Matrix Analysis

G V V Sharma*

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*The author is with the Department of Electrical Engineering, Indian Institute of Technology, Hyderabad 502285 India e-mail: gadepall@iith.ac.in.

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Abstract

This manual provides an introduction to vectors and their properties, based on the question papers, year 2020, from Class 10 and 12, CBSE.

1 2×1 VECTORS

1.1. Let

$$\mathbf{A} \equiv \overrightarrow{A} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \tag{1.1.1}$$

$$\equiv a_1 \overrightarrow{i} + a_2 \overrightarrow{j}, \tag{1.1.2}$$

$$\mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \tag{1.1.3}$$

be 2×1 vectors. Then, the determinant of the 2×2 matrix

$$\mathbf{M} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \end{pmatrix} \tag{1.1.4}$$

is defined as

$$\left|\mathbf{M}\right| = \left|\mathbf{A} \quad \mathbf{B}\right| \tag{1.1.5}$$

$$= \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1 \tag{1.1.6}$$

1.2. The value of the cross product of two vectors is given by (1.1.5).

1.3. The area of the triangle with vertices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ is given by the absolute value of

$$\frac{1}{2}\left|\mathbf{A} - \mathbf{B} \quad \mathbf{A} - \mathbf{C}\right| \tag{1.3.1}$$

1.4. The transpose of $\mathbf{A}$ is defined as

$$\mathbf{A}^\top = \begin{pmatrix} a_1 & a_2 \end{pmatrix} \tag{1.4.1}$$

1.5. The *inner product* or *dot product* is defined as

$$\mathbf{A}^\top\mathbf{B} \equiv \mathbf{A} \cdot \mathbf{B} \tag{1.5.1}$$

$$= \begin{pmatrix} a_1 & a_2 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = a_1 b_1 + a_2 b_2 \tag{1.5.2}$$

1.6. *norm* of $\mathbf{A}$ is defined as

$$\|A\| \equiv \left|\overrightarrow{A}\right| \tag{1.6.1}$$

$$= \sqrt{\mathbf{A}^\top\mathbf{A}} = \sqrt{a_1^2 + a_2^2} \tag{1.6.2}$$

Thus,

$$\|\lambda\mathbf{A}\| \equiv \left|\lambda\overrightarrow{A}\right| \tag{1.6.3}$$

$$= |\lambda| \, \|\mathbf{A}\| \tag{1.6.4}$$

1.7. The distance betwen the points $\mathbf{A}$ and $\mathbf{B}$ is given by

$$\|\mathbf{A} - \mathbf{B}\| \tag{1.7.1}$$

1.8. Let $\mathbf{x}$ be equidistant from the points $\mathbf{A}$ and $\mathbf{B}$. Then

$$(\mathbf{A}-\mathbf{B})^\top \mathbf{x} = \frac{\|\mathbf{A}\|^2 - \|\mathbf{B}\|^2}{2} \tag{1.8.1}$$

Solution:

$$\|\mathbf{x}-\mathbf{A}\| = \|\mathbf{A}-\mathbf{B}\| \tag{1.8.2}$$
$$\implies \|\mathbf{x}-\mathbf{A}\|^2 = \|\mathbf{x}-\mathbf{B}\|^2 \tag{1.8.3}$$

which can be expressed as

$$\begin{aligned}(\mathbf{x}-\mathbf{A})^\top(\mathbf{x}-\mathbf{A}) = (\mathbf{x}-\mathbf{B})^\top(\mathbf{x}-\mathbf{B}) \\ \implies \|\mathbf{x}\|^2 - 2\mathbf{x}^\top\mathbf{A} + \|\mathbf{A}\|^2 \\ = \|\mathbf{x}\|^2 - 2\mathbf{x}^\top\mathbf{B} + \|\mathbf{B}\|^2\end{aligned} \tag{1.8.4}$$

which can be simplified to obtain (1.8.1).

1.9. If $\mathbf{x}$ lies on the x-axis and is equidistant from the points $\mathbf{A}$ and $\mathbf{B}$,

$$\mathbf{x} = x\mathbf{e}_1 \tag{1.9.1}$$

where

$$x = \frac{\|\mathbf{A}\|^2 - \|\mathbf{B}\|^2}{2(\mathbf{A}-\mathbf{B})^\top \mathbf{e}_1} \tag{1.9.2}$$

Solution: From (1.8.1).

$$x(\mathbf{A}-\mathbf{B})^\top \mathbf{e}_1 = \frac{\|\mathbf{A}\|^2 - \|\mathbf{B}\|^2}{2} \tag{1.9.3}$$

yielding (1.9.2).

1.10. The angle between two vectors is given by

$$\theta = \cos^{-1}\frac{\mathbf{A}^\top\mathbf{B}}{\|A\|\,\|B\|} \tag{1.10.1}$$

1.11. If two vectors are orthogonal (perpendicular),

$$\mathbf{A}^\top\mathbf{B} = 0 \tag{1.11.1}$$

1.12. The *direction vector* of the line joining two points $\mathbf{A}, \mathbf{B}$ is given by

$$\mathbf{m} = \mathbf{A} - \mathbf{B} \tag{1.12.1}$$

1.13. The unit vector in the direction of $\mathbf{m}$ is defined as

$$\frac{\mathbf{m}}{\|\mathbf{m}\|} \tag{1.13.1}$$

1.14. If the direction vector of a line is expressed as

$$\mathbf{m} = \begin{pmatrix}1\\ m\end{pmatrix}, \tag{1.14.1}$$

the m is defined to be the slope of the line.

1.15. The *normal vector* to $\mathbf{m}$ is defined by

$$\mathbf{m}^\top\mathbf{n} = 0 \tag{1.15.1}$$

1.16. The point $\mathbf{P}$ that divides the line segment AB in the ratio $k:1$ is given by

$$\mathbf{P} = \frac{k\mathbf{B} + \mathbf{A}}{k+1} \tag{1.16.1}$$

1.17. The standard basis vectors are defined as

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{1.17.1}$$

$$\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{1.17.2}$$

2 3×1 VECTORS

2.1. Let

$$\mathbf{A} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \equiv a_1 \overrightarrow{i} + a_2 \overrightarrow{j} + a_3 \overrightarrow{j}, \tag{2.1.1}$$

$$\mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \tag{2.1.2}$$

and

$$\mathbf{A}_{ij} = \begin{pmatrix} a_i \\ a_j \end{pmatrix}, \tag{2.1.3}$$

$$\mathbf{B}_{ij} = \begin{pmatrix} b_i \\ b_j \end{pmatrix}. \tag{2.1.4}$$

2.2. The *cross product* or *vector product* of $\mathbf{A}, \mathbf{B}$ is defined as

$$\mathbf{A} \times \mathbf{B} = \begin{pmatrix} \begin{vmatrix} \mathbf{A}_{23} & \mathbf{B}_{23} \\ \mathbf{A}_{31} & \mathbf{B}_{31} \\ \mathbf{A}_{12} & \mathbf{B}_{12} \end{vmatrix} \end{pmatrix} \tag{2.2.1}$$

2.3. Verify that

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} \tag{2.3.1}$$

2.4. The area of a triangle is given by

$$\frac{1}{2} \left\| \mathbf{A} \times \mathbf{B} \right\| \tag{2.4.1}$$

3 MATRICES

3.1 Eigenvalues and Eigenvectors

3.1.1. The eigenvalue λ and the eigenvector $\mathbf{x}$ for a matrix $\mathbf{A}$ are defined as,

$$\mathbf{A}\mathbf{x} = \lambda \mathbf{x} \tag{3.1.1.1}$$

3.1.2. The eigenvalues are calculated by solving the equation

$$f(\lambda) = \left| \lambda \mathbf{I} - \mathbf{A} \right| = 0 \tag{3.1.2.1}$$

The above equation is known as the characteristic equation.

3.1.3. According to the Cayley-Hamilton theorem,

$$f(\lambda) = 0 \implies f(\mathbf{A}) = 0 \tag{3.1.3.1}$$

3.1.4. The trace of a square matrix is defined to be the sum of the diagonal elements.

$$\text{tr}(\mathbf{A}) = \sum_{i=1}^{N} a_{ii}. \tag{3.1.4.1}$$

where a_{ii} is the ith diagonal element of the matrix $\mathbf{A}$.

3.1.5. The trace of a matrix is equal to the sum of the eigenvalues

$$\text{tr}(\mathbf{A}) = \sum_{i=1}^{N} \lambda_i \tag{3.1.5.1}$$

3.2 Determinants

3.2.1. Let

$$\mathbf{A} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}. \tag{3.2.1.1}$$

be a 3×3 matrix. Then,

$$\left|\mathbf{A}\right| = a_1 \begin{pmatrix} b_2 & c_2 \\ b_3 & c_3 \end{pmatrix} - a_2 \begin{pmatrix} b_1 & c_1 \\ b_3 & c_3 \end{pmatrix} + a_3 \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}. \tag{3.2.1.2}$$

3.2.2. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of a matrix $\mathbf{A}$. Then, the product of the eigenvalues is equal to the determinant of $\mathbf{A}$.

$$\left|\mathbf{A}\right| = \prod_{i=1}^{n} \lambda_i \tag{3.2.2.1}$$

3.2.3.

$$\left|\mathbf{AB}\right| = \left|\mathbf{A}\right| \left|\mathbf{B}\right| \tag{3.2.3.1}$$

3.2.4. If $\mathbf{A}$ be an $n \times n$ matrix,

$$\left|k\mathbf{A}\right| = k^n \left|\mathbf{A}\right| \tag{3.2.4.1}$$

3.3 Rank of a Matrix

3.3.1. The rank of a matrix is defined as the number of linearly independent rows. This is also known as the row rank.

3.3.2. Row rank = Column rank.

3.3.3. The rank of a matrix is obtained as the number of nonzero rows obtained after row reduction.

3.3.4. An $n \times n$ matrix is invertible if and only if its rank is n.

3.4 Inverse of a Matrix

3.4.1. For a 2×2 matrix

$$\mathbf{A} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}, \tag{3.4.1.1}$$

the inverse is given by

$$\mathbf{A}^{-1} = \frac{1}{\left|\mathbf{A}\right|} \begin{pmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{pmatrix}, \tag{3.4.1.2}$$

3.4.2. For higher order matrices, the inverse should be calculated using row operations.

3.5 Orthogonality

3.5.1. The rotation matrix is defined as

$$\mathbf{R}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad \theta \in [0, 2\pi] \tag{3.5.1.1}$$

3.5.2. The rotation matrix is *orthogonal*

$$\mathbf{R}_\theta^\top \mathbf{R}_\theta = \mathbf{R}_\theta \mathbf{R}_\theta^\top = \mathbf{I} \tag{3.5.2.1}$$

3.5.3.

$$\mathbf{m}^\top \mathbf{n} = 0 \implies \mathbf{n} = \mathbf{R}_{\frac{\pi}{2}} \mathbf{m} \tag{3.5.3.1}$$

3.5.4.

$$\mathbf{n}^\top \mathbf{h} = 1 \implies \mathbf{n} = \frac{\mathbf{e}_1}{\mathbf{e}_1^\top \mathbf{h}} + \mu \mathbf{R}_{\frac{\pi}{2}} \mathbf{h}, \quad \mu \in \mathbb{R}. \tag{3.5.4.1}$$

4 Linear Forms

4.1 Two Dimensions

4.1.1. The equation of a line is given by

$$\mathbf{n}^\top \mathbf{x} = c \tag{4.1.1.1}$$

where $\mathbf{n}$ is the normal vector of the line.

