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# MATRIX ANALYSIS

## Through Coordinate Geometry

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# Introduction

This book links high school coordinate geometry to linear algebra and matrix analysis through solved problems.





# Chapter 1

## Vectors

1. Draw a quadrilateral in the Cartesian plane, whose vertices are  $(-4,5)$ ,  $(0,7)$ ,  $(5,-5)$ ,  $(-4,-2)$ . Also, find its area.

**Solution:** See Fig. 1.1.

$$ar(\triangle ABC) = \frac{1}{2} \|(\mathbf{B} - \mathbf{A}) \times (\mathbf{B} - \mathbf{C})\| \quad (1.1)$$

$$= \frac{1}{2} \begin{vmatrix} 4 & 2 \\ -5 & 12 \end{vmatrix} = 29 \quad (1.2)$$

Similarly,

$$ar(\triangle ADC) = \frac{1}{2} \|(\mathbf{D} - \mathbf{A}) \times (\mathbf{D} - \mathbf{C})\| \quad (1.3)$$

$$= \frac{1}{2} \begin{vmatrix} 0 & -7 \\ -9 & 3 \end{vmatrix} = 31.5 \quad (1.4)$$

Thus,

$$ar(ABCD) = ar(\triangle ABC) + ar(\triangle ADC) = 60.5 \quad (1.5)$$

2. The base of an equilateral triangle with side  $2a$  lies along the y-axis such that the

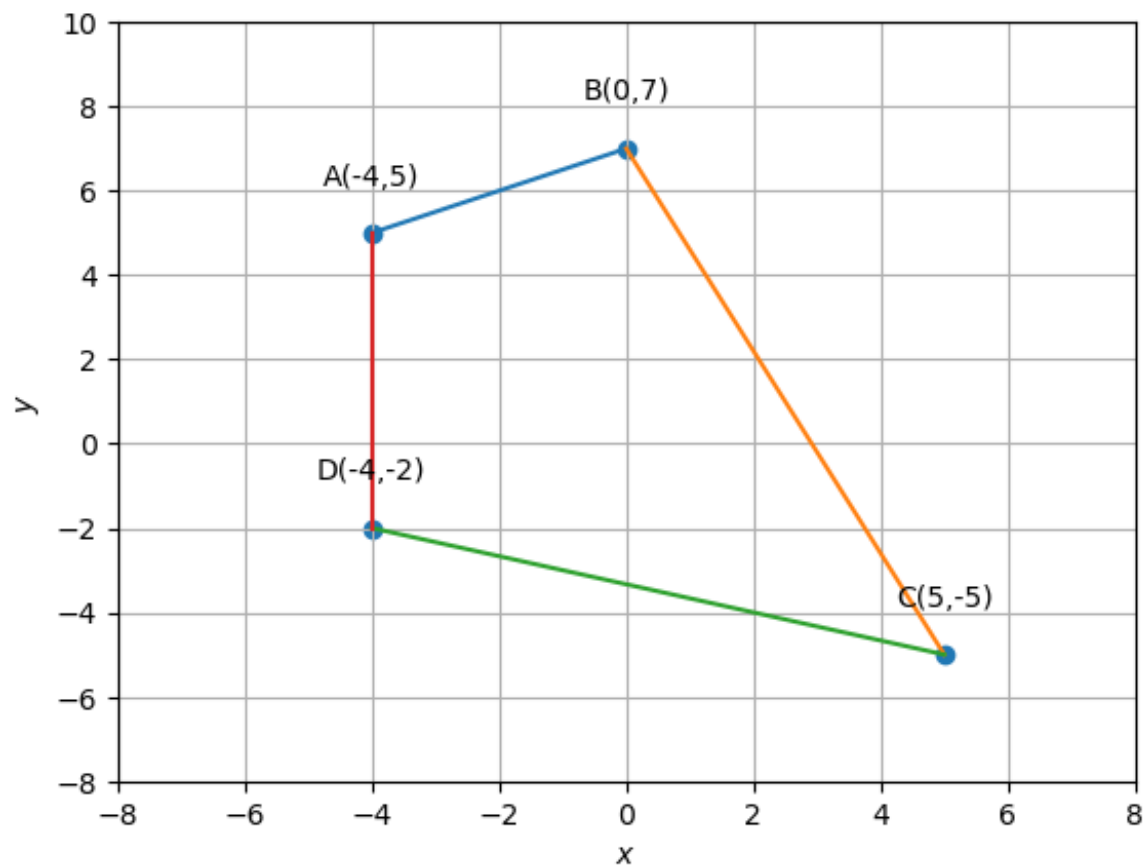


Figure 1.1:

mid-point of the base is at the origin. Find vertices of the triangle.

**Solution:** Let the base be  $BC$ . From the given information,

$$\mathbf{B} = a\mathbf{e}_2, \mathbf{C} = -a\mathbf{e}_2 \quad (1.6)$$

Since  $\mathbf{A}$  lies on the  $x$ -axis,

$$\mathbf{A} = k\mathbf{e}_1 \quad (1.7)$$

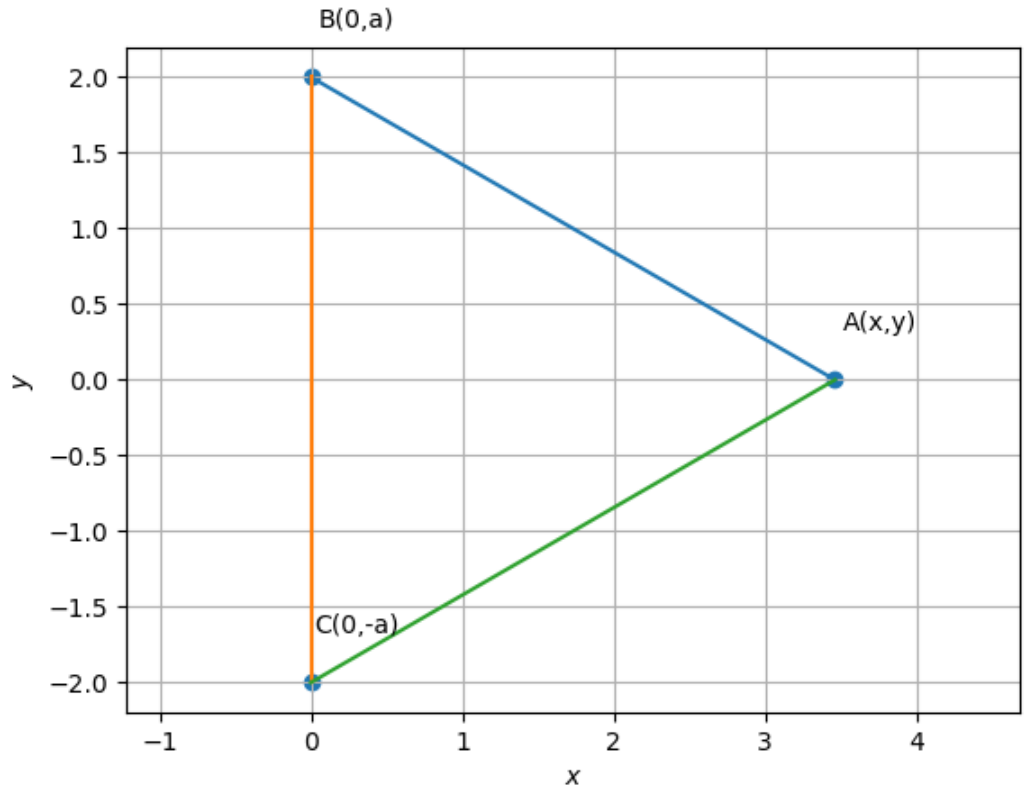


Figure 1.2:

and

$$\|\mathbf{A} - \mathbf{C}\|^2 = (2a)^2 \quad (1.8)$$

$$\Rightarrow \|\mathbf{A}\|^2 + \|\mathbf{C}\|^2 - 2\mathbf{A}^\top \mathbf{C} = 4a^2 \quad (1.9)$$

$$\Rightarrow k^2 + a^2 = 4a^2 \quad (1.10)$$

$$\text{or, } k = \pm a\sqrt{3} \quad (1.11)$$

Thus,

$$\mathbf{A} = \pm\sqrt{3}a\mathbf{e}_1 \quad (1.12)$$

Fig. 1.2 is plotted for  $a = 2$ .

3. Find a point on the x-axis, which is equidistant from the points  $\begin{pmatrix} 7 \\ 6 \end{pmatrix}$  and  $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ .

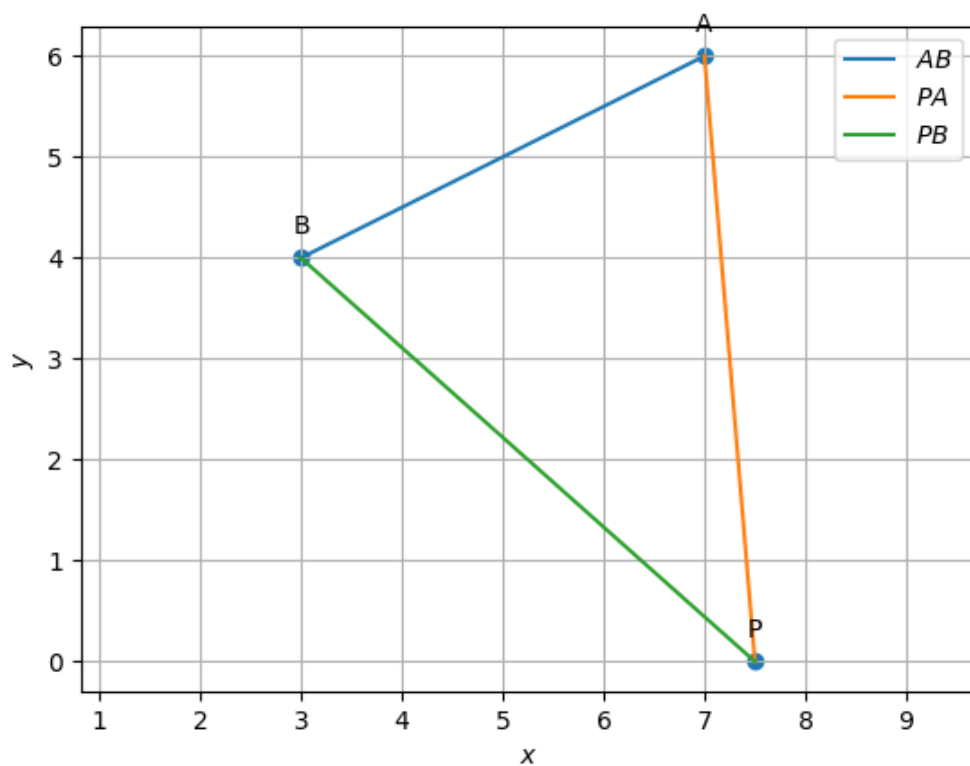


Figure 1.3:

**Solution:** From the given information

$$\|\mathbf{x} - \mathbf{A}\|^2 = \|\mathbf{x} - \mathbf{B}\|^2 \quad (1.13)$$

$$\implies (\mathbf{x} - \mathbf{A})^\top (\mathbf{x} - \mathbf{A}) = (\mathbf{x} - \mathbf{B})^\top (\mathbf{x} - \mathbf{B}) \quad (1.14)$$

$$\implies \|\mathbf{x}\|^2 - 2\mathbf{A}^\top \mathbf{x} + \|\mathbf{A}\|^2 = \|\mathbf{x}\|^2 - 2\mathbf{B}^\top \mathbf{x} + \|\mathbf{B}\|^2 \quad (1.15)$$

$$\text{or, } (\mathbf{A} - \mathbf{B})^\top \mathbf{x} = \frac{\|\mathbf{A}\|^2 - \|\mathbf{B}\|^2}{2} \quad (1.16)$$

Since  $\mathbf{x}$  lies on the  $x$ -axis,

$$\mathbf{x} = k\mathbf{e}_1 \quad (1.17)$$

which, upon substituting in (1.16) yields

$$k = \frac{15}{2} \quad (1.18)$$

4.

5. Find the slope of a line, which passes through the origin and the mid point of the line segment joining the points  $\mathbf{P}(0, -4)$  and  $\mathbf{B}(8, 0)$ .

**Solution:** The mid point of  $PB$  is

$$\mathbf{M} = \frac{1}{2}(\mathbf{P} + \mathbf{B}) = \begin{pmatrix} 4 \\ -2 \end{pmatrix} \quad (1.19)$$

The direction vector of line joining  $\mathbf{O}, \mathbf{M}$  is

$$\mathbf{m} = \mathbf{O} - \mathbf{M} = -\mathbf{M} \quad (1.20)$$

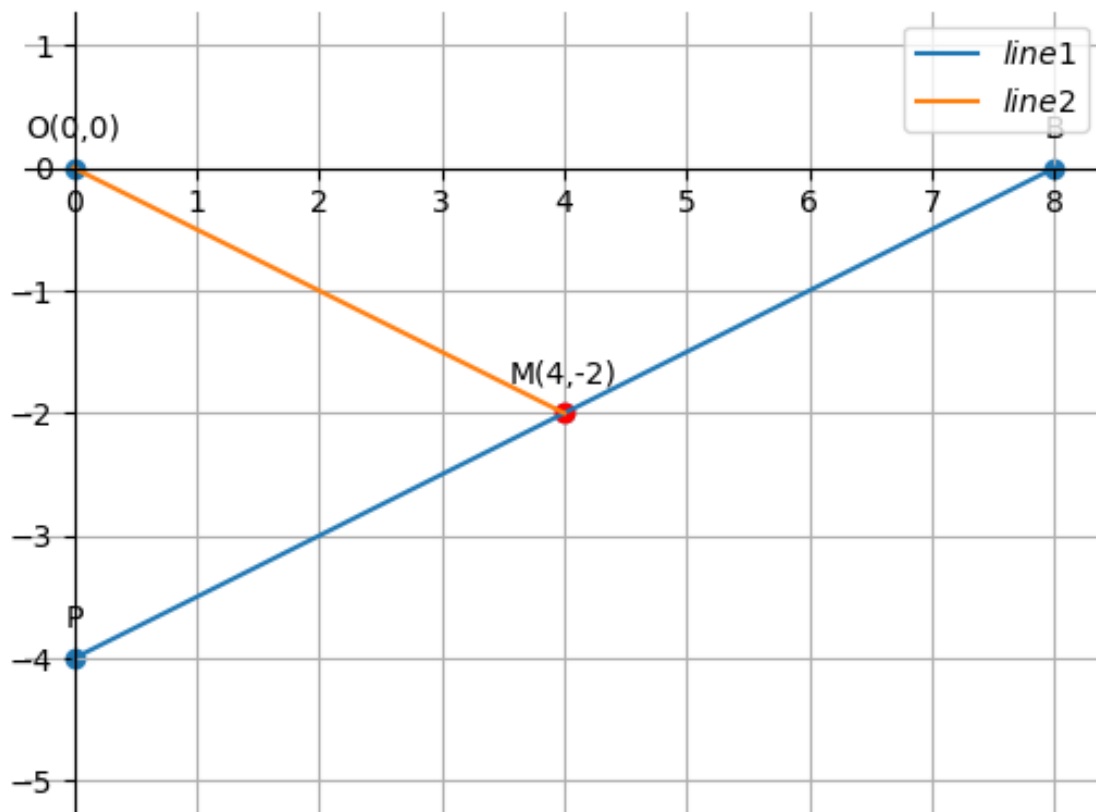


Figure 1.4:

which can be expressed as

$$\begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix} \quad (1.21)$$

Thus the slope is

$$m = -\frac{1}{2} \quad (1.22)$$

6. Without using the Baudhayana theorem, show that the points  $(4, 4)$ ,  $(3, 5)$  and  $(-1, -1)$  are the vertices of a right angled triangle.

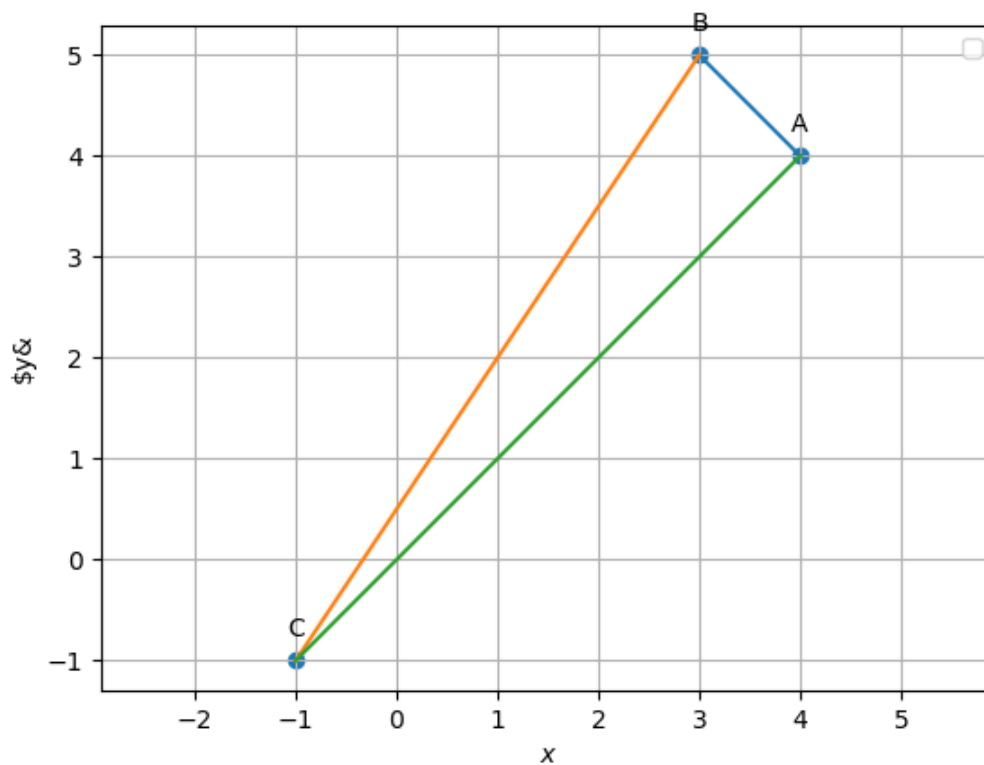


Figure 1.5:

$$\mathbf{C} - \mathbf{A} = \begin{pmatrix} -5 \\ -5 \end{pmatrix}, \quad (1.23)$$

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad (1.24)$$

$$\implies (\mathbf{C} - \mathbf{A})^\top (\mathbf{A} - \mathbf{B}) = 0 \quad (1.25)$$

Thus,  $AB \perp AC$ .

7. If three points  $(x, -1)$ ,  $(2, 1)$  and  $(4, 5)$  are collinear, find the value of  $x$ .

**Solution:** Let

$$\mathbf{A} = \begin{pmatrix} x \\ -1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}. \quad (1.26)$$

Then

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} x - 2 \\ -2 \end{pmatrix} \quad (1.27)$$

$$\mathbf{A} - \mathbf{C} = \begin{pmatrix} 4 - x \\ 6 \end{pmatrix} \quad (1.28)$$

Forming the collinearity matrix using (C.4),

$$\begin{pmatrix} x - 2 & -2 \\ 4 - x & 6 \end{pmatrix} \xleftrightarrow{R_1 = 3R_1 + R_2} \begin{pmatrix} 2x - 2 & 0 \\ 4 - x & 6 \end{pmatrix} \quad (1.29)$$



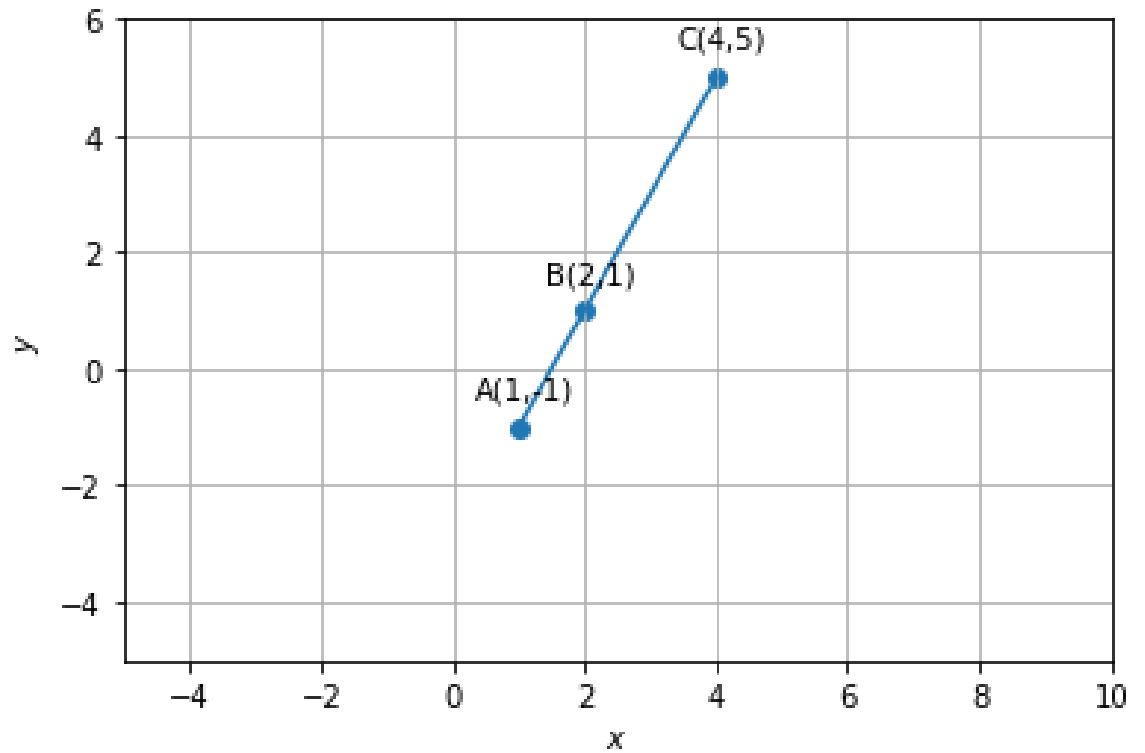


Figure 1.6:

If the rank of the matrix is 1, any one of the rows must be zero. So, making the first element in the above matrix 0,

$$x = 1 \quad (1.30)$$

8. Without using distance formula, show that points  $(-2, -1)$ ,  $(4, 0)$ ,  $(3, 3)$  and  $(-3, 2)$  are the vertices of a parallelogram.

**Solution:** See Fig. 1.7.

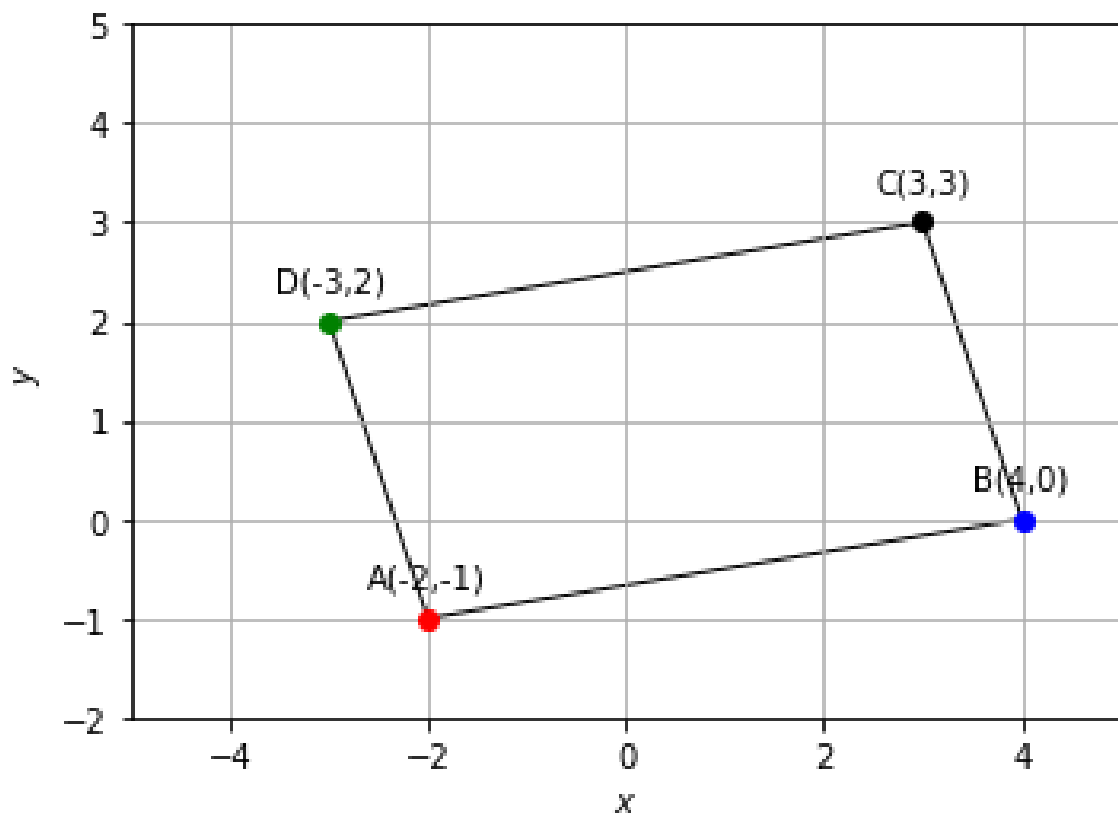


Figure 1.7:

$$\mathbf{A} = \begin{pmatrix} -2 \\ -1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} -3 \\ 2 \end{pmatrix} \quad (1.31)$$

and

$$\mathbf{P} = \mathbf{B} - \mathbf{A} = \begin{pmatrix} 6 \\ 1 \end{pmatrix} \quad (1.32)$$

$$\mathbf{Q} = \mathbf{C} - \mathbf{D} = \begin{pmatrix} 6 \\ 1 \end{pmatrix} \quad (1.33)$$

$$\mathbf{R} = \mathbf{A} - \mathbf{C} = \begin{pmatrix} 1 \\ -3 \end{pmatrix} \quad (1.34)$$

$$\mathbf{S} = \mathbf{A} - \mathbf{D} = \begin{pmatrix} 1 \\ -3 \end{pmatrix} \quad (1.35)$$

Since  $\mathbf{P} = \mathbf{Q}$  and  $\mathbf{R} = \mathbf{S}$ , from (A.38),  $ABCD$  is a parallelogram

9. Find the angle between x-axis and the line joining points (3,-1) and (4,-2)

**Solution:** See Fig. 1.8.

Let

$$\mathbf{P} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} 4 \\ -2 \end{pmatrix} \quad (1.36)$$

Then

$$\mathbf{C} = \mathbf{P} - \mathbf{Q} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad (1.37)$$

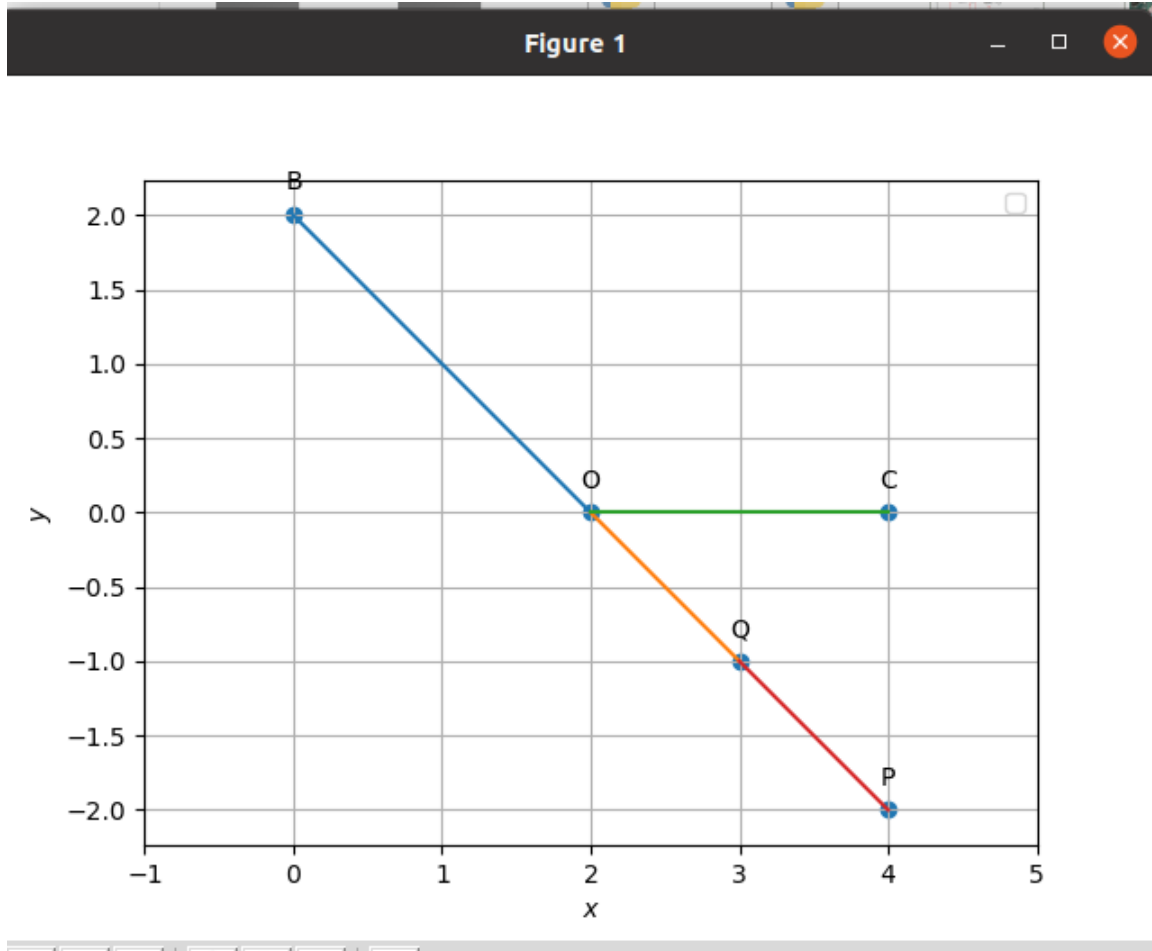


Figure 1.8:

The desired angle is given by

$$\cos \theta = \frac{\mathbf{C}^T \mathbf{e}_1}{\|\mathbf{C}\| \|\mathbf{e}_1\|} \quad (1.38)$$

$$= -\frac{1}{\sqrt{2}} \quad (1.39)$$

$$\Rightarrow \theta = 135^\circ \quad (1.40)$$

10. The slope of a line is double of the slope of another line. If tangent of the angle between them is  $1/3$ , find the slopes of the lines.

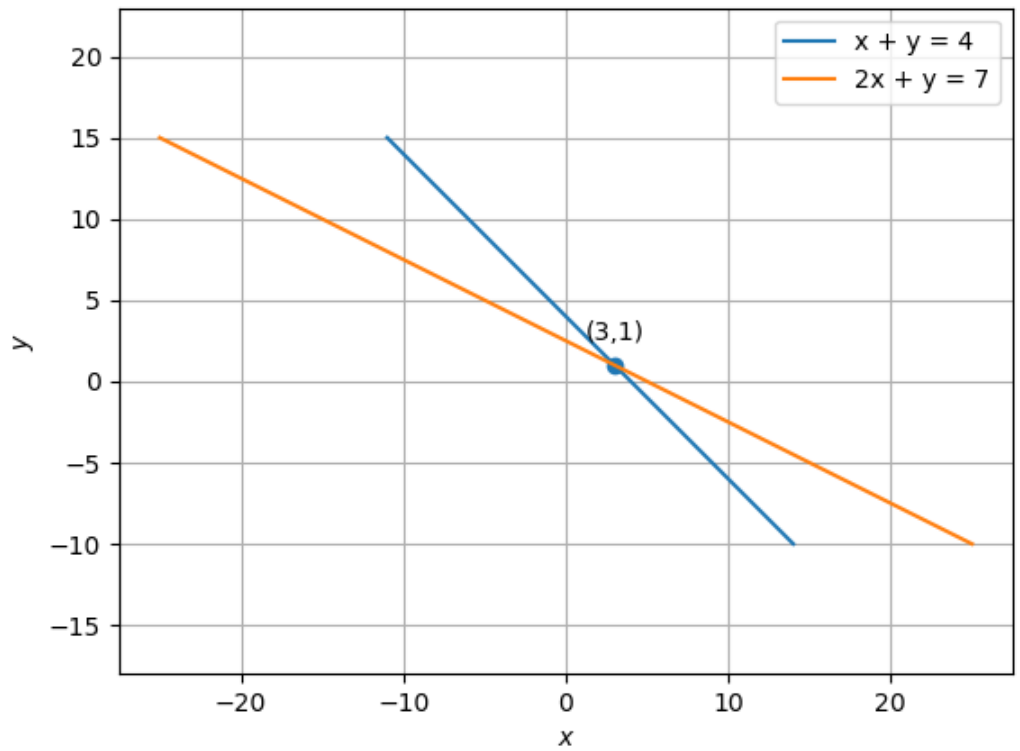


Figure 1.9:

**Solution:** The direction vector of a line is expressed as

$$\mathbf{m} = \begin{pmatrix} 1 \\ m \end{pmatrix} \quad (1.41)$$

where  $m$  is defined to be the slope of the line. If the angle between the lines be  $\theta$ ,

$$\tan \theta = \frac{1}{3} \implies \cos \theta = \frac{3}{\sqrt{10}} \quad (1.42)$$

The angle between two vectors is then expressed as

$$\frac{3}{\sqrt{10}} = \frac{\mathbf{m}_1^\top \mathbf{m}_2}{\|\mathbf{m}_1\| \|\mathbf{m}_2\|} \quad (1.43)$$

$$= \frac{\begin{pmatrix} 1 & m \end{pmatrix} \begin{pmatrix} 1 \\ 2m \end{pmatrix}}{\left\| \begin{pmatrix} 1 \\ m \end{pmatrix} \right\| \left\| \begin{pmatrix} 1 \\ 2m \end{pmatrix} \right\|} \quad (1.44)$$

$$= \frac{2m^2 + 1}{\sqrt{m^2 + 1} \sqrt{4m^2 + 1}} \quad (1.45)$$

$$\implies \frac{9}{10} = \frac{4m^4 + 4m^2 + 1}{4m^4 + 5m^2 + 1} \quad (1.46)$$

$$\text{or, } 4m^4 - 5m^2 + 1 = 0 \quad (1.47)$$

yielding

$$m = \pm \frac{1}{2}, \pm 1 \quad (1.48)$$

11. A line passes through  $(x_1, y_1)$  and  $(h, k)$ . If slope of the line is  $m$  show that

$$(k - y_1) = m(h - x_1).$$

**Solution:** Given

$$\mathbf{A} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} h \\ k \end{pmatrix} \quad (1.49)$$

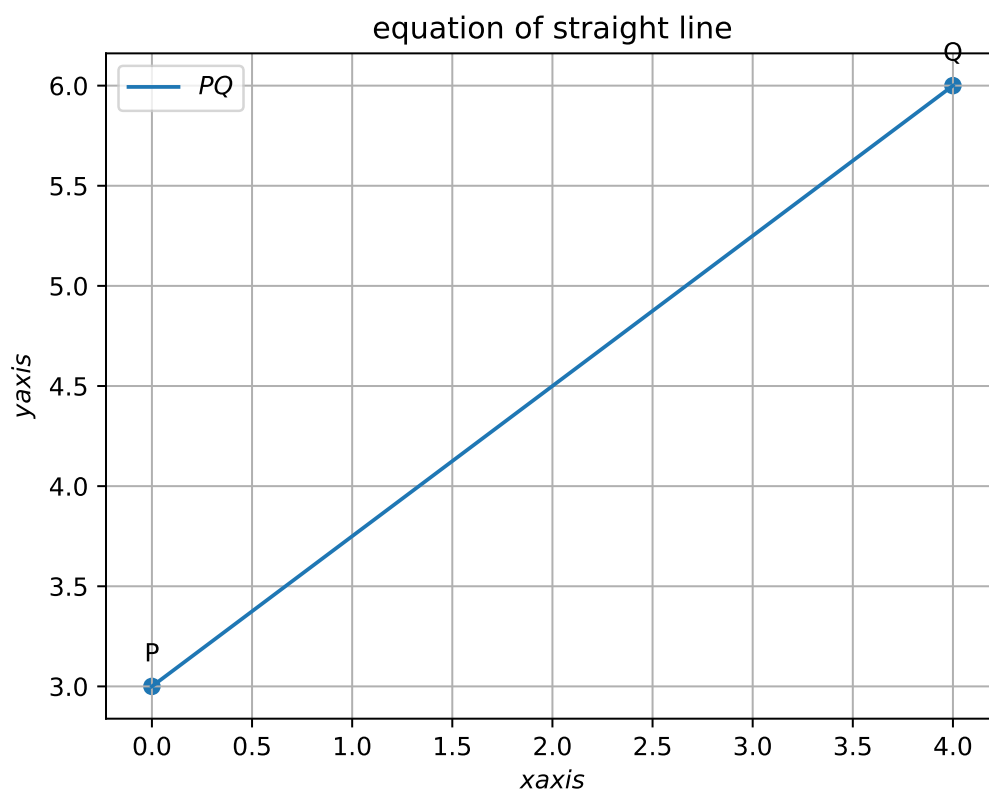


Figure 1.10:

The direction vector

$$\mathbf{m} = \mathbf{B} - \mathbf{A} \quad (1.50)$$

$$= \begin{pmatrix} h - x_1 \\ k - y_1 \end{pmatrix} \equiv \begin{pmatrix} 1 \\ \frac{k - y_1}{h - x_1} \end{pmatrix} \quad (1.51)$$

which yields the desired relation from (A.29).

12. If three points  $(h, 0)$ ,  $(a, b)$  and  $(0, k)$  lie on a line, show that

$$\frac{a}{h} + \frac{b}{k} = 1 \quad (1.52)$$

**Solution:** Let

$$\mathbf{A} = \begin{pmatrix} h \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} a \\ b \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 0 \\ k \end{pmatrix} \quad (1.53)$$

Forming the matrix in (C.4), we obtain, upon row reduction

$$\begin{pmatrix} h-a & -b \\ h & -k \end{pmatrix} \xleftrightarrow[\frac{h-a}{h}]{R_1} \begin{pmatrix} 1 & \frac{-b}{h-a} \\ h & -k \end{pmatrix} \quad (1.54)$$

$$\xleftrightarrow{R_2 \rightarrow R_2 - hR_1} \begin{pmatrix} 1 & \frac{-b}{h-a} \\ 0 & -k + \frac{bh}{h-a} \end{pmatrix} \quad (1.55)$$

For obtaining a rank 1 matrix,

$$-k + \frac{bh}{h-a} = 0 \quad (1.56)$$

$$\implies \frac{a}{h} + \frac{b}{k} = 1 \quad (1.57)$$

upon simplification.



## Chapter 2

# Quadrilaterals

### 2.1. Properties

1. The angles of quadrilateral are in the ratio 3:5:9:13. Find all the angles of the quadrilateral.
2. If diagonals of a parallelogram are equal then show that it is a rectangle.

**Solution:** See Fig. 2.1. From (A.38),

$$\mathbf{B} - \mathbf{A} = \mathbf{C} - \mathbf{D} \quad (2.1)$$

$$\implies \mathbf{B} - \mathbf{C} = \mathbf{A} - \mathbf{D} \quad (2.2)$$

Also, it is given that the diagonals of  $ABCD$  are equal. Hence,

$$\|\mathbf{C} - \mathbf{A}\|^2 = \|\mathbf{D} - \mathbf{B}\|^2 \quad (2.3)$$

$$\implies \|(\mathbf{C} - \mathbf{B}) + (\mathbf{B} - \mathbf{A})\|^2 = \|(\mathbf{D} - \mathbf{C}) + (\mathbf{C} - \mathbf{B})\|^2 \quad (2.4)$$

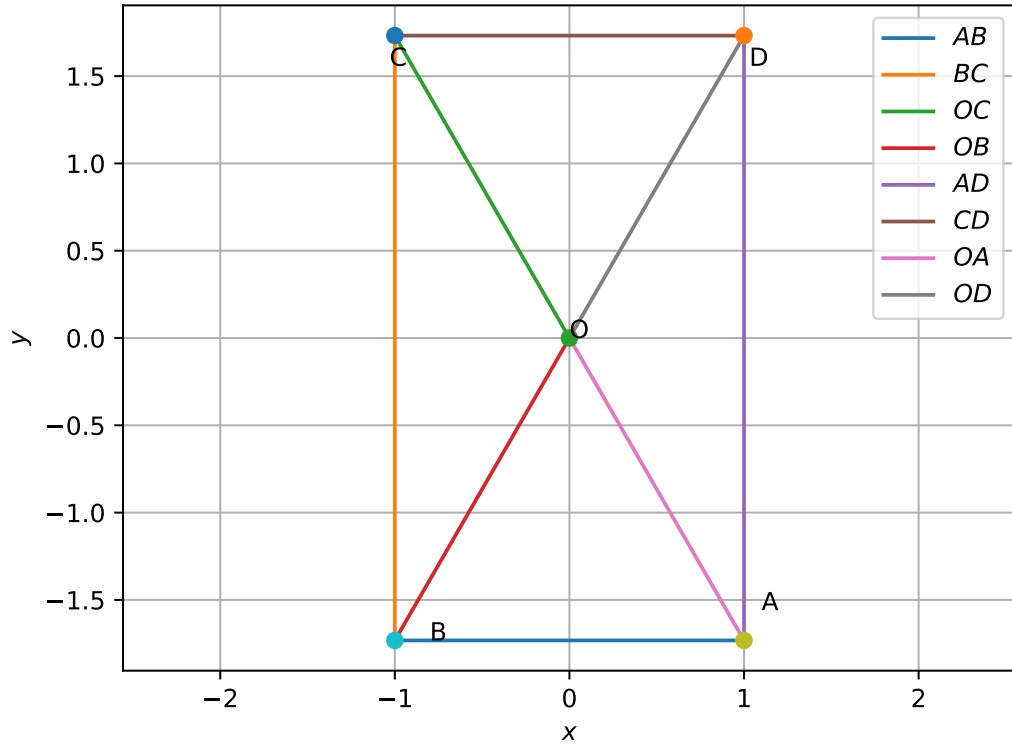


Figure 2.1:

which can be expressed as

$$\begin{aligned} & \| \mathbf{C} - \mathbf{B} \|^2 + \| \mathbf{B} - \mathbf{A} \|^2 + 2(\mathbf{C} - \mathbf{B})^\top (\mathbf{B} - \mathbf{A}) \\ &= \| \mathbf{D} - \mathbf{C} \|^2 + \| \mathbf{C} - \mathbf{B} \|^2 + 2(\mathbf{D} - \mathbf{C})^\top (\mathbf{C} - \mathbf{B}) \end{aligned} \quad (2.5)$$

which, can be simplified to obtain

$$(\mathbf{C} - \mathbf{B})^\top (\mathbf{B} - \mathbf{A}) = (\mathbf{D} - \mathbf{C})^\top (\mathbf{C} - \mathbf{B}) \quad (2.6)$$

since

$$\|\mathbf{D} - \mathbf{C}\| = \|\mathbf{B} - \mathbf{A}\| \quad (2.7)$$

yielding

$$(\mathbf{A} - \mathbf{B})^\top (\mathbf{B} - \mathbf{C}) = \mathbf{0} \quad (2.8)$$

from (2.1).

