

0.1 Foundations

0.1.1 Derivation of EOM of many body system

Hamiltonian of single atom dispersively coupled to single cavity mode by a running-wave laser drive

$$\hat{H}_{SP} = \frac{\hat{p}^2}{2M} - \hbar\omega_z \hat{F}_z + \hbar q \hat{F}_z^2 + \hbar\omega_c \hat{a}^\dagger \hat{a} - i \frac{\alpha_\nu}{2F} \left[\hat{\mathbf{E}}^{(+)} \times \hat{\mathbf{E}}^{(-)} \right] \cdot \hat{\mathbf{F}}. \quad (0.1.1)$$

Operator \hat{a} creates photon in z-polarized cavity mode of frequency ω_c . Second and third term are Zeeman splittings. $\hat{\mathbf{F}}$ is spin operator.

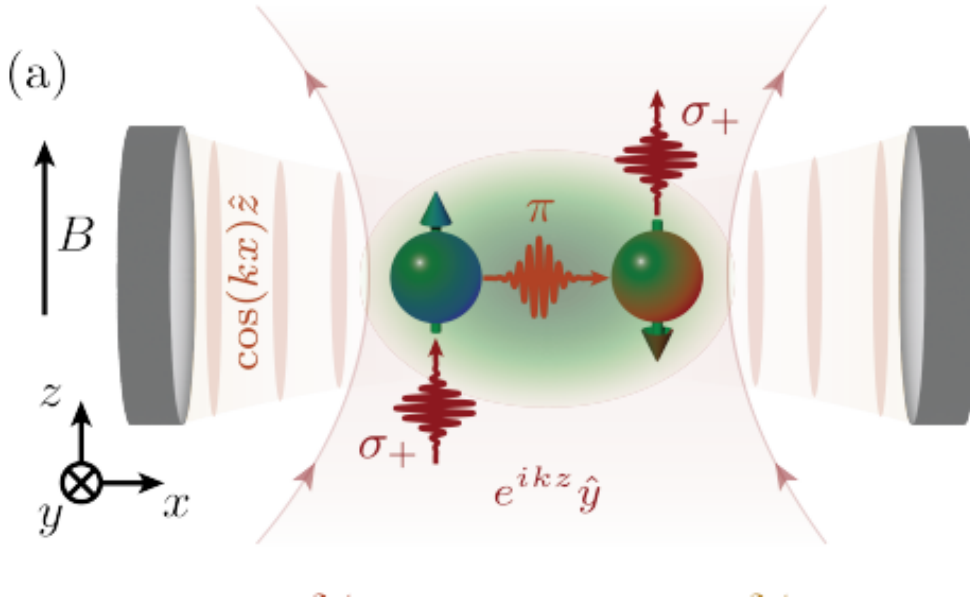


Figure 1: Pair production

Second quantization

Spinor field operator RHS "5-mode-expansion" doesnt contain second summand

$$\hat{\Psi}(\mathbf{x}) = \begin{pmatrix} \frac{k}{\sqrt{2\pi}} \cos(kx) (e^{ikz} \hat{c}_{+k,+1} + e^{-ikz} \hat{c}_{-k,+1}) \\ \frac{k}{2\pi} \hat{c}_{0,0} + \frac{\sqrt{2}k}{\sqrt{3\pi}} \cos^2(kx) \hat{c}_{\pm 2k,0} \\ \frac{k}{\sqrt{2\pi}} \cos(kx) (e^{-ikz} \hat{c}_{-k,-1} + e^{ikz} \hat{c}_{+k,-1}) \end{pmatrix} = \begin{pmatrix} \hat{c}_{+1,+k} \psi_{+1,+k} + \hat{c}_{+1,-k} \psi_{+1,-k} \\ \hat{c}_{0,0} \psi_{0,0} \\ \hat{c}_{-1,k} \psi_{-1,+k} + \hat{c}_{-1,-k} \psi_{-1,-k} \end{pmatrix} \quad (0.1.2)$$

Where the respective functions have to be normed

$$\int_{-\frac{\pi}{k}}^{\frac{\pi}{k}} \psi_{+1,+k}^* \psi_{+1,+k} dz dx = 1 \quad (0.1.3)$$

