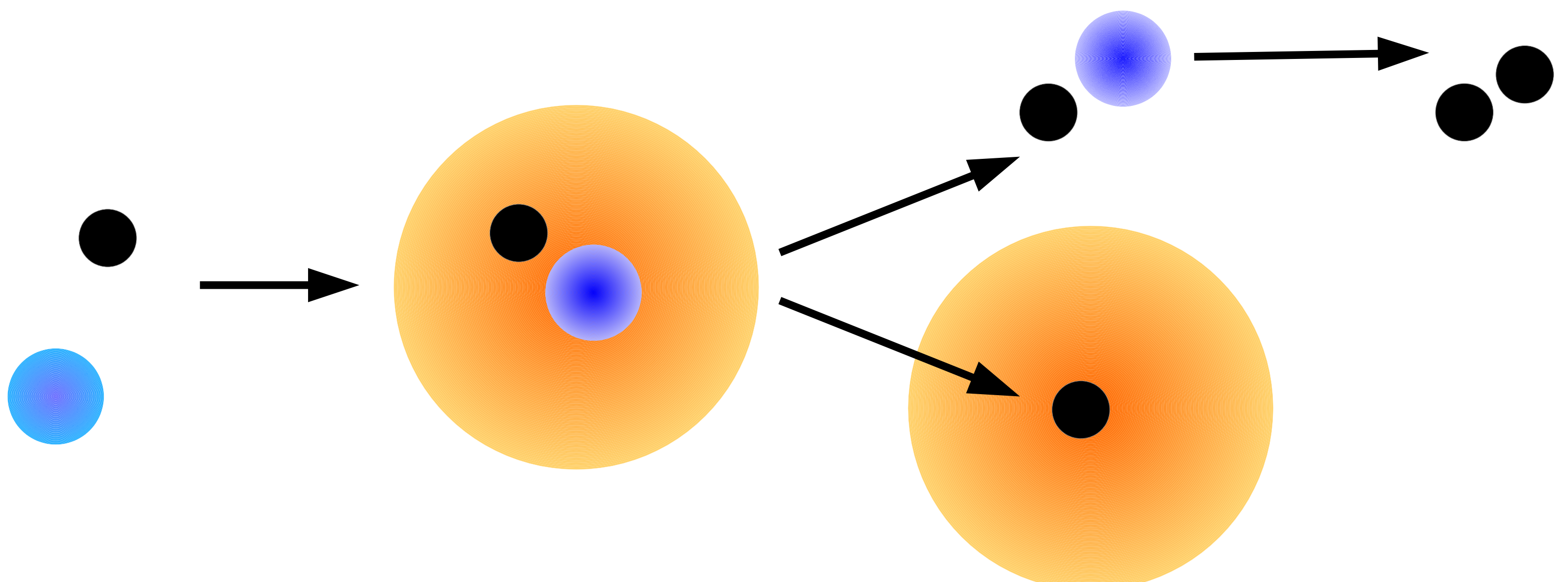
Binary evolution processes



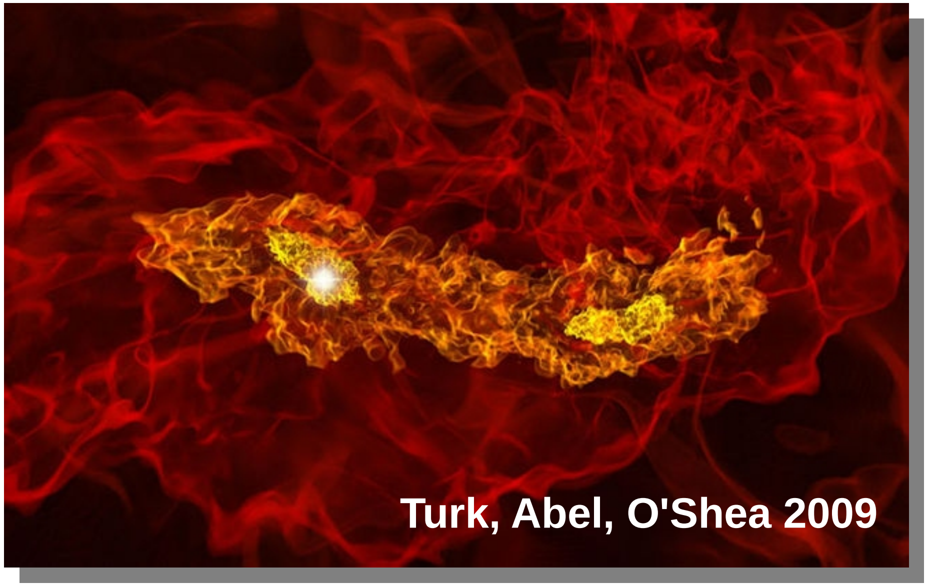
Formation of black hole and neutron star binaries

Previous lecture: how black holes and neutron stars form from SINGLE massive stars

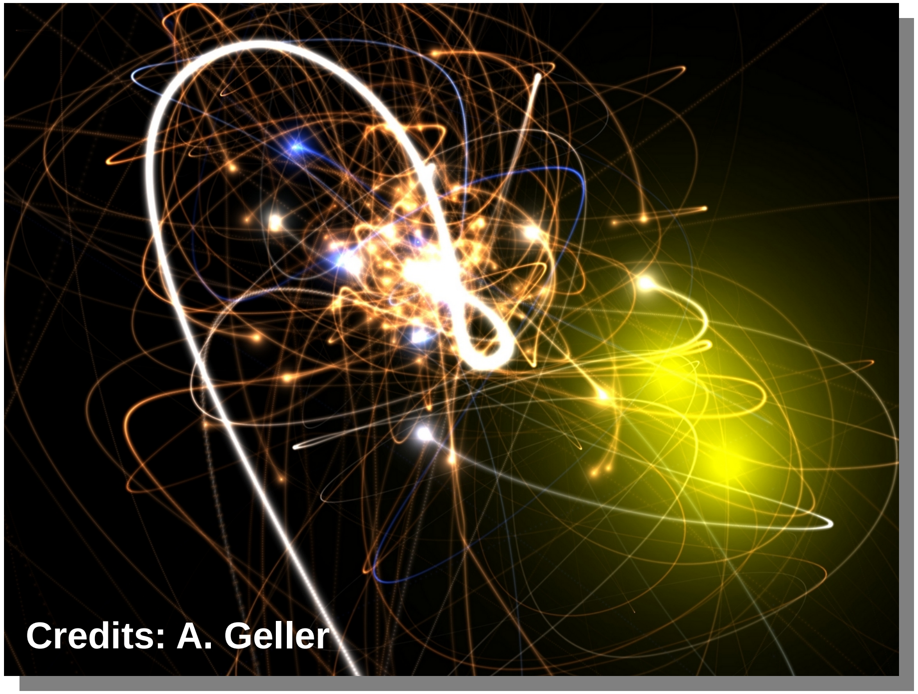
BUT LIGO-Virgo observed the merger of black holes and neutron stars in binaries

WHAT ARE THE FORMATION CHANNELS OF BINARY BLACK HOLES & NEUTRON STARS?

Formation of binary compact objects

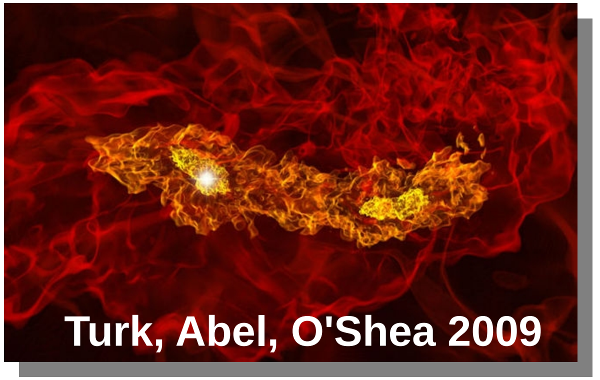


1) PRIMORDIAL BINARIES or ISOLATED BINARIES:
two stars form from same cloud and evolve into two gravitationally bound compact objects



2) DYNAMICAL BINARIES:
Binary compact object forms and/or evolves by dynamical processes. Especially true for binary black holes (BBHs)

Isolated binary evolution



MOST MASSIVE STARS ARE IN BINARY SYSTEMS
(e.g. Sana et al. 2012; Moe & Di Stefano 2017)

→ **It seems natural that a BBH forms from the evolution of a massive binary star**

BUT NOT SO EASY:
Many evolutionary processes can affect the binary

wind mass transfer	**tidal evolution**
Roche lobe mass transfer	**magnetic braking**
common envelope	**gravitational wave decay**
	SN kick

Final fate of two stars in a TIGHT binary system very different from that of two single stars or two stars in a loose binary system (semi-major axis >10^5 Rsun ~ 500 AU)

Population-synthesis codes

Binary evolution studied via POPULATION SYNTHESIS CODES:

* include models of stellar evolution in a simplified way
* include prescriptions for supernova explosions
* include treatment of binary evolution processes

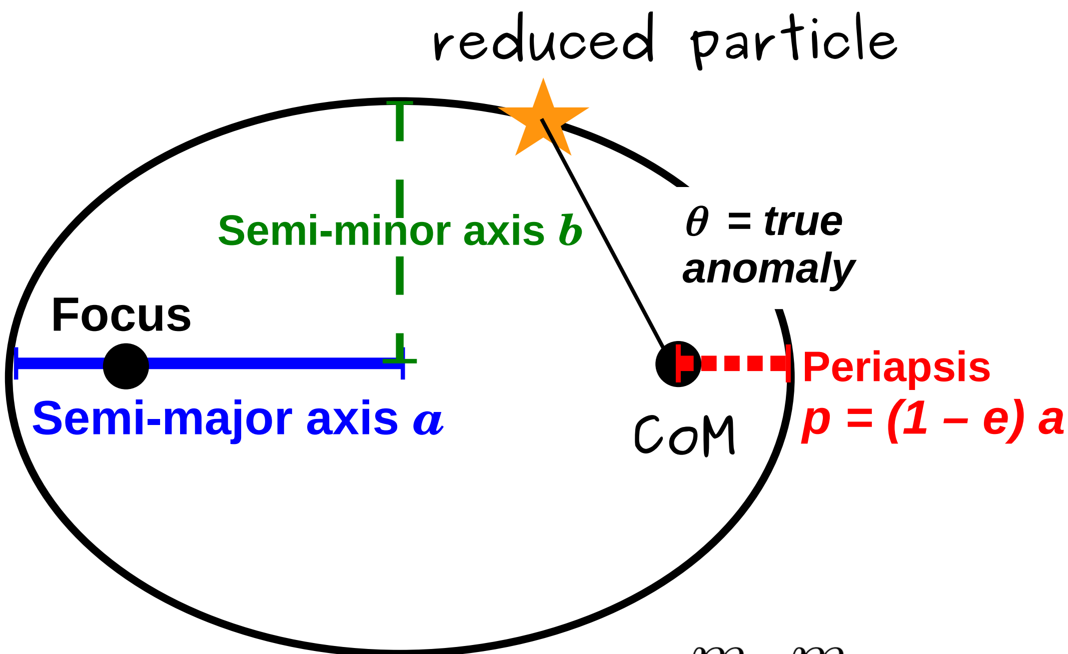
Examples of used population-synthesis codes

BSE	**(Hurley+ 2000, 2002)**
COMBINE	**(Kruckow+ 2018)**
COMPAS	**(Stevenson+ 2017; Chattopadhyay+ 2021)**
MOBSE	**(MM+ 2017; Giacobbo+ 2018; Giacobbo & MM 2018)**
Seba	**(Portegies Zwart & Verbunt 1996; MM+ 2013)**
SEVN	**(Spera+ 2015; Spera & MM 2017; Spera+ 2019; MM+ 2020)**
StarTrack	**(Belczynski+ 2007, 2008, 2010, 2016; Chruslinska+ 2018)**

During these lectures you will learn how to use MOBSE(or SEVN, if you prefer)

Orbital properties

In the reduced particle + Center of Mass (CoM) frame



Angular frequency

$$\omega = \sqrt{\frac{G\,(m_1 + m_2)}{a^3}}$$

Energy

$$E = \frac{1}{2}\,\mu\,v^2 - \frac{G\,m_1\,m_2}{r}$$

Angular momentum

$$L = \mu\,a^2\,\omega = \mu\,\sqrt{G\,(m_1 + m_2)\,a}$$

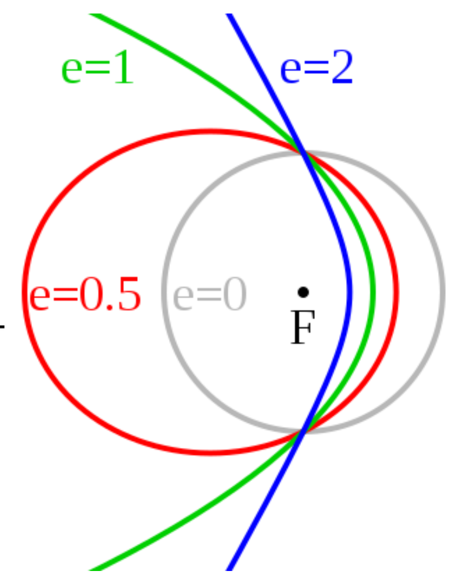
reduced mass: $\mu = \dfrac{m_1\,m_2}{(m_1 + m_2)}$