3. Show that if the diagonals of a quadrilateral bisect each other at right angles, then it is a rhombus.

**Solution:** See Fig. 2.2. From the given information,

$$\frac{\mathbf{B} + \mathbf{D}}{2} = \frac{\mathbf{A} + \mathbf{C}}{2} \quad (2.9)$$

$$(\mathbf{B} - \mathbf{D})^\top (\mathbf{A} - \mathbf{C}) = 0 \quad (2.10)$$

From (2.9),

$$\mathbf{B} - \mathbf{A} = \mathbf{C} - \mathbf{D} \quad (2.11)$$

which, from (A.38), is the definition of a parallelogram. Further, substituting

$$\mathbf{B} - \mathbf{D} = (\mathbf{B} - \mathbf{A}) + (\mathbf{A} - \mathbf{D}) \quad (2.12)$$

$$\mathbf{A} - \mathbf{C} = (\mathbf{A} - \mathbf{B}) + (\mathbf{B} - \mathbf{C}) \quad (2.13)$$

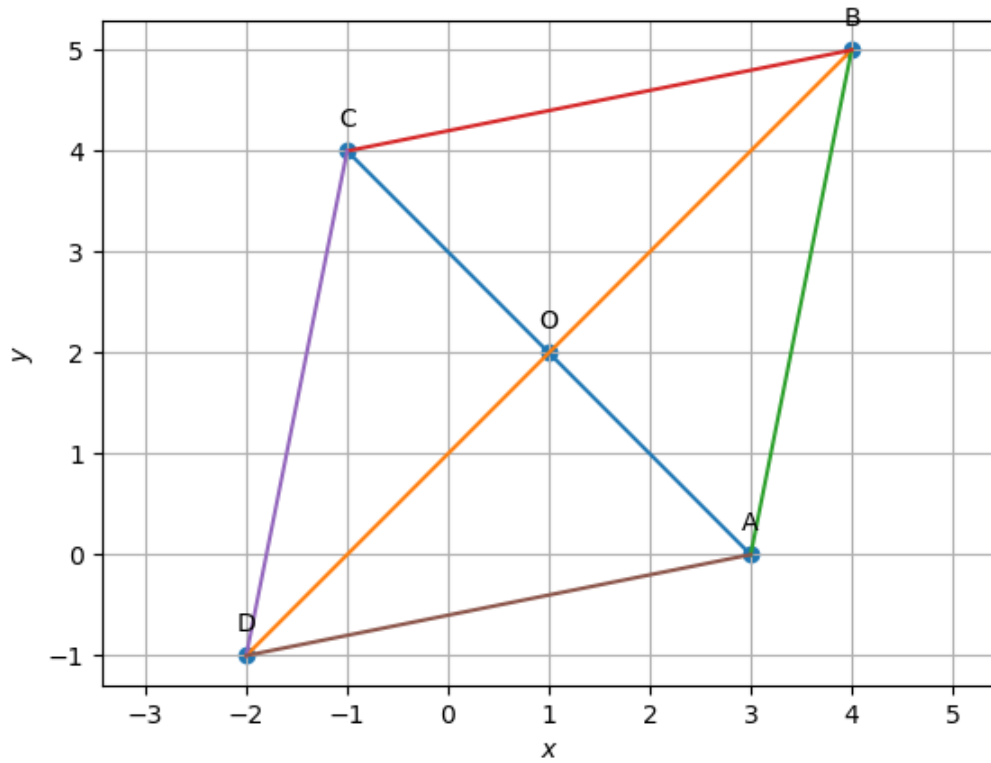


Figure 2.2: Rhombus

in (2.10),

$$\begin{aligned}
 & [(\mathbf{B} - \mathbf{A}) + (\mathbf{A} - \mathbf{D})]^\top [(\mathbf{A} - \mathbf{B}) + (\mathbf{B} - \mathbf{C})] = 0 \\
 & \implies -\|\mathbf{B} - \mathbf{A}\|^2 + (\mathbf{B} - \mathbf{A})^\top (\mathbf{B} - \mathbf{C}) + \\
 & \quad (\mathbf{A} - \mathbf{D})^\top (\mathbf{A} - \mathbf{B}) + (\mathbf{A} - \mathbf{D})^\top (\mathbf{B} - \mathbf{C}) = 0 \quad (2.14)
 \end{aligned}$$

From (2.11),

$$\mathbf{B} - \mathbf{C} = \mathbf{A} - \mathbf{D} \quad (2.15)$$

$$\implies (\mathbf{B} - \mathbf{A})^\top (\mathbf{B} - \mathbf{C}) + (\mathbf{A} - \mathbf{D})^\top (\mathbf{A} - \mathbf{B}) = \mathbf{0} \quad (2.16)$$

and

$$(\mathbf{A} - \mathbf{D})^\top (\mathbf{B} - \mathbf{C}) = \|\mathbf{B} - \mathbf{C}\|^2 \quad (2.17)$$

Substituting from

(2.16) and (2.17) in (2.14),

$$\|\mathbf{A} - \mathbf{B}\|^2 = \|\mathbf{B} - \mathbf{C}\|^2 \quad (2.18)$$

which means that the adjacent sides of the parallelogram are equal. Thus, the quadrilateral is a rhombus

4. Show that the diagonals of a square are equal and bisect each other at right angles.

**Solution:** This is obvious from Problems (2) and (3).

- 5.

6. Diagonal AC of a parallelogram ABCD bisects  $\angle A$  in Fig (2.3). Show that

- (a) it bisects  $\angle C$  also
- (b)  $ABCD$  is a rhombus

**Solution:**

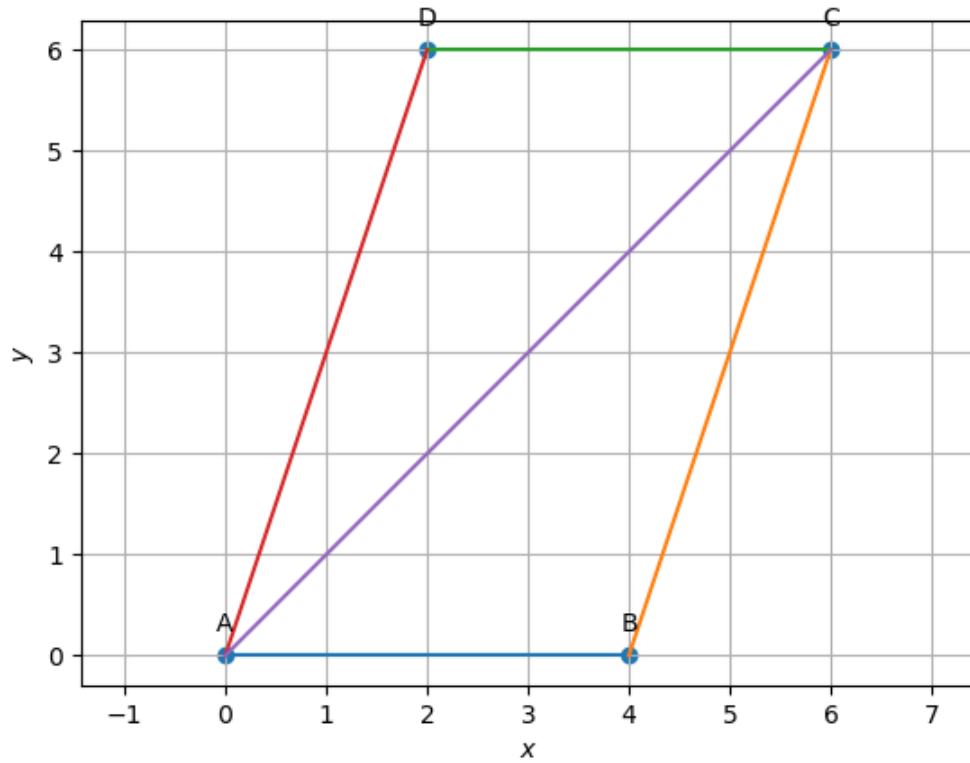


Figure 2.3:

(a) From (A.25),

$$\angle BAC = \angle DAC \quad (2.19)$$

$$\Rightarrow \frac{(\mathbf{A} - \mathbf{B})^T (\mathbf{A} - \mathbf{C})}{\|\mathbf{A} - \mathbf{B}\| \|\mathbf{A} - \mathbf{C}\|} = \frac{(\mathbf{A} - \mathbf{D})^T (\mathbf{A} - \mathbf{C})}{\|\mathbf{A} - \mathbf{D}\| \|\mathbf{A} - \mathbf{C}\|} \quad (2.20)$$

Also,

$$\cos \angle ACD = \frac{(\mathbf{C} - \mathbf{D})^T (\mathbf{C} - \mathbf{A})}{\|\mathbf{C} - \mathbf{D}\| \|\mathbf{C} - \mathbf{A}\|} \quad (2.21)$$

From Appendix A.38,

$$\mathbf{B} - \mathbf{A} = \mathbf{C} - \mathbf{D} \quad (2.22)$$

$$\Rightarrow \frac{(\mathbf{C} - \mathbf{D})^T(\mathbf{C} - \mathbf{A})}{\|\mathbf{C} - \mathbf{D}\| \|\mathbf{C} - \mathbf{A}\|} = \frac{(\mathbf{B} - \mathbf{A})^T(\mathbf{C} - \mathbf{A})}{\|\mathbf{B} - \mathbf{A}\| \|\mathbf{C} - \mathbf{A}\|} \quad (2.23)$$

upon substituting in (2.21). Thus, from (2.21) and (2.19),

$$\angle BAC = \angle DAC = \angle ACD \quad (2.24)$$

Similarly, it can be shown that

$$\angle ACD = \angle ACB \quad (2.25)$$

(b)

7.  $ABCD$  is a rhombus. Show that the diagonal  $AC$  bisects angle  $A$  as well as angle  $C$  and diagonal  $BD$  bisects angle  $B$  as well as angle  $D$ .

**Solution:** For the rhombus in Fig. 2.4,

$$\begin{aligned} \|\mathbf{A} - \mathbf{B}\| &= \|\mathbf{A} - \mathbf{D}\| \\ \mathbf{A} - \mathbf{B} &= \mathbf{D} - \mathbf{C} \end{aligned} \quad (2.26)$$

From (A.25),

$$\begin{aligned} \cos \angle BAC &= \frac{(\mathbf{A} - \mathbf{B})^T(\mathbf{A} - \mathbf{C})}{\|\mathbf{A} - \mathbf{B}\| \|\mathbf{A} - \mathbf{C}\|} \\ \cos \angle DAC &= \frac{(\mathbf{C} - \mathbf{D})^T(\mathbf{C} - \mathbf{A})}{\|\mathbf{C} - \mathbf{D}\| \|\mathbf{C} - \mathbf{A}\|} \end{aligned} \quad (2.27)$$

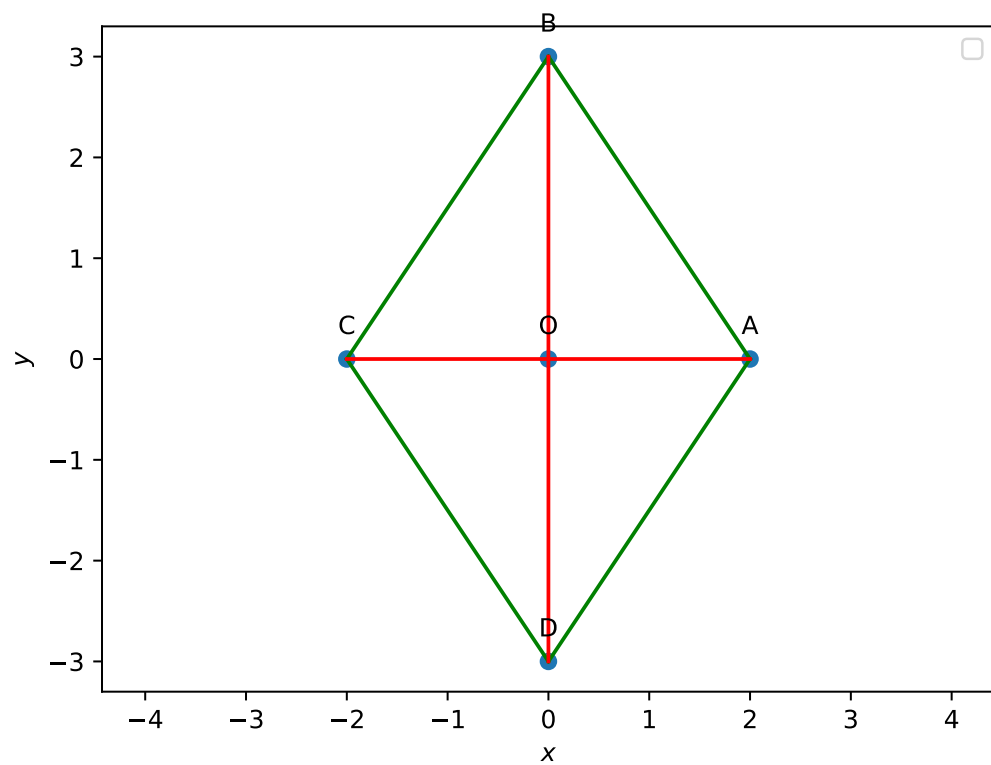


Figure 2.4:



From (2.26) and (2.27), we obtain

$$\cos \angle BAC = \cos \angle DAC \quad (2.28)$$

Thus,  $AC$  bisects  $\angle A$ . Similarly, the remaining results can be proved.

8.

9. In parallelogram  $ABCD$ , two points  $\mathbf{P}$  and  $\mathbf{Q}$  are taken on diagonal  $BD$  such that  $DP = BQ$ . Show that

$$(a) \triangle APD \cong \triangle CQB$$

$$(b) AP = CQ$$

$$(c) \triangle AQB \cong \triangle CPD$$

$$(d) AQ = CP$$

$$(e) APCQ \text{ is a parallelogram}$$

**Solution:** See Fig. 2.5.

From (A.25) and the given information,

$$\mathbf{A} - \mathbf{B} = \mathbf{D} - \mathbf{C} \quad (2.29)$$

$$\implies \mathbf{A} - \mathbf{D} = \mathbf{B} - \mathbf{C} \quad (2.30)$$

$$\mathbf{B} - \mathbf{Q} = \mathbf{P} - \mathbf{D} \quad (\text{given}) \quad (2.31)$$

From (2.29) and (2.31)

$$\mathbf{A} - \mathbf{P} = \mathbf{Q} - \mathbf{C} \quad (2.32)$$

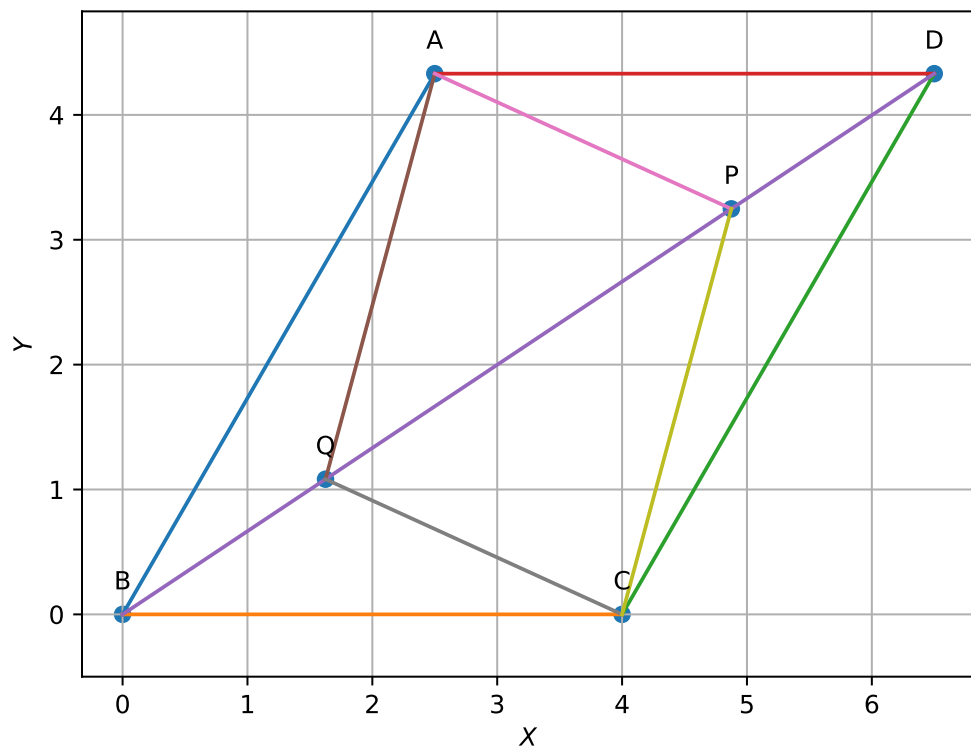


Figure 2.5:

(a) From (2.29), (2.31) and (2.32) taking the norms of the respective sides,

$$\triangle APD \cong \triangle CQB \quad (2.33)$$

(b) From (2.32), taking the norm,

$$AP = CQ \quad (2.34)$$

(c) From (2.29), (2.31) and (2.32) taking the norms of the respective sides,

$$\triangle AQB \cong \triangle CPD \quad (2.35)$$

(d) From (2.32),

$$AQ = CP \quad (2.36)$$

10.  $ABCD$  is a parallelogram and  $AP$  and  $CQ$  are perpendiculars from vertices  $\mathbf{A}$  and  $\mathbf{C}$  on diagonal  $BD$ . Show that

$$(a) \triangle APB \cong \triangle CQD$$

$$(b) AP = CQ$$

**Solution:** From Fig. 2.6, and (A.25),

$$\begin{aligned} \cos \angle ABD &= \frac{(\mathbf{A} - \mathbf{B})^T (\mathbf{D} - \mathbf{B})}{\|\mathbf{A} - \mathbf{B}\| \|\mathbf{D} - \mathbf{B}\|} \\ \cos \angle CDB &= \frac{(\mathbf{C} - \mathbf{D})^T (\mathbf{B} - \mathbf{D})}{\|\mathbf{C} - \mathbf{D}\| \|\mathbf{B} - \mathbf{D}\|} \end{aligned} \quad (2.37)$$

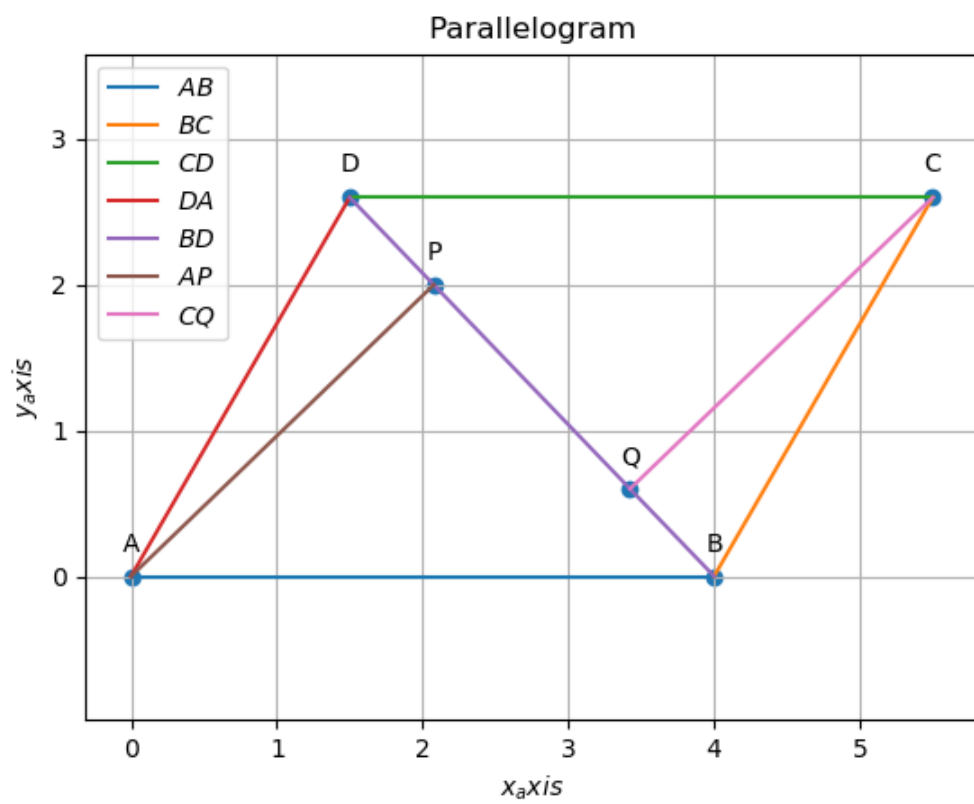


Figure 2.6:

From Appendix A.38,

$$\mathbf{A} - \mathbf{B} = \mathbf{D} - \mathbf{C} \quad (2.38)$$

Substituting in (2.37),

$$\cos \angle ABD = \cos \angle CDB \quad (2.39)$$

Using SAS congruence, 10a is proved. 10b follows from 10a.

11. In  $\triangle ABC$  and  $\triangle DEF$ ,  $AB = DE$ ,  $AB \parallel DE$ ,  $BC = EF$  and  $BC \parallel EF$ . Vertices **A**, **B** and **C** are joined to vertices **D**, **E** and **F** respectively (see Figure 2.7 ). Show that

- (a) quadrilateral  $ABED$  is a parallelogram
- (b) quadrilateral  $BEFC$  is a parallelogram
- (c)  $AD \parallel CF$  and  $AD = CF$
- (d) quadrilateral  $ACFD$  is a parallelogram
- (e)  $AC = DF$
- (f)  $\triangle ABC \cong \triangle DEF$ .

**Solution:** From the given information

$$\mathbf{A} - \mathbf{B} = \mathbf{D} - \mathbf{E} \quad (2.40)$$

$$\mathbf{B} - \mathbf{E} = \mathbf{C} - \mathbf{F} \quad (2.41)$$

- (a) From Appendix A.38, (2.40) defines the parallelogram  $ABED$ .
- (b) Similarly, (2.41) defines the parallelogram  $BEFC$ .

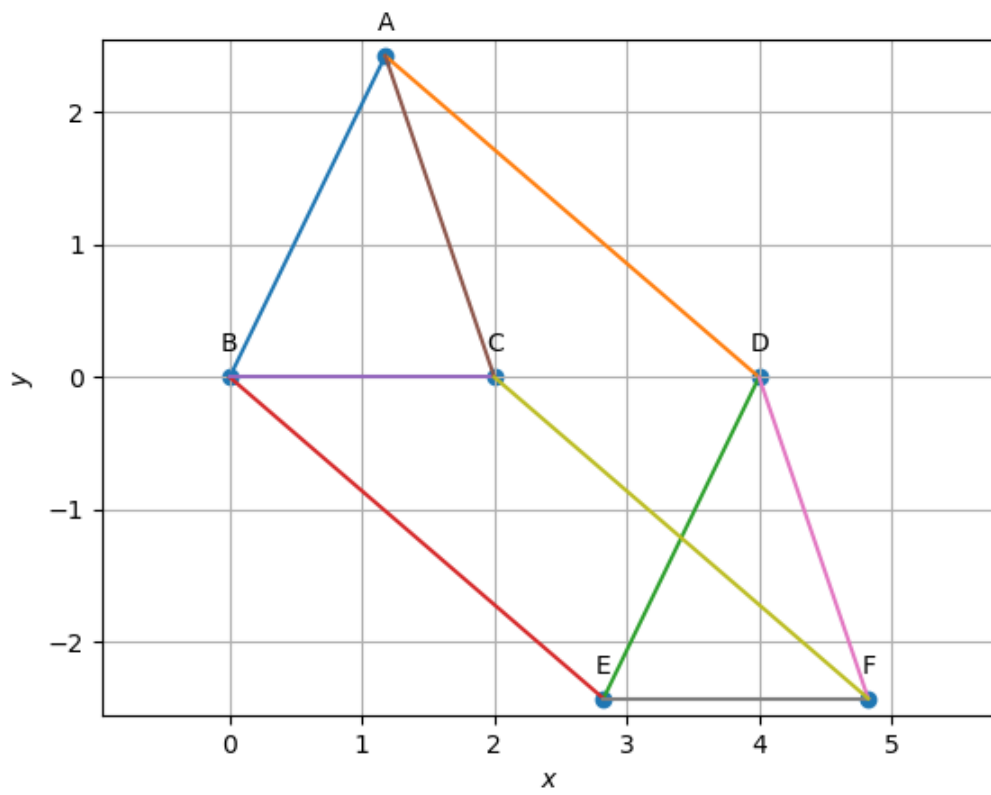


Figure 2.7:

(c) From (2.40) and (2.41),

$$\mathbf{A} - \mathbf{D} = \mathbf{C} - \mathbf{F} \quad (2.42)$$

which yields 11c.

(d) (2.42) implies that  $ACFD$  is a parallelogram.

(e) (2.42) implies  $AC = DF$ .

(f) Obvious from the fact the  $ABCD$ ,  $BEFC$  and  $ACFD$  are parallelograms.

12.  $ABCD$  is trapezium in which  $AB \parallel CD$  and  $AD = BC$ . Show that,

(a)  $\angle A = \angle B$

(b)  $\angle C = \angle D$

(c) Diagonal  $AC$  = Diagonal  $BD$

(d)  $\triangle ABC = \triangle BAD$

## 2.2. Mid Point Theorem

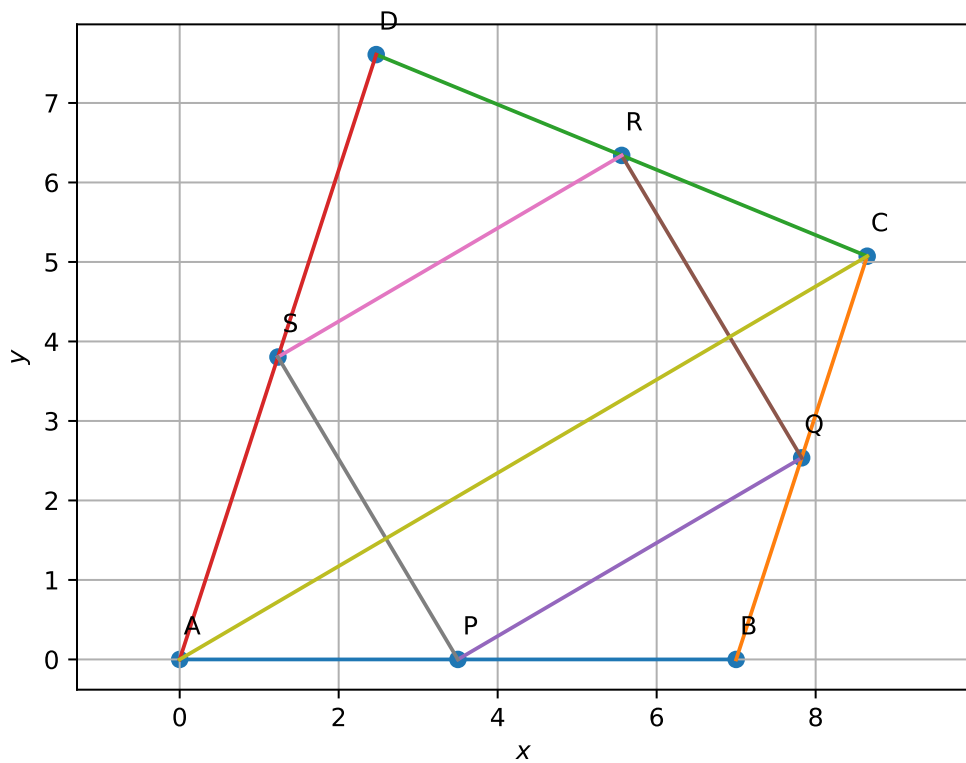


Figure 2.8:

1. ABCD is a quadrilateral in which **P**, **Q**, **R** and **S** are mid-points of the sides AB, BC, CD and DA (see Fig 2.8). AC is a diagonal.

Show that

(a)  $SR \parallel AC$  and  $SR = \frac{1}{2}AC$

(b)  $PQ = SR$

(c) PQRS is a parallelogram.



**Solution:** Using (A.35),

$$\begin{aligned}\mathbf{P} &= \frac{\mathbf{A} + \mathbf{B}}{2} \\ \mathbf{Q} &= \frac{\mathbf{C} + \mathbf{B}}{2} \\ \mathbf{R} &= \frac{\mathbf{C} + \mathbf{D}}{2} \\ \mathbf{S} &= \frac{\mathbf{D} + \mathbf{A}}{2}\end{aligned}\tag{2.43}$$

(a) Consequently,

$$\mathbf{R} - \mathbf{S} = \frac{\mathbf{C} - \mathbf{A}}{2}\tag{2.44}$$

$$\implies SR \parallel AC\tag{2.45}$$

Also,

$$\|\mathbf{R} - \mathbf{S}\| = \frac{\|\mathbf{C} - \mathbf{A}\|}{2}\tag{2.46}$$

$$\implies SR = \frac{1}{2}AC\tag{2.47}$$

(b) From (2.43),

$$\mathbf{R} - \mathbf{S} = \mathbf{Q} - \mathbf{P}\tag{2.48}$$

which means that  $PQRS$  is a parallelogram and  $PQ = SR$ .

2.

3.  $ABCD$  is a rectangle and  $\mathbf{P}$ ,  $\mathbf{Q}$ ,  $\mathbf{R}$  and  $\mathbf{S}$  are mid-points of the sides  $AB$ ,  $BC$ ,  $CD$  and  $DA$  respectively. Show that the quadrilateral  $PQRS$  is a rhombus.

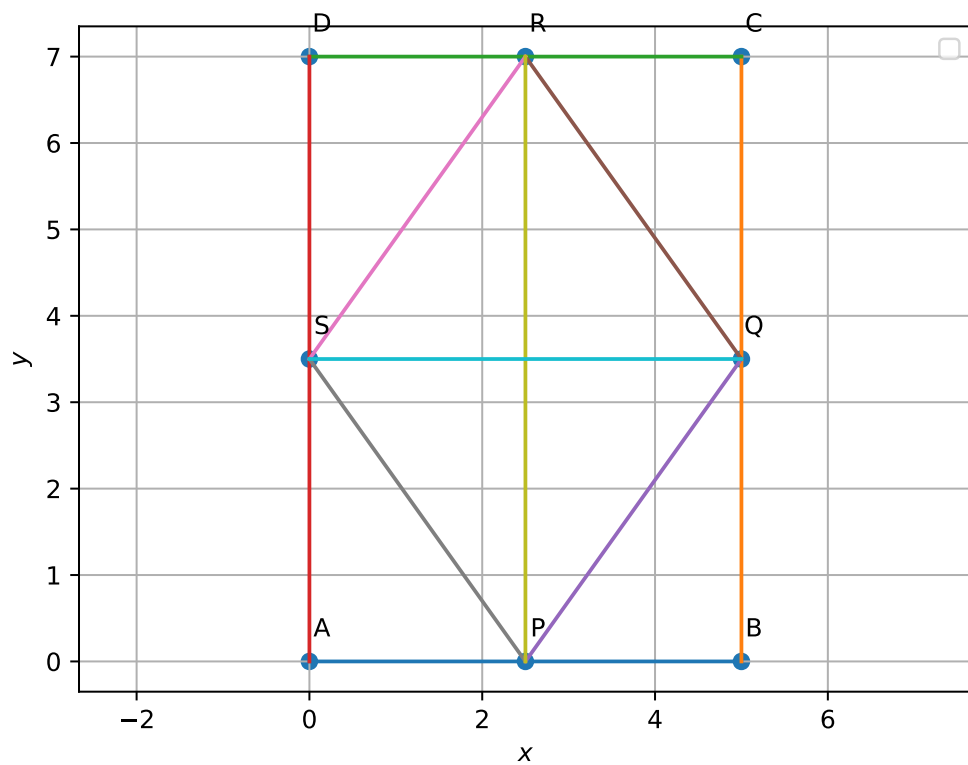


Figure 2.9:

**Solution:** From Problem 1, it is obvious that  $PQRS$  is a parallelogram. Further, from (2.43),

$$(\mathbf{P} - \mathbf{R})^\top (\mathbf{S} - \mathbf{Q}) = (\mathbf{A} + \mathbf{B} - \mathbf{C} - \mathbf{D})^\top (\mathbf{A} + \mathbf{D} - \mathbf{B} - \mathbf{C}) \quad (2.49)$$

$$= \mathbf{0} \quad (2.50)$$

since

$$(\mathbf{A} - \mathbf{D})^\top (\mathbf{A} - \mathbf{B}) = \mathbf{0} \quad (2.51)$$

$$\|\mathbf{A} - \mathbf{D}\| = \|\mathbf{A} - \mathbf{B}\| \quad (2.52)$$

as  $ABCD$  is a rectangle. Thus, the diagonals  $PR$  and  $SQ$  bisect each other proving that  $PQRS$  is a rhombus.

4.

5. In a parallelogram  $ABCD$ ,  $\mathbf{E}$  and  $\mathbf{F}$  are the mid-points of sides  $AB$  and  $CD$  respectively (see Fig. 2.10) Show that the line segments  $AF$  and  $EC$  trisect the diagonal  $BD$ .

*Proof.* From the given information,

$$\mathbf{E} = \frac{\mathbf{A} + \mathbf{B}}{2} \quad (2.53)$$

$$\mathbf{F} = \frac{\mathbf{C} + \mathbf{D}}{2} \quad (2.54)$$

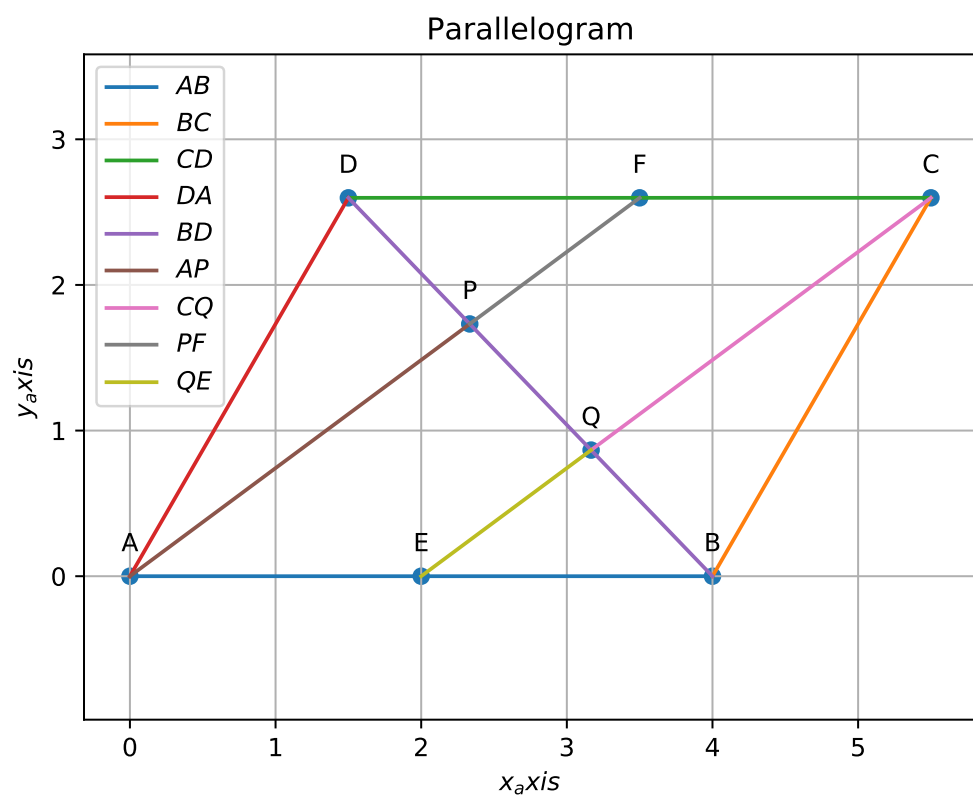


Figure 2.10:

Hence,

$$\mathbf{E} - \mathbf{C} = \frac{\mathbf{A} - \mathbf{C} + \mathbf{B} - \mathbf{C}}{2} \quad (2.55)$$

$$\mathbf{A} - \mathbf{F} = \frac{\mathbf{A} - \mathbf{C} + \mathbf{A} - \mathbf{D}}{2} \quad (2.56)$$

Since  $ABCD$  is a parallelogram,

$$\mathbf{B} - \mathbf{C} = \mathbf{A} - \mathbf{D} \quad (2.57)$$

$$\implies \mathbf{E} - \mathbf{C} = \mathbf{A} - \mathbf{F} \quad (2.58)$$

Thus,  $AF \parallel EC$ . From Appendix A.1.29, using the fact that  $\mathbf{F}$  is the mid point of  $CD$ , we conclude that  $\mathbf{P}$  is the mid point of  $DQ$ . Similarly, it can be shown that  $\mathbf{Q}$  is the mid point of  $BP$ .  $\square$

6.

7.  $ABC$  is a triangle right angled at  $\mathbf{C}$ . A line through the mid-point  $\mathbf{M}$  of hypotenuse  $AB$  and parallel to  $BC$  intersects  $AC$  at  $D$  (see Fig. 2.11). Show that

(a)  $D$  is the mid-point of  $AC$

(b)  $MD \perp AC$

(c)  $CM = MA = \frac{1}{2}AB$

**Solution:**

(a) Trivial from Appendix A.1.29.

(b) Since  $ABC$  is right angled at  $C$ ,

$$(\mathbf{C} - \mathbf{A})^\top (\mathbf{C} - \mathbf{B}) = 0 \quad (2.59)$$

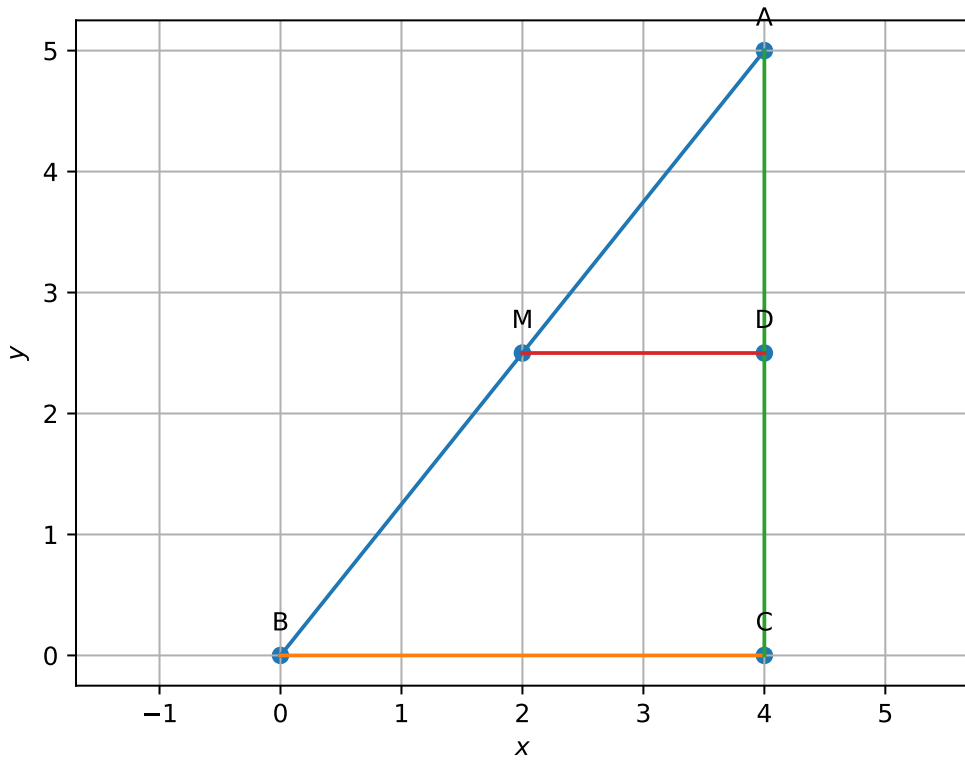


Figure 2.11:

Given that  $MD$  is parallel to  $BC$ , so

$$(\mathbf{C} - \mathbf{B}) = \lambda(\mathbf{M} - \mathbf{D}) \quad (2.60)$$

Substituting (2.60) in (2.59) and dividing by  $\lambda$ , we get

$$(\mathbf{C} - \mathbf{A})^\top (\mathbf{M} - \mathbf{D}) = 0 \quad (2.61)$$

From (2.61) it can be concluded that  $MD \perp AC$ .