(c.f. 2010 Dicke paper)

$$\Psi = \begin{pmatrix} \Psi_{+1} \\ \Psi_0 \\ \Psi_{-1} \end{pmatrix} \quad (0.1.4)$$

Next we find an effective many body hamiltonian.

$$H_{SP} = H_L + H_{AT} + H_{INT} \quad (0.1.5)$$

Where H_{AT} contains F_z and H_{INT} contains F_+, F_- .

$$H_{MB} = H_L + \int \hat{\Psi}^\dagger(\hat{x})(H_{AT} + H_{INT})\hat{\Psi}(\hat{x})d\hat{x} \quad (0.1.6)$$

e.g.

$$F_z = \begin{pmatrix} +1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad (0.1.7)$$

$$F_+ = \sqrt{2} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad (0.1.8)$$

This calculation is done in Rodrigos "Full derivation Hamiltonian" handwritten pdf. We do adiabatic elimination with effective operators and apply the rotating wave approximation. We obtain the effective many-body Hamiltonian

$$H = H_0 + H_+ + H_- \quad (0.1.9)$$

with e.g.

$$H_+ = \hbar\chi_+ (2\hat{c}_{-k,-1}^\dagger \hat{c}_{+k,+1}^\dagger \hat{c}_0 \hat{c}_0 + \hat{c}_0^\dagger \hat{c}_{+k,+1} \hat{c}_{+k,+1}^\dagger \hat{c}_0 + \hat{c}_{-k,-1}^\dagger \hat{c}_0 \hat{c}_0^\dagger \hat{c}_{-k,-1} + h.c.) \quad (0.1.10)$$

0.1.2 Further info to experiment: (rodrigo thesis p.103)

drive is operated in limit of large two-poton detunings

$$|\delta_\pm| \gg \kappa \quad (0.1.11)$$

$$\delta_\pm = \delta_c \pm \omega_z = (\omega_d \pm \omega_z) - \omega_c \quad (0.1.12)$$

We absorb drive photon, and go from

$$|0\rangle_0 \rightarrow |k\rangle_+ 1 \quad (0.1.13)$$

thus we need a energy conserving cavity photon with freq $\approx \omega_d - \omega_z$. (or $+\omega_z$?) Here we can still ignore the kinetic energy $\sim k$ of the atom since this energy is much smaller than κ .

Parametric amplification of pair production

look at 0.1.9 + assume mode $|0\rangle_0$ undepleted throughout the dynamics i.e. occupied by N atoms. set $\hat{c}_0 = \sqrt{N}$ and obtain

$$\hat{H}_{eff} = \hat{H}_{eff}^+ + \hat{H}_{eff}^- \quad (0.1.14)$$

with

$$\hat{H}_{eff}^\pm = \hbar(\omega_0 + 4N\chi_\pm)(\hat{K}_{z,\pm} - 1/2) + 4\hbar N\chi_\pm \hat{K}_{x,\pm} \quad (0.1.15)$$

Look at linear equations of motion

$$\frac{d}{dt} \begin{pmatrix} \hat{K}_{x,\pm} \\ \dots y \\ \dots z \end{pmatrix} = \mathbf{M}_\pm \begin{pmatrix} \hat{K}_{x,\pm} \\ \dots y \\ \dots z \end{pmatrix} \quad (0.1.16)$$

with three non-degenerate complex eigenvalues

$$\lambda_{1,\pm} = 0 \quad (0.1.17)$$

$$\lambda_{2,\pm} = \sqrt{-\omega_0(\omega_0 + 8N\chi_\pm)} =: +\lambda_\pm \quad (0.1.18)$$

$$\lambda_{3,\pm} = -\sqrt{-\omega_0(\omega_0 + 8N\chi_\pm)} \quad (0.1.19)$$

We also have

$$\langle N_{p,\pm} \rangle = \frac{1}{2}(\langle c_{1,\pm}^\dagger c_{1,\pm} \rangle + \langle c_{-1,\mp}^\dagger c_{-1,\mp} \rangle) \approx \langle K_{z,\pm} \rangle - \frac{1}{2} \approx A \cosh(\lambda_\pm t) + (const) \quad (0.1.20)$$