4.1.2. The equation of a line with normal vector $\mathbf{n}$ and passing through a point $\mathbf{A}$ is given by

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{A}) = 0 \tag{4.1.2.1}$$

4.1.3. The equation of a line L is also given by

$$\mathbf{n}^\top \mathbf{x} = \begin{cases} 0 & \mathbf{0} \in L \\ 1 & \text{otherwise} \end{cases} \tag{4.1.3.1}$$

4.1.4. The parametric equation of a line is given by

$$\mathbf{x} = \mathbf{A} + \lambda \mathbf{m} \tag{4.1.4.1}$$

where $\mathbf{m}$ is the direction vector of the line and $\mathbf{A}$ is any point on the line.

4.1.5. Let $\mathbf{A}$ and $\mathbf{B}$ be two points on a straight line and let $\mathbf{P} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$ be any point on it. If p_2 is known, then

$$\mathbf{P} = \mathbf{A} + \frac{p_2 - \mathbf{e}_2^\top \mathbf{A}}{\mathbf{e}_2^\top (\mathbf{B} - \mathbf{A})} (\mathbf{B} - \mathbf{A}) \tag{4.1.5.1}$$

Solution: The equation of the line can be expressed in parametric from as

$$\mathbf{x} = \mathbf{A} + \lambda\left(\mathbf{B} - \mathbf{A}\right) \tag{4.1.5.2}$$
$$\implies \mathbf{P} = \mathbf{A} + \lambda\left(\mathbf{B} - \mathbf{A}\right) \tag{4.1.5.3}$$
$$\implies \mathbf{e}_2^\top \mathbf{P} = \mathbf{e}_2^\top \mathbf{A} + \lambda \mathbf{e}_2^\top\left(\mathbf{B} - \mathbf{A}\right) \tag{4.1.5.4}$$
$$\implies p_2 = \mathbf{e}_2^\top \mathbf{A} + \lambda \mathbf{e}_2^\top\left(\mathbf{B} - \mathbf{A}\right) \tag{4.1.5.5}$$
$$\text{or, } \lambda = \frac{p_2 - \mathbf{e}_2^\top \mathbf{A}}{\mathbf{e}_2^\top\left(\mathbf{B} - \mathbf{A}\right)} \tag{4.1.5.6}$$

yielding (4.1.5.1).

4.1.6. The distance from a point $\mathbf{P}$ to the line in (4.1.1.1) is given by

$$d = \frac{\left|\mathbf{n}^\top \mathbf{P} - c\right|}{\left\|\mathbf{n}\right\|} \tag{4.1.6.1}$$

Solution: Without loss of generality, let $\mathbf{A}$ be the foot of the perpendicular from $\mathbf{P}$ to the line in (4.1.4.1). The equation of the normal to (4.1.1.1) can then be expressed as

$$\mathbf{x} = \mathbf{A} + \lambda \mathbf{n} \tag{4.1.6.2}$$
$$\implies \mathbf{P} - \mathbf{A} = \lambda \mathbf{n} \tag{4.1.6.3}$$

$\because$ $\mathbf{P}$ lies on (4.1.6.2). From the above, the desired distance can be expressed as

$$d = \left\|\mathbf{P} - \mathbf{A}\right\| = \left|\lambda\right| \left\|\mathbf{n}\right\| \tag{4.1.6.4}$$

From (4.1.6.3),

$$\mathbf{n}^\top\left(\mathbf{P} - \mathbf{A}\right) = \lambda \mathbf{n}^\top \mathbf{n} = \lambda \left\|\mathbf{n}\right\|^2 \tag{4.1.6.5}$$
$$\implies \left|\lambda\right| = \frac{\left|\mathbf{n}^\top\left(\mathbf{P} - \mathbf{A}\right)\right|}{\left\|\mathbf{n}\right\|^2} \tag{4.1.6.6}$$

Substituting the above in (4.1.6.4) and using the fact that

$$\mathbf{n}^\top \mathbf{A} = c \tag{4.1.6.7}$$

from (4.1.1.1), yields (4.1.6.1)

4.1.7. The distance from the origin to the line in (4.1.1.1) is given by

$$d = \frac{\left|c\right|}{\left\|\mathbf{n}\right\|} \tag{4.1.7.1}$$

4.1.8. The distance between the parallel lines

$$\begin{aligned} \mathbf{n}^\top \mathbf{x} &= c_1 \\ \mathbf{n}^\top \mathbf{x} &= c_2 \end{aligned} \tag{4.1.8.1}$$

is given by

$$d = \frac{\left|c_1 - c_2\right|}{\left\|\mathbf{n}\right\|} \tag{4.1.8.2}$$

4.1.9. The equation of the line perpendicular to (4.1.1.1) and passing through the point $\mathbf{P}$ is given by

$$\mathbf{m}^\top\left(\mathbf{x} - \mathbf{P}\right) = 0 \tag{4.1.9.1}$$

4.1.10. The foot of the perpendicular from $\mathbf{P}$ to the line in (4.1.1.1) is given by

$$\begin{pmatrix}\mathbf{m} & \mathbf{n}\end{pmatrix}^\top \mathbf{x} = \begin{pmatrix}\mathbf{m}^\top \mathbf{P} \\ c\end{pmatrix} \tag{4.1.10.1}$$

Solution: From (4.1.1.1) and (4.1.2.1) the foot of the perpendicular satisfies the equations

$$\mathbf{n}^\top \mathbf{x} = c \tag{4.1.10.2}$$
$$\mathbf{m}^\top (\mathbf{x} - \mathbf{P}) = 0 \tag{4.1.10.3}$$

where $\mathbf{m}$ is the direction vector of the given line. Combining the above into a matrix equation results in (4.1.10.1).

4.2 Three Dimensions

4.2.1. The area of a triangle with vertices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ is given by

$$\frac{1}{2} \left\|(\mathbf{A} - \mathbf{B}) \times (\mathbf{A} - \mathbf{C})\right\| \tag{4.2.1.1}$$

4.2.2. Points $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are on a line if

$$\text{rank} \begin{pmatrix}\mathbf{A} \\ \mathbf{B} \\ \mathbf{C}\end{pmatrix} = 1 \tag{4.2.2.1}$$

4.2.3. Points $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ form a paralelogram if

$$\text{rank} \begin{pmatrix}\mathbf{A} \\ \mathbf{B} \\ \mathbf{C} \\ \mathbf{D}\end{pmatrix} = 1, \text{rank} \begin{pmatrix}\mathbf{A} \\ \mathbf{B} \\ \mathbf{C}\end{pmatrix} = 2 \tag{4.2.3.1}$$

4.2.4. The equation of a line is given by (4.1.4.1)
4.2.5. The equation of a plane is given by (4.1.1.1)
4.2.6. The distance from the origin to the line in (4.1.1.1) is given by (4.1.7.1)
4.2.7. The distance from a point $\mathbf{P}$ to the line in (4.1.4.1) is given by

$$d = \left\|\mathbf{A} - \mathbf{P}\right\|^2 - \frac{\left\{\mathbf{m}^\top (\mathbf{A} - \mathbf{P})\right\}^2}{\left\|\mathbf{m}\right\|^2} \tag{4.2.7.1}$$

Solution:

$$d(\lambda) = \left\|\mathbf{A} + \lambda \mathbf{m} - \mathbf{P}\right\| \tag{4.2.7.2}$$
$$\implies d^2(\lambda) = \left\|\mathbf{A} + \lambda \mathbf{m} - \mathbf{P}\right\|^2 \tag{4.2.7.3}$$

which can be simplified to obtain

$$d^2(\lambda) = \lambda^2 \left\|\mathbf{m}\right\|^2 + 2\lambda \mathbf{m}^\top (\mathbf{A} - \mathbf{P}) + \left\|\mathbf{A} - \mathbf{P}\right\|^2 \tag{4.2.7.4}$$

which is of the form

$$d^2(\lambda) = a\lambda^2 + 2b\lambda + c \tag{4.2.7.5}$$
$$= a\left\{\left(\lambda + \frac{b}{a}\right)^2 + \left[\frac{c}{a} - \left(\frac{b}{a}\right)^2\right]\right\} \tag{4.2.7.6}$$

with

$$a = \|\mathbf{m}\|^2, b = \mathbf{m}^\top (\mathbf{A} - \mathbf{P}), c = \|\mathbf{A} - \mathbf{P}\|^2 \tag{4.2.7.7}$$

which can be expressed as From the above, $d^2(\lambda)$ is smallest when upon substituting from (4.2.7.7)

$$\lambda + \frac{b}{2a} = 0 \implies \lambda = -\frac{b}{2a} \tag{4.2.7.8}$$

$$= -\frac{\mathbf{m}^\top (\mathbf{A} - \mathbf{P})}{\|\mathbf{m}\|^2} \tag{4.2.7.9}$$

and consequently,

$$d_{\min}(\lambda) = a\left(\frac{c}{a} - \left(\frac{b}{a}\right)^2\right) \tag{4.2.7.10}$$

$$= c - \frac{b^2}{a} \tag{4.2.7.11}$$

yielding (4.2.7.1) after substituting from (4.2.7.7).

4.2.8. The distance between the parallel planes (4.1.8.1) is given by (4.1.8.2).

4.2.9. The plane

$$\mathbf{n}^\top \mathbf{x} = c \tag{4.2.9.1}$$

contains the line

$$\mathbf{x} = \mathbf{A} + \lambda \mathbf{m} \tag{4.2.9.2}$$

if

$$\mathbf{m}^\top \mathbf{n} = 0 \tag{4.2.9.3}$$

Solution: Any point on the line (4.2.9.2) should also satisfy (4.2.9.1). Hence,

$$\mathbf{n}^\top (\mathbf{A} + \lambda \mathbf{m}) = \mathbf{n}^\top \mathbf{A} = c \tag{4.2.9.4}$$

which can be simplified to obtain (4.2.9.3)

4.2.10. The foot of the perpendicular from a point $\mathbf{P}$ to the plane

$$\mathbf{n}^\top \mathbf{x} = c \tag{4.2.10.1}$$

is given by

Solution: The equation of the line perpendicular to the given plane and passing through $\mathbf{P}$ is

$$\mathbf{x} = \mathbf{P} + \lambda \mathbf{n} \tag{4.2.10.2}$$

From (4.2.13.1), the intersection of the above line with the given plane is

$$\mathbf{x} = \mathbf{P} + \frac{c - \mathbf{n}^\top \mathbf{P}}{\|\mathbf{n}\|^2}\mathbf{n} \tag{4.2.10.3}$$

4.2.11. The image of a point $\mathbf{P}$ with respect to the plane

$$\mathbf{n}^\top \mathbf{x} = c \tag{4.2.11.1}$$

is given by

$$\mathbf{R} = \mathbf{P} + 2\frac{c - \mathbf{n}^\top \mathbf{P}}{\|\mathbf{n}\|^2} \tag{4.2.11.2}$$

Solution: Let $\mathbf{R}$ be the desired image. Then, subtituting the expression for the foot of the perpendicular from $\mathbf{P}$ to the given plane using (4.2.10.3),

$$\frac{\mathbf{P}+\mathbf{R}}{2} = \mathbf{P} + \frac{c - \mathbf{n}^\top\mathbf{P}}{\left\|\mathbf{n}\right\|^2} \tag{4.2.11.3}$$

4.2.12. Let a plane pass through the points $\mathbf{A}, \mathbf{B}$ and be perpendicular to the plane

$$\mathbf{n}^\top\mathbf{x} = c \tag{4.2.12.1}$$

Then the equation of this plane is given by

$$\mathbf{p}^\top\mathbf{x} = 1 \tag{4.2.12.2}$$

where

$$\mathbf{p} = \begin{pmatrix}\mathbf{A} & \mathbf{B} & \mathbf{n}\end{pmatrix}^{-\top}\begin{pmatrix}1\\1\\0\end{pmatrix} \tag{4.2.12.3}$$

Solution: From the given information,

$$\mathbf{p}^\top\mathbf{A} = d \tag{4.2.12.4}$$
$$\mathbf{p}^\top\mathbf{B} = d \tag{4.2.12.5}$$
$$\mathbf{p}^\top\mathbf{n} = 0 \tag{4.2.12.6}$$

$\because$ the normal vectors to the two planes will also be perpendicular. The system of equations in (4.2.12.6) can be expressed as the matrix equation

$$\begin{pmatrix}\mathbf{A} & \mathbf{B} & \mathbf{n}\end{pmatrix}^{\top}\mathbf{p} = d\begin{pmatrix}1\\1\\0\end{pmatrix} \tag{4.2.12.7}$$

which yields (4.2.12.3) upon normalising with d.