Orbit as function of true anomaly:

$$r(\theta) = \frac{a\,(1 - e^2)}{1 + e\,\cos\theta}$$

Eccentricity:

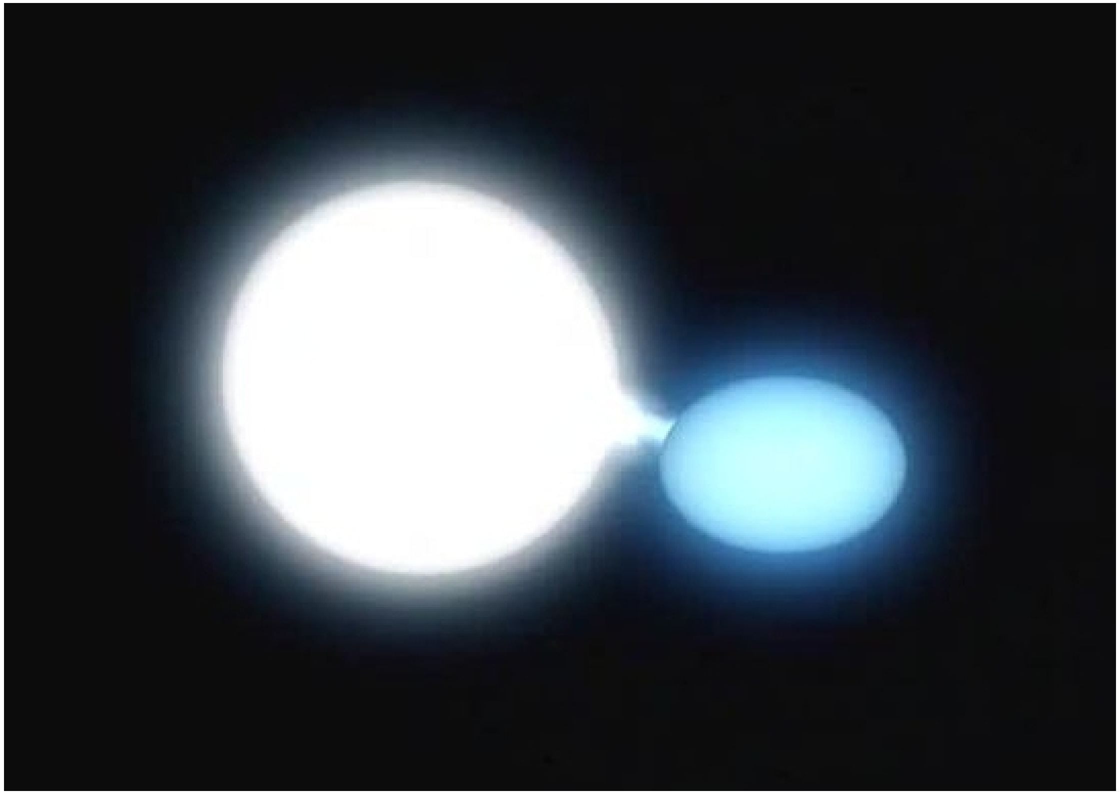
$$e = \sqrt{1 + \frac{2\,E\,L^2}{\mu^3\,G^2\,(m_1 + m_2)^2}}$$



Mass transfer

Two stars in a binary might exchange mass

1. wind mass transfer
2. Roche lobe overflow
3. common envelope



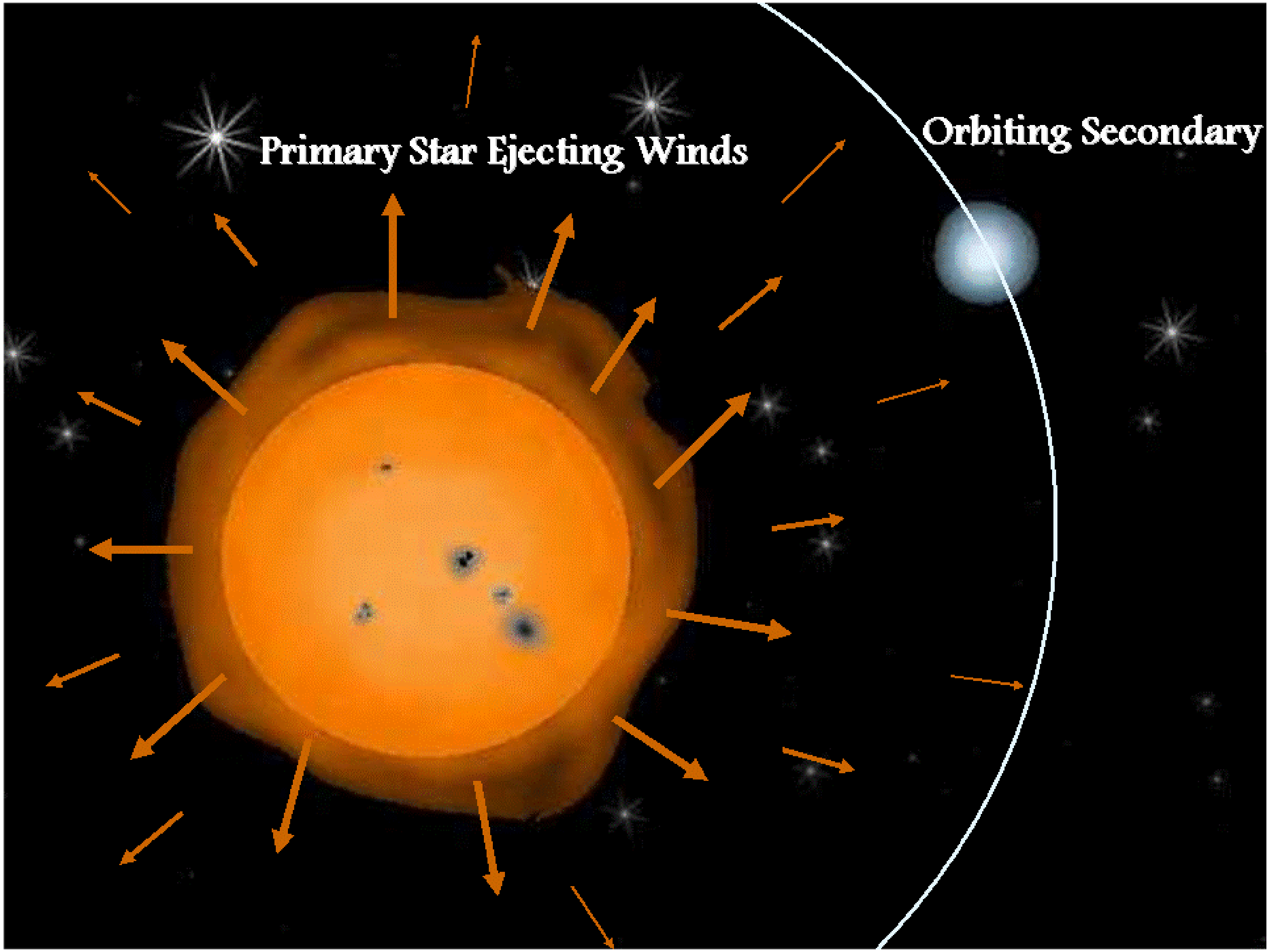
Credit: ESO/L. Calçada/M. Kornmesser/S.E. de Mink

If two stars exchange mass (and some mass is lost from the system), the final mass of the black holes will be completely different from two single stars

Mass transfer

Credits: ESO/L. Calçada/M. Kornmesser/S.E. de Mink

Wind mass transfer



Wind mass transfer

*** primary loses mass by stellar winds as** $\dot{M}_{1W}$

*** secondary acquires a part of it as (Bondi & Hoyle 1944)**

$$\left|\frac{\dot{M}_{2A}}{\dot{M}_{1W}}\right| \propto \left(\frac{v_{\rm orb}}{v_{\rm W}}\right)^4$$

VERY INEFFICIENT

In pop. synth. codes more detailed formula (Hurley et al. 2002)

$$\dot{M}_{2A} = -\frac{1}{\sqrt{1-e^2}}\left(\frac{G\,M_2}{v_W^2}\right)^2\frac{\alpha_W}{2\,a^2}\,\frac{1}{(1+v^2)^{3/2}}\,\dot{M}_{1W}$$

where

$$v^2 = \frac{v_{\rm orb}^2}{v_W^2} \qquad v_{\rm orb}^2 = \frac{G\,(M_1+M_2)}{a} \qquad v_W^2 = 2\,\beta_W\,\frac{G\,M_1}{R_1}$$

$$\alpha_W \sim 1.5 \quad \beta_W \sim 0.125-7$$

*** non-conservative mass transfer induces orbital angular momentum loss**

$$L = \mu\sqrt{G\,(M_1+M_2)\,a}$$

Roche potential

Assume the binary is CIRCULAR (ecc = 0) and
the two stars are point-mass or perfectly spherical

A STATIONARY TEST PARTICLE IN A FRAME THAT CO-ROTATES with the binary feels an effective acceleration: $\vec{a} = -\nabla\phi_R$

where the effective potential (Roche potential) is

$$\phi_R = \frac{G\,m_1}{|\vec{r} - \vec{d}_1|} + \frac{G\,m_2}{|\vec{r} - \vec{d}_2|} + \underbrace{\frac{1}{2}|\vec{\omega} \times \vec{r}|^2}_{\text{Centrifugal term}}$$

r is the position of the test particle
d1, d2 the positions of stars 1 and 2
ω is the angular frequency of the binary

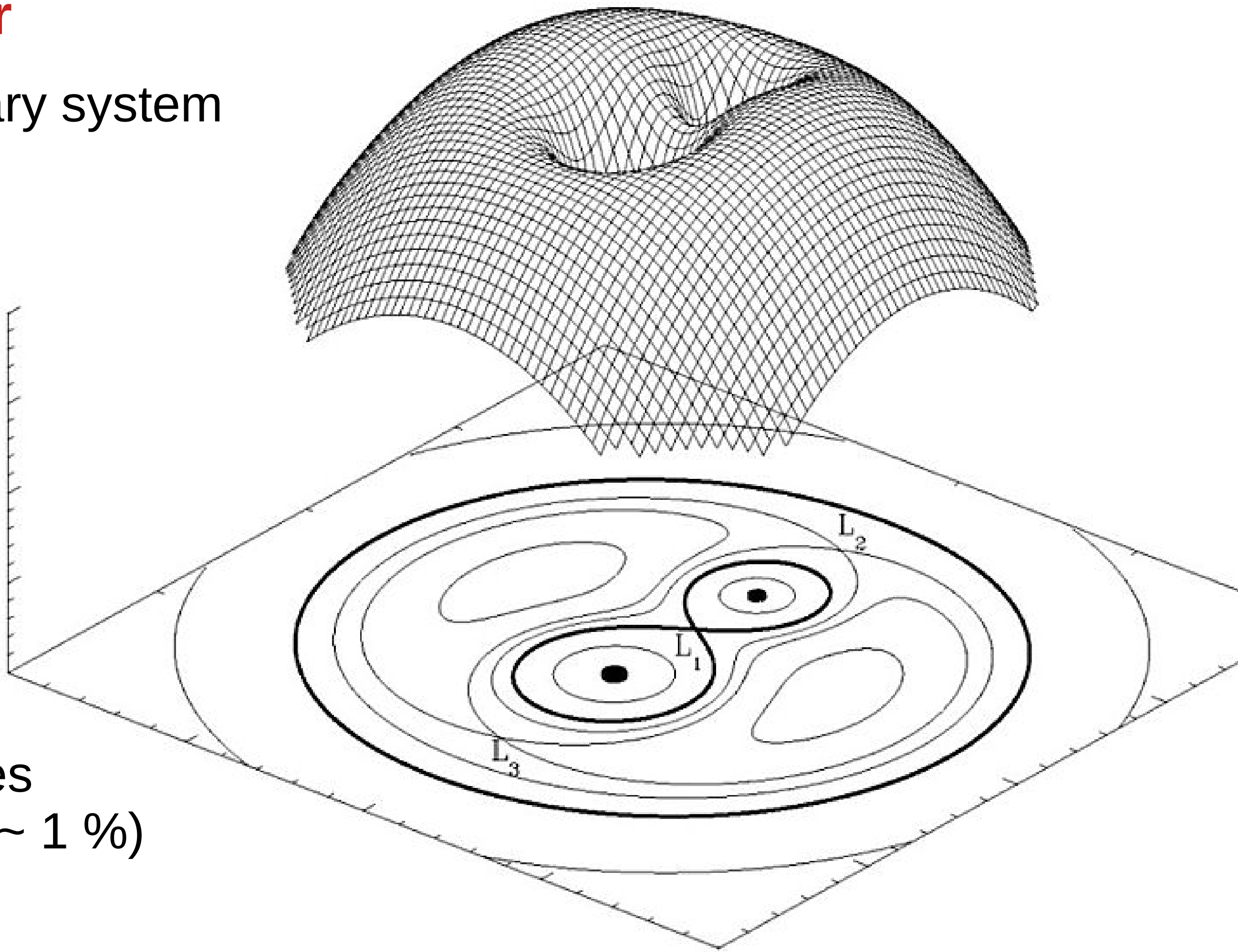
Note that, because of CoM definition $\vec{d}_1/m_2 = -\vec{d}_2/m_1 = \vec{d}/(m_1 + m_2)$
where $\vec{d} \equiv \vec{d}_1 - \vec{d}_2$

