0.1 Foundations

0.1.1 Derivation of EOM of many body system

Hamiltonian of single atom dispersively coupled to single cavity mode by a running-wave laser drive

$$\hat{H}_{SP} = \frac{\hat{p}^2}{2M} - \hbar\omega_z \hat{F}_z + \hbar q \hat{F}_z^2 + \hbar\omega_c \hat{a}^\dagger \hat{a} - i\frac{\alpha_\nu}{2F}\left[\hat{\boldsymbol{E}}^{(+)} \times \hat{\boldsymbol{E}}^{(-)}\right] \cdot \hat{\boldsymbol{F}}. \tag{0.1.1}$$

Operator $\hat{a}$ creates photon in z-polarized cavity mode of frequency ω_c. Second and third term are Zeeman splittings. $\hat{\boldsymbol{F}}$ is spin operator.

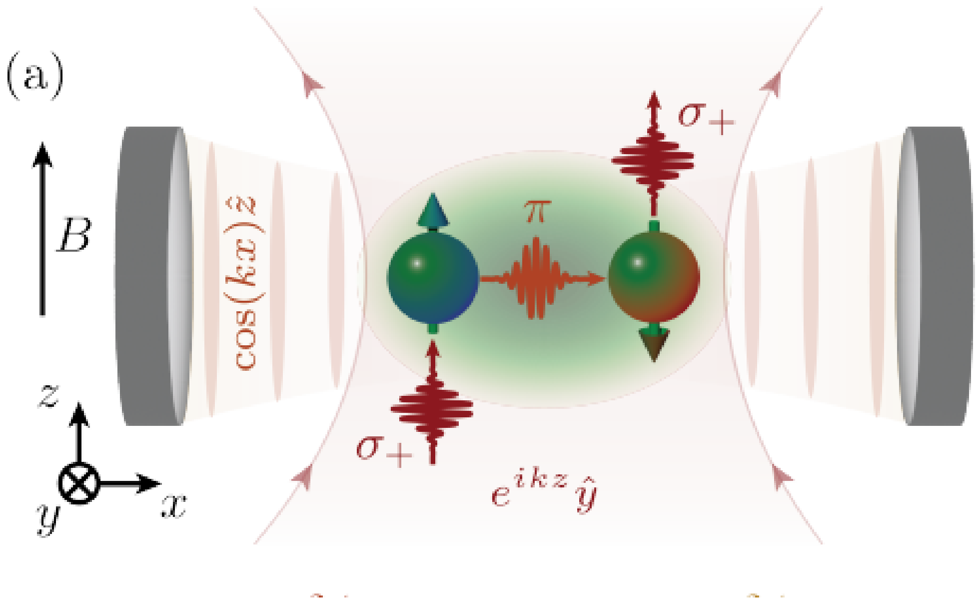

Figure 1: Pair production

Second quantization

Spinor field operator RHS "5-mode-expansion" doesnt contain second summand

$$\hat{\Psi}(\boldsymbol{x}) = \begin{pmatrix} \frac{k}{\sqrt{2\pi}}\cos(kx)(e^{ikz}\hat{c}_{+k,+1} + e^{-ikz}\hat{c}_{-k,+1}) \\ \frac{k}{2\pi}\hat{c}_{0,0} + \frac{\sqrt{2}k}{\sqrt{3}\pi}\cos^2(kx)\hat{c}_{\pm 2k_x,0} \\ \frac{k}{\sqrt{2\pi}}\cos(kx)(e^{-ikz}\hat{c}_{-k,-1} + e^{ikz}\hat{c}_{+k,-1}) \end{pmatrix} = \begin{pmatrix} \hat{c}_{+1,+k}\psi_{+1,+k} + \hat{c}_{+1,-k}\psi_{+1,-k} \\ \hat{c}_{0,0}\psi_{0,0} \\ \hat{c}_{-1,k}\psi_{-1,+k} + \hat{c}_{-1,-k}\psi_{-1,-k} \end{pmatrix} \tag{0.1.2}$$

Where the respective functions have to be normed

$$\int_{\frac{-\pi}{k}}^{\frac{\pi}{k}} \psi^*_{+1,+k}\psi_{+1,+k} dzdx = 1 \tag{0.1.3}$$

(c.f. 2010 Dicke paper)

$$\Psi = \begin{pmatrix} \Psi_{+1} \\ \Psi_0 \\ \Psi_{-1} \end{pmatrix} \tag{0.1.4}$$

Next we find an effective many body hamiltonian.

$$H_{SP} = H_L + H_{AT} + H_{INT} \tag{0.1.5}$$

Where H_{AT} contains F_z and H_{INT} contains F_+, F_-.

$$H_{MB} = H_L + \int \hat{\Psi}^\dagger(\hat{x})(H_{AT} + H_{INT})\hat{\Psi}(\hat{x})\hat{dx} \tag{0.1.6}$$

e.g.

$$F_z = \begin{pmatrix} +1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{0.1.7}$$

$$F_+ = \sqrt{2} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{0.1.8}$$

This calculation is done in Rodrigos "Full derivation Hamiltonian" handwritten pdf. We do adiabatic elimination with effective operators and apply the rotating wave approximation. We obtain the effective many-body Hamiltonian

$$H = H_0 + H_+ + H_- \tag{0.1.9}$$

with e.g.

