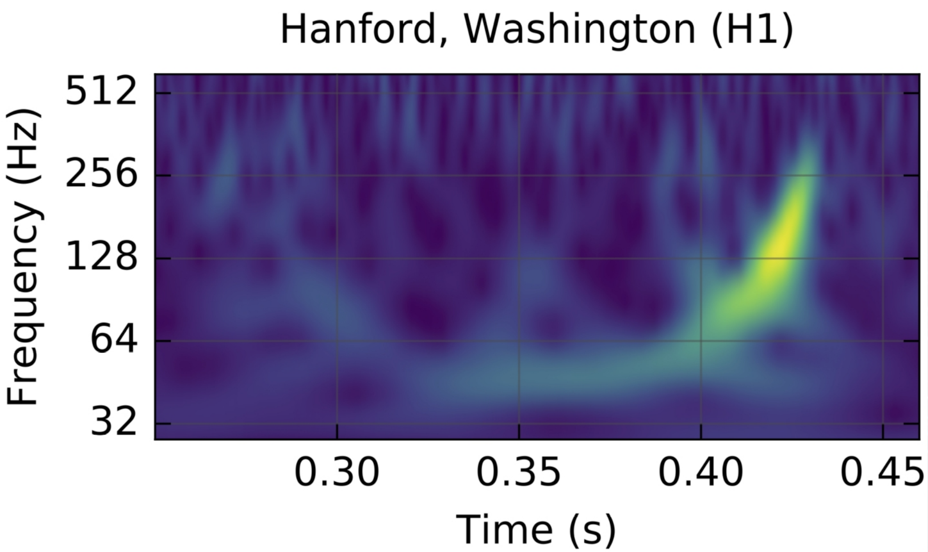
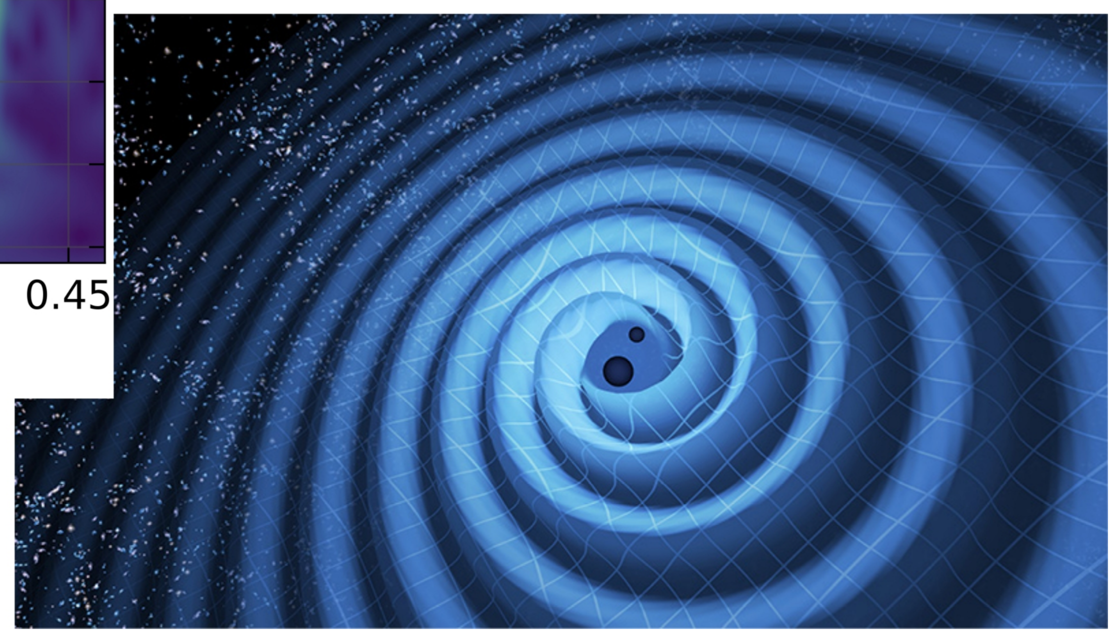
Gravitational Wave Astrophysics





GSSI, March 2022

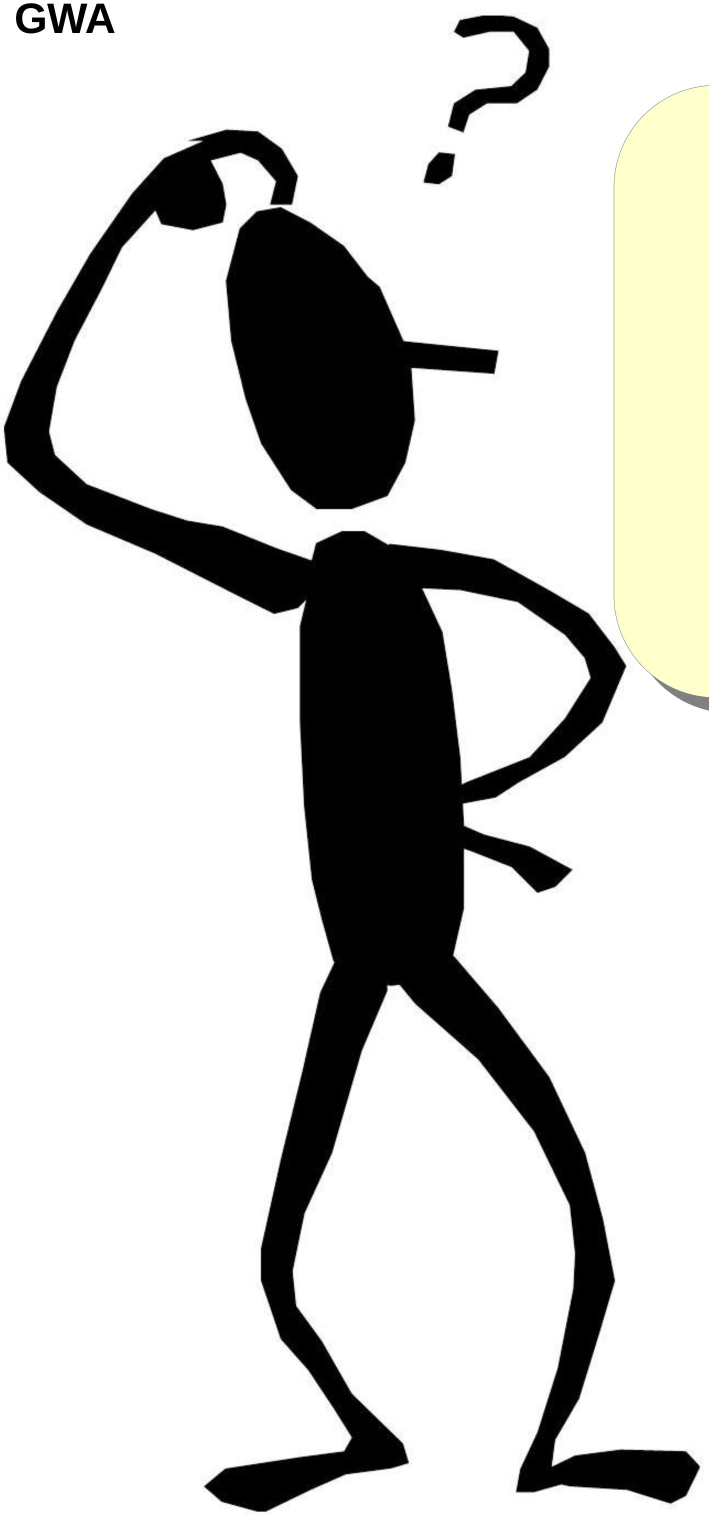
What is Gravitational – Wave (GW) Astrophysics?

* Astrophysical characterization of GW sources

* Young and fast evolving

* Boosted by GW detections

* Mostly (but not only) about binary black holes (BBHs), binary neutron stars (BNSs) and neutron star – black hole binaries (NSBHs)



OPEN QUESTION:

What are the formation channels of merging binaries observed by gravitational-wave interferometers?

Gravitational Wave (GW) Theory’s Extreme Summary

Some math:

Einstein's field equations

$$\overbrace{R_{\alpha\beta} - \frac{1}{2}\, R\, g_{\alpha\beta}}^{\text{Geometry of space-time}} = \overbrace{\frac{8\,\pi\, G}{c^4}\, T_{\alpha\beta}}^{\text{source term}}$$

where $R_{\alpha\beta} := R^{\mu}_{\alpha\mu\beta} = \sum_{\mu=0}^{3} R^{\mu}_{\alpha\mu\beta}$ **RICCI TENSOR,**

obtained contracting indices of Riemann curvature tensor

defined as $R^{\delta}_{\alpha\beta\gamma} w^{\gamma} := -(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha)\, w^{\delta}$

for an arbitrary vector *w*

(NOTE: Riemann curvature tensor is the measure of curvature in GR)

$R := g^{\mu\nu} R_{\mu\nu}$ **RICCI SCALAR,**

$g_{\alpha\beta}$ **METRIC,**

and $T_{\alpha\beta}$ **STRESS – ENERGY TENSOR (SOURCES of curvature)**

$$T^{\alpha\beta} := \left[\begin{array}{cc} \text{mass} - \text{energy density} & \text{energy flux} \\ \text{momentum density} & \text{stress tensor} \end{array} \right]$$

How to derive gravitational waves from Einstein eqs?

1. if strong-curvature, rapidly moving sources
   (e.g. during merger of two black holes):
   Numerical solution of fully nonlinear Einstein equations

2. if weak field (almost flat curvature) and “slow” moving sources
   (e.g. two black holes orbiting about each other but still binary systems, not yet merging):
   Analytic solution of LINEARIZED Einstein equations

Solving linearized Einstein’s equation reduces to solving this simplified equation (WAVE EQUATION):

$$\Box\, \bar{h}_{\alpha\beta} = \frac{16\,\pi\, G}{c^4} T_{\alpha\beta}$$

Summary:

By integrating equation $\Box\, \bar{h}_{\alpha\beta} = \frac{16\,\pi\,G}{c^4} T_{\alpha\beta}$

for distant and slowly moving sources, we obtain the solution:

$$\boxed{\overline{h}^{ij}(t,\vec{x}) \sim \frac{2\,G}{r\,c^4}\frac{d^2}{dt^2} I^{ij}(t - r/c)}$$

Distance source-observer

Second mass moment, or quadrupole moment of mass (similar but not exactly moment of inertia)

$$I^{ij} = \int dx^3\, \rho(t,\vec{x})\, x^i\, x^j$$

Retarded time

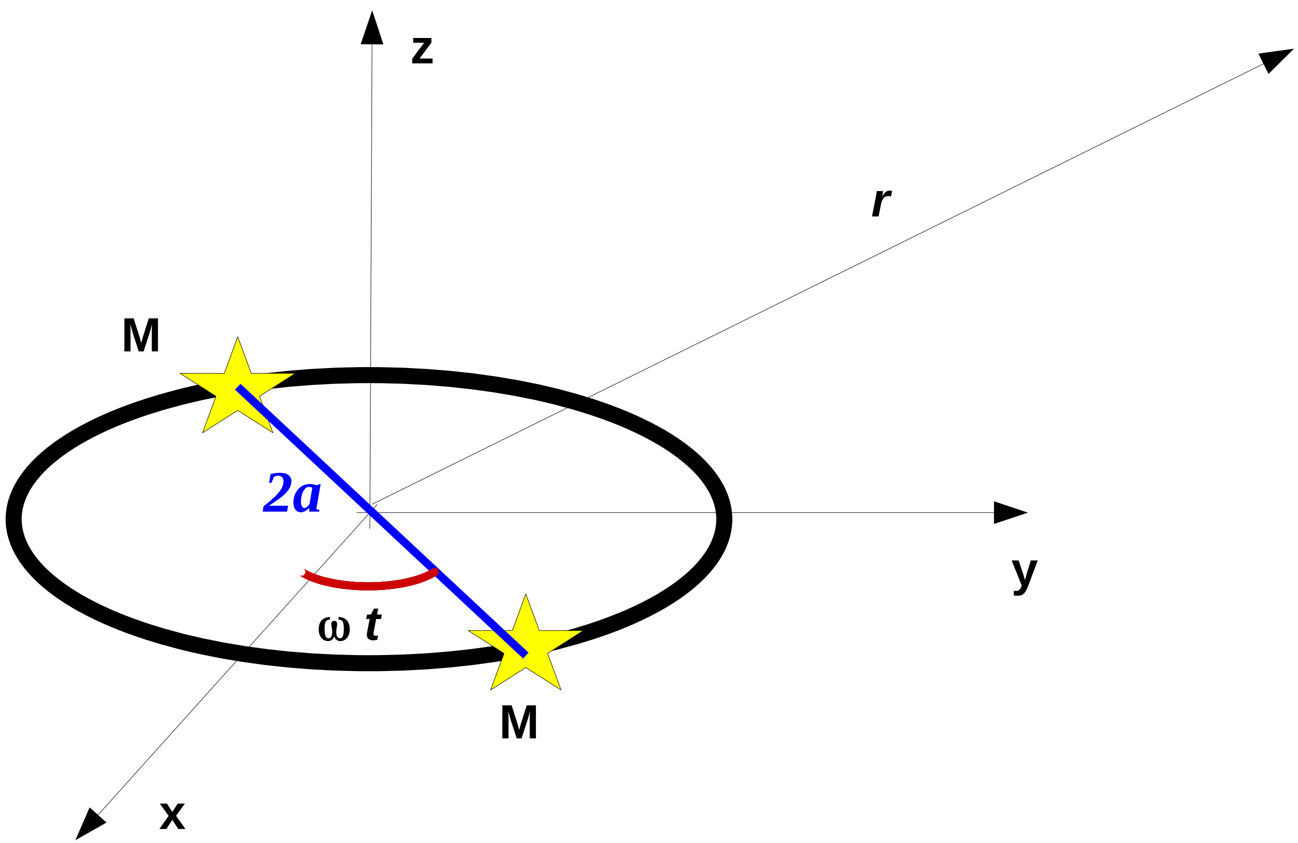
→ **not all accelerating masses produce gravitational waves, but only those with non-zero <u>QUADRUPOLE moment of mass</u>**

→ **for a gravitational wave to form, there must be an ASYMMETRY IN MASS DISTRIBUTION**

Binary system:

Now we restrict to the interesting case of a binary system

Two stars of <u>equal mass M</u> orbiting about each other with <u>CIRCULAR orbit</u> and relative distance (= semi-major axis) a



See Hartle, Gravity: An Introduction to Einstein's General Relativity

Binary system:

For the assumed geometry: x(t) = *a* cos(ωt), y(t) = *a* sin(ωt), z(t) = 0

The second mass moment is

$$I^{xx} = 2\,M\,a^2\cos^2(\omega\,t) = M\,a^2\,\left[1+\cos(2\,\omega\,t)\right]$$
$$I^{xy} = 2\,M\,a^2\sin(\omega\,t)\,\cos\omega\,t = M\,a^2\,\sin(2\,\omega\,t)$$
$$I^{yy} = 2\,M\,a^2\sin^2(\omega\,t) = M\,a^2\,\left[1-\cos(2\,\omega\,t)\right]$$
$$I^{zz} = I^{zx} = I^{zy} = 0$$

By taking the second derivatives and considering retarded time:

$$\bar{h}^{ij} \sim\ -\frac{2}{r}\frac{G}{c^4}(2\,\omega)^2\,M\,a^2\begin{pmatrix}\cos(2\,\omega\,(t-r/c)) & \sin(2\,\omega\,(t-r/c)) & 0\\ \sin(2\,\omega\,(t-r/c)) & -\cos(2\,\omega\,(t-r/c)) & 0\\ 0 & 0 & 0\end{pmatrix}$$

