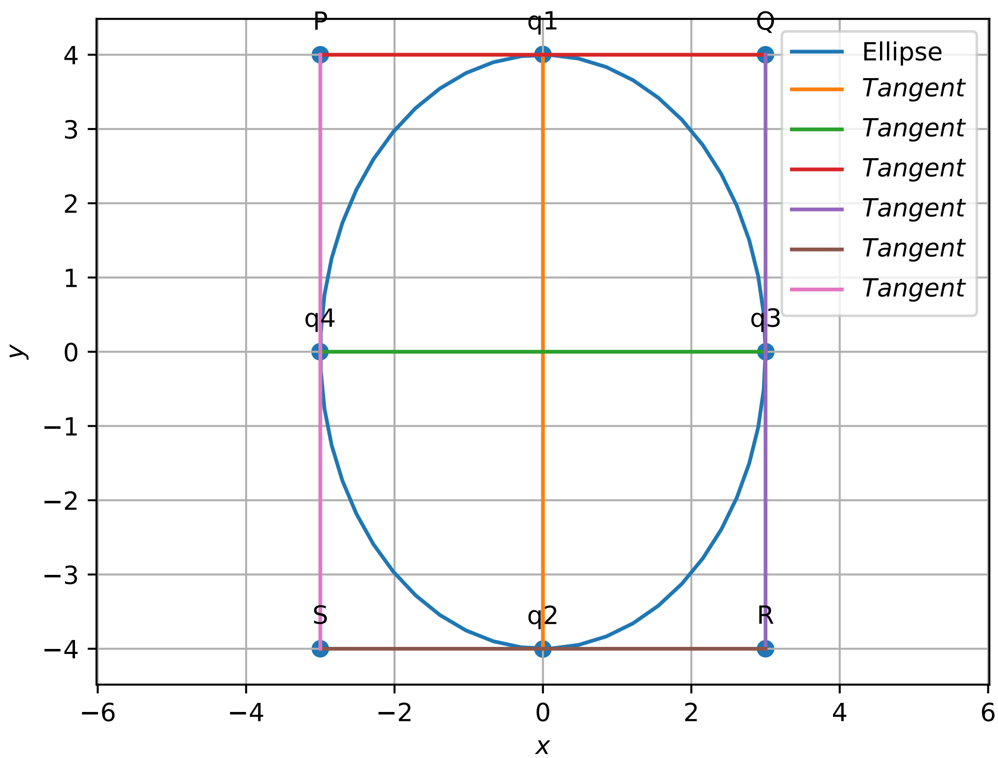


Figure 9.1.5.1:

(b) Simlarly, choosing

$$\mathbf{n_2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{9.1.5.5}$$

$$\mathbf{q_3} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \mathbf{q_4} = \begin{pmatrix} -3 \\ 0 \end{pmatrix} \tag{9.1.5.6}$$

9.1.6 Find the equation of the tangent line to the curve

$$y = x^2 - 2x + 7 \tag{9.1.6.1}$$

(a) parallel to the line $2x - y + 9 = 0$.

(b) perpendicular to the line $5y - 15x = 13$.

**Solution:** The parameters of the given conic are

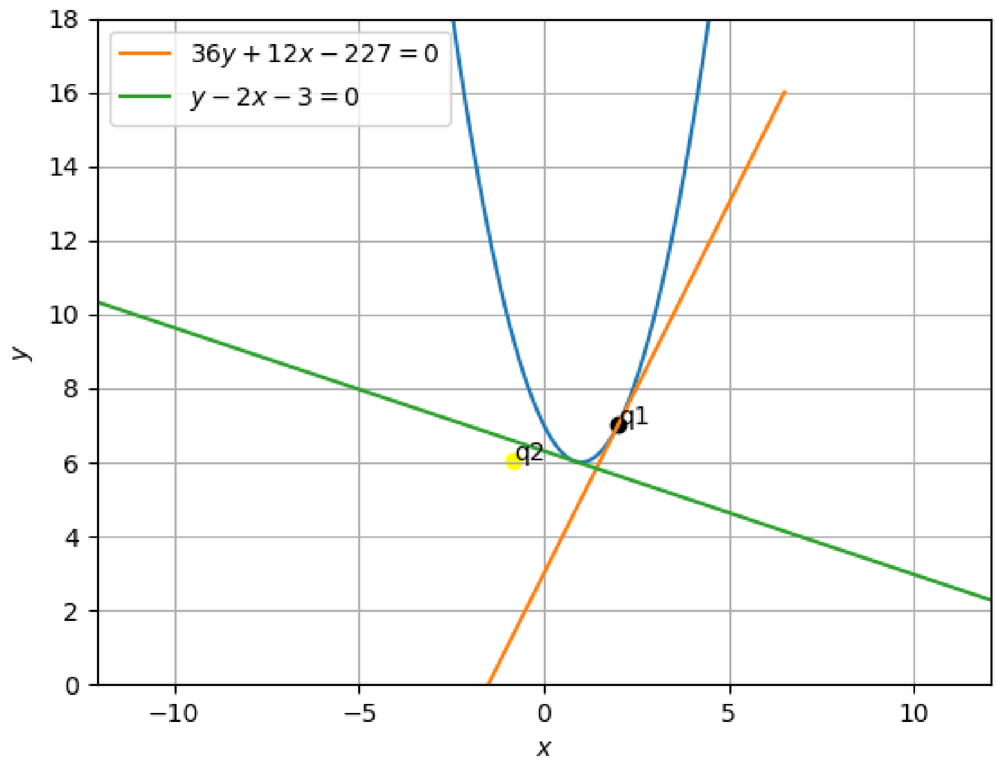


Figure 9.1.6.1:

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u} = -\begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix}, f = 7 \tag{9.1.6.2}$$

(a) In this case, the normal vector

$$\mathbf{n}_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \tag{9.1.6.3}$$

Since $\mathbf{V}$ is not invertible, the point of contact is given by (F.4.7.1) resulting in

$$\begin{pmatrix} \left(\begin{pmatrix} -1 \\ -\frac{1}{2} \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 2 \\ -1 \end{pmatrix}\right)^\top \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \mathbf{q}_1 = \begin{pmatrix} -7 \\ \frac{1}{2}\begin{pmatrix} 2 \\ -1 \end{pmatrix} - \begin{pmatrix} -1 \\ -\frac{1}{2} \end{pmatrix} \end{pmatrix} \tag{9.1.6.4}$$

By solving the above equation, we can get the point of contact as

$$\mathbf{q}_1 = \begin{pmatrix} 2 \\ 7 \end{pmatrix} \tag{9.1.6.5}$$

The tangent equation is then obtained as

$$\mathbf{n}_1^\top(\mathbf{x} - \mathbf{q}_1) = 0 \tag{9.1.6.6}$$

$$\implies \begin{pmatrix} 2 & -1 \end{pmatrix} \mathbf{x} + 3 = 0 \tag{9.1.6.7}$$

(b) In this case,

$$\mathbf{n}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \tag{9.1.6.8}$$

resulting in

$$\begin{pmatrix} \left(\begin{pmatrix} -1 \\ -\frac{1}{2} \end{pmatrix} + -\frac{1}{6}\begin{pmatrix} 1 \\ 3 \end{pmatrix}\right)^\top \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \mathbf{q}_2 = \begin{pmatrix} -7 \\ -\frac{1}{6}\begin{pmatrix} 1 \\ 3 \end{pmatrix} - \begin{pmatrix} -1 \\ -\frac{1}{2} \end{pmatrix} \end{pmatrix} \tag{9.1.6.9}$$

$$\text{or, } \mathbf{q}_2 = \begin{pmatrix} \frac{5}{6} \\ \frac{217}{36} \end{pmatrix} \tag{9.1.6.10}$$

The tangent equation is

$$\mathbf{n}_2^\top(\mathbf{x} - \mathbf{q}_2) = 0 \tag{9.1.6.11}$$

$$\text{or, } \begin{pmatrix} 1 & 3 \end{pmatrix} \mathbf{x} = \frac{227}{12} \tag{9.1.6.12}$$

9.1.7

9.1.8 Find the equation of the tangent to the curve

$$y = \sqrt{3x - 2} \tag{9.1.8.1}$$

which is parallel to the line

$$4x - 2y + 5 = 0 \tag{9.1.8.2}$$

**Solution:** The parameters for the given conic are

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{9.1.8.3}$$

$$\mathbf{u} = \begin{pmatrix} -3/2 \\ 0 \end{pmatrix}, \tag{9.1.8.4}$$

$$f = 2 \tag{9.1.8.5}$$

which represent a parabola. Following the approach in problem 9.1.6,

$$\mathbf{p_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{9.1.8.6}$$

$$\mathbf{n} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \tag{9.1.8.7}$$

yielding the matrix equation

$$\begin{pmatrix} -3 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{q} = \begin{pmatrix} -41/16 \\ 0 \\ 3/4 \end{pmatrix} \tag{9.1.8.8}$$

$$\tag{9.1.8.9}$$

The augmented matrix for (9.1.8.8) can be expressed as

$$\xleftrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{cc|c} -3 & 0 & -41/16 \\ 0 & 1 & 0 \\ 0 & 0 & 3/4 \end{array}\right) \tag{9.1.8.10}$$

$$\xleftrightarrow{-\frac{R_1}{-3} \leftarrow R_2} \left(\begin{array}{cc|c} 1 & 0 & 41/48 \\ 0 & 1 & 0 \\ 0 & 0 & 3/4 \end{array}\right) \tag{9.1.8.11}$$

$$\implies \mathbf{q} = \begin{pmatrix} \frac{41}{48} \\ \frac{3}{4} \end{pmatrix} \tag{9.1.8.12}$$

The equation of tangent is then obtained as

$$\begin{pmatrix} -2 & 1 \end{pmatrix} \mathbf{x} + \frac{23}{24} = 0 \tag{9.1.8.13}$$

See Fig. 9.1.8.1.

9.1.9 Find the point at which the line $y = x + 1$ is a tangent to the curve $y^2 = 4x$.

**Solution:** The parameters of the conic are

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} -2 & 0 \end{pmatrix}, f = 0 \tag{9.1.9.1}$$

Following the approach in Problem 9.1.6, since

$$\mathbf{n} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{9.1.9.2}$$

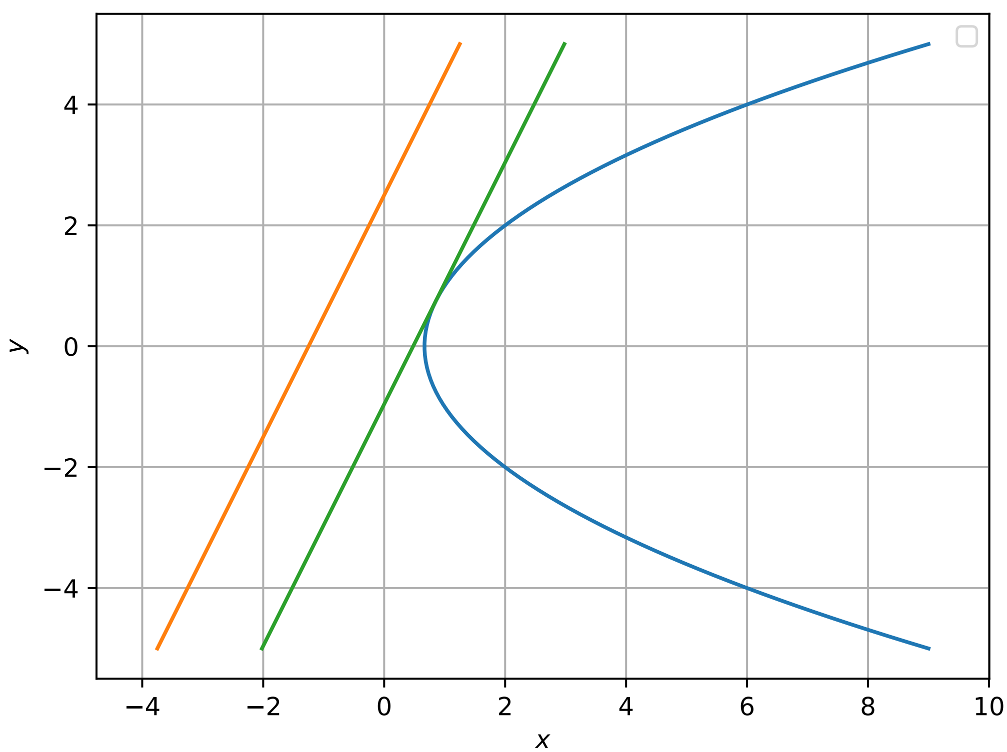


Figure 9.1.8.1:

we obtain

$$\mathbf{q} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{9.1.9.3}$$

See Fig. 9.1.9.1,

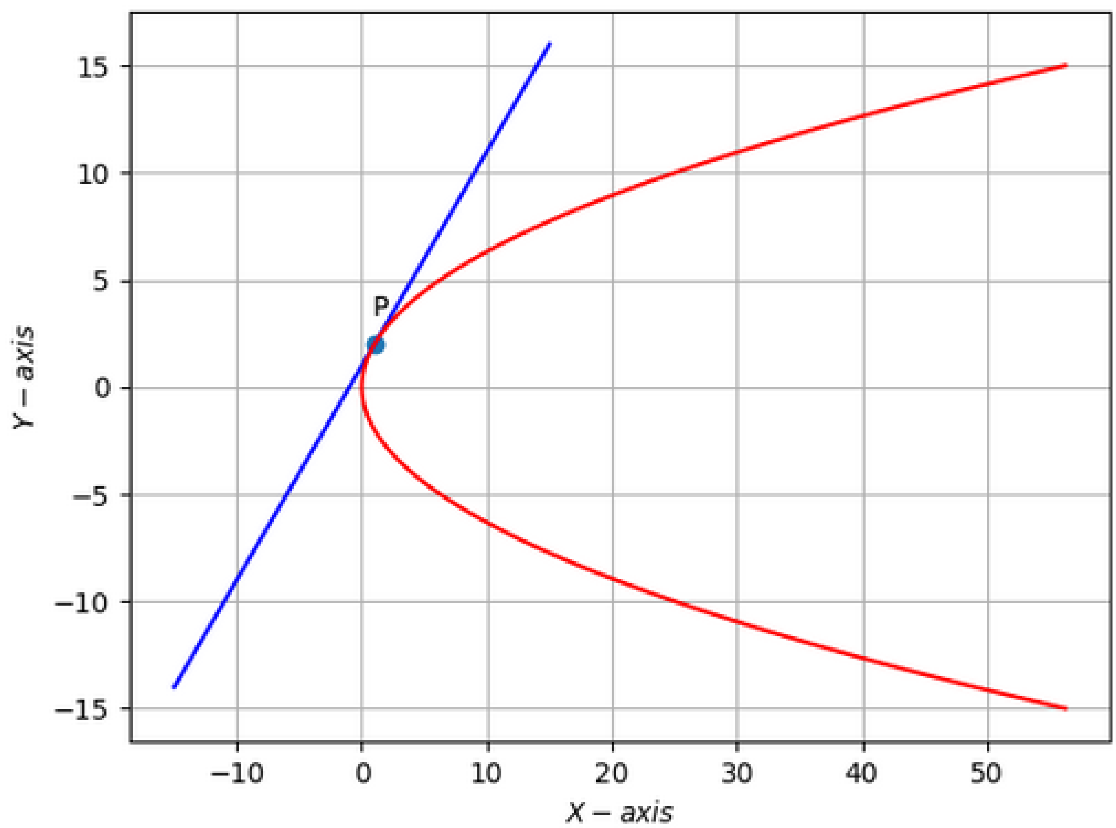


Figure 9.1.9.1:

## 9.2. Miscellaneous

9.2.1 Find the equation of the normal to curve $x^2 = 4y$ which passes through the point (1, 2).

**Solution:** The conic parameters are

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}, f = 0 \tag{9.2.1.1}$$

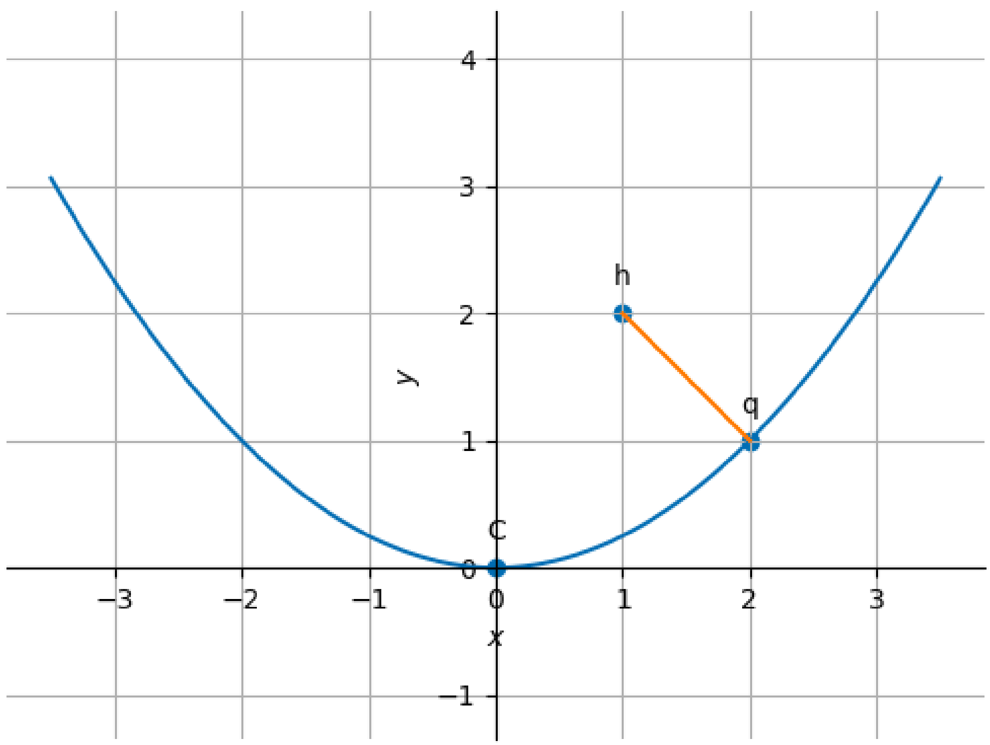


Figure 9.2.1.1:

Substituting these values in (F.4.10.1), we obtain

$$m = 1 \tag{9.2.1.2}$$

as the only real solution. Thus,

$$\mathbf{m} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{9.2.1.3}$$

and the equation of the normal is then obtained as

$$\mathbf{m}^{\top}(\mathbf{x}-\mathbf{h}) = 0 \tag{9.2.1.4}$$

$$\implies \begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{9.2.1.5}$$

$$= 3 \tag{9.2.1.6}$$

9.2.2 The line $y = mx + 1$ is a tangent to the curve $y^2 = 4x$, find the value of $m$.

**Solution:** The parameters for the given conic are

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}, f = 0 \tag{9.2.2.1}$$

The given tangent can be expressed in parametric form as

$$\mathbf{x} = \mathbf{e}_2 + \mu\mathbf{m} \tag{9.2.2.2}$$

Substituting from (9.2.2.2) and (9.2.2.1) in (F.4.8.1) and solving, we obtain

$$m = 1. \tag{9.2.2.3}$$

9.2.3 Find the normal at the point (1,1) on the curve

$$2y + x^2 = 3 \tag{9.2.3.1}$$

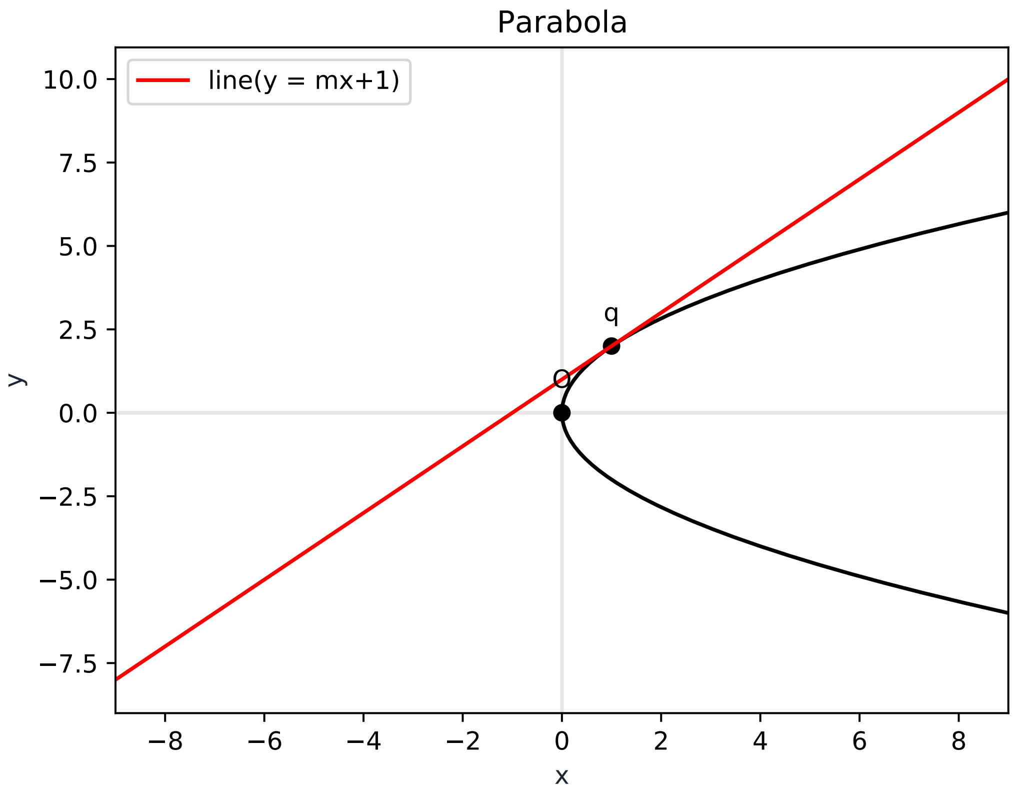


Figure 9.2.2.1:

**Solution:** Use (F.3.2.1) with

$$\mathbf{m} = \begin{pmatrix} 1 \\ m \end{pmatrix} \tag{9.2.3.2}$$

# Appendix A

# Vectors

## A.1. $2 \times 1$ vectors

A.1.1. Let

$$\mathbf{A} \equiv \overrightarrow{A} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \tag{A.1.1.1}$$

$$\equiv a_1 \overrightarrow{i} + a_2 \overrightarrow{j}, \tag{A.1.1.2}$$

$$\mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \tag{A.1.1.3}$$

be $2 \times 1$ vectors. Then, the determinant of the $2 \times 2$ matrix

$$\mathbf{M} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \end{pmatrix} \tag{A.1.1.4}$$

is defined as

$$\left|\mathbf{M}\right| = \left|\mathbf{A} \quad \mathbf{B}\right| \tag{A.1.1.5}$$

$$= \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1 \tag{A.1.1.6}$$

A.1.2. The value of the cross product of two vectors is given by (A.1.1.5).

A.1.3. The area of the triangle with vertices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ is given by

$$\frac{1}{2}\left\|(\mathbf{A}-\mathbf{B}) \times (\mathbf{A}-\mathbf{C})\right\| = \frac{1}{2}\left\|\mathbf{A}\times\mathbf{B} + \mathbf{B}\times\mathbf{C} + \mathbf{C}\times\mathbf{A}\right\| \tag{A.1.3.1}$$

A.1.4. If

$$\left\|\mathbf{A}\times\mathbf{B}\right\| = \left\|\mathbf{C}\times\mathbf{D}\right\|, \quad \text{then} \tag{A.1.4.1}$$

$$\mathbf{A}\times\mathbf{B} = \pm\left(\mathbf{C}\times\mathbf{D}\right) \tag{A.1.4.2}$$

where the sign depends on the orientation of the vectors.

A.1.5. The median divides the triangle into two triangles of equal area.

A.1.6. The transpose of $\mathbf{A}$ is defined as

$$\mathbf{A}^\top = \begin{pmatrix} a_1 & a_2 \end{pmatrix} \tag{A.1.6.1}$$

A.1.7. The <u>inner product</u> or <u>dot product</u> is defined as

$$\mathbf{A}^{\top}\mathbf{B} \equiv \mathbf{A} \cdot \mathbf{B} \tag{A.1.7.1}$$

$$= \begin{pmatrix} a_1 & a_2 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = a_1 b_1 + a_2 b_2 \tag{A.1.7.2}$$

A.1.8. <u>norm</u> of **A** is defined as

$$\|A\| \equiv \left|\overrightarrow{A}\right| \tag{A.1.8.1}$$

$$= \sqrt{\mathbf{A}^{\top}\mathbf{A}} = \sqrt{a_1^2 + a_2^2} \tag{A.1.8.2}$$

Thus,

$$\|\lambda \mathbf{A}\| \equiv \left|\lambda \overrightarrow{A}\right| \tag{A.1.8.3}$$

$$= |\lambda| \, \|\mathbf{A}\| \tag{A.1.8.4}$$

A.1.9. The distance betwen the points **A** and **B** is given by

$$\|\mathbf{A} - \mathbf{B}\| \tag{A.1.9.1}$$

A.1.10. Let **x** be equidistant from the points **A** and **B**. Then

$$(\mathbf{A} - \mathbf{B})^{\top} \mathbf{x} = \frac{\|\mathbf{A}\|^2 - \|\mathbf{B}\|^2}{2} \tag{A.1.10.1}$$

**Solution:**

$$\|\mathbf{x}-\mathbf{A}\| = \|\mathbf{A}-\mathbf{B}\| \tag{A.1.10.2}$$

$$\implies \|\mathbf{x}-\mathbf{A}\|^2 = \|\mathbf{x}-\mathbf{B}\|^2 \tag{A.1.10.3}$$

which can be expressed as

$$\begin{aligned}(\mathbf{x}-\mathbf{A})^\top(\mathbf{x}-\mathbf{A}) = (\mathbf{x}-\mathbf{B})^\top(\mathbf{x}-\mathbf{B}) \\ \implies \|\mathbf{x}\|^2 - 2\mathbf{x}^\top\mathbf{A} + \|\mathbf{A}\|^2 \\ = \|\mathbf{x}\|^2 - 2\mathbf{x}^\top\mathbf{B} + \|\mathbf{B}\|^2\end{aligned} \tag{A.1.10.4}$$

which can be simplified to obtain (A.1.10.1).

A.1.11. If $\mathbf{x}$ lies on the $x$-axis and is equidistant from the points $\mathbf{A}$ and $\mathbf{B}$,

$$\mathbf{x} = x\mathbf{e}_1 \tag{A.1.11.1}$$

where

$$x = \frac{\|\mathbf{A}\|^2 - \|\mathbf{B}\|^2}{2(\mathbf{A}-\mathbf{B})^\top\mathbf{e}_1} \tag{A.1.11.2}$$

**Solution:** From (A.1.10.1).

$$x(\mathbf{A}-\mathbf{B})^\top\mathbf{e}_1 = \frac{\|\mathbf{A}\|^2 - \|\mathbf{B}\|^2}{2} \tag{A.1.11.3}$$

yielding (A.1.11.2).

A.1.12. The angle between two vectors is given by

$$\theta = \cos^{-1} \frac{\mathbf{A}^\top \mathbf{B}}{\|A\| \, \|B\|} \tag{A.1.12.1}$$

A.1.13. If two vectors are orthogonal (perpendicular),

$$\mathbf{A}^\top \mathbf{B} = 0 \tag{A.1.13.1}$$

A.1.14. For an isoceles triangle $ABC$ ith $AB = AC$, the median $AD \perp BC$.

A.1.15. The <u>direction vector</u> of the line joining two points $\mathbf{A}, \mathbf{B}$ is given by

$$\mathbf{m} = \mathbf{A} - \mathbf{B} \tag{A.1.15.1}$$

A.1.16. The points $\mathbf{AAA}$

A.1.17. The unit vector in the direction of $\mathbf{m}$ is defined as

$$\frac{\mathbf{m}}{\|\mathbf{m}\|} \tag{A.1.17.1}$$

A.1.18. If the direction vector of a line is expressed as

$$\mathbf{m} = \begin{pmatrix} 1 \\ m \end{pmatrix}, \tag{A.1.18.1}$$

the $m$ is defined to be the slope of the line.

A.1.19. $AB \parallel CD$ if

$$\mathbf{A} - \mathbf{B} = k\left(\mathbf{C} - \mathbf{D}\right) \tag{A.1.19.1}$$

A.1.20. The normal vector to $\mathbf{m}$ is defined by

$$\mathbf{m}^\top \mathbf{n} = 0 \tag{A.1.20.1}$$

A.1.21. If

$$\mathbf{m}^\top \mathbf{n}_1 = 0 \tag{A.1.21.1}$$

$$\mathbf{m}^\top \mathbf{n}_2 = 0, \tag{A.1.21.2}$$

$$\mathbf{n}_1 \parallel \mathbf{n}_2 \tag{A.1.21.3}$$

A.1.22. The point $\mathbf{P}$ that divides the line segment $AB$ in the ratio $k : 1$ is given by

$$\mathbf{P} = \frac{k\mathbf{B} + \mathbf{A}}{k+1} \tag{A.1.22.1}$$

A.1.23. The standard basis vectors are defined as

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{A.1.23.1}$$

$$\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{A.1.23.2}$$

A.1.24. If $ABCD$ be a parallelogram,

$$\mathbf{B} - \mathbf{A} = \mathbf{C} - \mathbf{D} \tag{A.1.24.1}$$

A.1.25. Diagonals of a parallelogram bisect each other.