Hence ϕ_R depends only on *q = m1/m2* and *d*
(in the circular case *d* = semimajor axis, which we will indicate with a)

Roche lobe mass transfer

Equipotential surfaces in a binary system

For all $q > 0$ exists a 8-shaped critical potential surface



The connecting point is L1 (inner Lagrangian point)

The two lobes are called **ROCHE LOBES**

Approximation assuming 2 lobes are perfect circles (accurate to ~ 1 %)

$$\boxed{\frac{r_1}{a} = \frac{0.49\, q^{2/3}}{0.6\, q^{2/3} + \ln\left(1 + q^{1/3}\right)}}$$

where a = semi-major axis
$q = m_1/m_2$
1 = donor, 2 = accretor

Eggleton 1983

Roche lobe overflow

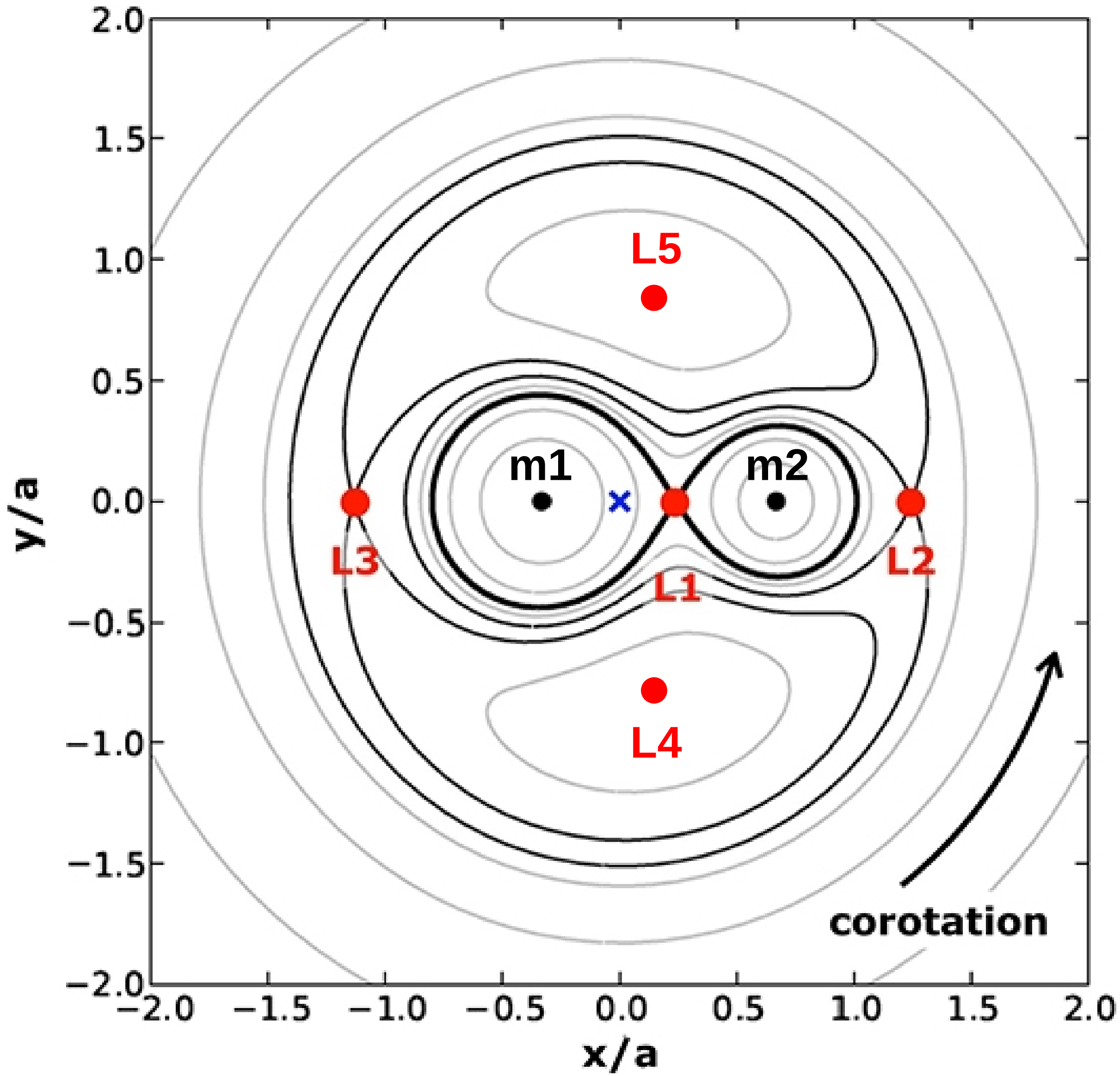
If a star fills its Roche lobe
(:= if star radius is equal or larger than Roche lobe),

matter flows without energy change to the other star
→ MASS TRANSFER

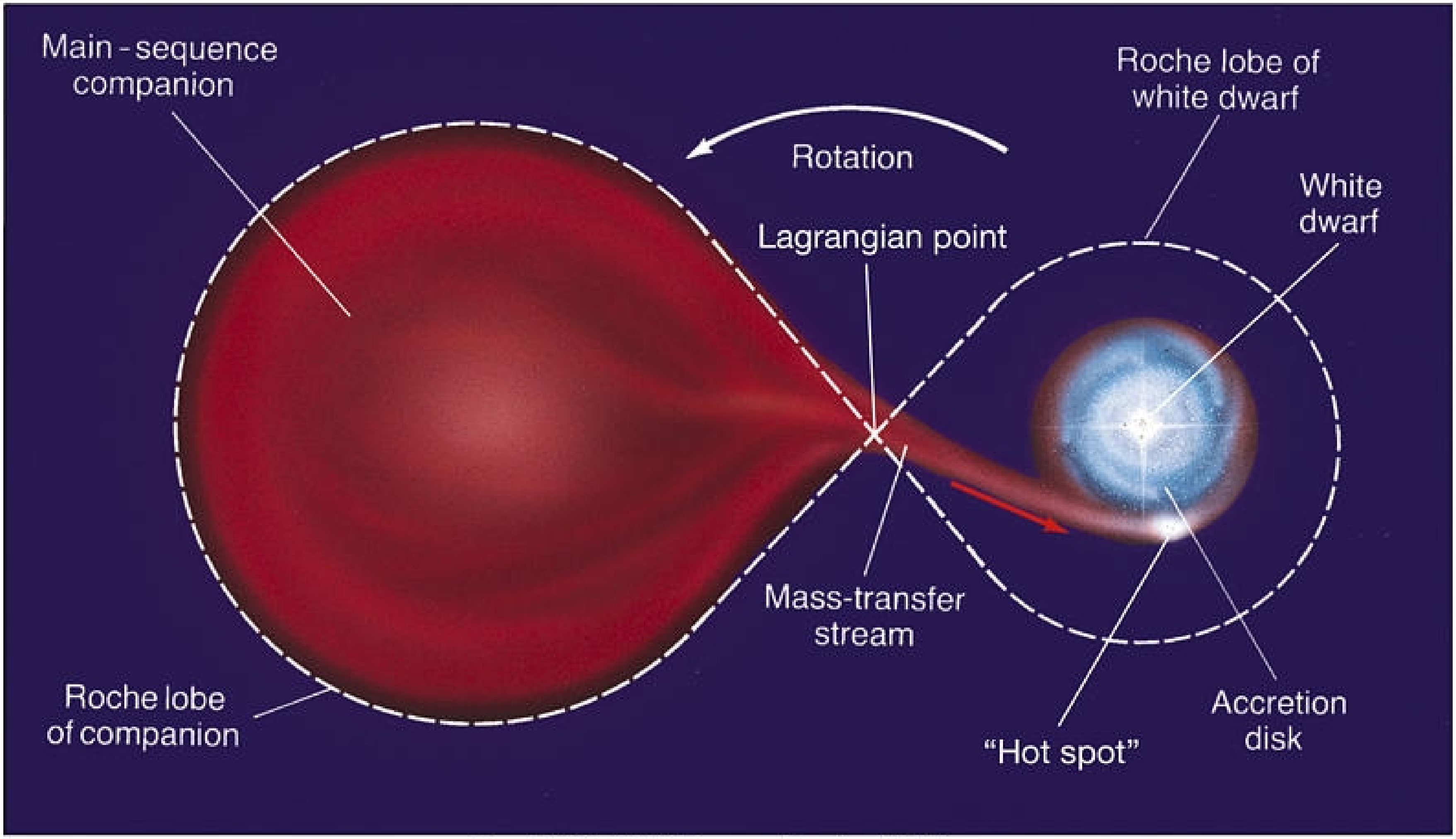
called
ROCHE LOBE OVERFLOW

Star that fills the RL:
DONOR (usually m1)

Star that receives (accretes) mass:
ACCRETOR (usually m2)



Roche lobe overflow



Note: accreting material gains angular momentum from **Coriolis** force and can form an **accretion DISK** around the accretor

Roche lobe overflow (un)stability

important to assess whether mass transfer is stable or unstable
and on which timescale

DYNAMICAL TIMESCALE

$$t_{\rm dyn} \sim \sqrt{\frac{\frac{4}{3}\pi\,R^3}{G\,M}} \sim 3300\,{\rm s}\,\left(\frac{R}{R_\odot}\right)^{3/2}\,\left(\frac{M_\odot}{M}\right)^{1/2}$$

Timescale for a star to collapse by its own gravity (in absence of pressure)

KELVIN-HELMHOLTZ TIMESCALE

$$t_{\rm KH} = \frac{\text{gravitational potential energy}}{\text{rate of energy loss}} \sim \frac{G\,M^2}{R\,L} \sim 3\times 10^7\,{\rm yr}\,\left(\frac{M}{M_\odot}\right)^2\,\frac{L_\odot}{L}\,\frac{R_\odot}{R}$$

Timescale for a star to contract by radiating away all of its gravitatonal potential energy (gas pressure provides the radiation)

NUCLEAR TIMESCALE

$$t_{\rm nuc,\,H} \sim 6\times 10^{18}\,\frac{f\,X\,M}{L} \sim 10^{10}{\rm yr}\,\frac{f}{0.1}\,\frac{X}{0.73}\,\frac{M}{M_\odot}\,\frac{L_\odot}{L}$$

Timescale for nuclear burning to change stellar properties (depends on nuclear reactions)

Roche lobe overflow (un)stability

IS ROCHE LOBE UNSTABLE

i) OVER A DYNAMICAL TIMESCALE?

ii) OR A K-H TIMESCALE?

iii) OR A NUCLEAR TIMESCALE
(i.e. it is stable)?

Roche lobe overflow (un)stability

Radius and mass of a star are connected by a simple relation:

$$R \propto m^{\zeta}$$

Variation of the donor's radius during Roche lobe (RL) mass transfer is

$$\frac{\mathrm{d}R_1}{\mathrm{d}t} = \frac{\partial R_1}{\partial t} + \zeta\,\frac{R_1}{m_1}\,\frac{\mathrm{d}m_1}{\mathrm{d}t}$$

Nuclear burning term

Adiabatic or thermal response of the star to mass loss

$$\frac{\mathrm{d}m1}{\mathrm{d}t} < 0$$

Variation of Roche lobe during RL mass transfer

$$\frac{\mathrm{d}R_{\mathrm{L},1}}{\mathrm{d}t} = \frac{\partial R_{\mathrm{L},1}}{\partial t} + \zeta_{\mathrm{L}}\,\frac{R_{\mathrm{L},1}}{m_1}\,\frac{\mathrm{d}m_1}{\mathrm{d}t}$$

tides and GW radiation

response of the Roche lobe to mass loss

→ MASS TRANSFER UNSTABLE (over dynamical or KH timescale) if

$$\zeta < \zeta_{\mathrm{L}}$$

Roche lobe SHRINKS faster than star does

Roche lobe overflow (un)stability

Assume mass transfer is CONSERVATIVE
(:= star 2 acquires all mass lost by star 1)

$$L = \frac{m_1\, m_2}{m_1 + m_2} \sqrt{G\, (m_1 + m_2)\, a}$$

Ang. Mom. is conserved ***L* = const**

***m*$_1$+*m*$_2$ = const**

$\rightarrow$ **$(m_1 m_2)^2\, a$ = const**

$\rightarrow$ Semi-major axis has a minimum for $m_1 = m_2$

If initially **Mdonor > Maccretor** **orbital separation decreases**

If initially **Mdonor < Maccretor** **orbital separation increases**

Since $R_{\rm L} \propto a$ when orbital sep. decreases RL shrinks
when orbital sep. increases RL expands

EXERCISE (not for exam):