To conclude: we see that we have eigenvectors of M. Those are perpendicular, since the eigenvalues are different. The time development of those is given by 0.1.16. Thus, its either phase oszillation for a complex eigenvalue or exponential growth for a real eigenvalue. 0.1.20 looks at the expectation value of the occupation of the modes that are not $|0\rangle_0$ (occupation of pairs). we see that the time development of those depends on $\langle K_{z,\pm} \rangle$, therefore on the eigenvectors of M, therefore on the eigenvalues of M. We see, that for a real λ_\pm the occuopation of those modes get macroscopic. So we say that for a critical coupling a second order phase transition occurs (lambda get real) featuring pairs. this fast change of coupling is called quench (faster than any period of oscillations happening in system e.g. $1/\omega_0$).

Note: if the number of pairs gets high, the undepleted approximation doesn't hold anymore. thus this equations can just predict the initial growth of pairs. (later there is saturation)

quench: the system jumps from one set of eigenstates to another set of eigenstates. before system was in single eigenstate, after quench it is superposition of different eigenstates \rightarrow oscillation of those. this can't be expressed analytically, thus calculations are done numerically.

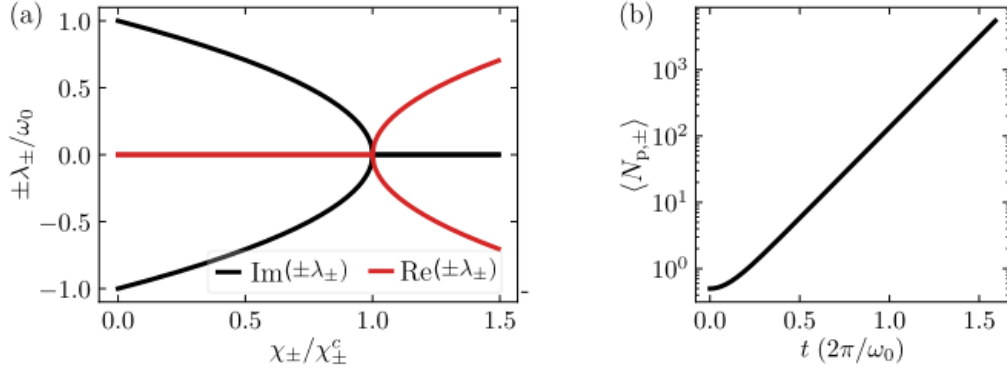


Figure 2: Parametric amplification

0.1.3 The energy scales

not sure, whether all plus /minus are chosen correctly in sketch, but the scheme looks approximately like this: We start in the Rb87 5 2 S1/2 F=1 (F... total angular momentum) fine structure) (<https://steck.us/alkalidata/rubidium87numbers.1.6.pdf>). We send light ω_d with $\lambda 790.02nm$. The transition to the 5 2 P3/2 (D2 transition) is around 980nm the transition to 5 2 P1/2 (D1) is around 995nm. thus, we are blue detuned with D1 and red detuned with D2 (check whether its not other way around)(one of the wavelength in Rb is wrong). This is called Tune out-wavelength. Even though it is not exactly the middle it is the effective middle. So the scalar polarization α_s is zero and there is no dipole potential trapping the atom. We are off resonant by THz and therefore won't reach any population in the 5 2 P levels (dispersive regime). N I S H A N T Dogra phdthesis ch3.1. explains this detailed. the energy of the π photon is $\omega_{\pm} = \omega_d \pm w_z$. So something like THz \pm e.g. $\omega_z = 2\pi 7MHz$.

The cavity wavelength is chosen to be similar to the drive wavelength: $|\delta_c| = |\omega_d - \omega_c| < 2\pi 10MHz$. But the cavity wavelength is off resonant to the π photon wavelength