A.1.26. The area of the parallelogram with vertices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and $\mathbf{D}$ is given by

$$\|(\mathbf{A} - \mathbf{B}) \times (\mathbf{A} - \mathbf{D})\| = \|\mathbf{A} \times \mathbf{B} + \mathbf{B} \times \mathbf{C} + \mathbf{C} \times \mathbf{A}\| \tag{A.1.26.1}$$

A.1.27. Points $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ form a triangle if

$$p\left(\mathbf{A} - \mathbf{B}\right) + q\left(\mathbf{A} - \mathbf{C}\right) = 0 \tag{A.1.27.1}$$

$$\text{or,}\ \ (p+q)\,\mathbf{A} - p\mathbf{B} - q\mathbf{C} = 0 \tag{A.1.27.2}$$

$$\implies p = 0, q = 0 \tag{A.1.27.3}$$

are linearly independent.

A.1.28. In $\triangle ABC$, if $\mathbf{D}, \mathbf{E}$ divide the lines $AB, AC$ in the ratio $k : 1$ respectively, then $DE \parallel BC$.

*Proof.* From (A.1.22.1),

$$\mathbf{D} = \frac{k\mathbf{B} + \mathbf{A}}{k+1} \tag{A.1.28.1}$$

$$\mathbf{E} = \frac{k\mathbf{C} + \mathbf{A}}{k+1} \tag{A.1.28.2}$$

$$\implies \mathbf{D} - \mathbf{E} = \frac{k}{k+1}(\mathbf{B} - \mathbf{C}) \tag{A.1.28.3}$$

Thus, from Appendix A.1.18, $DE \parallel BC$.

□

A.1.29. In $\triangle ABC$, if $DE \parallel BC$, $\mathbf{D}$ and $\mathbf{E}$ divide the lines $AB, AC$ in the same ratio.

*Proof.* If $DE \parallel BC$, from (A.1.19.1)

$$(\mathbf{B} - \mathbf{C}) = k(\mathbf{D} - \mathbf{E}) \tag{A.1.29.1}$$

Using (A.1.22.1), let

$$\mathbf{D} = \frac{k_1\mathbf{B} + \mathbf{A}}{k_1+1} \tag{A.1.29.2}$$

$$\mathbf{E} = \frac{k_2\mathbf{C} + \mathbf{A}}{k_2+1} \tag{A.1.29.3}$$

Subtituting the above in (A.1.29.1), after some algebra, we obtain

$$(p+q)\mathbf{A} - p\mathbf{B} - q\mathbf{C} = 0 \tag{A.1.29.4}$$

where

$$p = \frac{1}{k} - \frac{k_1}{k_1+1}, q = \frac{1}{k} - \frac{k_1}{k_1+1} \tag{A.1.29.5}$$

From (A.1.27.2),

$$p = q = 0 \tag{A.1.29.6}$$

$$\implies k_1 = k_2 = \frac{1}{k-1} \tag{A.1.29.7}$$

$\square$

## A.2. $3 \times 1$ vectors

A.2.1. Let

$$\mathbf{A} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \equiv a_1 \overrightarrow{i} + a_2 \overrightarrow{j} + a_3 \overrightarrow{j}, \tag{A.2.1.1}$$

$$\mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \tag{A.2.1.2}$$

and

$$
\mathbf{A}_{ij} = \begin{pmatrix} a_i \\ a_j \end{pmatrix}, \tag{A.2.1.3}
$$

$$
\mathbf{B}_{ij} = \begin{pmatrix} b_i \\ b_j \end{pmatrix}. \tag{A.2.1.4}
$$

A.2.2. The <u>cross product</u> or <u>vector product</u> of $\mathbf{A}, \mathbf{B}$ is defined as

$$
\mathbf{A} \times \mathbf{B} = \begin{pmatrix} \begin{vmatrix} \mathbf{A}_{23} & \mathbf{B}_{23} \end{vmatrix} \\ \begin{vmatrix} \mathbf{A}_{31} & \mathbf{B}_{31} \end{vmatrix} \\ \begin{vmatrix} \mathbf{A}_{12} & \mathbf{B}_{12} \end{vmatrix} \end{pmatrix} \tag{A.2.2.1}
$$

A.2.3. Verify that

$$
\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} \tag{A.2.3.1}
$$

A.2.4. The area of a triangle is given by

$$
\frac{1}{2} \left\| \mathbf{A} \times \mathbf{B} \right\| \tag{A.2.4.1}
$$

# Appendix B

# Matrices

## B.1. Eigenvalues and Eigenvectors

B.1.1. The eigenvalue $\lambda$ and the eigenvector $\mathbf{x}$ for a matrix $\mathbf{A}$ are defined as,

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x} \tag{B.1.1.1}$$

B.1.2. The eigenvalues are calculated by solving the equation

$$f(\lambda) = \left|\lambda\mathbf{I} - \mathbf{A}\right| = 0 \tag{B.1.2.1}$$

The above equation is known as the characteristic equation.

B.1.3. According to the Cayley-Hamilton theorem,

$$f(\lambda) = 0 \implies f(\mathbf{A}) = 0 \tag{B.1.3.1}$$

B.1.4. The trace of a square matrix is defined to be the sum of the diagonal elements.

$$\operatorname{tr}(\mathbf{A}) = \sum_{i=1}^{N} a_{ii}. \tag{B.1.4.1}$$

where $a_{ii}$ is the $i$th diagonal element of the matrix $\mathbf{A}$.

B.1.5. The trace of a matrix is equal to the sum of the eigenvalues

$$\operatorname{tr}(\mathbf{A}) = \sum_{i=1}^{N} \lambda_i \tag{B.1.5.1}$$

## B.2. Determinants

B.2.1. Let

$$\mathbf{A} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}. \tag{B.2.1.1}$$

be a $3 \times 3$ matrix. Then,

$$\begin{aligned} \left|\mathbf{A}\right| = a_1 \begin{pmatrix} b_2 & c_2 \\ b_3 & c_3 \end{pmatrix} - a_2 \begin{pmatrix} b_1 & c_1 \\ b_3 & c_3 \end{pmatrix} \\ + a_3 \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}. \end{aligned} \tag{B.2.1.2}$$

B.2.2. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of a matrix $\mathbf{A}$. Then, the product of the eigenvalues is equal to the determinant of $\mathbf{A}$.

$$\left|\mathbf{A}\right| = \prod_{i=1}^{n} \lambda_i \tag{B.2.2.1}$$

B.2.3.

$$\left|\mathbf{AB}\right| = \left|\mathbf{A}\right|\left|\mathbf{B}\right| \tag{B.2.3.1}$$

B.2.4. If $\mathbf{A}$ be an $n \times n$ matrix,

$$\left|k\mathbf{A}\right| = k^n\left|\mathbf{A}\right| \tag{B.2.4.1}$$

## B.3. Rank of a Matrix

B.3.1. The rank of a matrix is defined as the number of linearly independent rows. This is also known as the row rank.

B.3.2. Row rank = Column rank.

B.3.3. The rank of a matrix is obtained as the number of nonzero rows obtained after row reduction.

B.3.4. An $n \times n$ matrix is invertible if and only if its rank is $n$.

B.3.5. Points $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are on a line if

$$\text{rank}\begin{pmatrix}\mathbf{A}\\ \mathbf{B}\\ \mathbf{C}\end{pmatrix} = 1 \tag{B.3.5.1}$$

B.3.6. Points $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ form a paralelogram if

$$\operatorname{rank}\begin{pmatrix}\mathbf{A}\\ \mathbf{B}\\ \mathbf{C}\\ \mathbf{D}\end{pmatrix} = 1, \operatorname{rank}\begin{pmatrix}\mathbf{A}\\ \mathbf{B}\\ \mathbf{C}\end{pmatrix} = 2 \tag{B.3.6.1}$$

# B.4. Inverse of a Matrix

B.4.1. For a $2 \times 2$ matrix

$$\mathbf{A} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}, \tag{B.4.1.1}$$

the inverse is given by

$$\mathbf{A}^{-1} = \frac{1}{\left|\mathbf{A}\right|}\begin{pmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{pmatrix}, \tag{B.4.1.2}$$

B.4.2. For higher order matrices, the inverse should be calculated using row operations.

# B.5. Orthogonality

B.5.1. The rotation matrix is defined as

$$\mathbf{R}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad \theta \in [0, 2\pi] \tag{B.5.1.1}$$

B.5.2. The rotation matrix is orthogonal

$$\mathbf{R}_\theta^\top \mathbf{R}_\theta = \mathbf{R}_\theta \mathbf{R}_\theta^\top = \mathbf{I} \tag{B.5.2.1}$$

B.5.3. If the angle of rotation is $\frac{\pi}{2}$,

$$\mathbf{m}^\top \mathbf{n} = 0 \implies \mathbf{n} = \mathbf{R}_{\frac{\pi}{2}} \mathbf{m} \tag{B.5.3.1}$$

B.5.4.

$$\mathbf{n}^\top \mathbf{h} = 1 \implies \mathbf{n} = \frac{\mathbf{e}_1}{\mathbf{e}_1^\top \mathbf{h}} + \mu \mathbf{R}_{\frac{\pi}{2}} \mathbf{h}, \quad \mu \in \mathbb{R}. \tag{B.5.4.1}$$

B.5.5. The affine transformation is given by

$$\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c} \quad \text{(Affine Transformation)} \tag{B.5.5.1}$$

where $\mathbf{P}$ is invertible.

B.5.6. The eigenvalue decomposition of a symmetric matrix $\mathbf{V}$ is given by

$$\mathbf{P}^\top \mathbf{V} \mathbf{P} = \mathbf{D}. \quad \text{(Eigenvalue Decomposition)} \tag{B.5.6.1}$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \tag{B.5.6.2}$$

$$\mathbf{P} = \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix}, \quad \mathbf{P}^\top = \mathbf{P}^{-1}, \tag{B.5.6.3}$$

# Appendix C

# Linear Forms

## C.1. Two Dimensions

C.1.1. The equation of a line is given by

$$\mathbf{n}^\top \mathbf{x} = c \tag{C.1.1.1}$$

where $\mathbf{n}$ is the normal vector of the line.

C.1.2. The equation of a line with normal vector $\mathbf{n}$ and passing through a point $\mathbf{A}$ is given by

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{A}) = 0 \tag{C.1.2.1}$$

C.1.3. The equation of a line $L$ is also given by

$$\mathbf{n}^\top \mathbf{x} = \begin{cases} 0 & \mathbf{0} \in L \\ 1 & \text{otherwise} \end{cases} \tag{C.1.3.1}$$

C.1.4. Points $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are collinear if

$$\operatorname{rank}\begin{pmatrix}\mathbf{B}-\mathbf{A} & \mathbf{C}-\mathbf{A}\end{pmatrix} < 2 \tag{C.1.4.1}$$

*Proof.* From (C.1.1.1),

$$\mathbf{n}^\top \mathbf{A} = c \tag{C.1.4.2}$$

$$\mathbf{n}^\top \mathbf{B} = c \tag{C.1.4.3}$$

$$\mathbf{n}^\top \mathbf{C} = c \tag{C.1.4.4}$$

which can be expressed as

$$\begin{pmatrix}\mathbf{A} & \mathbf{B} & \mathbf{C}\end{pmatrix}^\top \mathbf{n} = c\begin{pmatrix}1\\1\\1\end{pmatrix} \tag{C.1.4.5}$$

The above set of equations are consistent if

$$\operatorname{rank}\begin{pmatrix}1 & 1 & 1\\ \mathbf{A} & \mathbf{B} & \mathbf{C}\end{pmatrix} < 3 \tag{C.1.4.6}$$

$$\implies \operatorname{rank}\begin{pmatrix}1 & 0 & 0\\ \mathbf{A} & \mathbf{B}-\mathbf{A} & \mathbf{C}-\mathbf{A}\end{pmatrix} < 3 \tag{C.1.4.7}$$

using the fact that row rank = column rank. The above condition can then be expressed as (C.1.4.1).

□

C.1.5. The parametric equation of a line is given by

$$\mathbf{x} = \mathbf{A} + \lambda \mathbf{m} \tag{C.1.5.1}$$

where $\mathbf{m}$ is the direction vector of the line and $\mathbf{A}$ is any point on the line.

C.1.6. Let $\mathbf{A}$ and $\mathbf{B}$ be two points on a straight line and let $\mathbf{P} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$ be any point on it. If $p_2$ is known, then

$$\mathbf{P} = \mathbf{A} + \frac{p_2 - \mathbf{e}_2^\top \mathbf{A}}{\mathbf{e}_2^\top (\mathbf{B} - \mathbf{A})} (\mathbf{B} - \mathbf{A}) \tag{C.1.6.1}$$

**Solution:** The equation of the line can be expressed in parametric from as

$$\mathbf{x} = \mathbf{A} + \lambda (\mathbf{B} - \mathbf{A}) \tag{C.1.6.2}$$

$$\implies \mathbf{P} = \mathbf{A} + \lambda (\mathbf{B} - \mathbf{A}) \tag{C.1.6.3}$$

$$\implies \mathbf{e}_2^\top \mathbf{P} = \mathbf{e}_2^\top \mathbf{A} + \lambda \mathbf{e}_2^\top (\mathbf{B} - \mathbf{A}) \tag{C.1.6.4}$$

$$\implies p_2 = \mathbf{e}_2^\top \mathbf{A} + \lambda \mathbf{e}_2^\top (\mathbf{B} - \mathbf{A}) \tag{C.1.6.5}$$

$$\text{or, } \lambda = \frac{p_2 - \mathbf{e}_2^\top \mathbf{A}}{\mathbf{e}_2^\top (\mathbf{B} - \mathbf{A})} \tag{C.1.6.6}$$

yielding (C.1.6.1).

C.1.7. The distance from a point $\mathbf{P}$ to the line in (C.1.1.1) is given by

$$d = \frac{\left|\mathbf{n}^\top \mathbf{P} - c\right|}{\|\mathbf{n}\|} \tag{C.1.7.1}$$

**Solution:** Without loss of generality, let $\mathbf{A}$ be the foot of the perpendicular from $\mathbf{P}$ to the line in (C.1.5.1). The equation of the normal to (C.1.1.1) can then be expressed as

$$\mathbf{x} = \mathbf{A} + \lambda \mathbf{n} \tag{C.1.7.2}$$

$$\implies \mathbf{P} - \mathbf{A} = \lambda \mathbf{n} \tag{C.1.7.3}$$

$\because$ $\mathbf{P}$ lies on (C.1.7.2). From the above, the desired distance can be expressed as

$$d = \left\|\mathbf{P} - \mathbf{A}\right\| = \left|\lambda\right| \left\|\mathbf{n}\right\| \tag{C.1.7.4}$$

From (C.1.7.3),

$$\mathbf{n}^\top \left(\mathbf{P} - \mathbf{A}\right) = \lambda \mathbf{n}^\top \mathbf{n} = \lambda \left\|\mathbf{n}\right\|^2 \tag{C.1.7.5}$$

$$\implies \left|\lambda\right| = \frac{\left|\mathbf{n}^\top \left(\mathbf{P} - \mathbf{A}\right)\right|}{\left\|\mathbf{n}\right\|^2} \tag{C.1.7.6}$$

Substituting the above in (C.1.7.4) and using the fact that

$$\mathbf{n}^\top \mathbf{A} = c \tag{C.1.7.7}$$

from (C.1.1.1), yields (C.1.7.1)

C.1.8. The distance from the origin to the line in (C.1.1.1) is given by

$$d = \frac{\left|c\right|}{\left\|\mathbf{n}\right\|} \tag{C.1.8.1}$$

C.1.9. The distance between the parallel lines

$$\begin{aligned} \mathbf{n}^\top \mathbf{x} &= c_1 \\ \mathbf{n}^\top \mathbf{x} &= c_2 \end{aligned} \tag{C.1.9.1}$$

is given by

$$d = \frac{|c_1 - c_2|}{\|\mathbf{n}\|} \tag{C.1.9.2}$$

C.1.10. The equation of the line perpendicular to (C.1.1.1) and passing through the point $\mathbf{P}$ is given by

$$\mathbf{m}^\top (\mathbf{x} - \mathbf{P}) = 0 \tag{C.1.10.1}$$

C.1.11. The foot of the perpendicular from $\mathbf{P}$ to the line in (C.1.1.1) is given by

$$\begin{pmatrix} \mathbf{m} & \mathbf{n} \end{pmatrix}^\top \mathbf{x} = \begin{pmatrix} \mathbf{m}^\top \mathbf{P} \\ c \end{pmatrix} \tag{C.1.11.1}$$

**Solution:** From (C.1.1.1) and (C.1.2.1) the foot of the perpendicular satisfies the equations

$$\mathbf{n}^\top \mathbf{x} = c \tag{C.1.11.2}$$

$$\mathbf{m}^\top (\mathbf{x} - \mathbf{P}) = 0 \tag{C.1.11.3}$$

where $\mathbf{m}$ is the direction vector of the given line. Combining the above into a matrix equation results in (C.1.11.1).