$$H_+ = \hbar\chi_+(2\hat{c}^\dagger_{-k,-1}\hat{c}^\dagger_{+k,+1}\hat{c}_0\hat{c}_0 + \hat{c}^\dagger_0\hat{c}_{+k,+1}\hat{c}^\dagger_{+k,+1}\hat{c}_0 + \hat{c}^\dagger_{-k,-1}\hat{c}_0\hat{c}^\dagger_0\hat{c}_{-k,-1} + h.c.) \tag{0.1.10}$$

0.1.2 Further info to experiment: (rodrigo thesis p.103)

drive is operated in limit of large two-poton detunings

$$|\delta_\pm| \gg \kappa \tag{0.1.11}$$

$$\delta_\pm = \delta_c \pm \omega_z = (\omega_d \pm \omega_z) - \omega_c \tag{0.1.12}$$

We absorb drive photon, and go from

$$|0\rangle_0 \rightarrow |+k\rangle_+ 1 \tag{0.1.13}$$

thus we need a energy conserving cavity photon with freq $\approx \omega_d - \omega_z$. (or $+\omega_z$?) Here we can still ignore the kinetic energy $\sim k$ of the atom since this energy is much smaller than κ.
Parametric amplification of pair production
look at 0.1.9 + assume mode $|0\rangle_0$ undepleted throughout the dynamics i.e. occupied by N atoms. set $\hat{c}_0 = \sqrt{N}$ and obtain

$$\hat{H}_{eff} = \hat{H}^+_{eff} + \hat{H}^-_{eff} \tag{0.1.14}$$

with

$$\hat{H}^{\pm}_{eff} = \hbar(\omega_0 + 4N\chi_\pm)(\hat{K}_{z,\pm} - 1/2) + 4\hbar N\chi_\pm \hat{K}_{x,\pm} \tag{0.1.15}$$

Look at linear equations of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\hat{K}_{x,\pm}\\ ....y\\ ...z\end{pmatrix} = \boldsymbol{M}_\pm \begin{pmatrix}\hat{K}_{x,\pm}\\ ....y\\ ...z\end{pmatrix} \tag{0.1.16}$$

with three non-degenerate complex eigenvalues

$$\lambda_{1,\pm} = 0 \tag{0.1.17}$$

$$\lambda_{2,\pm} = \sqrt{-\omega_0(\omega_0 + 8N\chi_\pm)} =: +\lambda_\pm \tag{0.1.18}$$

$$\lambda_{3,\pm} = -\sqrt{-\omega_0(\omega_0 + 8N\chi_\pm)} \tag{0.1.19}$$

We also have

$$\langle N_{p,\pm}\rangle = \frac{1}{2}(\langle c^\dagger_{1,\pm}c_{1,\pm}\rangle + \langle c^\dagger_{-1,\mp}c_{-1,\mp}\rangle) \approx \langle K_{z,\pm}\rangle - \frac{1}{2} \approx A\cosh(\lambda_\pm t) + (const) \tag{0.1.20}$$

To conclude: we see that we have eigenvectors of M. Those are perpendicular, since the eigenvalues are different. The time development of those is given by 0.1.16. Thus, its either phase oszillation for a complex eigenvalue or exponential growth for a real eigenvalue. 0.1.20 looks at the expectation value of the occupation of the modes that are not $|0\rangle_0$ (occupation of pairs). we see that the time development of those depends on $\langle K_{z,\pm}\rangle$, therefore on the eigenvectors of M, therefore on the eigenvalues of M. We see, that for a real $\lambda_\pm$ the occuopation of those modes get macroscopic. So we say that for a critical coupling a second order phase transition occurs (lambda get real) featuring pairs. this fast change of coupling is called quench (faster that any period of oscillations happening in system e.g. $1/\omega_0$).

Note: if the number of pairs gets high, the undepleted approximation doesnt hold anymore. thus this equations can just predict the inital growth of pairs. (later there is saturation)
quench: the system jumps from one set of eigenstates to another set of eigenstates. before system was in single eigenstate, after quench it is superposition of different eigenstates → oscillation of those. this cant be expressed analytically, thus calculations are done numerically.

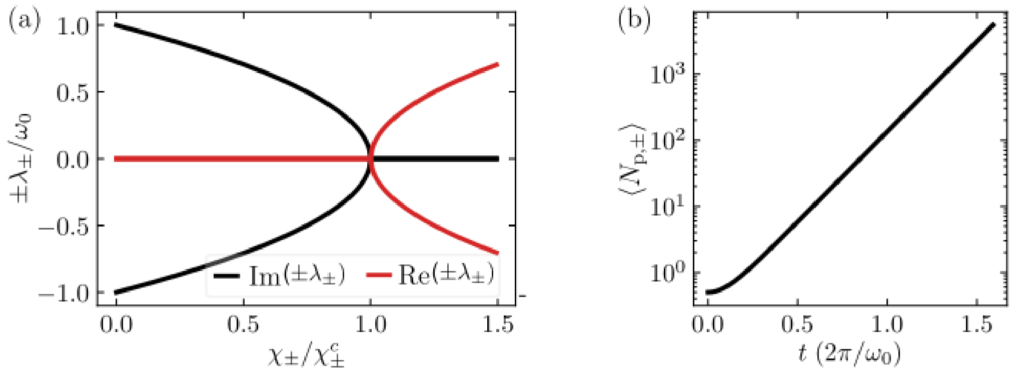

Figure 2: Parametric amplicication