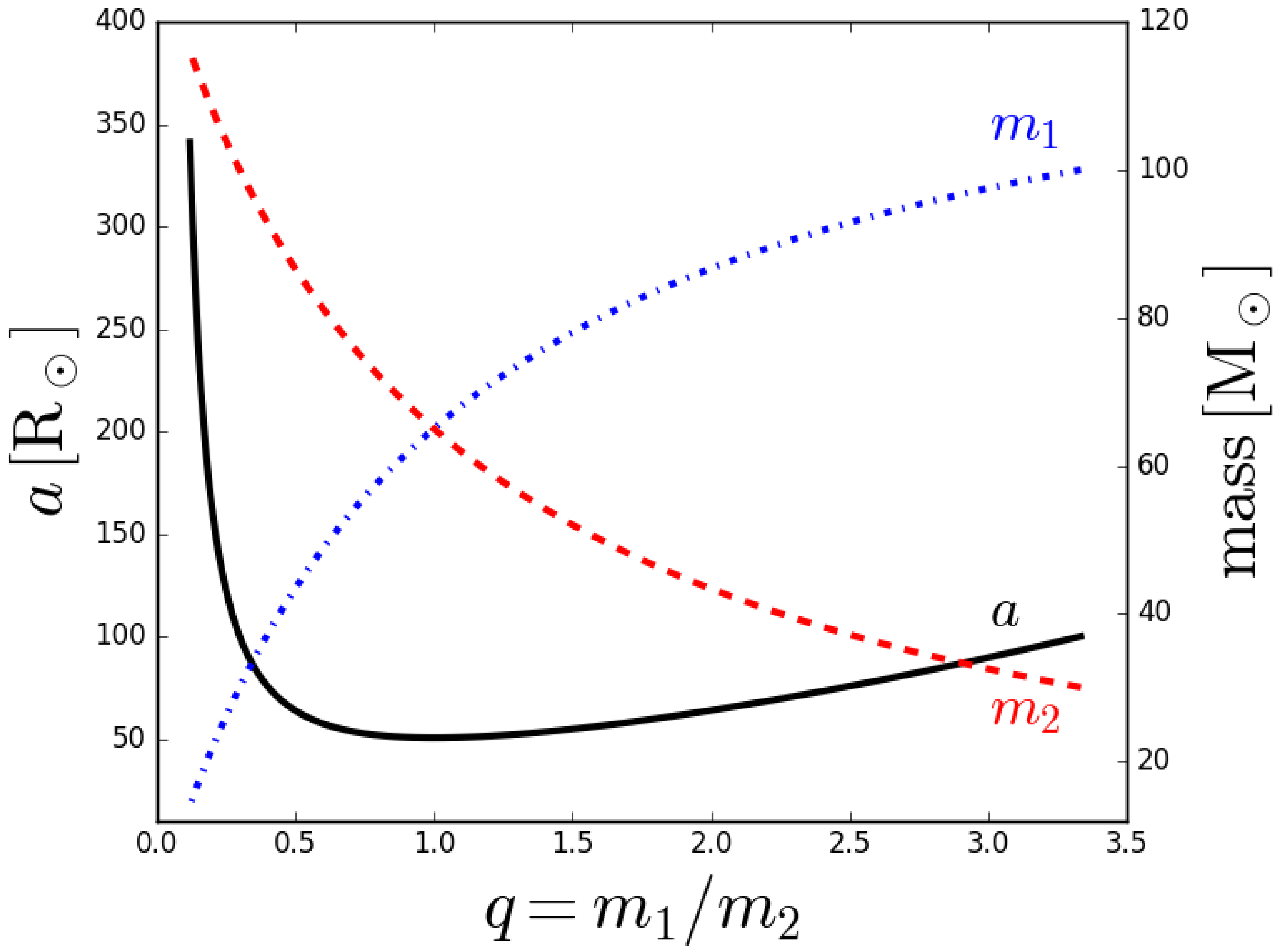
Calculate semi-major axis as a function of time for a RLO system with

* initial semi-major axis a = 100 Rsun (1 Rsun = 6.95e10 cm)
* initial donor mass m1 = 100 Msun (1 Msun = 1.989e33 g)
* initial accretor mass m2 = 30 Msun
* conservative mass transfer L = const, m1+m2 = const
* for semplicity, consider steps of mass dm = 1 Msun untill m1 = 15 Msun. Then mass transfer stops.

Plot change of a as a function of q=m1/m2

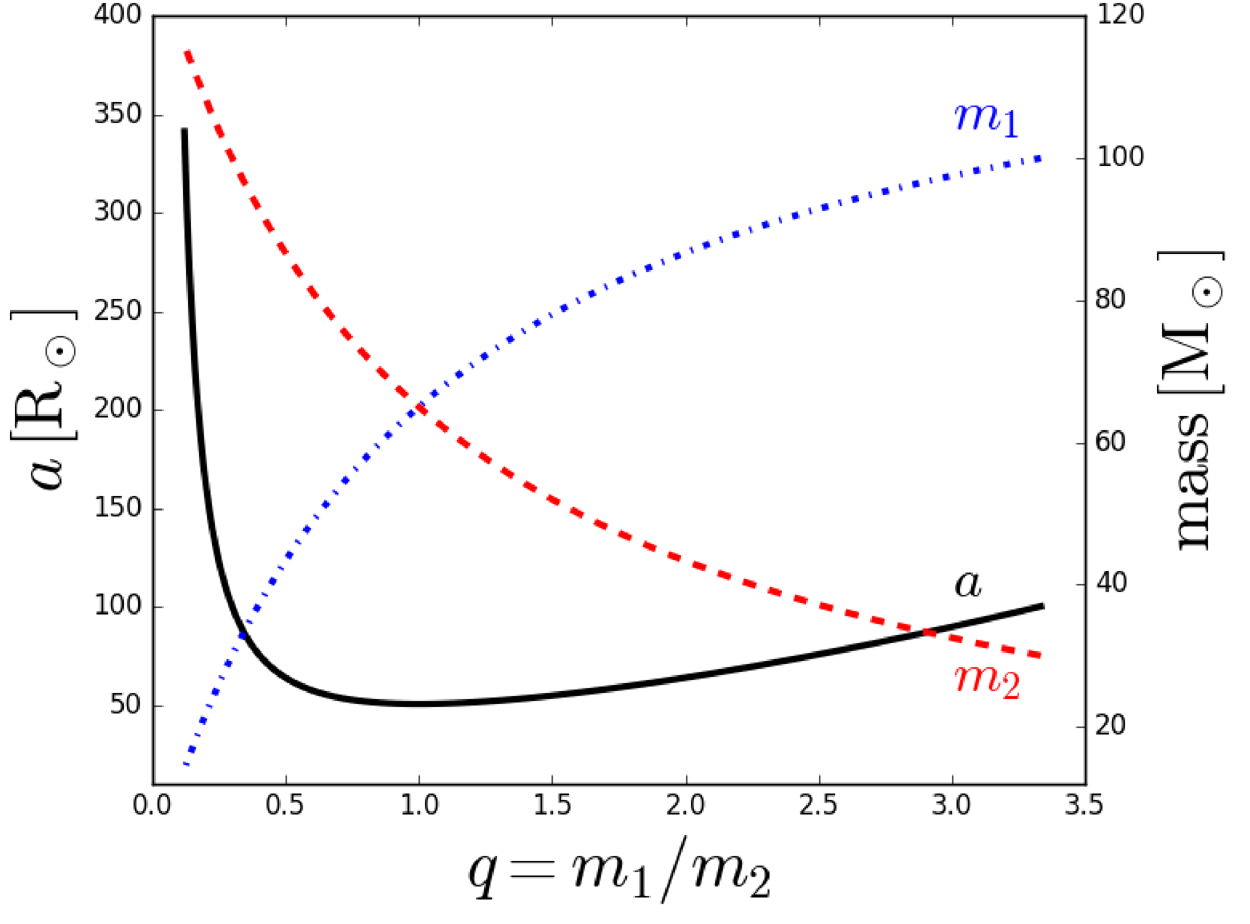
Plot change of m1 and m2 as a function of q

EXERCISE (not for exam):



Roche lobe overflow (un)stability

* If initially **m1 > m2** (assuming star 1 is the donor star),
 as star 1 transfers mass to star 2, the **orbital separation shrinks.**



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  * If it cannot shrink fast enough to keep hydrostatic equilibrium
    → **DYNAMICALLY UNSTABLE RLO → common env. or merger**

  * If it can shrink fast enough to keep hydrostatic equilibrium, it might still be that its new radius is SMALLER than the star's thermal equilibrium radius
    → **DONOR IS NOT IN THERMAL EQUILIBRIUM**

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* This expansion drives MORE MASS ACROSS THE RL
 Since expansion occurs on thermal timescale, the mass transfer is

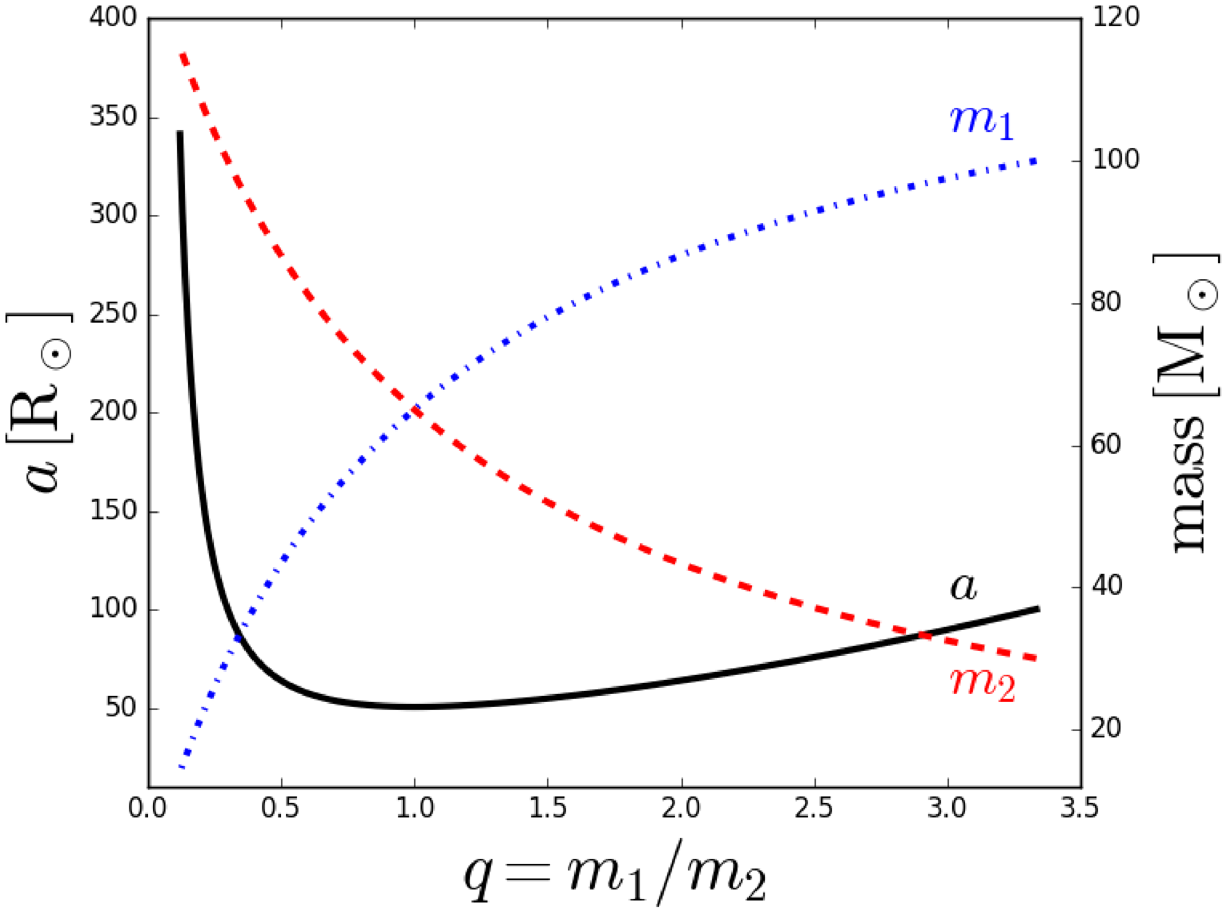
$$\dot{M}_1 \sim \frac{M_1}{t_{\rm KH}}$$

where t_{KH} := thermal (Kelvin-Helmholtz) timescale

– thermal timescale mass transfer –

Roche lobe overflow (un)stability

* Once the two masses have equalised and **m1 < m2 ,**
the orbital separation and the donor’s RL start to increase



Roche lobe overflow (un)stability

* Once the two masses have equalised and **$m1 < m2$,**
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*** after t_{KH}, the donor re-establishes thermal equilibrium**

→ **mass transfer continues on nuclear timescale**
– nuclear driven mass transfer –

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Notes:

Thermal timescale generally << than the nuclear timescale.