C.1.12. The equations of the angle bisectors of the lines

$$\mathbf{n}_1^\top \mathbf{x} = c_1 \tag{C.1.12.1}$$

$$\mathbf{n}_2^\top \mathbf{x} = c_2 \tag{C.1.12.2}$$

are given by

$$\frac{\mathbf{n}_1^\top \mathbf{x} - c_1}{\|\mathbf{n}_1\|} = \pm \frac{\mathbf{n}_2^\top \mathbf{x} - c_2}{\|\mathbf{n}_2\|} \tag{C.1.12.3}$$

*Proof.* Any point on the angle bisector is equidistant from the lines. □

## C.2. Three Dimensions

C.2.1. Points $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are on a line if

$$\operatorname{rank}\begin{pmatrix}\mathbf{A}\\ \mathbf{B}\\ \mathbf{C}\end{pmatrix} = 1 \tag{C.2.1.1}$$

C.2.2. Points $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ form a paralelogram if

$$\operatorname{rank}\begin{pmatrix}\mathbf{A}\\ \mathbf{B}\\ \mathbf{C}\\ \mathbf{D}\end{pmatrix} = 1, \operatorname{rank}\begin{pmatrix}\mathbf{A}\\ \mathbf{B}\\ \mathbf{C}\end{pmatrix} = 2 \tag{C.2.2.1}$$

C.2.3. The equation of a line is given by (C.1.5.1)

C.2.4. The equation of a plane is given by (C.1.1.1)

C.2.5. The distance from the origin to the line in (C.1.1.1) is given by (C.1.8.1)

C.2.6. The distance from a point $\mathbf{P}$ to the line in (C.1.5.1) is given by

$$d = \|\mathbf{A}-\mathbf{P}\|^2 - \frac{\left\{\mathbf{m}^\top\left(\mathbf{A}-\mathbf{P}\right)\right\}^2}{\|\mathbf{m}\|^2} \tag{C.2.6.1}$$

**Solution:**

$$d\left(\lambda\right) = \|\mathbf{A}+\lambda\mathbf{m}-\mathbf{P}\| \tag{C.2.6.2}$$

$$\implies d^2\left(\lambda\right) = \|\mathbf{A}+\lambda\mathbf{m}-\mathbf{P}\|^2 \tag{C.2.6.3}$$

which can be simplified to obtain

$$d^2\left(\lambda\right) = \lambda^2\|\mathbf{m}\|^2 + 2\lambda\mathbf{m}^\top\left(\mathbf{A}-\mathbf{P}\right) + \|\mathbf{A}-\mathbf{P}\|^2 \tag{C.2.6.4}$$

which is of the form

$$d^2\left(\lambda\right) = a\lambda^2 + 2b\lambda + c \tag{C.2.6.5}$$

$$= a\left\{\left(\lambda+\frac{b}{a}\right)^2 + \left[\frac{c}{a} - \left(\frac{b}{a}\right)^2\right]\right\} \tag{C.2.6.6}$$

with

$$a = \|\mathbf{m}\|^2, b = \mathbf{m}^\top\left(\mathbf{A}-\mathbf{P}\right), c = \|\mathbf{A}-\mathbf{P}\|^2 \tag{C.2.6.7}$$

which can be expressed as From the above, $d^2\left(\lambda\right)$ is smallest when upon substituting from (C.2.6.7)

$$\lambda + \frac{b}{2a} = 0 \implies \lambda = -\frac{b}{2a} \tag{C.2.6.8}$$

$$= -\frac{\mathbf{m}^\top (\mathbf{A} - \mathbf{P})}{\|\mathbf{m}\|^2} \tag{C.2.6.9}$$

and consequently,

$$d_{\min}(\lambda) = a\left(\frac{c}{a} - \left(\frac{b}{a}\right)^2\right) \tag{C.2.6.10}$$

$$= c - \frac{b^2}{a} \tag{C.2.6.11}$$

yielding (C.2.6.1) after substituting from (C.2.6.7).

C.2.7. The distance between the parallel planes (C.1.9.1) is given by (C.1.9.2).

C.2.8. The plane

$$\mathbf{n}^\top \mathbf{x} = c \tag{C.2.8.1}$$

contains the line

$$\mathbf{x} = \mathbf{A} + \lambda \mathbf{m} \tag{C.2.8.2}$$

if

$$\mathbf{m}^\top \mathbf{n} = 0 \tag{C.2.8.3}$$

**Solution:** Any point on the line (C.2.8.2) should also satisfy (C.2.8.1). Hence,

$$\mathbf{n}^\top (\mathbf{A} + \lambda \mathbf{m}) = \mathbf{n}^\top \mathbf{A} = c \tag{C.2.8.4}$$

which can be simplified to obtain (C.2.8.3)

C.2.9. The foot of the perpendicular from a point $\mathbf{P}$ to the plane

$$\mathbf{n}^\top \mathbf{x} = c \tag{C.2.9.1}$$

is given by

$$\mathbf{x} = \mathbf{P} + \frac{c - \mathbf{n}^\top \mathbf{P}}{\|\mathbf{n}\|^2}\mathbf{n} \tag{C.2.9.2}$$

**Solution:** The equation of the line perpendicular to the given plane and passing through $\mathbf{P}$ is

$$\mathbf{x} = \mathbf{P} + \lambda \mathbf{n} \tag{C.2.9.3}$$

From (C.2.12.1), the intersection of the above line with the given plane is (C.2.9.2).

C.2.10. The image of a point $\mathbf{P}$ with respect to the plane

$$\mathbf{n}^\top \mathbf{x} = c \tag{C.2.10.1}$$

is given by

$$\mathbf{R} = \mathbf{P} + 2\frac{c - \mathbf{n}^\top \mathbf{P}}{\|\mathbf{n}\|^2} \tag{C.2.10.2}$$

**Solution:** Let $\mathbf{R}$ be the desired image. Then, subtituting the expression for the foot of the perpendicular from $\mathbf{P}$ to the given plane using (C.2.9.2),

$$\frac{\mathbf{P} + \mathbf{R}}{2} = \mathbf{P} + \frac{c - \mathbf{n}^\top \mathbf{P}}{\|\mathbf{n}\|^2} \tag{C.2.10.3}$$

C.2.11. Let a plane pass through the points $\mathbf{A}, \mathbf{B}$ and be perpendicular to the plane

$$\mathbf{n}^\top \mathbf{x} = c \tag{C.2.11.1}$$

Then the equation of this plane is given by

$$\mathbf{p}^\top \mathbf{x} = 1 \tag{C.2.11.2}$$

where

$$\mathbf{p} = \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{n} \end{pmatrix}^{-\top} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \tag{C.2.11.3}$$

**Solution:** From the given information,

$$\mathbf{p}^\top \mathbf{A} = d \tag{C.2.11.4}$$

$$\mathbf{p}^\top \mathbf{B} = d \tag{C.2.11.5}$$

$$\mathbf{p}^\top \mathbf{n} = 0 \tag{C.2.11.6}$$

$\because$ the normal vectors to the two planes will also be perpendicular. The system of equations in (C.2.11.6) can be expressed as the matrix equation

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{n} \end{pmatrix}^\top \mathbf{p} = d \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \tag{C.2.11.7}$$

which yields (C.2.11.3) upon normalising with $d$.

C.2.12. The intersection of the line represented by (C.1.5.1) with the plane represented by (C.1.1.1) is given by

$$\mathbf{x} = \mathbf{A} + \frac{c - \mathbf{n}^\top \mathbf{A}}{\mathbf{n}^\top \mathbf{m}} \mathbf{m} \tag{C.2.12.1}$$

**Solution:** From (C.1.5.1) and (C.1.1.1),

$$\mathbf{x} = \mathbf{A} + \lambda \mathbf{m} \tag{C.2.12.2}$$

$$\mathbf{n}^\top \mathbf{x} = c \tag{C.2.12.3}$$

$$\implies \mathbf{n}^\top (\mathbf{A} + \lambda \mathbf{m}) = c \tag{C.2.12.4}$$

which can be simplified to obtain

$$\mathbf{n}^\top \mathbf{A} + \lambda \mathbf{n}^\top \mathbf{m} = c \tag{C.2.12.5}$$

$$\implies \lambda = \frac{c - \mathbf{n}^\top \mathbf{A}}{\mathbf{n}^\top \mathbf{m}} \tag{C.2.12.6}$$

Substituting the above in (C.2.12.4) yields (C.2.12.1).

C.2.13. The foot of the perpendicular from the point $\mathbf{P}$ to the line represented by (C.1.5.1) is given by

$$\mathbf{x} = \mathbf{A} + \frac{\mathbf{m}^\top (\mathbf{P} - \mathbf{A})}{\|\mathbf{m}\|^2} \mathbf{m} \tag{C.2.13.1}$$

**Solution:** Let the equation of the line be

$$\mathbf{x} = \mathbf{A} + \lambda \mathbf{m} \tag{C.2.13.2}$$

The equation of the plane perpendicular to the given line passing through $\mathbf{P}$ is given by

$$\mathbf{m}^\top (\mathbf{x} - \mathbf{P}) = 0 \tag{C.2.13.3}$$

$$\implies \mathbf{m}^\top \mathbf{x} = \mathbf{m}^\top \mathbf{P} \tag{C.2.13.4}$$

The desired foot of the perpendicular is the intersection of (C.2.13.2) with (C.2.13.3) which can be obtained from (C.2.12.1) as (C.2.13.1)

C.2.14. The foot of the perpendicular from a point $\mathbf{P}$ to a plane is $\mathbf{Q}$. The equation of the plane is given by

$$(\mathbf{P}-\mathbf{Q})^\top(\mathbf{x}-\mathbf{Q}) = 0 \tag{C.2.14.1}$$

**Solution:** The normal vector to the plane is given by

$$\mathbf{n} = \mathbf{P}-\mathbf{Q} \tag{C.2.14.2}$$

Hence, the equation of the plane is (C.2.14.1).

C.2.15. Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be points on a plane. The equation of the plane is then given by

$$\begin{pmatrix}\mathbf{A} & \mathbf{B} & \mathbf{C}\end{pmatrix}^\top \mathbf{n} = \begin{pmatrix}1\\1\\1\end{pmatrix} \tag{C.2.15.1}$$

**Solution:** Let the equation of the plane be

$$\mathbf{n}^\top\mathbf{x} = 1 \tag{C.2.15.2}$$

Then

$$\mathbf{n}^\top\mathbf{A} = 1 \tag{C.2.15.3}$$

$$\mathbf{n}^\top\mathbf{B} = 1 \tag{C.2.15.4}$$

$$\mathbf{n}^\top\mathbf{C} = 1 \tag{C.2.15.5}$$

which can be combined to obtain (C.2.15.1).

C.2.16. (Parallelogram Law) Let $\mathbf{A}, \mathbf{B}, \mathbf{D}$ be three vertices of a parallelogram. Then the vertex

$\mathbf{C}$ is given by

$$\mathbf{C} = \mathbf{B} + \mathbf{C} - \mathbf{A} \tag{C.2.16.1}$$

**Solution:** Shifting $\mathbf{A}$ to the origin, we obtain a parallelogram with corresponding vertices

$$\mathbf{0}, \mathbf{B} - \mathbf{A}, \mathbf{D} - \mathbf{A} \tag{C.2.16.2}$$

The fourth vertex of this parallelogram is then obtained as

$$(\mathbf{B} - \mathbf{A}) + (\mathbf{D} - \mathbf{A}) = \mathbf{D} + \mathbf{B} - 2\mathbf{A} \tag{C.2.16.3}$$

Shifting the origin to $\mathbf{A}$, the fourth vertex is obtained as

$$\mathbf{C} = \mathbf{D} + \mathbf{B} - 2\mathbf{A} + \mathbf{A} \tag{C.2.16.4}$$

$$= \mathbf{D} + \mathbf{B} - \mathbf{A} \tag{C.2.16.5}$$

C.2.17. (Affine Transformation) Let $\mathbf{A}, \mathbf{C}$, be opposite vertices of a square. The other two points can be obtained as

$$\mathbf{B} = \frac{\|\mathbf{A} - \mathbf{C}\|}{\sqrt{2}}\mathbf{P}\mathbf{e}_1 + \mathbf{A} \tag{C.2.17.1}$$

$$\mathbf{D} = \frac{\|\mathbf{A} - \mathbf{C}\|}{\sqrt{2}}\mathbf{P}\mathbf{e}_2 + \mathbf{A} \tag{C.2.17.2}$$

where

$$\mathbf{P} = \begin{pmatrix} \cos\left(\theta - \frac{\pi}{4}\right) & \sin\left(\theta - \frac{\pi}{4}\right) \\ \sin\left(\theta - \frac{\pi}{4}\right) & \cos\left(\theta - \frac{\pi}{4}\right) \end{pmatrix} \tag{C.2.17.3}$$

and

$$\cos\theta = \frac{\left(\mathbf{C} - \mathbf{A}\right)^{\top} \mathbf{e}_1}{\|\mathbf{A} - \mathbf{C}\| \, \|\mathbf{e}_1\|} \tag{C.2.17.4}$$

# Appendix D

# Quadratic Forms

## D.1. Conic equation

D.1.1. Let $\mathbf{q}$ be a point such that the ratio of its distance from a fixed point $\mathbf{F}$ and the distance $(d)$ from a fixed line

$$L : \mathbf{n}^\top \mathbf{x} = c \tag{D.1.1.1}$$

is constant, given by

$$\frac{\|\mathbf{q} - \mathbf{F}\|}{d} = e \tag{D.1.1.2}$$

The locus of $\mathbf{q}$ is known as a conic section. The line $L$ is known as the directrix and the point $\mathbf{F}$ is the focus. $e$ is defined to be the eccentricity of the conic.