Using Newton description of binary star: $\omega^2 = \dfrac{G\,2\,M}{a^3}$

We find

$$\bar{h}^{ij} \sim\ -\frac{16\,G^2}{c^4}\frac{M^2}{r\,a}\begin{pmatrix}\cos(2\,\omega\,(t-r/c)) & \sin(2\,\omega\,(t-r/c)) & 0\\ \sin(2\,\omega\,(t-r/c)) & -\cos(2\,\omega\,(t-r/c)) & 0\\ 0 & 0 & 0\end{pmatrix}$$

Binary system:

Gravitational waves for a circular, equal-mass binary system:

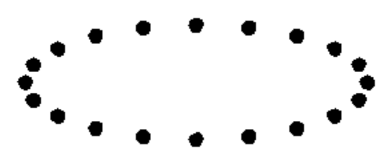
$$\bar{h}^{ij} \sim -\frac{16\,G^2}{c^4}\frac{M^2}{r\,a}\begin{pmatrix} \cos\left(2\,\omega\,(t-r/c)\right) & \sin\left(2\,\omega\,(t-r/c)\right) & 0 \\ \sin\left(2\,\omega\,(t-r/c)\right) & -\cos\left(2\,\omega\,(t-r/c)\right) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

1. GRAVITATIONAL WAVES from a BINARY STAR are MONOCHROMATIC with frequency = 2 orbital frequency

$$\omega_{GW} = 2\,\omega_{orb} = 2\,\sqrt{\frac{G\,(m_1+m_2)}{a^3}}$$

2. Two POLARIZATIONS, PLUS and CROSS (general, not only for binaries)

$$h_+ \sim \frac{16\,G^2}{c^4}\frac{M^2}{r\,a}\,\cos\left(2\,\omega\,(t-r/c)\right)$$
$$h_\times \sim \frac{16\,G^2}{c^4}\frac{M^2}{r\,a}\,\sin\left(2\,\omega\,(t-r/c)\right)$$



Binary system:

3. GRAVITATIONAL WAVE AMPLITUDE

$$h = \frac{1}{2}\,(h_+^2 + h_\times^2)^{0.5} \sim \frac{8\,G^2}{c^4}\,\frac{M^2}{r\,a}$$

$$h \sim 10^{-21}\,\left(\frac{M}{M_\odot}\right)^2\,\left(\frac{1\,\mathrm{kpc}}{r}\right)\,\left(\frac{1\,R_\odot}{a}\right)$$

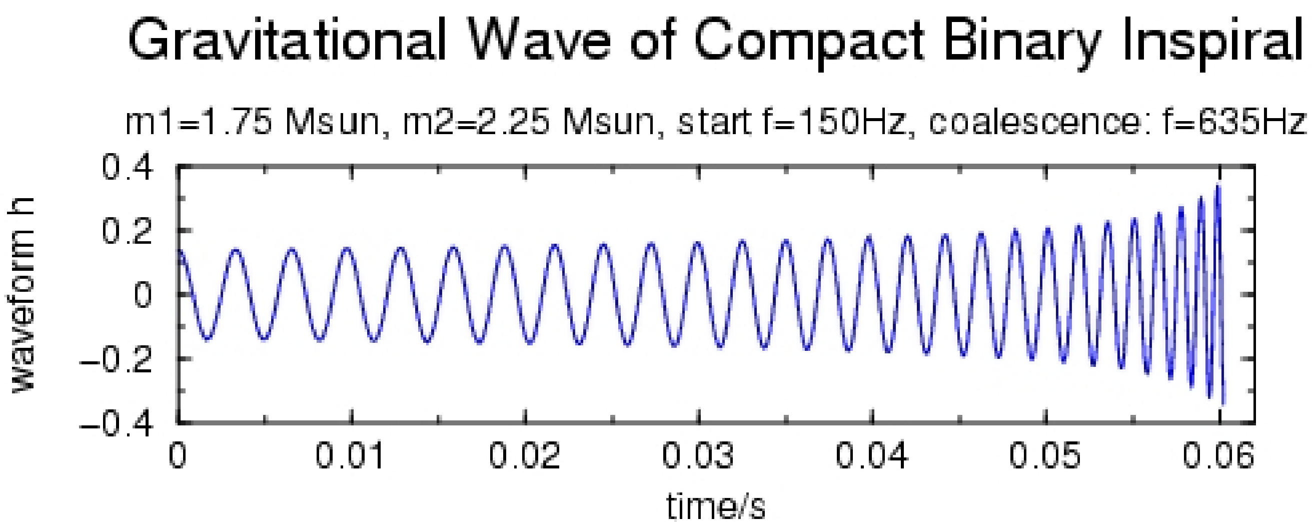
or in terms of period $P^2 = (2\,\pi)^2\,\frac{a^3}{G\,2\,M} \longrightarrow h \sim \frac{8\,(2\,\pi^2)^{1/3}\,G^{5/3}}{c^4}\,\frac{M^{5/3}}{r\,P^{2/3}}$

$$h \sim 10^{-21}\,\left(\frac{M}{M_\odot}\right)^{5/3}\,\left(\frac{1\,\mathrm{kpc}}{r}\right)\,\left(\frac{1\,\mathrm{h}}{P}\right)^{2/3}$$

AMPLITUDE of GWs from binary systems:

$$h = \frac{1}{2}\,(h_+^2 + h_\times^2)^{0.5} \sim \frac{8\,G^2}{c^4}\,\frac{M^2}{r\,a}$$

*** the bigger the amplitude (strain), the easier the detection**



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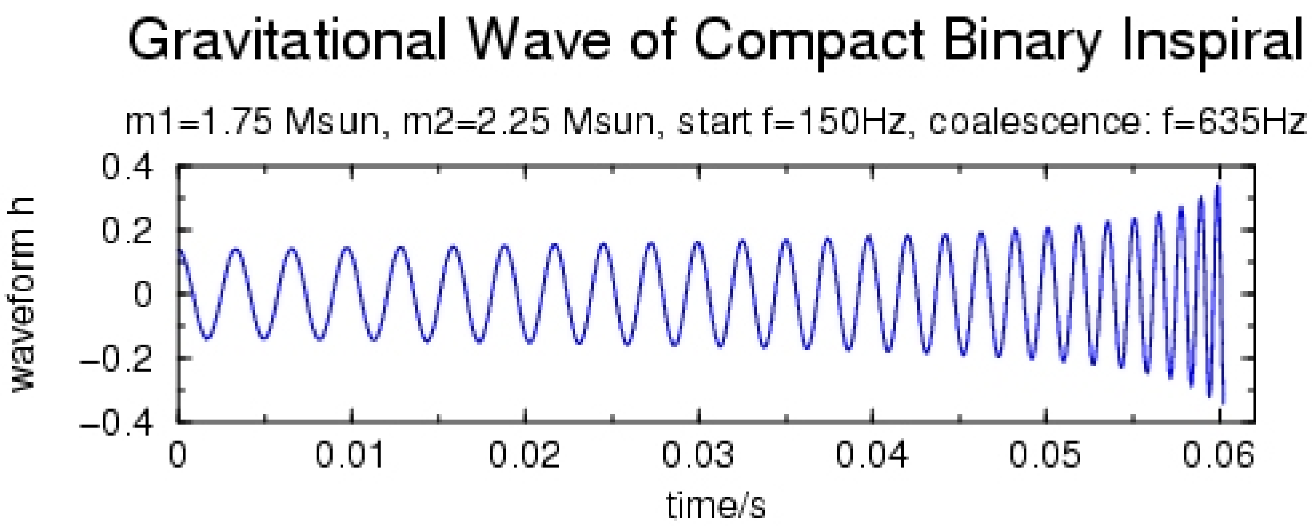
*** the bigger the amplitude (strain), the easier the detection**

*** the farther the binary, the smaller the amplitude**



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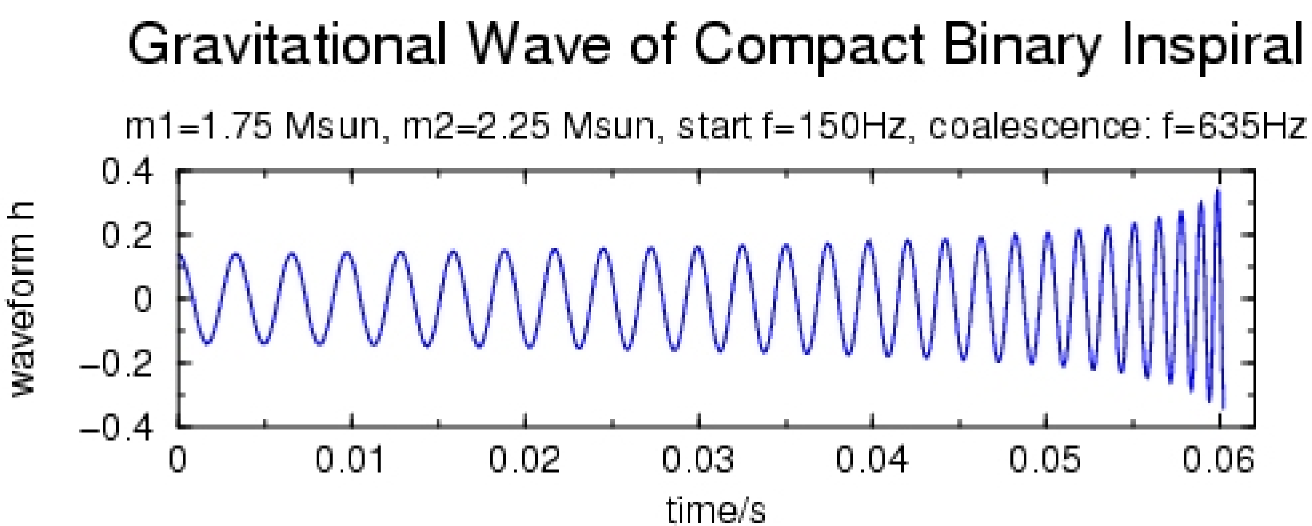
* the bigger the amplitude (strain), the easier the detection
* the farther the binary, the smaller the amplitude
* the larger the masses, the larger the amplitude



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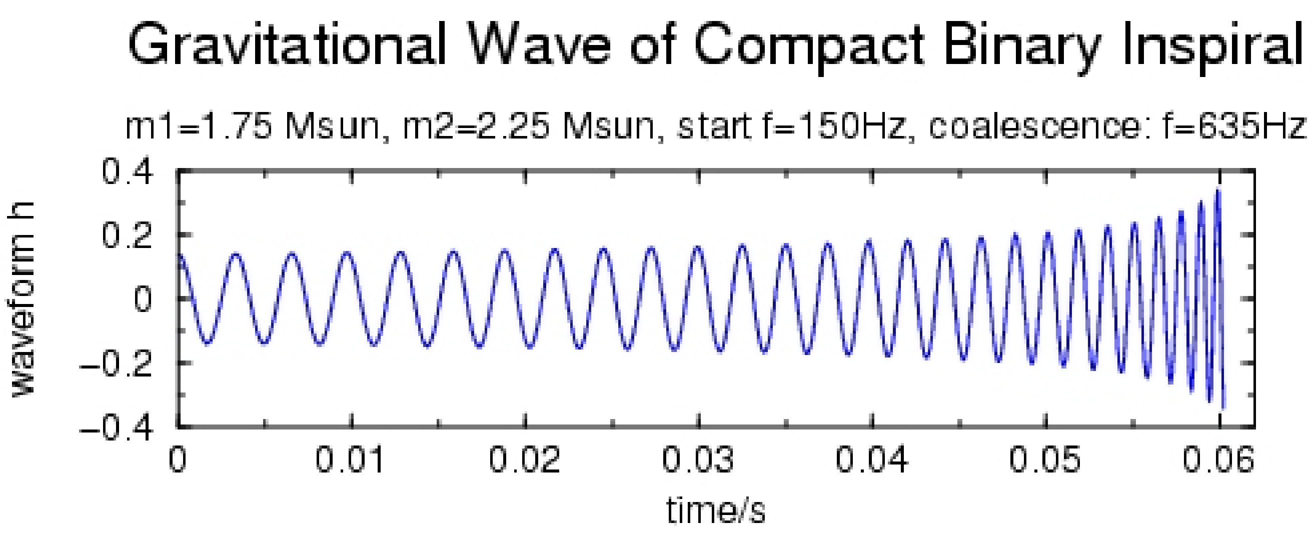
$$h = \frac{1}{2}\,(h_+^2 + h_\times^2)^{0.5} \sim \frac{8\,G^2}{c^4}\,\frac{M^2}{r\,a}$$

* the bigger the amplitude (strain), the easier the detection
* the farther the binary, the smaller the amplitude
* the larger the masses, the larger the amplitude
* the smaller the semi-major axis, the larger the amplitude



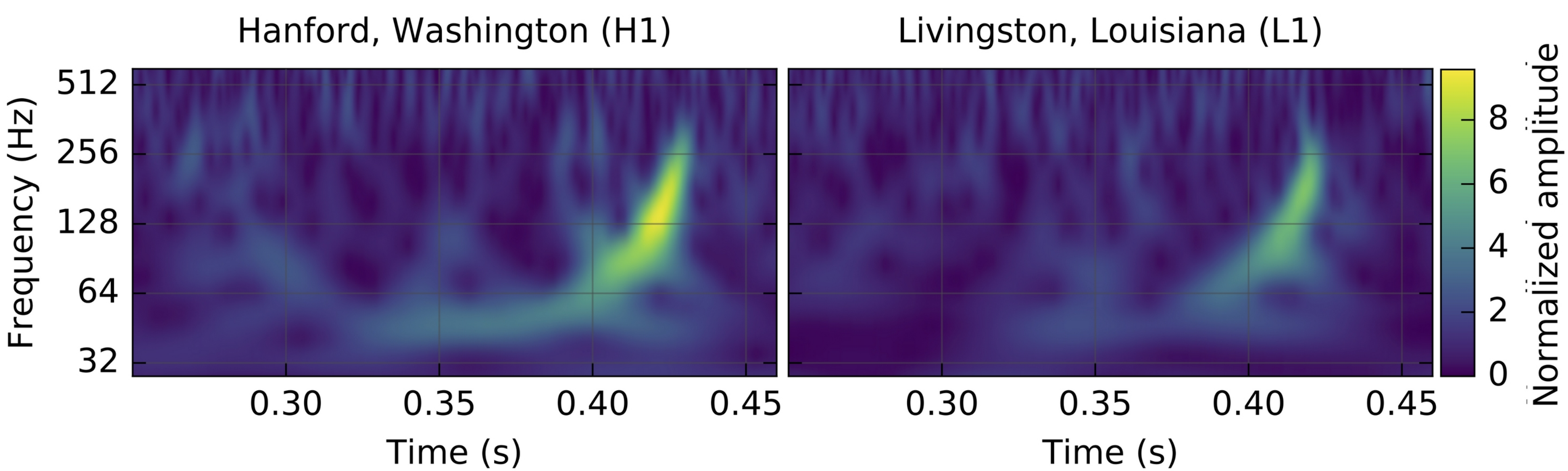
EMISSION of GWs implies LOSS of ORBITAL ENERGY:

THE BINARY SHRINKS WHILE EMITTING GWs TILL IT MERGES

$$E_{orb} = -\frac{G\,m_1\,m_2}{2\,a}$$

– If the binary shrinks ($a \rightarrow 0$), frequency becomes higher

$$\omega_{GW} = 2\,\omega_{orb} = 2\,\sqrt{\frac{G\,(m_1+m_2)}{a^3}}$$



Abbott et al. 2016

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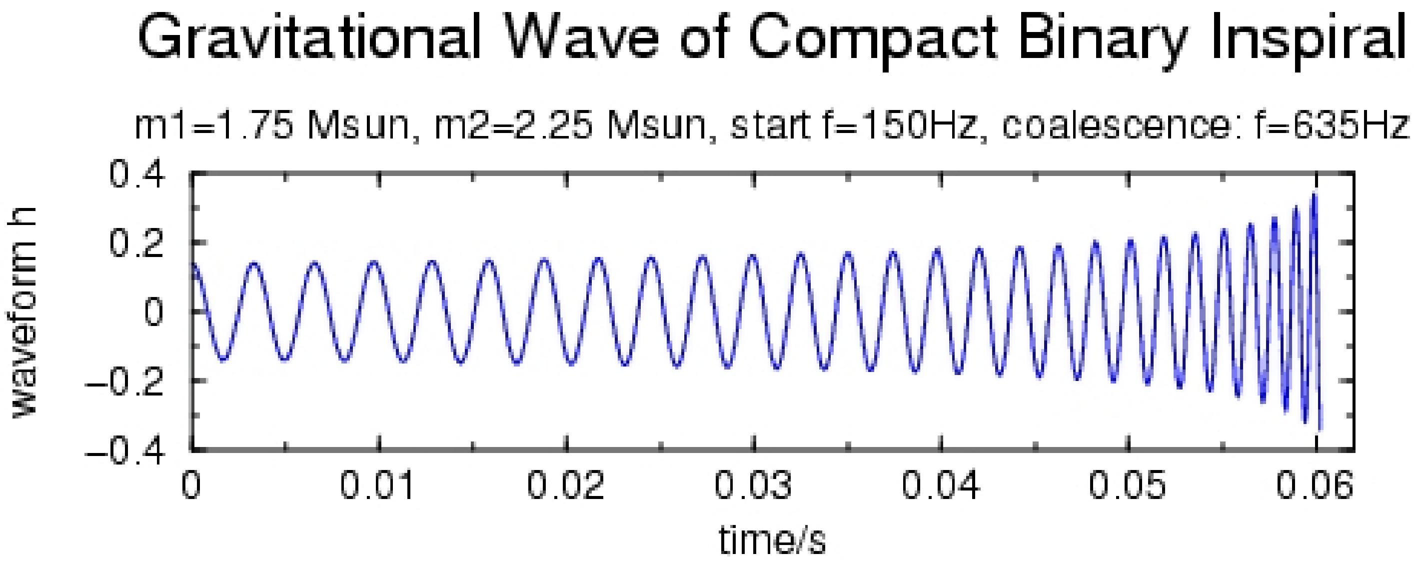
THE BINARY SHRINKS WHILE EMITTING GWs TILL IT MERGES

$$E_{orb} = -\frac{G\, m_1\, m_2}{2\, a}$$

– If the binary shrinks ($a \to 0$), frequency becomes higher

– If the binary shrinks amplitude increases

$$h \propto \frac{1}{a}$$



EMISSION of GWs implies LOSS of ORBITAL ENERGY:

Power radiated by GWs (circular binary):

From GR $$P_{GW} = \frac{32}{5}\frac{G^4}{c^5}\frac{1}{a^5}m_1^2\,m_2^2\,(m_1+m_2)$$

$$P_{GW} = \frac{dE_{orb}}{dt} = \frac{G\,m_1\,m_2}{2\,a^2}\frac{da}{dt}$$ From Kepler and Newton

$$\frac{da}{dt} = \frac{64}{5}\frac{G^3}{c^5}a^{-3}\,m_1\,m_2\,(m_1+m_2)$$

Integrating differential equation:

$$\boxed{t_{GW} = \frac{5}{256}\frac{c^5}{G^3}\frac{a^4}{m_1\,m_2\,(m_1+m_2)}}$$

Timescale for a system to merge by GW emission

Timescale extremely long

$$\boxed{t_{GW} = \frac{5}{256}\,\frac{c^5}{G^3}\,\frac{a^4}{m_1\,m_2\,(m_1+m_2)}}$$

EXERCISE: calculate tGW for 2 neutron stars
with mass equal to the Sun mass (1 Msun)
orbiting at the distance
between Sun and Earth (1 AU)

If m1 = m2 = Msun, a = 1 AU, eccentricity = 0

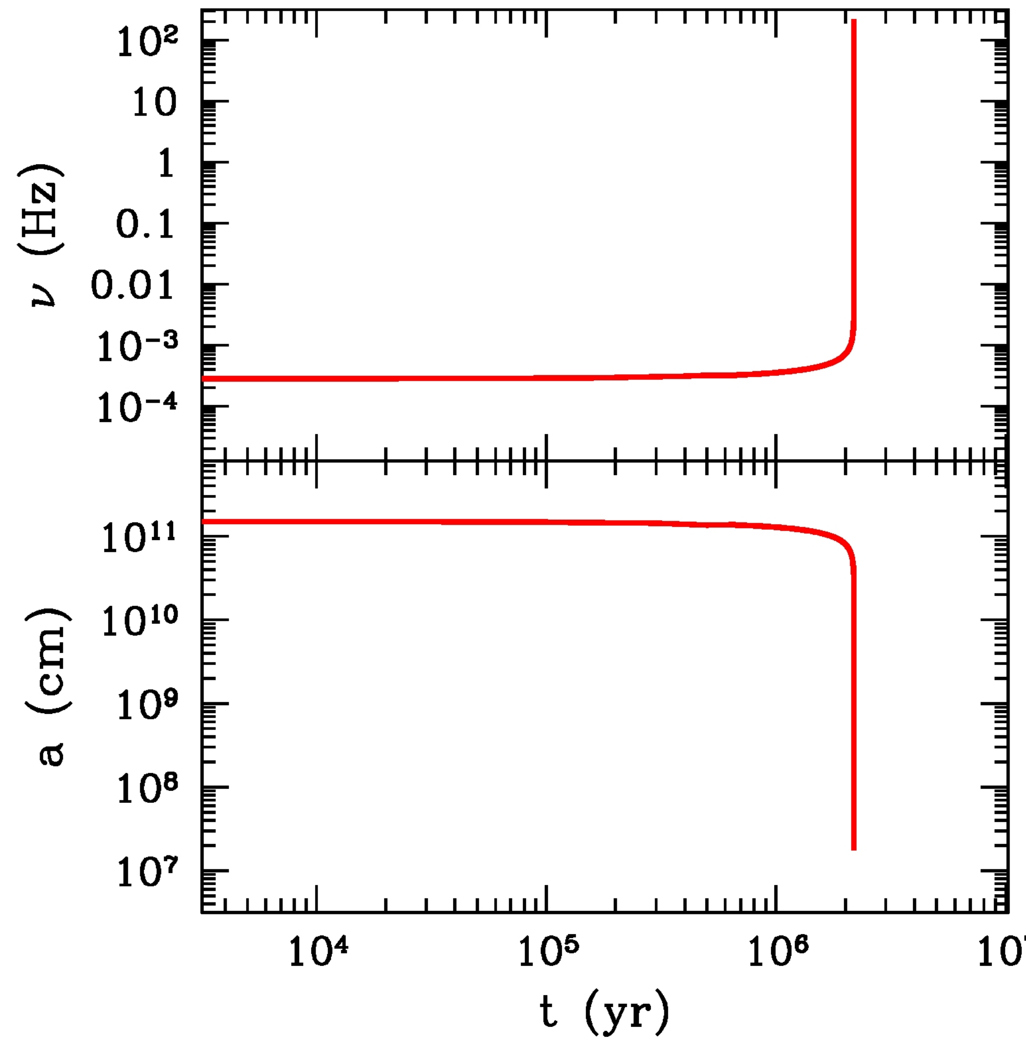
→ tGW ~ 2 x 10^17 yr

Life of the Universe ~ 13 x 10^9 yr

$$\frac{da}{dt} = \frac{64}{5}\,\frac{G^3}{c^5}\,a^{-3}\,m_1\,m_2\,(m_1+m_2)$$

$$\longrightarrow \quad a(t) = a_0\,\left[1 - \frac{256/5\,G^3\,m_1\,m_2\,(m_1+m_2)\,t}{c^5\,a_0^4}\right]^{1/4}$$

m1 = m2 = 10 Msun
a_0 = 0.01 AU
ecc = 0



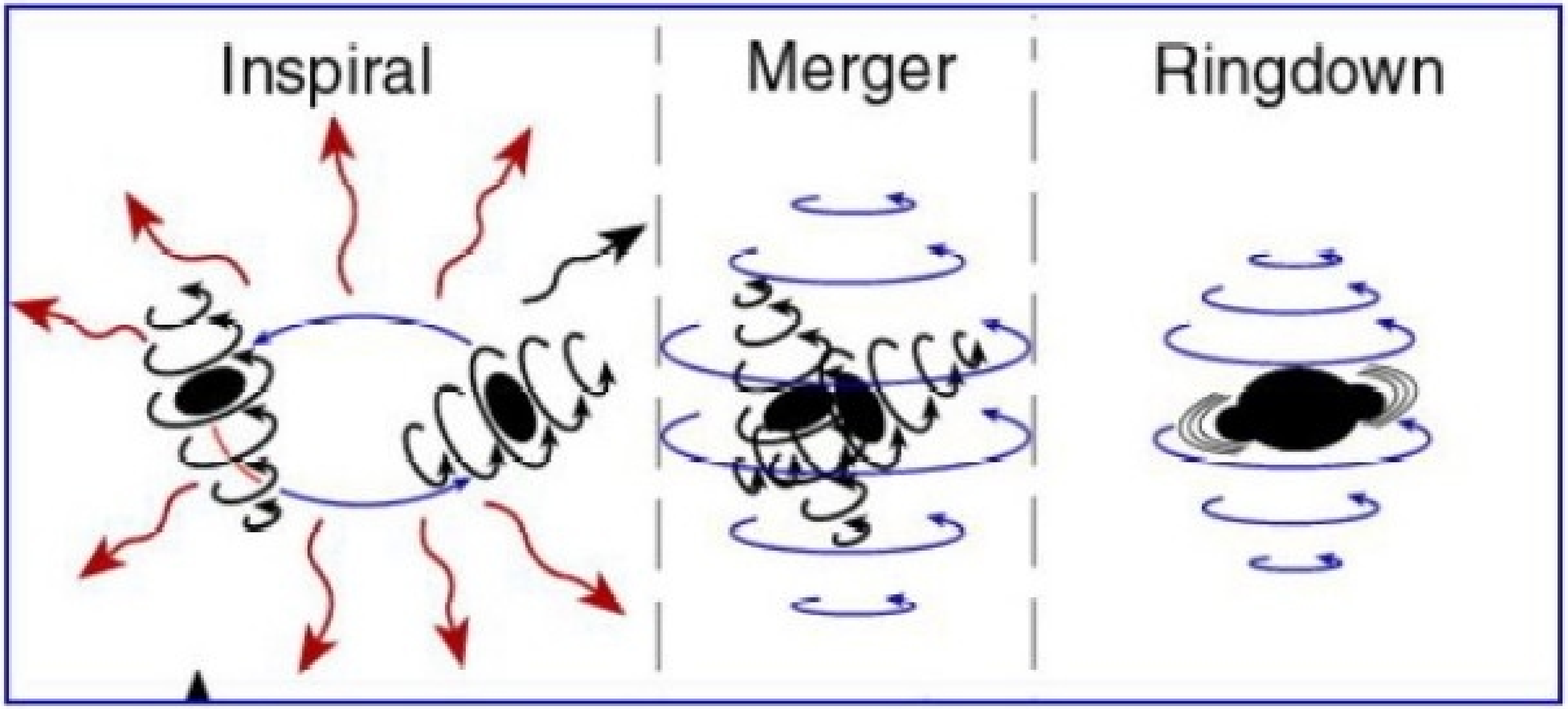
Note:
above formula neglects evolution of eccentricity via GW emission

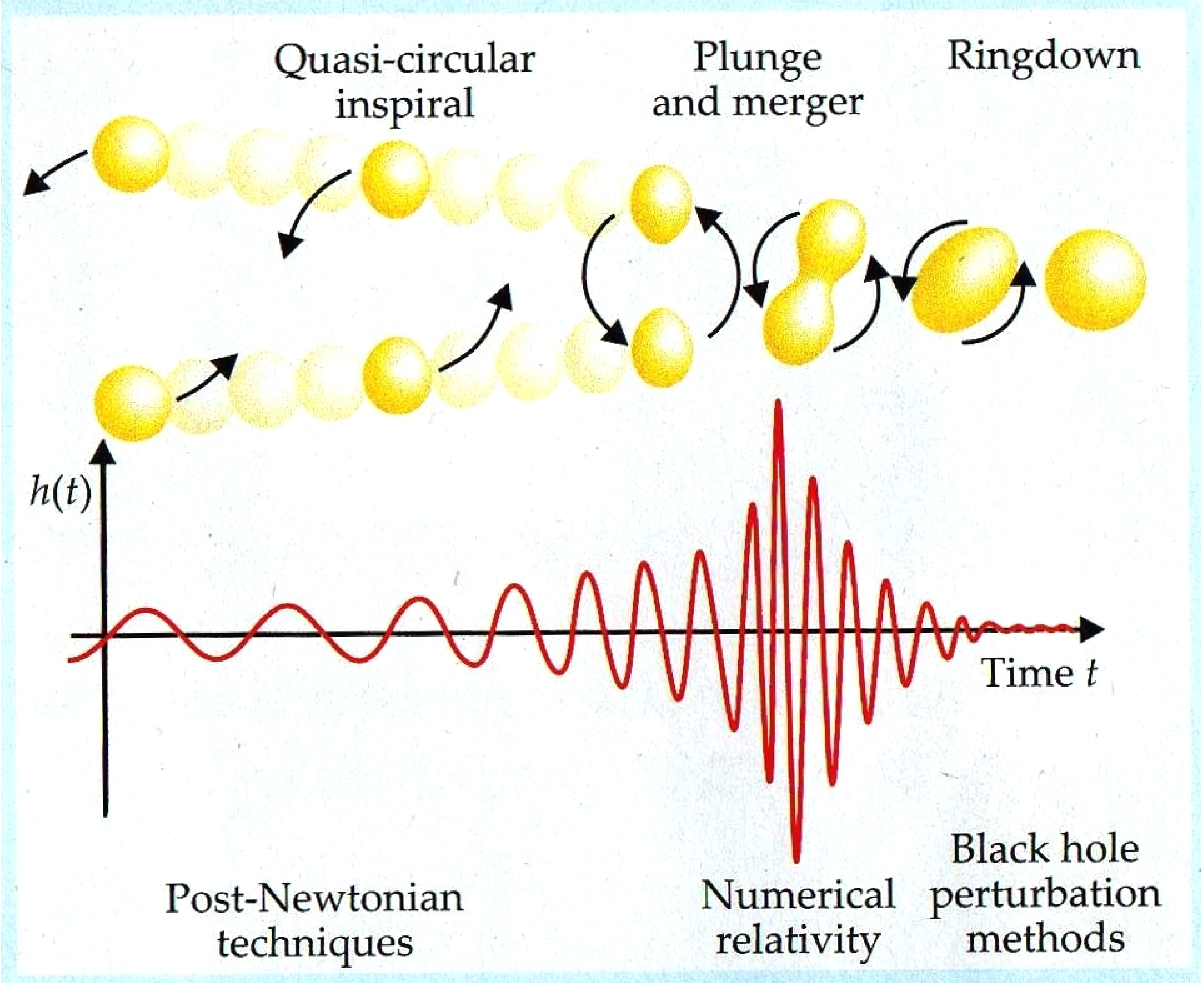
correct only if initial eccentricity is zero

Previous equations are not always true!