This implies that:

1. mass-transfer rate with $M_1 > M_2$ is larger than with $M_1 < M_2$
2. mass-transfer episode with $M_1 > M_2$ is shorter than with $M_1 < M_2$

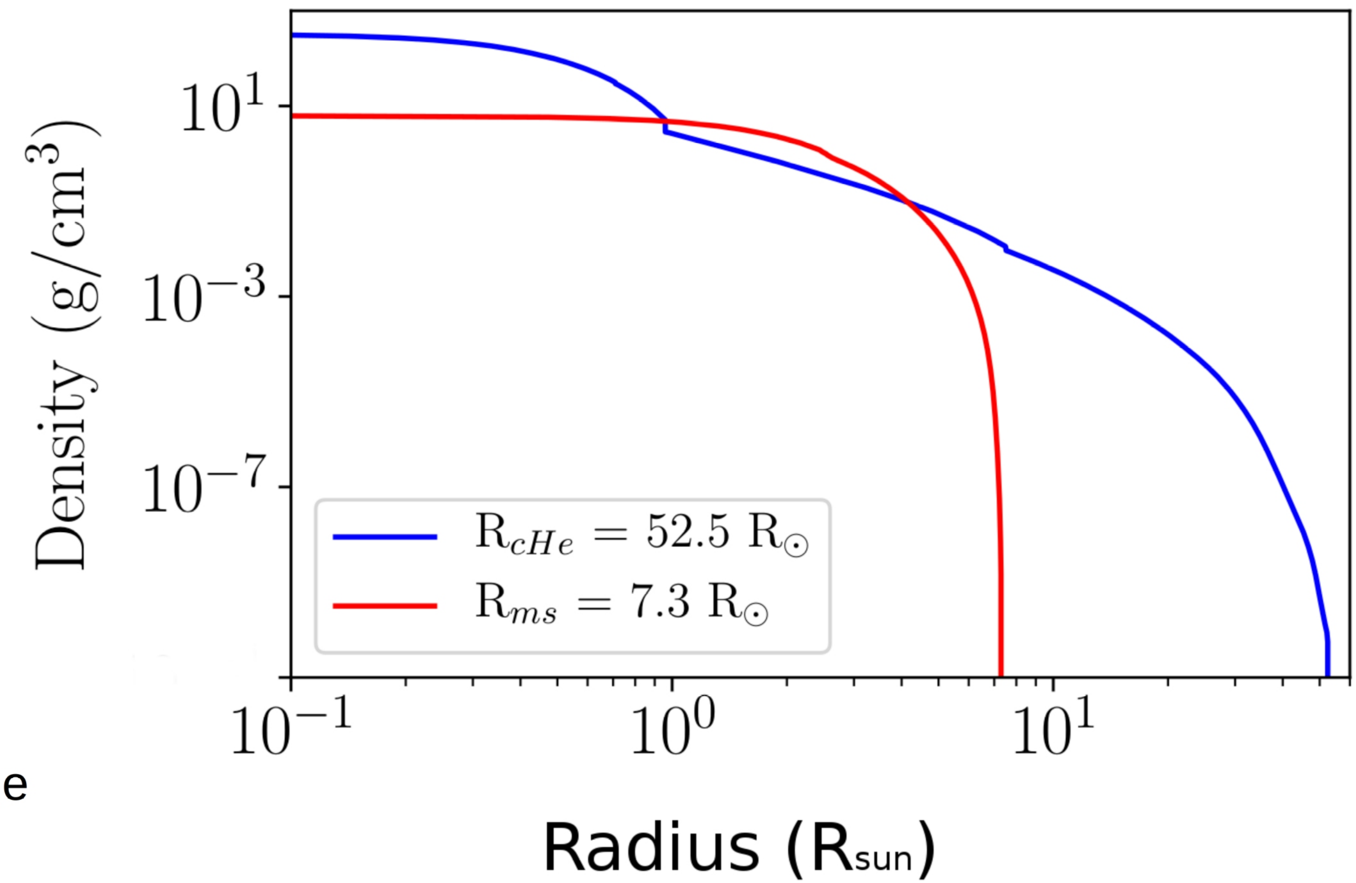
→ it is more likely to observe mass transfer
in the later LOW-Mdot and LONG-LIVED phase
than in the early HIGH-Mdot and SHORT-LIVED phase

WHAT HAPPENS IF ROCHE LOBE BECOMES DYNAMICALLY UNSTABLE?

The fate of a star is different
if it has (or not) a dense core
separated from a looser envelope

Blue: giant star
with He core

Red: main sequence
star (no core)



Courtesy: Alessandro Ballone
& Guglielmo Costa

WHAT HAPPENS IF ROCHE LOBE BECOMES DYNAMICALLY UNSTABLE?

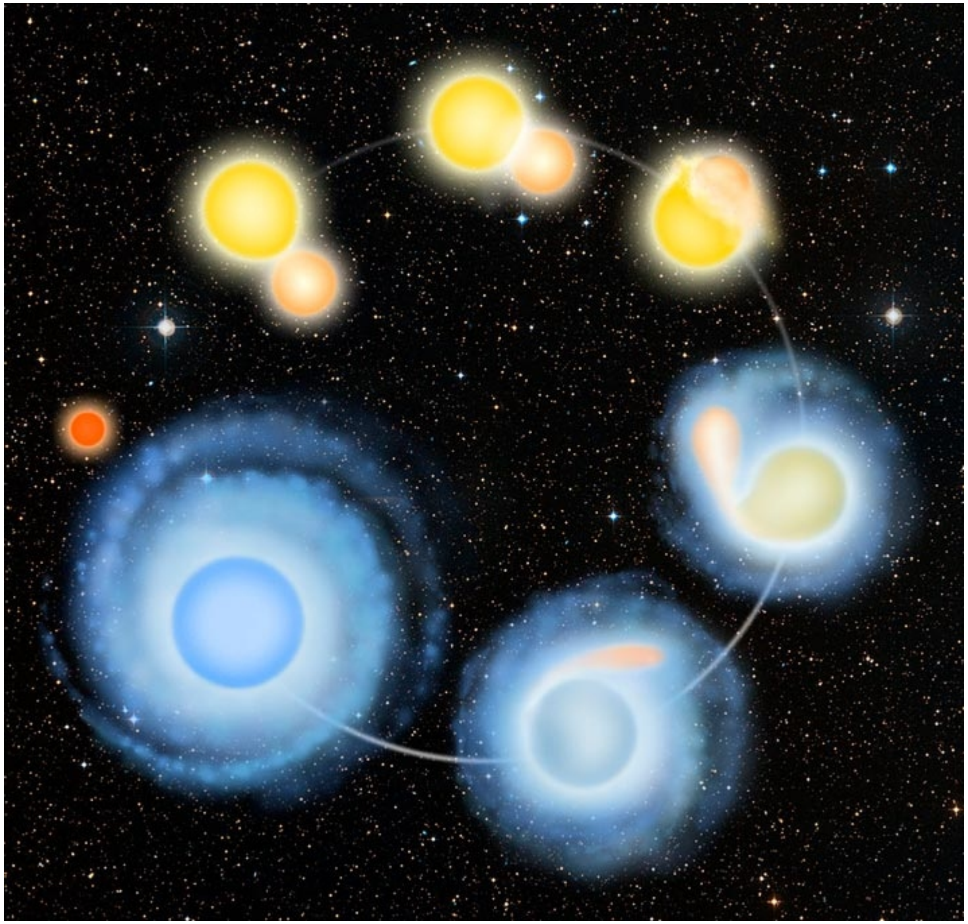
If mass transfer becomes unstable over dynamical time-scale,

- two stars without core **MERGE**

- if at least one of the two stars has a helium or carbon-oxygen CORE (= strong density gradient between core and envelope),

or the accretor is a compact object,

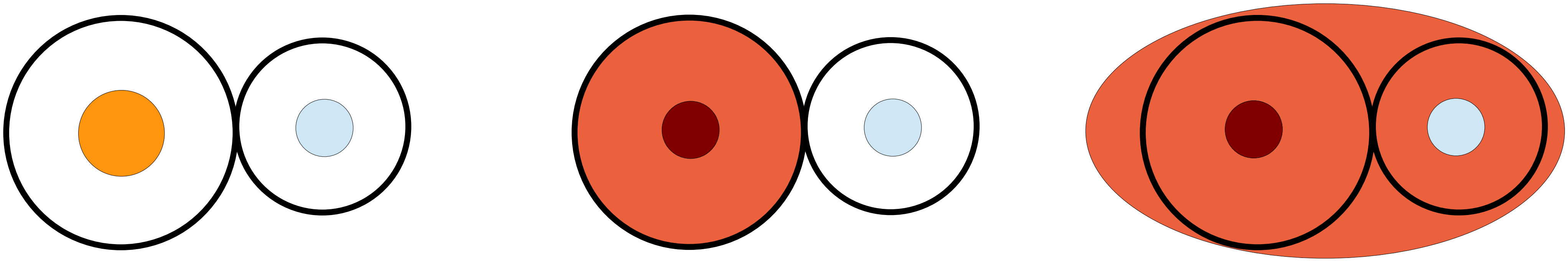
the two stars enter **COMMON ENVELOPE**



Star – star merger, visualization by

Credits: Barry Roal Carlsen

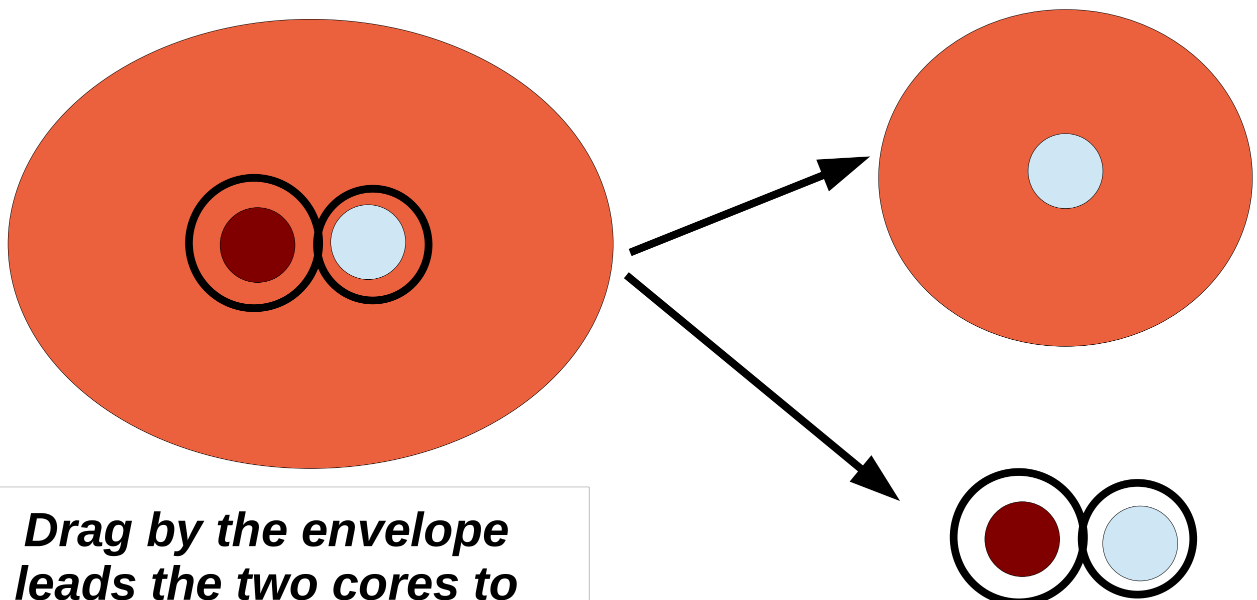
Common Envelope



Two massive stars initially under-filling Roche lobe

The first one evolves out of MS expands and start mass transfer onto the second

Mass transfer becomes unstable: CE phase



Drag by the envelope leads the two cores to spiral in

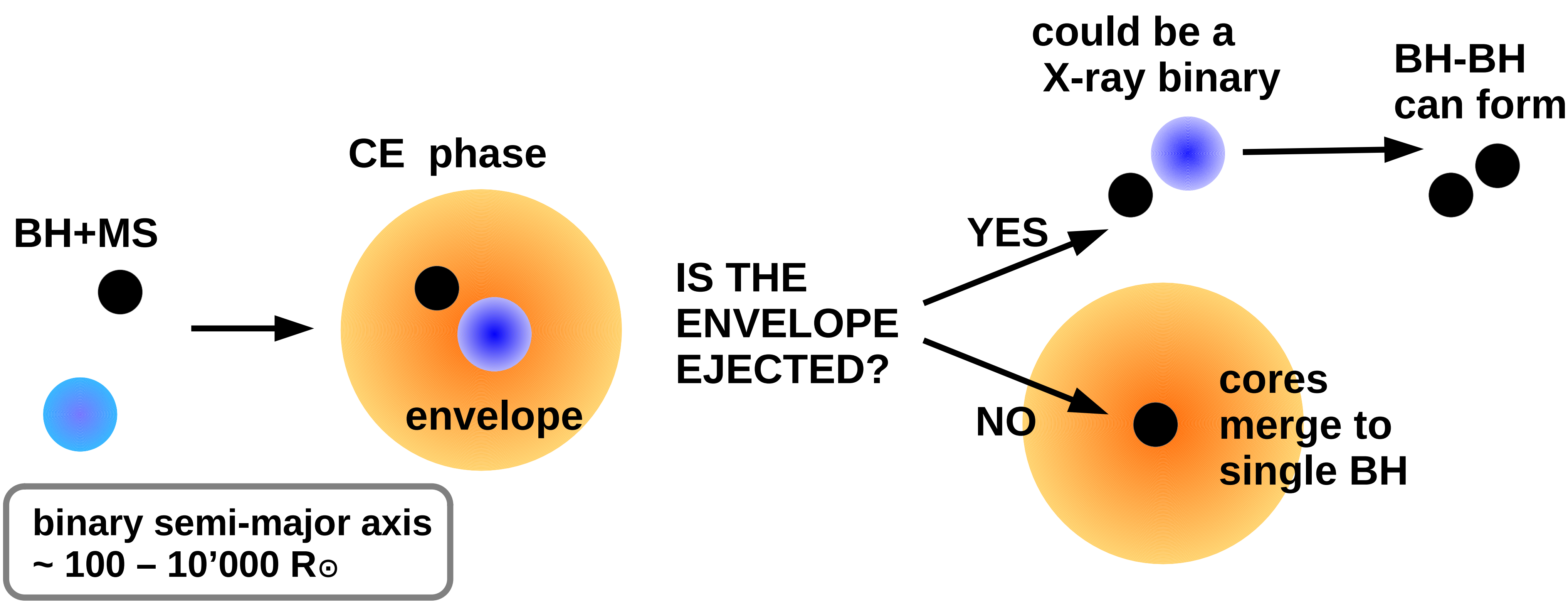
The two cores spiral in till they merge becoming a single star