(a) For $e = 1$, the conic is a parabola

(b) For $e < 1$, the conic is an ellipse

(c) For $e > 1$, the conic is a hyperbola

D.1.2. The equation of a conic with directrix $\mathbf{n}^\top \mathbf{x} = c$, eccentricity $e$ and focus $\mathbf{F}$ is given by

$$\mathrm{g}\left(\mathbf{x}\right) = \mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{D.1.2.1}$$

where

$$\mathbf{V} = \left\|\mathbf{n}\right\|^2 \mathbf{I} - e^2 \mathbf{n}\mathbf{n}^\top, \tag{D.1.2.2}$$

$$\mathbf{u} = ce^2\mathbf{n} - \left\|\mathbf{n}\right\|^2 \mathbf{F}, \tag{D.1.2.3}$$

$$f = \left\|\mathbf{n}\right\|^2 \left\|\mathbf{F}\right\|^2 - c^2 e^2 \tag{D.1.2.4}$$

*Proof.* Using Definition D.1.1 and Lemma C.1.7.1, for any point $\mathbf{x}$ on the conic,

$$\left\|\mathbf{x} - \mathbf{F}\right\|^2 = e^2 \frac{\left(\mathbf{n}^\top \mathbf{x} - c\right)^2}{\left\|\mathbf{n}\right\|^2} \tag{D.1.2.5}$$

$$\implies \left\|\mathbf{n}\right\|^2 \left(\mathbf{x} - \mathbf{F}\right)^\top \left(\mathbf{x} - \mathbf{F}\right) = e^2 \left(\mathbf{n}^\top \mathbf{x} - c\right)^2 \tag{D.1.2.6}$$

$$\implies \left\|\mathbf{n}\right\|^2 \left(\mathbf{x}^\top \mathbf{x} - 2\mathbf{F}^\top \mathbf{x} + \left\|\mathbf{F}\right\|^2\right) = e^2 \left(c^2 + \left(\mathbf{n}^\top \mathbf{x}\right)^2 - 2c\mathbf{n}^\top \mathbf{x}\right) \tag{D.1.2.7}$$

$$= e^2 \left(c^2 + \left(\mathbf{x}^\top \mathbf{n}\mathbf{n}^\top \mathbf{x}\right) - 2c\mathbf{n}^\top \mathbf{x}\right) \tag{D.1.2.8}$$

which can be expressed as (D.1.2.1) after simplification.

□

D.1.3. The eccentricity, directrices and foci of (D.1.2.1) are given by

$$e = \sqrt{1 - \frac{\lambda_1}{\lambda_2}} \tag{D.1.3.1}$$

$$\mathbf{n} = \sqrt{\lambda_2}\mathbf{p}_1,$$

$$c = \begin{cases} \frac{e\mathbf{u}^\top \mathbf{n} \pm \sqrt{e^2\left(\mathbf{u}^\top \mathbf{n}\right)^2 - \lambda_2(e^2-1)\left(\|\mathbf{u}\|^2 - \lambda_2 f\right)}}{\lambda_2 e(e^2-1)} & e \neq 1 \\ \frac{\|\mathbf{u}\|^2 - \lambda_2 f}{2\mathbf{u}^\top \mathbf{n}} & e = 1 \end{cases} \tag{D.1.3.2}$$

$$\mathbf{F} = \frac{ce^2\mathbf{n} - \mathbf{u}}{\lambda_2} \tag{D.1.3.3}$$

*Proof.* From (D.1.2.2), using the fact that $\mathbf{V}$ is symmetric with $\mathbf{V} = \mathbf{V}^\top$,

$$\mathbf{V}^\top \mathbf{V} = \left(\|\mathbf{n}\|^2 \mathbf{I} - e^2 \mathbf{n}\mathbf{n}^\top\right)^\top \left(\|\mathbf{n}\|^2 \mathbf{I} - e^2 \mathbf{n}\mathbf{n}^\top\right) \tag{D.1.3.4}$$

$$\implies \mathbf{V}^2 = \|\mathbf{n}\|^4 \mathbf{I} + e^4 \mathbf{n}\mathbf{n}^\top \mathbf{n}\mathbf{n}^\top - 2e^2 \|\mathbf{n}\|^2 \mathbf{n}\mathbf{n}^\top \tag{D.1.3.5}$$

$$= \|\mathbf{n}\|^4 \mathbf{I} + e^4 \|\mathbf{n}\|^2 \mathbf{n}\mathbf{n}^\top - 2e^2 \|\mathbf{n}\|^2 \mathbf{n}\mathbf{n}^\top \tag{D.1.3.6}$$

$$= \|\mathbf{n}\|^4 \mathbf{I} + e^2\left(e^2 - 2\right) \|\mathbf{n}\|^2 \mathbf{n}\mathbf{n}^\top \tag{D.1.3.7}$$

$$= \|\mathbf{n}\|^4 \mathbf{I} + \left(e^2 - 2\right) \|\mathbf{n}\|^2 \left(\|\mathbf{n}\|^2 \mathbf{I} - \mathbf{V}\right) \tag{D.1.3.8}$$

which can be expressed as

$$\mathbf{V}^2 + \left(e^2 - 2\right) \|\mathbf{n}\|^2 \mathbf{V} - \left(e^2 - 1\right) \|\mathbf{n}\|^4 \mathbf{I} = 0 \tag{D.1.3.9}$$

Using the Cayley-Hamilton theorem, (D.1.3.9) results in the characteristic equation,

$$\lambda^2 - \left(2 - e^2\right) \|\mathbf{n}\|^2 \lambda + \left(1 - e^2\right) \|\mathbf{n}\|^4 = 0 \tag{D.1.3.10}$$

which can be expressed as

$$\left(\frac{\lambda}{\|\mathbf{n}\|^2}\right)^2 - \left(2-e^2\right)\left(\frac{\lambda}{\|\mathbf{n}\|^2}\right) + \left(1-e^2\right) = 0 \tag{D.1.3.11}$$

$$\implies \frac{\lambda}{\|\mathbf{n}\|^2} = 1-e^2, 1 \tag{D.1.3.12}$$

$$\text{or, } \lambda_2 = \|\mathbf{n}\|^2, \lambda_1 = \left(1-e^2\right)\lambda_2 \tag{D.1.3.13}$$

From (D.1.3.13), the eccentricity of (D.1.2.1) is given by (D.1.3.1). Multiplying both sides of (D.1.2.2) by $\mathbf{n}$,

$$\mathbf{V}\mathbf{n} = \|\mathbf{n}\|^2 \mathbf{n} - e^2\mathbf{n}\mathbf{n}^\top\mathbf{n} \tag{D.1.3.14}$$

$$= \|\mathbf{n}\|^2 \left(1-e^2\right)\mathbf{n} \tag{D.1.3.15}$$

$$= \lambda_1\mathbf{n} \tag{D.1.3.16}$$

$$\tag{D.1.3.17}$$

from (D.1.3.13). Thus, $\lambda_1$ is the corresponding eigenvalue for $\mathbf{n}$. From (B.5.6.3) and (D.1.3.17), this implies that

$$\mathbf{p}_1 = \frac{\mathbf{n}}{\|\mathbf{n}\|} \tag{D.1.3.18}$$

$$\text{or, } \mathbf{n} = \|\mathbf{n}\|\,\mathbf{p}_1 = \sqrt{\lambda_2}\mathbf{p}_1 \tag{D.1.3.19}$$

from (D.1.3.13) . From (D.1.2.3) and (D.1.3.13),

$$\mathbf{F} = \frac{ce^2\mathbf{n} - \mathbf{u}}{\lambda_2} \tag{D.1.3.20}$$

$$\implies \|\mathbf{F}\|^2 = \frac{\left(ce^2\mathbf{n} - \mathbf{u}\right)^\top \left(ce^2\mathbf{n} - \mathbf{u}\right)}{\lambda_2^2} \tag{D.1.3.21}$$

$$\implies \lambda_2^2 \|\mathbf{F}\|^2 = c^2e^4\lambda_2 - 2ce^2\mathbf{u}^\top\mathbf{n} + \|\mathbf{u}\|^2 \tag{D.1.3.22}$$

Also, (D.1.2.4) can be expressed as

$$\lambda_2 \|\mathbf{F}\|^2 = f + c^2e^2 \tag{D.1.3.23}$$

From (D.1.3.22) and (D.1.3.23),

$$c^2e^4\lambda_2 - 2ce^2\mathbf{u}^\top\mathbf{n} + \|\mathbf{u}\|^2 = \lambda_2\left(f + c^2e^2\right) \tag{D.1.3.24}$$

$$\implies \lambda_2 e^2\left(e^2 - 1\right)c^2 - 2ce^2\mathbf{u}^\top\mathbf{n} + \|\mathbf{u}\|^2 - \lambda_2 f = 0 \tag{D.1.3.25}$$

yielding (D.1.3.3). □

D.1.4. (D.1.2.1) represents

(a) a parabola for $\left|\mathbf{V}\right| = 0$,

(b) ellipse for $\left|\mathbf{V}\right| > 0$ and

(c) hyperbola for $\left|\mathbf{V}\right| < 0$.

*Proof.* From (D.1.3.1),

$$\frac{\lambda_1}{\lambda_2} = 1 - e^2 \tag{D.1.4.1}$$

Also,

$$\left|\mathbf{V}\right| = \lambda_1\lambda_2 \tag{D.1.4.2}$$

yielding Table D.1.4.2 □

| **Eccentricity** | **Conic** | **Eigenvalue** | **Determinant** |
|---|---|---|---|
| $e = 1$ | Parabola | $\lambda_1 = 0$ | $\left\|\mathbf{V}\right\| = 0$ |
| $e < 1$ | Ellipse | $\lambda_1 > 0, \lambda_2 > 0$ | $\left\|\mathbf{V}\right\| > 0$ |
| $e > 1$ | Hyperbola | $\lambda_1 < 0, \lambda_2 > 0$ | $\left\|\mathbf{V}\right\| < 0$ |

Table D.1.4.2:

# D.2. Circles

D.2.1. The equation of a circle is given by

$$\left\|\mathbf{x}\right\|^2 + 2\mathbf{u}^\top\mathbf{x} + f = 0 \tag{D.2.1.1}$$

D.2.2. For a circle with centre $\mathbf{c}$ and radius r,

$$\mathbf{u} = -\mathbf{c}, f = \left\|\mathbf{u}\right\|^2 - r^2 \tag{D.2.2.1}$$

D.2.3. Any point $\mathbf{x}$ on a circle can be expressed as

$$\mathbf{x} = \mathbf{c} + r \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}. \tag{D.2.3.1}$$

D.2.4. The equation of the common chord of intersection of two circles is given by

$$\mathbf{u}_1^\top \mathbf{x} - \mathbf{u}_2^\top \mathbf{x} + f_1 - f_2 = 0 \tag{D.2.4.1}$$

D.2.5. The line joining the centre of a circle to the mid point of any chord is perpendicular to the chord.

*Proof.* Let $AB$ be any chord of a circle with centre $\mathbf{O} = \mathbf{0}$ and radius $r$. Then,

$$\|\mathbf{A}\|^2 = \|\mathbf{B}\|^2 = r^2 \tag{D.2.5.1}$$

$$\implies \|\mathbf{A}\|^2 - \|\mathbf{B}\|^2 = \mathbf{0} \tag{D.2.5.2}$$

$$\text{or, } (\mathbf{A} - \mathbf{B})^\top (\mathbf{A} + \mathbf{B}) = \mathbf{0} \tag{D.2.5.3}$$

which can be expressed as

$$(\mathbf{A} - \mathbf{B})^\top \left( \frac{\mathbf{A} + \mathbf{B}}{2} - \mathbf{O} \right) = \mathbf{0} \tag{D.2.5.4}$$

□

D.2.6. Let

$$\mathbf{A} = \begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} \cos\theta_2 \\ \sin\theta_2 \end{pmatrix}, \tag{D.2.6.1}$$

be points on a unit circle with centre $\mathbf{O}$ at the origin. Then

$$\cos AOB = \mathbf{A}^\top \mathbf{B} \tag{D.2.6.2}$$

D.2.7. Let

$$\mathbf{A} = \begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} \cos\theta_2 \\ \sin\theta_2 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \tag{D.2.7.1}$$

be points on a unit circle. Then

$$\cos ACB = \frac{(\mathbf{C}-\mathbf{A})^\top (\mathbf{C}-\mathbf{B})}{\|\mathbf{C}-\mathbf{A}\| \|\mathbf{C}-\mathbf{B}\|} \tag{D.2.7.2}$$

$$= \cos\left(\frac{\theta_1-\theta_2}{2}\right) \tag{D.2.7.3}$$

*Proof.* Since

$$(\mathbf{C}-\mathbf{A})^\top (\mathbf{C}-\mathbf{B}) = \|\mathbf{C}\|^2 - \mathbf{C}^\top (\mathbf{A}+\mathbf{B}) + \mathbf{A}^\top \mathbf{B} \tag{D.2.7.4}$$

$$= 1 - \cos(\theta-\theta_1) - \cos(\theta-\theta_2) + \cos(\theta_1-\theta_2) \tag{D.2.7.5}$$

$$= 2\cos^2\left(\frac{\theta_1-\theta_2}{2}\right) - 2\cos\left(\frac{\theta_1-\theta_2}{2}\right)\cos\left(\theta - \frac{\theta_1+\theta_2}{2}\right) \tag{D.2.7.6}$$

$$= 4\cos\left(\frac{\theta_1-\theta_2}{2}\right)\sin\left(\frac{\theta-\theta_1}{2}\right)\sin\left(\frac{\theta-\theta_2}{2}\right), \tag{D.2.7.7}$$

and

$$\|\mathbf{C}-\mathbf{A}\|^2 = \|\mathbf{C}\|^2 + \|\mathbf{A}\|^2 - 2\mathbf{C}^\top\mathbf{A}, \tag{D.2.7.8}$$

$$= 4\sin^2\left(\frac{\theta-\theta_1}{2}\right), \tag{D.2.7.9}$$

$$\|\mathbf{C}-\mathbf{B}\|^2 = \|\mathbf{C}\|^2 + \|\mathbf{B}\|^2 - 2\mathbf{C}^\top\mathbf{B}, \tag{D.2.7.10}$$

$$= 4\sin^2\left(\frac{\theta-\theta_2}{2}\right), \tag{D.2.7.11}$$

(D.2.7.2) can be expressed as

$$\frac{\cos\left(\frac{\theta_1-\theta_2}{2}\right)\sin\left(\frac{\theta-\theta_1}{2}\right)\sin\left(\frac{\theta-\theta_2}{2}\right)}{\sin\left(\frac{\theta-\theta_1}{2}\right)\sin\left(\frac{\theta-\theta_1}{2}\right)} \tag{D.2.7.12}$$

yielding (D.2.7.3) □

D.2.8. From (D.2.6.2) and (D.2.7.3),

$$\angle AOB = 2\angle AOC \tag{D.2.8.1}$$

# D.3. Standard Form

D.3.1. Using the affine transformation in (B.5.5.1), the conic in (D.1.2.1) can be expressed in standard form as

$$\mathbf{y}^\top\left(\frac{\mathbf{D}}{f_0}\right)\mathbf{y} = 1 \qquad |\mathbf{V}| \neq 0 \tag{D.3.1.1}$$

$$\mathbf{y}^\top\mathbf{D}\mathbf{y} = -\eta\mathbf{e}_1^\top\mathbf{y} \qquad |\mathbf{V}| = 0 \tag{D.3.1.2}$$

where

$$f_0 = \mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} - f \neq 0 \tag{D.3.1.3}$$

$$\eta = 2\mathbf{u}^\top \mathbf{p}_1 \tag{D.3.1.4}$$

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{D.3.1.5}$$

*Proof.* Using (B.5.5.1) (D.1.2.1) can be expressed as

$$(\mathbf{P}\mathbf{y} + \mathbf{c})^\top \mathbf{V}(\mathbf{P}\mathbf{y} + \mathbf{c}) + 2\mathbf{u}^\top (\mathbf{P}\mathbf{y} + \mathbf{c}) + f = 0, \tag{D.3.1.6}$$

yielding

$$\mathbf{y}^\top \mathbf{P}^\top \mathbf{V}\mathbf{P}\mathbf{y} + 2(\mathbf{V}\mathbf{c} + \mathbf{u})^\top \mathbf{P}\mathbf{y} + \mathbf{c}^\top \mathbf{V}\mathbf{c} + 2\mathbf{u}^\top \mathbf{c} + f = 0 \tag{D.3.1.7}$$