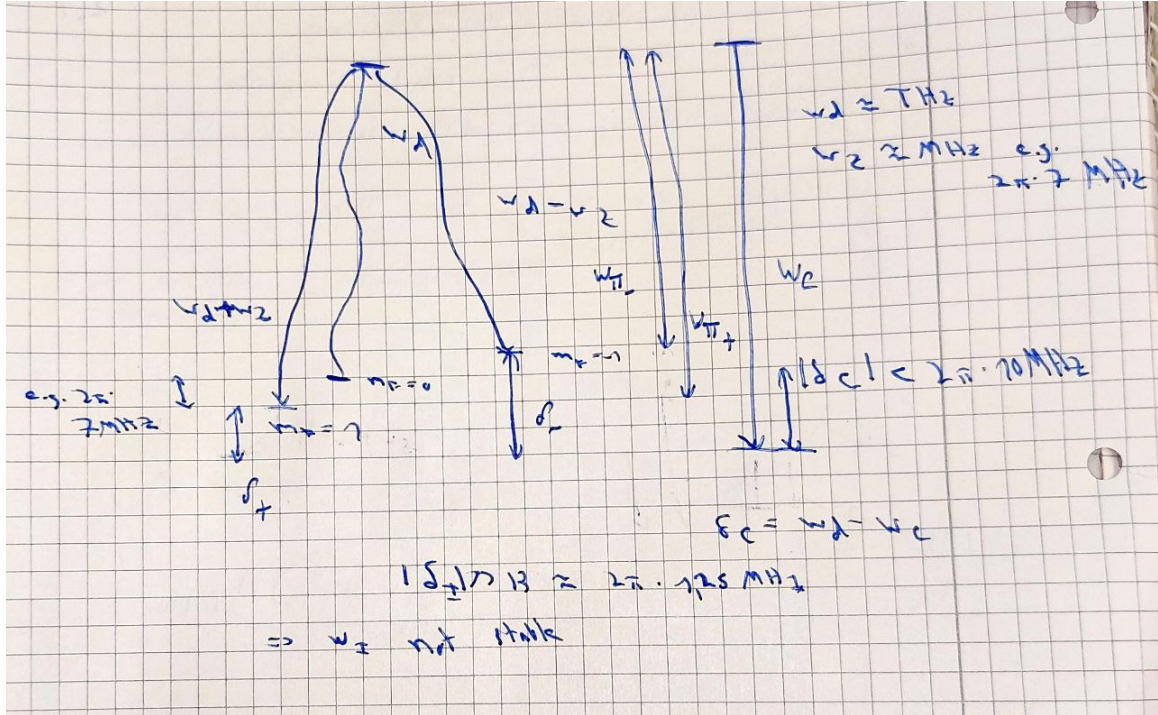


Figure 3: Energy sketch

if we compare this detuning δ_{\pm} (= diff ω_{\pm} and ω_c) with the cavity loss κ . $|\delta_{\pm}| \gg \kappa$
 From somewhere we get

$$\chi_{\pm} = \eta^2 \frac{\delta_{\pm}}{\delta_{\pm}^2 + \kappa^2} \quad (0.1.21)$$

and

$$\gamma_{\pm} = \frac{\eta^2 \kappa}{\delta_{\pm}^2 + \kappa^2}. \quad (0.1.22)$$

$$\eta \sim \alpha_{\nu} E_d E_0 \quad (0.1.23)$$

where α_{ν} vectorial polarisierbarkeit, E_d intensity of drive, E_0 electric field of vacuum of cavity mode \Rightarrow given by geometry. 0.1.21 shows, that for high ω_z the first atom will be most likely in state $|+k, +1\rangle$. This is because high ω_z implies high $|\delta_-|$ and small $|\delta_+|$ which leads to a stronger coupling for the + channel.

Experimentally we can change δ_c and ω_z .

how is k defined? $k = k_z$?. own consideration: let $\omega_c = \omega_d + x$, where x is order of MHz. We get by Taylor

$$\lambda_c = \frac{c}{\omega_d + x} = \frac{c}{\omega_d} - x \frac{c}{\omega_d^2} + O(x^2) \approx \frac{c}{\omega_d} = \lambda_d. \quad (0.1.24)$$

Thats the reason, why we use for python just one single wavelength λ_M . So we can conclude that $k_z \approx k_x$ (=k, right?).

With this result we look at the energies of different states:

$$\psi_{\pm 2k,0} \approx N \cos(2kx) \otimes |m = 0\rangle \quad (0.1.25)$$

and therefore

$$E_{rec} = \frac{\hbar^2(2k)^2}{2m} = \hbar 4\omega_{rec} \quad (0.1.26)$$

with

$$\omega_{rec} = \frac{\hbar k^2}{2m} \quad (0.1.27)$$

(check formulas, rodrigo gave them to me.).