(c) Since

$$\|\mathbf{C} - \mathbf{M}\|^2 - \|\mathbf{A} - \mathbf{M}\|^2 = \|\mathbf{C}\|^2 - \|\mathbf{A}\|^2 - 2(\mathbf{C} - \mathbf{A})^\top \mathbf{M} \quad (2.62)$$

$$= (\mathbf{C} - \mathbf{A})^\top (\mathbf{C} + \mathbf{A} - 2\mathbf{M}) \quad (2.63)$$

$$= (\mathbf{C} - \mathbf{A})^\top (\mathbf{C} - \mathbf{B}) = \mathbf{0} \quad (2.64)$$

upon substituting from Property 7a and (2.59). Thus,  $CM = AM$ .





## Chapter 3

# Areas

### 3.1. Parallelograms

1. In the Figure 3.1,  $ABCD$  is a parallelogram,  $AE \perp DC$  and  $CF \perp AD$ . If  $AB = 16cm$ ,  $AE = 8cm$ , and  $CF = 10cm$ , find  $AD$ .
2. If  $\mathbf{E}$ ,  $\mathbf{F}$ ,  $\mathbf{G}$  and  $\mathbf{H}$  are respectively the mid-points of the sides of a parallelogram  $ABCD$ , show that

$$ar(EFGH) = \frac{1}{2}ar(ABCD) \quad (3.1)$$

*Proof.* From Problem 1,  $EFGH$  is also a parallelogram and

$$\mathbf{E} - \mathbf{F} = \frac{\mathbf{A} - \mathbf{C}}{2} \quad (3.2)$$

$$\mathbf{E} - \mathbf{H} = \frac{\mathbf{A} - \mathbf{D}}{2} \quad (3.3)$$

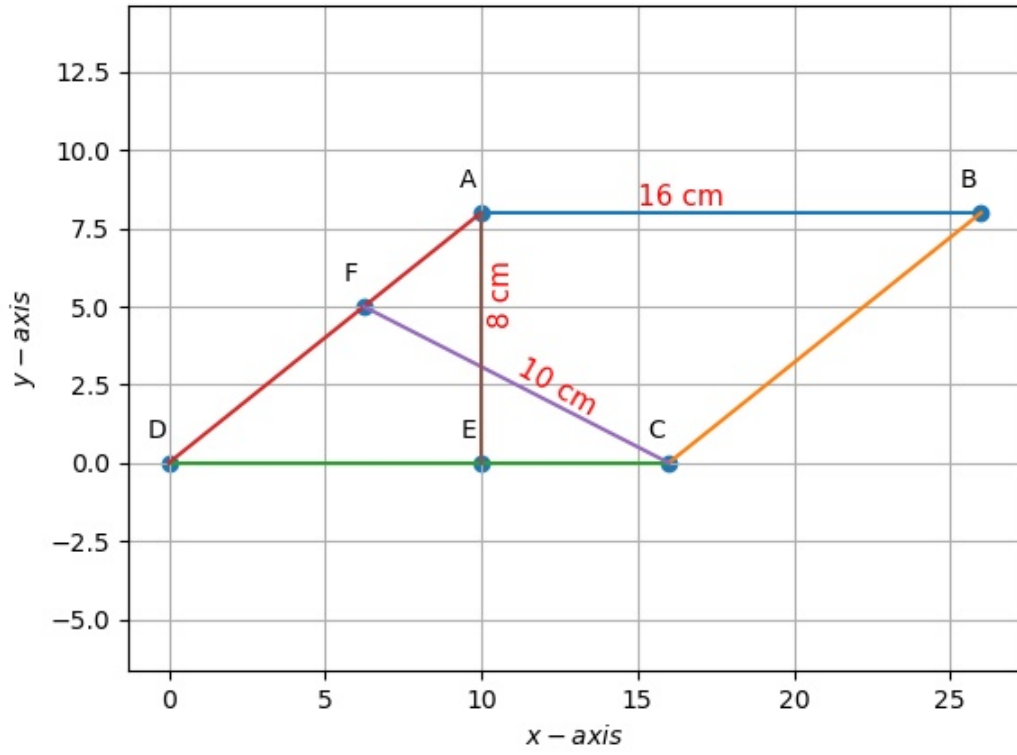


Figure 3.1:

Thus, the area off  $EFGH$  is obtained from (A.39) as

$$\|(\mathbf{E} - \mathbf{F}) \times (\mathbf{E} - \mathbf{H})\| = \frac{1}{4} \|(\mathbf{A} - \mathbf{C}) \times (\mathbf{B} - \mathbf{D})\| \quad (3.4)$$

From Appendix A.38,

$$\mathbf{D} = \mathbf{C} - \mathbf{B} + \mathbf{A} \quad (3.5)$$

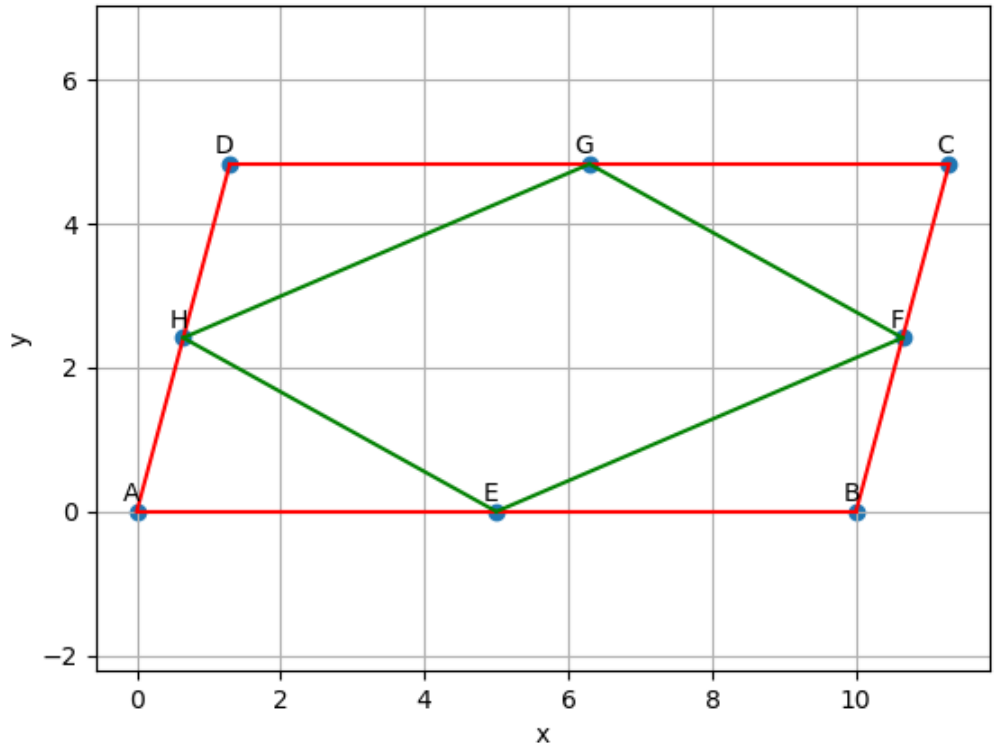


Figure 3.2:

which,

$$(\mathbf{A} - \mathbf{C}) \times (\mathbf{B} - \mathbf{D}) = (\mathbf{A} - \mathbf{C}) \times (2\mathbf{B} - \mathbf{C} - \mathbf{A}) \quad (3.6)$$

$$= 2(\mathbf{A} \times \mathbf{B} + \mathbf{B} \times \mathbf{C} + \mathbf{C} \times \mathbf{A}) \quad (3.7)$$

Substituting (3.7) in (3.4) yields

$$\|(\mathbf{E} - \mathbf{F}) \times (\mathbf{E} - \mathbf{H})\| = \frac{1}{2} \|\mathbf{A} \times \mathbf{B} + \mathbf{B} \times \mathbf{C} + \mathbf{C} \times \mathbf{A}\| \quad (3.8)$$

The area of  $ABCD$  is

$$\|(\mathbf{A} - \mathbf{B}) \times (\mathbf{A} - \mathbf{D})\| = \|\mathbf{A} \times \mathbf{B} + \mathbf{B} \times \mathbf{C} + \mathbf{C} \times \mathbf{A}\| \quad (3.9)$$

upon substituting from Appendix A.38 and simplifying. From (3.8) and (3.9) we obtain (3.1).  $\square$

3.

4. For a given Parallelogram  $ABCD$ , show that for any point  $\mathbf{P}$  inside the parallelogram,

$$(a) \quad Ar(APD) + Ar(PBC) = \frac{1}{2}Ar(ABCD)$$

$$(b) \quad Ar(APD) + Ar(PBC) = Ar(APB) + Ar(PCD)$$

5. In Fig.1,  $PQRS$  and  $ABRS$  are parallelograms and  $\mathbf{X}$  is any point on side  $BR$ . Show that

$$(a) \quad ar(PQRS) = ar(ABRS)$$

$$(b) \quad ar(AXS) = \frac{1}{2}ar(PQRS)$$

*Proof.* (a) From Appendix A.38,

$$\mathbf{A} - \mathbf{B} = \mathbf{S} - \mathbf{R} = \mathbf{P} - \mathbf{Q} \quad (3.10)$$

and from Appendix A.1.26, using (3.10), we obtain Property 5a.

(b) Using section formula, let

$$\mathbf{X} = \frac{\mathbf{R} + k\mathbf{B}}{1 + k}. \quad (3.11)$$

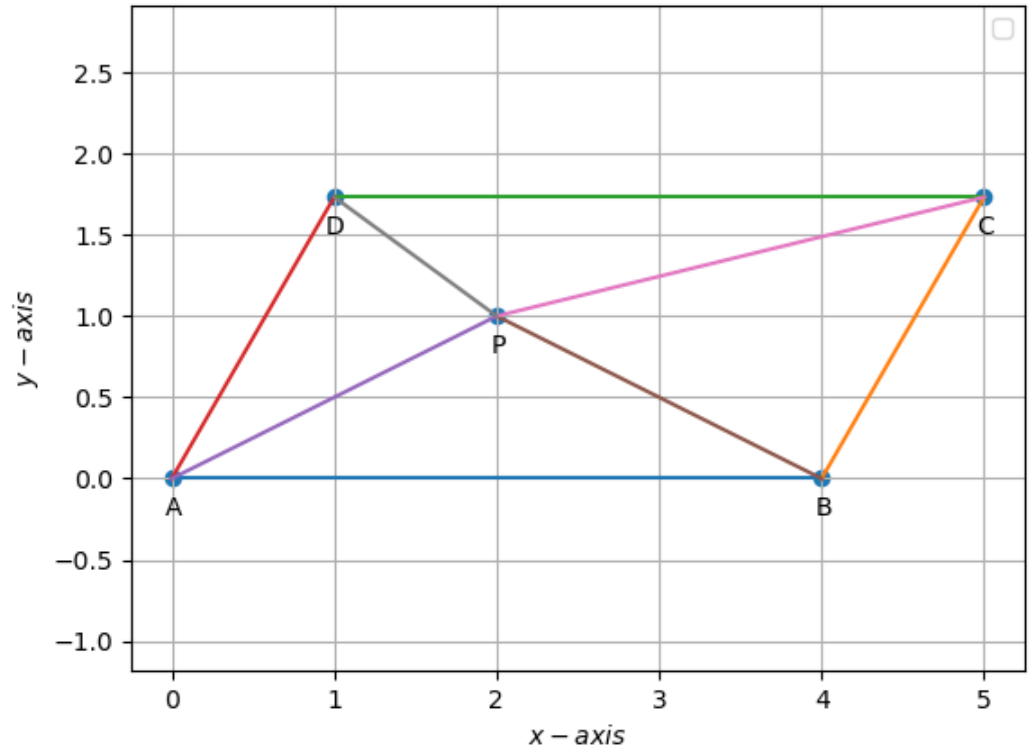


Figure 3.3:

Then,

$$ar(AXS) = \frac{1}{2} \|\mathbf{S} \times \mathbf{X} + \mathbf{X} \times \mathbf{A} + \mathbf{A} \times \mathbf{S}\| \quad (3.12)$$

$$= \frac{1}{2} \left\| \frac{\mathbf{S} \times \mathbf{R} + k\mathbf{S} \times \mathbf{B} + \mathbf{R} \times \mathbf{A} + k\mathbf{B} \times \mathbf{A}}{k+1} + \mathbf{A} \times \mathbf{S} \right\| \quad (3.13)$$

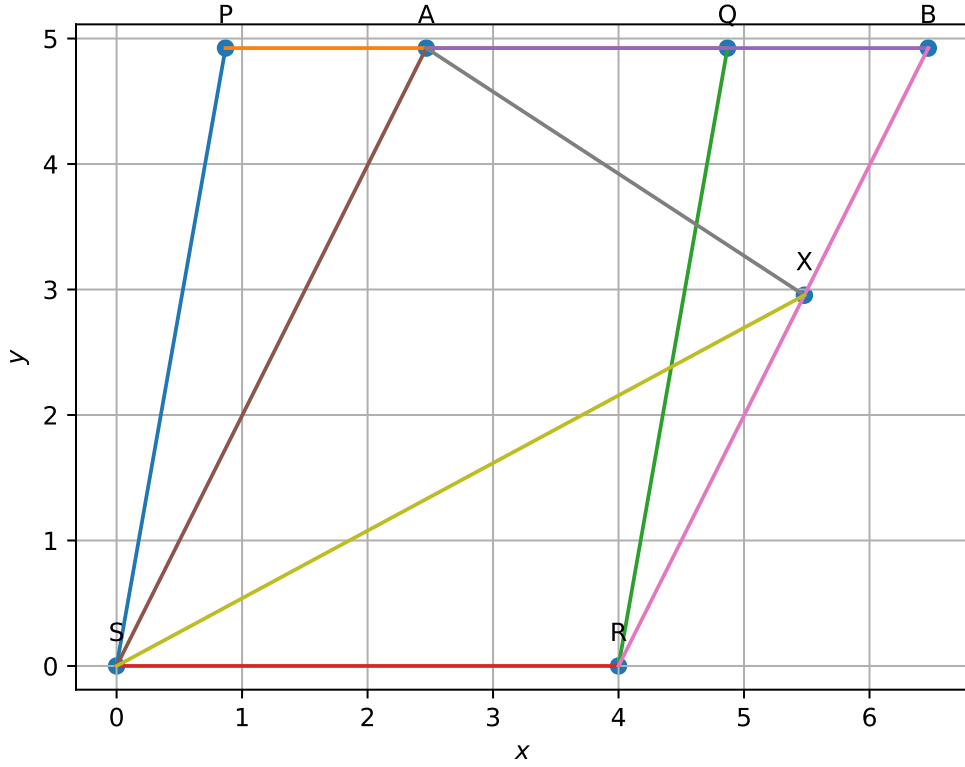


Figure 3.4:

Substituting for  $\mathbf{B}$  from (3.10) in the above,

$$ar(AXS) = \frac{1}{2} \left\| \frac{\mathbf{S} \times \mathbf{R} + \mathbf{R} \times \mathbf{A} + k(\mathbf{S} - \mathbf{A}) \times (\mathbf{A} - \mathbf{S} + \mathbf{R})}{k+1} + \mathbf{A} \times \mathbf{S} \right\| \quad (3.14)$$

$$= \frac{1}{2} \left\| \frac{\mathbf{S} \times \mathbf{R} + \mathbf{R} \times \mathbf{A} + k(\mathbf{S} - \mathbf{A}) \times \mathbf{R}}{k+1} + \mathbf{A} \times \mathbf{S} \right\| \quad (3.15)$$

$$= \frac{1}{2} \|\mathbf{S} \times \mathbf{R} + \mathbf{R} \times \mathbf{A} + \mathbf{A} \times \mathbf{S}\| \quad (3.16)$$

$$= \frac{1}{2} ar(ABRS) \quad (3.17)$$

□

## 3.2. Triangles

1. In the Figure 3.5, **E** is any point on median  $AD$  of a  $\triangle ABC$ . Show that

$$ar(ABE) = ar(ACE). \quad (3.18)$$

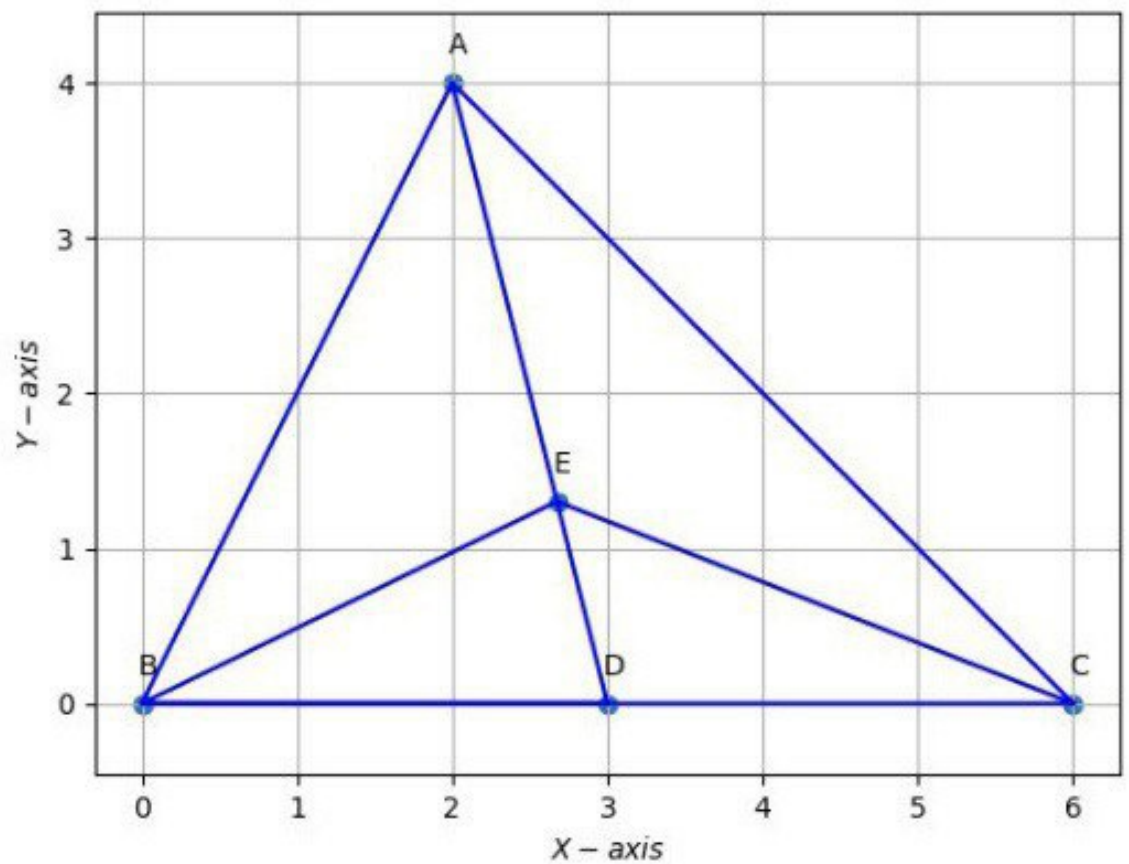


Figure 3.5:

*Proof.* From (A.7)

$$ar(BDE) = \frac{1}{2} \|\mathbf{B} \times \mathbf{D} + \mathbf{D} \times \mathbf{E} + \mathbf{E} \times \mathbf{B}\| \quad (3.19)$$

$$= \frac{1}{2} \left\| \mathbf{B} \times \left( \frac{\mathbf{B} + \mathbf{C}}{2} \right) + \left( \frac{\mathbf{B} + \mathbf{C}}{2} \right) \times \mathbf{E} + \mathbf{E} \times \mathbf{B} \right\| \quad (3.20)$$

$$= \frac{1}{4} \|\mathbf{B} \times \mathbf{C} + \mathbf{C} \times \mathbf{E} + \mathbf{E} \times \mathbf{B}\| \quad (3.21)$$

after simplification. Similarly, it can be shown that

$$ar(EDC) = \frac{1}{4} \|\mathbf{B} \times \mathbf{C} + \mathbf{C} \times \mathbf{E} + \mathbf{E} \times \mathbf{B}\| \quad (3.22)$$

$$= ar(BDE) \quad (3.23)$$

The same approach can be used to show that

$$ar(ADB) = ar(ADC) \quad (3.24)$$

Subtracting (3.23) from (3.24) yields (3.18)

□

2. In  $\triangle ABC$ ,  $\mathbf{E}$  is the mid-point of median  $AD$ . Show that

$$ar(\triangle BED) = \frac{1}{4} ar(\triangle ABC) \quad (3.25)$$

*Proof.* From Problem 2,

$$ar(\triangle BED) = \frac{1}{4} \|\mathbf{B} \times \mathbf{C} + \mathbf{C} \times \mathbf{E} + \mathbf{E} \times \mathbf{B}\| \quad (3.26)$$



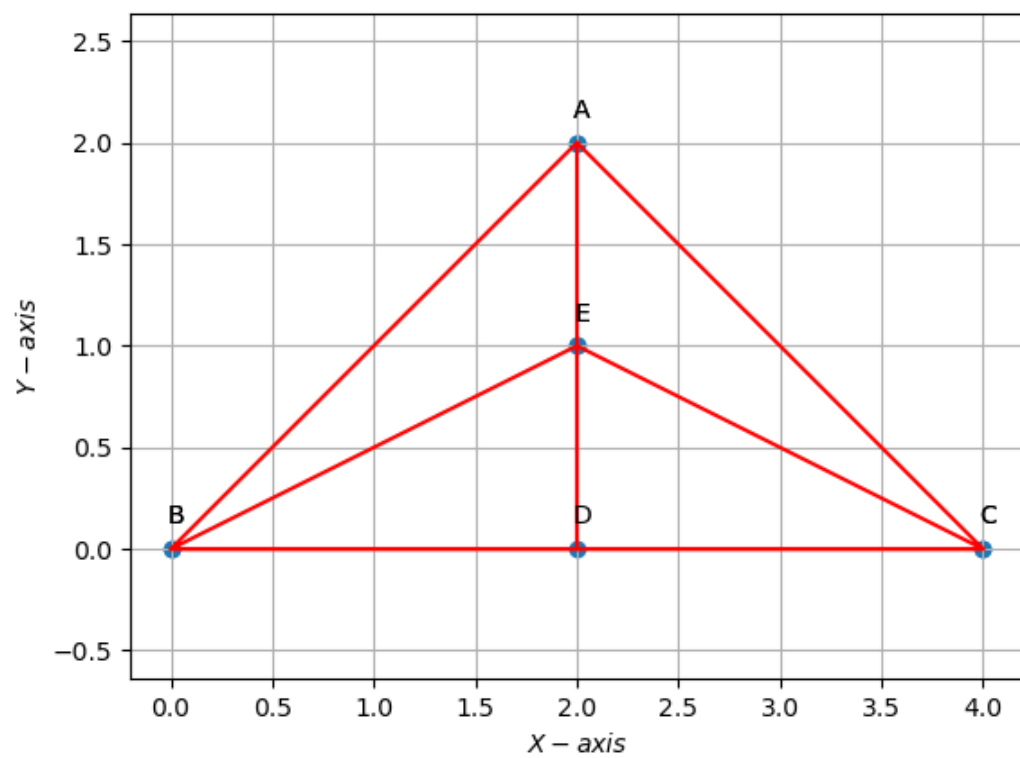


Figure 3.6:

Since

$$\mathbf{E} = \frac{\mathbf{A} + \mathbf{D}}{2} \quad (3.27)$$

$$= \frac{2\mathbf{A} + \mathbf{B} + \mathbf{C}}{4}, \quad (3.28)$$

substituting the above in (3.26) yields

$$ar(\triangle BED) = \frac{1}{4} \left\| \mathbf{B} \times \mathbf{C} + \mathbf{C} \times \frac{2\mathbf{A} + \mathbf{B} + \mathbf{C}}{4} + \frac{2\mathbf{A} + \mathbf{B} + \mathbf{C}}{4} \times \mathbf{B} \right\| \quad (3.29)$$

$$= \frac{1}{8} \left\| \mathbf{A} \times \mathbf{B} + \mathbf{B} \times \mathbf{C} + \mathbf{C} \times \mathbf{A} \right\| \quad (3.30)$$

resulting in (3.25).  $\square$

3. Show that the diagonals of a parallelogram divide it into four triangles of equal area.

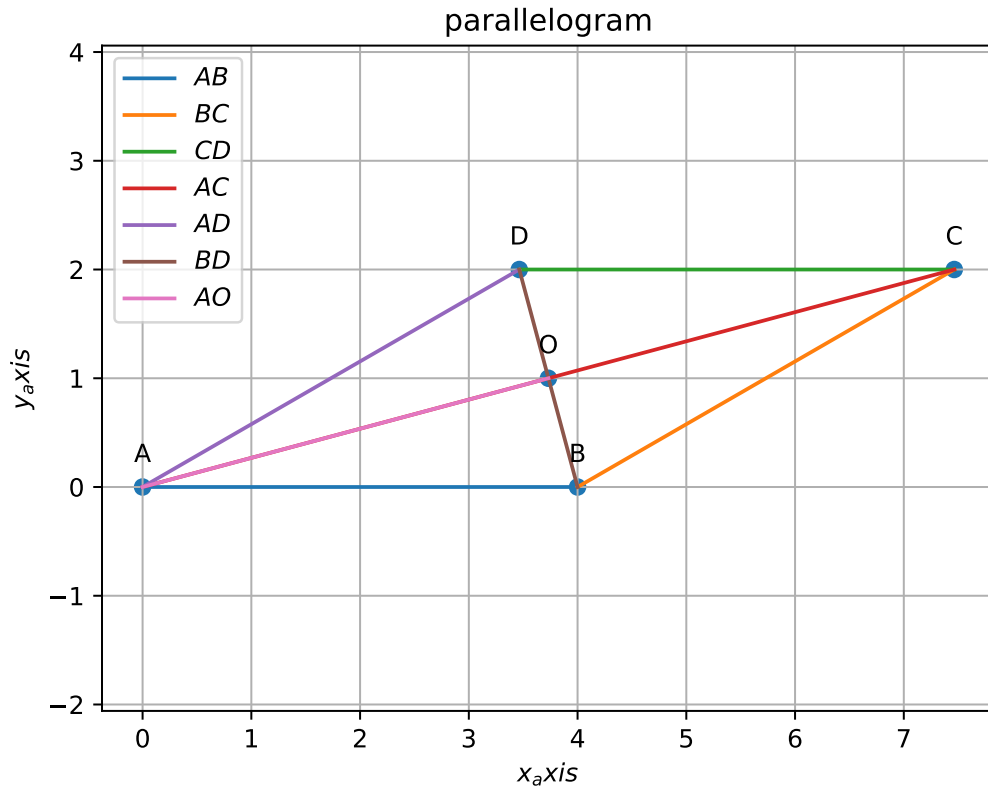


Figure 3.7:

*Proof.* See Fig. 3.7. From Appendix A.1.25 and A.1.3

$$ar(AOB) = \frac{1}{2} \|\mathbf{A} \times \mathbf{O} + \mathbf{O} \times \mathbf{B} + \mathbf{B} \times \mathbf{A}\| \quad (3.31)$$

$$= \frac{1}{2} \left\| \mathbf{A} \times \left( \frac{\mathbf{A} + \mathbf{C}}{2} \right) + \left( \frac{\mathbf{A} + \mathbf{C}}{2} \right) \times \mathbf{B} + \mathbf{B} \times \mathbf{A} \right\| \quad (3.32)$$

$$= \frac{1}{4} \|\mathbf{A} \times \mathbf{C} + \mathbf{C} \times \mathbf{B} + \mathbf{B} \times \mathbf{A}\| \quad (3.33)$$

yielding the desired result from Appendix A.1.26 □

4.  $ABC, ABD$  are 2 triangles on same base  $AB$ , if line segment  $CD$  is bisected by  $AB$  at  $\mathbf{O}$ , show that

$$ar(ABC) = ar(ABD) \quad (3.34)$$

*Proof.* See Fig. 3.8.  $AO$  and  $OB$  are medians of triangles  $ADC$  and  $BDC$ . From Appendix A.1.5, (3.34) is trivial. □

5.

6.

7.

8.

9. The side  $AB$  of a parallelogram  $ABCD$  is produced to any point  $\mathbf{P}$ . A line through  $\mathbf{A}$  and parallel to  $CP$  meets  $CB$  produced at  $\mathbf{Q}$  and then parallelogram  $PBQR$  is

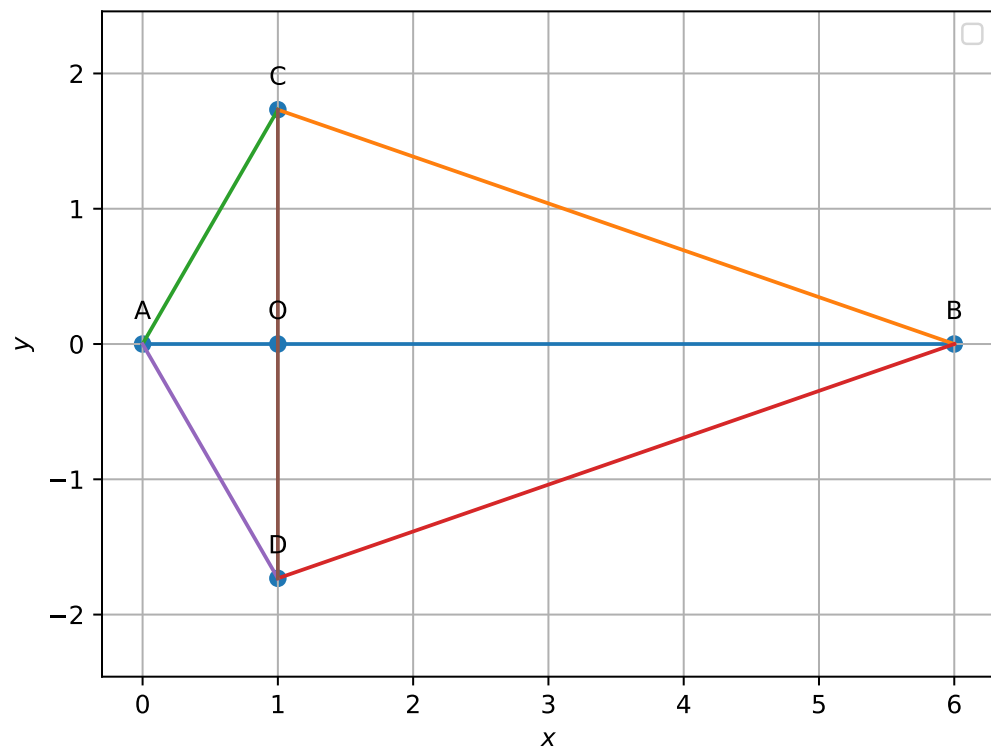


Figure 3.8:

completed. Show that

$$ar(ABCD) = ar(PBQR) \quad (3.35)$$

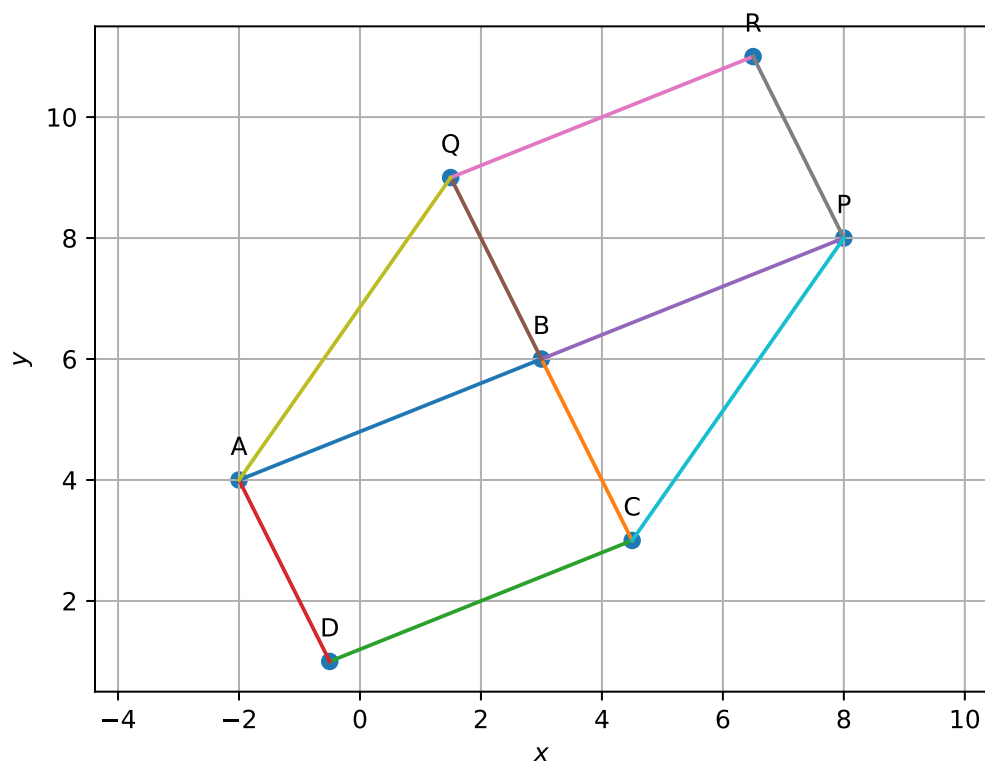


Figure 3.9:

*Proof.* From the given information, using section formula,

$$\mathbf{Q} = \frac{k_1 \mathbf{C} + \mathbf{B}}{k_1 + 1} \quad (3.36)$$

$$\mathbf{P} = \frac{k_2 \mathbf{A} + \mathbf{B}}{k_2 + 1} \quad (3.37)$$

Also, since  $AQ \parallel CP$ ,

$$\mathbf{A} - \mathbf{Q} = k(\mathbf{C} - \mathbf{P}) \quad (3.38)$$

Substituting from (3.36) and (3.37) in the above,

$$\mathbf{A} - \frac{k_1 \mathbf{C} + \mathbf{B}}{k_1 + 1} = k \left( \mathbf{C} - \frac{k_2 \mathbf{A} + \mathbf{B}}{k_2 + 1} \right) \quad (3.39)$$

which, after some algebra, yields

$$\left( 1 + \frac{k k_2}{k_2 + 1} \right) \mathbf{A} + \left( \frac{k}{k_2 + 1} - \frac{1}{k_1 + 1} \right) \mathbf{B} - \left( \frac{k_1}{k_1 + 1} + k \right) \mathbf{C} = \mathbf{0} \quad (3.40)$$

From Appendix A.1.27, (3.40) results in

$$\left( \frac{k}{k_2 + 1} - \frac{1}{k_1 + 1} \right) = \left( \frac{k_1}{k_1 + 1} + k \right) = 0 \quad (3.41)$$

$$\text{or, } k_1 + k_2 = -1 \quad (3.42)$$

From Appendix A.1.26

$$ar(PBQR) = \|\mathbf{P} \times \mathbf{B} + \mathbf{B} \times \mathbf{Q} + \mathbf{Q} \times \mathbf{P}\| \quad (3.43)$$

The R.H.S. in the above can be expressed as

$$\frac{k_2 \mathbf{A} + \mathbf{B}}{k_2 + 1} \times \mathbf{B} + \mathbf{B} \times \frac{k_1 \mathbf{C} + \mathbf{B}}{k_1 + 1} + \frac{k_1 \mathbf{C} + \mathbf{B}}{k_1 + 1} \times \frac{k_2 \mathbf{A} + \mathbf{B}}{k_2 + 1} \quad (3.44)$$

leading to

$$\begin{aligned} & \left( \frac{k_2}{k_2 + 1} - \frac{k_2}{(k_1 + 1)(k_2 + 1)} \right) \mathbf{A} \times \mathbf{B} \\ & + \mathbf{B} \times \mathbf{C} \left( \frac{k_1}{k_1 + 1} - \frac{k_1}{(k_1 + 1)(k_2 + 1)} \right) \\ & + \frac{k_1 k_2}{(k_1 + 1)(k_2 + 1)} \mathbf{C} \times \mathbf{A} \end{aligned} \quad (3.45)$$

that can be simplified to obtain

$$ar(PBQR) = \frac{k_1 k_2}{(k_1 + 1)(k_2 + 1)} \|(\mathbf{A} \times \mathbf{B} + \mathbf{B} \times \mathbf{C} + \mathbf{C} \times \mathbf{A})\| \quad (3.46)$$

$$= \|(\mathbf{A} \times \mathbf{B} + \mathbf{B} \times \mathbf{C} + \mathbf{C} \times \mathbf{A})\| \quad (3.47)$$

using the fact that

$$\frac{k_1 k_2}{(k_1 + 1)(k_2 + 1)} = 1 \quad (3.48)$$

from (3.42). Also, from Appendix A.1.26,

$$ar(ABCD) = \|(\mathbf{A} \times \mathbf{B} + \mathbf{B} \times \mathbf{C} + \mathbf{C} \times \mathbf{A})\| \quad (3.49)$$

yielding (3.35) from (3.47). □

10.

11.  $ABCDE$  is a pentagon. A line through  $\mathbf{B}$  parallel to  $AC$  meets  $DC$  produced at  $F$ .

Show that

$$ar(ACB) = ar(ACF) \quad (3.50)$$

$$ar(AEDF) = ar(ABCDE) \quad (3.51)$$

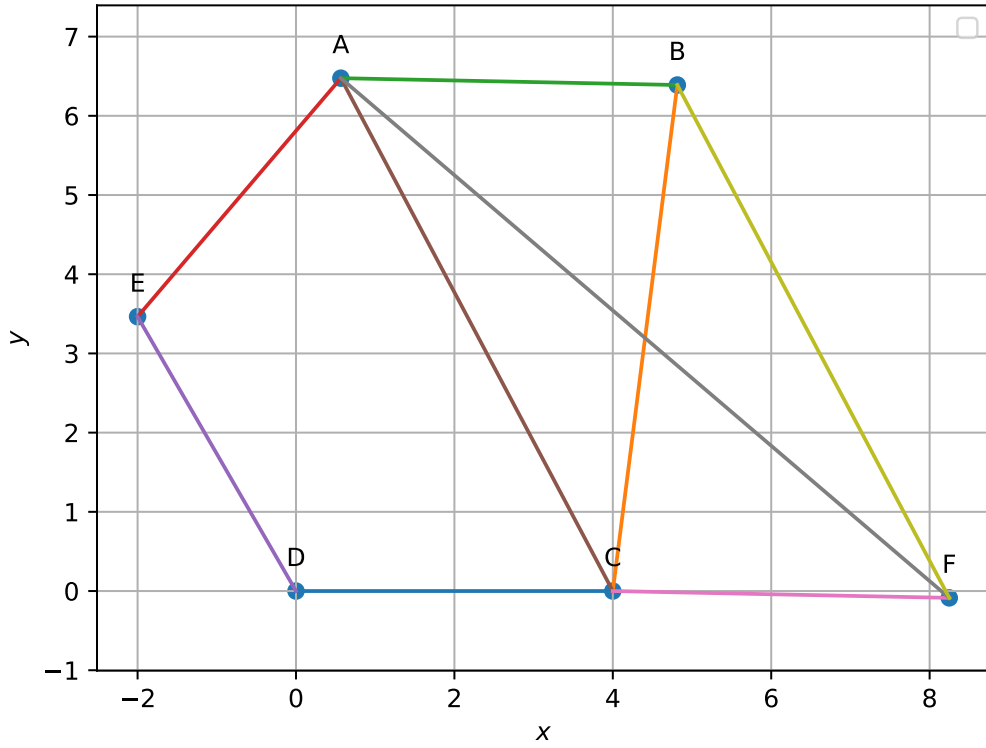


Figure 3.10:

*Proof.* Since  $BF \parallel AC$ ,

$$\mathbf{F} - \mathbf{B} = k(\mathbf{C} - \mathbf{A}) \quad (3.52)$$

$$\implies \mathbf{F} = \mathbf{B} + k(\mathbf{C} - \mathbf{A}) \quad (3.53)$$

Thus, from Appendix A.1.3,

$$ar(ACF) = \frac{1}{2} \|\mathbf{F} \times \mathbf{A} + \mathbf{A} \times \mathbf{C} + \mathbf{C} \times \mathbf{F}\| \quad (3.54)$$



Substituting from (3.53) in (3.54),

$$ar(ACF) = \frac{1}{2} \|\{\mathbf{B} + k(\mathbf{C} - \mathbf{A})\} \times \mathbf{A} + \mathbf{A} \times \mathbf{C} + \mathbf{C} \times \{\mathbf{B} + k(\mathbf{C} - \mathbf{A})\}\| \quad (3.55)$$

$$= \frac{1}{2} \|\mathbf{B} \times \mathbf{A} + \mathbf{A} \times \mathbf{C} + \mathbf{C} \times \mathbf{B}\| \quad (3.56)$$

$$= ar(ACB) \quad (3.57)$$

upon substituting from from Appendix A.1.3. (3.51) follows from (3.50).