4.2.13. The intersection of the line represented by (4.1.4.1) with the plane represented by (4.1.1.1) is given by

$$\mathbf{x} = \mathbf{A} + \frac{c - \mathbf{n}^\top\mathbf{A}}{\mathbf{n}^\top\mathbf{m}}\mathbf{m} \tag{4.2.13.1}$$

Solution: From (4.1.4.1) and (4.1.1.1),

$$\mathbf{x} = \mathbf{A} + \lambda\mathbf{m} \tag{4.2.13.2}$$
$$\mathbf{n}^\top\mathbf{x} = c \tag{4.2.13.3}$$
$$\implies \mathbf{n}^\top\left(\mathbf{A} + \lambda\mathbf{m}\right) = c \tag{4.2.13.4}$$

which can be simplified to obtain

$$\mathbf{n}^\top\mathbf{A} + \lambda\mathbf{n}^\top\mathbf{m} = c \tag{4.2.13.5}$$
$$\implies \lambda = \frac{c - \mathbf{n}^\top\mathbf{A}}{\mathbf{n}^\top\mathbf{m}} \tag{4.2.13.6}$$

Substituting the above in (4.2.13.4) yields (4.2.13.1).

4.2.14. The foot of the perpendicular from the point $\mathbf{P}$ to the line represented by (4.1.4.1) is given by

$$\mathbf{x} = \mathbf{A} + \frac{\mathbf{m}^\top\left(\mathbf{P} - \mathbf{A}\right)}{\left\|\mathbf{m}\right\|^2}\mathbf{m} \tag{4.2.14.1}$$

Solution: Let the equation of the line be

$$\mathbf{x} = \mathbf{A} + \lambda \mathbf{m} \tag{4.2.14.2}$$

The equation of the plane perpendicular to the given line passing through $\mathbf{P}$ is given by

$$\mathbf{m}^\top (\mathbf{x} - \mathbf{P}) = 0 \tag{4.2.14.3}$$

$$\implies \mathbf{m}^\top \mathbf{x} = \mathbf{m}^\top \mathbf{P} \tag{4.2.14.4}$$

The desired foot of the perpendicular is the intersection of (4.2.14.2) with (4.2.14.3) which can be obtained from (4.2.13.1) as (4.2.14.1)

4.2.15. The foot of the perpendicular from a point $\mathbf{P}$ to a plane is $\mathbf{Q}$. The equation of the plane is given by

$$(\mathbf{P} - \mathbf{Q})^\top (\mathbf{x} - \mathbf{Q}) = 0 \tag{4.2.15.1}$$

Solution: The normal vector to the plane is given by

$$\mathbf{n} = \mathbf{P} - \mathbf{Q} \tag{4.2.15.2}$$

Hence, the equation of the plane is (4.2.15.1).

4.2.16. Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be points on a plane. The equation of the plane is then given by

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix}^\top \mathbf{n} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{4.2.16.1}$$

Solution: Let the equation of the plane be

$$\mathbf{n}^\top \mathbf{x} = 1 \tag{4.2.16.2}$$

Then

$$\mathbf{n}^\top \mathbf{A} = 1 \tag{4.2.16.3}$$

$$\mathbf{n}^\top \mathbf{B} = 1 \tag{4.2.16.4}$$

$$\mathbf{n}^\top \mathbf{C} = 1 \tag{4.2.16.5}$$

which can be combined to obtain (4.2.16.1).

4.2.17. (Parallelogram Law) Let $\mathbf{A}, \mathbf{B}, \mathbf{D}$ be three vertices of a parallelogram. Then the vertex $\mathbf{C}$ is given by

$$\mathbf{C} = \mathbf{B} + \mathbf{C} - \mathbf{A} \tag{4.2.17.1}$$

Solution: Shifting $\mathbf{A}$ to the origin, we obtain a parallelogram with corresponding vertices

$$\mathbf{0}, \mathbf{B} - \mathbf{A}, \mathbf{D} - \mathbf{A} \tag{4.2.17.2}$$

The fourth vertex of this parallelogram is then obtained as

$$(\mathbf{B} - \mathbf{A}) + (\mathbf{D} - \mathbf{A}) = \mathbf{D} + \mathbf{B} - 2\mathbf{A} \tag{4.2.17.3}$$

Shifting the origin to $\mathbf{A}$, the fourth vertex is obtained as

$$\mathbf{C} = \mathbf{D} + \mathbf{B} - 2\mathbf{A} + \mathbf{A} \tag{4.2.17.4}$$

$$= \mathbf{D} + \mathbf{B} - \mathbf{A} \tag{4.2.17.5}$$

4.2.18. (Affine Transformation) Let $\mathbf{A}, \mathbf{C}$, be opposite vertices of a square. The other two points can be obtained as

$$\mathbf{B} = \frac{\|\mathbf{A}-\mathbf{C}\|}{\sqrt{2}}\mathbf{P}\mathbf{e}_1 + \mathbf{A} \tag{4.2.18.1}$$

$$\mathbf{D} = \frac{\|\mathbf{A}-\mathbf{C}\|}{\sqrt{2}}\mathbf{P}\mathbf{e}_2 + \mathbf{A} \tag{4.2.18.2}$$

where

$$\mathbf{P} = \begin{pmatrix} \cos\left(\theta - \frac{\pi}{4}\right) & \sin\left(\theta - \frac{\pi}{4}\right) \\ \sin\left(\theta - \frac{\pi}{4}\right) & \cos\left(\theta - \frac{\pi}{4}\right) \end{pmatrix} \tag{4.2.18.3}$$

and

$$\cos\theta = \frac{\left(\mathbf{C}-\mathbf{A}\right)^\top \mathbf{e}_1}{\|\mathbf{A}-\mathbf{C}\| \, \|\mathbf{e}_1\|} \tag{4.2.18.4}$$

5 Quadratic Forms

5.1 Definitions

Definition 1. *The* affine *transformation is given by*

$$\mathbf{x} = \mathbf{P}\mathbf{y} + \mathbf{c} \quad \textit{(Affine Transformation)} \tag{5.1.1}$$

where $\mathbf{P}$ *is invertible.*

Definition 2. *The eigenvalue decomposition of a symmetric matrix* $\mathbf{V}$ *is given by*

$$\mathbf{P}^\top \mathbf{V} \mathbf{P} = \mathbf{D}. \quad \textit{(Eigenvalue Decomposition)} \tag{5.1.2}$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \tag{5.1.3}$$

$$\mathbf{P} = \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix}, \quad \mathbf{P}^\top = \mathbf{P}^{-1}, \tag{5.1.4}$$

5.2 The Quadratic Form

Definition 3. *Let* $\mathbf{q}$ *be a point such that the ratio of its distance from a fixed point* $\mathbf{F}$ *and the distance* (d) *from a fixed line*

$$L : \mathbf{n}^\top \mathbf{x} = c \tag{5.2.1}$$

is constant, given by

$$\frac{\|\mathbf{q}-\mathbf{F}\|}{d} = e \tag{5.2.2}$$

The locus of $\mathbf{q}$ *is known as a conic section. The line* L *is known as the directrix and the point* $\mathbf{F}$ *is the focus.* e *is defined to be the eccentricity of the conic.*

1) *For* $e = 1$, *the conic is a parabola*
2) *For* $e < 1$, *the conic is an ellipse*
3) *For* $e > 1$, *the conic is a hyperbola*

Theorem 5.1. *The equation of a conic with directrix* $\mathbf{n}^\top \mathbf{x} = c$, *eccentricity* e *and focus* $\mathbf{F}$ *is given by*

$$\mathbf{x}^\top \mathbf{V} \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{5.2.1}$$

where

$$\mathbf{V} = \|\mathbf{n}\|^2 \mathbf{I} - e^2\mathbf{n}\mathbf{n}^\top, \tag{5.2.2}$$
$$\mathbf{u} = ce^2\mathbf{n} - \|\mathbf{n}\|^2 \mathbf{F}, \tag{5.2.3}$$
$$f = \|\mathbf{n}\|^2 \|\mathbf{F}\|^2 - c^2e^2 \tag{5.2.4}$$

Proof. Using Definition 3 and Lemma 4.1.6.1, for any point $\mathbf{x}$ on the conic,

$$\|\mathbf{x} - \mathbf{F}\|^2 = e^2\frac{\left(\mathbf{n}^\top\mathbf{x} - c\right)^2}{\|\mathbf{n}\|^2} \tag{5.2.5}$$
$$\implies \|\mathbf{n}\|^2 (\mathbf{x} - \mathbf{F})^\top (\mathbf{x} - \mathbf{F}) = e^2 \left(\mathbf{n}^\top\mathbf{x} - c\right)^2 \tag{5.2.6}$$
$$\implies \|\mathbf{n}\|^2 \left(\mathbf{x}^\top\mathbf{x} - 2\mathbf{F}^\top\mathbf{x} + \|\mathbf{F}\|^2\right) = e^2 \left(c^2 + \left(\mathbf{n}^\top\mathbf{x}\right)^2 - 2c\mathbf{n}^\top\mathbf{x}\right) \tag{5.2.7}$$
$$= e^2 \left(c^2 + \left(\mathbf{x}^\top\mathbf{n}\mathbf{n}^\top\mathbf{x}\right) - 2c\mathbf{n}^\top\mathbf{x}\right) \tag{5.2.8}$$

which can be expressed as (5.2.1) after simplification.

□

Theorem 5.2. *The eccentricity, directrices and foci of* (5.2.1) *are given by*

$$e = \sqrt{1 - \frac{\lambda_1}{\lambda_2}} \tag{5.2.9}$$
$$\mathbf{n} = \sqrt{\lambda_2}\mathbf{p}_1,$$
$$c = \begin{cases} \frac{e\mathbf{u}^\top\mathbf{n} \pm \sqrt{e^2\left(\mathbf{u}^\top\mathbf{n}\right)^2 - \lambda_2(e^2-1)\left(\|\mathbf{u}\|^2 - \lambda_2 f\right)}}{\lambda_2 e(e^2-1)} & e \neq 1 \\ \frac{\|\mathbf{u}\|^2 - \lambda_2 f}{2\mathbf{u}^\top\mathbf{n}} & e = 1 \end{cases} \tag{5.2.10}$$
$$\mathbf{F} = \frac{ce^2\mathbf{n} - \mathbf{u}}{\lambda_2} \tag{5.2.11}$$

Proof. See Appendix A □

Theorem 5.3. (5.2.1) *represents*

1) *a parabola for* $\left|\mathbf{V}\right| = 0$,
2) *ellipse for* $\left|\mathbf{V}\right| > 0$ *and*
3) *hyperbola for* $\left|\mathbf{V}\right| < 0$.