From (D.3.1.7) and (B.5.6.1),

$$\mathbf{y}^\top \mathbf{D}\mathbf{y} + 2(\mathbf{V}\mathbf{c} + \mathbf{u})^\top \mathbf{P}\mathbf{y} + \mathbf{c}^\top (\mathbf{V}\mathbf{c} + \mathbf{u}) + \mathbf{u}^\top \mathbf{c} + f = 0 \tag{D.3.1.8}$$

When $\mathbf{V}^{-1}$ exists, choosing

$$\mathbf{V}\mathbf{c} + \mathbf{u} = \mathbf{0}, \quad \text{or}, \mathbf{c} = -\mathbf{V}^{-1}\mathbf{u}, \tag{D.3.1.9}$$

and substituting (D.3.1.9) in (D.3.1.8) yields (D.3.1.1). When $|\mathbf{V}| = 0, \lambda_1 = 0$ and

$$\mathbf{V}\mathbf{p}_1 = 0, \mathbf{V}\mathbf{p}_2 = \lambda_2 \mathbf{p}_2. \tag{D.3.1.10}$$

where $\mathbf{p}_1, \mathbf{p}_2$ are the eigenvectors of $\mathbf{V}$ such that (B.5.6.1)

$$\mathbf{P} = \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix}, \tag{D.3.1.11}$$

Substituting (D.3.1.11) in (D.3.1.8),

$$\mathbf{y}^\top \mathbf{D} \mathbf{y} + 2 \left( \mathbf{c}^\top \mathbf{V} + \mathbf{u}^\top \right) \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix} \mathbf{y} + \mathbf{c}^\top \left( \mathbf{V} \mathbf{c} + \mathbf{u} \right) + \mathbf{u}^\top \mathbf{c} + f = 0 \tag{D.3.1.12}$$

$$\implies \mathbf{y}^\top \mathbf{D} \mathbf{y} + 2 \begin{pmatrix} \left( \mathbf{c}^\top \mathbf{V} + \mathbf{u}^\top \right) \mathbf{p}_1 & \left( \mathbf{c}^\top \mathbf{V} + \mathbf{u}^\top \right) \mathbf{p}_2 \end{pmatrix} \mathbf{y} + \mathbf{c}^\top \left( \mathbf{V} \mathbf{c} + \mathbf{u} \right) + \mathbf{u}^\top \mathbf{c} + f = 0 \tag{D.3.1.13}$$

$$\implies \mathbf{y}^\top \mathbf{D} \mathbf{y} + 2 \begin{pmatrix} \mathbf{u}^\top \mathbf{p}_1 & \left( \lambda_2 \mathbf{c}^\top + \mathbf{u}^\top \right) \mathbf{p}_2 \end{pmatrix} \mathbf{y} + \mathbf{c}^\top \left( \mathbf{V} \mathbf{c} + \mathbf{u} \right) + \mathbf{u}^\top \mathbf{c} + f = 0 \tag{D.3.1.14}$$

upon substituting from (D.3.1.10) yielding

$$\lambda_2 y_2^2 + 2 \left( \mathbf{u}^\top \mathbf{p}_1 \right) y_1 + 2 y_2 \left( \lambda_2 \mathbf{c} + \mathbf{u} \right)^\top \mathbf{p}_2 + \mathbf{c}^\top \left( \mathbf{V} \mathbf{c} + \mathbf{u} \right) + \mathbf{u}^\top \mathbf{c} + f = 0 \tag{D.3.1.15}$$

Thus, (D.3.1.15) can be expressed as (D.3.1.2) by choosing

$$\eta = 2 \mathbf{u}^\top \mathbf{p}_1 \tag{D.3.1.16}$$

and $\mathbf{c}$ in (D.3.1.8) such that

$$2 \mathbf{P}^\top \left( \mathbf{V} \mathbf{c} + \mathbf{u} \right) = \eta \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{D.3.1.17}$$

$$\mathbf{c}^\top \left( \mathbf{V} \mathbf{c} + \mathbf{u} \right) + \mathbf{u}^\top \mathbf{c} + f = 0 \tag{D.3.1.18}$$

$\because \mathbf{P}^\top \mathbf{P} = \mathbf{I}$, multiplying (D.3.1.17) by $\mathbf{P}$ yields

$$(\mathbf{V}\mathbf{c} + \mathbf{u}) = \frac{\eta}{2}\mathbf{p}_1, \tag{D.3.1.19}$$

which, upon substituting in (D.3.1.18) results in

$$\frac{\eta}{2}\mathbf{c}^\top \mathbf{p}_1 + \mathbf{u}^\top \mathbf{c} + f = 0 \tag{D.3.1.20}$$

(D.3.1.19) and (D.3.1.20) can be clubbed together to obtain (E.2.1.2). $\square$

D.3.2. For the standard conic,

$$\mathbf{P} = \mathbf{I} \tag{D.3.2.1}$$

$$\mathbf{u} = \begin{cases} 0 & e \neq 1 \\ \frac{\eta}{2}\mathbf{e}_1 & e = 1 \end{cases} \tag{D.3.2.2}$$

$$\lambda_1 \begin{cases} = 0 & e = 1 \\ \neq 0 & e \neq 1 \end{cases} \tag{D.3.2.3}$$

where

$$\mathbf{I} = \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 \end{pmatrix} \tag{D.3.2.4}$$

is the identity matrix.

D.3.3.

(a) The directrices for the standard conic are given by

$$\mathbf{e}_1^\top \mathbf{y} = \pm\frac{1}{e}\sqrt{\frac{|f_0|}{\lambda_2\left(1-e^2\right)}} \qquad e \neq 1 \tag{D.3.3.1}$$

$$\mathbf{e}_1^\top \mathbf{y} = \frac{\eta}{2\lambda_2} \qquad e = 1 \tag{D.3.3.2}$$

(b) The foci of the standard ellipse and hyperbola are given by

$$\mathbf{F} = \begin{cases} \pm e\sqrt{\frac{|f_0|}{\lambda_2(1-e^2)}}\mathbf{e}_1 & e \neq 1 \\ -\frac{\eta}{4\lambda_2}\mathbf{e}_1 & e = 1 \end{cases} \tag{D.3.3.3}$$

*Proof.* (a) For the standard hyperbola/ellipse in (D.3.1.1), from (D.3.2.1), (D.1.3.2) and (D.3.2.2),

$$\mathbf{n} = \sqrt{\frac{\lambda_2}{f_0}}\mathbf{e}_1 \tag{D.3.3.4}$$

$$c = \pm\frac{\sqrt{-\frac{\lambda_2}{f_0}\left(e^2-1\right)\left(\frac{\lambda_2}{f_0}\right)}}{\frac{\lambda_2}{f_0}e\left(e^2-1\right)} \tag{D.3.3.5}$$

$$= \pm\frac{1}{e\sqrt{1-e^2}} \tag{D.3.3.6}$$

yielding (D.3.3.1) upon substituting from (D.1.3.1) and simplifying. For the standard parabola in (D.3.1.2), from (D.3.2.1), (D.1.3.2) and (D.3.2.2), noting that $f = 0$,

$$\mathbf{n} = \sqrt{\lambda_2}\mathbf{e}_1 \tag{D.3.3.7}$$

$$c = \frac{\left\| \frac{\eta}{2}\mathbf{e}_1 \right\|^2}{2\left(\frac{\eta}{\mathbf{2}}\right)\left(\mathbf{e_1}\right)^\top \mathbf{n}} \tag{D.3.3.8}$$

$$\tag{D.3.3.9}$$

$$= \frac{\eta}{4\sqrt{\lambda_2}} \tag{D.3.3.10}$$

yielding (D.3.3.2).

(b) For the standard ellipse/hyperbola, substituting from (D.3.3.6), (D.3.3.4), (D.3.2.2) and (D.1.3.1) in (D.1.3.3),

$$\mathbf{F} = \pm \frac{\left(\frac{1}{e\sqrt{1-e^2}}\right)\left(e^2\right)\sqrt{\frac{\lambda_2}{f_0}}\mathbf{e}_1}{\frac{\lambda_2}{f_0}} \tag{D.3.3.11}$$

yielding (D.3.3.3) after simplification. For the standard parabola, substituting from (D.3.3.10), (D.3.3.7), (D.3.2.2) and (D.1.3.1) in (D.1.3.3),

$$\mathbf{F} = \frac{\left(\frac{\eta}{4\sqrt{\lambda_2}}\right)\sqrt{\lambda_2}\mathbf{e}_1 - \frac{\eta}{\mathbf{2}}\mathbf{e_1}}{\lambda_2} \tag{D.3.3.12}$$

$$\tag{D.3.3.13}$$

yielding (D.3.3.3) after simplification.

□

# Appendix E

# Conic Parameters

## E.1. Standard Form

E.1.1. The center of the standard ellipse/hyperbola, defined to be the mid point of the line joining the foci, is the origin.

E.1.2. The principal (major) axis of the standard ellipse/hyperbola, defined to be the line joining the two foci is the $x$-axis.

*Proof.* From (D.3.3.3), it is obvious that the line joining the foci passes through the origin. Also, the direction vector of this line is $\mathbf{e}_1$. Thus, the principal axis is the $x$-axis. □

E.1.3. The minor axis of the standard ellipse/hyperbola, defined to be the line orthogonal to the $x$-axis is the $y$-axis.

E.1.4. The axis of symmetry of the standard parabola, defined to be the line perpendicular to the directrix and passing through the focus, is the $x$- axis.

*Proof.* From (D.3.3.7) and (D.3.3.3), the axis of the parabola can be expressed using

(C.1.2.1) as

$$\mathbf{e}_2^\top \left(\mathbf{y} + \frac{\eta}{4\lambda_2}\mathbf{e}_1\right) = 0 \tag{E.1.4.1}$$

$$\implies \mathbf{e}_2^\top \mathbf{y} = 0, \tag{E.1.4.2}$$

which is the equation of the $x$-axis. □

E.1.5. The point where the parabola intersects its axis of symmetry is called the vertex. For the standard parabola, the vertex is the origin.

*Proof.* (E.1.4.2) can be expressed as

$$\mathbf{y} = \alpha \mathbf{e}_1, \tag{E.1.5.1}$$

using (C.1.2.1). Substituting (E.1.5.1) in (D.3.1.2),

$$\alpha^2 \mathbf{e}_1^\top \mathbf{D} \mathbf{e}_1 = -\eta\alpha \mathbf{e}_1^\top \mathbf{e}_1 \tag{E.1.5.2}$$

$$\implies \alpha = 0, \text{ or, } \mathbf{y} = \mathbf{0}. \tag{E.1.5.3}$$

□

E.1.6. The focal length of the standard parabola, , defined to be the distance between the vertex and the focus, measured along the axis of symmetry, is $\left|\frac{\eta}{4\lambda_2}\right|$

# E.2. Quadratic Form

E.2.1. The center/vertex of a conic section are given by

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \qquad \left|\mathbf{V}\right| \neq 0 \tag{E.2.1.1}$$

$$\begin{pmatrix} \mathbf{u}^\top + \frac{\eta}{2}\mathbf{p}_1^\top \\ \mathbf{V} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -f \\ \frac{\eta}{2}\mathbf{p}_1 - \mathbf{u} \end{pmatrix} \qquad \left|\mathbf{V}\right| = 0 \tag{E.2.1.2}$$

*Proof.* In (B.5.5.1), substituting $\mathbf{y} = \mathbf{0}$, the center/vertex for the quadratic form is obtained as

$$\mathbf{x} = \mathbf{c}, \tag{E.2.1.3}$$

where $\mathbf{c}$ is derived as (E.2.1.1) and (E.2.1.2) in Appendix D.3.1. □

E.2.2. The equation of the minor and major axes for the ellipse/hyperbola are respectively given by

$$\mathbf{p}_i^\top (\mathbf{x} - \mathbf{c}) = 0, i = 1, 2 \tag{E.2.2.1}$$

The axis of symmetry for the parabola is also given by (E.2.2.1).

*Proof.* From (E.1.2), the major/symmetry axis for the hyperbola/ellipse/parabola can be expressed using (B.5.5.1) as

$$\mathbf{e}_2^\top \mathbf{P}^\top (\mathbf{x} - \mathbf{c}) = 0 \tag{E.2.2.2}$$

$$\implies (\mathbf{P}\mathbf{e}_2)^\top (\mathbf{x} - \mathbf{c}) = 0 \tag{E.2.2.3}$$

yielding (E.2.2.1), and the proof for the minor axis is similar. $\square$

# Appendix F

# Conic Lines

## F.1. Pair of Straight Lines

F.1.1. The asymptotes of the hyperbola in (D.3.1.1), defined to be the lines that do not intersect the hyperbola, are given by

$$\left(\sqrt{|\lambda_1|} \quad \pm\sqrt{|\lambda_2|}\right)\mathbf{y} = 0 \tag{F.1.1.1}$$

*Proof.* From (D.3.1.1), it is obvious that the pair of lines represented by

$$\mathbf{y}^\top \mathbf{D}\mathbf{y} = 0 \tag{F.1.1.2}$$

do not intersect the conic

$$\mathbf{y}^\top \mathbf{D}\mathbf{y} = f_0 \tag{F.1.1.3}$$

Thus, (F.1.1.2) represents the asysmptotes of the hyperbola in (D.3.1.1) and can be expressed as

$$\lambda_1 y_1^2 + \lambda_2 y_1^2 = 0, \tag{F.1.1.4}$$

which can then be simplified to obtain (F.1.1.1).