□

12.

13.

14.

15.

16. In the Figure 3.11,

$$ar(DRC) = ar(DPC) \quad (3.58)$$

$$ar(BDP) = ar(ARC). \quad (3.59)$$

Show that the quadrilaterals  $ABCD$  and  $DCPR$  are trapeziums.

*Proof.* From Appendix A.1.4 and (3.58),

$$\frac{1}{2} \|(\mathbf{D} - \mathbf{R}) \times (\mathbf{D} - \mathbf{C})\| = \frac{1}{2} \|(\mathbf{C} - \mathbf{D}) \times (\mathbf{C} - \mathbf{P})\| \quad (3.60)$$

$$\implies (\mathbf{D} - \mathbf{R}) \times (\mathbf{D} - \mathbf{C}) = (\mathbf{C} - \mathbf{D}) \times (\mathbf{C} - \mathbf{P}) \quad (3.61)$$

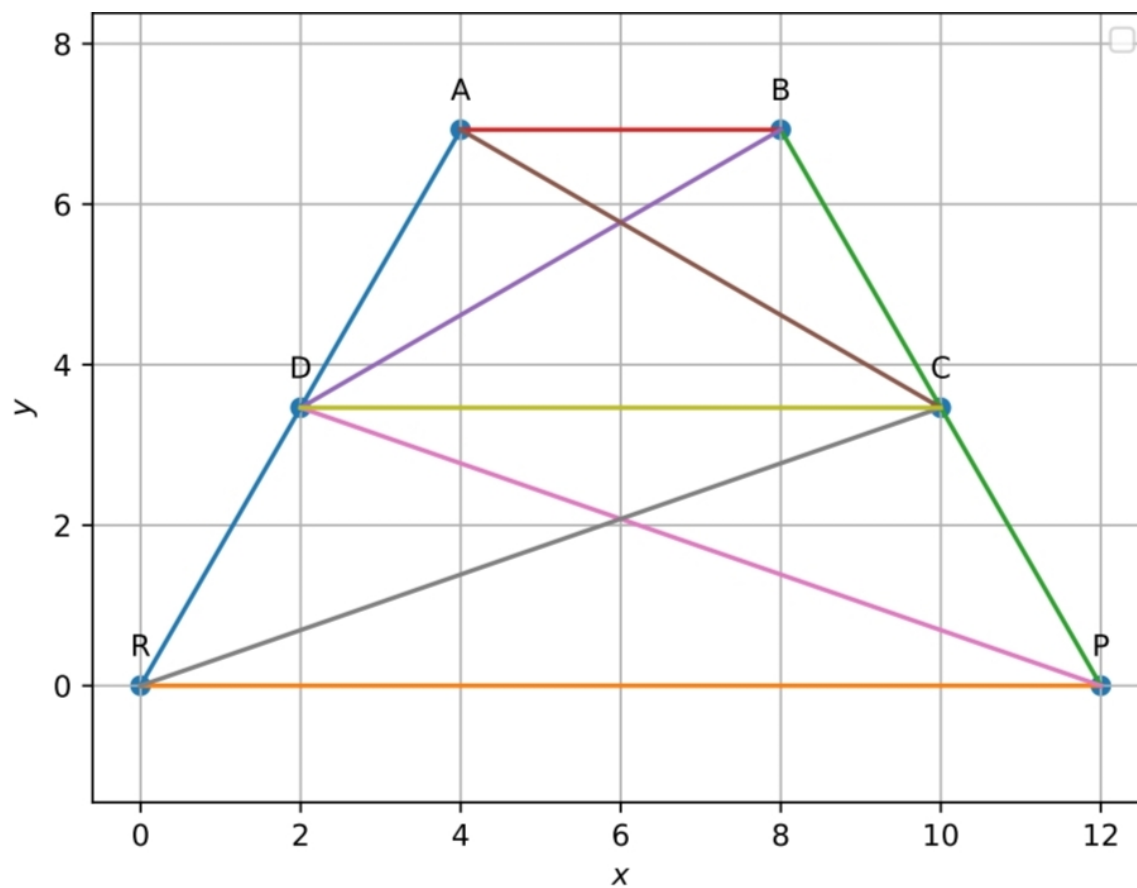


Figure 3.11:

which can be expressed as

$$(\mathbf{C} - \mathbf{D}) \times (\mathbf{C} - \mathbf{D} + \mathbf{R} - \mathbf{P}) = \mathbf{0} \quad (3.62)$$

$$\implies (\mathbf{C} - \mathbf{D}) \times (\mathbf{R} - \mathbf{P}) = \mathbf{0} \quad (3.63)$$

$$\text{or, } CD \parallel RP \quad (3.64)$$

Hence,  $DCPR$  is a trapezium. Similarly, it can be shown that  $ABCD$  is also a trapezium.

□

### 3.3. Triangles and Parallelograms

- 1.
- 2.
3. In Fig. 3.12  $ABCD$ ,  $DCFE$  and  $ABFE$  are parallelograms. Show that

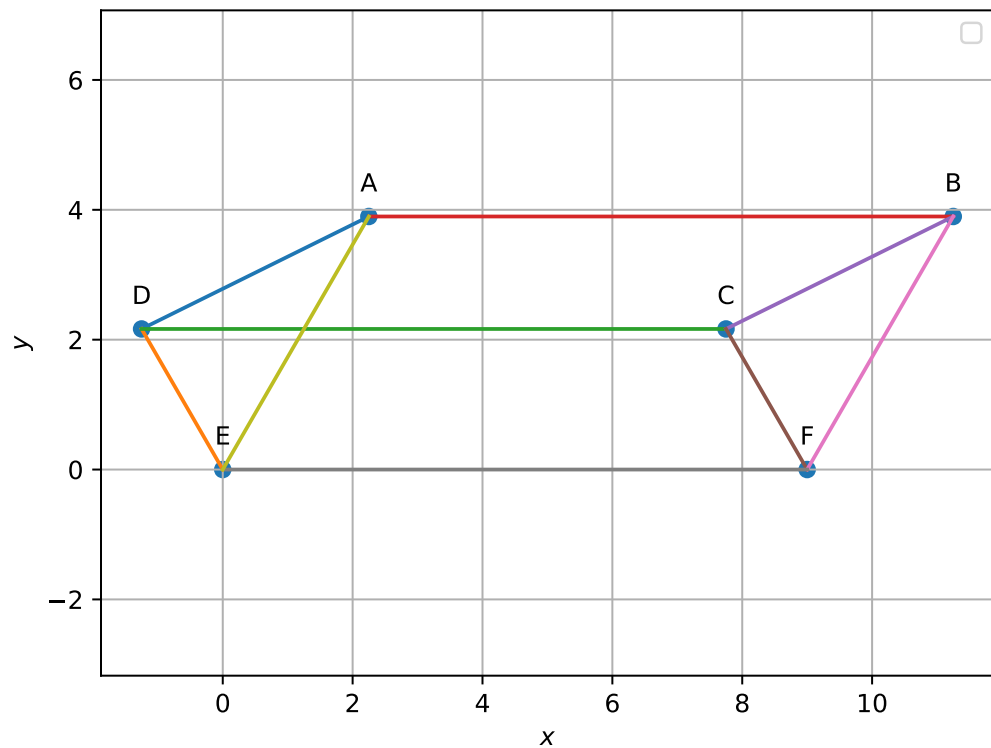


Figure 3.12:

$$ar(ADE) = ar(BCF) \quad (3.65)$$

*Proof.* From the given information and Appendix A.38,

$$\mathbf{B} - \mathbf{A} = \mathbf{C} - \mathbf{D} \quad (3.66)$$

$$\mathbf{C} - \mathbf{D} = \mathbf{F} - \mathbf{E} \quad (3.67)$$

$$\mathbf{B} - \mathbf{A} = \mathbf{F} - \mathbf{E} \quad (3.68)$$

Thus, from Appendix A.1.26,

$$ar(ADE) = \|(\mathbf{D} - \mathbf{E}) \times (\mathbf{D} - \mathbf{A})\| \quad (3.69)$$

$$= \|(\mathbf{C} - \mathbf{F}) \times (\mathbf{C} - \mathbf{B})\| \quad (3.70)$$

$$= ar(ADE) \quad (3.71)$$

upon substituting from (3.66) and (3.67).  $\square$

4. In figure below,  $ABCD$  is a parallelogram and  $BC$  is produced to a point  $\mathbf{Q}$  such that  $AD = CQ$ . If  $AQ$  intersect  $DC$  at  $\mathbf{P}$ , show that

$$ar(BPC) = ar(DPQ). \quad (3.72)$$

5. In Fig. 3.14,  $ABC$  and  $BDE$  are two equilateral triangles such that  $\mathbf{D}$  is the mid-point



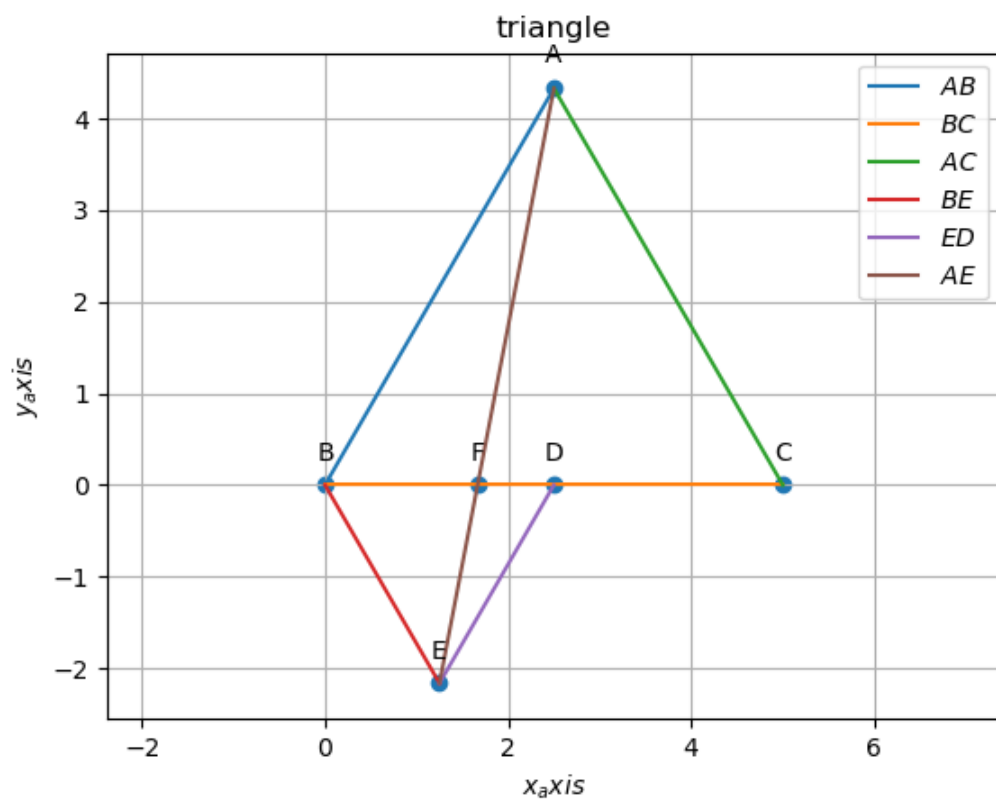


Figure 3.14:

6.

7.

8.

## Chapter 4

# Line

### 4.1. Equation of a Line

1.

2.

3.

4.

5.

6.

7.

8.

9. The Vertices of Triangle  $PQR$  is  $\mathbf{P}(2, 1)$ ,  $\mathbf{Q}(-2, 3)$ ,  $\mathbf{R}(4, 5)$ . Find the equation of the Median Through  $\mathbf{R}$ .

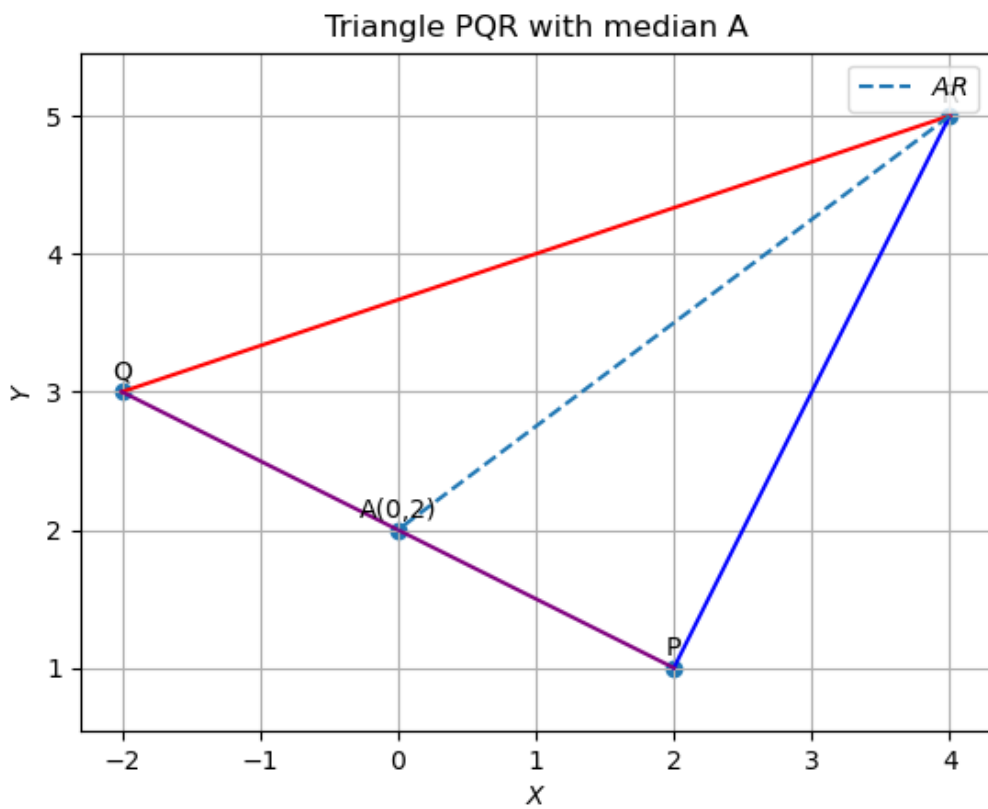


Figure 4.1:

**Solution:** See Fig. 4.1. Using Section Formula,

$$\mathbf{A} = \frac{\mathbf{P} + \mathbf{Q}}{2} \quad (4.1)$$

$$= \begin{pmatrix} 0 \\ 2 \end{pmatrix} \quad (4.2)$$



So , the Direction Vector of  $AR$  is

$$\mathbf{m} = \mathbf{R} - \mathbf{A} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \quad (4.3)$$

$$\Rightarrow \mathbf{n} = \begin{pmatrix} 3 \\ -4 \end{pmatrix} \quad (4.4)$$

which is the normal vector. Thus, from (C.2), the equation of the line is

$$\begin{pmatrix} 3 & -4 \end{pmatrix} (\mathbf{x} - \mathbf{R}) = 0 \quad (4.5)$$

$$\Rightarrow \begin{pmatrix} 3 & -4 \end{pmatrix} \mathbf{x} = 8 \quad (4.6)$$

10. Find the equation of the line passing through  $(-3,5)$  and perpendicular to the line through the points  $(2,5)$  and  $(-3,6)$ .

**Solution:** Let

$$\mathbf{A} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -3 \\ 6 \end{pmatrix}, \mathbf{P} = \begin{pmatrix} -3 \\ 5 \end{pmatrix} \quad (4.7)$$

The normal vector of the desired line is then given by

$$\mathbf{n} = \mathbf{B} - \mathbf{A} = \begin{pmatrix} 5 \\ -1 \end{pmatrix} \quad (4.8)$$

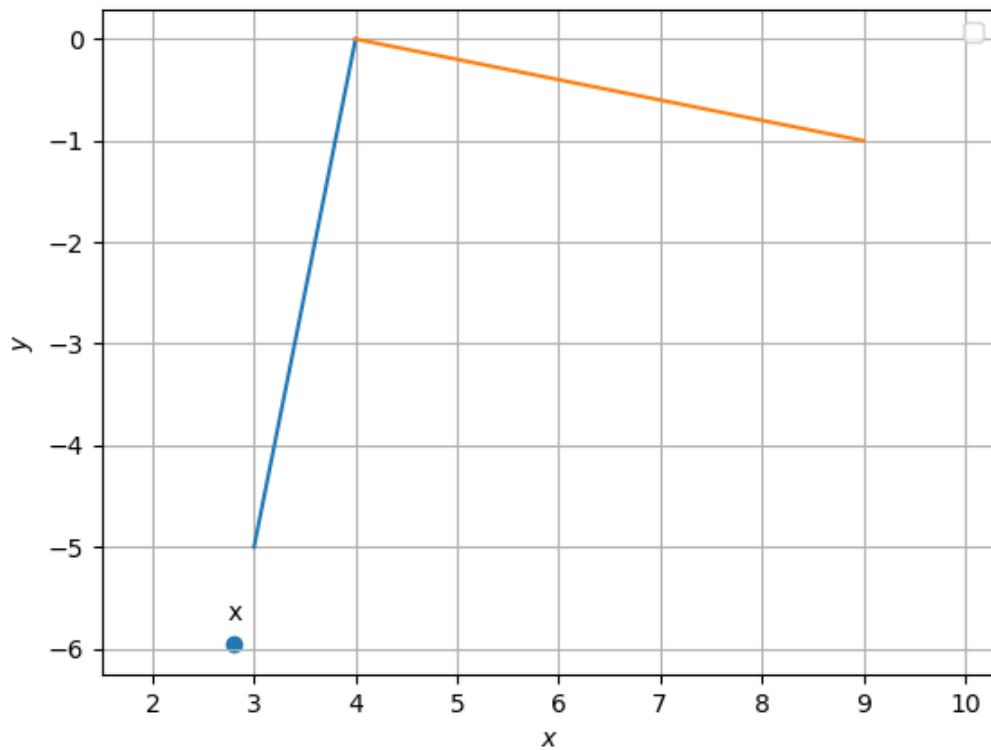


Figure 4.2:

Thus, the equation of the line is

$$\begin{pmatrix} 5 & -1 \end{pmatrix} \left( \mathbf{x} - \begin{pmatrix} -3 \\ 5 \end{pmatrix} \right) = 0 \quad (4.9)$$

$$\implies \begin{pmatrix} 5 & -1 \end{pmatrix} \mathbf{x} = -20 \quad (4.10)$$

11. A line perpendicular to the line segment joining the points (1,0) and (2,3) divides it in the ratio 1 :  $n$ . Find the equation of the line.

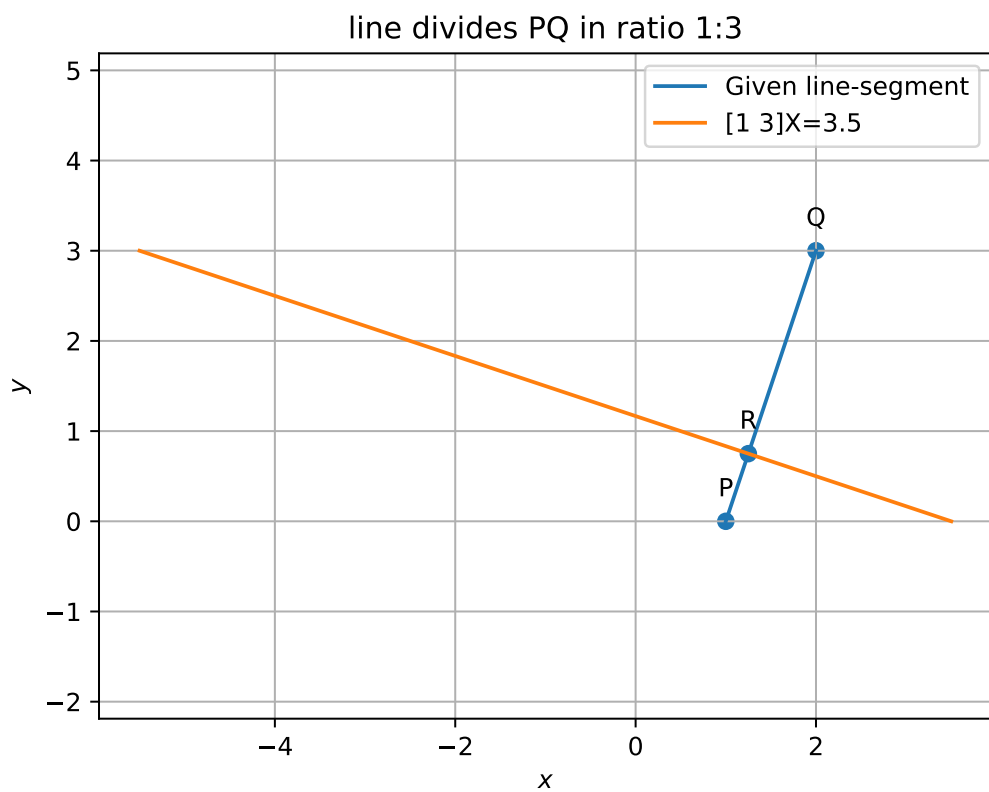


Figure 4.3:

**Solution:** Let

$$\mathbf{P} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad (4.11)$$

The direction vector of  $PQ$  is

$$\mathbf{m} = \mathbf{Q} - \mathbf{P} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad (4.12)$$

Also, using section formula,

$$\mathbf{R} = \frac{\mathbf{Q} + n\mathbf{P}}{1 + n} \quad (4.13)$$

and the equation of line passing through  $\mathbf{R}$  is

$$\mathbf{m}^\top (\mathbf{x} - \mathbf{R}) = 0 \quad (4.14)$$

$$\Rightarrow \begin{pmatrix} 1 & 3 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 & 3 \end{pmatrix} \begin{pmatrix} \frac{2+n}{1+n} \\ \frac{3}{1+n} \end{pmatrix} \quad (4.15)$$

$$= \frac{11 + n}{1 + n} \quad (4.16)$$

12.

13. Find equation of a line passing through a point (2,2) and cutting off intercepts on the axes whose sum is 9.

**Solution:** Let the  $x$  intercept be  $a$  and the  $y$  intercept be  $b$ . Then

$$a + b = 9 \quad (4.17)$$

Let

$$\mathbf{P} = \begin{pmatrix} a \\ 0 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} 0 \\ b \end{pmatrix}, \mathbf{R} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \quad (4.18)$$

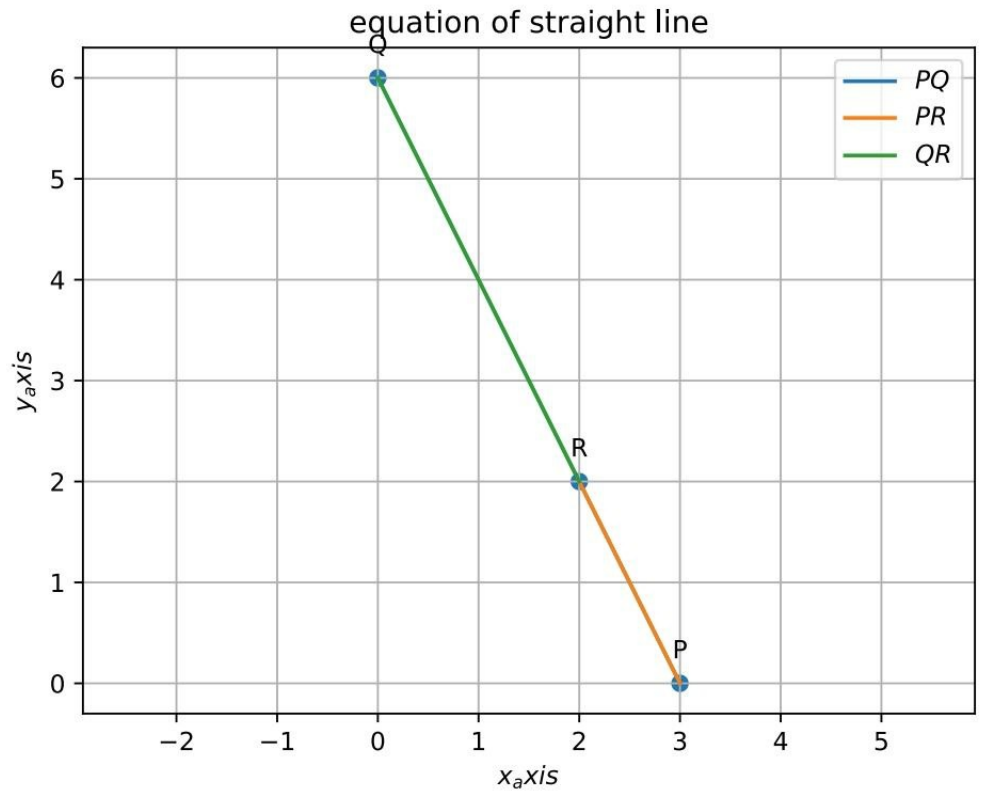


Figure 4.4:

Since the points are collinear, from (C.4), we obtain the matrix

$$\begin{pmatrix} \mathbf{P} - \mathbf{Q} & \mathbf{P} - \mathbf{R} \end{pmatrix} = \begin{pmatrix} a & a - 2 \\ -b & -2 \end{pmatrix} \quad (4.19)$$

which is singular if the determinant

$$-2a + b(a - 2) = ab - 2(a + b) = 0 \quad (4.20)$$

yielding

$$ab = 18 \quad (4.21)$$

upon substituting from (4.17). (4.21) and (4.17) form

$$x^2 - 9x + 18 = 0 \quad (4.22)$$

with roots

$$x = 6, 3 \quad (4.23)$$

$$\text{or, } \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \end{pmatrix} \quad (4.24)$$

Since the direction vector of the line is

$$\mathbf{P} - \mathbf{Q} = \begin{pmatrix} a \\ -b \end{pmatrix}, \quad (4.25)$$

the normal vector is

$$\mathbf{n} = \begin{pmatrix} b \\ a \end{pmatrix} \equiv \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad (4.26)$$

Thus, the possible equations of the line are

$$\begin{pmatrix} 1 & 2 \end{pmatrix} \mathbf{x} = 6 \quad (4.27)$$

$$\begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} = 6 \quad (4.28)$$

14. Find the equation of the line through the point (0,2) making an angle

$$2\pi/3 \quad (4.29)$$

with the positive X-axis. Also find the equation of the line parallel to it and crossing the Y-axis at a distance of 2 units below the origin

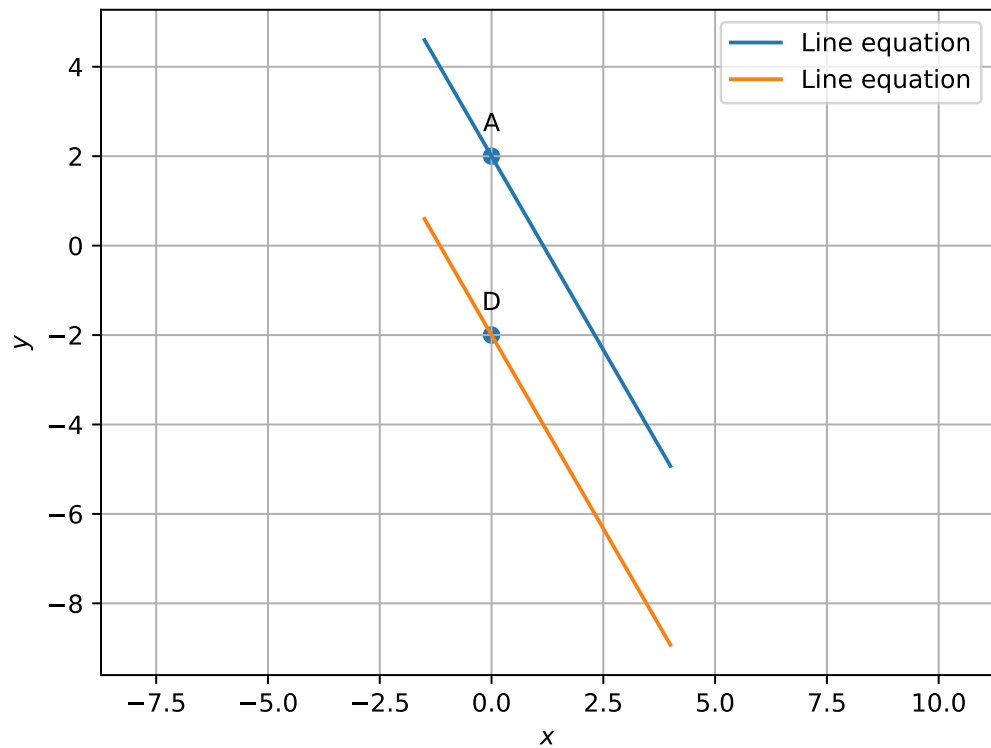


Figure 4.5:

**Solution:** From the given information, the direction vector is

$$\mathbf{m} = \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} \quad (4.30)$$

Thus, the normal vector is

$$\mathbf{n} = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix} \quad (4.31)$$

and the equation of the line is

$$\begin{pmatrix} \sqrt{3} & 1 \end{pmatrix} \left( \mathbf{x} - \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right) = 0 \quad (4.32)$$

$$\implies \begin{pmatrix} \sqrt{3} & 1 \end{pmatrix} \mathbf{x} = 2 \quad (4.33)$$

The equation of the parallel crossing the Y-axis at a distance of 2 units below the origin is given by

$$\begin{pmatrix} \sqrt{3} & 1 \end{pmatrix} \left( \mathbf{x} - \begin{pmatrix} 0 \\ -2 \end{pmatrix} \right) = 0 \quad (4.34)$$

$$\implies \begin{pmatrix} \sqrt{3} & 1 \end{pmatrix} \mathbf{x} = -2 \quad (4.35)$$

15. The perpendicular from the origin to a line meets it at the point (-2,9). Find the equation of the line.

**Solution:**



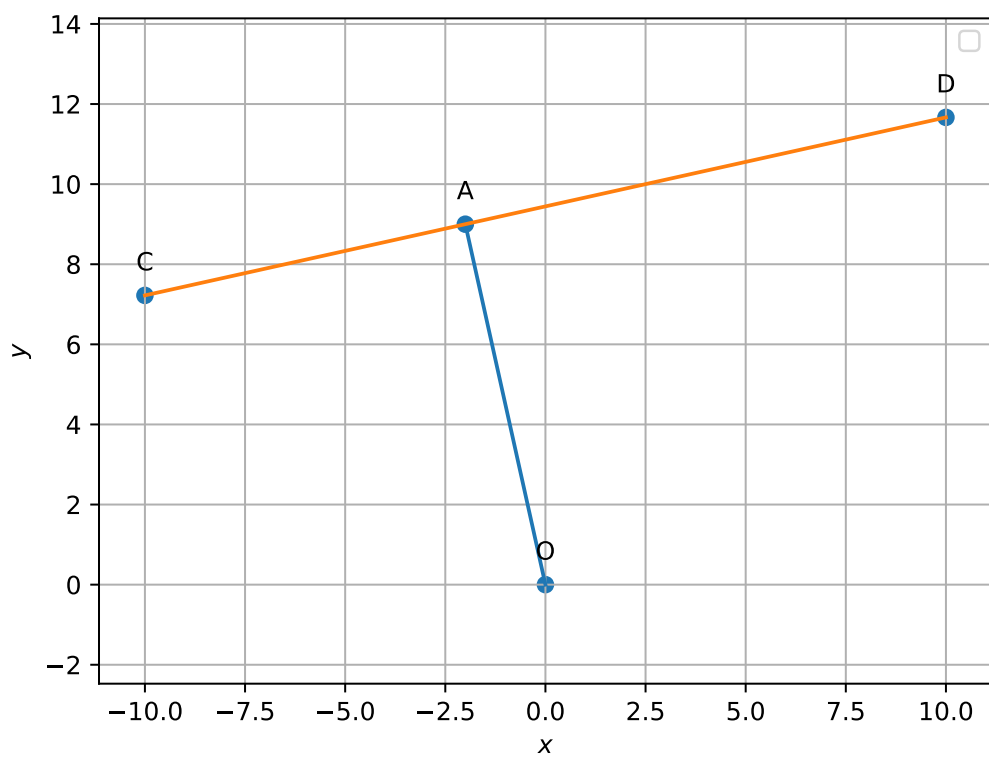


Figure 4.6:

Given

$$\mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{A} = \begin{pmatrix} -2 \\ 9 \end{pmatrix} \quad (4.36)$$

The normal vector is

$$\mathbf{n} = \mathbf{O} - \mathbf{A} \quad (4.37)$$

$$= \begin{pmatrix} 2 \\ -9 \end{pmatrix} \quad (4.38)$$

yielding the equation of the line as

$$\begin{pmatrix} 2 & -9 \end{pmatrix} \left( \mathbf{x} - \begin{pmatrix} 2 \\ -9 \end{pmatrix} \right) = 0 \quad (4.39)$$

$$\implies \begin{pmatrix} 2 & -9 \end{pmatrix} \mathbf{x} = 85 \quad (4.40)$$

16.

17.

18.

19.

20. By using the concept of equation of a line, prove that the three points  $(3, 0)$ ,  $(-2, -2)$  and  $(8, 2)$  are collinear.

**Solution:** The collinearity matrix can be expressed as

$$\begin{pmatrix} -5 & -2 \\ 5 & 2 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_1 + R_2} = \begin{pmatrix} -5 & -2 \\ 0 & 0 \end{pmatrix} \quad (4.41)$$

which is a rank 1 matrix.

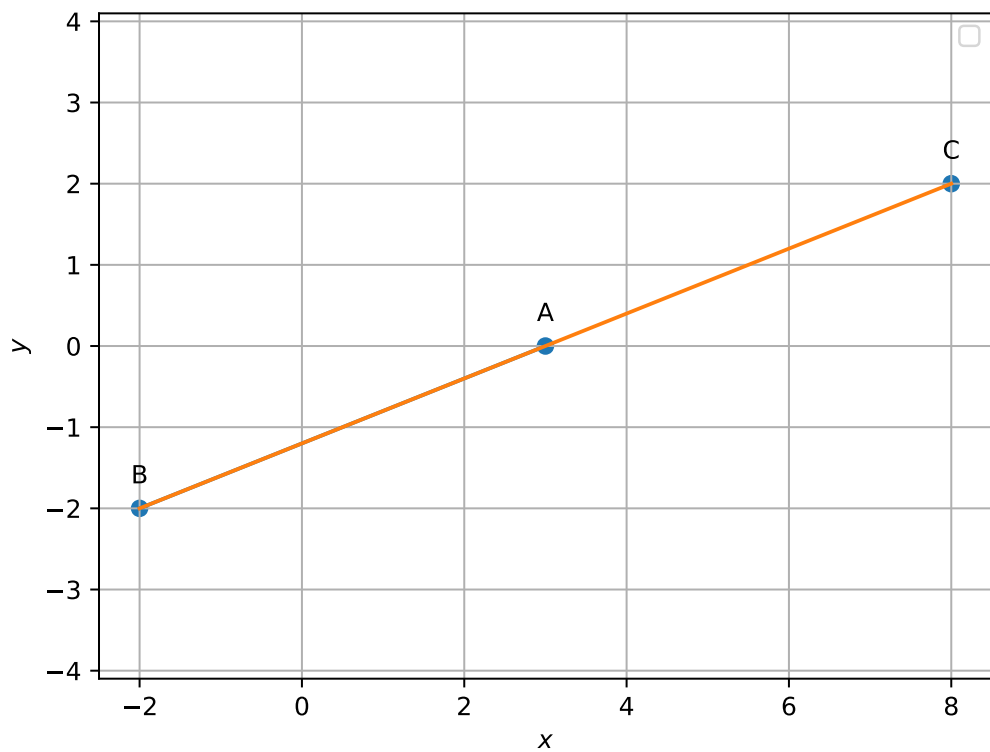


Figure 4.7:

## 4.2. General Equation of a Line

- 1.
- 2.
- 3.
- 4.
- 5.

6.

7. Find the equation of the line parallel to the line  $3x-4y+2=0$  and passing through the point  $(-2,3)$ .

**Solution:** From the given information,

$$\mathbf{n} = \begin{pmatrix} 3 \\ -4 \end{pmatrix} \quad (4.42)$$

$$\Rightarrow \begin{pmatrix} 3 & -4 \end{pmatrix} \left\{ \mathbf{x} - \begin{pmatrix} -2 \\ 3 \end{pmatrix} \right\} = 0 \quad (4.43)$$

$$= -18 \quad (4.44)$$

which is the required equation of the line.

8.

9. Find angle between the lines,  $\sqrt{3}x + y = 1$  and  $x + \sqrt{3}y=1$ .

**Solution:** From the given equations, the normal vectors can be expressed as

$$\mathbf{n}_1 = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}, \mathbf{n}_2 = \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} \quad (4.45)$$

The angle between the lines can then be expressed as

$$\cos\theta = \frac{\mathbf{n}_1^T \mathbf{n}_2}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} \quad (4.46)$$

$$= \frac{\sqrt{3}}{2} \quad (4.47)$$

$$\text{or, } \theta = 30^\circ \quad (4.48)$$

10.

11.

12.

13.

14.

15.

16.

17.

18.