Proof. From (5.2.9),

$$\frac{\lambda_1}{\lambda_2} = 1 - e^2 \tag{5.2.1}$$

Also,

$$\left|\mathbf{V}\right| = \lambda_1\lambda_2 \tag{5.2.2}$$

yielding Table 3 □

Eccentricity	Conic	Eigenvalue	Determinant
$e = 1$	Parabola	$\lambda_1 = 0$	$\left\|\mathbf{V}\right\| = 0$
$e < 1$	Ellipse	$\lambda_1 > 0, \lambda_2 > 0$	$\left\|\mathbf{V}\right\| > 0$
$e > 1$	Hyperbola	$\lambda_1 < 0, \lambda_2 > 0$	$\left\|\mathbf{V}\right\| < 0$

TABLE 3

5.3 Standard Form

Theorem 5.4. *Using the affine transformation in* (5.1.1)*, the conic in* (5.2.1) *can be expressed in standard form as*

$$\mathbf{y}^\top \left(\frac{\mathbf{D}}{f_0}\right)\mathbf{y} = 1 \qquad |\mathbf{V}| \neq 0 \tag{5.3.1}$$

$$\mathbf{y}^\top \mathbf{D}\mathbf{y} = -\eta \mathbf{e}_1^\top \mathbf{y} \qquad |\mathbf{V}| = 0 \tag{5.3.2}$$

where

$$f_0 = \mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} - f \neq 0 \tag{5.3.3}$$

$$\eta = 2\mathbf{u}^\top \mathbf{p}_1 \tag{5.3.4}$$

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{5.3.5}$$

Proof. See Appendix B. $\square$

Corollary 5.5. *For the standard conic,*

$$\mathbf{P} = \mathbf{I} \tag{5.3.6}$$

$$\mathbf{u} = \begin{cases} 0 & e \neq 1 \\ \frac{\eta}{2}\mathbf{e}_1 & e = 1 \end{cases} \tag{5.3.7}$$

$$\lambda_1 \begin{cases} = 0 & e = 1 \\ \neq 0 & e \neq 1 \end{cases} \tag{5.3.8}$$

where

$$\mathbf{I} = \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 \end{pmatrix} \tag{5.3.9}$$

is the identity matrix.

Theorem 5.6.

1) *The directrices for the standard conic are given by*

$$\mathbf{e}_1^\top \mathbf{y} = \pm\frac{1}{e}\sqrt{\frac{|f_0|}{\lambda_2\left(1 - e^2\right)}} \qquad e \neq 1 \tag{5.3.1}$$

$$\mathbf{e}_1^\top \mathbf{y} = \frac{\eta}{2\lambda_2} \qquad e = 1 \tag{5.3.2}$$

2) *The foci of the standard ellipse and hyperbola are given by*

$$\mathbf{F} = \begin{cases} \pm e\sqrt{\frac{|f_0|}{\lambda_2(1-e^2)}}\mathbf{e}_1 & e \neq 1 \\ -\frac{\eta}{4\lambda_2}\mathbf{e}_1 & e = 1 \end{cases} \tag{5.3.1}$$

Proof. See Appendix C. $\square$

5.4 Corollaries-Standard Form

Corollary 5.7. *The center of the standard ellipse/hyperbola, defined to be the mid point of the line joining the foci, is the origin.*

Corollary 5.8. *The principal (major) axis of the standard ellipse/hyperbola, defined to be the line joining the two foci is the x-axis.*

Proof. From (5.3.1), it is obvious that the line joining the foci passes through the origin. Also, the direction vector of this line is $\mathbf{e}_1$. Thus, the principal axis is the x-axis. □

Corollary 5.9. *The minor axis of the standard ellipse/hyperbola, defined to be the line orthogonal to the x-axis is the y-axis.*

Corollary 5.10. *The axis of symmetry of the standard parabola, defined to be the line perpendicular to the directrix and passing through the focus, is the x- axis.*

Proof. From (3.0.2.4) and (5.3.1), the axis of the parabola can be expressed using (4.1.2.1) as

$$\mathbf{e}_2^\top \left(\mathbf{y} + \frac{\eta}{4\lambda_2}\mathbf{e}_1\right) = 0 \tag{5.4.1}$$

$$\implies \mathbf{e}_2^\top \mathbf{y} = 0, \tag{5.4.2}$$

which is the equation of the x-axis. □

Corollary 5.11. *The point where the parabola intersects its axis of symmetry is called the vertex. For the standard parabola, the vertex is the origin.*

Proof. (5.4.2) can be expressed as

$$\mathbf{y} = \alpha\mathbf{e}_1, \tag{5.4.3}$$

using (4.1.2.1). Substituting (5.4.3) in (5.3.2),

$$\alpha^2\mathbf{e}_1^\top\mathbf{D}\mathbf{e}_1 = -\eta\alpha\mathbf{e}_1^\top\mathbf{e}_1 \tag{5.4.4}$$

$$\implies \alpha = 0, \text{ or, } \mathbf{y} = \mathbf{0}. \tag{5.4.5}$$

□

Corollary 5.12. *The* focal length *of the standard parabola, , defined to be the distance between the vertex and the focus, measured along the axis of symmetry, is* $\left|\frac{\eta}{4\lambda_2}\right|$

5.5 Corollaries-Quadratic Form

Corollary 5.13. *The center/vertex of a conic section are given by*

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \qquad \left|\mathbf{V}\right| \neq 0 \tag{5.5.1}$$

$$\begin{pmatrix}\mathbf{u}^\top + \frac{\eta}{2}\mathbf{p}_1^\top \\ \mathbf{V}\end{pmatrix}\mathbf{c} = \begin{pmatrix}-f \\ \frac{\eta}{2}\mathbf{p}_1 - \mathbf{u}\end{pmatrix} \qquad \left|\mathbf{V}\right| = 0 \tag{5.5.2}$$

Proof. In (5.1.1), substituting $\mathbf{y} = \mathbf{0}$, the center/vertex for the quadratic form is obtained as

$$\mathbf{x} = \mathbf{c}, \tag{5.5.3}$$

where $\mathbf{c}$ is derived as (5.5.1) and (5.5.2) in Appendix B. □

Corollary 5.14. *The equation of the minor and major axes for the ellipse/hyperbola are respectively given by*

$$\mathbf{p}_i^\top\left(\mathbf{x} - \mathbf{c}\right) = 0, i = 1, 2 \tag{5.5.4}$$

The axis of symmetry for the parabola is also given by (5.5.4).

Proof. From (5.8), the major/symmetry axis for the hyperbola/ellipse/parabola can be expressed using (5.1.1) as

$$\mathbf{e}_2^\top \mathbf{P}^\top (\mathbf{x} - \mathbf{c}) = 0 \tag{5.5.5}$$

$$\implies (\mathbf{P}\mathbf{e}_2)^\top (\mathbf{x} - \mathbf{c}) = 0 \tag{5.5.6}$$

yielding (5.5.4), and the proof for the minor axis is similar. □

5.6 Pair of Straight Lines

Lemma 5.1 (Asymptotes)**.** *The asymptotes of the hyperbola in* (5.3.1)*, defined to be the lines that do not intersect the hyperbola, are given by*

$$\begin{pmatrix} \sqrt{|\lambda_1|} & \pm\sqrt{|\lambda_2|} \end{pmatrix} \mathbf{y} = 0 \tag{5.6.1}$$

Proof. From (5.3.1), it is obvious that the pair of lines represented by

$$\mathbf{y}^\top \mathbf{D} \mathbf{y} = 0 \tag{5.6.2}$$

do not intersect the conic

$$\mathbf{y}^\top \mathbf{D} \mathbf{y} = f_0 \tag{5.6.3}$$

Thus, (5.6.2) represents the asysmptotes of the hyperbola in (5.3.1) and can be expressed as

$$\lambda_1 y_1^2 + \lambda_2 y_1^2 = 0, \tag{5.6.4}$$

which can then be simplified to obtain (5.6.1).

□

Corollary 5.15. (5.2.1) *represents a pair of straight lines if*

$$\mathbf{u}^\top \mathbf{V}^{-1} \mathbf{u} - f = 0 \tag{5.6.5}$$

Theorem 5.16. (5.2.1) *represents a pair of straight lines if the matrix*

$$\begin{pmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u} & f \end{pmatrix} \tag{5.6.6}$$

is singular.

Proof. Let

$$\begin{pmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u} & f \end{pmatrix} \mathbf{x} = \mathbf{0} \tag{5.6.7}$$

Expressing

$$\mathbf{x} = \begin{pmatrix} \mathbf{y} \\ y_3 \end{pmatrix}, \tag{5.6.8}$$

$$\begin{pmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^\top & f \end{pmatrix} \begin{pmatrix} \mathbf{y} \\ y_3 \end{pmatrix} = \mathbf{0} \tag{5.6.9}$$

$$\implies \mathbf{V}\mathbf{y} + y_3 \mathbf{u} = \mathbf{0} \quad \text{and} \tag{5.6.10}$$

$$\mathbf{u}^\top \mathbf{y} + f y_3 = 0 \tag{5.6.11}$$

From (5.6.10) we obtain,

$$\mathbf{y}^\top \mathbf{V}\mathbf{y} + y_3\mathbf{y}^\top \mathbf{u} = \mathbf{0} \tag{5.6.12}$$

$$\implies \mathbf{y}^\top \mathbf{V}\mathbf{y} + y_3\mathbf{u}^\top \mathbf{y} = \mathbf{0} \tag{5.6.13}$$

yielding (5.6.5) upon substituting from (5.6.11). □

Corollary 5.17. *Using the affine transformation,* (5.6.1) *can be expressed as the lines*

$$\left(\sqrt{|\lambda_1|} \quad \pm\sqrt{|\lambda_2|}\right) \mathbf{P}^\top (\mathbf{x} - \mathbf{c}) = 0 \tag{5.6.14}$$

Corollary 5.18. *The angle between the asymptotes can be expressed as*

$$\cos\theta = \frac{|\lambda_1| - |\lambda_2|}{|\lambda_1| + |\lambda_2|} \tag{5.6.15}$$

Proof. The normal vectors of the lines in (5.6.14) are

$$\begin{aligned} \mathbf{n}_1 &= \mathbf{P}\begin{pmatrix} \sqrt{|\lambda_1|} \\ \sqrt{|\lambda_2|} \end{pmatrix} \\ \mathbf{n}_2 &= \mathbf{P}\begin{pmatrix} \sqrt{|\lambda_1|} \\ -\sqrt{|\lambda_2|} \end{pmatrix} \end{aligned} \tag{5.6.16}$$

The angle between the asymptotes is given by

$$\cos\theta = \frac{\mathbf{n_1}^\top \mathbf{n_2}}{\|\mathbf{n_1}\| \, \|\mathbf{n_2}\|} \tag{5.6.17}$$

The orthogonal matrix $\mathbf{P}$ preserves the norm, i.e.

$$\|\mathbf{n_1}\| = \left\|\mathbf{P}\begin{pmatrix} \sqrt{|\lambda_1|} \\ \sqrt{|\lambda_2|} \end{pmatrix}\right\| = \left\|\begin{pmatrix} \sqrt{|\lambda_1|} \\ \sqrt{|\lambda_2|} \end{pmatrix}\right\| \tag{5.6.18}$$

$$= \sqrt{|\lambda_1| + |\lambda_2|} = \|\mathbf{n_2}\| \tag{5.6.19}$$

It is easy to verify that

$$\mathbf{n_1}^\top \mathbf{n_2} = |\lambda_1| - |\lambda_2| \tag{5.6.20}$$

Thus, the angle between the asymptotes is obtained from (5.6.17) as (5.6.15). □

5.7 Intersection of Conics

Lemma 5.2. *Let*

$$\mathbf{x}^\top \mathbf{V}_i\mathbf{x} + 2\mathbf{u}_i^\top \mathbf{x} + f_i = 0, \quad i = 1, 2 \tag{5.7.1}$$

be the equation of two conics. The locus of their intersection is a pair of straight lines if

$$\begin{vmatrix} \mathbf{V}_1 + \mu\mathbf{V}_2 & \mathbf{u}_1 + \mu\mathbf{u}_2 \\ (\mathbf{u}_1 + \mu\mathbf{u}_2)^\top & f \end{vmatrix} = 0, \left|\mathbf{V}_1 + \mu\mathbf{V}_2\right| < 0 \tag{5.7.2}$$

Proof. The intersection of the conics in (5.7.1) is given by the curve

$$\mathbf{x}^\top (\mathbf{V}_1 + \mu\mathbf{V}_2)\mathbf{x} + 2(\mathbf{u}_1 + \mu\mathbf{u}_2)^\top \mathbf{x} + f_1 + \mu f_2 = 0, \tag{5.7.3}$$

which, from Theorem 5.16 represents a pair of straight lines if (5.7.2) is satisfied. □

Corollary 5.19. *The points of intersection of the conics in* (5.7.1) *are the points of the intersection of the lines in* (5.7.3).