Only before merger when binary can be considered Keplerian

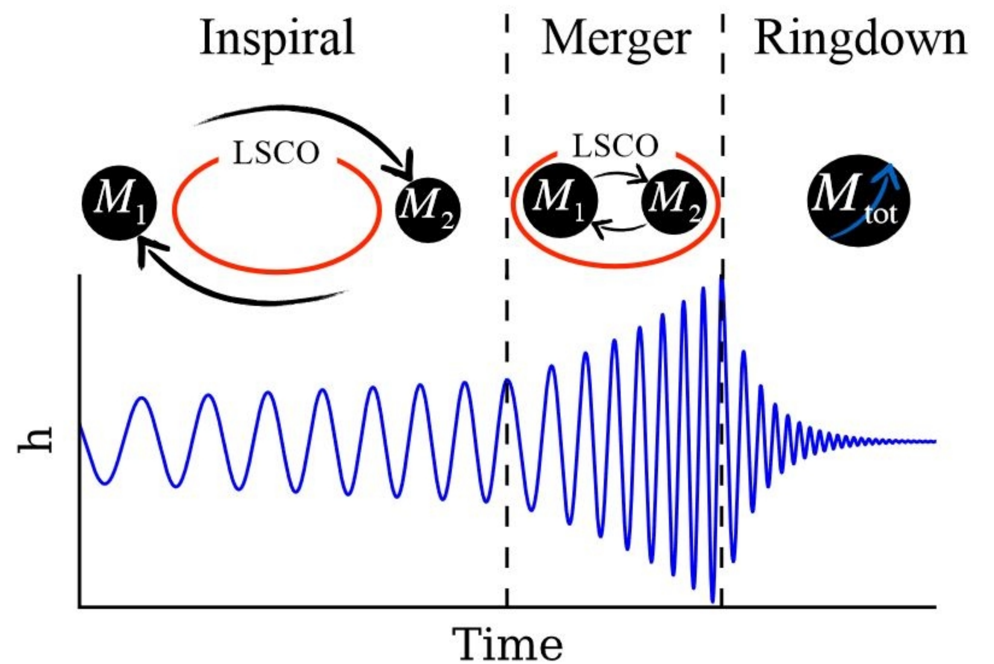
i.e. only during inspiral




Quasi-circular inspiral
Plunge and merger
Ringdown
h(t)
Time t
Post-Newtonian techniques
Numerical relativity
Black hole perturbation methods

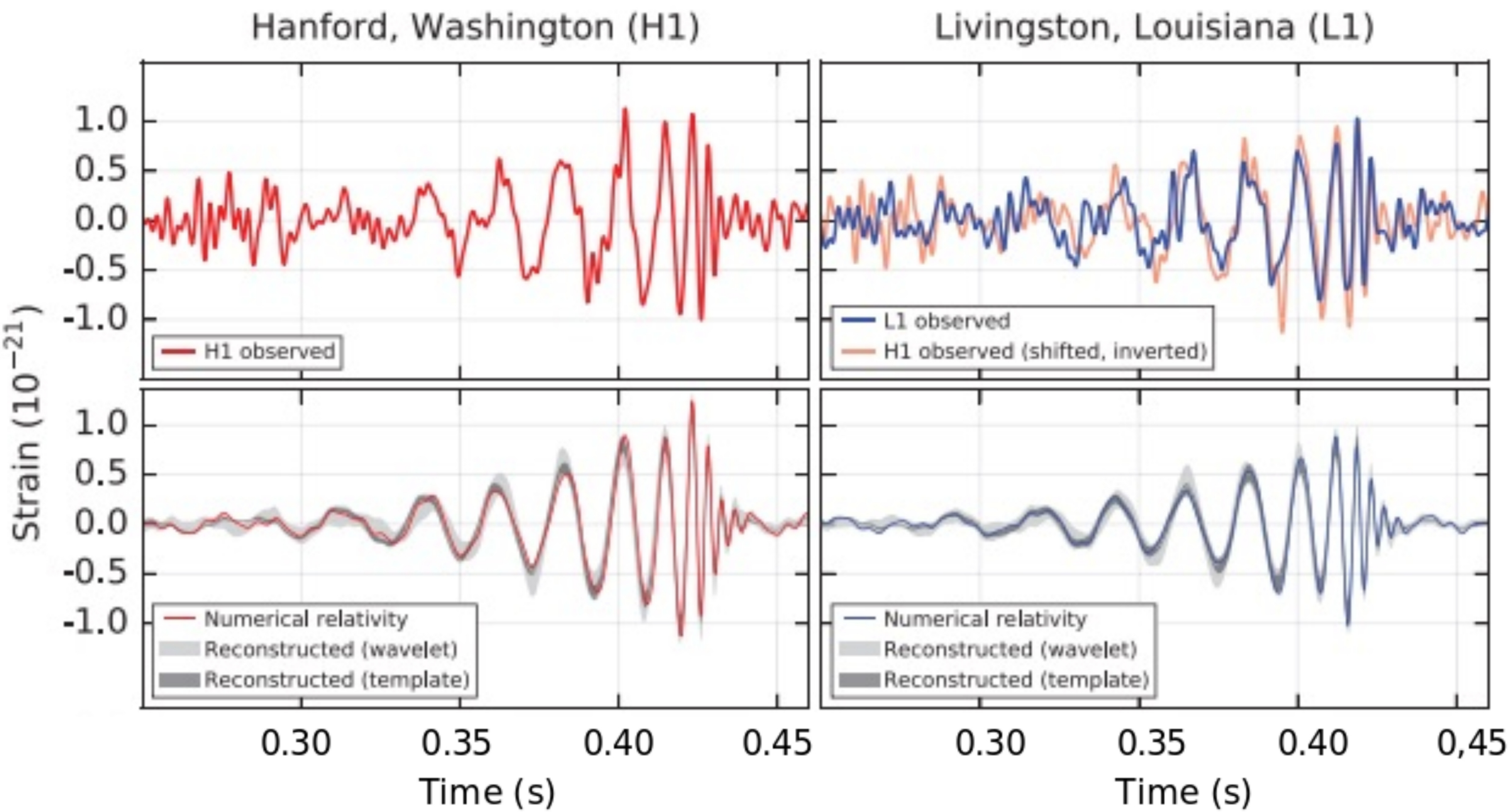
Simple way to estimate frequency at merger:
Last stable circular orbit around a non-spinning black hole

$$r_{\rm LSCO} = 6\,\frac{G\,(m_1+m_2)}{c^2}$$



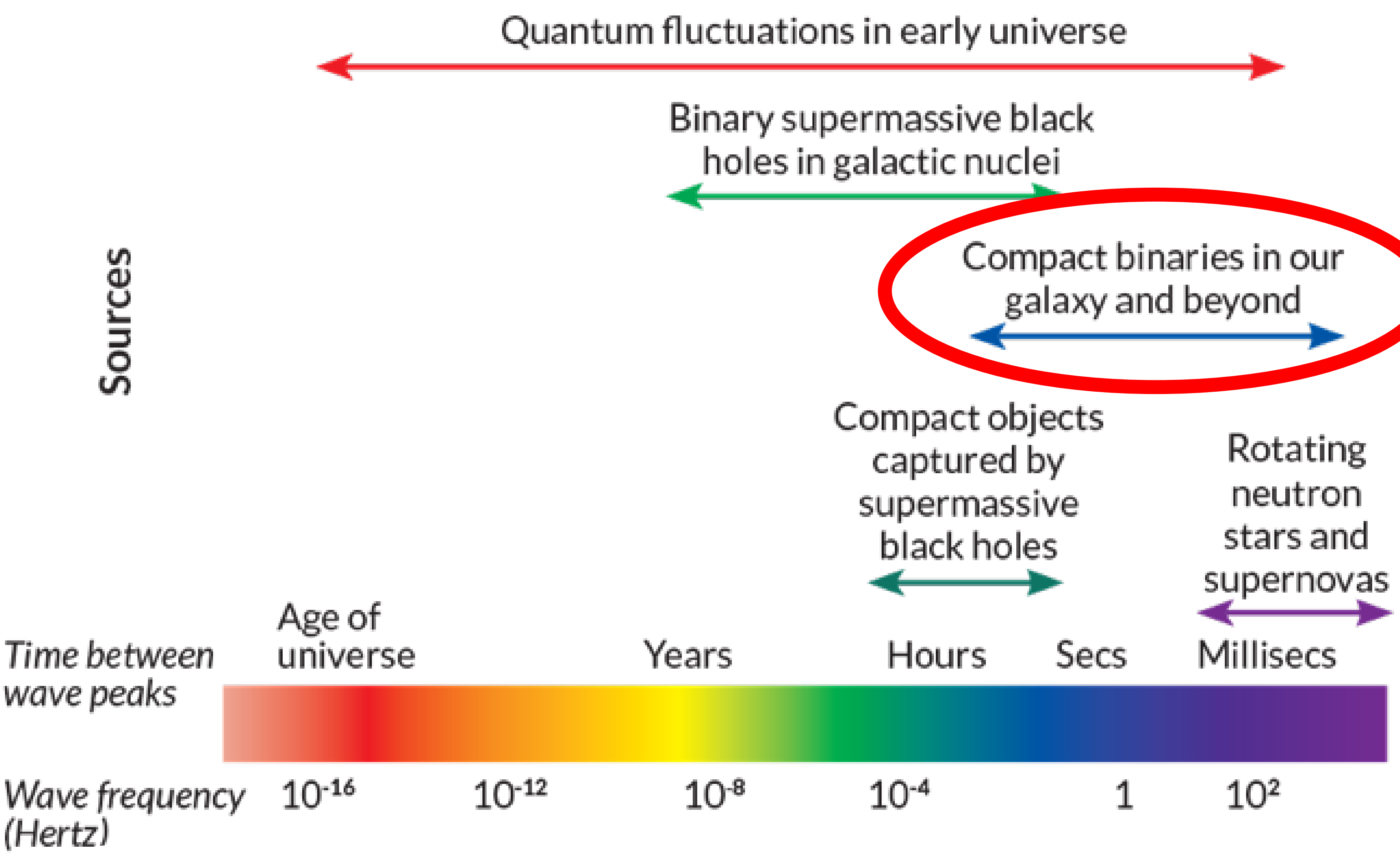
$$\omega_{\rm GW,LSCO} = 2\sqrt{\frac{G\,(m_1+m_2)}{r^3_{\rm LSCO}}} = \frac{2\,c^3}{6^{3/2}\,G\,(m_1+m_2)}$$

$$\boxed{\omega_{\rm GW,LSCO} = 460\,{\rm Hz}\,\frac{60\,M_{\rm sun}}{(m_1+m_2)}}$$



Abbott et al. 2016

What are the astrophysical objects with non-zero quadrupole?



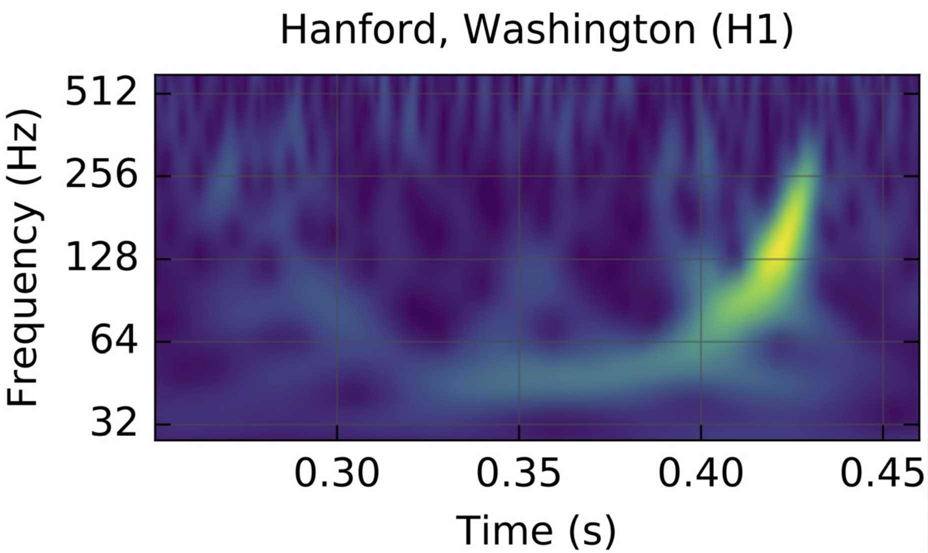
Compact binaries will be the focus of this course!

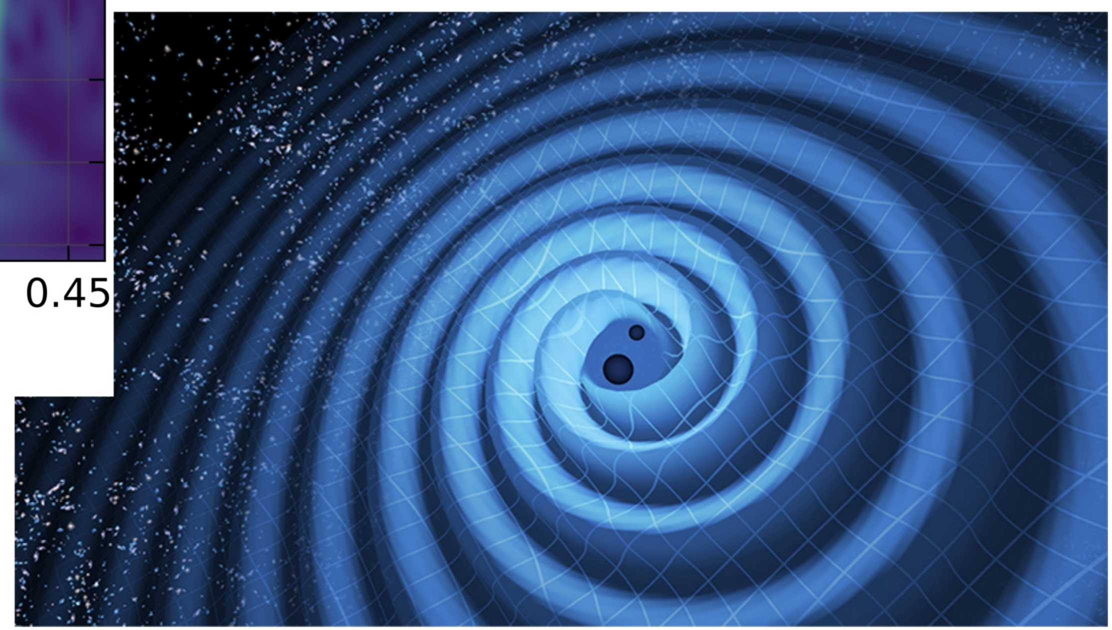
References:

* James B. Hartle, Gravity: An Introduction to Einstein's General Relativity, Pearson