### **4.3. Miscellaneous Exercises**



## Chapter 5

# Circles

### 5.1. Equation

- 1.
- 2.
- 3.
- 4.
- 5.
- 6.
- 7.
- 8.
- 9.
- 10.
11. Find the equation of the circle passing through the points (2,3) and (-1,1) and whose centre is on the line  $x - 3y - 11 = 0$

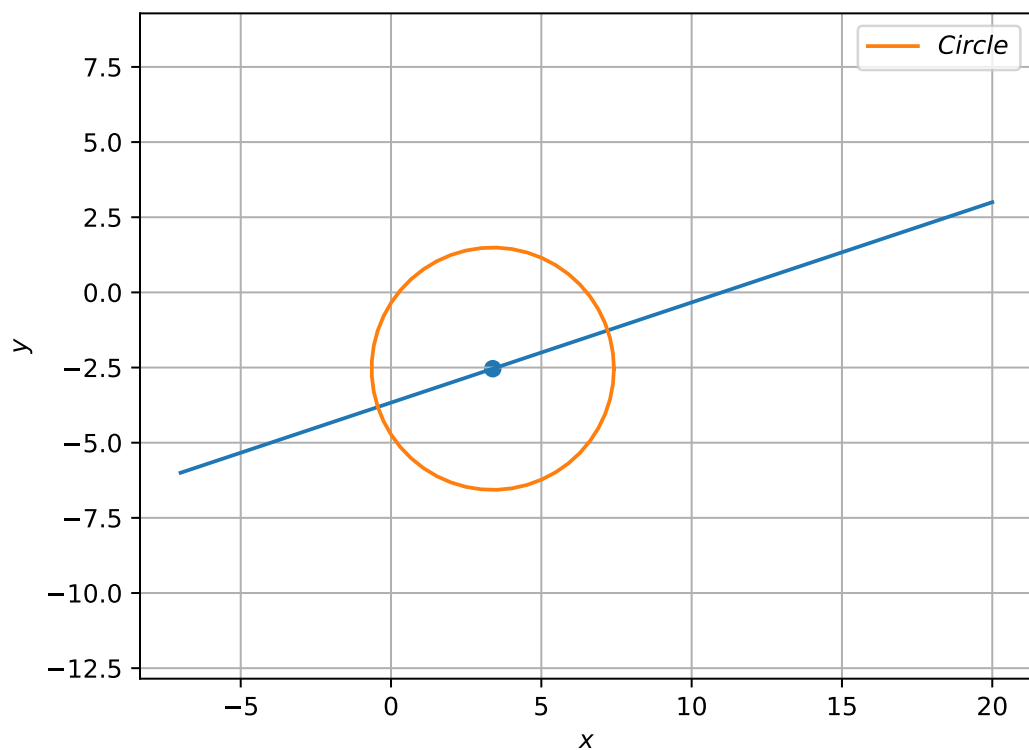


Figure 5.1:

**Solution:** See Fig. From (D.38), and the given information,

$$\|\mathbf{P}\|^2 + 2\mathbf{u}^\top \mathbf{P} + f = 0 \quad (5.1)$$

$$\|\mathbf{Q}\|^2 + 2\mathbf{u}^\top \mathbf{Q} + f = 0 \quad (5.2)$$

$$-\mathbf{n}^\top \mathbf{u} = c \quad (5.3)$$

by noting that the centre of the circle is  $-\mathbf{u}$ . Substituting numerical values, we obtain



the matrix equation

$$\begin{pmatrix} 4 & 6 & 1 \\ -2 & 2 & 1 \\ -1 & 3 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ f \end{pmatrix} = \begin{pmatrix} -13 \\ -2 \\ 11 \end{pmatrix} \quad (5.4)$$

(5.5)

The augmented matrix for (5.4) can be expressed as

$$\xleftrightarrow{1/4R_1 \leftrightarrow R_1} \left( \begin{array}{ccc|c} 1 & 3/2 & .1/4 & -13/4 \\ -2 & 2 & 1 & -2 \\ -1 & 3 & 0 & 11 \end{array} \right) \quad (5.6)$$

which can be reduced to echelon form using row operations to obtain

$$\mathbf{u} = \begin{pmatrix} -7/2 \\ 5/2 \end{pmatrix}, f = -14 \quad (5.7)$$

12. Find the equation of circle with radius 5 whose center lies on x-axis and passes through point  $(2, 3)$ .

**Solution:** See Fig. 5.2. From the given information, the following equations can be formulated using (D.38).

$$\|\mathbf{P}\|^2 + 2\mathbf{u}^\top \mathbf{P} + f = 0 \quad (5.8)$$

$$\mathbf{u} = k\mathbf{e}_1 \quad (5.9)$$

$$\|\mathbf{u}\|^2 - f = r^2 \quad (5.10)$$

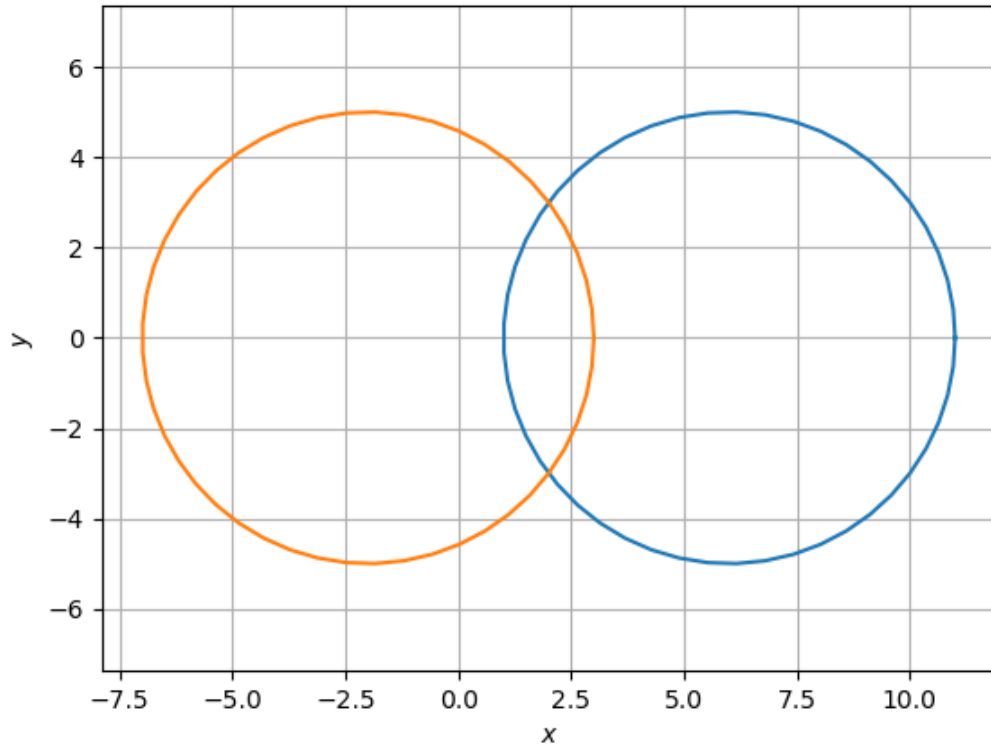


Figure 5.2:

where

$$\mathbf{P} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \text{ and } r = 5 \quad (5.11)$$

From (5.8) and (5.10),

$$\|\mathbf{P}\|^2 + 2\mathbf{u}^\top \mathbf{P} + \|\mathbf{u}\|^2 = r^2 \quad (5.12)$$

Substituting from (5.9) in the above,

$$k^2 + 2k\mathbf{e}_1^\top \mathbf{P} + \|\mathbf{P}\|^2 - r^2 = 0 \quad (5.13)$$

resulting in

$$k = -\mathbf{e}_1^\top \mathbf{P} \pm \sqrt{(\mathbf{e}_1^\top \mathbf{P})^2 + r^2 - \|\mathbf{P}\|^2} \quad (5.14)$$

Substituting numerical values,

$$k = 2, -6 \quad (5.15)$$

resulting in circles with centre

$$-\mathbf{u} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} 6 \\ 0 \end{pmatrix}. \quad (5.16)$$

This is verified in Fig. (5.2).

## 5.2. Equal Chords

1. Two circles of radii 5cm and 3cm intersect at two points and the distance between their center is 4cm. Find the length of the common chord.

**Solution:** See Fig. 5.3. and

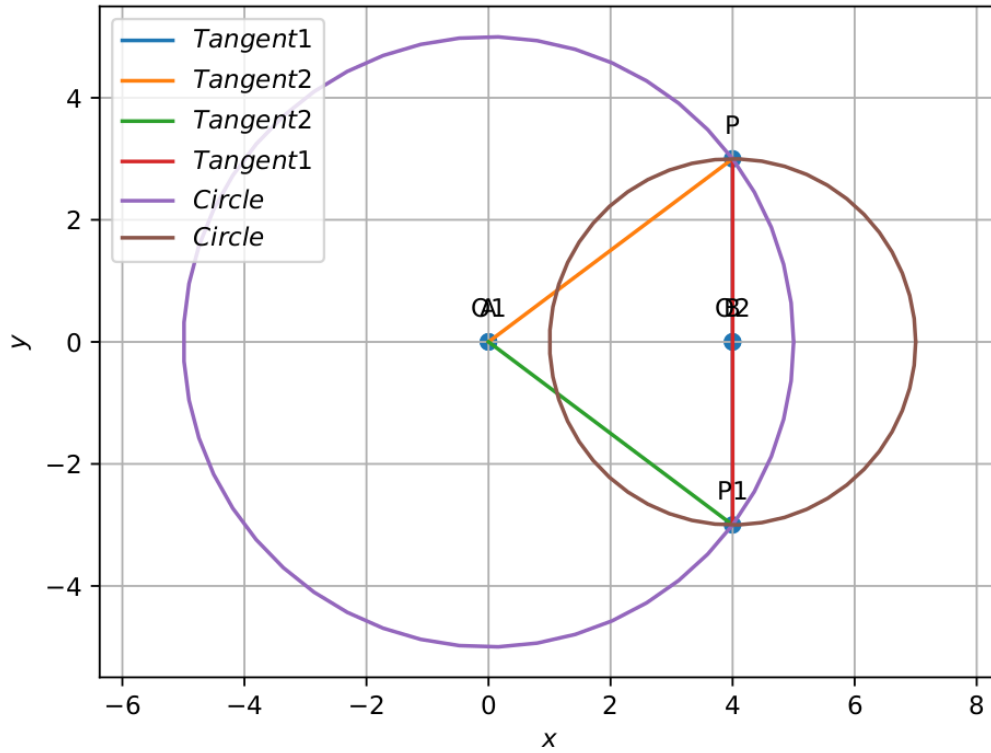


Figure 5.3:

Parameter	Value	Description
$\mathbf{c}_1$	$\mathbf{0}$	Center of Circle 1
$\mathbf{c}_2$	$4\mathbf{e}_1$	Center of Circle 2
$r_1$	5	Radius of Circle 1
$r_2$	3	Radius of Circle 2

Table 5.2:

From Table 5.2, (D.38) and (D.39), the equations of the two circles are

$$\begin{aligned}
 \|\mathbf{x}\|^2 - 25 &= 0 \\
 \|\mathbf{x}\|^2 - 8\mathbf{e}_1^\top \mathbf{x} + 7 &= 0
 \end{aligned}
 \tag{5.17}$$

From (5.17) and (D.41) the equation of the common chord is

$$\mathbf{e}_1^\top \mathbf{x} = 4 \quad (5.18)$$

It is easy to verify that

$$\mathbf{q} = 4\mathbf{e}_1 \quad (5.19)$$

is a point on (5.18). Substituting

$$\mathbf{m} = \mathbf{e}_2, \mathbf{q} = 4\mathbf{e}_1, \mathbf{V} = \mathbf{I}, \mathbf{u} = \mathbf{0}, f = -25 \quad (5.20)$$

in (F.32), the length of the chord in (F.24) is given by

$$\frac{2\sqrt{[\mathbf{e}_2^\top (4\mathbf{e}_1)]^2 - (16\mathbf{e}_1^\top \mathbf{e}_1 - 25)(\mathbf{e}_2^\top \mathbf{e}_2)}}{\mathbf{e}_2^\top \mathbf{e}_2} \|\mathbf{e}_2\| = 6 \quad (5.21)$$

2.

3.

4.

5.

6.

## 5.3. Inscribed Polygons

1. In Fig. 5.4, **A**, **B** and **C** are three points with centre **O** such that  $\angle BOC = 30^\circ$  and  $\angle AOB = 60^\circ$ . If **D** is a point on the circle other than the arc ABC, find  $\angle ADC$ .

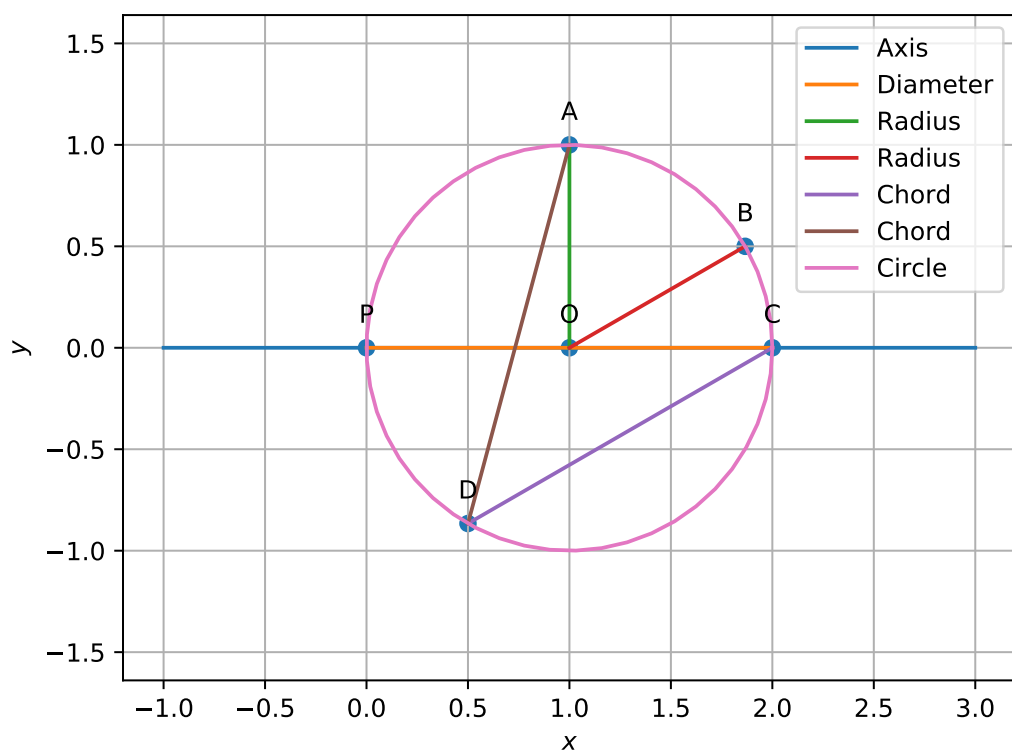


Figure 5.4:

**Solution:** See Fig. (5.4).

$$\mathbf{A} = \mathbf{e}_2, \mathbf{B} = \begin{pmatrix} \cos 30 \\ \sin 30 \end{pmatrix}, \mathbf{C} = \mathbf{e}_1 \text{ and } \mathbf{D} = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}. \quad (5.22)$$

2.

3. Let  $\angle PQR = 100^\circ$  where  $\mathbf{P}, \mathbf{Q}, \mathbf{R}$  are points on a circle with centre  $\mathbf{O}$ . Find  $\angle OPR$ .

**Solution:** In Fig. 5.5,

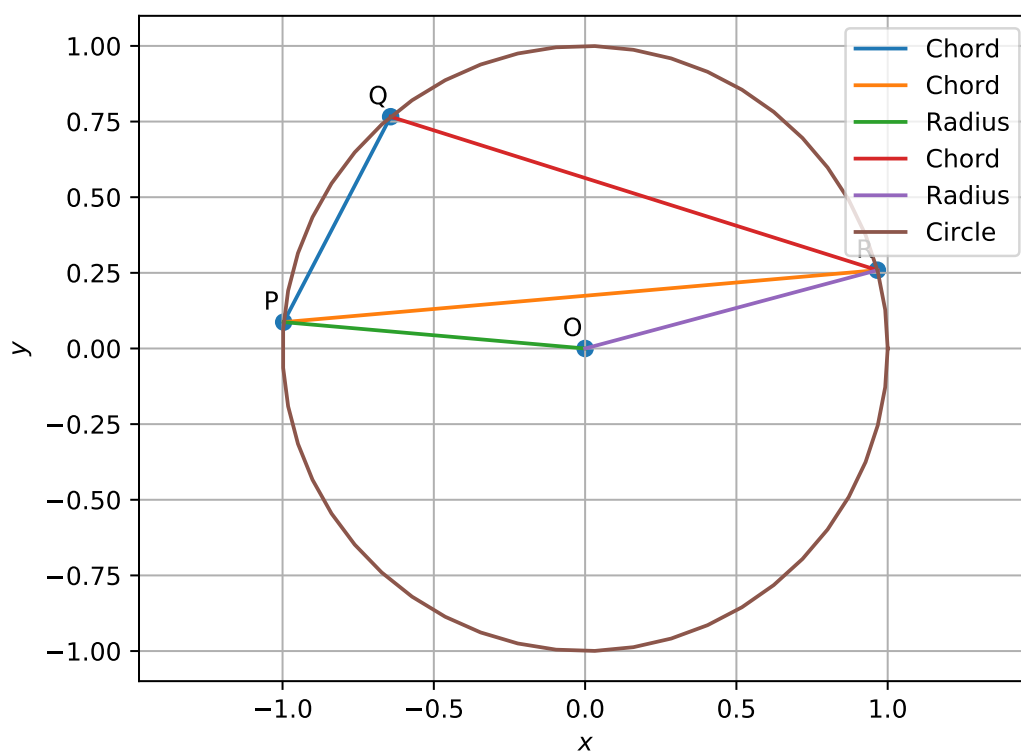


Figure 5.5:

$$\mathbf{P} = \begin{pmatrix} \cos(\theta + 160) \\ \sin(\theta + 160) \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix}, \mathbf{R} = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}. \quad (5.23)$$

## 5.4. Tangent to a Circle

- 1.

2. Draw a circle and two lines parallel to a given line such that one is a tangent and the other is a secant to the circle

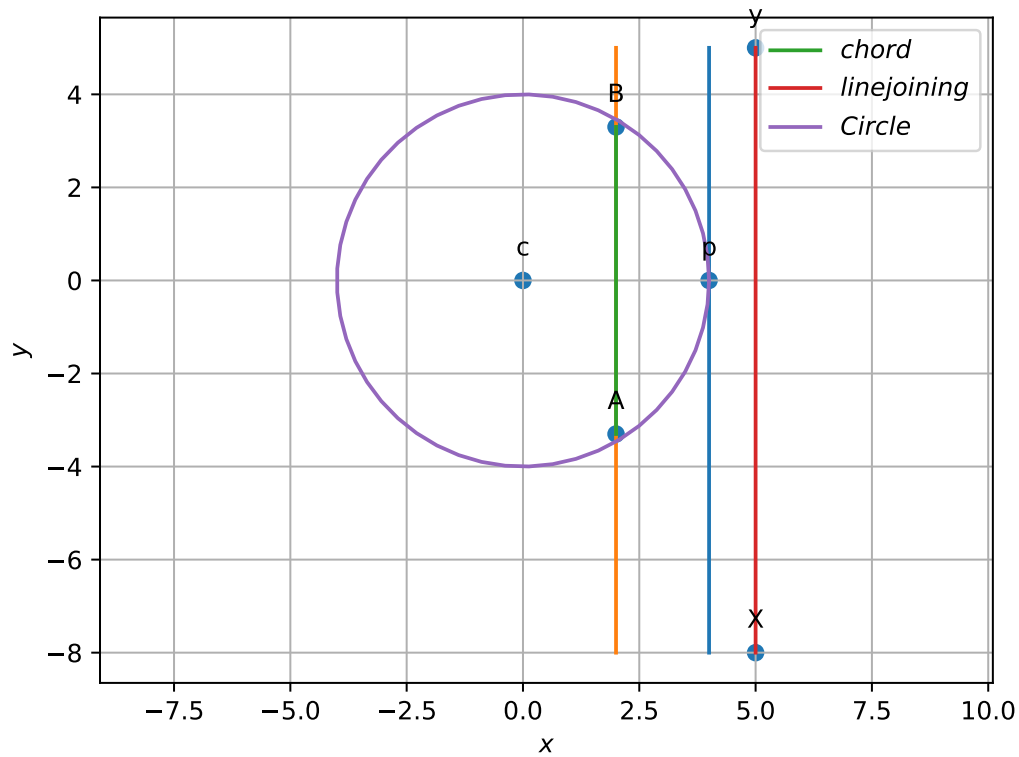


Figure 5.6:

**Solution:** The parameters of the circle in Fig. 5.6 are

$$\mathbf{u} = \mathbf{0}, f = -16 \quad (5.24)$$

Considering the given line to be

$$\mathbf{e}_1^\top \mathbf{x} = 5 \quad (5.25)$$



the tangent to the circle will be

$$\mathbf{e}_1^\top \mathbf{x} = 4 \quad (5.26)$$

and the secant will be

$$\mathbf{e}_1^\top \mathbf{x} = c \quad (5.27)$$

where

$$|c| < 4 \quad (5.28)$$

## 5.5. Tangents from a Point

- 1.
- 2.
- 3.
4. Show that the tangents of circle drawn at the ends of diameter are parallel.

**Solution:** See Fig. 5.7. Let  $\mathbf{A}, \mathbf{B}$  be the end points of the diameter of the circle through which the tangents are drawn. From (D.39),

$$\frac{\mathbf{A} + \mathbf{B}}{2} = -\mathbf{u} \quad (5.29)$$

$$\implies \mathbf{A} + \mathbf{B} = -2\mathbf{u} \quad (5.30)$$

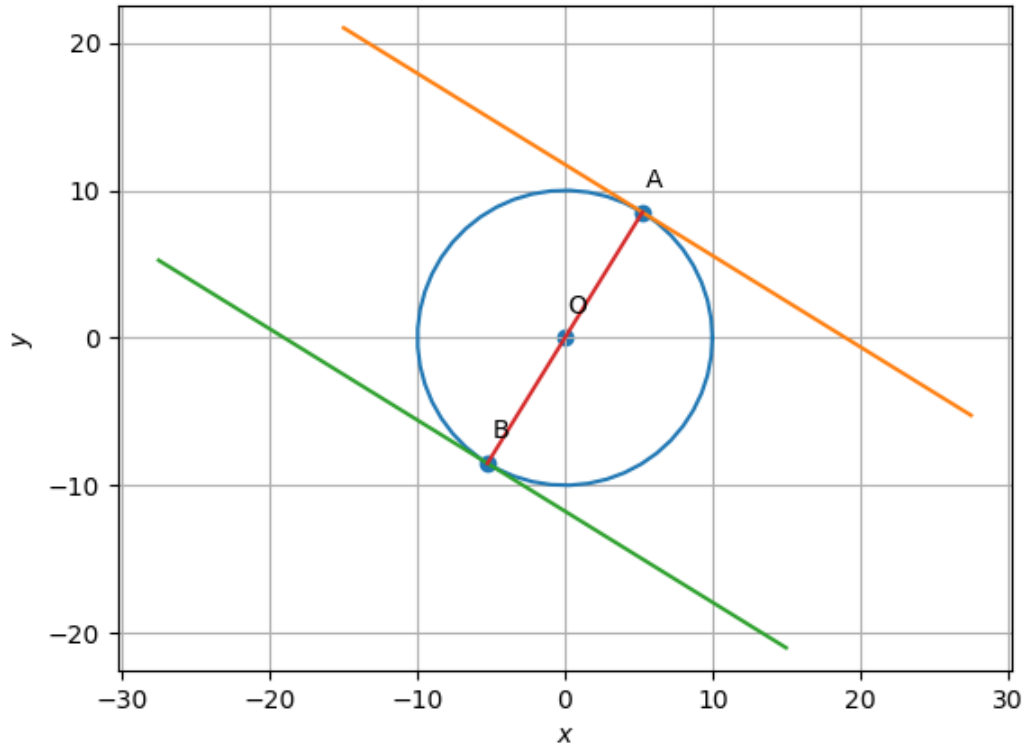


Figure 5.7:

From (F.29),

$$\mathbf{m}_1^\top (\mathbf{A} + \mathbf{u}) = 0 \quad (5.31)$$

$$\mathbf{m}_2^\top (\mathbf{B} + \mathbf{u}) = 0 \quad (5.32)$$

where  $\mathbf{m}_1, \mathbf{m}_2$  are the direction vectors of the tangents at  $\mathbf{A}, \mathbf{B}$  respectively. Then,

the normal vectors at the point of contact of targets are

$$\mathbf{A} + \mathbf{u} = k_1 \mathbf{n}_1 \quad (5.33)$$

$$\mathbf{B} + \mathbf{u} = k_2 \mathbf{n}_2 \quad (5.34)$$

Adding (5.33) and (5.34),

$$k_1 \mathbf{n}_1 + k_2 \mathbf{n}_2 = \mathbf{A} + \mathbf{B} + 2\mathbf{u} \quad (5.35)$$

$$= \mathbf{0} \quad (5.36)$$

from (5.30), (5.36) can be expressed as

$$k_1 \mathbf{n}_1 + k_2 \mathbf{n}_2 = \mathbf{0} \quad (5.37)$$

$$k_1 \mathbf{n}_1 = -k_2 \mathbf{n}_2 \quad (5.38)$$

Since

$$\mathbf{n}_1 \times \mathbf{n}_2 = \mathbf{0}, \quad (5.39)$$

$$\mathbf{n}_1 \parallel \mathbf{n}_2 \implies \mathbf{m}_1 \parallel \mathbf{m}_2 \quad (5.40)$$

5.

6. The length of a tangent from a point  $\mathbf{A}$  at distance 5 cm from the centre of the circle is 4 cm. Find the radius of the circle.

**Solution:** From the Baudhayana theorem, the radius

$$r = 3 \quad (5.41)$$

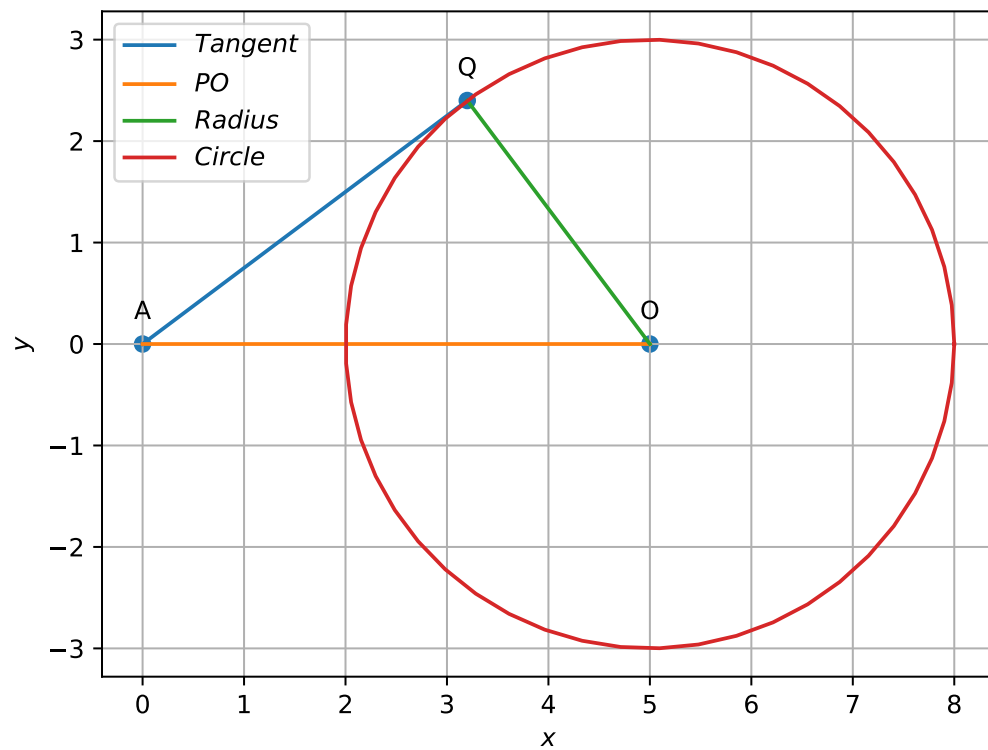


Figure 5.8:

Let

$$\mathbf{A} = \mathbf{O} \text{ and } \mathbf{O} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} \quad (5.42)$$

The equation of the circle can then be expressed as

$$\|\mathbf{x}\|^2 + 2\mathbf{u}^\top \mathbf{x} + f = 0 \quad (5.43)$$

where

$$\mathbf{u} = -\mathbf{O} = -\begin{pmatrix} 5 \\ 0 \end{pmatrix} \quad (5.44)$$

$$f = \|\mathbf{u}\|^2 - r^2 = 16 \quad (5.45)$$

From (F.85),

$$\boldsymbol{\Sigma} = (\mathbf{A} + \mathbf{u})(\mathbf{A} + \mathbf{u})^\top - (\mathbf{A}^\top \mathbf{A} + 2\mathbf{u}^\top \mathbf{A} + f)\mathbf{I} \quad (5.46)$$

$$= \begin{pmatrix} 9 & 0 \\ 0 & -16 \end{pmatrix} \quad (5.47)$$

Thus, from (F.84),

$$\mathbf{P} = \mathbf{I}, \lambda_1 = 9, \lambda_2 = -16 \quad (5.48)$$

$$\Rightarrow \mathbf{n}_1 = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \quad \text{and} \quad \mathbf{n}_2 = \begin{pmatrix} 3 \\ -4 \end{pmatrix} \quad (5.49)$$

Substituting from the above in (F.68),

$$\mathbf{q}_{22} = \frac{1}{5} \begin{pmatrix} 16 \\ 12 \end{pmatrix} = \mathbf{Q} \quad (5.50)$$

in Fig. 5.8.

7. Two concentric circles are of radii 5cm and 3cm. Find the length of the chord of the larger circle which touches the smaller circle.

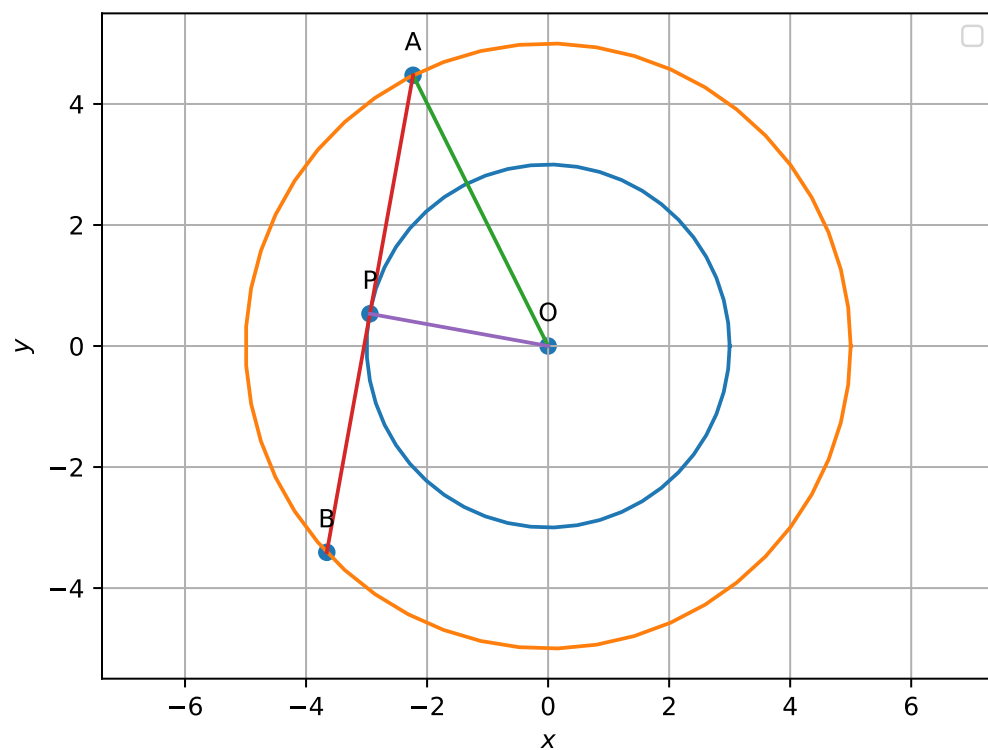


Figure 5.9:

**Solution:** See Fig. 5.9. Let

$$\mathbf{O} = \mathbf{0} \quad (5.51)$$

$$r_1 = 5, r_2 = 3. \quad (5.52)$$

Choosing

$$\mathbf{A} = r_1 \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, \quad (5.53)$$

$\mathbf{P}$  can be obtained following the approach in Problem 7. From Appendix D.2.5,  $\mathbf{P}$  is the mid point of  $AB$ . This can be used to obtain  $\mathbf{B}$ .

8. A quadrilateral  $ABCD$  is drawn to circumscribe a circle. Show that  $AB + CD$  is equal to  $BC + AD$

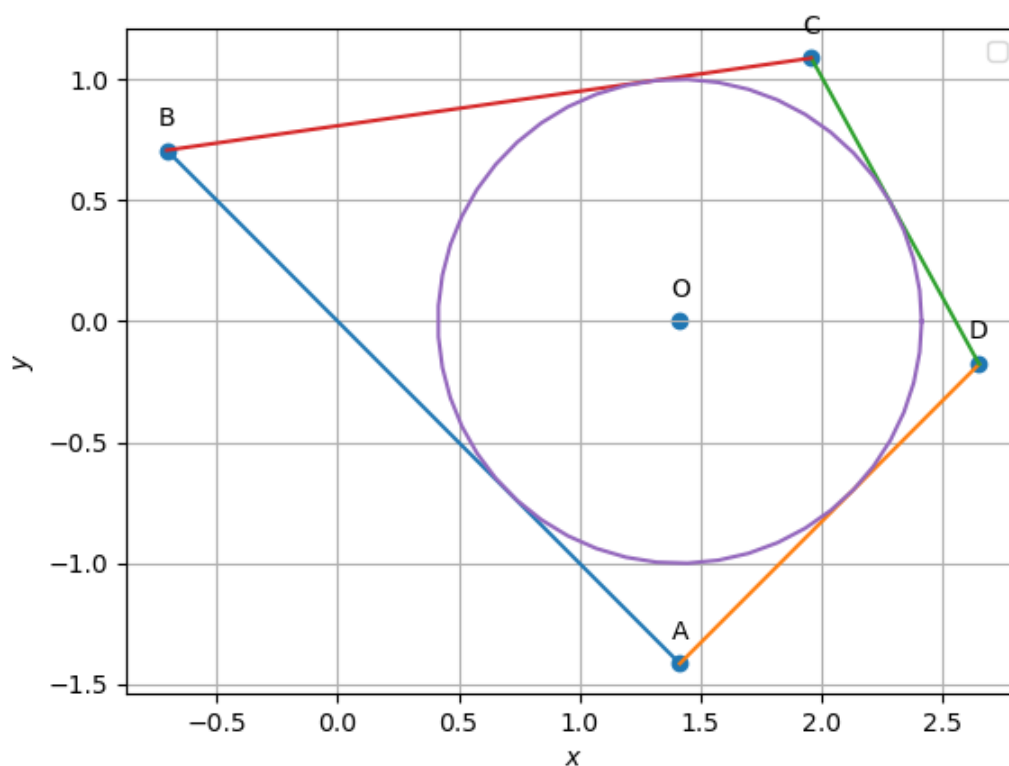


Figure 5.10:

**Solution:**

- (a) Draw the circle.
- (b) Choose the point  $\mathbf{A}$ .
- (c) Draw the tangents from  $\mathbf{A}$  to the circle.

- (d) Choose points **B**, **D** on the tangents.
- (e) From **B**, **D**, draw tangents to the circle intersecting at **C**.
9. In Fig. 5.11,  $XY$  and  $EF$  are two parallel tangents to a circle with centre **O** and another tangent  $AB$  with point of contact **C** intersecting  $XY$  at **A** and  $EF$  at **B**. Prove that  $\angle AOB = 90^\circ$ .

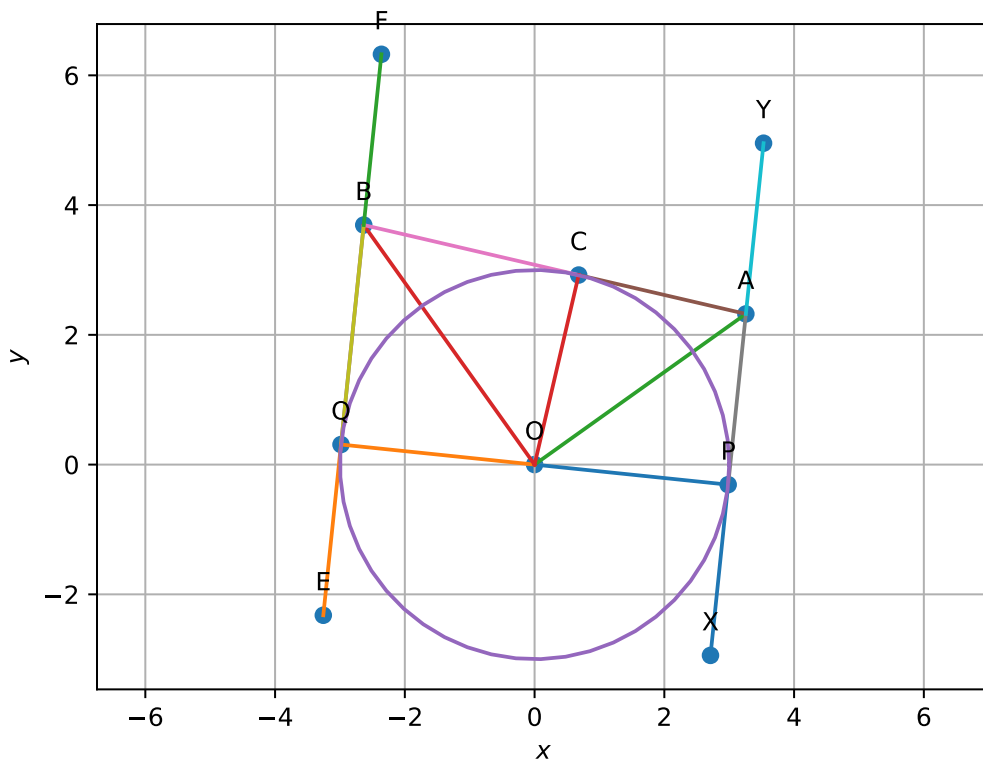


Figure 5.11:

**Solution:**

10. Prove that the angle between the two tangents drawn from an external point to a circle is supplementary to the angle subtended by the line-segment joining the points



of contact at the centre.

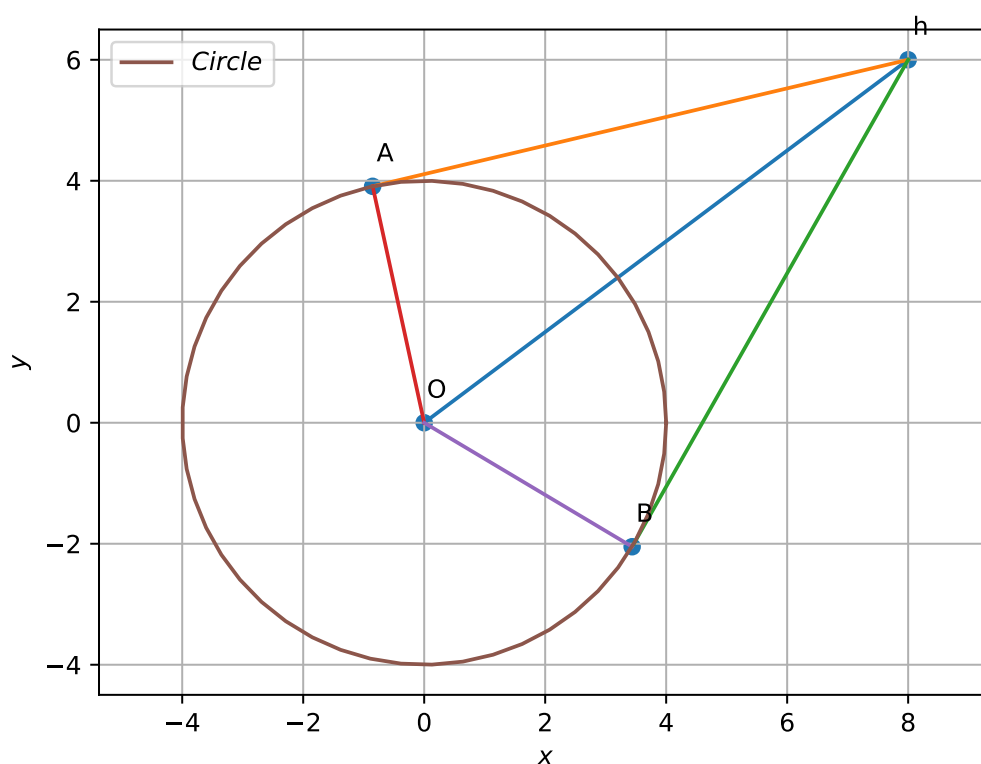


Figure 5.12:

**Solution:** Follow the approach in Problem 6 for constructing the tangents to the circle.

11.

12. A triangle  $ABC$  is drawn to circumscribe a circle of radius 4cm such that the segments  $BD$  and  $DC$  into which  $BC$  is divided by the point of contact  $D$  are of lengths 8cm and 6cm respectively. Find the sides  $AB$  and  $AC$ .

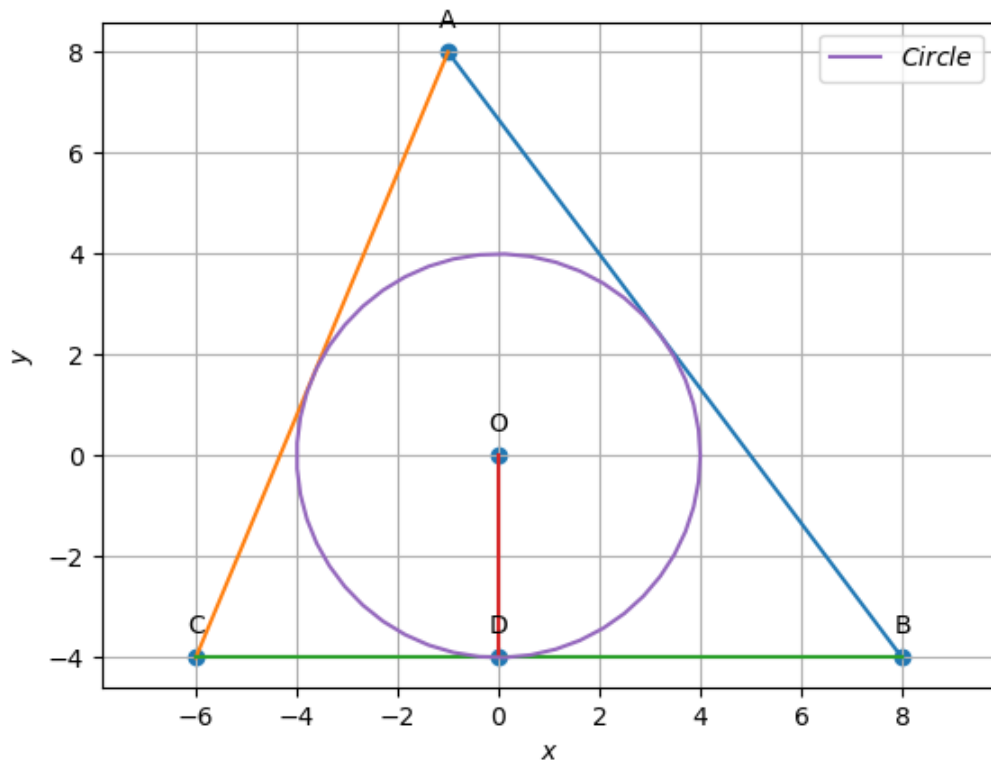


Figure 5.13:

13.

## 5.6. Construction of Tangents to a Circle

1. Draw a circle of radius 6 cm. From a point 10 cm away from its centre, construct the pair of tangents to the circle and measure their lengths.