6 Conic Lines

6.1 Chords of a Conic

Theorem 6.1 (Chord)**.** *The points of intersection of the line*

$$L: \quad \mathbf{x} = \mathbf{q} + \mu \mathbf{m} \quad \mu \in \mathbb{R} \tag{6.1.1}$$

with the conic section in (5.2.1) *are given by*

$$\mathbf{x}_i = \mathbf{q} + \mu_i \mathbf{m} \tag{6.1.2}$$

where

$$\mu_i = \frac{1}{\mathbf{m}^\top \mathbf{V} \mathbf{m}} \Big(-\mathbf{m}^\top \left(\mathbf{V}\mathbf{q} + \mathbf{u} \right) \\ \pm \sqrt{\left[\mathbf{m}^\top \left(\mathbf{V}\mathbf{q} + \mathbf{u} \right) \right]^2 - \left(\mathbf{q}^\top \mathbf{V} \mathbf{q} + 2\mathbf{u}^\top \mathbf{q} + f \right) \left(\mathbf{m}^\top \mathbf{V} \mathbf{m} \right)} \Big) \tag{6.1.3}$$

Proof. Substituting (6.1.1) in (5.2.1),

$$\left(\mathbf{q} + \mu \mathbf{m} \right)^\top \mathbf{V} \left(\mathbf{q} + \mu \mathbf{m} \right) + 2\mathbf{u}^\top \left(\mathbf{q} + \mu \mathbf{m} \right) + f = 0 \tag{6.1.4}$$

$$\implies \mu^2 \mathbf{m}^\top \mathbf{V} \mathbf{m} + 2\mu \mathbf{m}^\top \left(\mathbf{V}\mathbf{q} + \mathbf{u} \right) + \mathbf{q}^\top \mathbf{V} \mathbf{q} + 2\mathbf{u}^\top \mathbf{q} + f = 0 \tag{6.1.5}$$

Solving the above quadratic in (6.1.5) yields (6.1.3). □

Corollary 6.2. *If L in* (6.1.1) *touches* (5.2.1) *at exactly one point* $\mathbf{q}$*,*

$$\mathbf{m}^\top \left(\mathbf{V}\mathbf{q} + \mathbf{u} \right) = 0 \tag{6.1.6}$$

Proof. In this case, (6.1.5) has exactly one root. Hence, in (6.1.3)

$$\left[\mathbf{m}^\top \left(\mathbf{V}\mathbf{q} + \mathbf{u} \right) \right]^2 - \left(\mathbf{m}^\top \mathbf{V} \mathbf{m} \right) \left(\mathbf{q}^\top \mathbf{V} \mathbf{q} + 2\mathbf{u}^\top \mathbf{q} + f \right) = 0 \tag{6.1.7}$$

$\because$ $\mathbf{q}$ is the point of contact, $\mathbf{q}$ satisfies (5.2.1) and

$$\mathbf{q}^\top \mathbf{V} \mathbf{q} + 2\mathbf{u}^\top \mathbf{q} + f = 0 \tag{6.1.8}$$

Substituting (6.1.8) in (6.1.7) and simplifying, we obtain (6.1.6). □

Theorem 6.3. *The length of the chord in* (6.1.1) *is given by*

$$\frac{2\sqrt{\left[\mathbf{m}^\top \left(\mathbf{V}\mathbf{q} + \mathbf{u} \right) \right]^2 - \left(\mathbf{q}^\top \mathbf{V} \mathbf{q} + 2\mathbf{u}^\top \mathbf{q} + f \right) \left(\mathbf{m}^\top \mathbf{V} \mathbf{m} \right)}}{\mathbf{m}^\top \mathbf{V} \mathbf{m}} \left\| \mathbf{m} \right\| \tag{6.1.9}$$

Proof. The distance between the points in (6.1.2) is given by

$$\left\| \mathbf{x}_1 - \mathbf{x}_2 \right\| = \left| \mu_1 - \mu_2 \right| \left\| \mathbf{m} \right\| \tag{6.1.10}$$

Substituing μ_i from (6.1.3) in (6.1.10) yields (6.1.9). □

Theorem 6.4. *The affine transform for the conic section, preserves the norm. This implies that the length of any chord of a conic is invariant to translation and/or rotation.*

Proof. Let

$$\mathbf{x}_i = \mathbf{P} \mathbf{y}_i + \mathbf{c} \tag{6.1.11}$$

be any two points on the conic. Then the distance between the points is given by

$$\left\| \mathbf{x}_1 - \mathbf{x}_2 \right\| = \left\| \mathbf{P} \left(\mathbf{y}_1 - \mathbf{y}_2 \right) \right\| \tag{6.1.12}$$

which can be expressed as

$$\|\mathbf{x}_1 - \mathbf{x}_2\|^2 = (\mathbf{y}_1 - \mathbf{y}_2)^\top \mathbf{P}^\top \mathbf{P} (\mathbf{y}_1 - \mathbf{y}_2) \tag{6.1.13}$$

$$= \|\mathbf{y}_1 - \mathbf{y}_2\|^2 \tag{6.1.14}$$

since

$$\mathbf{P}^\top \mathbf{P} = \mathbf{I} \tag{6.1.15}$$

□

Corollary 6.5. *For the standard hyperbola/ellipse, the length of the major axis is*

$$2\sqrt{\left|\frac{f_0}{\lambda_1}\right|} \tag{6.1.16}$$

and the minor axis is

$$2\sqrt{\left|\frac{f_0}{\lambda_2}\right|} \tag{6.1.17}$$

Proof. See Appendix D □

Theorem 6.6 (latus rectum)**.** *The latus rectum of a conic section is the chord that passes through the focus and is perpendicular to the major axis. The length of the latus rectum for a conic is given by*

$$l = \begin{cases} 2\frac{\sqrt{|f_0\lambda_1|}}{\lambda_2} & e \neq 1 \\ \frac{\eta}{\lambda_2} & e = 1 \end{cases} \tag{6.1.18}$$

Proof. See Appendix E. □

6.2 Tangent and Normal

Theorem 6.7 (Tangent)**.** *Given the point of contact* $\mathbf{q}$*, the equation of a tangent to* (5.2.1) *is*

$$(\mathbf{V}\mathbf{q} + \mathbf{u})^\top \mathbf{x} + \mathbf{u}^\top \mathbf{q} + f = 0 \tag{6.2.1}$$

Proof. The normal vector is obtained from (6.1.6) and (1.15.1) as

$$\kappa\mathbf{n} = \mathbf{V}\mathbf{q} + \mathbf{u}, \kappa \in \mathbb{R} \tag{6.2.2}$$

From (6.2.2) and (4.1.2.1), the equation of the tangent is

$$(\mathbf{V}\mathbf{q} + \mathbf{u})^\top (\mathbf{x} - \mathbf{q}) = 0 \tag{6.2.3}$$

$$\implies (\mathbf{V}\mathbf{q} + \mathbf{u})^\top \mathbf{x} - \mathbf{q}^\top \mathbf{V}\mathbf{q} - \mathbf{u}^\top \mathbf{q} = 0 \tag{6.2.4}$$

which, upon substituting from (6.1.8) and simplifying yields (6.2.1) □

Theorem 6.8. *If* $\mathbf{V}^{-1}$ *exists, given the normal vector* $\mathbf{n}$*, the tangent points of contact to* (5.2.1) *are given by*

$$\begin{gathered} \mathbf{q}_i = \mathbf{V}^{-1}(\kappa_i\mathbf{n} - \mathbf{u}), i = 1, 2 \\ \text{where } \kappa_i = \pm\sqrt{\frac{f_0}{\mathbf{n}^\top \mathbf{V}^{-1}\mathbf{n}}} \end{gathered} \tag{6.2.5}$$

Proof. From (6.2.2),

$$\mathbf{q} = \mathbf{V}^{-1}(\kappa\mathbf{n} - \mathbf{u}), \quad \kappa \in \mathbb{R} \tag{6.2.6}$$

Substituting (6.2.6) in (6.1.8),

$$\left(\kappa\mathbf{n}-\mathbf{u}\right)^\top\mathbf{V}^{-1}\left(\kappa\mathbf{n}-\mathbf{u}\right)+2\mathbf{u}^\top\mathbf{V}^{-1}\left(\kappa\mathbf{n}-\mathbf{u}\right)+f=0 \tag{6.2.7}$$

$$\implies \kappa^2\mathbf{n}^\top\mathbf{V}^{-1}\mathbf{n}-\mathbf{u}^\top\mathbf{V}^{-1}\mathbf{u}+f=0 \tag{6.2.8}$$

$$\text{or, } \kappa=\pm\sqrt{\frac{f_0}{\mathbf{n}^\top\mathbf{V}^{-1}\mathbf{n}}} \tag{6.2.9}$$

Substituting (6.2.9) in (6.2.6) yields (6.2.5). □

Theorem 6.9. *If $\mathbf{V}$ is not invertible, given the normal vector $\mathbf{n}$, the point of contact to* (5.2.1) *is given by the matrix equation*

$$\begin{pmatrix}\left(\mathbf{u}+\kappa\mathbf{n}\right)^\top\\ \mathbf{V}\end{pmatrix}\mathbf{q}=\begin{pmatrix}-f\\ \kappa\mathbf{n}-\mathbf{u}\end{pmatrix} \tag{6.2.10}$$

$$\textit{where } \kappa=\frac{\mathbf{p}_1^\top\mathbf{u}}{\mathbf{p}_1^\top\mathbf{n}},\quad \mathbf{V}\mathbf{p}_1=0 \tag{6.2.11}$$

Proof. If $\mathbf{V}$ is non-invertible, it has a zero eigenvalue. If the corresponding eigenvector is $\mathbf{p}_1$, then,

$$\mathbf{V}\mathbf{p}_1=0 \tag{6.2.12}$$

From (6.2.2),

$$\kappa\mathbf{n}=\mathbf{V}\mathbf{q}+\mathbf{u},\quad \kappa\in\mathbb{R} \tag{6.2.13}$$

$$\implies \kappa\mathbf{p}_1^\top\mathbf{n}=\mathbf{p}_1^\top\mathbf{V}\mathbf{q}+\mathbf{p}_1^\top\mathbf{u} \tag{6.2.14}$$

$$\text{or, } \kappa\mathbf{p}_1^\top\mathbf{n}=\mathbf{p}_1^\top\mathbf{u},\quad \because \mathbf{p}_1^\top\mathbf{V}=0,\quad (\text{ from (6.2.12)}) \tag{6.2.15}$$

yielding κ in (6.2.11). From (6.2.13),

$$\kappa\mathbf{q}^\top\mathbf{n}=\mathbf{q}^\top\mathbf{V}\mathbf{q}+\mathbf{q}^\top\mathbf{u} \tag{6.2.16}$$

$$\implies \kappa\mathbf{q}^\top\mathbf{n}=-f-\mathbf{q}^\top\mathbf{u}\quad\text{from (6.1.8),} \tag{6.2.17}$$

$$\text{or, } \left(\kappa\mathbf{n}+\mathbf{u}\right)^\top\mathbf{q}=-f \tag{6.2.18}$$

(6.2.13) can be expressed as

$$\mathbf{V}\mathbf{q}=\kappa\mathbf{n}-\mathbf{u}. \tag{6.2.19}$$

(6.2.18) and (6.2.19) clubbed together result in (6.2.10). □

Theorem 6.10. *The normal vectors of the tangents from a point $\mathbf{h}$ to the conic in* (5.2.1) *are given by*

$$\mathbf{n}=\frac{\mathbf{e}_1}{\mathbf{e}_1^\top\mathbf{h}}+\mu_i\mathbf{R}\mathbf{h}, \tag{6.2.20}$$

where

$$\mathbf{R}=\mathbf{R}_{\frac{\pi}{2}}=\begin{pmatrix}0&-1\\1&0\end{pmatrix} \tag{6.2.21}$$

and μ_i are obtained from (6.1.3) *by substituting*

$$\mathbf{m}=\mathbf{R}\mathbf{h},\mathbf{u}=0,\mathbf{q}=\frac{\mathbf{e}_1}{\mathbf{e}_1^\top\mathbf{h}} \tag{6.2.22}$$

and replacing $\mathbf{V}$ with $\mathbf{V}^{-1}$.