The energy released during the spiral in removes the envelope: The two cores form a new tighter binary

Common Envelope

WHY is important for BBH demography?

binary semi-major axis
~ 1 – 100 R⊙



Common Envelope

Probably the least understood process in binary evolution

Four STAGES (with different physics):

1. loss of COROTATION: instable mass transfer prevents the envelope to co-rotate with the core

NOT YET MODELLED SELF-CONSISTENTLY (Ivanova et al. 2013)

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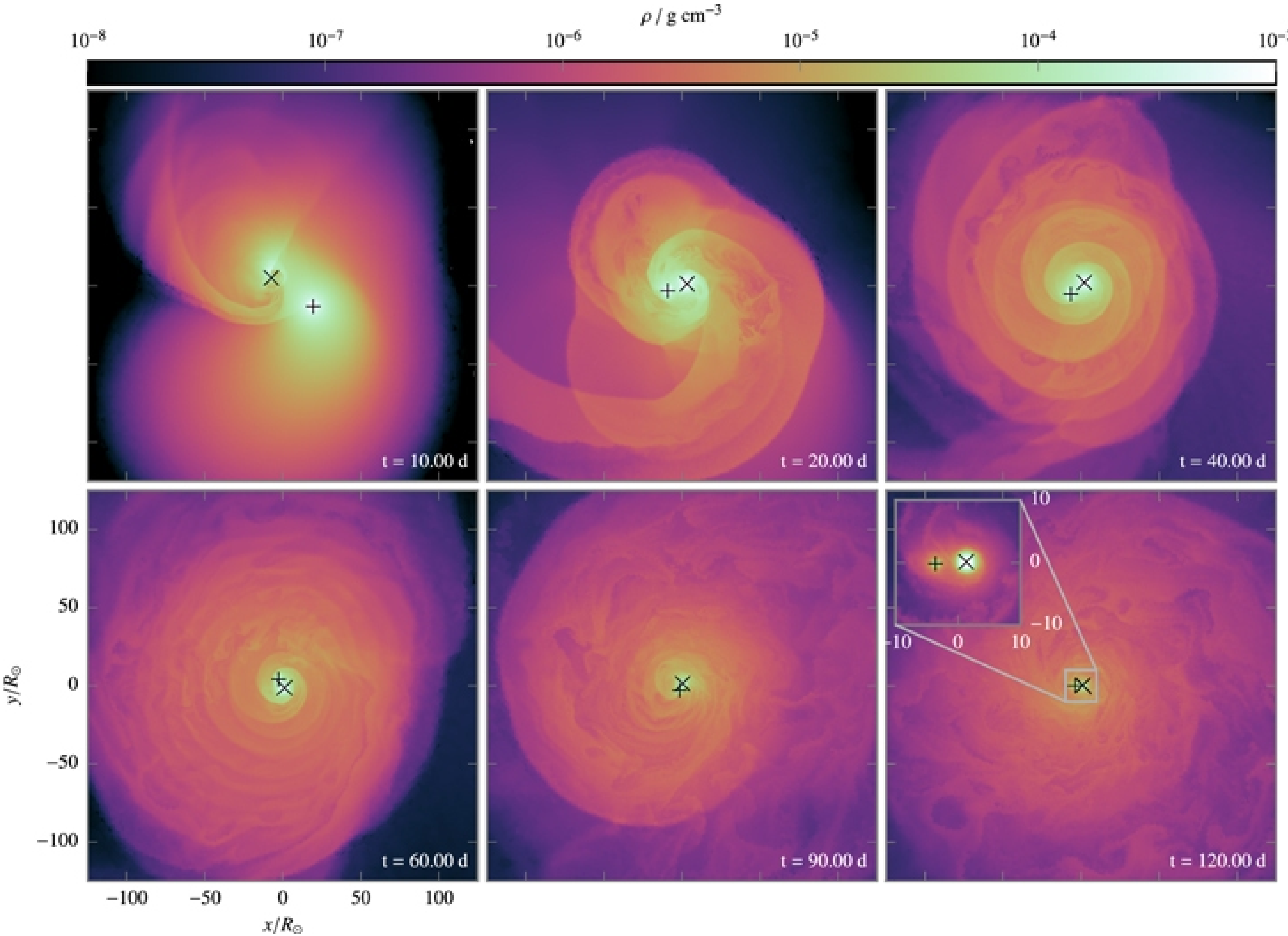
2. fast SPIRAL IN: two cores spiral in – they lose kinetic energy by drag
with the gas and heat the gaseous envelope –
on dynamical time scale (~100d) – SIMULATED IN 3D
(Ricker & Taam 2008, 2012; Passy et al. 2012; Ohlmann+ 2016)



From Ohlmann et al. 2016, ApJ, 816, L9

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POORLY UNDERSTOOD!!! WHAT REMOVES THE ENVELOPE?

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POORLY UNDERSTOOD!!! WHAT REMOVES THE ENVELOPE?

4. MERGER of the cores **or EJECTION** of ENVELOPE

SEE IVANOVA ET AL. 2013, A&ARv, 21, 59 for a review

Common Envelope

Most used analytic formalism (α formalism, Webbink 1984) does not capture physics. In its version by Hurley+ (2002, MNRAS, 329, 897) the α formalism is:

1. initial binding energy of envelope (λ = geometrical factor)

$$E_{\rm bind,i} = -\frac{G}{\lambda}\left(\frac{M_1\,M_{\rm env,1}}{r_1} + \frac{M_2\,M_{\rm env,2}}{r_2}\right)$$

2. orbital binding energy of the cores

$$E_{\rm orb} = \frac{1}{2}\,\frac{G\,M_{\rm c,\,1}\,M_{\rm c,\,2}}{a}$$

3. change of orbital energy needed to unbind the envelope:

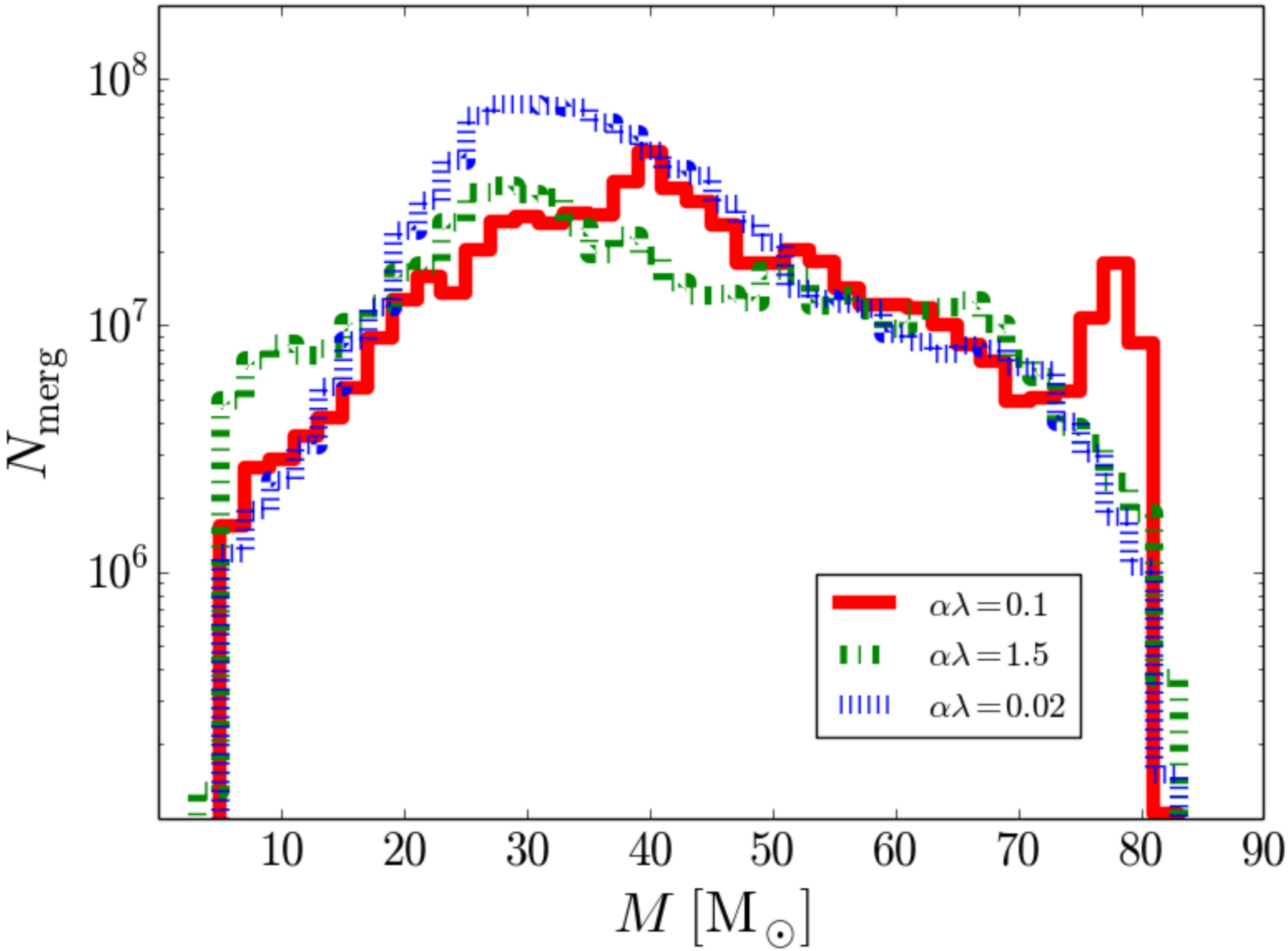
$$E_{\rm bind,i} = \Delta E_{\rm orb} = \alpha\,(E_{\rm orb,f} - E_{\rm orb,i})$$

α is free parameter (energy removal efficiency)

4. if final semi-major axis smaller than the sum of core radii $a_{\rm f} < (r_{\rm c,1} + r_{\rm c,2})$

THEN the cores merge (Hurley+ 2002, MNRAS, 329, 897)