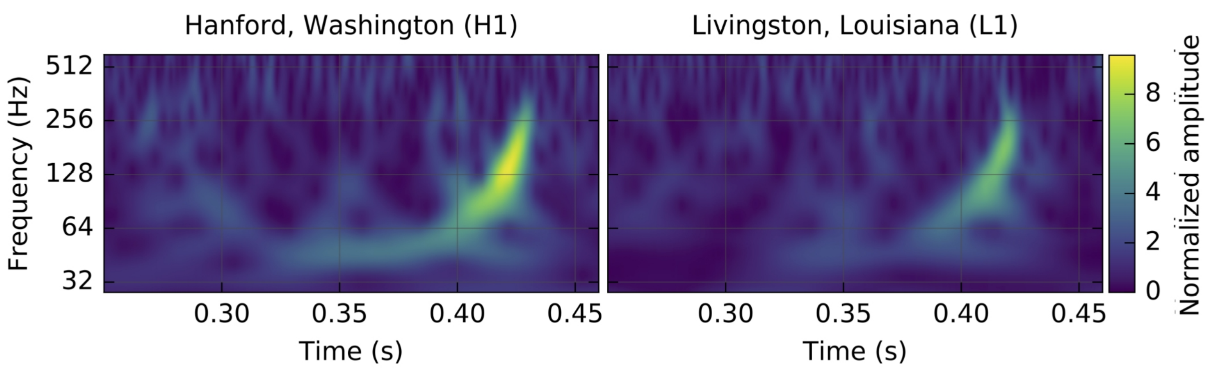
* Jolien D. E. Creighton & Warren G. Anderson, Gravitational-Wave Physics and Astronomy: An Introduction to Theory, Experiment and Data Analysis, Wiley Series in Cosmology

Observational Facts





GW150914: the first binary black hole (BBH)

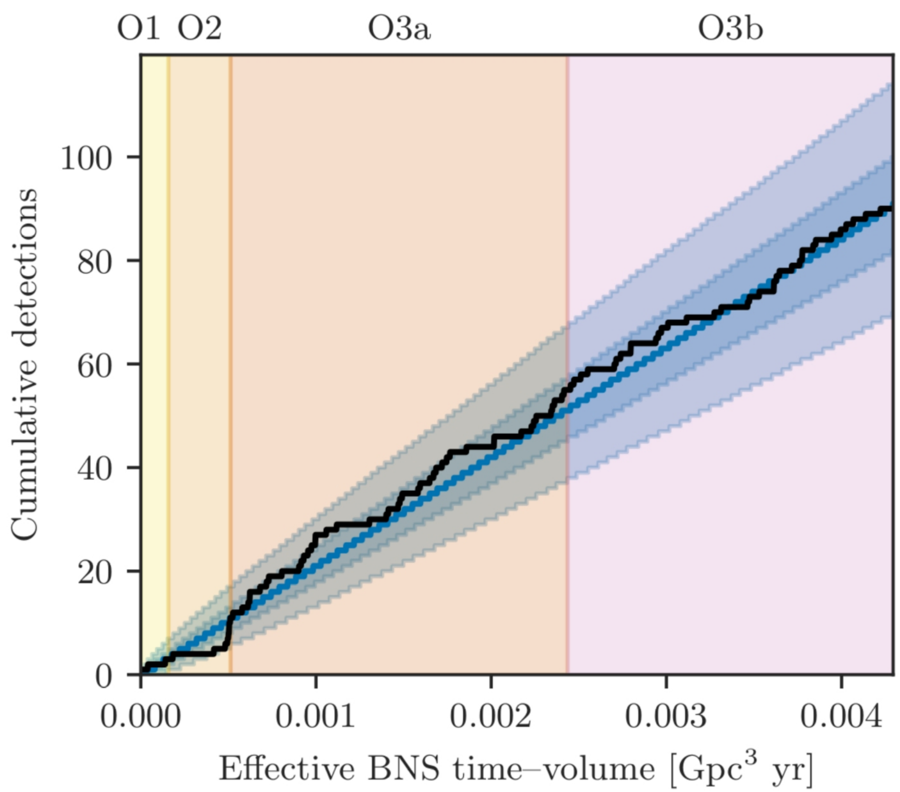


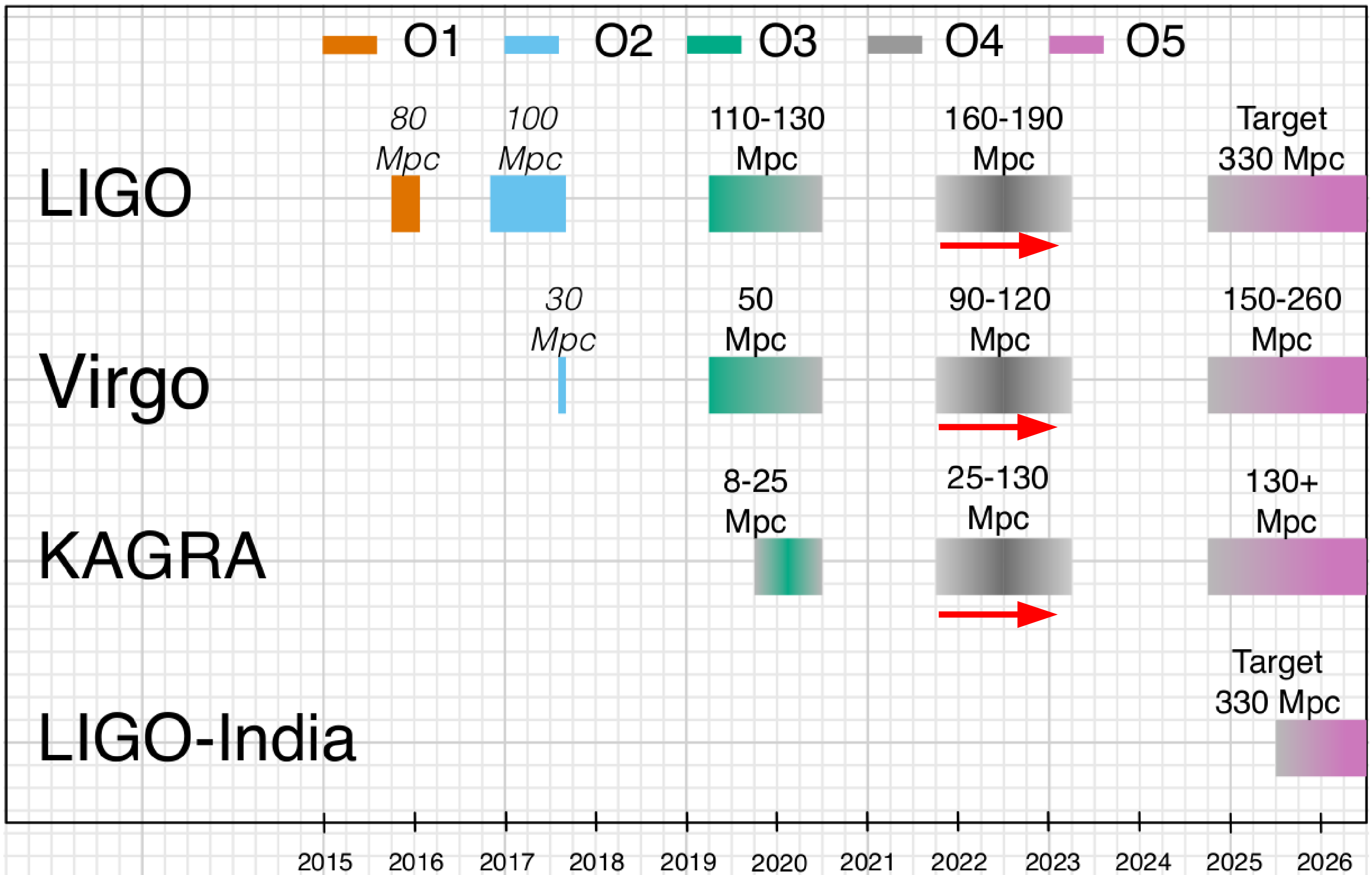
Abbott et al. 2016, PhRvL, 116, 1102

O1 + O2 + O3:

90 GW event candidates
most of them BBHs

(Abbott et al. 2021, GWTC-2;
Abbott et al. 2022, GWTC-2.1;
Abbott et al. 2022, GWTC-3)





See Abbott et al. 2019, Observing scenarios paper
https://dcc.ligo.org/LIGO-P1200087-V57/public

15 Observables (+ eccentricity):

- 2 masses
- 6 spin components
- redshift of merger
- polarization
- inclination of binary wrt interferometers
- 2 sky positions (RA, DEC)
- reference time
- phase at a reference time

Mass and Spins:

black holes (BHs) are uniquely defined by mass & spin
(electric charge deemed to be negligible)

Masses:

If black hole (BH) is in binary system, we have 2 masses:

m_1, m_2 = Mass of first BH, Mass of second BH

but LIGO-Virgo measured two combinations of m_1, m_2 :

Chirp mass:

$$\boxed{m_{\rm chirp} = \frac{(m_1\, m_2)^{3/5}}{(m_1 + m_2)^{1/5}}}$$

change of frequency during inspiral scales with it

Total mass:

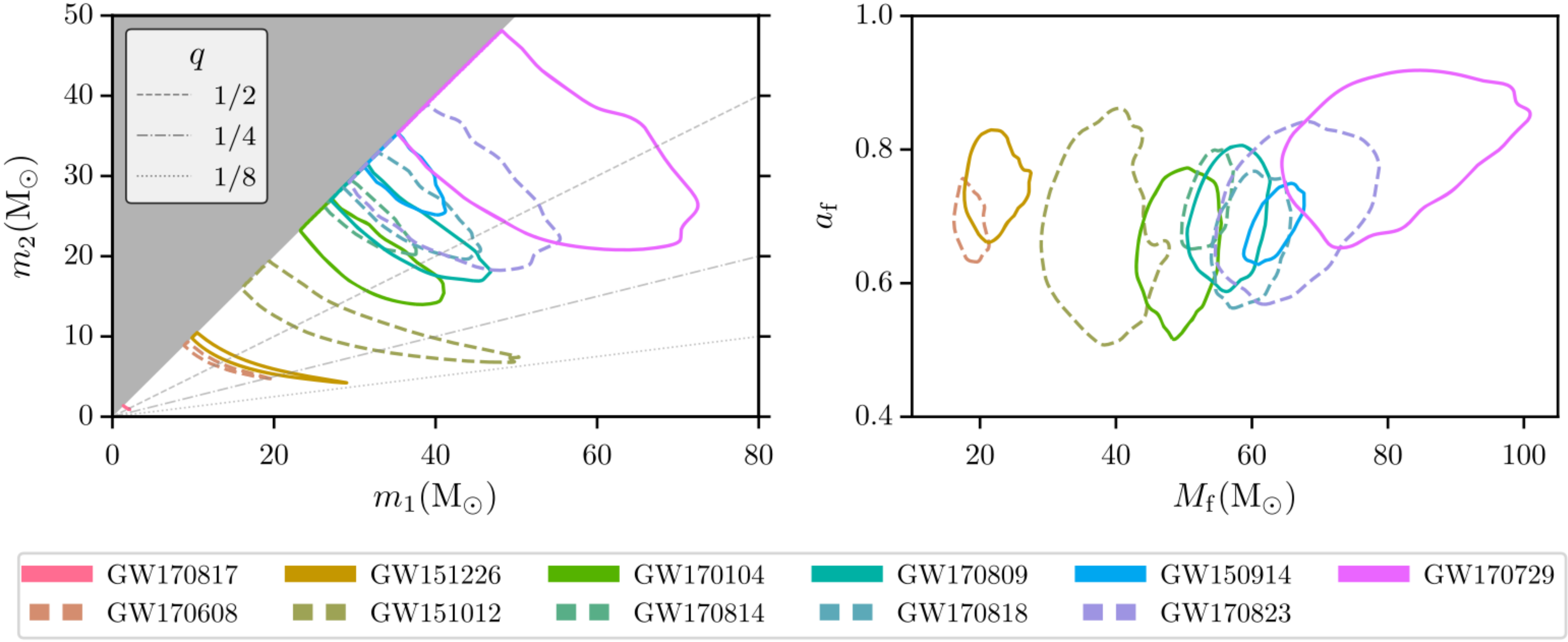
$$\boxed{M = (m_1 + m_2)}$$

frequency at merger scales with it

Other relevant mass (for phase of GWs): mass ratio

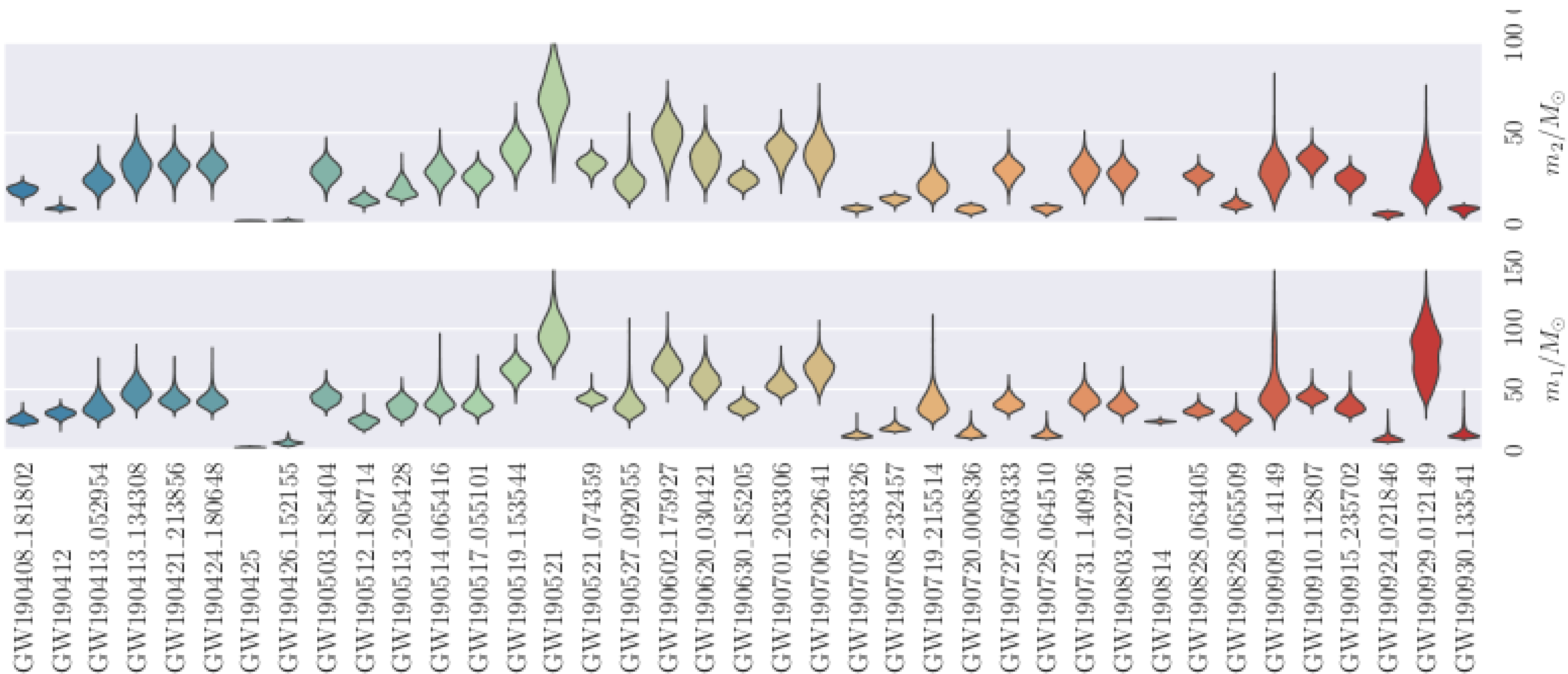
$$\boxed{q = \frac{m_2}{m_1}}$$

Masses in the first GW transient catalog (GWTC-1):



Abbott et al. 2019, GWTC1, https://ui.adsabs.harvard.edu/abs/2019PhRvX...9c1040A/