Proof. From (6.1.6), the normal vector to the tangent at $\mathbf{q}$ can be expressed as

$$\mathbf{n} = \mathbf{V}\mathbf{q} + \mathbf{u} \tag{6.2.23}$$

$$\implies \mathbf{q} = \mathbf{V}^{-1}\left(\mathbf{n} - \mathbf{u}\right) \tag{6.2.24}$$

which upon substituting in (5.2.1) yields

$$\left(\mathbf{n} - \mathbf{u}\right)^\top \mathbf{V}^{-1}\mathbf{V}\mathbf{V}^{-1}\left(\mathbf{n} - \mathbf{u}\right) + 2\mathbf{u}^\top\mathbf{V}^{-1}\left(\mathbf{n} - \mathbf{u}\right) + f = 0 \tag{6.2.25}$$

$$\implies \mathbf{n}^\top\mathbf{V}^{-1}\mathbf{n} - f_0 = 0 \tag{6.2.26}$$

Since the tangents pass through the point $\mathbf{h}$,

$$\mathbf{n}^\top\mathbf{h} = 1 \tag{6.2.27}$$

and from (3.5.4.1), we obtain (6.2.20). □

Theorem 6.11. *The normal vectors of the tangents from a point* $\mathbf{h}$ *to the conic in* (5.2.1) *are given by*

$$\begin{aligned}\mathbf{n}_1 &= \mathbf{P}\begin{pmatrix}\sqrt{\left|\lambda_1\right|}\\ \sqrt{\left|\lambda_2\right|}\end{pmatrix}\\ \mathbf{n}_2 &= \mathbf{P}\begin{pmatrix}\sqrt{\left|\lambda_1\right|}\\ -\sqrt{\left|\lambda_2\right|}\end{pmatrix}\end{aligned} \tag{6.2.28}$$

where $\lambda_i, \mathbf{P}$ *are the eigenparameters of*

$$\mathbf{\Sigma} = \left(\mathbf{V}\mathbf{h} + \mathbf{u}\right)\left(\mathbf{V}\mathbf{h} + \mathbf{u}\right)^\top - \mathbf{V}\left(\mathbf{h}^\top\mathbf{V}\mathbf{h} + 2\mathbf{u}^\top\mathbf{h} + f\right). \tag{6.2.29}$$

Proof. From (6.1.3), and (6.1.7)

$$\left[\mathbf{m}^\top\left(\mathbf{V}\mathbf{h} + \mathbf{u}\right)\right]^2 - \left(\mathbf{m}^\top\mathbf{V}\mathbf{m}\right)\left(\mathbf{h}^\top\mathbf{V}\mathbf{h} + 2\mathbf{u}^\top\mathbf{h} + f\right) = 0 \tag{6.2.30}$$

$$\implies \mathbf{m}^\top\left[\left(\mathbf{V}\mathbf{h} + \mathbf{u}\right)\left(\mathbf{V}\mathbf{h} + \mathbf{u}\right)^\top - \mathbf{V}\left(\mathbf{h}^\top\mathbf{V}\mathbf{h} + 2\mathbf{u}^\top\mathbf{h} + f\right)\right]\mathbf{m} = 0 \tag{6.2.31}$$

yielding (6.2.29). Consequently, from (5.6.16), (6.2.28) can be obtained. □

7 Examples

7.1 Loney

Example 7.1 (parabola)**.** *To show that*

$$9x^2 - 24xy + 16y^2 - 18x - 101y + 19 = 0 \tag{7.1.1}$$

is the equation of a parabola with latus rectum of length 3, vertex

$$\frac{1}{25}\begin{pmatrix}-29\\ 25\end{pmatrix} \tag{7.1.2}$$

and axis

$$3x - 4y + 7 = 0 \tag{7.1.3}$$

Solution: Comparing (7.1.1) with (5.2.1),

$$\mathbf{V} = \begin{pmatrix}9 & -12\\ -12 & 16\end{pmatrix} \tag{7.1.4}$$

$$\mathbf{u} = -\frac{1}{2}\begin{pmatrix}18\\ 101\end{pmatrix} \tag{7.1.5}$$

$$f = 19 \tag{7.1.6}$$

The eigenvalues of $\mathbf{V}$ are obtained as

$$\left|\lambda\mathbf{I} - \mathbf{V}\right| = 0 \tag{7.1.7}$$

$$\implies \lambda_1 = 0, \lambda_2 = 25 \tag{7.1.8}$$

Since the $\mathbf{V}$ matrix has a 0 eigenvalue, (7.1.1) is a parabola. The eigenvector corresponding to the 0 eigenalue is given by

$$\begin{pmatrix} 9 & -12 \\ -12 & 16 \end{pmatrix} \mathbf{p}_1 = \mathbf{0} \tag{7.1.9}$$

yielding

$$\mathbf{p}_1 = \frac{1}{5}\begin{pmatrix} 4 \\ 3 \end{pmatrix} \tag{7.1.10}$$

Substituting from (7.1.4), (7.1.5), (7.1.6), (7.1.10) and (7.1.8) in (5.3.3), (5.3.4) and (6.1.18) the latus rectum is obtained as

$$\eta = \frac{\left|2\mathbf{u}^\top \mathbf{p}_1\right|}{\lambda_2} = 3 \tag{7.1.11}$$

The vertex of the parabola is obtained from (5.5.2) as

$$\begin{pmatrix} -39 & -73 \\ 9 & -12 \\ -12 & 16 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -19 \\ -21 \\ 28 \end{pmatrix} \tag{7.1.12}$$

yielding (7.1.2). The second eigenvector of $\mathbf{V}$ is orthogonal to $\mathbf{p}_1$ and obtained as

$$\mathbf{p}_2 = \frac{1}{5}\begin{pmatrix} 3 \\ -4 \end{pmatrix} \tag{7.1.13}$$

Substituting from (7.1.2) and (7.1.12) in (5.5.4), the equation of the axis of symmetry for the parabola can be expressed as (7.1.3).

Example 7.2 (ellipse)**.** *To show that the equation*

$$14x^2 - 4xy + 11y^2 - 44x - 58y + 71 = 0 \tag{7.1.14}$$

represents an ellipse with centre

$$\mathbf{c} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \tag{7.1.15}$$

and lengths of semi-axes

$$\sqrt{6} \text{ and } 2 \tag{7.1.16}$$

Solution: The parameters for (7.1.14), are

$$\mathbf{V} = \begin{pmatrix} 14 & -2 \\ -2 & 11 \end{pmatrix} \tag{7.1.17}$$

$$\mathbf{u} = \begin{pmatrix} 22 \\ 29 \end{pmatrix} \tag{7.1.18}$$

$$f = 71 \tag{7.1.19}$$

Since

$$\left|\mathbf{V}\right| = 150 > 0, \tag{7.1.20}$$

(7.1.14) is an ellipse. Substituting from (7.1.17) and (7.1.18) in (5.5.1), the center of the ellipse is obtained as (7.1.15). Also, the eigenvalues of $\mathbf{V}$ are

$$\lambda_1 = 10, \lambda_2 = 15 \tag{7.1.21}$$

Substituting from (7.1.17), (7.1.18) and (7.1.19) in (5.5.4),

$$f_0 = 60 \tag{7.1.22}$$

Substituting from (7.1.21) and (7.1.22) in (6.1.16) and (6.1.17), the lengths of the semi-axes are obtained as (7.1.16).

Example 7.3 (hyperbola)**.** *To show that the equation*

$$x^2 - 3xy + y^2 + 10x - 10y + 21 = 0 \tag{7.1.23}$$

represents a hyperbola with centre

$$\mathbf{c} = \begin{pmatrix} -2 \\ 2 \end{pmatrix} \tag{7.1.24}$$

and length of semi-axes

$$\sqrt{2} \text{ and } \sqrt{\frac{2}{5}} \tag{7.1.25}$$

Solution: The conic parameters for (7.1.23), are

$$\mathbf{V} = \frac{1}{2}\begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix} \tag{7.1.26}$$

$$\mathbf{u} = \begin{pmatrix} 5 \\ -5 \end{pmatrix} \tag{7.1.27}$$

$$f = 21 \tag{7.1.28}$$

Since

$$\left|\mathbf{V}\right| = -\frac{5}{4} < 0, \tag{7.1.29}$$

(7.1.23) is a hyperbola. Substituting from (7.1.26) and (7.1.26) in (5.5.1), the center of the hyperbola is obtained as (7.1.15). Also, the eigenvalues of $\mathbf{V}$ are

$$\lambda_1 = -\frac{1}{2}, \lambda_2 = \frac{5}{2} \tag{7.1.30}$$

From (5.5.4),

$$f_0 = -1 \tag{7.1.31}$$

Substituting from (7.1.30) and (7.1.31) in (6.1.16) and (6.1.17), the lengths of the semi-axes are then obtained as (7.1.25).

Example 7.4 (tangents)**.** *To show that the tangents to the curve*

$$x^2 + 4xy + 3y^2 - 5x - 6y + 3 = 0 \tag{7.1.32}$$

parallel to the line

$$x + 4y + c = 0 \tag{7.1.33}$$

are

$$\begin{aligned} x + 4y - 5 &= 0 \\ x + 4y - 8 &= 0 \end{aligned} \tag{7.1.34}$$

Solution: The conic parameters for (7.1.32) are

$$\mathbf{V} = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \tag{7.1.35}$$

$$\mathbf{u} = \begin{pmatrix} -\frac{5}{2} \\ -3 \end{pmatrix} \tag{7.1.36}$$

$$f = 3 \tag{7.1.37}$$

Since

$$\left|\mathbf{V}\right| = -1 < 0, \tag{7.1.38}$$

(7.1.32) is a hyperbola. From (7.1.33), the normal vector to the tangent is

$$\mathbf{n} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \tag{7.1.39}$$

The equation of the tangent can be expressed as

$$\mathbf{n}^\top \left(\mathbf{x} - \mathbf{q}_i\right) = 0 \tag{7.1.40}$$

where $\mathbf{q}_i$ are the points of contact. Comparing the above with (7.1.33),

$$c = -\mathbf{n}^\top \mathbf{q}_i \tag{7.1.41}$$

which, upon substituting from (6.2.5) can be expressed as

$$c = -\mathbf{n}^\top \left\{\mathbf{V}^{-1}\left(\kappa_i \mathbf{n} - \mathbf{u}\right)\right\} \tag{7.1.42}$$

$$= -\mathbf{n}^\top \left\{\mathbf{V}^{-1}\left(\pm\sqrt{\frac{f_0}{\mathbf{n}^\top\mathbf{V}^{-1}\mathbf{n}}}\mathbf{n} - \mathbf{u}\right)\right\} \tag{7.1.43}$$

$$= \pm\sqrt{f_0 \mathbf{n}^\top\mathbf{V}^{-1}\mathbf{n}} + \mathbf{n}^\top\mathbf{V}^{-1}\mathbf{u}. \tag{7.1.44}$$

Substituting from (7.1.35), (7.1.36) and (7.1.37) in (5.3.3),

$$\mathbf{f}_0 = -\frac{3}{4} \tag{7.1.45}$$

Substituing from (7.1.35), (7.1.36), (7.1.39) and (7.1.45) in (6.2.9),

$$c = -5 \text{ or } c = -8 \tag{7.1.46}$$

yielding (7.1.34).