$\square$

F.1.2. (D.1.2.1) represents a pair of straight lines if

$$\mathbf{u}^{\top}\mathbf{V}^{-1}\mathbf{u} - f = 0 \tag{F.1.2.1}$$

F.1.3. (D.1.2.1) represents a pair of straight lines if the matrix

$$\begin{pmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u} & f \end{pmatrix} \tag{F.1.3.1}$$

is singular.

*Proof.* Let

$$\begin{pmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u} & f \end{pmatrix} \mathbf{x} = \mathbf{0} \tag{F.1.3.2}$$

Expressing

$$\mathbf{x} = \begin{pmatrix} \mathbf{y} \\ y_3 \end{pmatrix}, \tag{F.1.3.3}$$

$$
\begin{pmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^\top & f \end{pmatrix} \begin{pmatrix} \mathbf{y} \\ y_3 \end{pmatrix} = \mathbf{0} \tag{F.1.3.4}
$$

$$
\implies \mathbf{V}\mathbf{y} + y_3\mathbf{u} = \mathbf{0} \quad \text{and} \tag{F.1.3.5}
$$

$$
\mathbf{u}^\top \mathbf{y} + f y_3 = 0 \tag{F.1.3.6}
$$

From (F.1.3.5) we obtain,

$$
\mathbf{y}^\top \mathbf{V}\mathbf{y} + y_3\mathbf{y}^\top \mathbf{u} = \mathbf{0} \tag{F.1.3.7}
$$

$$
\implies \mathbf{y}^\top \mathbf{V}\mathbf{y} + y_3\mathbf{u}^\top \mathbf{y} = \mathbf{0} \tag{F.1.3.8}
$$

yielding (F.1.2.1) upon substituting from (F.1.3.6). □

F.1.4. Using the affine transformation, (F.1.1.1) can be expressed as the lines

$$
\begin{pmatrix} \sqrt{|\lambda_1|} & \pm\sqrt{|\lambda_2|} \end{pmatrix} \mathbf{P}^\top (\mathbf{x} - \mathbf{c}) = 0 \tag{F.1.4.1}
$$

F.1.5. The angle between the asymptotes can be expressed as

$$
\cos\theta = \frac{|\lambda_1| - |\lambda_2|}{|\lambda_1| + |\lambda_2|} \tag{F.1.5.1}
$$

*Proof.* The normal vectors of the lines in (F.1.4.1) are

$$
\begin{aligned}
\mathbf{n}_1 &= \mathbf{P} \begin{pmatrix} \sqrt{|\lambda_1|} \\ \sqrt{|\lambda_2|} \end{pmatrix} \\
\mathbf{n}_2 &= \mathbf{P} \begin{pmatrix} \sqrt{|\lambda_1|} \\ -\sqrt{|\lambda_2|} \end{pmatrix}
\end{aligned} \tag{F.1.5.2}
$$

The angle between the asymptotes is given by

$$\cos\theta = \frac{\mathbf{n_1}^\top \mathbf{n_2}}{\|\mathbf{n_1}\| \, \|\mathbf{n_2}\|} \tag{F.1.5.3}$$

The orthogonal matrix $\mathbf{P}$ preserves the norm, i.e.

$$\|\mathbf{n_1}\| = \left\| \mathbf{P} \begin{pmatrix} \sqrt{|\lambda_1|} \\ \sqrt{|\lambda_2|} \end{pmatrix} \right\| = \left\| \begin{pmatrix} \sqrt{|\lambda_1|} \\ \sqrt{|\lambda_2|} \end{pmatrix} \right\| \tag{F.1.5.4}$$

$$= \sqrt{|\lambda_1| + |\lambda_2|} = \|\mathbf{n_2}\| \tag{F.1.5.5}$$

It is easy to verify that

$$\mathbf{n_1}^\top \mathbf{n_2} = |\lambda_1| - |\lambda_2| \tag{F.1.5.6}$$

Thus, the angle between the asymptotes is obtained from (F.1.5.3) as (F.1.5.1). $\square$

# F.2. Intersection of Conics

F.2.1. Let

$$\mathbf{x}^\top \mathbf{V}_i \mathbf{x} + 2\mathbf{u}_i^\top \mathbf{x} + f_i = 0, \quad i = 1, 2 \tag{F.2.1.1}$$

be the equation of two conics. The locus of their intersection is a pair of straight lines if

$$\begin{vmatrix} \mathbf{V}_1 + \mu \mathbf{V}_2 & \mathbf{u}_1 + \mu \mathbf{u}_2 \\ (\mathbf{u}_1 + \mu \mathbf{u}_2)^\top & f \end{vmatrix} = 0, \left| \mathbf{V}_1 + \mu \mathbf{V}_2 \right| < 0 \tag{F.2.1.2}$$

*Proof.* The intersection of the conics in (F.2.1.1) is given by the curve

$$\mathbf{x}^\top (\mathbf{V}_1 + \mu \mathbf{V}_2) \mathbf{x} + 2 (\mathbf{u}_1 + \mu \mathbf{u}_2)^\top \mathbf{x} + f_1 + \mu f_2 = 0, \tag{F.2.1.3}$$

which, from Theorem F.1.3 represents a pair of straight lines if (F.2.1.2) is satisfied. □

F.2.2. The points of intersection of the conics in (F.2.1.1) are the points of the intersection of the lines in (F.2.1.3).

# F.3. Chords of a Conic

F.3.1. The points of intersection of the line

$$L : \quad \mathbf{x} = \mathbf{h} + \mu \mathbf{m} \quad \mu \in \mathbb{R} \tag{F.3.1.1}$$

with the conic section in (D.1.2.1) are given by

$$\mathbf{x}_i = \mathbf{h} + \mu_i \mathbf{m} \tag{F.3.1.2}$$

where

$$\mu_i = \frac{1}{\mathbf{m}^\top \mathbf{V} \mathbf{m}} \left( -\mathbf{m}^\top \left( \mathbf{V}\mathbf{h} + \mathbf{u} \right) \right.$$
$$\left. \pm \sqrt{\left[ \mathbf{m}^\top \left( \mathbf{V}\mathbf{h} + \mathbf{u} \right) \right]^2 - \mathrm{g}\left(\mathbf{h}\right) \left( \mathbf{m}^\top \mathbf{V} \mathbf{m} \right)} \right) \tag{F.3.1.3}$$

*Proof.* Substituting (F.3.1.1) in (D.1.2.1),

$$\left( \mathbf{h} + \mu \mathbf{m} \right)^\top \mathbf{V} \left( \mathbf{h} + \mu \mathbf{m} \right) + 2\mathbf{u}^\top \left( \mathbf{h} + \mu \mathbf{m} \right) + f = 0 \tag{F.3.1.4}$$

$$\implies \mu^2 \mathbf{m}^\top \mathbf{V} \mathbf{m} + 2\mu \mathbf{m}^\top \left( \mathbf{V}\mathbf{h} + \mathbf{u} \right) + \mathbf{h}^\top \mathbf{V} \mathbf{h} + 2\mathbf{u}^\top \mathbf{h} + f = 0 \tag{F.3.1.5}$$

$$\text{or, } \mu^2 \mathbf{m}^\top \mathbf{V} \mathbf{m} + 2\mu \mathbf{m}^\top \left( \mathbf{V}\mathbf{h} + \mathbf{u} \right) + \mathrm{g}\left(\mathbf{h}\right) = 0 \tag{F.3.1.6}$$

for g defined in (D.1.2.1). Solving the above quadratic in (F.3.1.6) yields (F.3.1.3). □

F.3.2. If $L$ in (F.3.1.1) touches (D.1.2.1) at exactly one point $\mathbf{q}$,

$$\mathbf{m}^\top \left( \mathbf{V}\mathbf{q} + \mathbf{u} \right) = 0 \tag{F.3.2.1}$$

*Proof.* In this case, (F.3.1.6) has exactly one root. Hence, in (F.3.1.3)

$$\left[ \mathbf{m}^\top \left( \mathbf{V}\mathbf{q} + \mathbf{u} \right) \right]^2 - \left( \mathbf{m}^\top \mathbf{V} \mathbf{m} \right) \mathrm{g}\left(\mathbf{q}\right) = 0 \tag{F.3.2.2}$$

$\because$ $\mathbf{q}$ is the point of contact,

$$\mathrm{g}\left(\mathbf{q}\right) = 0 \tag{F.3.2.3}$$

Substituting (F.3.2.3) in (F.3.2.2) and simplifying, we obtain (F.3.2.1). □

F.3.3. The length of the chord in (F.3.1.1) is given by

$$\frac{2\sqrt{\left[\mathbf{m}^\top\left(\mathbf{Vh}+\mathbf{u}\right)\right]^2-\left(\mathbf{h}^\top\mathbf{Vh}+2\mathbf{u}^\top\mathbf{h}+f\right)\left(\mathbf{m}^\top\mathbf{Vm}\right)}}{\mathbf{m}^\top\mathbf{Vm}}\left\|\mathbf{m}\right\| \tag{F.3.3.1}$$

*Proof.* The distance between the points in (F.3.1.2) is given by

$$\left\|\mathbf{x}_1-\mathbf{x}_2\right\| = \left|\mu_1-\mu_2\right|\left\|\mathbf{m}\right\| \tag{F.3.3.2}$$

Substituing $\mu_i$ from (F.3.1.3) in (F.3.3.2) yields (F.3.3.1). □

F.3.4. The affine transform for the conic section, preserves the norm. This implies that the length of any chord of a conic is invariant to translation and/or rotation.

*Proof.* Let

$$\mathbf{x}_i = \mathbf{P}\mathbf{y}_i + \mathbf{c} \tag{F.3.4.1}$$

be any two points on the conic. Then the distance between the points is given by

$$\left\|\mathbf{x}_1-\mathbf{x}_2\right\| = \left\|\mathbf{P}\left(\mathbf{y}_1-\mathbf{y}_2\right)\right\| \tag{F.3.4.2}$$

which can be expressed as

$$\left\|\mathbf{x}_1-\mathbf{x}_2\right\|^2 = \left(\mathbf{y}_1-\mathbf{y}_2\right)^\top\mathbf{P}^\top\mathbf{P}\left(\mathbf{y}_1-\mathbf{y}_2\right) \tag{F.3.4.3}$$

$$= \left\|\mathbf{y}_1-\mathbf{y}_2\right\|^2 \tag{F.3.4.4}$$

since

$$\mathbf{P}^\top \mathbf{P} = \mathbf{I} \tag{F.3.4.5}$$

$\square$

F.3.5. For the standard hyperbola/ellipse, the length of the major axis is

$$2\sqrt{\left|\frac{f_0}{\lambda_1}\right|} \tag{F.3.5.1}$$

and the minor axis is

$$2\sqrt{\left|\frac{f_0}{\lambda_2}\right|} \tag{F.3.5.2}$$

*Proof.* Since the major axis passes through the origin,

$$\mathbf{q} = \mathbf{0} \tag{F.3.5.3}$$

Further, from Corollary (E.1.2),

$$\mathbf{m} = \mathbf{e}_2, \tag{F.3.5.4}$$

and from (D.3.1.1),

$$\mathbf{V} = \frac{\mathbf{D}}{f_0}, \mathbf{u} = 0, f = -1 \tag{F.3.5.5}$$

Substituting the above in (F.3.3.1),

$$\frac{2\sqrt{\mathbf{e}_1^\top \frac{\mathbf{D}}{f_0}\mathbf{e}_1}}{\mathbf{e}_1^\top \frac{\mathbf{D}}{f_0}\mathbf{e}_1}\|\mathbf{e}_1\| \tag{F.3.5.6}$$

yielding (F.3.5.1). Similarly, for the minor axis, the only different parameter is

$$\mathbf{m} = \mathbf{e}_2, \tag{F.3.5.7}$$

Substituting the above in (F.3.3.1),

$$\frac{2\sqrt{\mathbf{e}_2^\top \frac{\mathbf{D}}{f_0}\mathbf{e}_2}}{\mathbf{e}_2^\top \frac{\mathbf{D}}{f_0}\mathbf{e}_2}\|\mathbf{e}_2\| \tag{F.3.5.8}$$

yielding (F.3.5.2).

$\square$

F.3.6. The latus rectum of a conic section is the chord that passes through the focus and is perpendicular to the major axis. The length of the latus rectum for a conic is given by

$$l = \begin{cases} 2\frac{\sqrt{|f_0\lambda_1|}}{\lambda_2} & e \neq 1 \\ \frac{\eta}{\lambda_2} & e = 1 \end{cases} \tag{F.3.6.1}$$

*Proof.* The latus rectum is perpendicular to the major axis for the standard conic. Hence, from Corollary (E.1.2),

$$\mathbf{m} = \mathbf{e}_2, \tag{F.3.6.2}$$

Since it passes through the focus, from (D.3.3.3)

$$\mathbf{q} = \mathbf{F} = \pm e\sqrt{\frac{f_0}{\lambda_2\left(1-e^2\right)}}\mathbf{e}_1 \tag{F.3.6.3}$$

for the standard hyperbola/ellipse. Also, from (D.3.1.1),

$$\mathbf{V} = \frac{\mathbf{D}}{f_0}, \mathbf{u} = 0, f = -1 \tag{F.3.6.4}$$

Substituting the above in (F.3.3.1),

$$\frac{2\sqrt{\left[\mathbf{e}_2^\top\left(\frac{\mathbf{D}}{f_0}e\sqrt{\frac{f_0}{\lambda_2(1-e^2)}}\mathbf{e}_1\right)\right]^2 - \left(e\sqrt{\frac{f_0}{\lambda_2(1-e^2)}}\mathbf{e}_1^\top\frac{\mathbf{D}}{f_0}e\sqrt{\frac{f_0}{\lambda_2(1-e^2)}}\mathbf{e}_1 - 1\right)\left(\mathbf{e}_2^\top\frac{\mathbf{D}}{f_0}\mathbf{e}_2\right)}}{\mathbf{e}_2^\top\frac{\mathbf{D}}{f_0}\mathbf{e}_2}\|\mathbf{e}_2\| \tag{F.3.6.5}$$

Since

$$\mathbf{e}_2^\top\mathbf{D}\mathbf{e}_1 = 0, \mathbf{e}_1^\top\mathbf{D}\mathbf{e}_1 = \lambda_1, \mathbf{e}_1^\top\mathbf{e}_1 = 1, \|\mathbf{e}_2\| = 1, \mathbf{e}_2^\top\mathbf{D}\mathbf{e}_2 = \lambda_2, \tag{F.3.6.6}$$

(F.3.6.5) can be expressed as

$$\frac{2\sqrt{\left(1-\frac{\lambda_1 e^2}{\lambda_2(1-e^2)}\right)\left(\frac{\lambda_2}{f_0}\right)}}{\frac{\lambda_2}{f_0}} \tag{F.3.6.7}$$

$$= 2\frac{\sqrt{f_0\lambda_1}}{\lambda_2} \qquad \left(\because e^2 = 1-\frac{\lambda_1}{\lambda_2}\right) \tag{F.3.6.8}$$