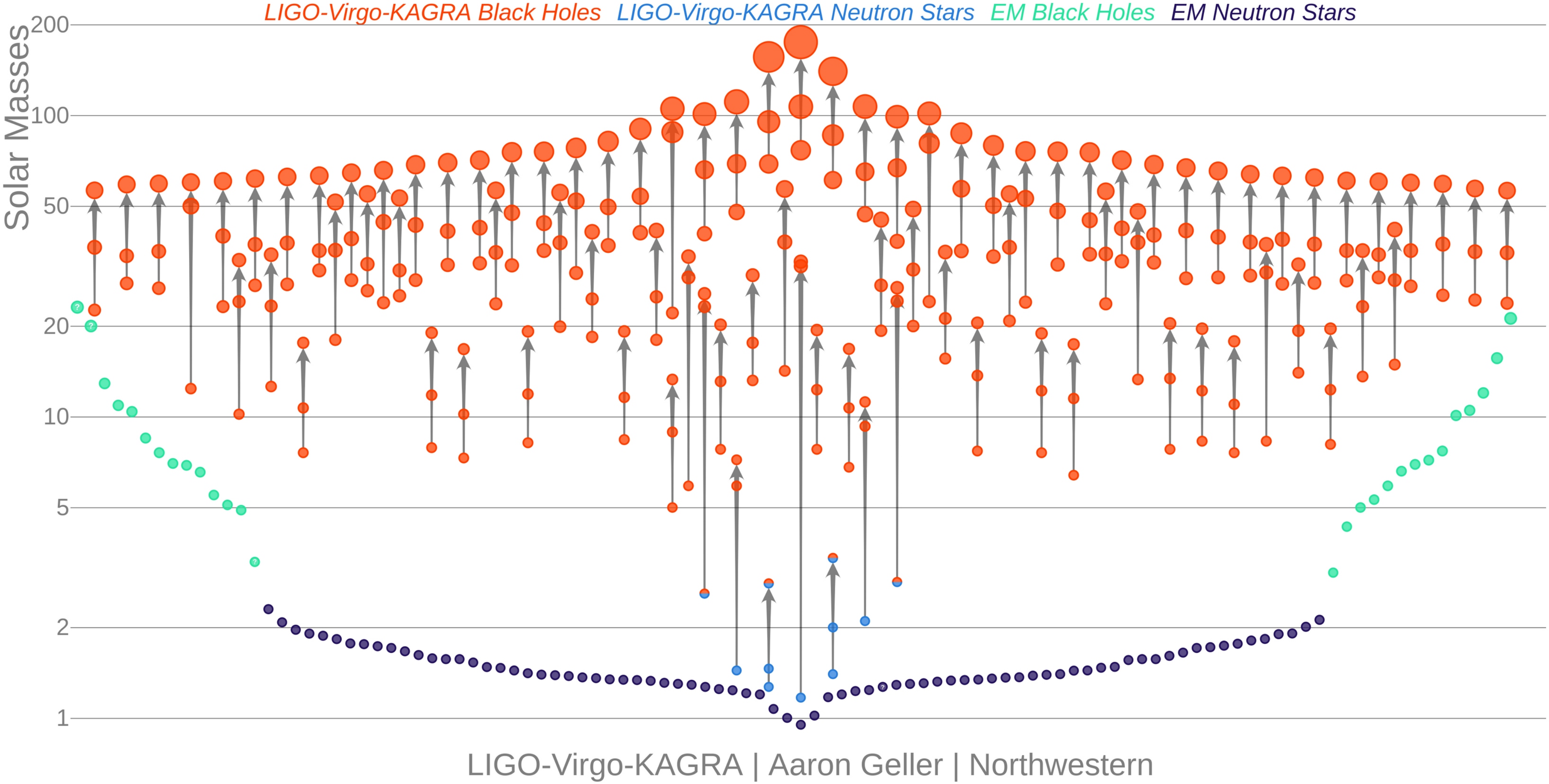
Masses in the second GW transient catalog (GWTC-2):



Masses of the 39 new event candidates in O3a

Abbott et al. 2020, GWTC-2, 2020, https://arxiv.org/abs/2010.14527

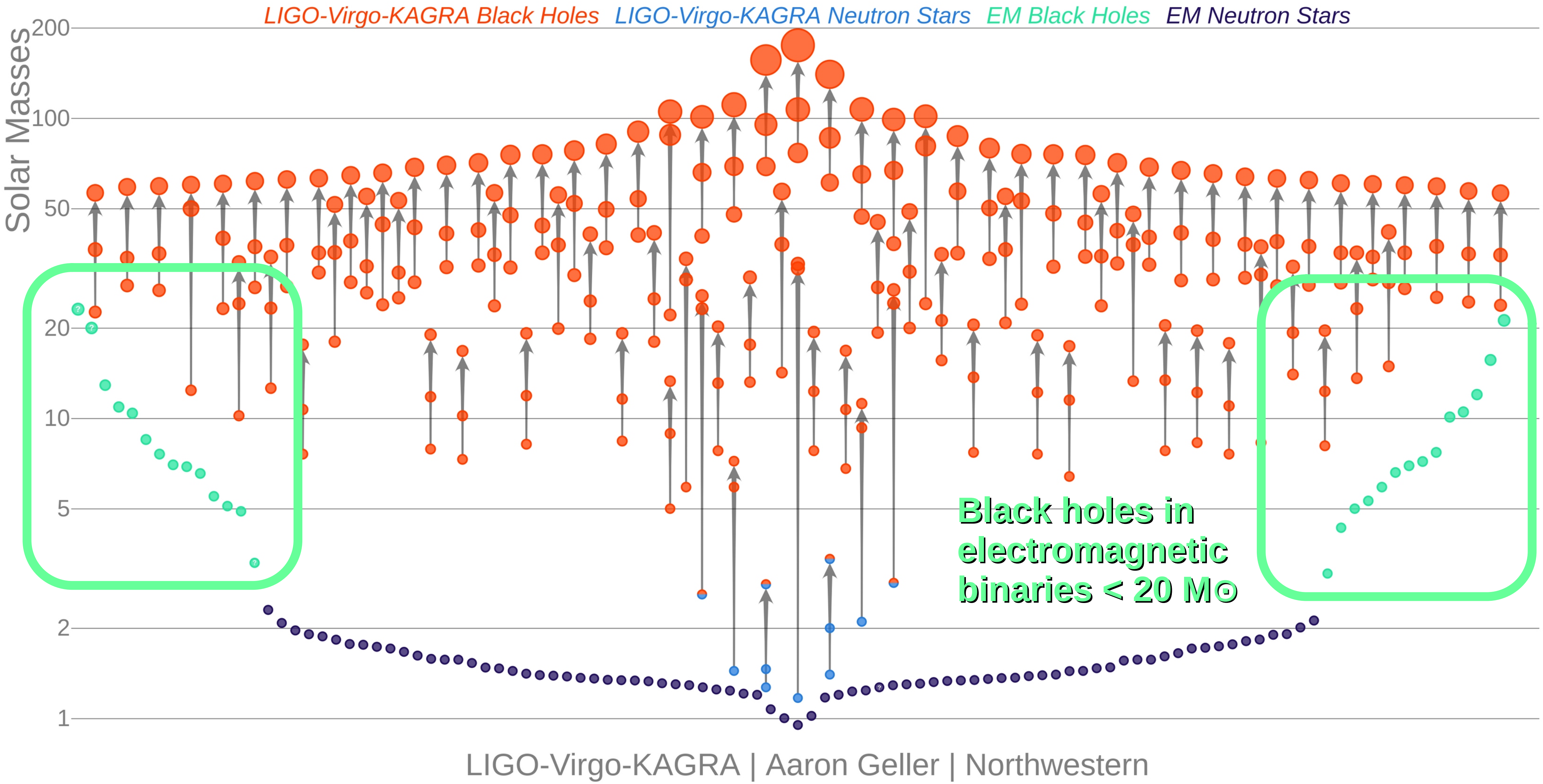
Masses in the Stellar Graveyard



Abbott et al. 2022, GWTC-3

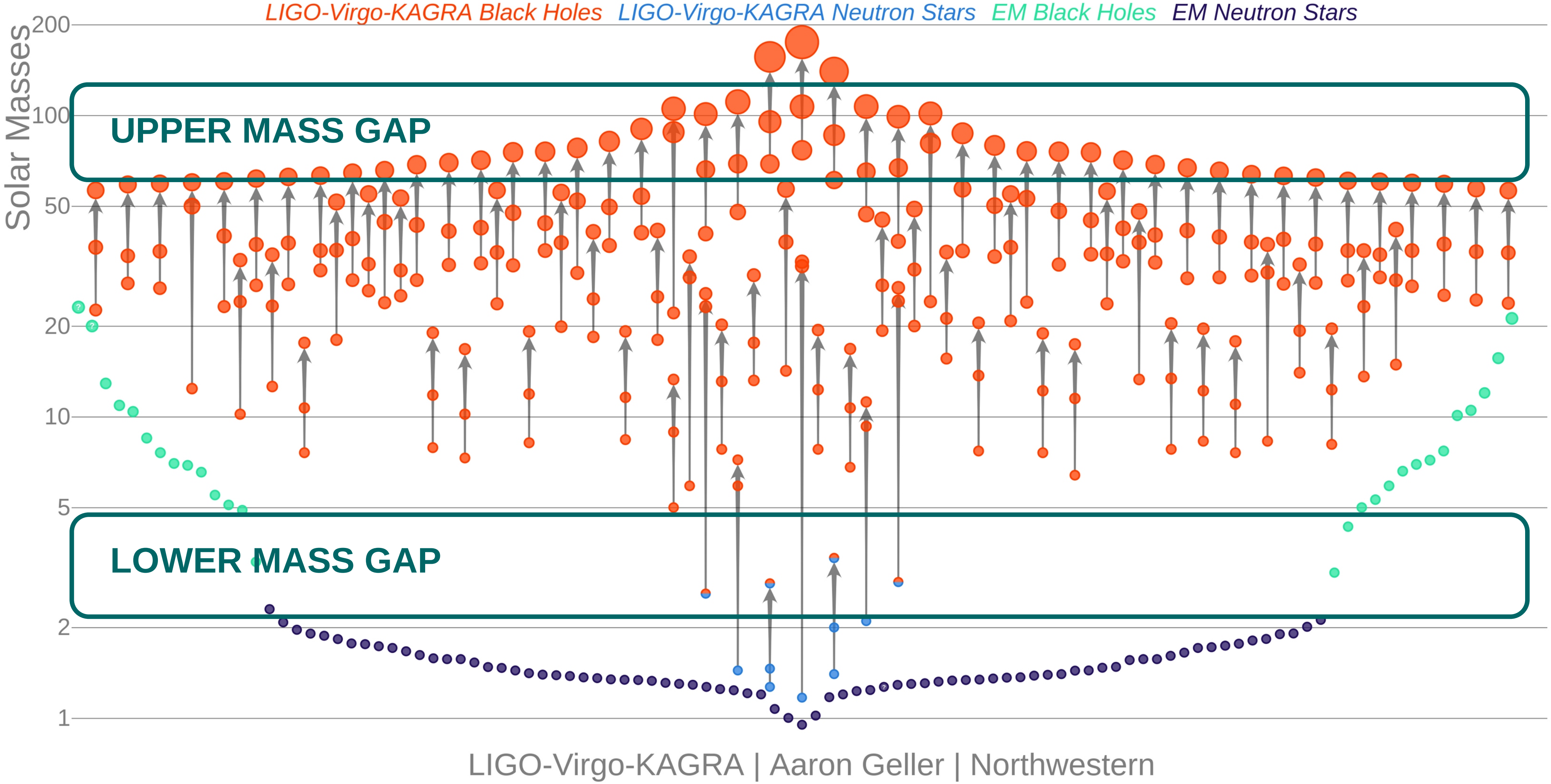
Michela Mapelli

Masses in the Stellar Graveyard



Abbott et al. 2022, GWTC-3

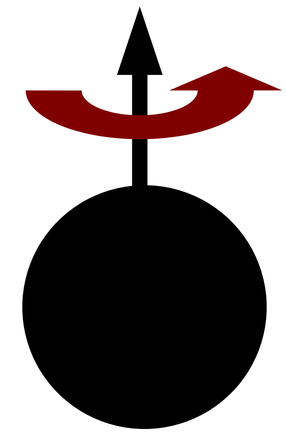
Masses in the Stellar Graveyard



Abbott et al. 2022, GWTC-3

Spins:

$$\vec{\chi} = \frac{\vec{J}\,c}{G\,m_{\rm BH}^2}$$

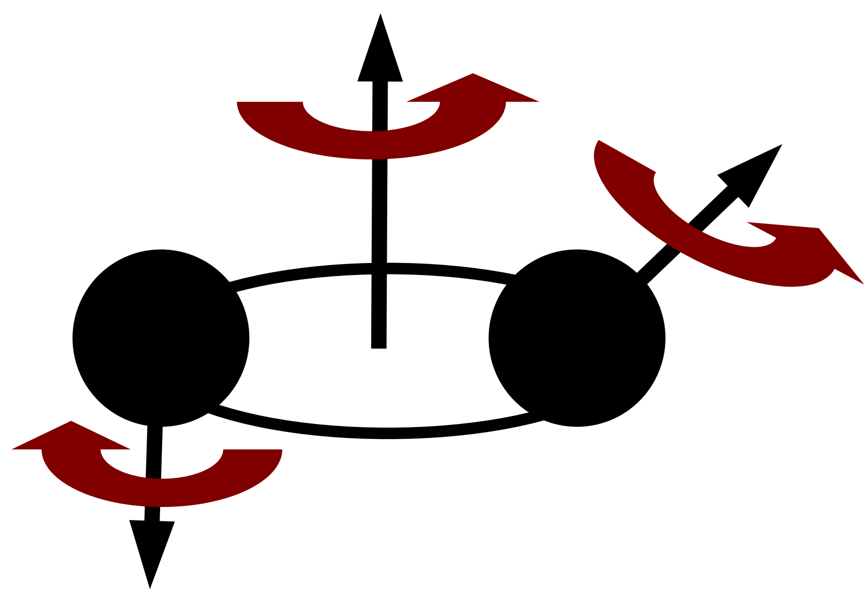


$$\chi \;= 0 \;\rightarrow \mathrm{Schwarzschild\,BH}$$

$$\chi = 0.998 \;\rightarrow \mathrm{Maximally\,Rotating\,BH}$$

If black hole is in binary
we have 6 spin components:

3 per each black hole



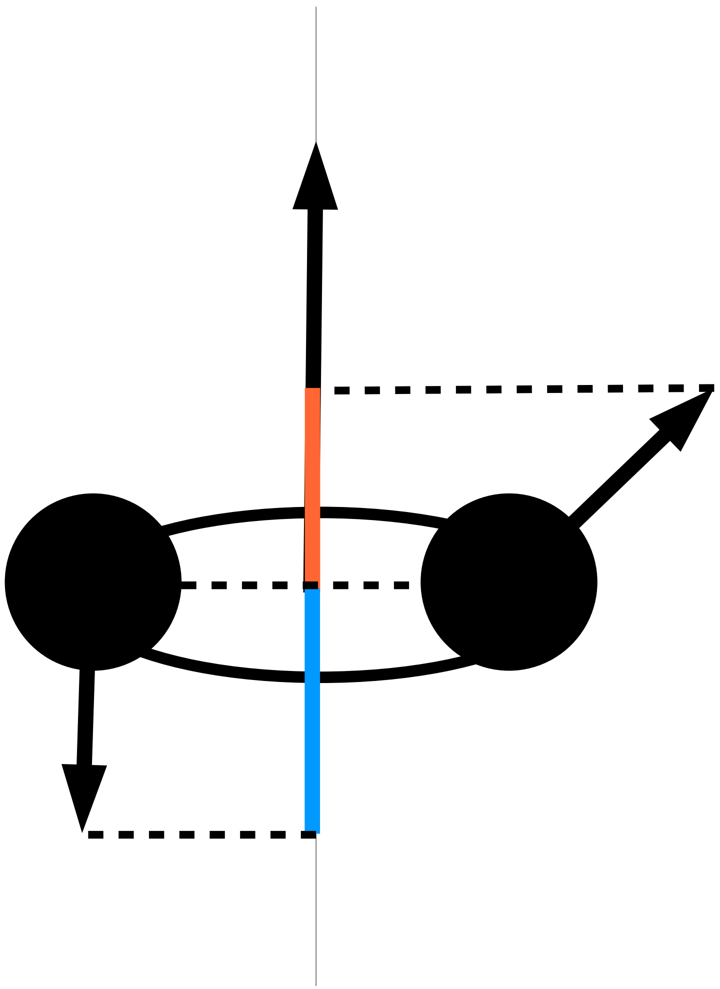
Spins:

LIGO – Virgo not enough to measure 6 spins, hence alternative params

1. EFFECTIVE SPIN:

$$\chi_{\rm eff} = \frac{(m_1\,\vec{\chi}_1 + m_2\,\vec{\chi}_2)}{(m_1 + m_2)} \cdot \frac{\vec{L}}{L}$$

$$-1 \leq \chi_{\rm eff} \leq 1$$



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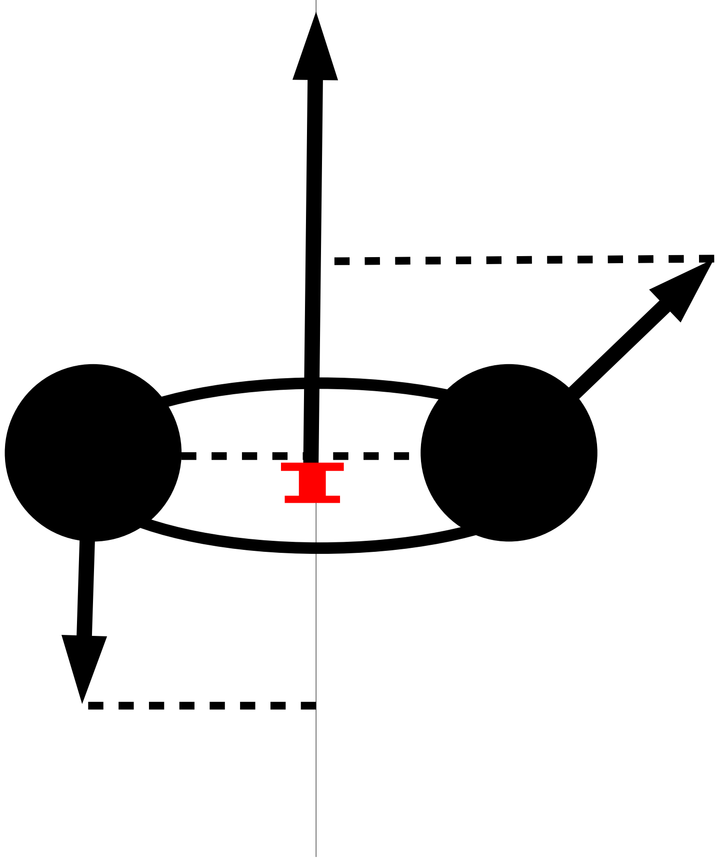
$$\chi_{\rm eff} = \frac{(m_1\,\vec{\chi}_1 + m_2\,\vec{\chi}_2)}{(m_1 + m_2)} \cdot \frac{\vec{L}}{L}$$

BBH orbital angular momentum

$$-1 \le \chi_{\rm eff} \le 1$$

Measured because affects phase of GWs
while orthogonal spin to binary ang. mom.
measures precession



Spins:

LIGO – Virgo not enough to measure 6 spins, hence alternative params

1. EFFECTIVE SPIN

$$\chi_{\rm eff} = \frac{(m_1\,\vec{\chi}_1 + m_2\,\vec{\chi}_2)}{(m_1+m_2)} \cdot \frac{\vec{L}}{L}$$

Components of spins
perpendicular to BBH
orbital angular momentum

2. Effective precession spin:

$$\chi_{\rm p} = \frac{1}{B_1\,m_1^2}\,\max\left(B_1\,\chi_{1,\perp},\; B_2\,\chi_{2,\perp}\right)$$

$$B_1 = 2 + 3\,\frac{q}{2} \qquad B_2 = 2 + \frac{3}{2\,q}$$

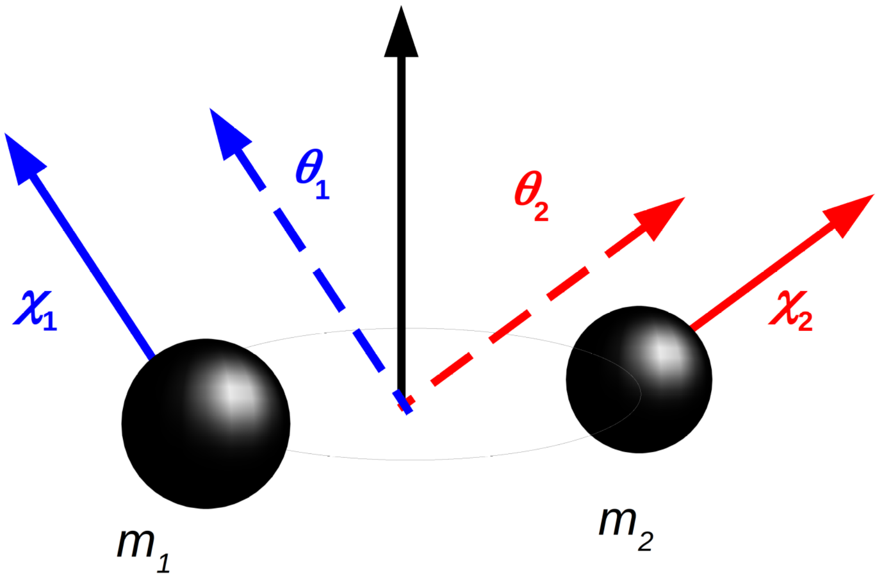
Spin components perpendicular to binary ang. mom.
measured by precession

Spins:

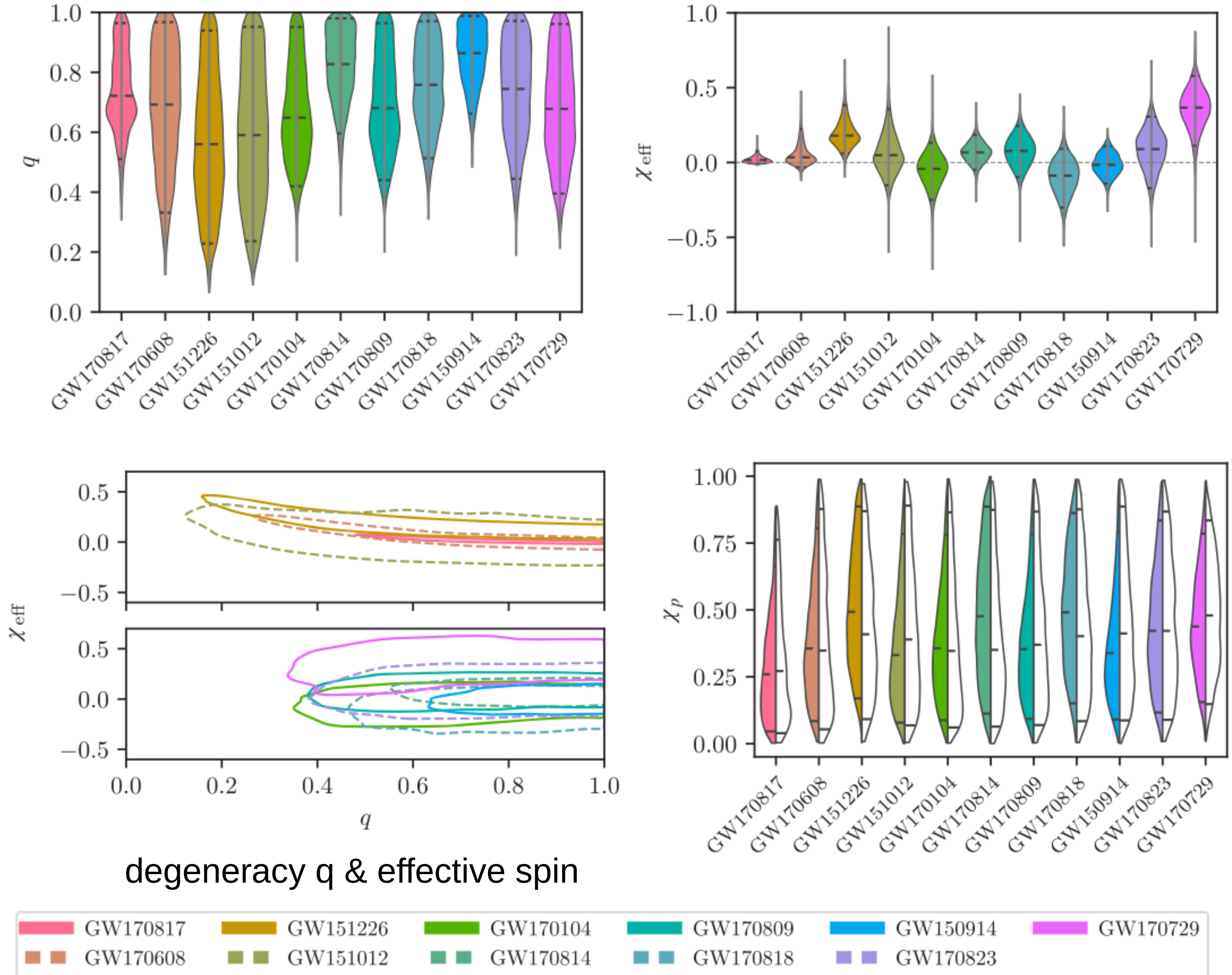
LIGO – Virgo not enough to measure 6 spins, hence alternative params

3. SPIN MAGNITUDE χ

4. Spin TILT $\cos\theta$



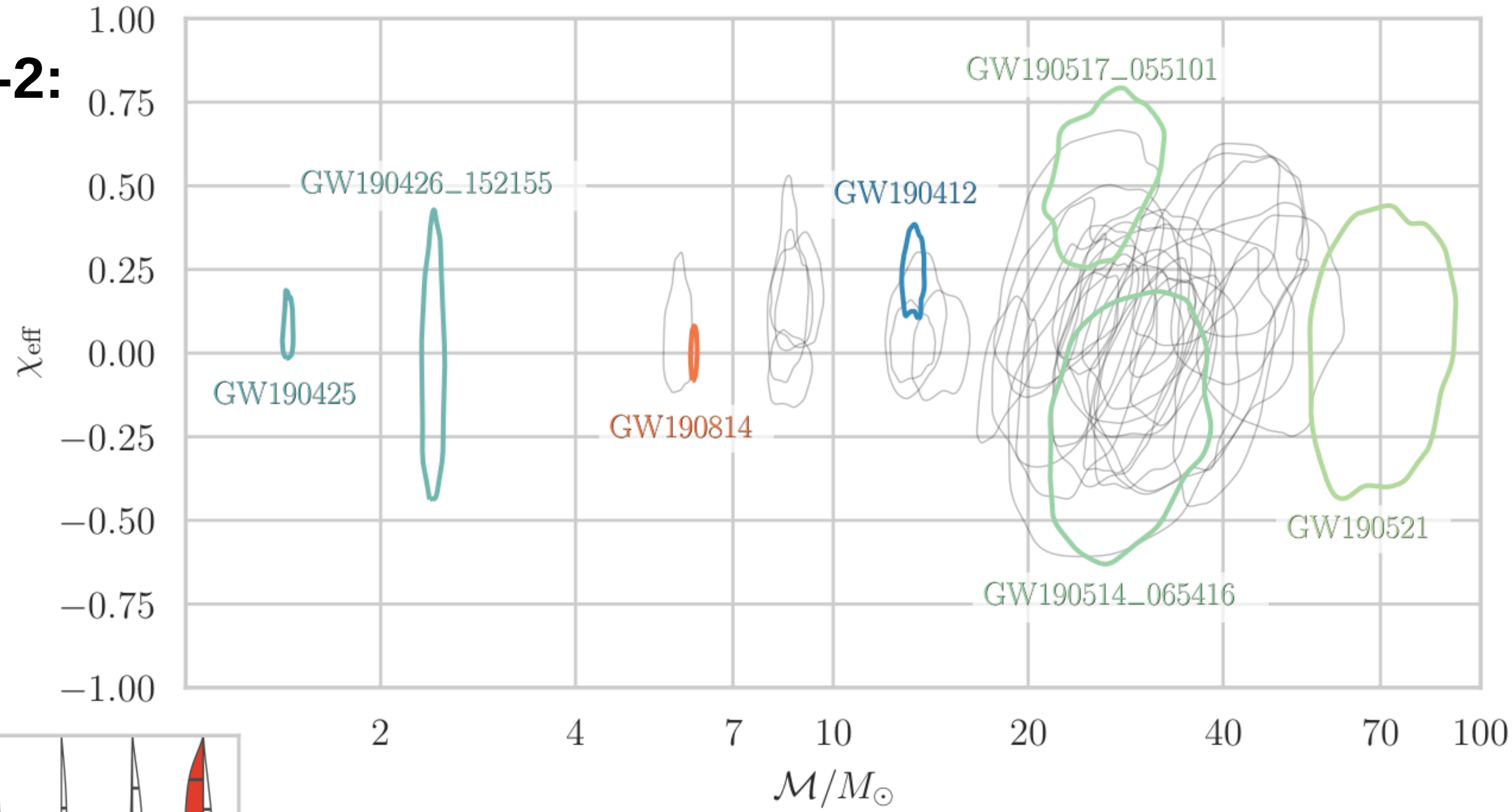
Mass ratios and spins in GWTC-1:

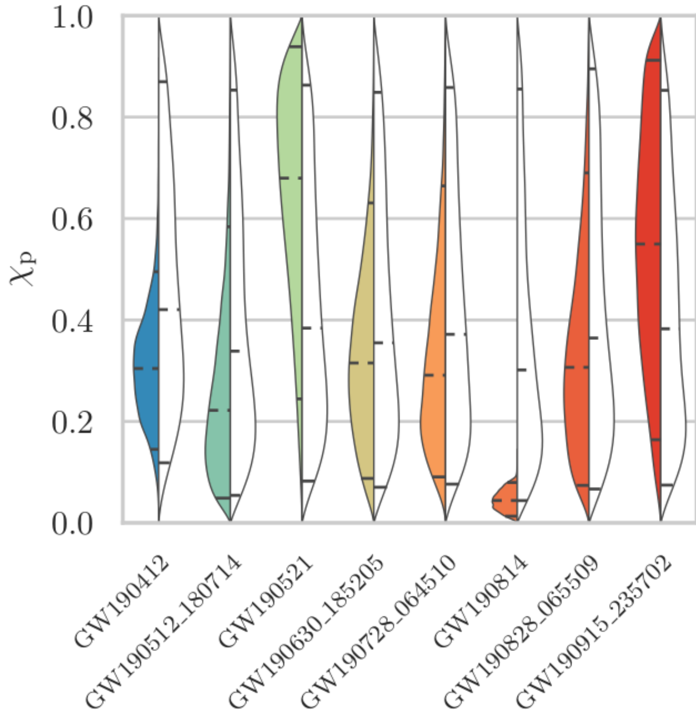


degeneracy q & effective spin

Abbott et al. 2019, GWTC1, https://ui.adsabs.harvard.edu/abs/2019PhRvX...9c1040A/