7.2 Miscellaneous

7.2.1. Given unit basis vectors $\mathbf{a}, \mathbf{b}$, with angle θ between them, the locus of the coordinates of a unit vector $\mathbf{c}$ in the space spanned by $\mathbf{a}, \mathbf{b}$ is given by

$$\mathbf{x}^\top \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \mathbf{x} = 1 \tag{7.2.1.1}$$

with $\rho = \cos\theta$.
Solution: Let

$$\mathbf{c} = x_1\mathbf{a} + x_2\mathbf{b} = \begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix} \mathbf{x} \tag{7.2.1.2}$$

Then,

$$\left\|\mathbf{c}\right\|^2 = \mathbf{x}^\top \begin{pmatrix} \mathbf{a}^\top \\ \mathbf{b}^\top \end{pmatrix} \begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix} \mathbf{x} \tag{7.2.1.3}$$

$$= \mathbf{x}^\top \begin{pmatrix} 1 & \mathbf{a}^\top\mathbf{b} \\ \mathbf{a}^\top\mathbf{b} & 1 \end{pmatrix} \mathbf{x} \tag{7.2.1.4}$$

which can be expressed as (7.2.1.1).

7.2.2. Given the coordinates of $\mathbf{c}$, the angle θ between the basis vectors is given by

$$\rho = \frac{1 - \left\|\mathbf{x}\right\|^2}{\mathbf{x}^\top\mathbf{R}\mathbf{x}} \tag{7.2.2.1}$$

where

$$\mathbf{R} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{7.2.2.2}$$

Solution: Let

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{7.2.2.3}$$

For

$$\mathbf{V} = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}, \mathbf{u} = 0, f = -1 \tag{7.2.2.4}$$

in (5.2.1),
Since

$$\left|\mathbf{V}\right| = 1 - \rho^2, 0 < \left|\mathbf{V}\right| < 1, \tag{7.2.2.5}$$

(7.2.1.1) represents the equation of an ellipse. Using eigenvalue decomposition,

$$\mathbf{V} = \mathbf{P}^\top\mathbf{D}\mathbf{P} \tag{7.2.2.6}$$

where

$$\mathbf{P} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 1+\rho & 0 \\ 0 & 1-\rho \end{pmatrix} \tag{7.2.2.7}$$

Using the affine transformation,

$$\mathbf{x} = \mathbf{P}\mathbf{y} \tag{7.2.2.8}$$

(7.2.1.1) can be expressed as

$$\mathbf{y}^\top\mathbf{D}\mathbf{y} = 1 \implies y_1^2\left(1+\rho\right) + y_2^2\left(1-\rho\right) = 1 \tag{7.2.2.9}$$

which can be simplified to obtain

$$\rho = \frac{1 - y_1^2 - y_2^2}{y_1^2 - y_2^2} \tag{7.2.2.10}$$

$$= \frac{1 - \|\mathbf{y}\|^2}{\mathbf{y}^\top \mathbf{Q} \mathbf{y}} \tag{7.2.2.11}$$

where

$$\mathbf{Q} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{7.2.2.12}$$

From (7.2.2.8), (7.2.2.11) can be expressed as (7.2.2.1)

$$\because \|\mathbf{a}\| = \|\mathbf{b}\| = 1. \tag{7.2.2.13}$$

Appendix A

From (5.2.2), using the fact that $\mathbf{V}$ is symmetric with $\mathbf{V} = \mathbf{V}^\top$,

$$\mathbf{V}^\top \mathbf{V} = \left(\|\mathbf{n}\|^2 \mathbf{I} - e^2 \mathbf{n}\mathbf{n}^\top\right)^\top \left(\|\mathbf{n}\|^2 \mathbf{I} - e^2 \mathbf{n}\mathbf{n}^\top\right) \tag{1.0.2.1}$$

$$\implies \mathbf{V}^2 = \|\mathbf{n}\|^4 \mathbf{I} + e^4 \mathbf{n}\mathbf{n}^\top \mathbf{n}\mathbf{n}^\top - 2e^2 \|\mathbf{n}\|^2 \mathbf{n}\mathbf{n}^\top \tag{1.0.2.2}$$

$$= \|\mathbf{n}\|^4 \mathbf{I} + e^4 \|\mathbf{n}\|^2 \mathbf{n}\mathbf{n}^\top - 2e^2 \|\mathbf{n}\|^2 \mathbf{n}\mathbf{n}^\top \tag{1.0.2.3}$$

$$= \|\mathbf{n}\|^4 \mathbf{I} + e^2 \left(e^2 - 2\right) \|\mathbf{n}\|^2 \mathbf{n}\mathbf{n}^\top \tag{1.0.2.4}$$

$$= \|\mathbf{n}\|^4 \mathbf{I} + \left(e^2 - 2\right) \|\mathbf{n}\|^2 \left(\|\mathbf{n}\|^2 \mathbf{I} - \mathbf{V}\right) \tag{1.0.2.5}$$

which can be expressed as

$$\mathbf{V}^2 + \left(e^2 - 2\right) \|\mathbf{n}\|^2 \mathbf{V} - \left(e^2 - 1\right) \|\mathbf{n}\|^4 \mathbf{I} = 0 \tag{1.0.2.6}$$

Using the Cayley-Hamilton theorem, (1.0.2.6) results in the characteristic equation,

$$\lambda^2 - \left(2 - e^2\right) \|\mathbf{n}\|^2 \lambda + \left(1 - e^2\right) \|\mathbf{n}\|^4 = 0 \tag{1.0.2.7}$$

which can be expressed as

$$\left(\frac{\lambda}{\|\mathbf{n}\|^2}\right)^2 - \left(2 - e^2\right) \left(\frac{\lambda}{\|\mathbf{n}\|^2}\right) + \left(1 - e^2\right) = 0 \tag{1.0.2.8}$$

$$\implies \frac{\lambda}{\|\mathbf{n}\|^2} = 1 - e^2, 1 \tag{1.0.2.9}$$

$$\text{or, } \lambda_2 = \|\mathbf{n}\|^2, \lambda_1 = \left(1 - e^2\right) \lambda_2 \tag{1.0.2.10}$$

From (1.0.2.10), the eccentricity of (5.2.1) is given by (5.2.9). Multiplying both sides of (5.2.2) by $\mathbf{n}$,

$$\mathbf{V}\mathbf{n} = \|\mathbf{n}\|^2 \mathbf{n} - e^2 \mathbf{n}\mathbf{n}^\top \mathbf{n} \tag{1.0.2.11}$$

$$= \|\mathbf{n}\|^2 \left(1 - e^2\right) \mathbf{n} \tag{1.0.2.12}$$

$$= \lambda_1 \mathbf{n} \tag{1.0.2.13}$$

$$\tag{1.0.2.14}$$

from (1.0.2.10). Thus, λ_1 is the corresponding eigenvalue for $\mathbf{n}$. From (5.1.4) and (1.0.2.14), this implies that

$$\mathbf{p}_1 = \frac{\mathbf{n}}{\|\mathbf{n}\|} \tag{1.0.2.15}$$

$$\text{or, } \mathbf{n} = \|\mathbf{n}\| \mathbf{p}_1 = \sqrt{\lambda_2} \mathbf{p}_1 \tag{1.0.2.16}$$

from (1.0.2.10) . From (5.2.3) and (1.0.2.10),

$$\mathbf{F} = \frac{ce^2\mathbf{n} - \mathbf{u}}{\lambda_2} \tag{1.0.2.17}$$

$$\implies \|\mathbf{F}\|^2 = \frac{\left(ce^2\mathbf{n} - \mathbf{u}\right)^\top \left(ce^2\mathbf{n} - \mathbf{u}\right)}{\lambda_2^2} \tag{1.0.2.18}$$

$$\implies \lambda_2^2 \|\mathbf{F}\|^2 = c^2e^4\lambda_2 - 2ce^2\mathbf{u}^\top\mathbf{n} + \|\mathbf{u}\|^2 \tag{1.0.2.19}$$

Also, (5.2.4) can be expressed as

$$\lambda_2 \|\mathbf{F}\|^2 = f + c^2e^2 \tag{1.0.2.20}$$

From (1.0.2.19) and (1.0.2.20),

$$c^2e^4\lambda_2 - 2ce^2\mathbf{u}^\top\mathbf{n} + \|\mathbf{u}\|^2 = \lambda_2\left(f + c^2e^2\right) \tag{1.0.2.21}$$

$$\implies \lambda_2 e^2\left(e^2 - 1\right)c^2 - 2ce^2\mathbf{u}^\top\mathbf{n} + \|\mathbf{u}\|^2 - \lambda_2 f = 0 \tag{1.0.2.22}$$

yielding (5.2.11).

Appendix B

Using (5.1.1) (5.2.1) can be expressed as

$$\left(\mathbf{P}\mathbf{y} + \mathbf{c}\right)^\top \mathbf{V}\left(\mathbf{P}\mathbf{y} + \mathbf{c}\right) + 2\mathbf{u}^\top\left(\mathbf{P}\mathbf{y} + \mathbf{c}\right) + f = 0, \tag{2.0.2.1}$$

yielding

$$\mathbf{y}^\top\mathbf{P}^\top\mathbf{V}\mathbf{P}\mathbf{y} + 2\left(\mathbf{V}\mathbf{c} + \mathbf{u}\right)^\top\mathbf{P}\mathbf{y} + \mathbf{c}^\top\mathbf{V}\mathbf{c} + 2\mathbf{u}^\top\mathbf{c} + f = 0 \tag{2.0.2.2}$$

From (2.0.2.2) and (5.1.2),

$$\mathbf{y}^\top\mathbf{D}\mathbf{y} + 2\left(\mathbf{V}\mathbf{c} + \mathbf{u}\right)^\top\mathbf{P}\mathbf{y} + \mathbf{c}^\top\left(\mathbf{V}\mathbf{c} + \mathbf{u}\right) + \mathbf{u}^\top\mathbf{c} + f = 0 \tag{2.0.2.3}$$

When $\mathbf{V}^{-1}$ exists, choosing

$$\mathbf{V}\mathbf{c} + \mathbf{u} = \mathbf{0}, \quad \text{or}, \mathbf{c} = -\mathbf{V}^{-1}\mathbf{u}, \tag{2.0.2.4}$$

and substituting (2.0.2.4) in (2.0.2.3) yields (5.3.1).