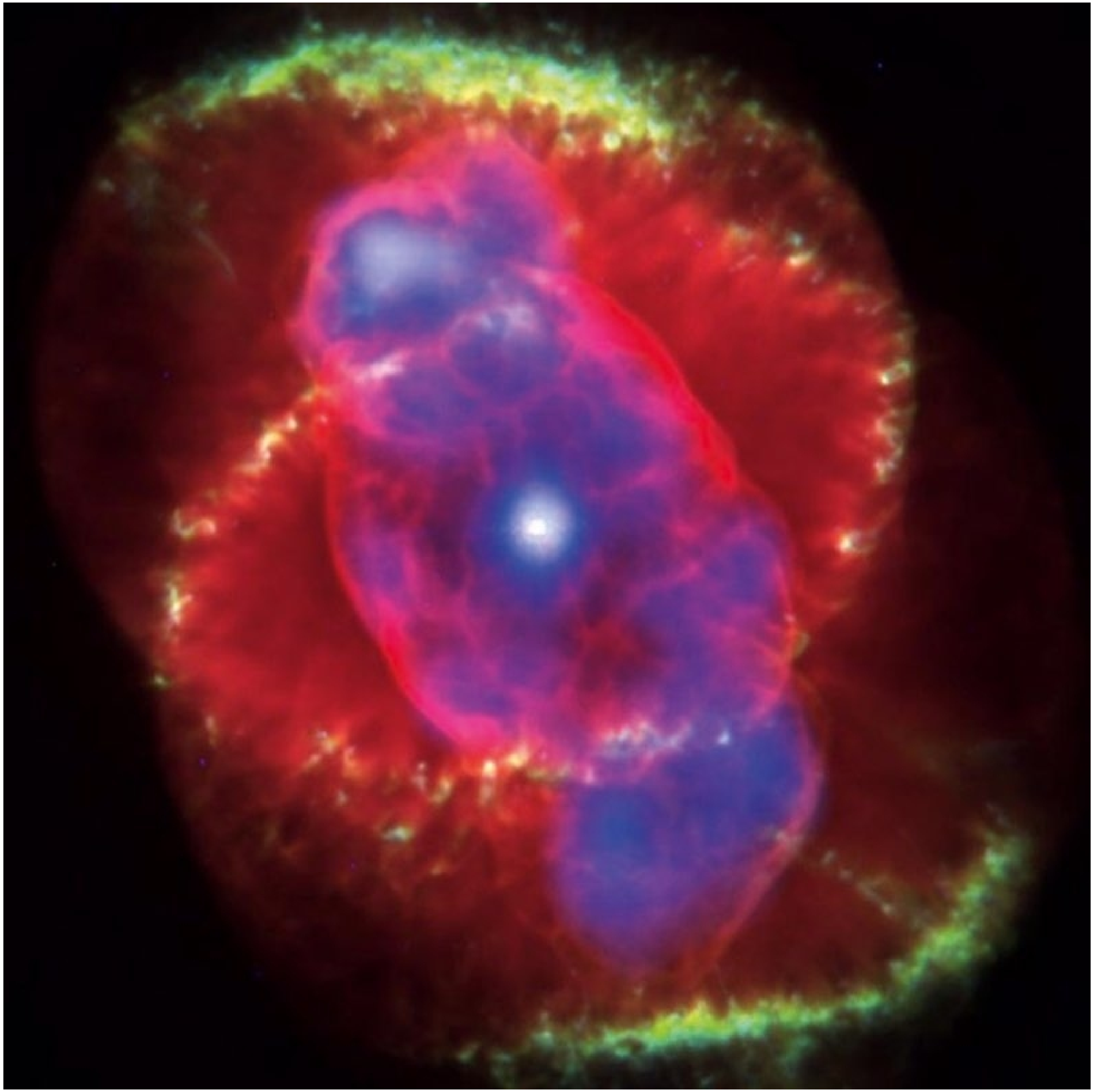
Common Envelope



updated version of BSE (MOBSE, MM+ 2017, Giacobbo+ 2018)

Common Envelope

observed post-CE systems



e.g. Cat's eye nebula

Alternatives to common envelope:

1. stable mass transfer

(Giacobbo+ 2018; Bavera+ 2020)

BASIC IDEA:
a few BBH (up to 20% of all mergers)
can become tight enough to merge via GW emission
via stable Roche lobe overflow, no common envelope

May be many more according to recent calculations
– e.g. Marchant et al. 2021

2. chemically homogeneous evolution

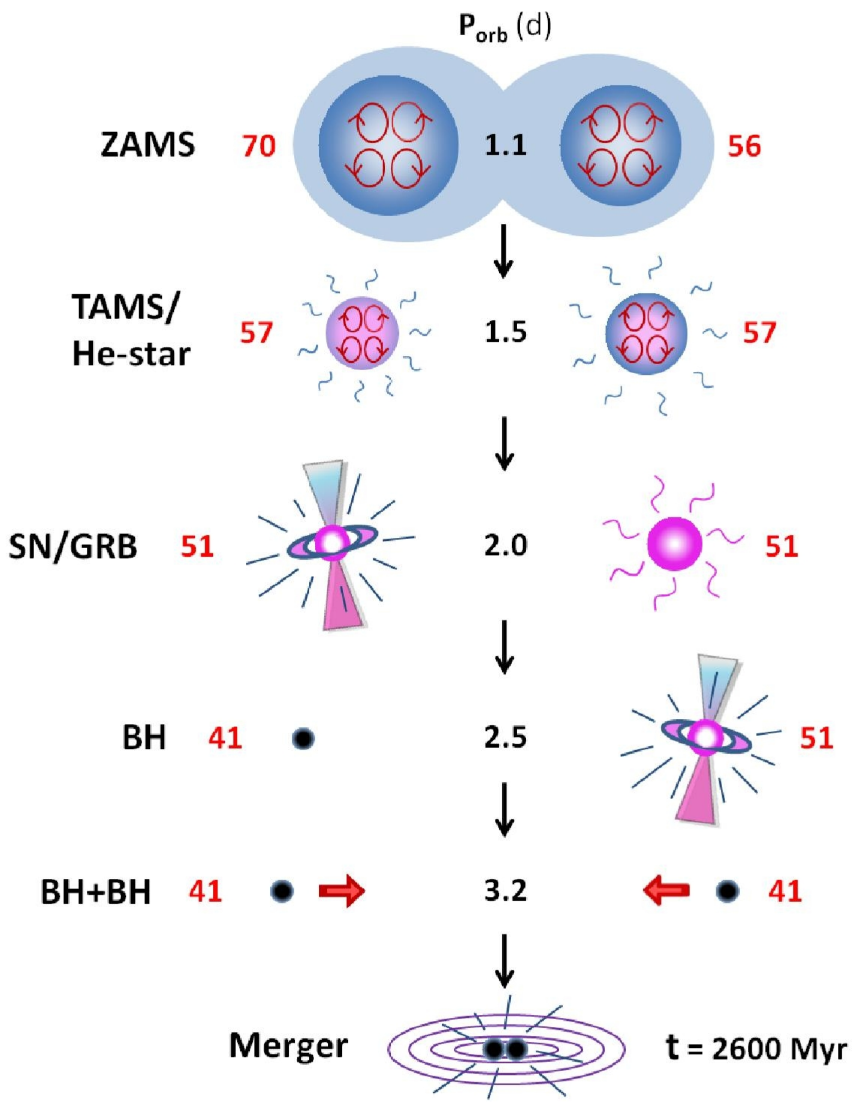
(Marchant+ 2016; Mandel & de Mink 2016; de Mink & Mandel 2016)

BASIC IDEA:
if stars are chemically homogeneous, their radii are smaller
→ close binaries avoid common envelope and premature merger

To be chemically homogeneous, stars need to ROTATE fast

3. OVER-CONTACT BINARIES (Marchant+ 2016):

Metal-poor fast rotating stars may OVERFILL ROCHE LOBE WITHOUT ENTERING COMMON ENVELOPE



Why?

Star rotation induces chemical mixing

Chemical mixing prevents star radius from growing significantly (efficient only if star is metal poor)

Predictions of this model:

* nearly equal-mass BH-BH

* BH masses ~25 – 60, 130 – 230 M$_\odot$ increasing with decreasing metallicity (no low-mass BHs!)

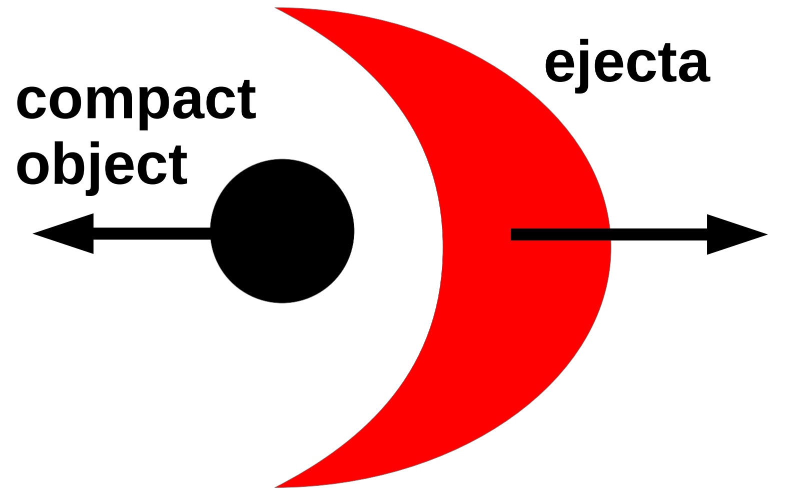
* aligned spins unless SN reset them

BUT YOU STILL NEED COMMON ENVELOPE FOR NEUTRON STARS AND WHITE DWARFS

Supernova kicks and BH binaries:

A massive-star binary can become a BH-BH binary only if it is not unbound by SN kicks

WHY KICKS?



* asymmetry in mass ejection
  during core collapse

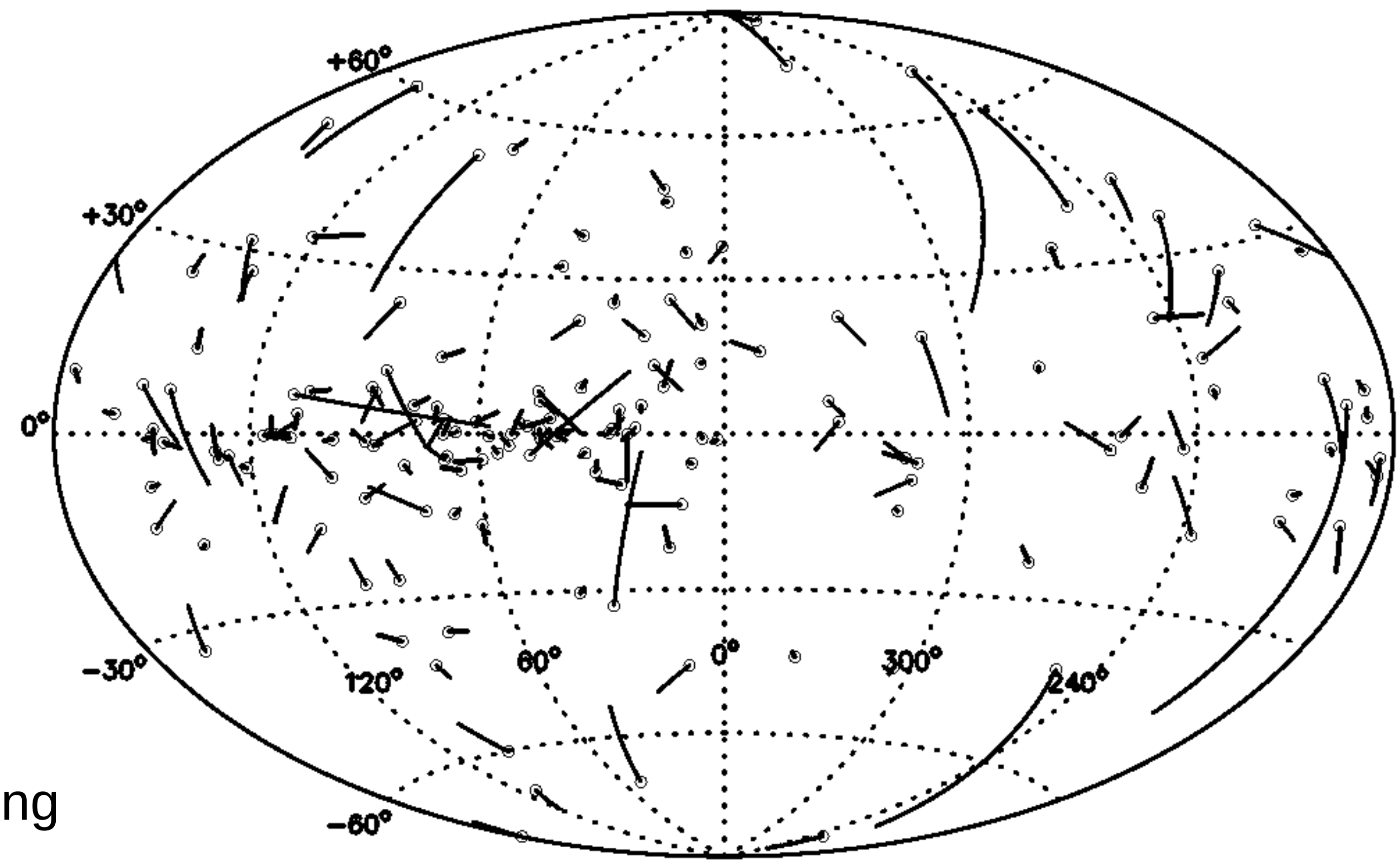
* asymmetry in neutrino emission
  during core collapse

* symmetric mass loss in a binary:
  breaks the binary only if pre-SN mass > companion mass
  (Blaauw mechanism, Blaauw 1961)

Supernova kicks and compact binaries:

SN kicks for NSs constrained from velocity of PULSARS

Hobbs+ (2005):
sample of 233 pulsars
with proper motion
measurements



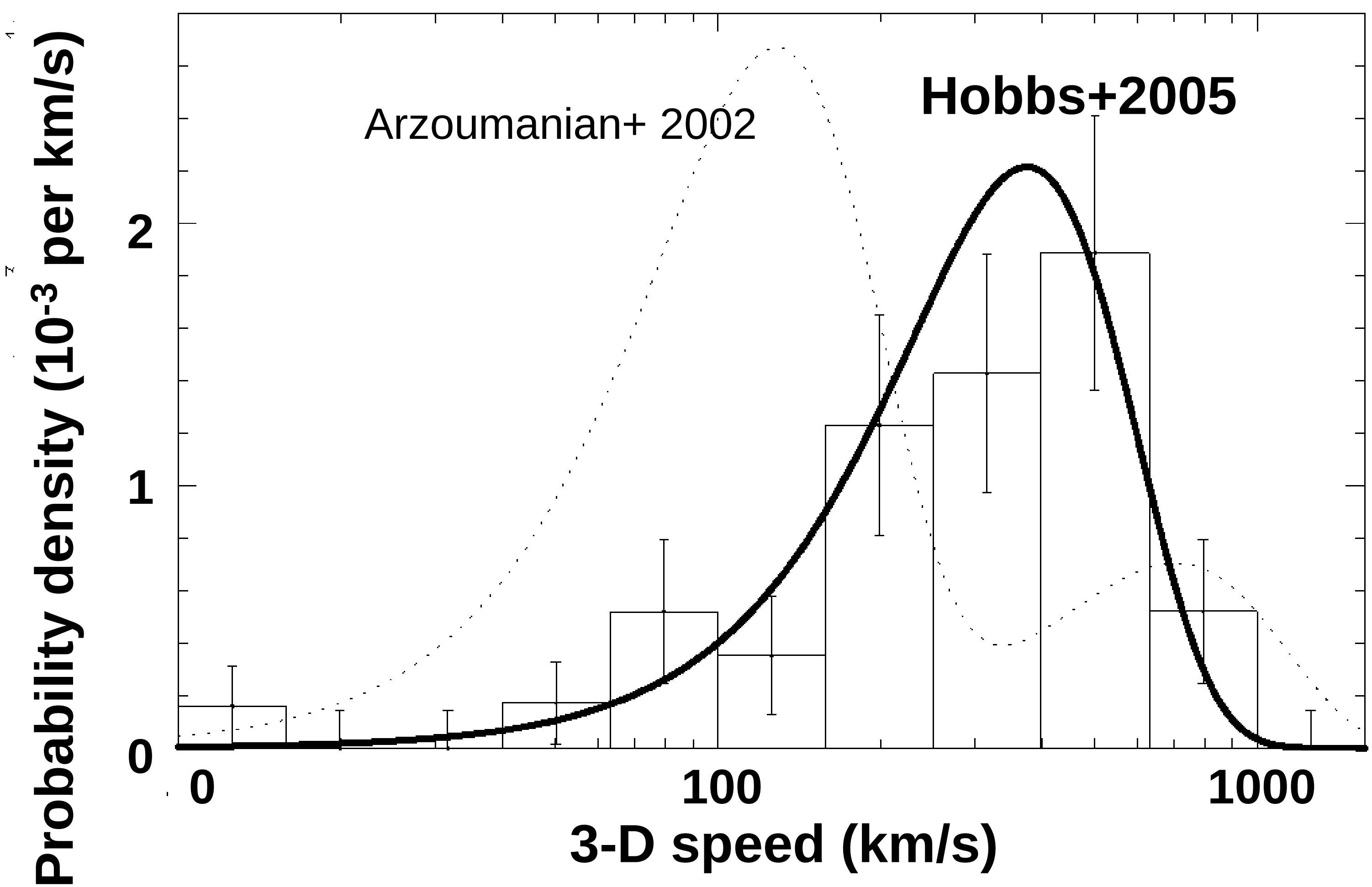
A pulsar is currently
at the position
indicated by a circle

The track is its motion
for the last 1 Myr assuming
no radial velocity.

Supernova kicks and compact binaries:

Hobbs+ (2005): 3-D velocity distribution of pulsars obtained from the observed 2-D distributions of pulsars

→ Maxwellian distribution with sigma ~ 265 km/s



Supernova kicks and compact binaries:

High (>100 km/s) velocity kicks for NSs (with caveats!)

WHAT ABOUT BHs?

No reliable methods to measure. Then people assume

0. same Maxwellian kick distribution for BHs and Nss

1. conservation of linear momentum

$$\boxed{v_{\rm kick,\,BH} = \frac{m_{\rm NS}}{m_{\rm BH}}\,v_{\rm kick,\,NS}}$$

2. BHs formed without SN (failed or direct collapse) get NO KICK + kick modulated by FALLBACK (e.g. Fryer et al. 2012)

$$\boxed{v_{\rm kick,\,BH} = (1 - f_{\rm fb})\,v_{\rm kick,\,NS}}$$

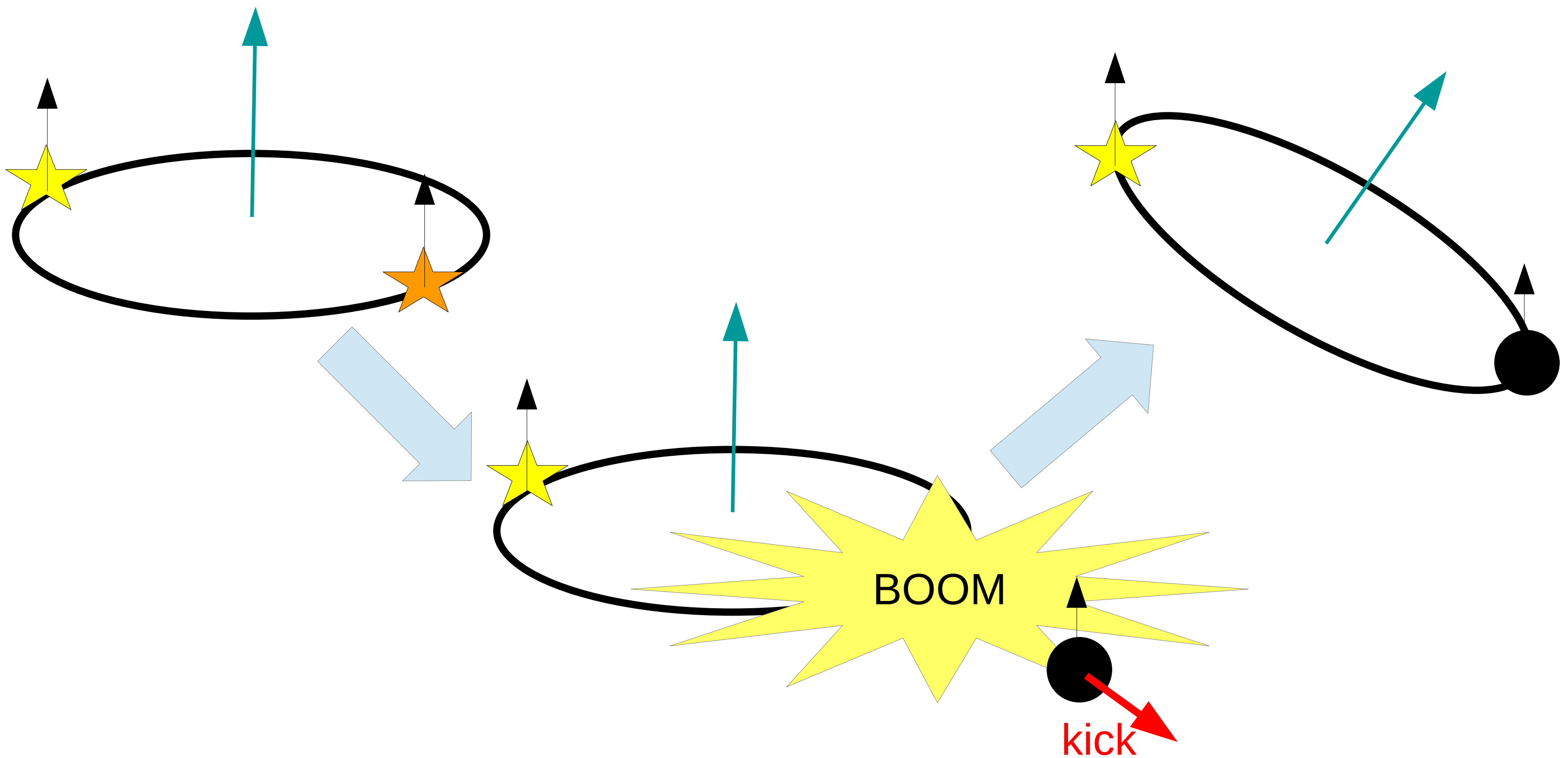
3. Momentum conservation + asimmetries depend on ejecta mass (e.g. Bray & Eldridge 2016, 2018; Giacobbo & MM 2019)

$$\boxed{v_{\rm kick,\,rem} \propto m_{\rm rem}^{-1}\,m_{\rm ej}}$$

WHAT ABOUT THE impact of supernova kick on SPIN?

* **Tides tend to align the spins with the orbital angular momentum**

* **Spin direction can be changed by SN kick**
  SN kick tilts the orbital plane of the binary



* **If NO SN kick spin direction DOES NOT CHANGE**

Gravitational wave decay

Implemented in the simplest possible way with Peters 1964 formalism
(see lecture 1)

produces both circularization and orbital decay

$$\frac{da}{dt} = -\frac{64}{5}\,\frac{G^3\,M\,m\,(M+m)}{c^5\,a^3\,(1-e^2)^{7/2}}\,\left(1+\frac{73}{24}\,e^2+\frac{37}{96}\,e^4\right)$$

$$\frac{de}{dt} = -\frac{304}{15}\,e\,\frac{G^3\,M\,m\,(M+m)}{c^5\,a^4\,(1-e^2)^{5/2}}\,\left(1+\frac{121}{304}\,e^2\right)$$

Isolated binary evolution summary:

* possible Roche lobe

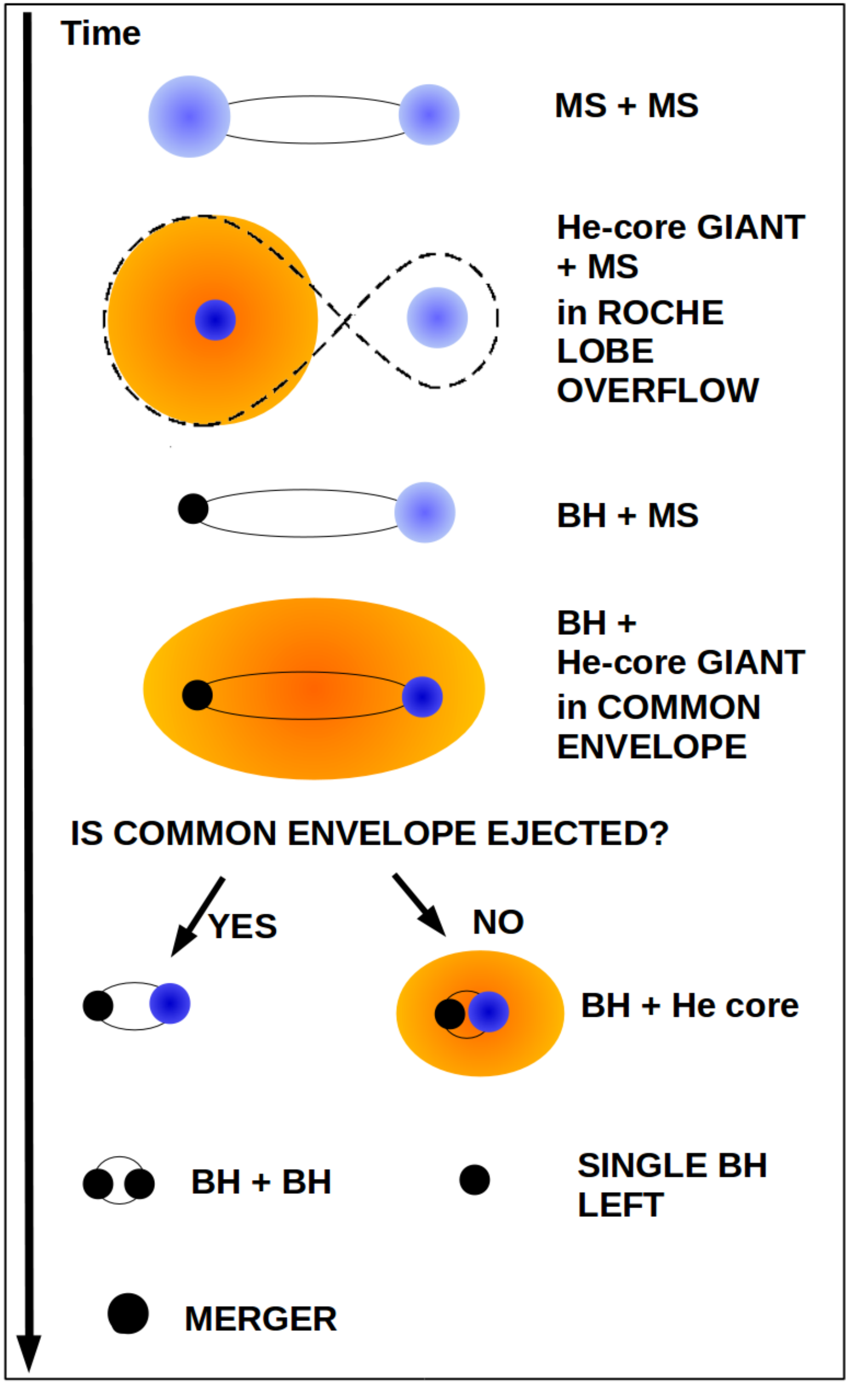
* 1st BH formation

* Common envelope
BH – giant
crucial to shrink the binary from >>100 R⊙ to <100 R⊙

* If binary survives common envelope, formation of second BH

* BH – BH merger

cartoon from MM2021



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* P. Eggleton, Evolutionary processes in Binary and Multiple Stars, Cambridge Astrophysics Series