**Solution:** Follow the approach in Problem 6.

2. Construct a tangent to a circle of radius 4cm from a point on the concentric circle of

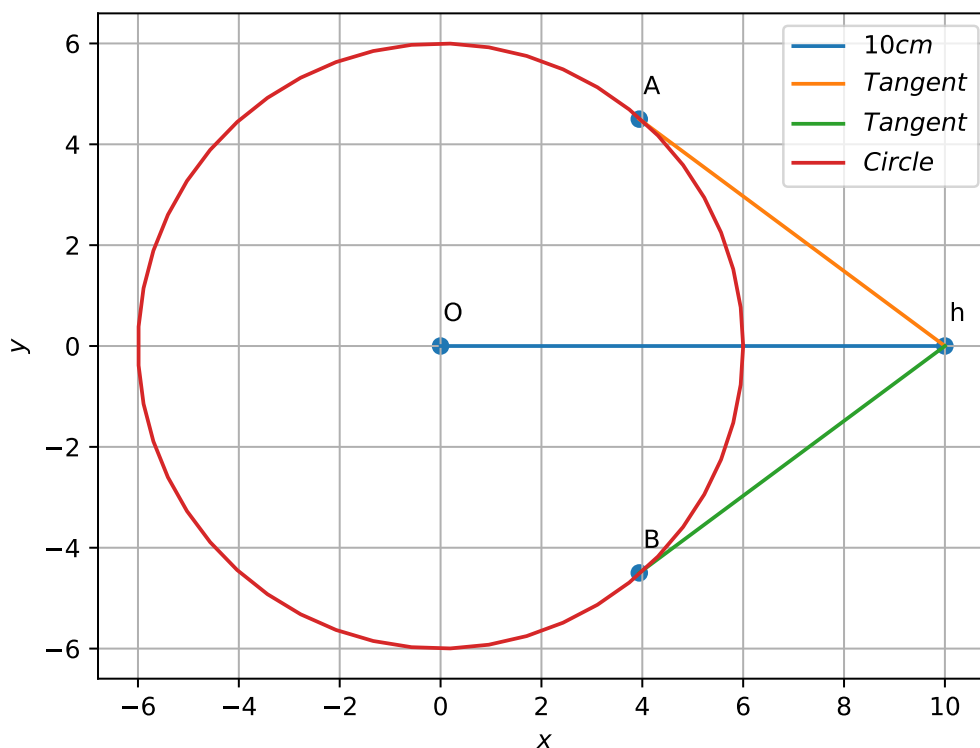


Figure 5.14:

radius 6cm and measure its length. Also verify the measurement by actual calculation.

**Solution:** See Fig. 5.15.

3. Draw a circle of radius 3 cm. Take two points **P** and **Q** on one of its extended diameter each at a distance of 7 cm from its centre. Draw tangents to the circle from these two points **P** and **Q**.

**Solution:** See Fig. 5.16.

4. Draw a pair of tangents to a circle of radius 5 cm which are inclined to each other at an angle of  $60^\circ$ .

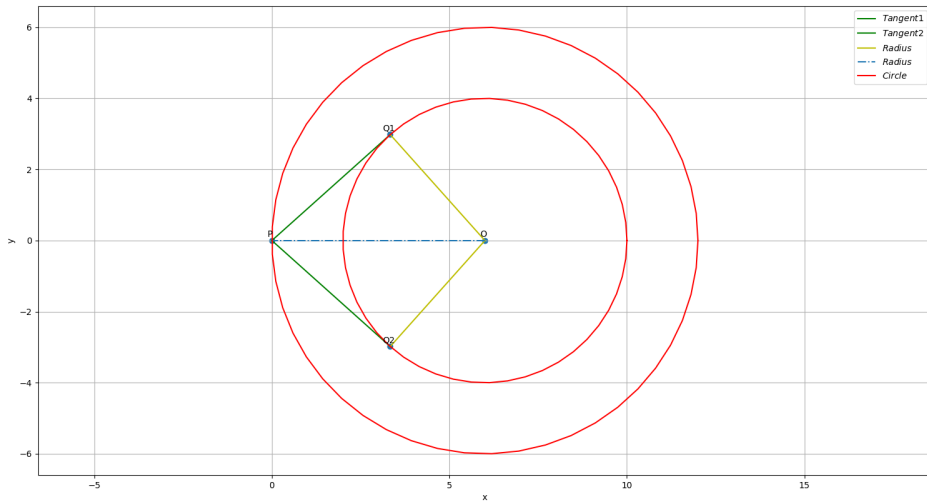


Figure 5.15:

**Solution:** See Fig. 5.17.

5. Draw a line segment  $AB$  of length 8cm. Taking **A** as centre, draw a circle of radius 4cm and taking **B** as centre, draw another circle of radius 3cm. Construct tangents to each circle from the centre of the circle.

**Solution:** See Fig. 5.18.

6. Let  $ABC$  be a right triangle in which  $AB = 6\text{cm}$ ,  $BC = 8\text{cm}$  and  $\angle B = 90^\circ$ .  $BD$  is the perpendicular from **B** on  $AC$ . The circle through **B**, **C**, **D** is drawn. Construct the tangents from **A** to this circle.

**Solution:** See Fig. 5.19.

$$BD \perp AC \implies \mathbf{O} = \frac{\mathbf{B} + \mathbf{C}}{2} \quad (5.54)$$

From (C.29), the coordinates of **D** can be obtained.

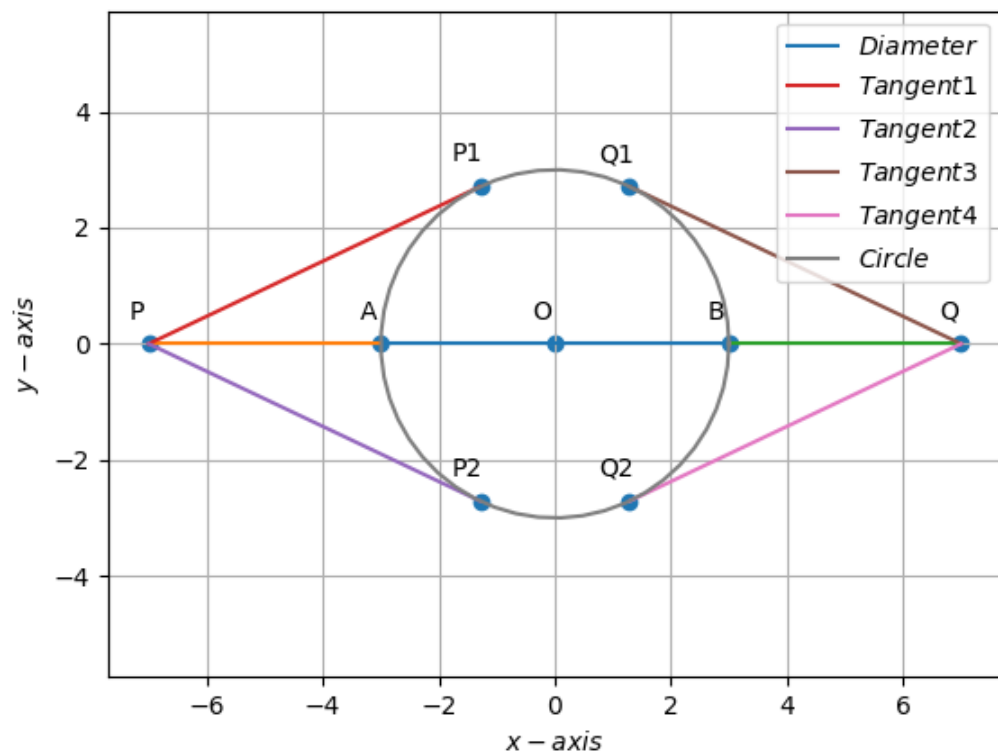


Figure 5.16:

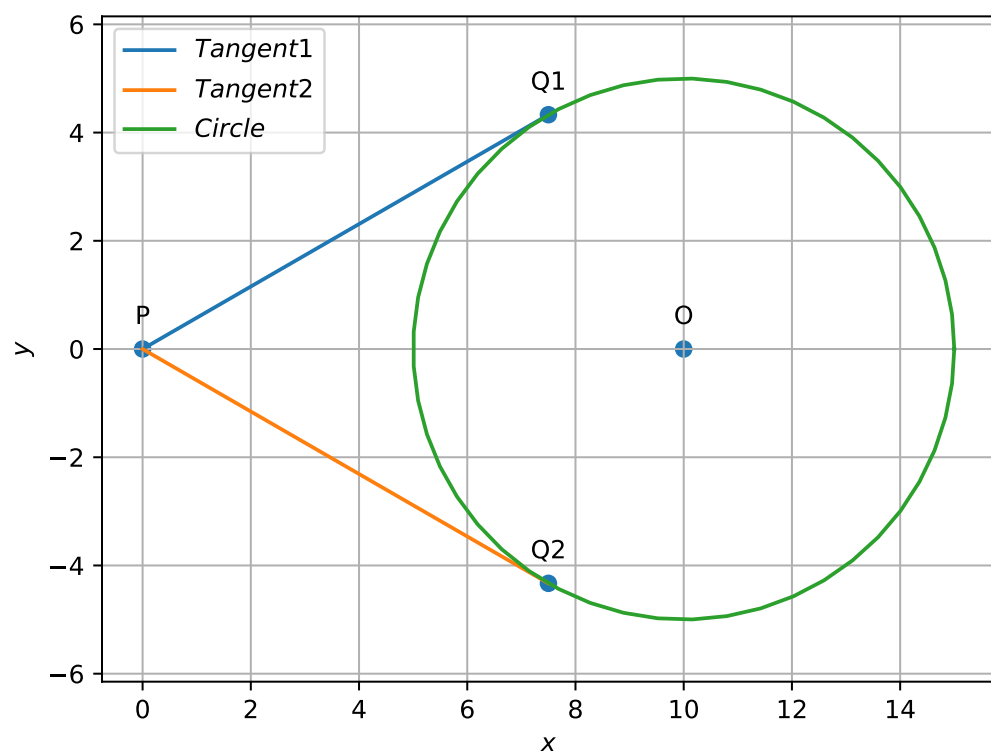


Figure 5.17:

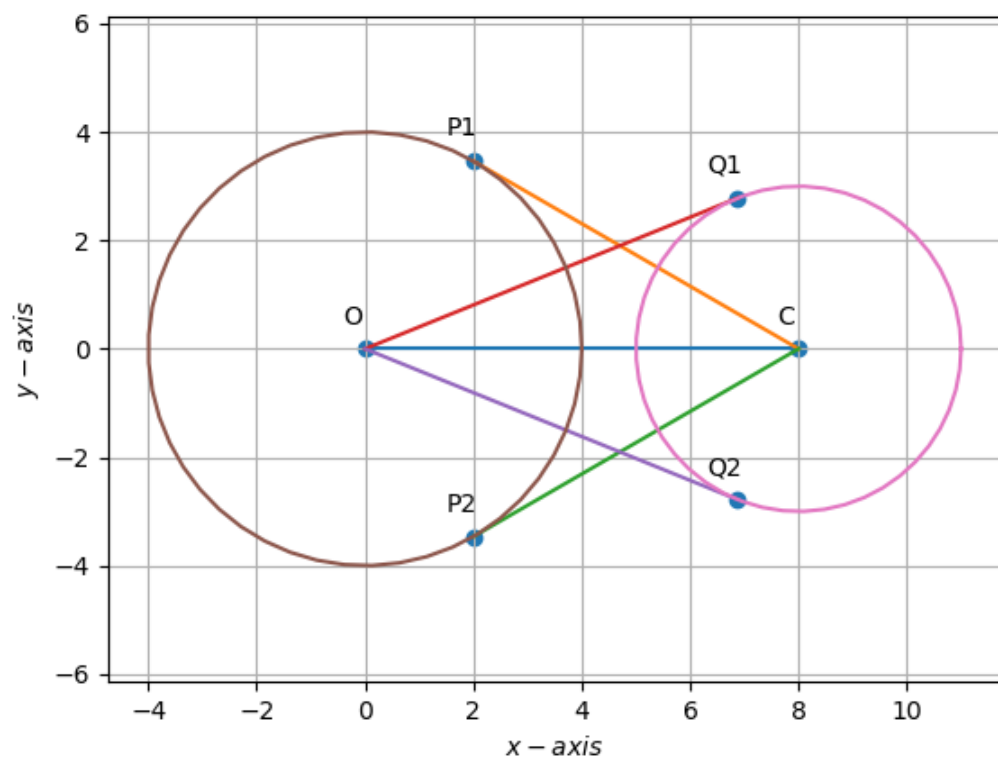


Figure 5.18:

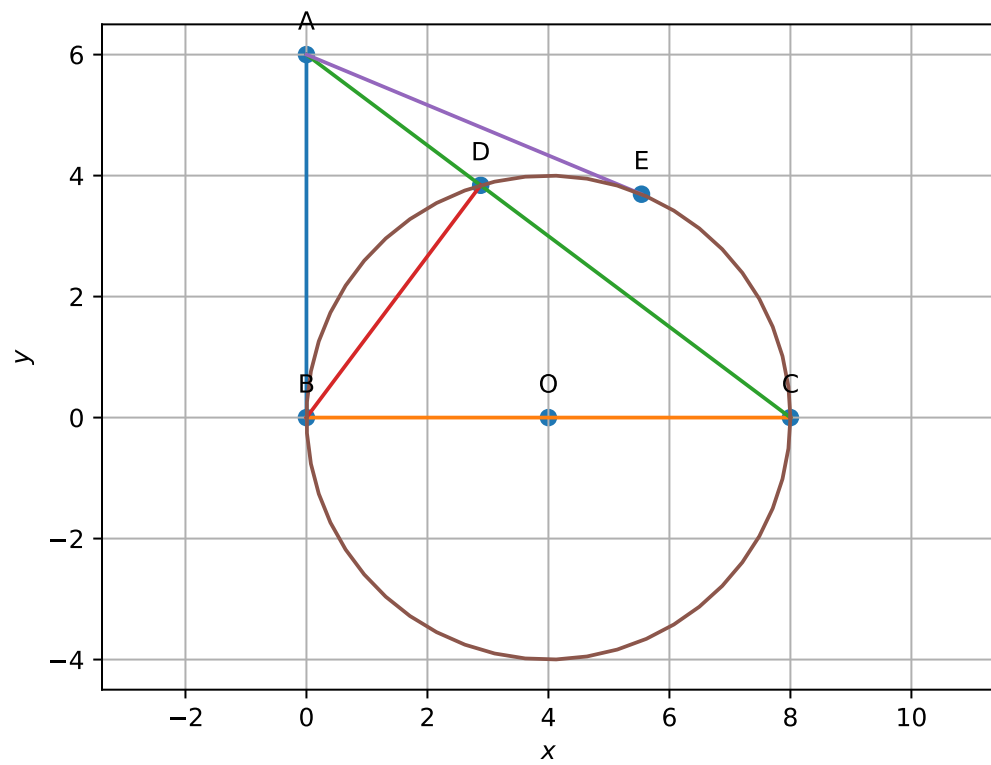


Figure 5.19:



## Chapter 6

# Conics

### 6.1. Miscellaneous

- 1.
2. An arch is in the form of a parabola with its axis vertical. The arch is 10m high and 5m wide at the base. How wide is it 2m from the vertex of the parabola?

**Solution:**

- 3.
4. An arch is in the form of a semi-ellipse. It is 8 m wide and 2 m high at the centre. Find the height of the arch at a point 1.5 m from one end.

**Solution:**

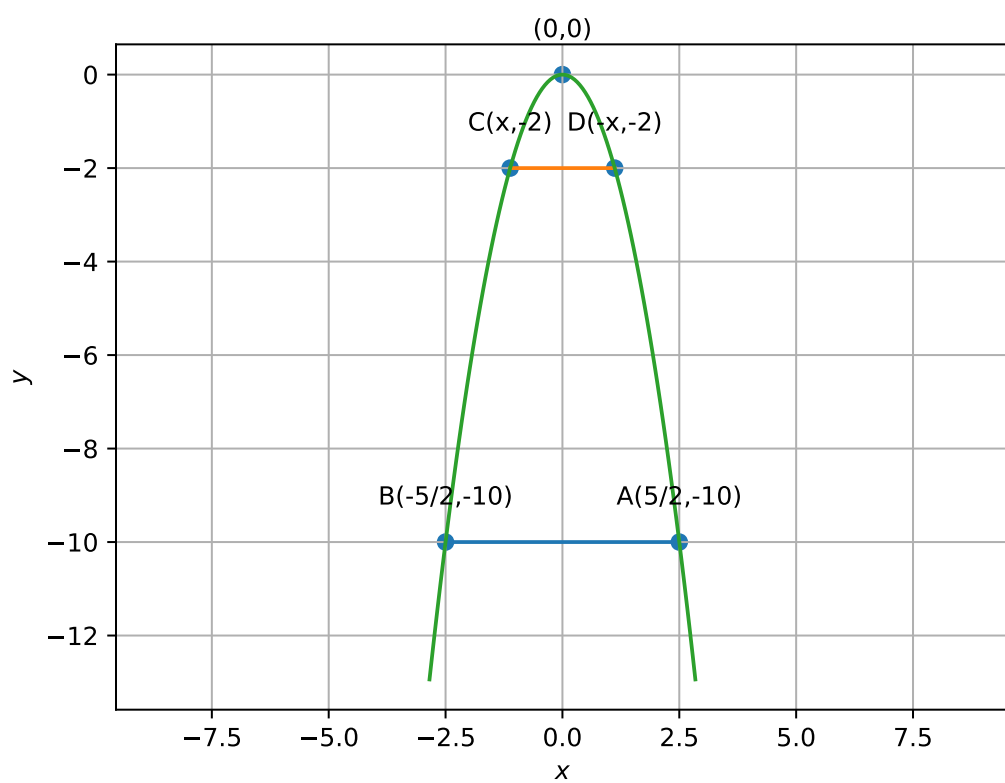


Figure 6.1:

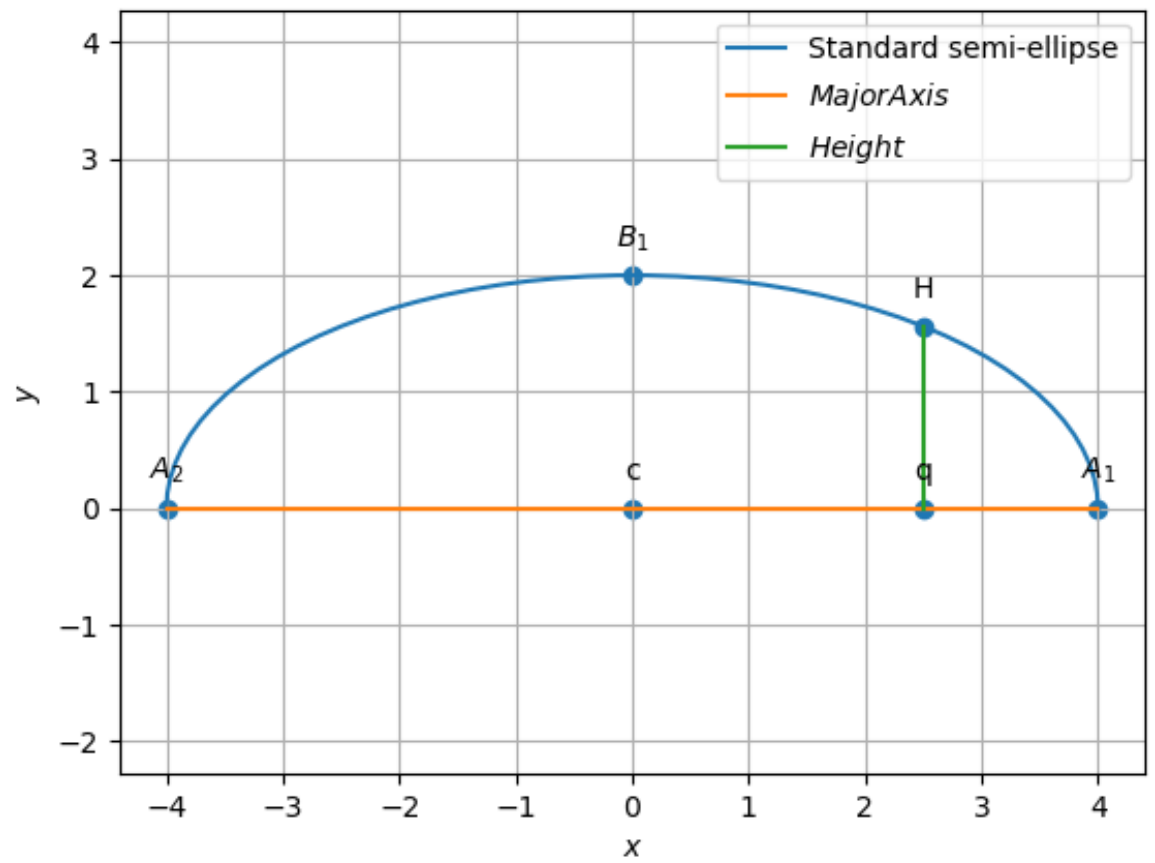


Figure 6.2:



## Chapter 7

# Triangle

### 7.1. Triangle Constructions

1. Construct a triangle  $ABC$  in which  $BC = 7cm$ ,  $\angle B = 75^\circ$  and  $AB + AC = 13cm$ .

**Solution:** See Fig. 7.1.

Using the cosine formula in  $\triangle ABC$ ,

$$b^2 = a^2 + c^2 - 2ac \cos B \quad (7.1)$$

$$\implies (b + c)(b - c) = a^2 - 2ac \cos B \quad (7.2)$$

$$\text{or, } K(b - c) = a^2 - 2ac \cos B \quad (7.3)$$

where

$$K = b + c \quad (7.4)$$

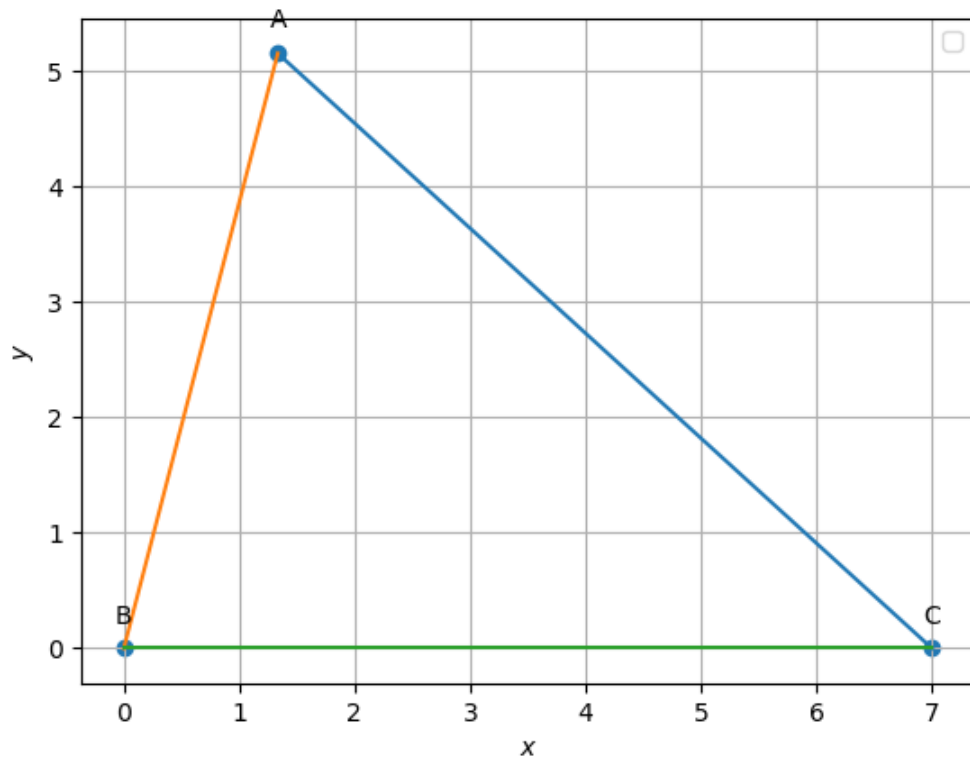


Figure 7.1:

From (7.3) and (7.4),

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} b \\ c \end{pmatrix} = \begin{pmatrix} \frac{a^2 - 2ac \cos B}{K} \\ K \end{pmatrix} \quad (7.5)$$

$$\Rightarrow \begin{pmatrix} b \\ c \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \frac{a^2 - 2ac \cos B}{K} \\ K \end{pmatrix} \quad (7.6)$$

$$\therefore \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = 2\mathbf{I} \quad (7.7)$$

From (7.6)

$$c = \frac{1}{2} \mathbf{e}_2^\top \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \frac{a^2}{K} \\ K \end{pmatrix} - \frac{2ac \cos B}{K} \quad (7.8)$$

$$\Rightarrow c = \frac{1}{2 \left(1 + \frac{2a \cos B}{K}\right)} \mathbf{e}_2^\top \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \frac{a^2}{K} \\ K \end{pmatrix} \quad (7.9)$$

The coordinates of  $\triangle ABC$  can then be expressed as

$$\mathbf{A} = c \begin{pmatrix} \cos B \\ \sin B \end{pmatrix}, \mathbf{B} = \mathbf{0}, \mathbf{C} = \begin{pmatrix} a \\ 0 \end{pmatrix}. \quad (7.10)$$

2. Construct a triangle  $ABC$  in which  $BC = 8\text{cm}$ ,  $\angle B = 45^\circ$  and  $AB - AC = 3.5\text{cm}$ .

**Solution:** See Fig. 7.2. Using the cosine formula in  $\triangle ABC$ ,

$$b^2 = a^2 + c^2 - 2ac \cos B \quad (7.11)$$

$$\Rightarrow (b+c)(b-c) = a^2 - 2ac \cos B \quad (7.12)$$

$$\text{or, } K(b+c) = a^2 - 2ac \cos B \quad (7.13)$$

where

$$-K = b - c \quad (7.14)$$

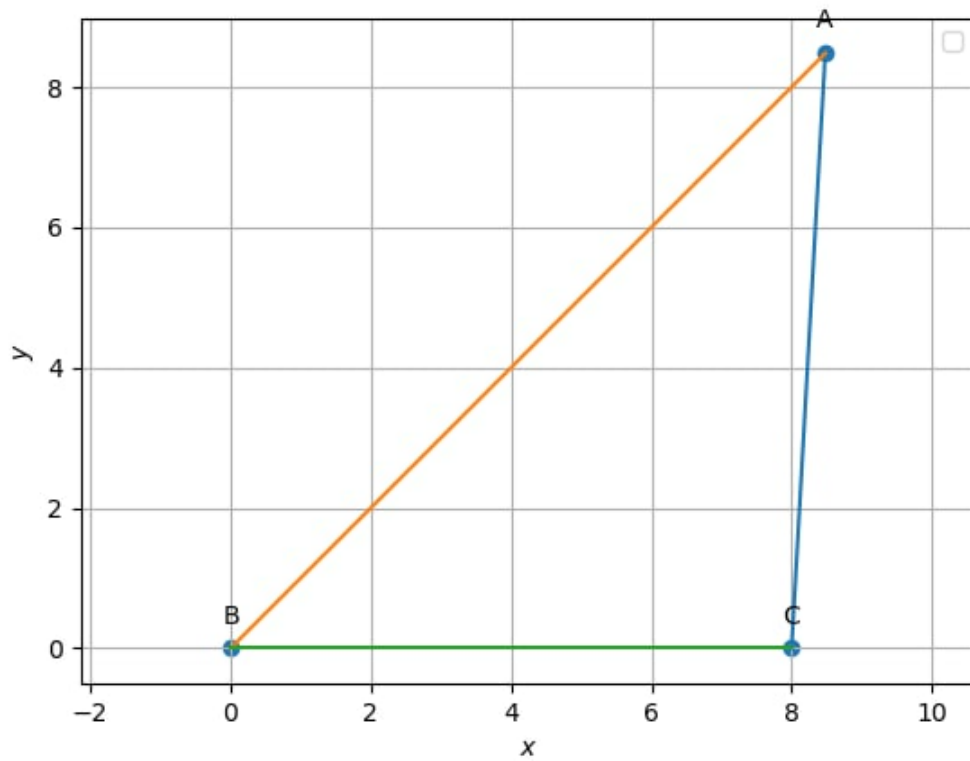


Figure 7.2:

From (7.13) and (7.14),

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} b \\ c \end{pmatrix} = \begin{pmatrix} \frac{a^2 - 2ac \cos B}{K} \\ -K \end{pmatrix} \quad (7.15)$$

$$\Rightarrow \begin{pmatrix} b \\ c \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \frac{a^2 - 2ac \cos B}{K} \\ -K \end{pmatrix} \quad (7.16)$$

$$\therefore \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = 2\mathbf{I} \quad (7.17)$$



From (7.16)

$$c = \frac{1}{2} \mathbf{e}_2^\top \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \frac{a^2}{K} \\ -K \end{pmatrix} - \frac{2ac \cos B}{K} \quad (7.18)$$

$$\Rightarrow c = \frac{1}{2 \left(1 + \frac{2a \cos B}{K}\right)} \mathbf{e}_2^\top \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \frac{a^2}{K} \\ -K \end{pmatrix} \quad (7.19)$$

The coordinates of  $\triangle ABC$  can then be expressed as

$$\mathbf{A} = c \begin{pmatrix} \cos B \\ \sin B \end{pmatrix}, \mathbf{B} = \mathbf{0}, \mathbf{C} = \begin{pmatrix} a \\ 0 \end{pmatrix}. \quad (7.20)$$

3. Construct a triangle  $PQR$  in which  $QR = 6cm$ ,  $\angle Q = 60^\circ$  and  $PR - PQ = 2cm$ .

**Solution:** Same as Problem 1 with

$$\angle Q = \angle B, QR = a, PR = b, PQ = c \quad (7.21)$$

4. Construct a triangle  $XYZ$  in which  $\angle Y = 30^\circ$ ,  $\angle Z = 90^\circ$  and  $XY + YZ + ZX = 11cm$ .

**Solution:** From the given information,

$$x + y + z = K \quad (7.22)$$

$$y \cos Z + z \cos Y - x = 0 \quad (7.23)$$

$$y \sin Z - z \sin Y = 0 \quad (7.24)$$

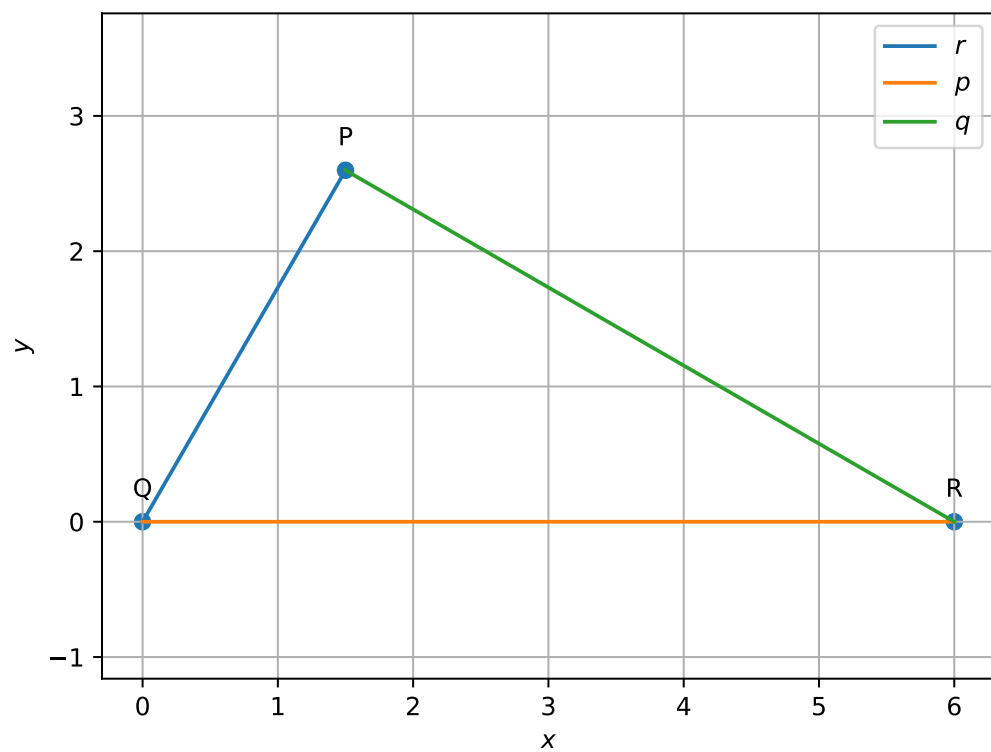


Figure 7.3:

resulting in the matrix equation

$$\begin{pmatrix} 1 & 1 & 1 \\ \cos Z & \cos Y & -1 \\ \sin Z & -\sin Y & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = K \mathbf{e}_1 \quad (7.25)$$

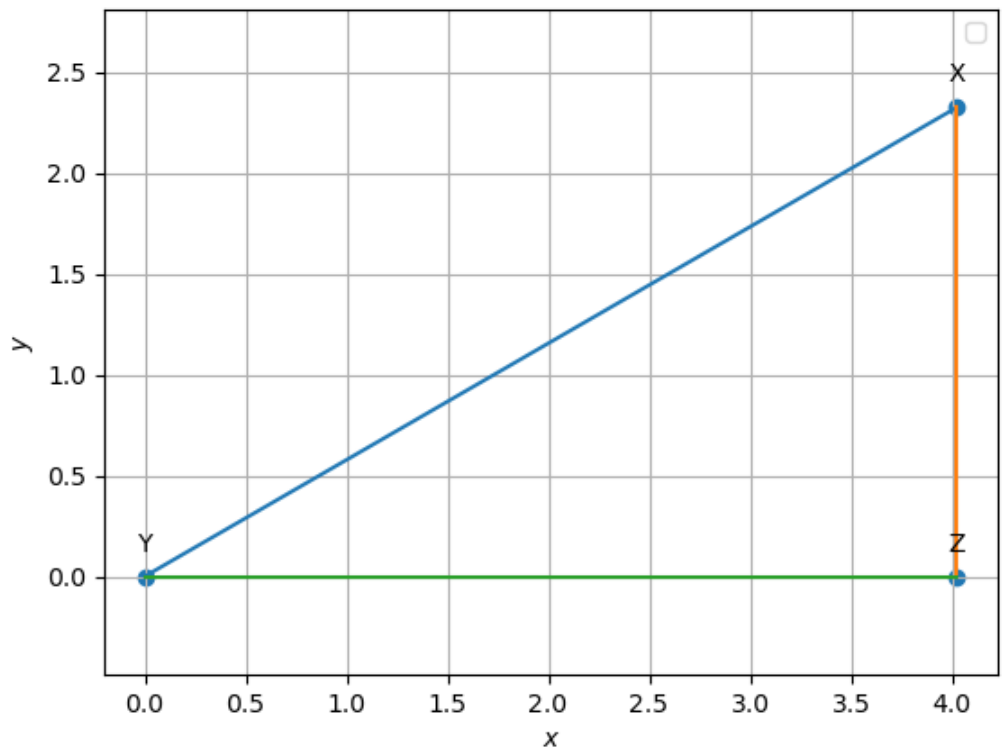


Figure 7.4:

which can be solved to obtain all the sides.  $\triangle XYZ$  can then be plotted using

$$\mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix}, \mathbf{Y} = \mathbf{0}, \mathbf{Z} = \begin{pmatrix} x \\ 0 \end{pmatrix} \quad (7.26)$$

5. Construct a right triangle whose base is 12cm and sum of its hypotenuse and other side is 18cm.

**Solution:** From the given information, let

$$a = 12, \angle B = 90^\circ, b + c = 18 \tag{7.27}$$

We need to find  $b$ . This is similar to Problem 1.

# Appendix A

## Vectors

### A.1. $2 \times 1$ vectors

A.1.1. Let

$$\mathbf{A} \equiv \vec{A} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \tag{A.1}$$

$$\equiv a_1 \vec{i} + a_2 \vec{j}, \tag{A.2}$$

$$\mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \tag{A.3}$$

be  $2 \times 1$  vectors. Then, the determinant of the  $2 \times 2$  matrix

$$\mathbf{M} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \end{pmatrix} \tag{A.4}$$

is defined as

$$|\mathbf{M}| = \begin{vmatrix} \mathbf{A} & \mathbf{B} \end{vmatrix} \quad (\text{A.5})$$

$$= \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1 \quad (\text{A.6})$$

A.1.2. The value of the cross product of two vectors is given by (A.5).

A.1.3. The area of the triangle with vertices  $\mathbf{A}, \mathbf{B}, \mathbf{C}$  is given by

$$\frac{1}{2} \|(\mathbf{A} - \mathbf{B}) \times (\mathbf{A} - \mathbf{C})\| = \frac{1}{2} \|\mathbf{A} \times \mathbf{B} + \mathbf{B} \times \mathbf{C} + \mathbf{C} \times \mathbf{A}\| \quad (\text{A.7})$$

A.1.4. If

$$\|\mathbf{A} \times \mathbf{B}\| = \|\mathbf{C} \times \mathbf{D}\|, \quad \text{then} \quad (\text{A.8})$$

$$\mathbf{A} \times \mathbf{B} = \pm (\mathbf{C} \times \mathbf{D}) \quad (\text{A.9})$$

where the sign depends on the orientation of the vectors.

A.1.5. The median divides the triangle into two triangles of equal area.

A.1.6. The transpose of  $\mathbf{A}$  is defined as

$$\mathbf{A}^\top = \begin{pmatrix} a_1 & a_2 \end{pmatrix} \quad (\text{A.10})$$

A.1.7. The inner product or dot product is defined as

$$\mathbf{A}^\top \mathbf{B} \equiv \mathbf{A} \cdot \mathbf{B} \quad (\text{A.11})$$

$$= \begin{pmatrix} a_1 & a_2 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = a_1 b_1 + a_2 b_2 \quad (\text{A.12})$$

A.1.8. norm of  $\mathbf{A}$  is defined as

$$\|\mathbf{A}\| \equiv \left| \vec{A} \right| \quad (\text{A.13})$$

$$= \sqrt{\mathbf{A}^\top \mathbf{A}} = \sqrt{a_1^2 + a_2^2} \quad (\text{A.14})$$

Thus,

$$\|\lambda \mathbf{A}\| \equiv \left| \lambda \vec{A} \right| \quad (\text{A.15})$$

$$= |\lambda| \|\mathbf{A}\| \quad (\text{A.16})$$

A.1.9. The distance between the points  $\mathbf{A}$  and  $\mathbf{B}$  is given by

$$\|\mathbf{A} - \mathbf{B}\| \quad (\text{A.17})$$

A.1.10. Let  $\mathbf{x}$  be equidistant from the points  $\mathbf{A}$  and  $\mathbf{B}$ . Then

$$(\mathbf{A} - \mathbf{B})^\top \mathbf{x} = \frac{\|\mathbf{A}\|^2 - \|\mathbf{B}\|^2}{2} \quad (\text{A.18})$$

**Solution:**

$$\|\mathbf{x} - \mathbf{A}\| = \|\mathbf{A} - \mathbf{B}\| \quad (\text{A.19})$$

$$\implies \|\mathbf{x} - \mathbf{A}\|^2 = \|\mathbf{x} - \mathbf{B}\|^2 \quad (\text{A.20})$$

which can be expressed as

$$\begin{aligned} (\mathbf{x} - \mathbf{A})^\top (\mathbf{x} - \mathbf{A}) &= (\mathbf{x} - \mathbf{B})^\top (\mathbf{x} - \mathbf{B}) \\ \implies \|\mathbf{x}\|^2 - 2\mathbf{x}^\top \mathbf{A} + \|\mathbf{A}\|^2 &= \|\mathbf{x}\|^2 - 2\mathbf{x}^\top \mathbf{B} + \|\mathbf{B}\|^2 \quad (\text{A.21}) \end{aligned}$$

which can be simplified to obtain (A.18).

A.1.11. If  $\mathbf{x}$  lies on the  $x$ -axis and is equidistant from the points  $\mathbf{A}$  and  $\mathbf{B}$ ,

$$\mathbf{x} = x\mathbf{e}_1 \quad (\text{A.22})$$

where

$$x = \frac{\|\mathbf{A}\|^2 - \|\mathbf{B}\|^2}{2(\mathbf{A} - \mathbf{B})^\top \mathbf{e}_1} \quad (\text{A.23})$$

**Solution:** From (A.18).

$$x(\mathbf{A} - \mathbf{B})^\top \mathbf{e}_1 = \frac{\|\mathbf{A}\|^2 - \|\mathbf{B}\|^2}{2} \quad (\text{A.24})$$

yielding (A.23).



A.1.12. The angle between two vectors is given by

$$\theta = \cos^{-1} \frac{\mathbf{A}^\top \mathbf{B}}{\|\mathbf{A}\| \|\mathbf{B}\|} \quad (\text{A.25})$$

A.1.13. If two vectors are orthogonal (perpendicular),

$$\mathbf{A}^\top \mathbf{B} = 0 \quad (\text{A.26})$$

A.1.14. For an isocles triangle  $ABC$  ith  $AB = AC$ , the median  $AD \perp BC$ .

A.1.15. The direction vector of the line joining two points  $\mathbf{A}, \mathbf{B}$  is given by

$$\mathbf{m} = \mathbf{A} - \mathbf{B} \quad (\text{A.27})$$

A.1.16. The points  $\mathbf{A}, \mathbf{A}, \mathbf{A}$

A.1.17. The unit vector in the direction of  $\mathbf{m}$  is defined as

$$\frac{\mathbf{m}}{\|\mathbf{m}\|} \quad (\text{A.28})$$

A.1.18. If the direction vector of a line is expressed as

$$\mathbf{m} = \begin{pmatrix} 1 \\ m \end{pmatrix}, \quad (\text{A.29})$$

the  $m$  is defined to be the slope of the line.