B.1

When $|\mathbf{V}| = 0, \lambda_1 = 0$ and

$$\mathbf{V}\mathbf{p}_1 = 0, \mathbf{V}\mathbf{p}_2 = \lambda_2\mathbf{p}_2. \tag{2.1.2.1}$$

where $\mathbf{p}_1, \mathbf{p}_2$ are the eigenvectors of $\mathbf{V}$ such that (5.1.2)

$$\mathbf{P} = \begin{pmatrix}\mathbf{p}_1 & \mathbf{p}_2\end{pmatrix}, \tag{2.1.2.2}$$

Substituting (2.1.2.2) in (2.0.2.3),

$$\mathbf{y}^\top\mathbf{D}\mathbf{y} + 2\left(\mathbf{c}^\top\mathbf{V} + \mathbf{u}^\top\right)\begin{pmatrix}\mathbf{p}_1 & \mathbf{p}_2\end{pmatrix}\mathbf{y} + \mathbf{c}^\top\left(\mathbf{V}\mathbf{c} + \mathbf{u}\right) + \mathbf{u}^\top\mathbf{c} + f = 0 \tag{2.1.2.3}$$

$$\implies \mathbf{y}^\top\mathbf{D}\mathbf{y} + 2\left(\left(\mathbf{c}^\top\mathbf{V} + \mathbf{u}^\top\right)\mathbf{p}_1\ \left(\mathbf{c}^\top\mathbf{V} + \mathbf{u}^\top\right)\mathbf{p}_2\right)\mathbf{y} + \mathbf{c}^\top\left(\mathbf{V}\mathbf{c} + \mathbf{u}\right) + \mathbf{u}^\top\mathbf{c} + f = 0 \tag{2.1.2.4}$$

$$\implies \mathbf{y}^\top\mathbf{D}\mathbf{y} + 2\left(\mathbf{u}^\top\mathbf{p}_1 \quad \left(\lambda_2\mathbf{c}^\top + \mathbf{u}^\top\right)\mathbf{p}_2\right)\mathbf{y} + \mathbf{c}^\top\left(\mathbf{V}\mathbf{c} + \mathbf{u}\right) + \mathbf{u}^\top\mathbf{c} + f = 0 \tag{2.1.2.5}$$

upon substituting from (2.1.2.1) yielding

$$\lambda_2 y_2^2 + 2\left(\mathbf{u}^\top \mathbf{p}_1\right) y_1 + 2y_2\left(\lambda_2 \mathbf{c} + \mathbf{u}\right)^\top \mathbf{p}_2 + \mathbf{c}^\top\left(\mathbf{V}\mathbf{c} + \mathbf{u}\right) + \mathbf{u}^\top \mathbf{c} + f = 0 \tag{2.1.2.6}$$

Thus, (2.1.2.6) can be expressed as (5.3.2) by choosing

$$\eta = 2\mathbf{u}^\top \mathbf{p}_1 \tag{2.1.2.7}$$

and $\mathbf{c}$ in (2.0.2.3) such that

$$2\mathbf{P}^\top\left(\mathbf{V}\mathbf{c} + \mathbf{u}\right) = \eta \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.1.2.8}$$

$$\mathbf{c}^\top\left(\mathbf{V}\mathbf{c} + \mathbf{u}\right) + \mathbf{u}^\top \mathbf{c} + f = 0 \tag{2.1.2.9}$$

$\because \mathbf{P}^\top \mathbf{P} = \mathbf{I}$, multiplying (2.1.2.8) by $\mathbf{P}$ yields

$$\left(\mathbf{V}\mathbf{c} + \mathbf{u}\right) = \frac{\eta}{2}\mathbf{p}_1, \tag{2.1.2.10}$$

which, upon substituting in (2.1.2.9) results in

$$\frac{\eta}{2}\mathbf{c}^\top \mathbf{p}_1 + \mathbf{u}^\top \mathbf{c} + f = 0 \tag{2.1.2.11}$$

(2.1.2.10) and (2.1.2.11) can be clubbed together to obtain (5.5.2).

Appendix C

a) For the standard hyperbola/ellipse in (5.3.1), from (5.3.6), (5.2.10) and (5.3.7),

$$\mathbf{n} = \sqrt{\frac{\lambda_2}{f_0}}\mathbf{e}_1 \tag{3.0.2.1}$$

$$c = \pm\frac{\sqrt{-\frac{\lambda_2}{f_0}\left(e^2 - 1\right)\left(\frac{\lambda_2}{f_0}\right)}}{\frac{\lambda_2}{f_0}e\left(e^2 - 1\right)} \tag{3.0.2.2}$$

$$= \pm\frac{1}{e\sqrt{1 - e^2}} \tag{3.0.2.3}$$

yielding (5.3.1) upon substituting from (5.2.9) and simplifying. For the standard parabola in (5.3.2), from (5.3.6), (5.2.10) and (5.3.7), noting that $f = 0$,

$$\mathbf{n} = \sqrt{\lambda_2}\mathbf{e}_1 \tag{3.0.2.4}$$

$$c = \frac{\left\|\frac{\eta}{2}\mathbf{e}_1\right\|^2}{2\left(\frac{\eta}{\mathbf{2}}\right)\left(\mathbf{e_1}\right)^\top \mathbf{n}} \tag{3.0.2.5}$$

$$\tag{3.0.2.6}$$

$$= \frac{\eta}{4\sqrt{\lambda_2}} \tag{3.0.2.7}$$

yielding (5.3.2).

b) For the standard ellipse/hyperbola, substituting from (3.0.2.3), (3.0.2.1), (5.3.7) and (5.2.9) in (5.2.11),

$$\mathbf{F} = \pm\frac{\left(\frac{1}{e\sqrt{1-e^2}}\right)\left(e^2\right)\sqrt{\frac{\lambda_2}{f_0}}\mathbf{e}_1}{\frac{\lambda_2}{f_0}} \tag{3.0.2.8}$$

yielding (5.3.1) after simplification. For the standard parabola, substituting from (3.0.2.7), (3.0.2.4), (5.3.7) and (5.2.9) in (5.2.11),

$$\mathbf{F} = \frac{\left(\frac{\eta}{4\sqrt{\lambda_2}}\right)\sqrt{\lambda_2}\mathbf{e}_1 - \frac{\eta}{\mathbf{2}}\mathbf{e_1}}{\lambda_2} \tag{3.0.2.9}$$

$$\tag{3.0.2.10}$$

yielding (5.3.1) after simplification.

Appendix D

Since the major axis passes through the origin,

$$\mathbf{q} = \mathbf{0} \tag{4.0.2.1}$$

Further, from Corollary (5.8),

$$\mathbf{m} = \mathbf{e}_2, \tag{4.0.2.2}$$

and from (5.3.1),

$$\mathbf{V} = \frac{\mathbf{D}}{f_0}, \mathbf{u} = 0, f = -1 \tag{4.0.2.3}$$

Substituting the above in (6.1.9),

$$\frac{2\sqrt{\mathbf{e}_1^\top \frac{\mathbf{D}}{f_0}\mathbf{e}_1}}{\mathbf{e}_1^\top \frac{\mathbf{D}}{f_0}\mathbf{e}_1}\left\|\mathbf{e}_1\right\| \tag{4.0.2.4}$$

yielding (6.1.16). Similarly, for the minor axis, the only different parameter is

$$\mathbf{m} = \mathbf{e}_2, \tag{4.0.2.5}$$

Substituting the above in (6.1.9),

$$\frac{2\sqrt{\mathbf{e}_2^\top \frac{\mathbf{D}}{f_0}\mathbf{e}_2}}{\mathbf{e}_2^\top \frac{\mathbf{D}}{f_0}\mathbf{e}_2}\left\|\mathbf{e}_2\right\| \tag{4.0.2.6}$$

yielding (6.1.17).

Appendix E

The latus rectum is perpendicular to the major axis for the standard conic. Hence, from Corollary (5.8),

$$\mathbf{m} = \mathbf{e}_2, \tag{5.0.2.1}$$

Since it passes through the focus, from (5.3.1)

$$\mathbf{q} = \mathbf{F} = \pm e\sqrt{\frac{f_0}{\lambda_2\left(1-e^2\right)}}\mathbf{e}_1 \tag{5.0.2.2}$$

for the standard hyperbola/ellipse. Also, from (5.3.1),

$$\mathbf{V} = \frac{\mathbf{D}}{f_0}, \mathbf{u} = 0, f = -1 \tag{5.0.2.3}$$

Substituting the above in (6.1.9),

$$\frac{2\sqrt{\left[\mathbf{e}_2^\top\left(\frac{\mathbf{D}}{f_0}e\sqrt{\frac{f_0}{\lambda_2(1-e^2)}}\mathbf{e}_1\right)\right]^2-\left(e\sqrt{\frac{f_0}{\lambda_2(1-e^2)}}\mathbf{e}_1^\top\frac{\mathbf{D}}{f_0}e\sqrt{\frac{f_0}{\lambda_2(1-e^2)}}\mathbf{e}_1-1\right)\left(\mathbf{e}_2^\top\frac{\mathbf{D}}{f_0}\mathbf{e}_2\right)}}{\mathbf{e}_2^\top\frac{\mathbf{D}}{f_0}\mathbf{e}_2}\left\|\mathbf{e}_2\right\| \tag{5.0.2.4}$$

Since

$$\mathbf{e}_2^\top\mathbf{D}\mathbf{e}_1=0,\mathbf{e}_1^\top\mathbf{D}\mathbf{e}_1=\lambda_1,\mathbf{e}_1^\top\mathbf{e}_1=1,\left\|\mathbf{e}_2\right\|=1,\mathbf{e}_2^\top\mathbf{D}\mathbf{e}_2=\lambda_2, \tag{5.0.2.5}$$

(5.0.2.4) can be expressed as

$$\frac{2\sqrt{\left(1-\frac{\lambda_1e^2}{\lambda_2(1-e^2)}\right)\left(\frac{\lambda_2}{f_0}\right)}}{\frac{\lambda_2}{f_0}} \tag{5.0.2.6}$$

$$=2\frac{\sqrt{f_0\lambda_1}}{\lambda_2} \qquad \left(\because e^2=1-\frac{\lambda_1}{\lambda_2}\right) \tag{5.0.2.7}$$

For the standard parabola, the parameters in (6.1.9) are

$$\mathbf{q}=\mathbf{F}=-\frac{\eta}{4\lambda_2}\mathbf{e}_1,\mathbf{m}=\mathbf{e}_1,\mathbf{V}=\mathbf{D},\mathbf{u}=\frac{\eta}{2}\mathbf{e}_1^\top,f=0 \tag{5.0.2.8}$$

Substituting the above in (6.1.9), the length of the latus rectum can be expressed as

$$\frac{2\sqrt{\left[\mathbf{e}_2^\top\left(\mathbf{D}\left(-\frac{\eta}{4\lambda_2}\mathbf{e}_1\right)+\frac{\eta}{2}\mathbf{e}_1\right)\right]^2-\left(\left(-\frac{\eta}{4\lambda_2}\mathbf{e}_1\right)^\top\mathbf{D}\left(-\frac{\eta}{4\lambda_2}\mathbf{e}_1\right)+2\frac{\eta}{2}\mathbf{e}_1^\top\left(-\frac{\eta}{4\lambda_2}\mathbf{e}_1\right)\right)\left(\mathbf{e}_2^\top\mathbf{D}\mathbf{e}_2\right)}}{\mathbf{e}_2^\top\mathbf{D}\mathbf{e}_2}\left\|\mathbf{e}_2\right\| \tag{5.0.2.9}$$

Since

$$\mathbf{e}_2^\top\mathbf{D}\mathbf{e}_1=0,\mathbf{e}_2^\top\mathbf{e}_2=0,\mathbf{e}_1^\top\mathbf{D}\mathbf{e}_1=0,\mathbf{e}_1^\top\mathbf{e}_1=1,\left\|\mathbf{e}_1\right\|=1,\mathbf{e}_2^\top\mathbf{D}\mathbf{e}_2=\lambda_2, \tag{5.0.2.10}$$

(5.0.2.9) can be expressed as

$$2\frac{\sqrt{\frac{\eta^2}{4\lambda_2}\lambda_2}}{\lambda_2}=\frac{\eta}{\lambda_2} \tag{5.0.2.11}$$