For the standard parabola, the parameters in (F.3.3.1) are

$$\mathbf{q} = \mathbf{F} = -\frac{\eta}{4\lambda_2}\mathbf{e}_1, \mathbf{m} = \mathbf{e}_1, \mathbf{V} = \mathbf{D}, \mathbf{u} = \frac{\eta}{2}\mathbf{e}_1^\top, f = 0 \tag{F.3.6.9}$$

Substituting the above in (F.3.3.1), the length of the latus rectum can be expressed as

$$2\frac{\sqrt{\left[\mathbf{e}_2^\top\left(\mathbf{D}\left(-\frac{\eta}{4\lambda_2}\mathbf{e}_1\right)+\frac{\eta}{2}\mathbf{e}_1\right)\right]^2-\left(\left(-\frac{\eta}{4\lambda_2}\mathbf{e}_1\right)^\top\mathbf{D}\left(-\frac{\eta}{4\lambda_2}\mathbf{e}_1\right)+2\frac{\eta}{2}\mathbf{e}_1^\top\left(-\frac{\eta}{4\lambda_2}\mathbf{e}_1\right)\right)\left(\mathbf{e}_2^\top\mathbf{D}\mathbf{e}_2\right)}}{\mathbf{e}_2^\top\mathbf{D}\mathbf{e}_2}\left\|\mathbf{e}_2\right\| \tag{F.3.6.10}$$

Since

$$\mathbf{e}_2^\top\mathbf{D}\mathbf{e}_1 = 0, \mathbf{e}_2^\top\mathbf{e}_2 = 0, \mathbf{e}_1^\top\mathbf{D}\mathbf{e}_1 = 0, \mathbf{e}_1^\top\mathbf{e}_1 = 1, \left\|\mathbf{e}_1\right\| = 1, \mathbf{e}_2^\top\mathbf{D}\mathbf{e}_2 = \lambda_2, \tag{F.3.6.11}$$

(F.3.6.10) can be expressed as

$$2\frac{\sqrt{\frac{\eta^2}{4\lambda_2}\lambda_2}}{\lambda_2} = \frac{\eta}{\lambda_2} \tag{F.3.6.12}$$

□

# F.4. Tangent and Normal

F.4.1. Given the point of contact $\mathbf{q}$, the equation of a tangent to (D.1.2.1) is

$$\left(\mathbf{V}\mathbf{q}+\mathbf{u}\right)^\top\mathbf{x}+\mathbf{u}^\top\mathbf{q}+f = 0 \tag{F.4.1.1}$$

*Proof.* The normal vector is obtained from (F.3.2.1) and (A.1.20.1) as

$$\kappa\mathbf{n} = \mathbf{V}\mathbf{q}+\mathbf{u}, \kappa \in \mathbb{R} \tag{F.4.1.2}$$

From (F.4.1.2) and (C.1.2.1), the equation of the tangent is

$$(\mathbf{V}\mathbf{q}+\mathbf{u})^\top(\mathbf{x}-\mathbf{q}) = 0 \tag{F.4.1.3}$$

$$\implies (\mathbf{V}\mathbf{q}+\mathbf{u})^\top\mathbf{x}-\mathbf{q}^\top\mathbf{V}\mathbf{q}-\mathbf{u}^\top\mathbf{q} = 0 \tag{F.4.1.4}$$

which, upon substituting from (F.3.2.3) and simplifying yields (F.4.1.1) □

F.4.2. Given the point of contact $\mathbf{q}$, the equation of the normal to (D.1.2.1) is

$$(\mathbf{V}\mathbf{q}+\mathbf{u})^\top\mathbf{R}(\mathbf{x}-\mathbf{q}) = 0 \tag{F.4.2.1}$$

*Proof.* The direction vector of the tangent is obtained from (F.4.1.2) as as

$$\mathbf{m} = \mathbf{R}(\mathbf{V}\mathbf{q}+\mathbf{u}), \tag{F.4.2.2}$$

where $\mathbf{R}$ is the rotation matrix. From (F.4.2.2) and (C.1.2.1), the equation of the normal is given by (F.4.2.1) □

F.4.3. Given the tangent

$$\mathbf{n}^\top\mathbf{x} = c, \tag{F.4.3.1}$$

the point of contact to the conic in (D.1.2.1) is given by

$$\begin{pmatrix}\mathbf{n}^\top\\ \mathbf{m}^\top\mathbf{V}\end{pmatrix}\mathbf{q} = \begin{pmatrix}c\\ -\mathbf{m}^\top\mathbf{u}\end{pmatrix} \tag{F.4.3.2}$$

*Proof.* From (F.3.2.1),

$$\mathbf{m}^\top(\mathbf{V}\mathbf{q}+\mathbf{u}) = 0 \tag{F.4.3.3}$$

$$\implies \mathbf{m}^\top\mathbf{V}\mathbf{q} = -\mathbf{m}^\top\mathbf{u} \tag{F.4.3.4}$$

Combining (F.4.3.1) and (F.4.3.4), (F.4.3.2) is obtained.

□

F.4.4. If $\mathbf{V}^{-1}$ exists, given the normal vector $\mathbf{n}$, the tangent points of contact to (D.1.2.1) are given by

$$\begin{aligned}\mathbf{q}_i &= \mathbf{V}^{-1}\left(\kappa_i\mathbf{n}-\mathbf{u}\right), i = 1, 2\\ \text{where } \kappa_i &= \pm\sqrt{\frac{f_0}{\mathbf{n}^\top\mathbf{V}^{-1}\mathbf{n}}}\end{aligned} \tag{F.4.4.1}$$

*Proof.* From (F.4.1.2),

$$\mathbf{q} = \mathbf{V}^{-1}\left(\kappa\mathbf{n}-\mathbf{u}\right), \quad \kappa \in \mathbb{R} \tag{F.4.4.2}$$

Substituting (F.4.4.2) in (F.3.2.3),

$$\left(\kappa\mathbf{n}-\mathbf{u}\right)^\top\mathbf{V}^{-1}\left(\kappa\mathbf{n}-\mathbf{u}\right) + 2\mathbf{u}^\top\mathbf{V}^{-1}\left(\kappa\mathbf{n}-\mathbf{u}\right) + f = 0 \tag{F.4.4.3}$$

$$\implies \kappa^2\mathbf{n}^\top\mathbf{V}^{-1}\mathbf{n} - \mathbf{u}^\top\mathbf{V}^{-1}\mathbf{u} + f = 0 \tag{F.4.4.4}$$

$$\text{or, } \kappa = \pm\sqrt{\frac{f_0}{\mathbf{n}^\top\mathbf{V}^{-1}\mathbf{n}}} \tag{F.4.4.5}$$

Substituting (F.4.4.5) in (F.4.4.2) yields (F.4.4.1). □

F.4.5. For a conic/hyperbola, a line with normal vector $\mathbf{n}$ cannot be a tangent if

$$\frac{\mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} - f}{\mathbf{n}^\top \mathbf{V}^{-1}\mathbf{n}} < 0 \tag{F.4.5.1}$$

F.4.6. For a circle,

$$\mathbf{q}_{ij} = \left(\pm r\frac{\mathbf{n}_j}{\|\mathbf{n}_j\|} - \mathbf{u}\right), \quad i, j = 1, 2 \tag{F.4.6.1}$$

*Proof.* From (F.4.4.1), and (D.2.2.1),

$$\kappa_{ij} = \pm\frac{r}{\|\mathbf{n}_j\|} \tag{F.4.6.2}$$

□

F.4.7. If $\mathbf{V}$ is not invertible, given the normal vector $\mathbf{n}$, the point of contact to (D.1.2.1) is given by the matrix equation

$$\begin{pmatrix} (\mathbf{u}+\kappa\mathbf{n})^\top \\ \mathbf{V} \end{pmatrix} \mathbf{q} = \begin{pmatrix} -f \\ \kappa\mathbf{n} - \mathbf{u} \end{pmatrix} \tag{F.4.7.1}$$

$$\text{where } \kappa = \frac{\mathbf{p}_1^\top \mathbf{u}}{\mathbf{p}_1^\top \mathbf{n}}, \quad \mathbf{V}\mathbf{p}_1 = 0 \tag{F.4.7.2}$$

*Proof.* If $\mathbf{V}$ is non-invertible, it has a zero eigenvalue. If the corresponding eigenvector is $\mathbf{p}_1$, then,

$$\mathbf{V}\mathbf{p}_1 = 0 \tag{F.4.7.3}$$

From (F.4.1.2),

$$\kappa \mathbf{n} = \mathbf{V}\mathbf{q} + \mathbf{u}, \quad \kappa \in \mathbb{R} \tag{F.4.7.4}$$

$$\implies \kappa \mathbf{p}_1^\top \mathbf{n} = \mathbf{p}_1^\top \mathbf{V}\mathbf{q} + \mathbf{p}_1^\top \mathbf{u} \tag{F.4.7.5}$$

$$\text{or, } \kappa \mathbf{p}_1^\top \mathbf{n} = \mathbf{p}_1^\top \mathbf{u}, \quad \because \mathbf{p}_1^\top \mathbf{V} = 0, \quad (\text{ from (F.4.7.3)}) \tag{F.4.7.6}$$

yielding $\kappa$ in (F.4.7.2). From (F.4.7.4),

$$\kappa \mathbf{q}^\top \mathbf{n} = \mathbf{q}^\top \mathbf{V}\mathbf{q} + \mathbf{q}^\top \mathbf{u} \tag{F.4.7.7}$$

$$\implies \kappa \mathbf{q}^\top \mathbf{n} = -f - \mathbf{q}^\top \mathbf{u} \quad \text{from (F.3.2.3),} \tag{F.4.7.8}$$

$$\text{or, } \left(\kappa \mathbf{n} + \mathbf{u}\right)^\top \mathbf{q} = -f \tag{F.4.7.9}$$

(F.4.7.4) can be expressed as

$$\mathbf{V}\mathbf{q} = \kappa \mathbf{n} - \mathbf{u}. \tag{F.4.7.10}$$

(F.4.7.9) and (F.4.7.10) clubbed together result in (F.4.7.1). □

F.4.8. A point $\mathbf{h}$ lies on a tangent to the conic in (D.1.2.1) if

$$\mathbf{m}^\top \left[\left(\mathbf{V}\mathbf{h} + \mathbf{u}\right)\left(\mathbf{V}\mathbf{h} + \mathbf{u}\right)^\top - \mathbf{V}\mathrm{g}\left(\mathbf{h}\right)\right] \mathbf{m} = 0 \tag{F.4.8.1}$$

*Proof.* From (F.3.1.3) and (F.3.2.2)

$$\left[\mathbf{m}^\top \left(\mathbf{V}\mathbf{h} + \mathbf{u}\right)\right]^2 - \left(\mathbf{m}^\top \mathbf{V}\mathbf{m}\right) \mathrm{g}\left(\mathbf{h}\right) = 0 \tag{F.4.8.2}$$

yielding (F.4.8.1). □

F.4.9. The normal vectors of the tangents to the conic in (D.1.2.1) from a point $\mathbf{h}$ are given by

$$\begin{aligned}\mathbf{n}_1 &= \mathbf{P}\begin{pmatrix}\sqrt{|\lambda_1|}\\ \sqrt{|\lambda_2|}\end{pmatrix}\\ \mathbf{n}_2 &= \mathbf{P}\begin{pmatrix}\sqrt{|\lambda_1|}\\ -\sqrt{|\lambda_2|}\end{pmatrix}\end{aligned} \tag{F.4.9.1}$$

where $\lambda_i, \mathbf{P}$ are the eigenparameters of

$$\mathbf{\Sigma} = (\mathbf{V}\mathbf{h}+\mathbf{u})(\mathbf{V}\mathbf{h}+\mathbf{u})^\top - (\mathrm{g}(\mathbf{h}))\,\mathbf{V}. \tag{F.4.9.2}$$

*Proof.* From (F.4.8.1) we obtain (F.4.9.2). Consequently, from (F.1.5.2), (F.4.9.1) can be obtained. □

F.4.10. A point $\mathbf{h}$ lies on a normal to the conic in (D.1.2.1) if

$$\begin{aligned}\left(\mathbf{m}^\top(\mathbf{V}\mathbf{h}+\mathbf{u})\right)^2\left(\mathbf{n}^\top\mathbf{V}\mathbf{n}\right) - 2\left(\mathbf{m}^\top\mathbf{V}\mathbf{n}\right)\left(\mathbf{m}^\top(\mathbf{V}\mathbf{h}+\mathbf{u})\mathbf{n}^\top(\mathbf{V}\mathbf{h}+\mathbf{u})\right)&\\ +\mathrm{g}(\mathbf{h})\left(\mathbf{m}^\top\mathbf{V}\mathbf{n}\right)^2 = 0&\end{aligned} \tag{F.4.10.1}$$

*Proof.* The point of contact for the normal passing through a point $\mathbf{h}$ is given by

$$\mathbf{q} = \mathbf{h} + \mu\mathbf{n} \tag{F.4.10.2}$$

From (F.3.2.1), the tangent at $\mathbf{q}$ satisfies

$$\mathbf{m}^\top(\mathbf{V}\mathbf{q}+\mathbf{u}) = 0 \tag{F.4.10.3}$$

Substituting (F.4.10.2) in (F.4.10.3),

$$\mathbf{m}^\top (\mathbf{V}(\mathbf{h} + \mu\mathbf{n}) + \mathbf{u}) = 0 \tag{F.4.10.4}$$

$$\implies \mu\mathbf{m}^\top \mathbf{V}\mathbf{n} = -\mathbf{m}^\top (\mathbf{V}\mathbf{h} + \mathbf{u}) \tag{F.4.10.5}$$

yielding

$$\mu = -\frac{\mathbf{m}^\top (\mathbf{V}\mathbf{h} + \mathbf{u})}{\mathbf{m}^\top \mathbf{V}\mathbf{n}}, \tag{F.4.10.6}$$

From (F.3.1.6),

$$\mu^2 \mathbf{n}^\top \mathbf{V}\mathbf{n} + 2\mu\mathbf{n}^\top (\mathbf{V}\mathbf{h} + \mathbf{u}) + \mathrm{g}(\mathbf{h}) = 0 \tag{F.4.10.7}$$

From (F.4.10.6), (F.4.10.7) can be expressed as

$$\left(-\frac{\mathbf{m}^\top (\mathbf{V}\mathbf{h} + \mathbf{u})}{\mathbf{m}^\top \mathbf{V}\mathbf{n}}\right)^2 \mathbf{n}^\top \mathbf{V}\mathbf{n} + 2\left(-\frac{\mathbf{m}^\top (\mathbf{V}\mathbf{h} + \mathbf{u})}{\mathbf{m}^\top \mathbf{V}\mathbf{n}}\right) \mathbf{n}^\top (\mathbf{V}\mathbf{h} + \mathbf{u}) + \mathrm{g}(\mathbf{h}) = 0 \tag{F.4.10.8}$$

yielding (F.4.10.1). □