A.1.19.  $AB \parallel CD$  if

$$\mathbf{A} - \mathbf{B} = k(\mathbf{C} - \mathbf{D}) \quad (\text{A.30})$$

A.1.20. The normal vector to  $\mathbf{m}$  is defined by

$$\mathbf{m}^\top \mathbf{n} = 0 \quad (\text{A.31})$$

A.1.21. If

$$\mathbf{m}^\top \mathbf{n}_1 = 0 \quad (\text{A.32})$$

$$\mathbf{m}^\top \mathbf{n}_2 = 0, \quad (\text{A.33})$$

$$\mathbf{n}_1 \parallel \mathbf{n}_2 \quad (\text{A.34})$$

A.1.22. The point  $\mathbf{P}$  that divides the line segment  $AB$  in the ratio  $k : 1$  is given by

$$\mathbf{P} = \frac{k\mathbf{B} + \mathbf{A}}{k + 1} \quad (\text{A.35})$$

A.1.23. The standard basis vectors are defined as

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad (\text{A.36})$$

$$\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (\text{A.37})$$

A.1.24. If  $ABCD$  be a parallelogram,

$$\mathbf{B} - \mathbf{A} = \mathbf{C} - \mathbf{D} \quad (\text{A.38})$$

A.1.25. Diagonals of a parallelogram bisect each other.

A.1.26. The area of the parallelogram with vertices  $\mathbf{A}, \mathbf{B}, \mathbf{C}$  and  $\mathbf{D}$  is given by

$$\|(\mathbf{A} - \mathbf{B}) \times (\mathbf{A} - \mathbf{D})\| = \|\mathbf{A} \times \mathbf{B} + \mathbf{B} \times \mathbf{C} + \mathbf{C} \times \mathbf{A}\| \quad (\text{A.39})$$

A.1.27. Points  $\mathbf{A}, \mathbf{B}$  and  $\mathbf{C}$  form a triangle if

$$p(\mathbf{A} - \mathbf{B}) + q(\mathbf{A} - \mathbf{C}) = 0 \quad (\text{A.40})$$

$$\text{or, } (p + q)\mathbf{A} - p\mathbf{B} - q\mathbf{C} = 0 \quad (\text{A.41})$$

$$\implies p = 0, q = 0 \quad (\text{A.42})$$

are linearly independent.

A.1.28. In  $\triangle ABC$ , if  $\mathbf{D}, \mathbf{E}$  divide the lines  $AB, AC$  in the ratio  $k : 1$  respectively, then  $DE \parallel BC$ .

*Proof.* From (A.35),

$$\mathbf{D} = \frac{k\mathbf{B} + \mathbf{A}}{k + 1} \quad (\text{A.43})$$

$$\mathbf{E} = \frac{k\mathbf{C} + \mathbf{A}}{k + 1} \quad (\text{A.44})$$

$$\implies \mathbf{D} - \mathbf{E} = \frac{k}{k + 1} (\mathbf{B} - \mathbf{C}) \quad (\text{A.45})$$

Thus, from Appendix A.1.18,  $DE \parallel BC$ .

□

A.1.29. In  $\triangle ABC$ , if  $DE \parallel BC$ ,  $\mathbf{D}$  and  $\mathbf{E}$  divide the lines  $AB, AC$  in the same ratio.

*Proof.* If  $DE \parallel BC$ , from (A.30)

$$(\mathbf{B} - \mathbf{C}) = k (\mathbf{D} - \mathbf{E}) \quad (\text{A.46})$$

Using (A.35), let

$$\mathbf{D} = \frac{k_1\mathbf{B} + \mathbf{A}}{k_1 + 1} \quad (\text{A.47})$$

$$\mathbf{E} = \frac{k_2\mathbf{C} + \mathbf{A}}{k_2 + 1} \quad (\text{A.48})$$

Substituting the above in (A.46), after some algebra, we obtain

$$(p + q) \mathbf{A} - p\mathbf{B} - q\mathbf{C} = 0 \quad (\text{A.49})$$

where

$$p = \frac{1}{k} - \frac{k_1}{k_1 + 1}, q = \frac{1}{k} - \frac{k_1}{k_1 + 1} \quad (\text{A.50})$$

From (A.41),

$$p = q = 0 \quad (\text{A.51})$$

$$\implies k_1 = k_2 = \frac{1}{k - 1} \quad (\text{A.52})$$

□

## A.2. $3 \times 1$ vectors

A.2.1. Let

$$\mathbf{A} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \equiv a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{j}, \quad (\text{A.53})$$

$$\mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \quad (\text{A.54})$$

and

$$\mathbf{A}_{ij} = \begin{pmatrix} a_i \\ a_j \end{pmatrix}, \quad (\text{A.55})$$

$$\mathbf{B}_{ij} = \begin{pmatrix} b_i \\ b_j \end{pmatrix}. \quad (\text{A.56})$$

A.2.2. The cross product or vector product of  $\mathbf{A}, \mathbf{B}$  is defined as

$$\mathbf{A} \times \mathbf{B} = \begin{pmatrix} \begin{vmatrix} \mathbf{A}_{23} & \mathbf{B}_{23} \\ \mathbf{A}_{31} & \mathbf{B}_{31} \\ \mathbf{A}_{12} & \mathbf{B}_{12} \end{vmatrix} \end{pmatrix} \quad (\text{A.57})$$

A.2.3. Verify that

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} \quad (\text{A.58})$$

A.2.4. The area of a triangle is given by

$$\frac{1}{2} \|\mathbf{A} \times \mathbf{B}\| \quad (\text{A.59})$$

## Appendix B

# Matrices

### B.1. Eigenvalues and Eigenvectors

B.1.1. The eigenvalue  $\lambda$  and the eigenvector  $\mathbf{x}$  for a matrix  $\mathbf{A}$  are defined as,

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x} \tag{B.1}$$

B.1.2. The eigenvalues are calculated by solving the equation

$$f(\lambda) = \left| \lambda\mathbf{I} - \mathbf{A} \right| = 0 \tag{B.2}$$

The above equation is known as the characteristic equation.

B.1.3. According to the Cayley-Hamilton theorem,

$$f(\lambda) = 0 \implies f(\mathbf{A}) = 0 \tag{B.3}$$

B.1.4. The trace of a square matrix is defined to be the sum of the diagonal elements.

$$\text{tr}(\mathbf{A}) = \sum_{i=1}^N a_{ii}. \tag{B.4}$$

where  $a_{ii}$  is the  $i$ th diagonal element of the matrix  $\mathbf{A}$ .

B.1.5. The trace of a matrix is equal to the sum of the eigenvalues

$$\text{tr}(\mathbf{A}) = \sum_{i=1}^N \lambda_i \quad (\text{B.5})$$

## B.2. Determinants

B.2.1. Let

$$\mathbf{A} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}. \quad (\text{B.6})$$

be a  $3 \times 3$  matrix. Then,

$$\begin{aligned} |\mathbf{A}| &= a_1 \begin{pmatrix} b_2 & c_2 \\ b_3 & c_3 \end{pmatrix} - a_2 \begin{pmatrix} b_1 & c_1 \\ b_3 & c_3 \end{pmatrix} \\ &\quad + a_3 \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}. \end{aligned} \quad (\text{B.7})$$

B.2.2. Let  $\lambda_1, \lambda_2, \dots, \lambda_n$  be the eigenvalues of a matrix  $\mathbf{A}$ . Then, the product of the eigenvalues is equal to the determinant of  $\mathbf{A}$ .

$$|\mathbf{A}| = \prod_{i=1}^n \lambda_i \quad (\text{B.8})$$



B.2.3.

$$\left| \mathbf{AB} \right| = \left| \mathbf{A} \right| \left| \mathbf{B} \right| \quad (\text{B.9})$$

B.2.4. If  $\mathbf{A}$  be an  $n \times n$  matrix,

$$\left| k\mathbf{A} \right| = k^n \left| \mathbf{A} \right| \quad (\text{B.10})$$

## B.3. Rank of a Matrix

B.3.1. The rank of a matrix is defined as the number of linearly independent rows. This is also known as the row rank.

B.3.2. Row rank = Column rank.

B.3.3. The rank of a matrix is obtained as the number of nonzero rows obtained after row reduction.

B.3.4. An  $n \times n$  matrix is invertible if and only if its rank is  $n$ .

B.3.5. Points  $\mathbf{A}, \mathbf{B}, \mathbf{C}$  are on a line if

$$\text{rank} \begin{pmatrix} \mathbf{A} \\ \mathbf{B} \\ \mathbf{C} \end{pmatrix} = 1 \quad (\text{B.11})$$

B.3.6. Points  $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$  form a paralelogram if

$$\text{rank} \begin{pmatrix} \mathbf{A} \\ \mathbf{B} \\ \mathbf{C} \\ \mathbf{D} \end{pmatrix} = 1, \text{rank} \begin{pmatrix} \mathbf{A} \\ \mathbf{B} \\ \mathbf{C} \end{pmatrix} = 2 \quad (\text{B.12})$$

## B.4. Inverse of a Matrix

B.4.1. For a  $2 \times 2$  matrix

$$\mathbf{A} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}, \quad (\text{B.13})$$

the inverse is given by

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|} \begin{pmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{pmatrix}, \quad (\text{B.14})$$

B.4.2. For higher order matrices, the inverse should be calculated using row operations.

## B.5. Orthogonality

B.5.1. The rotation matrix is defined as

$$\mathbf{R}_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad \theta \in [0, 2\pi] \quad (\text{B.15})$$

B.5.2. The rotation matrix is orthogonal

$$\mathbf{R}_\theta^\top \mathbf{R}_\theta = \mathbf{R}_\theta \mathbf{R}_\theta^\top = \mathbf{I} \quad (\text{B.16})$$

B.5.3. If the angle of rotation is  $\frac{\pi}{2}$ ,

$$\mathbf{m}^\top \mathbf{n} = 0 \implies \mathbf{n} = \mathbf{R}_{\frac{\pi}{2}} \mathbf{m} \quad (\text{B.17})$$

B.5.4.

$$\mathbf{n}^\top \mathbf{h} = 1 \implies \mathbf{n} = \frac{\mathbf{e}_1}{\mathbf{e}_1^\top \mathbf{h}} + \mu \mathbf{R}_{\frac{\pi}{2}} \mathbf{h}, \quad \mu \in \mathbb{R}. \quad (\text{B.18})$$

B.5.5. The affine transformation is given by

$$\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c} \quad (\text{Affine Transformation}) \quad (\text{B.19})$$

where  $\mathbf{P}$  is invertible.

B.5.6. The eigenvalue decomposition of a symmetric matrix  $\mathbf{V}$  is given by

$$\mathbf{P}^\top \mathbf{V} \mathbf{P} = \mathbf{D}. \quad (\text{Eigenvalue Decomposition}) \quad (\text{B.20})$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad (\text{B.21})$$

$$\mathbf{P} = \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix}, \quad \mathbf{P}^\top = \mathbf{P}^{-1}, \quad (\text{B.22})$$



## Appendix C

# Linear Forms

### C.1. Two Dimensions

C.1.1. The equation of a line is given by

$$\mathbf{n}^\top \mathbf{x} = c \tag{C.1}$$

where  $\mathbf{n}$  is the normal vector of the line.

C.1.2. The equation of a line with normal vector  $\mathbf{n}$  and passing through a point  $\mathbf{A}$  is given by

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{A}) = 0 \tag{C.2}$$

C.1.3. The equation of a line  $L$  is also given by

$$\mathbf{n}^\top \mathbf{x} = \begin{cases} 0 & \mathbf{0} \in L \\ 1 & \text{otherwise} \end{cases} \tag{C.3}$$

C.1.4. Points  $\mathbf{A}, \mathbf{B}, \mathbf{C}$  are collinear if

$$\text{rank} \begin{pmatrix} \mathbf{B} - \mathbf{A} & \mathbf{C} - \mathbf{A} \end{pmatrix} < 2 \quad (\text{C.4})$$

*Proof.* From (C.1),

$$\mathbf{n}^\top \mathbf{A} = c \quad (\text{C.5})$$

$$\mathbf{n}^\top \mathbf{B} = c \quad (\text{C.6})$$

$$\mathbf{n}^\top \mathbf{C} = c \quad (\text{C.7})$$

which can be expressed as

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix}^\top \mathbf{n} = c \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad (\text{C.8})$$

The above set of equations are consistent if

$$\text{rank} \begin{pmatrix} 1 & 1 & 1 \\ \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} < 3 \quad (\text{C.9})$$

$$\Rightarrow \text{rank} \begin{pmatrix} 1 & 0 & 0 \\ \mathbf{A} & \mathbf{B} - \mathbf{A} & \mathbf{C} - \mathbf{A} \end{pmatrix} < 3 \quad (\text{C.10})$$

using the fact that row rank = column rank. The above condition can then be expressed as (C.4).

□

C.1.5. The parametric equation of a line is given by

$$\mathbf{x} = \mathbf{A} + \lambda \mathbf{m} \quad (\text{C.11})$$

where  $\mathbf{m}$  is the direction vector of the line and  $\mathbf{A}$  is any point on the line.

C.1.6. Let  $\mathbf{A}$  and  $\mathbf{B}$  be two points on a straight line and let  $\mathbf{P} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$  be any point on it.

If  $p_2$  is known, then

$$\mathbf{P} = \mathbf{A} + \frac{p_2 - \mathbf{e}_2^\top \mathbf{A}}{\mathbf{e}_2^\top (\mathbf{B} - \mathbf{A})} (\mathbf{B} - \mathbf{A}) \quad (\text{C.12})$$

**Solution:** The equation of the line can be expressed in parametric form as

$$\mathbf{x} = \mathbf{A} + \lambda (\mathbf{B} - \mathbf{A}) \quad (\text{C.13})$$

$$\implies \mathbf{P} = \mathbf{A} + \lambda (\mathbf{B} - \mathbf{A}) \quad (\text{C.14})$$

$$\implies \mathbf{e}_2^\top \mathbf{P} = \mathbf{e}_2^\top \mathbf{A} + \lambda \mathbf{e}_2^\top (\mathbf{B} - \mathbf{A}) \quad (\text{C.15})$$

$$\implies p_2 = \mathbf{e}_2^\top \mathbf{A} + \lambda \mathbf{e}_2^\top (\mathbf{B} - \mathbf{A}) \quad (\text{C.16})$$

$$\text{or, } \lambda = \frac{p_2 - \mathbf{e}_2^\top \mathbf{A}}{\mathbf{e}_2^\top (\mathbf{B} - \mathbf{A})} \quad (\text{C.17})$$

yielding (C.12).

C.1.7. The distance from a point  $\mathbf{P}$  to the line in (C.1) is given by

$$d = \frac{|\mathbf{n}^\top \mathbf{P} - c|}{\|\mathbf{n}\|} \quad (\text{C.18})$$

**Solution:** Without loss of generality, let  $\mathbf{A}$  be the foot of the perpendicular from  $\mathbf{P}$

to the line in (C.11). The equation of the normal to (C.1) can then be expressed as

$$\mathbf{x} = \mathbf{A} + \lambda \mathbf{n} \quad (\text{C.19})$$

$$\implies \mathbf{P} - \mathbf{A} = \lambda \mathbf{n} \quad (\text{C.20})$$

$\because \mathbf{P}$  lies on (C.19). From the above, the desired distance can be expressed as

$$d = \|\mathbf{P} - \mathbf{A}\| = |\lambda| \|\mathbf{n}\| \quad (\text{C.21})$$

From (C.20),

$$\mathbf{n}^\top (\mathbf{P} - \mathbf{A}) = \lambda \mathbf{n}^\top \mathbf{n} = \lambda \|\mathbf{n}\|^2 \quad (\text{C.22})$$

$$\implies |\lambda| = \frac{|\mathbf{n}^\top (\mathbf{P} - \mathbf{A})|}{\|\mathbf{n}\|^2} \quad (\text{C.23})$$

Substituting the above in (C.21) and using the fact that

$$\mathbf{n}^\top \mathbf{A} = c \quad (\text{C.24})$$

from (C.1), yields (C.18)

C.1.8. The distance from the origin to the line in (C.1) is given by

$$d = \frac{|c|}{\|\mathbf{n}\|} \quad (\text{C.25})$$



C.1.9. The distance between the parallel lines

$$\begin{aligned}\mathbf{n}^\top \mathbf{x} &= c_1 \\ \mathbf{n}^\top \mathbf{x} &= c_2\end{aligned}\tag{C.26}$$

is given by

$$d = \frac{|c_1 - c_2|}{\|\mathbf{n}\|}\tag{C.27}$$

C.1.10. The equation of the line perpendicular to (C.1) and passing through the point  $\mathbf{P}$  is given by

$$\mathbf{m}^\top (\mathbf{x} - \mathbf{P}) = 0\tag{C.28}$$

C.1.11. The foot of the perpendicular from  $\mathbf{P}$  to the line in (C.1) is given by

$$\begin{pmatrix} \mathbf{m} & \mathbf{n} \end{pmatrix}^\top \mathbf{x} = \begin{pmatrix} \mathbf{m}^\top \mathbf{P} \\ c \end{pmatrix}\tag{C.29}$$

**Solution:** From (C.1) and (C.2) the foot of the perpendicular satisfies the equations

$$\mathbf{n}^\top \mathbf{x} = c\tag{C.30}$$

$$\mathbf{m}^\top (\mathbf{x} - \mathbf{P}) = 0\tag{C.31}$$

where  $\mathbf{m}$  is the direction vector of the given line. Combining the above into a matrix equation results in (C.29).

C.1.12. The equations of the angle bisectors of the lines

$$\mathbf{n}_1^\top \mathbf{x} = c_1 \quad (\text{C.32})$$

$$\mathbf{n}_2^\top \mathbf{x} = c_2 \quad (\text{C.33})$$

are given by

$$\frac{\mathbf{n}_1^\top \mathbf{x} - c_1}{\|\mathbf{n}_1\|} = \pm \frac{\mathbf{n}_2^\top \mathbf{x} - c_2}{\|\mathbf{n}_2\|} \quad (\text{C.34})$$

*Proof.* Any point on the angle bisector is equidistant from the lines.  $\square$

## C.2. Three Dimensions

C.2.1. Points  $\mathbf{A}, \mathbf{B}, \mathbf{C}$  are on a line if

$$\text{rank} \begin{pmatrix} \mathbf{A} \\ \mathbf{B} \\ \mathbf{C} \end{pmatrix} = 1 \quad (\text{C.35})$$

C.2.2. Points  $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$  form a parallelogram if

$$\text{rank} \begin{pmatrix} \mathbf{A} \\ \mathbf{B} \\ \mathbf{C} \\ \mathbf{D} \end{pmatrix} = 1, \text{rank} \begin{pmatrix} \mathbf{A} \\ \mathbf{B} \\ \mathbf{C} \end{pmatrix} = 2 \quad (\text{C.36})$$

C.2.3. The equation of a line is given by (C.11)

C.2.4. The equation of a plane is given by (C.1)

C.2.5. The distance from the origin to the line in (C.1) is given by (C.25)

C.2.6. The distance from a point  $\mathbf{P}$  to the line in (C.11) is given by

$$d = \|\mathbf{A} - \mathbf{P}\|^2 - \frac{\{\mathbf{m}^\top (\mathbf{A} - \mathbf{P})\}^2}{\|\mathbf{m}\|^2} \quad (\text{C.37})$$

**Solution:**

$$d(\lambda) = \|\mathbf{A} + \lambda \mathbf{m} - \mathbf{P}\| \quad (\text{C.38})$$

$$\implies d^2(\lambda) = \|\mathbf{A} + \lambda \mathbf{m} - \mathbf{P}\|^2 \quad (\text{C.39})$$

which can be simplified to obtain

$$d^2(\lambda) = \lambda^2 \|\mathbf{m}\|^2 + 2\lambda \mathbf{m}^\top (\mathbf{A} - \mathbf{P}) + \|\mathbf{A} - \mathbf{P}\|^2 \quad (\text{C.40})$$

which is of the form

$$d^2(\lambda) = a\lambda^2 + 2b\lambda + c \quad (\text{C.41})$$

$$= a \left\{ \left( \lambda + \frac{b}{a} \right)^2 + \left[ \frac{c}{a} - \left( \frac{b}{a} \right)^2 \right] \right\} \quad (\text{C.42})$$

with

$$a = \|\mathbf{m}\|^2, b = \mathbf{m}^\top (\mathbf{A} - \mathbf{P}), c = \|\mathbf{A} - \mathbf{P}\|^2 \quad (\text{C.43})$$

which can be expressed as From the above,  $d^2(\lambda)$  is smallest when upon substituting

from (C.43)

$$\lambda + \frac{b}{2a} = 0 \implies \lambda = -\frac{b}{2a} \quad (\text{C.44})$$

$$= -\frac{\mathbf{m}^\top (\mathbf{A} - \mathbf{P})}{\|\mathbf{m}\|^2} \quad (\text{C.45})$$

and consequently,

$$d_{\min}(\lambda) = a \left( \frac{c}{a} - \left( \frac{b}{a} \right)^2 \right) \quad (\text{C.46})$$

$$= c - \frac{b^2}{a} \quad (\text{C.47})$$

yielding (C.37) after substituting from (C.43).

C.2.7. The distance between the parallel planes (C.26) is given by (C.27).

C.2.8. The plane

$$\mathbf{n}^\top \mathbf{x} = c \quad (\text{C.48})$$

contains the line

$$\mathbf{x} = \mathbf{A} + \lambda \mathbf{m} \quad (\text{C.49})$$

if

$$\mathbf{m}^\top \mathbf{n} = 0 \quad (\text{C.50})$$

**Solution:** Any point on the line (C.49) should also satisfy (C.48). Hence,

$$\mathbf{n}^\top (\mathbf{A} + \lambda \mathbf{m}) = \mathbf{n}^\top \mathbf{A} = c \quad (\text{C.51})$$

which can be simplified to obtain (C.50)

C.2.9. The foot of the perpendicular from a point  $\mathbf{P}$  to the plane

$$\mathbf{n}^\top \mathbf{x} = c \quad (\text{C.52})$$

is given by

$$\mathbf{x} = \mathbf{P} + \frac{c - \mathbf{n}^\top \mathbf{P}}{\|\mathbf{n}\|^2} \mathbf{n} \quad (\text{C.53})$$

**Solution:** The equation of the line perpendicular to the given plane and passing through  $\mathbf{P}$  is

$$\mathbf{x} = \mathbf{P} + \lambda \mathbf{n} \quad (\text{C.54})$$

From (C.65), the intersection of the above line with the given plane is (C.53).

C.2.10. The image of a point  $\mathbf{P}$  with respect to the plane

$$\mathbf{n}^\top \mathbf{x} = c \quad (\text{C.55})$$

is given by

$$\mathbf{R} = \mathbf{P} + 2 \frac{c - \mathbf{n}^\top \mathbf{P}}{\|\mathbf{n}\|^2} \quad (\text{C.56})$$

**Solution:** Let  $\mathbf{R}$  be the desired image. Then, substituting the expression for the foot of the perpendicular from  $\mathbf{P}$  to the given plane using (C.53),

$$\frac{\mathbf{P} + \mathbf{R}}{2} = \mathbf{P} + \frac{c - \mathbf{n}^\top \mathbf{P}}{\|\mathbf{n}\|^2} \quad (\text{C.57})$$

C.2.11. Let a plane pass through the points  $\mathbf{A}, \mathbf{B}$  and be perpendicular to the plane

$$\mathbf{n}^\top \mathbf{x} = c \quad (\text{C.58})$$

Then the equation of this plane is given by

$$\mathbf{p}^\top \mathbf{x} = 1 \quad (\text{C.59})$$

where

$$\mathbf{p} = \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{n} \end{pmatrix}^{-\top} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad (\text{C.60})$$

**Solution:** From the given information,

$$\mathbf{p}^\top \mathbf{A} = d \quad (\text{C.61})$$

$$\mathbf{p}^\top \mathbf{B} = d \quad (\text{C.62})$$

$$\mathbf{p}^\top \mathbf{n} = 0 \quad (\text{C.63})$$

$\therefore$  the normal vectors to the two planes will also be perpendicular. The system of equations in (C.63) can be expressed as the matrix equation

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{n} \end{pmatrix}^\top \mathbf{p} = d \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad (\text{C.64})$$

which yields (C.60) upon normalising with  $d$ .

C.2.12. The intersection of the line represented by (C.11) with the plane represented by (C.1) is given by

$$\mathbf{x} = \mathbf{A} + \frac{c - \mathbf{n}^\top \mathbf{A}}{\mathbf{n}^\top \mathbf{m}} \mathbf{m} \quad (\text{C.65})$$

**Solution:** From (C.11) and (C.1),

$$\mathbf{x} = \mathbf{A} + \lambda \mathbf{m} \quad (\text{C.66})$$

$$\mathbf{n}^\top \mathbf{x} = c \quad (\text{C.67})$$

$$\implies \mathbf{n}^\top (\mathbf{A} + \lambda \mathbf{m}) = c \quad (\text{C.68})$$

which can be simplified to obtain

$$\mathbf{n}^\top \mathbf{A} + \lambda \mathbf{n}^\top \mathbf{m} = c \quad (\text{C.69})$$

$$\implies \lambda = \frac{c - \mathbf{n}^\top \mathbf{A}}{\mathbf{n}^\top \mathbf{m}} \quad (\text{C.70})$$

Substituting the above in (C.68) yields (C.65).

C.2.13. The foot of the perpendicular from the point  $\mathbf{P}$  to the line represented by (C.11) is given by

$$\mathbf{x} = \mathbf{A} + \frac{\mathbf{m}^\top (\mathbf{P} - \mathbf{A})}{\|\mathbf{m}\|^2} \mathbf{m} \quad (\text{C.71})$$

**Solution:** Let the equation of the line be

$$\mathbf{x} = \mathbf{A} + \lambda \mathbf{m} \quad (\text{C.72})$$

The equation of the plane perpendicular to the given line passing through  $\mathbf{P}$  is given by

$$\mathbf{m}^\top (\mathbf{x} - \mathbf{P}) = 0 \quad (\text{C.73})$$

$$\implies \mathbf{m}^\top \mathbf{x} = \mathbf{m}^\top \mathbf{P} \quad (\text{C.74})$$

The desired foot of the perpendicular is the intersection of (C.72) with (C.73) which can be obtained from (C.65) as (C.71)

C.2.14. The foot of the perpendicular from a point  $\mathbf{P}$  to a plane is  $\mathbf{Q}$ . The equation of the



plane is given by

$$(\mathbf{P} - \mathbf{Q})^\top (\mathbf{x} - \mathbf{Q}) = 0 \quad (\text{C.75})$$

**Solution:** The normal vector to the plane is given by

$$\mathbf{n} = \mathbf{P} - \mathbf{Q} \quad (\text{C.76})$$

Hence, the equation of the plane is (C.75).

C.2.15. Let  $\mathbf{A}, \mathbf{B}, \mathbf{C}$  be points on a plane. The equation of the plane is then given by

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix}^\top \mathbf{n} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad (\text{C.77})$$

**Solution:** Let the equation of the plane be

$$\mathbf{n}^\top \mathbf{x} = 1 \quad (\text{C.78})$$

Then

$$\mathbf{n}^\top \mathbf{A} = 1 \quad (\text{C.79})$$

$$\mathbf{n}^\top \mathbf{B} = 1 \quad (\text{C.80})$$

$$\mathbf{n}^\top \mathbf{C} = 1 \quad (\text{C.81})$$

which can be combined to obtain (C.77).

C.2.16. (Parallelogram Law) Let  $\mathbf{A}, \mathbf{B}, \mathbf{D}$  be three vertices of a parallelogram. Then the vertex

$\mathbf{C}$  is given by

$$\mathbf{C} = \mathbf{B} + \mathbf{C} - \mathbf{A} \quad (\text{C.82})$$

**Solution:** Shifting  $\mathbf{A}$  to the origin, we obtain a parallelogram with corresponding vertices

$$\mathbf{0}, \mathbf{B} - \mathbf{A}, \mathbf{D} - \mathbf{A} \quad (\text{C.83})$$

The fourth vertex of this parallelogram is then obtained as

$$(\mathbf{B} - \mathbf{A}) + (\mathbf{D} - \mathbf{A}) = \mathbf{D} + \mathbf{B} - 2\mathbf{A} \quad (\text{C.84})$$

Shifting the origin to  $\mathbf{A}$ , the fourth vertex is obtained as

$$\mathbf{C} = \mathbf{D} + \mathbf{B} - 2\mathbf{A} + \mathbf{A} \quad (\text{C.85})$$

$$= \mathbf{D} + \mathbf{B} - \mathbf{A} \quad (\text{C.86})$$

C.2.17. (Affine Transformation) Let  $\mathbf{A}, \mathbf{C}$ , be opposite vertices of a square. The other two points can be obtained as

$$\mathbf{B} = \frac{\|\mathbf{A} - \mathbf{C}\|}{\sqrt{2}} \mathbf{Pe}_1 + \mathbf{A} \quad (\text{C.87})$$

$$\mathbf{D} = \frac{\|\mathbf{A} - \mathbf{C}\|}{\sqrt{2}} \mathbf{Pe}_2 + \mathbf{A} \quad (\text{C.88})$$

where

$$\mathbf{P} = \begin{pmatrix} \cos\left(\theta - \frac{\pi}{4}\right) & \sin\left(\theta - \frac{\pi}{4}\right) \\ \sin\left(\theta - \frac{\pi}{4}\right) & \cos\left(\theta - \frac{\pi}{4}\right) \end{pmatrix} \quad (\text{C.89})$$

and

$$\cos \theta = \frac{(\mathbf{C} - \mathbf{A})^\top \mathbf{e}_1}{\|\mathbf{A} - \mathbf{C}\| \|\mathbf{e}_1\|} \quad (\text{C.90})$$



## Appendix D

# Quadratic Forms

### D.1. Conic equation

D.1.1. Let  $\mathbf{q}$  be a point such that the ratio of its distance from a fixed point  $\mathbf{F}$  and the distance ( $d$ ) from a fixed line

$$L : \mathbf{n}^\top \mathbf{x} = c \tag{D.1}$$

is constant, given by

$$\frac{\|\mathbf{q} - \mathbf{F}\|}{d} = e \tag{D.2}$$

The locus of  $\mathbf{q}$  is known as a conic section. The line  $L$  is known as the directrix and the point  $\mathbf{F}$  is the focus.  $e$  is defined to be the eccentricity of the conic.

(a) For  $e = 1$ , the conic is a parabola

(b) For  $e < 1$ , the conic is an ellipse

(c) For  $e > 1$ , the conic is a hyperbola

D.1.2. The equation of a conic with directrix  $\mathbf{n}^\top \mathbf{x} = c$ , eccentricity  $e$  and focus  $\mathbf{F}$  is given by

$$\mathbf{x}^\top \mathbf{V} \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \quad (\text{D.3})$$

where

$$\mathbf{V} = \|\mathbf{n}\|^2 \mathbf{I} - e^2 \mathbf{n} \mathbf{n}^\top, \quad (\text{D.4})$$

$$\mathbf{u} = ce^2 \mathbf{n} - \|\mathbf{n}\|^2 \mathbf{F}, \quad (\text{D.5})$$

$$f = \|\mathbf{n}\|^2 \|\mathbf{F}\|^2 - c^2 e^2 \quad (\text{D.6})$$

*Proof.* Using Definition D.1.1 and Lemma C.18, for any point  $\mathbf{x}$  on the conic,

$$\|\mathbf{x} - \mathbf{F}\|^2 = e^2 \frac{(\mathbf{n}^\top \mathbf{x} - c)^2}{\|\mathbf{n}\|^2} \quad (\text{D.7})$$

$$\implies \|\mathbf{n}\|^2 (\mathbf{x} - \mathbf{F})^\top (\mathbf{x} - \mathbf{F}) = e^2 (\mathbf{n}^\top \mathbf{x} - c)^2 \quad (\text{D.8})$$

$$\implies \|\mathbf{n}\|^2 (\mathbf{x}^\top \mathbf{x} - 2\mathbf{F}^\top \mathbf{x} + \|\mathbf{F}\|^2) = e^2 \left( c^2 + (\mathbf{n}^\top \mathbf{x})^2 - 2c\mathbf{n}^\top \mathbf{x} \right) \quad (\text{D.9})$$

$$= e^2 \left( c^2 + (\mathbf{x}^\top \mathbf{n} \mathbf{n}^\top \mathbf{x}) - 2c\mathbf{n}^\top \mathbf{x} \right) \quad (\text{D.10})$$

which can be expressed as (D.3) after simplification.

□

D.1.3. The eccentricity, directrices and foci of (D.3) are given by

$$e = \sqrt{1 - \frac{\lambda_1}{\lambda_2}} \quad (\text{D.11})$$

$$\mathbf{n} = \sqrt{\lambda_2} \mathbf{p}_1,$$

$$c = \begin{cases} \frac{e \mathbf{u}^\top \mathbf{n} \pm \sqrt{e^2 (\mathbf{u}^\top \mathbf{n})^2 - \lambda_2 (e^2 - 1) (\|\mathbf{u}\|^2 - \lambda_2 f)}}{\lambda_2 e (e^2 - 1)} & e \neq 1 \\ \frac{\|\mathbf{u}\|^2 - \lambda_2 f}{2 \mathbf{u}^\top \mathbf{n}} & e = 1 \end{cases} \quad (\text{D.12})$$

$$\mathbf{F} = \frac{ce^2 \mathbf{n} - \mathbf{u}}{\lambda_2} \quad (\text{D.13})$$

*Proof.* From (D.4), using the fact that  $\mathbf{V}$  is symmetric with  $\mathbf{V} = \mathbf{V}^\top$ ,

$$\mathbf{V}^\top \mathbf{V} = \left( \|\mathbf{n}\|^2 \mathbf{I} - e^2 \mathbf{n} \mathbf{n}^\top \right)^\top \left( \|\mathbf{n}\|^2 \mathbf{I} - e^2 \mathbf{n} \mathbf{n}^\top \right) \quad (\text{D.14})$$

$$\implies \mathbf{V}^2 = \|\mathbf{n}\|^4 \mathbf{I} + e^4 \mathbf{n} \mathbf{n}^\top \mathbf{n} \mathbf{n}^\top - 2e^2 \|\mathbf{n}\|^2 \mathbf{n} \mathbf{n}^\top \quad (\text{D.15})$$

$$= \|\mathbf{n}\|^4 \mathbf{I} + e^4 \|\mathbf{n}\|^2 \mathbf{n} \mathbf{n}^\top - 2e^2 \|\mathbf{n}\|^2 \mathbf{n} \mathbf{n}^\top \quad (\text{D.16})$$

$$= \|\mathbf{n}\|^4 \mathbf{I} + e^2 (e^2 - 2) \|\mathbf{n}\|^2 \mathbf{n} \mathbf{n}^\top \quad (\text{D.17})$$

$$= \|\mathbf{n}\|^4 \mathbf{I} + (e^2 - 2) \|\mathbf{n}\|^2 \left( \|\mathbf{n}\|^2 \mathbf{I} - \mathbf{V} \right) \quad (\text{D.18})$$

which can be expressed as

$$\mathbf{V}^2 + (e^2 - 2) \|\mathbf{n}\|^2 \mathbf{V} - (e^2 - 1) \|\mathbf{n}\|^4 \mathbf{I} = 0 \quad (\text{D.19})$$

Using the Cayley-Hamilton theorem, (D.19) results in the characteristic equation,

$$\lambda^2 - (2 - e^2) \|\mathbf{n}\|^2 \lambda + (1 - e^2) \|\mathbf{n}\|^4 = 0 \quad (\text{D.20})$$

which can be expressed as

$$\left(\frac{\lambda}{\|\mathbf{n}\|^2}\right)^2 - (2 - e^2) \left(\frac{\lambda}{\|\mathbf{n}\|^2}\right) + (1 - e^2) = 0 \quad (\text{D.21})$$

$$\implies \frac{\lambda}{\|\mathbf{n}\|^2} = 1 - e^2, 1 \quad (\text{D.22})$$

$$\text{or, } \lambda_2 = \|\mathbf{n}\|^2, \lambda_1 = (1 - e^2) \lambda_2 \quad (\text{D.23})$$

From (D.23), the eccentricity of (D.3) is given by (D.11). Multiplying both sides of (D.4) by  $\mathbf{n}$ ,

$$\mathbf{V}\mathbf{n} = \|\mathbf{n}\|^2 \mathbf{n} - e^2 \mathbf{n}\mathbf{n}^\top \mathbf{n} \quad (\text{D.24})$$

$$= \|\mathbf{n}\|^2 (1 - e^2) \mathbf{n} \quad (\text{D.25})$$

$$= \lambda_1 \mathbf{n} \quad (\text{D.26})$$

$$(\text{D.27})$$

from (D.23). Thus,  $\lambda_1$  is the corresponding eigenvalue for  $\mathbf{n}$ . From (B.22) and (D.27), this implies that

$$\mathbf{p}_1 = \frac{\mathbf{n}}{\|\mathbf{n}\|} \quad (\text{D.28})$$

$$\text{or, } \mathbf{n} = \|\mathbf{n}\| \mathbf{p}_1 = \sqrt{\lambda_2} \mathbf{p}_1 \quad (\text{D.29})$$



from (D.23) . From (D.5) and (D.23),

$$\mathbf{F} = \frac{ce^2 \mathbf{n} - \mathbf{u}}{\lambda_2} \quad (\text{D.30})$$

$$\implies \|\mathbf{F}\|^2 = \frac{(ce^2 \mathbf{n} - \mathbf{u})^\top (ce^2 \mathbf{n} - \mathbf{u})}{\lambda_2^2} \quad (\text{D.31})$$

$$\implies \lambda_2^2 \|\mathbf{F}\|^2 = c^2 e^4 \lambda_2 - 2ce^2 \mathbf{u}^\top \mathbf{n} + \|\mathbf{u}\|^2 \quad (\text{D.32})$$

Also, (D.6) can be expressed as

$$\lambda_2 \|\mathbf{F}\|^2 = f + c^2 e^2 \quad (\text{D.33})$$

From (D.32) and (D.33),

$$c^2 e^4 \lambda_2 - 2ce^2 \mathbf{u}^\top \mathbf{n} + \|\mathbf{u}\|^2 = \lambda_2 (f + c^2 e^2) \quad (\text{D.34})$$

$$\implies \lambda_2 e^2 (e^2 - 1) c^2 - 2ce^2 \mathbf{u}^\top \mathbf{n} + \|\mathbf{u}\|^2 - \lambda_2 f = 0 \quad (\text{D.35})$$

yielding (D.13). □

D.1.4. (D.3) represents

(a) a parabola for  $\left| \mathbf{V} \right| = 0$ ,

(b) ellipse for  $\left| \mathbf{V} \right| > 0$  and

(c) hyperbola for  $\left| \mathbf{V} \right| < 0$ .

*Proof.* From (D.11),

$$\frac{\lambda_1}{\lambda_2} = 1 - e^2 \quad (\text{D.36})$$

Also,

$$\left| \mathbf{V} \right| = \lambda_1 \lambda_2 \quad (\text{D.37})$$

yielding Table D.2

□

<b>Eccentricity</b>	<b>Conic</b>	<b>Eigenvalue</b>	<b>Determinant</b>		
$e = 1$	Parabola	$\lambda_1 = 0$		$\mathbf{V}$	$= 0$
$e < 1$	Ellipse	$\lambda_1 > 0, \lambda_2 > 0$		$\mathbf{V}$	$> 0$
$e > 1$	Hyperbola	$\lambda_1 < 0, \lambda_2 > 0$		$\mathbf{V}$	$< 0$

Table D.2:

## D.2. Circles

D.2.1. The equation of a circle is given by

$$\|\mathbf{x}\|^2 + 2\mathbf{u}^\top \mathbf{x} + f = 0 \quad (\text{D.38})$$

D.2.2. For a circle with centre  $\mathbf{c}$  and radius  $r$ ,

$$\mathbf{u} = -\mathbf{c}, f = \|\mathbf{u}\|^2 - r^2 \quad (\text{D.39})$$

D.2.3. Any point  $\mathbf{x}$  on a circle can be expressed as

$$\mathbf{x} = \mathbf{c} + r \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}. \quad (\text{D.40})$$

D.2.4. The equation of the common chord of intersection of two circles is given by

$$\mathbf{u}_1^\top \mathbf{x} - \mathbf{u}_2^\top \mathbf{x} + f_1 - f_2 = 0 \quad (\text{D.41})$$

D.2.5. The line joining the centre of a circle to the mid point of any chord is perpendicular to the chord.