We also get

$$\psi_{+k,+1} \approx N \cos(kx) e^{ikz}. \quad (0.1.28)$$

$$E_{kin} = \int \psi_{+k,+1} \hbar \left(\frac{\partial_x^2 + \partial_y^2}{2m} \right) \psi_{+k,+1} dx dz \quad (0.1.29)$$

$$= \frac{\hbar k_x^2 + \hbar k_z^2}{2m} = 2\omega_{rec}. \quad (0.1.30)$$

So if we consider the creation of a single pair, we get that this pair has energy

$$\omega_0 = 2q + 4\omega_{rec} \quad (0.1.31)$$

in comparison to the to atoms in the BEC. The first order zeeman splitting cancelled out such that only the 2nd order Zeeman splitting q and the kinetic energy ω_{rec} contribute. (check with rodrigo, whether this is correct understanding. for this understanding we dont need to consider rotating frame.). We see that for small q the $|\pm 2k, 0\rangle$ and $|+k, +1\rangle$ modes have the same energy scale.

Derivation of recoil energy:

$$E_{rec} = \frac{p_{rec}^2}{2m} = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2 4\pi^2}{2m\lambda^2} = \hbar \omega_{rec}. \quad (0.1.32)$$

0.1.4 Number squeezing

Following (thesis Finger)

Introduce imbalance operator

$$\hat{J}_z = \frac{1}{2}(\hat{N}_{+1} - \hat{N}_{-1}) \quad (0.1.33)$$

where $\hat{N}_{+1} = \hat{c}_{+k,+1}^\dagger \hat{c}_{+k,+1}$ and $\hat{N}_{-1} = \hat{c}_{-k,-1}^\dagger \hat{c}_{-k,-1}$. We introduce

$$\xi_N^2 = \frac{4\sigma^2(\hat{J}_z)}{N} \quad (0.1.34)$$

where $\sigma^2(\hat{J}_z)$ is variance of imbalance operator. Normalize to the squeezing parameter $\xi_{N,coh}^2$ of uncorrelated spin coherent state $\sigma^2(\hat{J}_z) = \langle N_p \rangle / 2$. If the expression

$$\frac{\xi_N^2}{\xi_{N,coh}^2} = \frac{2\sigma^2(\hat{J}_z)}{\langle N_p \rangle} \quad (0.1.35)$$

gets smaller than one, the N-atom state is squeezed below the standard quantum limit. (i.e. below the fluctuations associated with a coherent spin state).

We choose sound initial values: $N = 80000$, $\eta = 2 \times \pi \times 1.7e3$, $\kappa = 2 \times \pi \times 1.25e6$, $\omega_Z = 2 \times \pi \times 7.09e6$, $\delta C = -2 \times \pi \times 25.8e6$ and get $x_p = (\eta^2 \times 2 \times \delta C) / (\delta C^2 + (\kappa^2))$. Pair coupling for χ_+ Channel: $x_p = -0.0009662$. In the following we hold this coupling constant.

0.1.5 Bipartite entanglement

(following finger thesis)

Assume small Zeeman splittings $\omega_z \rightarrow 0$ (which means two-channel configuration), where both couplings $\mu_+ \approx \mu_- = \mu$ become equal (what is difference between μ and χ ?). We get

$$|\psi\rangle = (1 - \mu^2) \sum_{N_p^+, N_p^- = 0}^{\infty} \mu^{N_p^+ + N_p^-} |N_p^+, N_p^+; N_p^-, N_p^-\rangle \quad (0.1.36)$$

Thus, for high-gain limit $\mu \rightarrow 1$ our state consists of a superposition of many pair states with different pair numbers. State is called 'entangled bright squeezed vacuum state'

signal atoms go in +z direction(A), idler atoms in -z direction(B). Thus, we can see this as two subsystems A, B. Introduce collective pseudo-spin operator:

$$\mathbf{J}_{A,B} = \sum_n^{N_{A,B}} \mathbf{j}_n \quad (0.1.37)$$

fullfilling

$$\mathbf{J} = \mathbf{J}_A + \mathbf{J}_B \quad (0.1.38)$$