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Grupa 252

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Tema #5

Ex#2 Fie $[a,b] \subseteq \mathbb{R}$

$$\mathbb{P}(Z\cdot\sigma \in [a,b]) = \mathbb{P}\left(Z \in \left[\frac{a}{\sigma}, \frac{b}{\sigma}\right]\right) =$$

$$= \int_{\frac{a}{\sigma}}^{\frac{b}{\sigma}} \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{z^2}{2}} dz$$

$$\left.\text{S.V.} \begin{cases} x = z\cdot\sigma \\ dx = dz\cdot\sigma \end{cases}\right\} \Rightarrow$$

$$\Rightarrow \mathbb{P}(Z\cdot\sigma \in [a,b]) = \int_a^b \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{\left(\frac{x}{\sigma}\right)^2}{2}} \cdot \frac{1}{\sigma} dx =$$

$$= \int_a^b \underbrace{\frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{x^2}{2\sigma^2}}}_{p_x(x)} dx \Rightarrow$$

$$\Rightarrow p_x(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{x^2}{2\sigma^2}}, \quad X \sim N(0, \sigma^2)$$

Ex#3 Fie $[a,b] \subseteq \mathbb{R}$

$$\mathbb{P}(X\cdot\sigma \in [a,b]) = \mathbb{P}\left(X \in \left[\frac{a}{\sigma}, \frac{b}{\sigma}\right]\right) =$$

$$= \int_{\frac{a}{\sigma}}^{\frac{b}{\sigma}} \lambda \cdot e^{-\lambda x} dx$$

$$\left.\text{S.V.} \begin{cases} y = x\cdot\sigma \\ dy = dx\cdot\sigma \end{cases}\right\} \Rightarrow$$

$$\Rightarrow \mathbb{P}(X\cdot\sigma \in [a,b]) = \int_a^b \lambda \cdot e^{-\lambda\cdot\frac{y}{\sigma}} \cdot \frac{1}{\sigma} dy = \int_a^b \underbrace{\frac{\lambda}{\sigma} \cdot e^{-\frac{\lambda}{\sigma}\cdot y}}_{p_y(y)} dy \Rightarrow$$

$$\Rightarrow p_y(y) = \frac{\lambda}{\sigma} \cdot e^{-\frac{\lambda}{\sigma} y}, \quad Y \sim Exp\left(\frac{\lambda}{\sigma}\right)$$

$\mathbb{E}[X] = \frac{1}{\lambda}$

$\mathbb{E}[X^2] = \frac{2}{\lambda^2}$

$Z = X^2$

$\Bigg\} \Rightarrow \mathbb{E}[Z] = E[X^2] = \frac{2}{\lambda^2} = \frac{1}{\frac{\lambda^2}{2}} \Rightarrow$

$\Rightarrow Z \sim Exp\left(\frac{\lambda^2}{2}\right)$

Ex #4 Fie $[X] + \cancel{1} = k$

$P([X]+1) = \mathbb{P}(k-1 \le X < k)$

$= e^{-(k-1)\lambda} - e^{-k\lambda}$

$= e^{-(k-1)\lambda} - e^{-(k-1)\lambda} \cdot e^{-\lambda}$

$= \cancel{e^{-k\lambda}} e^{-(k-1)\lambda} \cdot (1 - e^{-\lambda})$

$= (e^{-\lambda})^{(k-1)} (1 - e^{-x})$

$= (1 - 1 + e^{-\lambda})^{(k-1)} (1 - e^{-\lambda})$

$= (1 - (1 - e^{-\lambda}))^{(k-1)} (1 - e^{-\lambda}) \Rightarrow$

$\Rightarrow [X] + 1 \sim Geom(1 - e^{-\lambda})$

Ex #5

$f(x) = \begin{cases} k - \frac{x}{50}, & 0 \le x \le 10 \\ 0, & \text{altfel} \end{cases}$

a) $\sum_{-\infty}^{+\infty} f(x) = 1 \Leftrightarrow$

$1\cancel{0}k - \frac{0+1+2+\dots+10}{50} = 1 \Leftrightarrow$

$\cancel{12k} -$

$11k - \frac{55}{50} = 1 \Leftrightarrow$

$550k - 55 = 50 \Leftrightarrow$

$550k = 105 \Rightarrow k = \frac{105}{550} = \frac{21}{110}$

b) $\mathbb{P}(X \le 5) = 6k - \frac{0+1+\dots+5}{50}$

$= 6 \cdot \frac{21}{110} - \frac{15}{50}$

$= \frac{630}{550} - \frac{165}{550}$

$= \frac{465}{550} = \frac{93}{110}$

c) $\mathbb{E}[X] = \int_0^{10} x \cdot \left(\frac{21}{110} - \frac{x}{50}\right) dx$

$= \int_0^{10} \left(\frac{21x}{110} - \frac{x^2}{50}\right) dx$

$= \frac{21 \cdot \frac{x^2}{2}}{110}\Big|_0^{10} - \frac{x^3}{3 \cdot 50}\Big|_0^{10}$

$= \frac{21(100-0)}{220} - \frac{1000-0}{150}$

$= \frac{21 \cdot \cancel{100}}{\cancel{220}} - \frac{\cancel{1000}}{\cancel{150}}$

$= \frac{315 - 220}{33} = \frac{95}{33}$

$\mathbb{E}[X^2] = \int_0^{10} x^2 \left(\frac{21}{110} - \frac{x}{50}\right) dx$

$= \int_0^{10} \left(\frac{21x^2}{110} - \frac{x^3}{50}\right) dx$

$= \frac{21 \cdot \frac{x^3}{3}}{110}\Big|_0^{10} - \frac{x^4}{4 \cdot 50}\Big|_0^{10}$

$= \frac{\cancel{21} \cdot 100\cancel{0}}{\cancel{330}} - \frac{1000\cancel{0}}{\cancel{200}}$

$= \frac{700 - 550}{11} = \frac{150}{11}$

$$\mathbb{V}\mathrm{ar}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

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$$= \frac{150}{11} - \left(\frac{95}{33}\right)^2 = \frac{150}{11} - \frac{95^2}{3^2 \cdot 11^2}$$

$$= \frac{14850 - 9025}{1089}$$

$$= \frac{5825}{1089}$$