*Proof.* Let  $AB$  be any chord of a circle with centre  $\mathbf{O} = \mathbf{0}$  and radius  $r$ . Then,

$$\|\mathbf{A}\|^2 = \|\mathbf{B}\|^2 = r^2 \quad (\text{D.42})$$

$$\implies \|\mathbf{A}\|^2 - \|\mathbf{B}\|^2 = 0 \quad (\text{D.43})$$

$$\text{or, } (\mathbf{A} - \mathbf{B})^\top (\mathbf{A} + \mathbf{B}) = 0 \quad (\text{D.44})$$

which can be expressed as

$$(\mathbf{A} - \mathbf{B})^\top \left( \frac{\mathbf{A} + \mathbf{B}}{2} - \mathbf{O} \right) = 0 \quad (\text{D.45})$$

□

D.2.6. Let

$$\mathbf{A} = \begin{pmatrix} \cos \theta_1 \\ \sin \theta_1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} \cos \theta_2 \\ \sin \theta_2 \end{pmatrix}, \quad (\text{D.46})$$

be points on a unit circle with centre  $\mathbf{O}$  at the origin. Then

$$\cos AOB = \mathbf{A}^\top \mathbf{B} \quad (\text{D.47})$$

D.2.7. Let

$$\mathbf{A} = \begin{pmatrix} \cos \theta_1 \\ \sin \theta_1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} \cos \theta_2 \\ \sin \theta_2 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, \quad (\text{D.48})$$

be points on a unit circle. Then

$$\cos ACB = \frac{(\mathbf{C} - \mathbf{A})^\top (\mathbf{C} - \mathbf{B})}{\|\mathbf{C} - \mathbf{A}\| \|\mathbf{C} - \mathbf{B}\|} \quad (\text{D.49})$$

$$= \cos \left( \frac{\theta_1 - \theta_2}{2} \right) \quad (\text{D.50})$$

*Proof.* Since

$$(\mathbf{C} - \mathbf{A})^\top (\mathbf{C} - \mathbf{B}) = \|\mathbf{C}\|^2 - \mathbf{C}^\top (\mathbf{A} + \mathbf{B}) + \mathbf{A}^\top \mathbf{B} \quad (\text{D.51})$$

$$= 1 - \cos(\theta - \theta_1) - \cos(\theta - \theta_2) + \cos(\theta_1 - \theta_2) \quad (\text{D.52})$$

$$= 2 \cos^2 \left( \frac{\theta_1 - \theta_2}{2} \right) - 2 \cos \left( \frac{\theta_1 - \theta_2}{2} \right) \cos \left( \theta - \frac{\theta_1 + \theta_2}{2} \right) \quad (\text{D.53})$$

$$= 4 \cos \left( \frac{\theta_1 - \theta_2}{2} \right) \sin \left( \frac{\theta - \theta_1}{2} \right) \sin \left( \frac{\theta - \theta_2}{2} \right), \quad (\text{D.54})$$

and

$$\|\mathbf{C} - \mathbf{A}\|^2 = \|\mathbf{C}\|^2 + \|\mathbf{A}\|^2 - 2\mathbf{C}^\top \mathbf{A}, \quad (\text{D.55})$$

$$= 4 \sin^2 \left( \frac{\theta - \theta_1}{2} \right), \quad (\text{D.56})$$

$$\|\mathbf{C} - \mathbf{B}\|^2 = \|\mathbf{C}\|^2 + \|\mathbf{B}\|^2 - 2\mathbf{C}^\top \mathbf{B}, \quad (\text{D.57})$$

$$= 4 \sin^2 \left( \frac{\theta - \theta_2}{2} \right), \quad (\text{D.58})$$

(D.49) can be expressed as

$$\frac{\cos \left( \frac{\theta_1 - \theta_2}{2} \right) \sin \left( \frac{\theta - \theta_1}{2} \right) \sin \left( \frac{\theta - \theta_2}{2} \right)}{\sin \left( \frac{\theta - \theta_1}{2} \right) \sin \left( \frac{\theta - \theta_2}{2} \right)} \quad (\text{D.59})$$

yielding (D.50)

□

D.2.8. From (D.47) and (D.50),

$$\angle AOB = 2\angle AOC \quad (\text{D.60})$$

## D.3. Standard Form

D.3.1. Using the affine transformation in (B.19), the conic in (D.3) can be expressed in standard form as

$$\mathbf{y}^\top \left( \frac{\mathbf{D}}{f_0} \right) \mathbf{y} = 1 \quad |\mathbf{V}| \neq 0 \quad (\text{D.61})$$

$$\mathbf{y}^\top \mathbf{D} \mathbf{y} = -\eta \mathbf{e}_1^\top \mathbf{y} \quad |\mathbf{V}| = 0 \quad (\text{D.62})$$

where

$$f_0 = \mathbf{u}^\top \mathbf{V}^{-1} \mathbf{u} - f \neq 0 \quad (\text{D.63})$$

$$\eta = 2\mathbf{u}^\top \mathbf{p}_1 \quad (\text{D.64})$$

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (\text{D.65})$$

*Proof.* Using (B.19) (D.3) can be expressed as

$$(\mathbf{P}\mathbf{y} + \mathbf{c})^\top \mathbf{V} (\mathbf{P}\mathbf{y} + \mathbf{c}) + 2\mathbf{u}^\top (\mathbf{P}\mathbf{y} + \mathbf{c}) + f = 0, \quad (\text{D.66})$$

yielding

$$\mathbf{y}^\top \mathbf{P}^\top \mathbf{V} \mathbf{P} \mathbf{y} + 2(\mathbf{V}\mathbf{c} + \mathbf{u})^\top \mathbf{P} \mathbf{y} + \mathbf{c}^\top \mathbf{V} \mathbf{c} + 2\mathbf{u}^\top \mathbf{c} + f = 0 \quad (\text{D.67})$$

From (D.67) and (B.20),

$$\mathbf{y}^\top \mathbf{D} \mathbf{y} + 2(\mathbf{V}\mathbf{c} + \mathbf{u})^\top \mathbf{P} \mathbf{y} + \mathbf{c}^\top (\mathbf{V}\mathbf{c} + \mathbf{u}) + \mathbf{u}^\top \mathbf{c} + f = 0 \quad (\text{D.68})$$

When  $\mathbf{V}^{-1}$  exists, choosing

$$\mathbf{V}\mathbf{c} + \mathbf{u} = \mathbf{0}, \quad \text{or, } \mathbf{c} = -\mathbf{V}^{-1}\mathbf{u}, \quad (\text{D.69})$$

and substituting (D.69) in (D.68) yields (D.61). When  $|\mathbf{V}| = 0, \lambda_1 = 0$  and

$$\mathbf{V}\mathbf{p}_1 = \mathbf{0}, \mathbf{V}\mathbf{p}_2 = \lambda_2 \mathbf{p}_2. \quad (\text{D.70})$$

where  $\mathbf{p}_1, \mathbf{p}_2$  are the eigenvectors of  $\mathbf{V}$  such that (B.20)

$$\mathbf{P} = \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix}, \quad (\text{D.71})$$

Substituting (D.71) in (D.68),

$$\mathbf{y}^\top \mathbf{D} \mathbf{y} + 2 \left( \mathbf{c}^\top \mathbf{V} + \mathbf{u}^\top \right) \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix} \mathbf{y} + \mathbf{c}^\top (\mathbf{V} \mathbf{c} + \mathbf{u}) + \mathbf{u}^\top \mathbf{c} + f = 0 \quad (\text{D.72})$$

$$\implies \mathbf{y}^\top \mathbf{D} \mathbf{y} + 2 \left( (\mathbf{c}^\top \mathbf{V} + \mathbf{u}^\top) \mathbf{p}_1 (\mathbf{c}^\top \mathbf{V} + \mathbf{u}^\top) \mathbf{p}_2 \right) \mathbf{y} + \mathbf{c}^\top (\mathbf{V} \mathbf{c} + \mathbf{u}) + \mathbf{u}^\top \mathbf{c} + f = 0 \quad (\text{D.73})$$

$$\implies \mathbf{y}^\top \mathbf{D} \mathbf{y} + 2 \left( \mathbf{u}^\top \mathbf{p}_1 \quad (\lambda_2 \mathbf{c}^\top + \mathbf{u}^\top) \mathbf{p}_2 \right) \mathbf{y} + \mathbf{c}^\top (\mathbf{V} \mathbf{c} + \mathbf{u}) + \mathbf{u}^\top \mathbf{c} + f = 0 \quad (\text{D.74})$$

upon substituting from (D.70) yielding

$$\lambda_2 y_2^2 + 2 \left( \mathbf{u}^\top \mathbf{p}_1 \right) y_1 + 2 y_2 (\lambda_2 \mathbf{c} + \mathbf{u})^\top \mathbf{p}_2 + \mathbf{c}^\top (\mathbf{V} \mathbf{c} + \mathbf{u}) + \mathbf{u}^\top \mathbf{c} + f = 0 \quad (\text{D.75})$$

Thus, (D.75) can be expressed as (D.62) by choosing

$$\eta = 2 \mathbf{u}^\top \mathbf{p}_1 \quad (\text{D.76})$$

and  $\mathbf{c}$  in (D.68) such that

$$2 \mathbf{P}^\top (\mathbf{V} \mathbf{c} + \mathbf{u}) = \eta \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (\text{D.77})$$

$$\mathbf{c}^\top (\mathbf{V} \mathbf{c} + \mathbf{u}) + \mathbf{u}^\top \mathbf{c} + f = 0 \quad (\text{D.78})$$

$\therefore \mathbf{P}^\top \mathbf{P} = \mathbf{I}$ , multiplying (D.77) by  $\mathbf{P}$  yields

$$(\mathbf{V}\mathbf{c} + \mathbf{u}) = \frac{\eta}{2}\mathbf{p}_1, \quad (\text{D.79})$$

which, upon substituting in (D.78) results in

$$\frac{\eta}{2}\mathbf{c}^\top \mathbf{p}_1 + \mathbf{u}^\top \mathbf{c} + f = 0 \quad (\text{D.80})$$

(D.79) and (D.80) can be clubbed together to obtain (E.7).  $\square$

D.3.2. For the standard conic,

$$\mathbf{P} = \mathbf{I} \quad (\text{D.81})$$

$$\mathbf{u} = \begin{cases} 0 & e \neq 1 \\ \frac{\eta}{2}\mathbf{e}_1 & e = 1 \end{cases} \quad (\text{D.82})$$

$$\lambda_1 \begin{cases} = 0 & e = 1 \\ \neq 0 & e \neq 1 \end{cases} \quad (\text{D.83})$$

where

$$\mathbf{I} = \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 \end{pmatrix} \quad (\text{D.84})$$

is the identity matrix.

D.3.3.



(a) The directrices for the standard conic are given by

$$\mathbf{e}_1^\top \mathbf{y} = \pm \frac{1}{e} \sqrt{\frac{|f_0|}{\lambda_2(1-e^2)}} \quad e \neq 1 \quad (\text{D.85})$$

$$\mathbf{e}_1^\top \mathbf{y} = \frac{\eta}{2\lambda_2} \quad e = 1 \quad (\text{D.86})$$

(b) The foci of the standard ellipse and hyperbola are given by

$$\mathbf{F} = \begin{cases} \pm e \sqrt{\frac{|f_0|}{\lambda_2(1-e^2)}} \mathbf{e}_1 & e \neq 1 \\ -\frac{\eta}{4\lambda_2} \mathbf{e}_1 & e = 1 \end{cases} \quad (\text{D.87})$$

*Proof.* (a) For the standard hyperbola/ellipse in (D.61), from (D.81), (D.12) and (D.82),

$$\mathbf{n} = \sqrt{\frac{\lambda_2}{f_0}} \mathbf{e}_1 \quad (\text{D.88})$$

$$c = \pm \frac{\sqrt{-\frac{\lambda_2}{f_0} (e^2 - 1) \left(\frac{\lambda_2}{f_0}\right)}}{\frac{\lambda_2}{f_0} e (e^2 - 1)} \quad (\text{D.89})$$

$$= \pm \frac{1}{e\sqrt{1-e^2}} \quad (\text{D.90})$$

yielding (D.85) upon substituting from (D.11) and simplifying. For the standard parabola in (D.62), from (D.81), (D.12) and (D.82), noting that  $f = 0$ ,

$$\mathbf{n} = \sqrt{\lambda_2} \mathbf{e}_1 \quad (\text{D.91})$$

$$c = \frac{\left\| \frac{\eta}{2} \mathbf{e}_1 \right\|^2}{2 \left( \frac{\eta}{2} \right) (\mathbf{e}_1)^\top \mathbf{n}} \quad (\text{D.92})$$

$$(\text{D.93})$$

$$= \frac{\eta}{4\sqrt{\lambda_2}} \quad (\text{D.94})$$

yielding (D.86).

- (b) For the standard ellipse/hyperbola, substituting from (D.90), (D.88), (D.82) and (D.11) in (D.13),

$$\mathbf{F} = \pm \frac{\left( \frac{1}{e\sqrt{1-e^2}} \right) (e^2) \sqrt{\frac{\lambda_2}{f_0}} \mathbf{e}_1}{\frac{\lambda_2}{f_0}} \quad (\text{D.95})$$

yielding (D.87) after simplification. For the standard parabola, substituting from (D.94), (D.91), (D.82) and (D.11) in (D.13),

$$\mathbf{F} = \frac{\left( \frac{\eta}{4\sqrt{\lambda_2}} \right) \sqrt{\lambda_2} \mathbf{e}_1 - \frac{\eta}{2} \mathbf{e}_1}{\lambda_2} \quad (\text{D.96})$$

$$(\text{D.97})$$

yielding (D.87) after simplification.

□

## Appendix E

# Conic Parameters

### E.1. Standard Form

E.1.1. The center of the standard ellipse/hyperbola, defined to be the mid point of the line joining the foci, is the origin.

E.1.2. The principal (major) axis of the standard ellipse/hyperbola, defined to be the line joining the two foci is the  $x$ -axis.

*Proof.* From (D.87), it is obvious that the line joining the foci passes through the origin. Also, the direction vector of this line is  $\mathbf{e}_1$ . Thus, the principal axis is the  $x$ -axis.  $\square$

E.1.3. The minor axis of the standard ellipse/hyperbola, defined to be the line orthogonal to the  $x$ -axis is the  $y$ -axis.

E.1.4. The axis of symmetry of the standard parabola, defined to be the line perpendicular to the directrix and passing through the focus, is the  $x$ -axis.

*Proof.* From (D.91) and (D.87), the axis of the parabola can be expressed using (C.2)

as

$$\mathbf{e}_2^\top \left( \mathbf{y} + \frac{\eta}{4\lambda_2} \mathbf{e}_1 \right) = 0 \quad (\text{E.1})$$

$$\implies \mathbf{e}_2^\top \mathbf{y} = 0, \quad (\text{E.2})$$

which is the equation of the  $x$ -axis.  $\square$

E.1.5. The point where the parabola intersects its axis of symmetry is called the vertex. For the standard parabola, the vertex is the origin.

*Proof.* (E.2) can be expressed as

$$\mathbf{y} = \alpha \mathbf{e}_1, \quad (\text{E.3})$$

using (C.2). Substituting (E.3) in (D.62),

$$\alpha^2 \mathbf{e}_1^\top \mathbf{D} \mathbf{e}_1 = -\eta \alpha \mathbf{e}_1^\top \mathbf{e}_1 \quad (\text{E.4})$$

$$\implies \alpha = 0, \text{ or, } \mathbf{y} = \mathbf{0}. \quad (\text{E.5})$$

$\square$

E.1.6. The focal length of the standard parabola, , defined to be the distance between the vertex and the focus, measured along the axis of symmetry, is  $\left| \frac{\eta}{4\lambda_2} \right|$

## E.2. Quadratic Form

E.2.1. The center/vertex of a conic section are given by

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \quad \left| \mathbf{V} \right| \neq 0 \quad (\text{E.6})$$

$$\begin{pmatrix} \mathbf{u}^\top + \frac{\eta}{2}\mathbf{p}_1^\top \\ \mathbf{v} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -f \\ \frac{\eta}{2}\mathbf{p}_1 - \mathbf{u} \end{pmatrix} \quad \left| \mathbf{V} \right| = 0 \quad (\text{E.7})$$

*Proof.* In (B.19), substituting  $\mathbf{y} = \mathbf{0}$ , the center/vertex for the quadratic form is obtained as

$$\mathbf{x} = \mathbf{c}, \quad (\text{E.8})$$

where  $\mathbf{c}$  is derived as (E.6) and (E.7) in Appendix D.3.1.  $\square$

E.2.2. The equation of the minor and major axes for the ellipse/hyperbola are respectively given by

$$\mathbf{p}_i^\top (\mathbf{x} - \mathbf{c}) = 0, i = 1, 2 \quad (\text{E.9})$$

The axis of symmetry for the parabola is also given by (E.9).

*Proof.* From (E.1.2), the major/symmetry axis for the hyperbola/ellipse/parabola can be expressed using (B.19) as

$$\mathbf{e}_2^\top \mathbf{P}^\top (\mathbf{x} - \mathbf{c}) = 0 \quad (\text{E.10})$$

$$\implies (\mathbf{P}\mathbf{e}_2)^\top (\mathbf{x} - \mathbf{c}) = 0 \quad (\text{E.11})$$

yielding (E.9), and the proof for the minor axis is similar.

□

## Appendix F

### Conic Lines

#### F.1. Pair of Straight Lines

F.1.1. The asymptotes of the hyperbola in (D.61), defined to be the lines that do not intersect the hyperbola, are given by

$$\left( \sqrt{|\lambda_1|} \pm \sqrt{|\lambda_2|} \right) \mathbf{y} = 0 \quad (\text{F.1})$$

*Proof.* From (D.61), it is obvious that the pair of lines represented by

$$\mathbf{y}^\top \mathbf{D} \mathbf{y} = 0 \quad (\text{F.2})$$

do not intersect the conic

$$\mathbf{y}^\top \mathbf{D} \mathbf{y} = f_0 \quad (\text{F.3})$$

Thus, (F.2) represents the asymptotes of the hyperbola in (D.61) and can be expressed as

$$\lambda_1 y_1^2 + \lambda_2 y_1^2 = 0, \quad (\text{F.4})$$

which can then be simplified to obtain (F.1).

□

F.1.2. (D.3) represents a pair of straight lines if

$$\mathbf{u}^\top \mathbf{V}^{-1} \mathbf{u} - f = 0 \quad (\text{F.5})$$

F.1.3. (D.3) represents a pair of straight lines if the matrix

$$\begin{pmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u} & f \end{pmatrix} \quad (\text{F.6})$$

is singular.

*Proof.* Let

$$\begin{pmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u} & f \end{pmatrix} \mathbf{x} = \mathbf{0} \quad (\text{F.7})$$

Expressing

$$\mathbf{x} = \begin{pmatrix} \mathbf{y} \\ y_3 \end{pmatrix}, \quad (\text{F.8})$$



$$\begin{pmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^\top & f \end{pmatrix} \begin{pmatrix} \mathbf{y} \\ y_3 \end{pmatrix} = \mathbf{0} \quad (\text{F.9})$$

$$\implies \mathbf{V}\mathbf{y} + y_3\mathbf{u} = \mathbf{0} \quad \text{and} \quad (\text{F.10})$$

$$\mathbf{u}^\top \mathbf{y} + f y_3 = 0 \quad (\text{F.11})$$

From (F.10) we obtain,

$$\mathbf{y}^\top \mathbf{V}\mathbf{y} + y_3 \mathbf{y}^\top \mathbf{u} = 0 \quad (\text{F.12})$$

$$\implies \mathbf{y}^\top \mathbf{V}\mathbf{y} + y_3 \mathbf{u}^\top \mathbf{y} = 0 \quad (\text{F.13})$$

yielding (F.5) upon substituting from (F.11). □

F.1.4. Using the affine transformation, (F.1) can be expressed as the lines

$$\left( \sqrt{|\lambda_1|} \quad \pm \sqrt{|\lambda_2|} \right) \mathbf{P}^\top (\mathbf{x} - \mathbf{c}) = 0 \quad (\text{F.14})$$

F.1.5. The angle between the asymptotes can be expressed as

$$\cos \theta = \frac{|\lambda_1| - |\lambda_2|}{|\lambda_1| + |\lambda_2|} \quad (\text{F.15})$$

*Proof.* The normal vectors of the lines in (F.14) are

$$\begin{aligned} \mathbf{n}_1 &= \mathbf{P} \begin{pmatrix} \sqrt{|\lambda_1|} \\ \sqrt{|\lambda_2|} \end{pmatrix} \\ \mathbf{n}_2 &= \mathbf{P} \begin{pmatrix} \sqrt{|\lambda_1|} \\ -\sqrt{|\lambda_2|} \end{pmatrix} \end{aligned} \quad (\text{F.16})$$

The angle between the asymptotes is given by

$$\cos \theta = \frac{\mathbf{n}_1^\top \mathbf{n}_2}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} \quad (\text{F.17})$$

The orthogonal matrix  $\mathbf{P}$  preserves the norm, i.e.

$$\|\mathbf{n}_1\| = \left\| \mathbf{P} \begin{pmatrix} \sqrt{|\lambda_1|} \\ \sqrt{|\lambda_2|} \end{pmatrix} \right\| = \left\| \begin{pmatrix} \sqrt{|\lambda_1|} \\ \sqrt{|\lambda_2|} \end{pmatrix} \right\| \quad (\text{F.18})$$

$$= \sqrt{|\lambda_1| + |\lambda_2|} = \|\mathbf{n}_2\| \quad (\text{F.19})$$

It is easy to verify that

$$\mathbf{n}_1^\top \mathbf{n}_2 = |\lambda_1| - |\lambda_2| \quad (\text{F.20})$$

Thus, the angle between the asymptotes is obtained from (F.17) as (F.15).  $\square$

## F.2. Intersection of Conics

F.2.1. Let

$$\mathbf{x}^\top \mathbf{V}_i \mathbf{x} + 2\mathbf{u}_i^\top \mathbf{x} + f_i = 0, \quad i = 1, 2 \quad (\text{F.21})$$

be the equation of two conics. The locus of their intersection is a pair of straight lines if

$$\begin{vmatrix} \mathbf{V}_1 + \mu \mathbf{V}_2 & \mathbf{u}_1 + \mu \mathbf{u}_2 \\ (\mathbf{u}_1 + \mu \mathbf{u}_2)^\top & f \end{vmatrix} = 0, \quad \left| \mathbf{V}_1 + \mu \mathbf{V}_2 \right| < 0 \quad (\text{F.22})$$

*Proof.* The intersection of the conics in (F.21) is given by the curve

$$\mathbf{x}^\top (\mathbf{V}_1 + \mu \mathbf{V}_2) \mathbf{x} + 2 (\mathbf{u}_1 + \mu \mathbf{u}_2)^\top \mathbf{x} + f_1 + \mu f_2 = 0, \quad (\text{F.23})$$

which, from Theorem F.1.3 represents a pair of straight lines if (F.22) is satisfied.  $\square$

F.2.2. The points of intersection of the conics in (F.21) are the points of the intersection of the lines in (F.23).

## F.3. Chords of a Conic

F.3.1. The points of intersection of the line

$$L : \quad \mathbf{x} = \mathbf{q} + \mu \mathbf{m} \quad \mu \in \mathbb{R} \quad (\text{F.24})$$

with the conic section in (D.3) are given by

$$\mathbf{x}_i = \mathbf{q} + \mu_i \mathbf{m} \quad (\text{F.25})$$

where

$$\mu_i = \frac{1}{\mathbf{m}^\top \mathbf{V} \mathbf{m}} \left( -\mathbf{m}^\top (\mathbf{V} \mathbf{q} + \mathbf{u}) \pm \sqrt{[\mathbf{m}^\top (\mathbf{V} \mathbf{q} + \mathbf{u})]^2 - (\mathbf{q}^\top \mathbf{V} \mathbf{q} + 2\mathbf{u}^\top \mathbf{q} + f)(\mathbf{m}^\top \mathbf{V} \mathbf{m})} \right) \quad (\text{F.26})$$

*Proof.* Substituting (F.24) in (D.3),

$$(\mathbf{q} + \mu \mathbf{m})^\top \mathbf{V} (\mathbf{q} + \mu \mathbf{m}) + 2\mathbf{u}^\top (\mathbf{q} + \mu \mathbf{m}) + f = 0 \quad (\text{F.27})$$

$$\implies \mu^2 \mathbf{m}^\top \mathbf{V} \mathbf{m} + 2\mu \mathbf{m}^\top (\mathbf{V} \mathbf{q} + \mathbf{u}) + \mathbf{q}^\top \mathbf{V} \mathbf{q} + 2\mathbf{u}^\top \mathbf{q} + f = 0 \quad (\text{F.28})$$

Solving the above quadratic in (F.28) yields (F.26).  $\square$

F.3.2. If  $L$  in (F.24) touches (D.3) at exactly one point  $\mathbf{q}$ ,

$$\mathbf{m}^\top (\mathbf{V} \mathbf{q} + \mathbf{u}) = 0 \quad (\text{F.29})$$

*Proof.* In this case, (F.28) has exactly one root. Hence, in (F.26)

$$\left[ \mathbf{m}^\top (\mathbf{V} \mathbf{q} + \mathbf{u}) \right]^2 - \left( \mathbf{m}^\top \mathbf{V} \mathbf{m} \right) \left( \mathbf{q}^\top \mathbf{V} \mathbf{q} + 2\mathbf{u}^\top \mathbf{q} + f \right) = 0 \quad (\text{F.30})$$

$\because \mathbf{q}$  is the point of contact,  $\mathbf{q}$  satisfies (D.3) and

$$\mathbf{q}^\top \mathbf{V} \mathbf{q} + 2\mathbf{u}^\top \mathbf{q} + f = 0 \quad (\text{F.31})$$

Substituting (F.31) in (F.30) and simplifying, we obtain (F.29).  $\square$

F.3.3. The length of the chord in (F.24) is given by

$$\frac{2\sqrt{[\mathbf{m}^\top (\mathbf{V}\mathbf{q} + \mathbf{u})]^2 - (\mathbf{q}^\top \mathbf{V}\mathbf{q} + 2\mathbf{u}^\top \mathbf{q} + f)(\mathbf{m}^\top \mathbf{V}\mathbf{m})}}{\mathbf{m}^\top \mathbf{V}\mathbf{m}} \|\mathbf{m}\| \quad (\text{F.32})$$

*Proof.* The distance between the points in (F.25) is given by

$$\|\mathbf{x}_1 - \mathbf{x}_2\| = |\mu_1 - \mu_2| \|\mathbf{m}\| \quad (\text{F.33})$$

Substituting  $\mu_i$  from (F.26) in (F.33) yields (F.32).  $\square$

F.3.4. The affine transform for the conic section, preserves the norm. This implies that the length of any chord of a conic is invariant to translation and/or rotation.

*Proof.* Let

$$\mathbf{x}_i = \mathbf{P}\mathbf{y}_i + \mathbf{c} \quad (\text{F.34})$$

be any two points on the conic. Then the distance between the points is given by

$$\|\mathbf{x}_1 - \mathbf{x}_2\| = \|\mathbf{P}(\mathbf{y}_1 - \mathbf{y}_2)\| \quad (\text{F.35})$$

which can be expressed as

$$\|\mathbf{x}_1 - \mathbf{x}_2\|^2 = (\mathbf{y}_1 - \mathbf{y}_2)^\top \mathbf{P}^\top \mathbf{P} (\mathbf{y}_1 - \mathbf{y}_2) \quad (\text{F.36})$$

$$= \|\mathbf{y}_1 - \mathbf{y}_2\|^2 \quad (\text{F.37})$$

since

$$\mathbf{P}^\top \mathbf{P} = \mathbf{I} \quad (\text{F.38})$$

□

F.3.5. For the standard hyperbola/ellipse, the length of the major axis is

$$2\sqrt{\left|\frac{f_0}{\lambda_1}\right|} \quad (\text{F.39})$$

and the minor axis is

$$2\sqrt{\left|\frac{f_0}{\lambda_2}\right|} \quad (\text{F.40})$$

*Proof.* Since the major axis passes through the origin,

$$\mathbf{q} = \mathbf{0} \quad (\text{F.41})$$

Further, from Corollary (E.1.2),

$$\mathbf{m} = \mathbf{e}_2, \quad (\text{F.42})$$

and from (D.61),

$$\mathbf{V} = \frac{\mathbf{D}}{f_0}, \mathbf{u} = 0, f = -1 \quad (\text{F.43})$$

Substituting the above in (F.32),

$$\frac{2\sqrt{\mathbf{e}_1^\top \frac{\mathbf{D}}{f_0} \mathbf{e}_1}}{\mathbf{e}_1^\top \frac{\mathbf{D}}{f_0} \mathbf{e}_1} \|\mathbf{e}_1\| \quad (\text{F.44})$$

yielding (F.39). Similarly, for the minor axis, the only different parameter is

$$\mathbf{m} = \mathbf{e}_2, \quad (\text{F.45})$$

Substituting the above in (F.32),

$$\frac{2\sqrt{\mathbf{e}_2^\top \frac{\mathbf{D}}{f_0} \mathbf{e}_2}}{\mathbf{e}_2^\top \frac{\mathbf{D}}{f_0} \mathbf{e}_2} \|\mathbf{e}_2\| \quad (\text{F.46})$$

yielding (F.40).

□

F.3.6. The latus rectum of a conic section is the chord that passes through the focus and is perpendicular to the major axis. The length of the latus rectum for a conic is given by

$$l = \begin{cases} 2\frac{\sqrt{|f_0\lambda_1|}}{\lambda_2} & e \neq 1 \\ \frac{\eta}{\lambda_2} & e = 1 \end{cases} \quad (\text{F.47})$$

*Proof.* The latus rectum is perpendicular to the major axis for the standard conic. Hence, from Corollary (E.1.2),

$$\mathbf{m} = \mathbf{e}_2, \quad (\text{F.48})$$

Since it passes through the focus, from (D.87)

$$\mathbf{q} = \mathbf{F} = \pm e \sqrt{\frac{f_0}{\lambda_2(1-e^2)}} \mathbf{e}_1 \quad (\text{F.49})$$

for the standard hyperbola/ellipse. Also, from (D.61),

$$\mathbf{V} = \frac{\mathbf{D}}{f_0}, \mathbf{u} = 0, f = -1 \quad (\text{F.50})$$

Substituting the above in (F.32),

$$\frac{2 \sqrt{\left[ \mathbf{e}_2^\top \left( \frac{\mathbf{D}}{f_0} e \sqrt{\frac{f_0}{\lambda_2(1-e^2)}} \mathbf{e}_1 \right) \right]^2 - \left( e \sqrt{\frac{f_0}{\lambda_2(1-e^2)}} \mathbf{e}_1^\top \frac{\mathbf{D}}{f_0} e \sqrt{\frac{f_0}{\lambda_2(1-e^2)}} \mathbf{e}_1 - 1 \right) \left( \mathbf{e}_2^\top \frac{\mathbf{D}}{f_0} \mathbf{e}_2 \right)}}{\mathbf{e}_2^\top \frac{\mathbf{D}}{f_0} \mathbf{e}_2} \|\mathbf{e}_2\| \quad (\text{F.51})$$

Since

$$\mathbf{e}_2^\top \mathbf{D} \mathbf{e}_1 = 0, \mathbf{e}_1^\top \mathbf{D} \mathbf{e}_1 = \lambda_1, \mathbf{e}_1^\top \mathbf{e}_1 = 1, \|\mathbf{e}_2\| = 1, \mathbf{e}_2^\top \mathbf{D} \mathbf{e}_2 = \lambda_2, \quad (\text{F.52})$$

(F.51) can be expressed as

$$\frac{2 \sqrt{\left( 1 - \frac{\lambda_1 e^2}{\lambda_2(1-e^2)} \right) \left( \frac{\lambda_2}{f_0} \right)}}{\frac{\lambda_2}{f_0}} \quad (\text{F.53})$$

$$= 2 \frac{\sqrt{f_0 \lambda_1}}{\lambda_2} \quad \left( \because e^2 = 1 - \frac{\lambda_1}{\lambda_2} \right) \quad (\text{F.54})$$

For the standard parabola, the parameters in (F.32) are

$$\mathbf{q} = \mathbf{F} = -\frac{\eta}{4\lambda_2} \mathbf{e}_1, \mathbf{m} = \mathbf{e}_1, \mathbf{V} = \mathbf{D}, \mathbf{u} = \frac{\eta}{2} \mathbf{e}_1^\top, f = 0 \quad (\text{F.55})$$



Substituting the above in (F.32), the length of the latus rectum can be expressed as

$$\frac{2\sqrt{\left[\mathbf{e}_2^\top \left(\mathbf{D} \left(-\frac{\eta}{4\lambda_2} \mathbf{e}_1\right) + \frac{\eta}{2} \mathbf{e}_1\right)\right]^2 - \left(\left(-\frac{\eta}{4\lambda_2} \mathbf{e}_1\right)^\top \mathbf{D} \left(-\frac{\eta}{4\lambda_2} \mathbf{e}_1\right) + 2\frac{\eta}{2} \mathbf{e}_1^\top \left(-\frac{\eta}{4\lambda_2} \mathbf{e}_1\right)\right) (\mathbf{e}_2^\top \mathbf{D} \mathbf{e}_2)}}{\mathbf{e}_2^\top \mathbf{D} \mathbf{e}_2} \|\mathbf{e}_2\| \quad (\text{F.56})$$

Since

$$\mathbf{e}_2^\top \mathbf{D} \mathbf{e}_1 = 0, \mathbf{e}_2^\top \mathbf{e}_2 = 0, \mathbf{e}_1^\top \mathbf{D} \mathbf{e}_1 = 0, \mathbf{e}_1^\top \mathbf{e}_1 = 1, \|\mathbf{e}_1\| = 1, \mathbf{e}_2^\top \mathbf{D} \mathbf{e}_2 = \lambda_2, \quad (\text{F.57})$$

(F.56) can be expressed as

$$2 \frac{\sqrt{\frac{\eta^2}{4\lambda_2} \lambda_2}}{\lambda_2} = \frac{\eta}{\lambda_2} \quad (\text{F.58})$$

□

## F.4. Tangent and Normal

F.4.1. Given the point of contact  $\mathbf{q}$ , the equation of a tangent to (D.3) is

$$(\mathbf{V} \mathbf{q} + \mathbf{u})^\top \mathbf{x} + \mathbf{u}^\top \mathbf{q} + f = 0 \quad (\text{F.59})$$

*Proof.* The normal vector is obtained from (F.29) and (A.31) as

$$\kappa \mathbf{n} = \mathbf{V} \mathbf{q} + \mathbf{u}, \kappa \in \mathbb{R} \quad (\text{F.60})$$

From (F.60) and (C.2), the equation of the tangent is

$$(\mathbf{V}\mathbf{q} + \mathbf{u})^\top (\mathbf{x} - \mathbf{q}) = 0 \quad (\text{F.61})$$

$$\implies (\mathbf{V}\mathbf{q} + \mathbf{u})^\top \mathbf{x} - \mathbf{q}^\top \mathbf{V}\mathbf{q} - \mathbf{u}^\top \mathbf{q} = 0 \quad (\text{F.62})$$

which, upon substituting from (F.31) and simplifying yields (F.59)  $\square$

F.4.2. If  $\mathbf{V}^{-1}$  exists, given the normal vector  $\mathbf{n}$ , the tangent points of contact to (D.3) are given by

$$\mathbf{q}_i = \mathbf{V}^{-1} (\kappa_i \mathbf{n} - \mathbf{u}), i = 1, 2 \quad (\text{F.63})$$

where  $\kappa_i = \pm \sqrt{\frac{f_0}{\mathbf{n}^\top \mathbf{V}^{-1} \mathbf{n}}}$

*Proof.* From (F.60),

$$\mathbf{q} = \mathbf{V}^{-1} (\kappa \mathbf{n} - \mathbf{u}), \quad \kappa \in \mathbb{R} \quad (\text{F.64})$$

Substituting (F.64) in (F.31),

$$(\kappa \mathbf{n} - \mathbf{u})^\top \mathbf{V}^{-1} (\kappa \mathbf{n} - \mathbf{u}) + 2\mathbf{u}^\top \mathbf{V}^{-1} (\kappa \mathbf{n} - \mathbf{u}) + f = 0 \quad (\text{F.65})$$

$$\implies \kappa^2 \mathbf{n}^\top \mathbf{V}^{-1} \mathbf{n} - \mathbf{u}^\top \mathbf{V}^{-1} \mathbf{u} + f = 0 \quad (\text{F.66})$$

$$\text{or, } \kappa = \pm \sqrt{\frac{f_0}{\mathbf{n}^\top \mathbf{V}^{-1} \mathbf{n}}} \quad (\text{F.67})$$

Substituting (F.67) in (F.64) yields (F.63).  $\square$

F.4.3. For a circle,

$$\mathbf{q}_{ij} = \left( \pm r \frac{\mathbf{n}_j}{\|\mathbf{n}_j\|} - \mathbf{u} \right), \quad i, j = 1, 2, \quad (\text{F.68})$$

*Proof.* From (F.63), and (D.39),

$$\kappa_{ij} = \pm \frac{r}{\|\mathbf{n}_j\|} \quad (\text{F.69})$$

□

F.4.4. If  $\mathbf{V}$  is not invertible, given the normal vector  $\mathbf{n}$ , the point of contact to (D.3) is given by the matrix equation

$$\begin{pmatrix} (\mathbf{u} + \kappa \mathbf{n})^\top \\ \mathbf{V} \end{pmatrix} \mathbf{q} = \begin{pmatrix} -f \\ \kappa \mathbf{n} - \mathbf{u} \end{pmatrix} \quad (\text{F.70})$$

$$\text{where } \kappa = \frac{\mathbf{p}_1^\top \mathbf{u}}{\mathbf{p}_1^\top \mathbf{n}}, \quad \mathbf{V} \mathbf{p}_1 = 0 \quad (\text{F.71})$$

*Proof.* If  $\mathbf{V}$  is non-invertible, it has a zero eigenvalue. If the corresponding eigenvector is  $\mathbf{p}_1$ , then,

$$\mathbf{V} \mathbf{p}_1 = 0 \quad (\text{F.72})$$

From (F.60),

$$\kappa \mathbf{n} = \mathbf{V} \mathbf{q} + \mathbf{u}, \quad \kappa \in \mathbb{R} \quad (\text{F.73})$$

$$\implies \kappa \mathbf{p}_1^\top \mathbf{n} = \mathbf{p}_1^\top \mathbf{V} \mathbf{q} + \mathbf{p}_1^\top \mathbf{u} \quad (\text{F.74})$$

$$\text{or, } \kappa \mathbf{p}_1^\top \mathbf{n} = \mathbf{p}_1^\top \mathbf{u}, \quad \because \mathbf{p}_1^\top \mathbf{V} = 0, \quad (\text{ from (F.72)}) \quad (\text{F.75})$$

yielding  $\kappa$  in (F.71). From (F.73),

$$\kappa \mathbf{q}^\top \mathbf{n} = \mathbf{q}^\top \mathbf{V} \mathbf{q} + \mathbf{q}^\top \mathbf{u} \quad (\text{F.76})$$

$$\implies \kappa \mathbf{q}^\top \mathbf{n} = -f - \mathbf{q}^\top \mathbf{u} \quad \text{from (F.31),} \quad (\text{F.77})$$

$$\text{or, } (\kappa \mathbf{n} + \mathbf{u})^\top \mathbf{q} = -f \quad (\text{F.78})$$

(F.73) can be expressed as

$$\mathbf{V} \mathbf{q} = \kappa \mathbf{n} - \mathbf{u}. \quad (\text{F.79})$$

(F.78) and (F.79) clubbed together result in (F.70).  $\square$

F.4.5. The normal vectors of the tangents to the conic in (D.3) satisfy

$$\mathbf{n}^\top \mathbf{V}^{-1} \mathbf{n} - f_0 = 0 \quad (\text{F.80})$$

*Proof.* From (F.29), the normal vector to the tangent at  $\mathbf{q}$  can be expressed as

$$\mathbf{n} = \mathbf{V} \mathbf{q} + \mathbf{u} \quad (\text{F.81})$$

$$\implies \mathbf{q} = \mathbf{V}^{-1} (\mathbf{n} - \mathbf{u}) \quad (\text{F.82})$$

which upon substituting in (D.3) yields

$$(\mathbf{n} - \mathbf{u})^\top \mathbf{V}^{-1} \mathbf{V} \mathbf{V}^{-1} (\mathbf{n} - \mathbf{u}) + 2\mathbf{u}^\top \mathbf{V}^{-1} (\mathbf{n} - \mathbf{u}) + f = 0 \quad (\text{F.83})$$

which can be simplified to obtain (F.80).  $\square$

F.4.6. The normal vectors of the tangents to the conic in (D.3) from a point  $\mathbf{h}$  are given by

$$\begin{aligned}\mathbf{n}_1 &= \mathbf{P} \begin{pmatrix} \sqrt{|\lambda_1|} \\ \sqrt{|\lambda_2|} \end{pmatrix} \\ \mathbf{n}_2 &= \mathbf{P} \begin{pmatrix} \sqrt{|\lambda_1|} \\ -\sqrt{|\lambda_2|} \end{pmatrix}\end{aligned}\tag{F.84}$$

where  $\lambda_i, \mathbf{P}$  are the eigenparameters of

$$\boldsymbol{\Sigma} = (\mathbf{V}\mathbf{h} + \mathbf{u})(\mathbf{V}\mathbf{h} + \mathbf{u})^\top - \left(\mathbf{h}^\top \mathbf{V}\mathbf{h} + 2\mathbf{u}^\top \mathbf{h} + f\right) \mathbf{V}.\tag{F.85}$$

*Proof.* From (F.26), and (F.30)

$$\left[\mathbf{m}^\top (\mathbf{V}\mathbf{h} + \mathbf{u})\right]^2 - \left(\mathbf{m}^\top \mathbf{V}\mathbf{m}\right) \left(\mathbf{h}^\top \mathbf{V}\mathbf{h} + 2\mathbf{u}^\top \mathbf{h} + f\right) = 0\tag{F.86}$$

$$\implies \mathbf{m}^\top \left[(\mathbf{V}\mathbf{h} + \mathbf{u})(\mathbf{V}\mathbf{h} + \mathbf{u})^\top - \mathbf{V} \left(\mathbf{h}^\top \mathbf{V}\mathbf{h} + 2\mathbf{u}^\top \mathbf{h} + f\right)\right] \mathbf{m} = 0\tag{F.87}$$

yielding (F.85). Consequently, from (F.16), (F.84) can be obtained.  $\square$