Spins in GWTC-2:





Abbott et al. 2020, GWTC-2, 2020,
https://arxiv.org/abs/2010.14527

Now it is time to remove observational biases: a population perspective

1) The data { *d* }

Bayesian framework (LALInference, Veitch et al. 2015)
to extract posterior samples of **parameters of each GW event (e.g., masses, spins)**

2) Simple astrophysical models $\pi(\theta|\Lambda)$

e.g. for primary mass a **power law + Gaussian**
with parameters θ (e.g., mass)
and hyper-parameters Λ (e.g., slope of power law)
to be fitted to the data

3) Hierarchical Bayesian approach

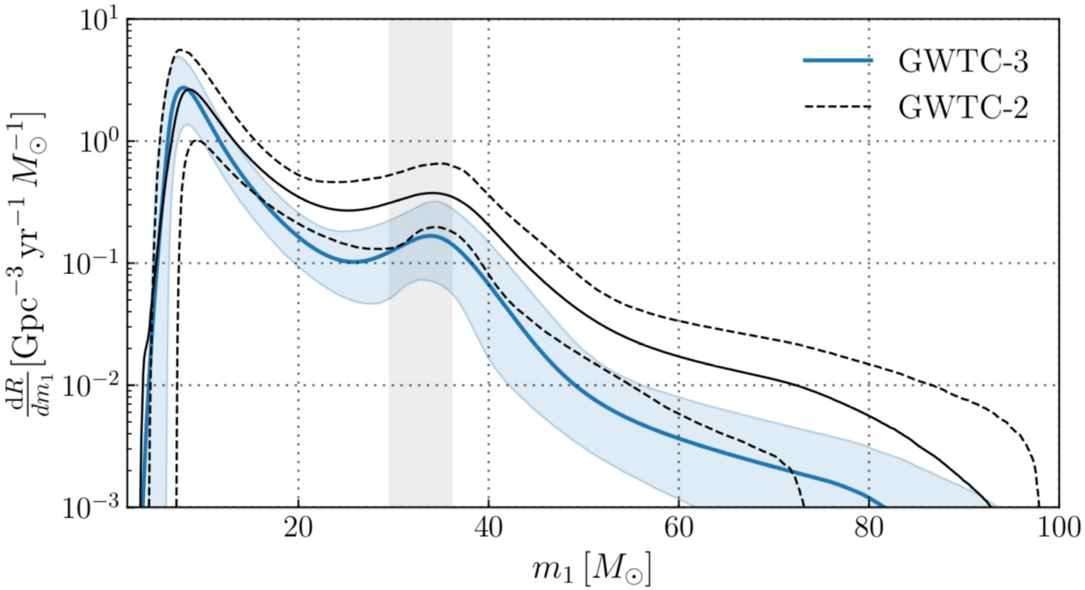
to measure the parameters θ of the population models
marginalizing over the properties of individual GW events

$$\mathcal{L}(\{d\}|\Lambda, N) \propto \underbrace{N^{N_{\rm det}} e^{-N\xi(\Lambda)}}_{\text{expected number of detections: encodes observational bias}} \prod_{i=1}^{N_{\rm det}} \int \underbrace{\mathcal{L}(d_i|\theta)}_{\text{single event likelihood}} \underbrace{\pi(\theta|\Lambda)}_{\text{PRIOR: astrophysical model}} d\theta$$

In practice, we use a log-uniform prior on *N* and then average over discrete samples

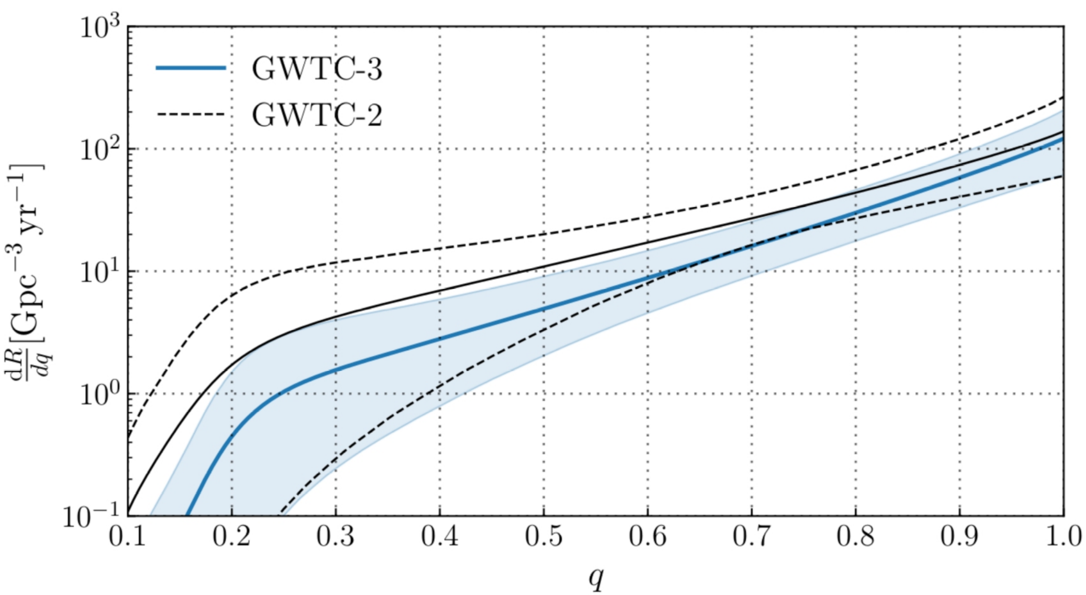
$$\mathcal{L}(\{d\}|\Lambda) \propto \Pi_{i=1}^{N_{det}} \frac{\int \mathcal{L}(d_i|\theta)\, \pi(\theta|\Lambda)\, d\theta}{\xi(\Lambda)} \propto \Pi_{i=1}^{N_{det}} \frac{1}{\xi(\Lambda)} \langle \frac{\pi(\theta|\Lambda)}{\pi_{\rm def}(\theta)} \rangle$$

Masses in GWTC-2 and GWTC-3 from a population study:



Reconstructed mass function (removing observational biases, assuming an astrophysical model):

possible peaks at ~ 10, 35 M⊙
tail up to ~ 80 M⊙

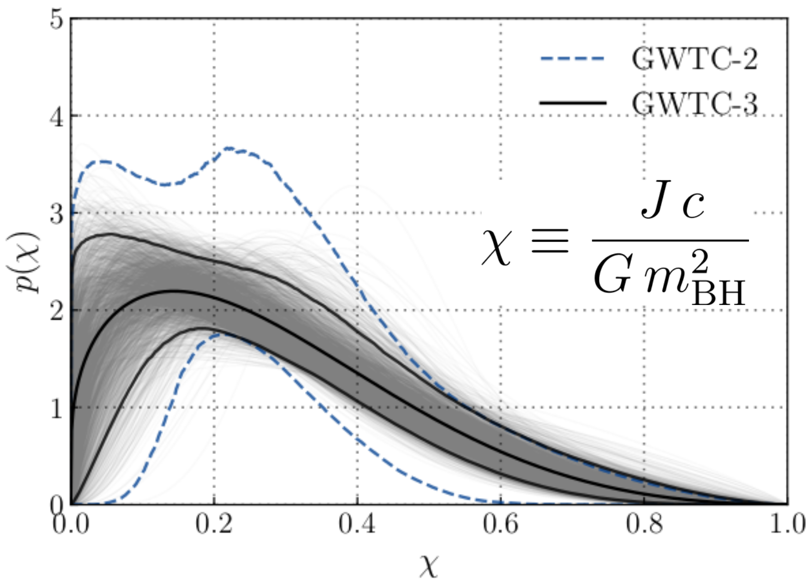


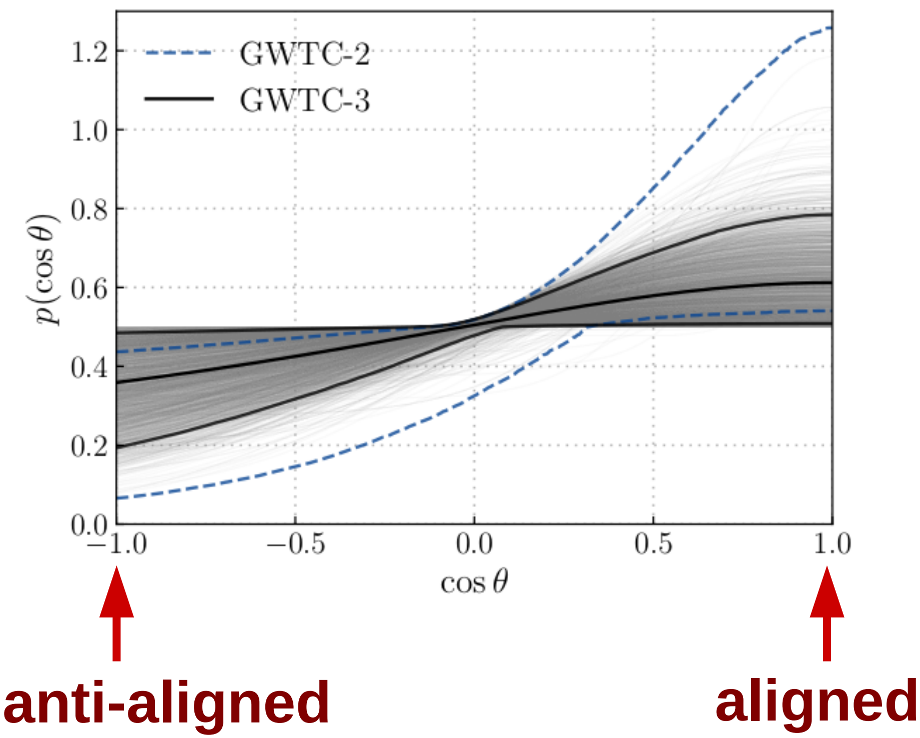
Reconstructed mass ratio (removing observational biases, assuming an astrophysical model):

preference for equal-mass systems

Abbott et al. 2022, population

Spins in GWTC-2 and GWTC-3 from a population study:





Reconstructed spin magnitude:

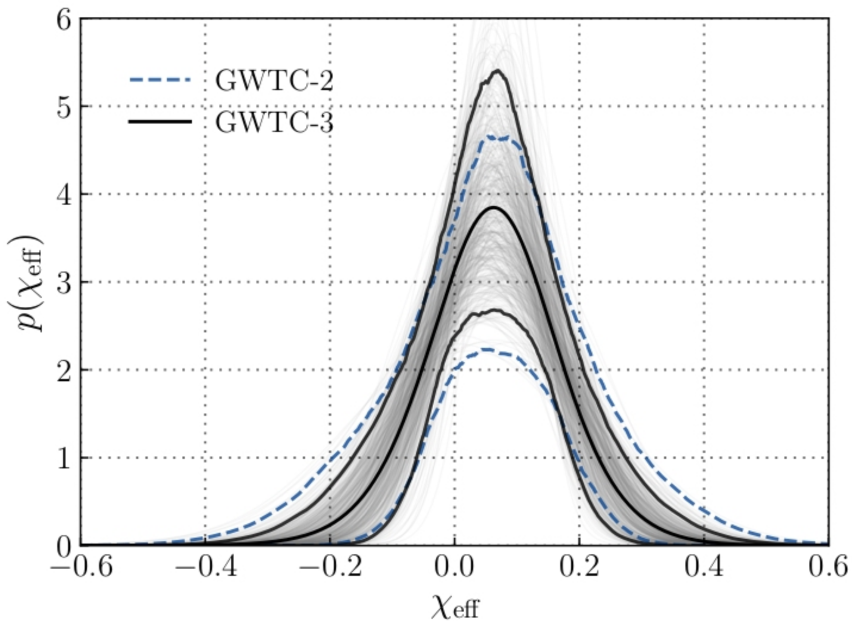
preference for **relatively low spins** with long tail

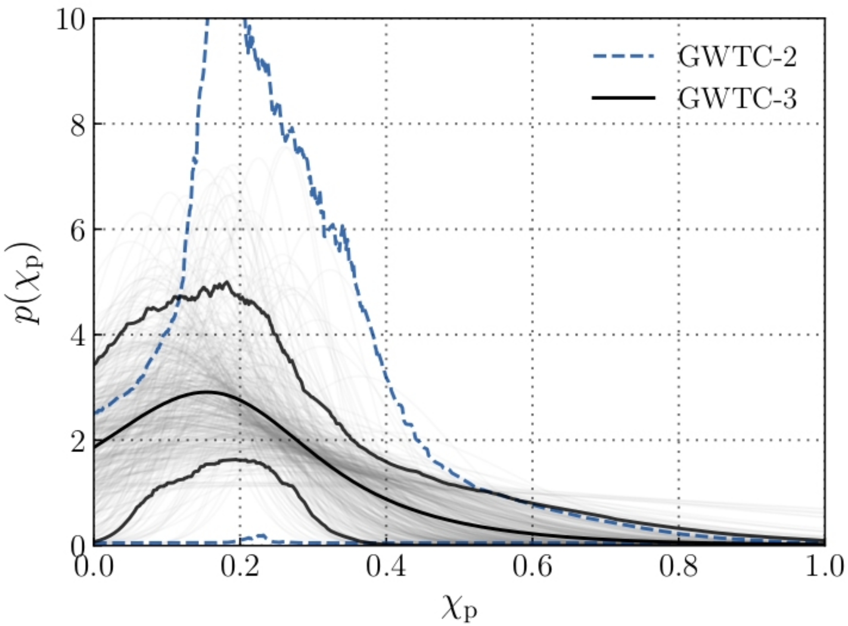
Reconstructed spin-orbit tilt:

mild preference for **aligned systems**

Abbott et al. 2022, population

Spins in GWTC-2 and GWTC-3 from a population study:





Reconstructed χ_{eff}:

preference for **positive values**

Reconstructed χ_p:

preference for **small values**
(still poor reconstruction)

Abbott et al. 2022, population

Rates from GWTC-1:

BNS Rate = 110 – 3840 Gpc^{-3} yr^{-1}

OLD

BHNS Upper Limit R < 610 Gpc^{-3} yr^{-1}

Abbott et al. 2019, GWTC1, https://ui.adsabs.harvard.edu/abs/2019PhRvX...9c1040A/

BBH Rate ~ 10 – 138 Gpc^{-3} yr^{-1}

Abbott et al. 2019, GWTC1, https://ui.adsabs.harvard.edu/abs/2019PhRvX...9c1040A/
Abbott et al. 2019, https://ui.adsabs.harvard.edu/abs/2019ApJ...882L..24A/

Intermediate-mass black holes (IMBHs) upper limit ~ 0.20 Gpc^{-3} yr^{-1}

Abbott et al. 2019, https://journals.aps.org/prd/abstract/10.1103/PhysRevD.100.064064

Note: every rate estimation requires assumptions about the population!

Rates from GWTC-2:

OLD

BNS Rate ~ 320 [+490, –240] ~ 80 – 810 Gpc^{-3} yr^{-1}

Abbott et al. 2020, https://ui.adsabs.harvard.edu/abs/2020arXiv201014533T/abstract

BHNS Upper Limit R < 610 Gpc^{-3} yr^{-1} (still the same as O2)

Abbott et al. 2019, GWTC1, https://ui.adsabs.harvard.edu/abs/2019PhRvX...9c1040A/

BBH Rate ~ 23.9 [+14.3, –8.6] ~ 15.3 – 38.2 Gpc^{-3} yr^{-1}
without GW190814

BBH Rate ~ 52 [+52, –26] ~ 26 – 104 Gpc^{-3} yr^{-1}
with GW190814

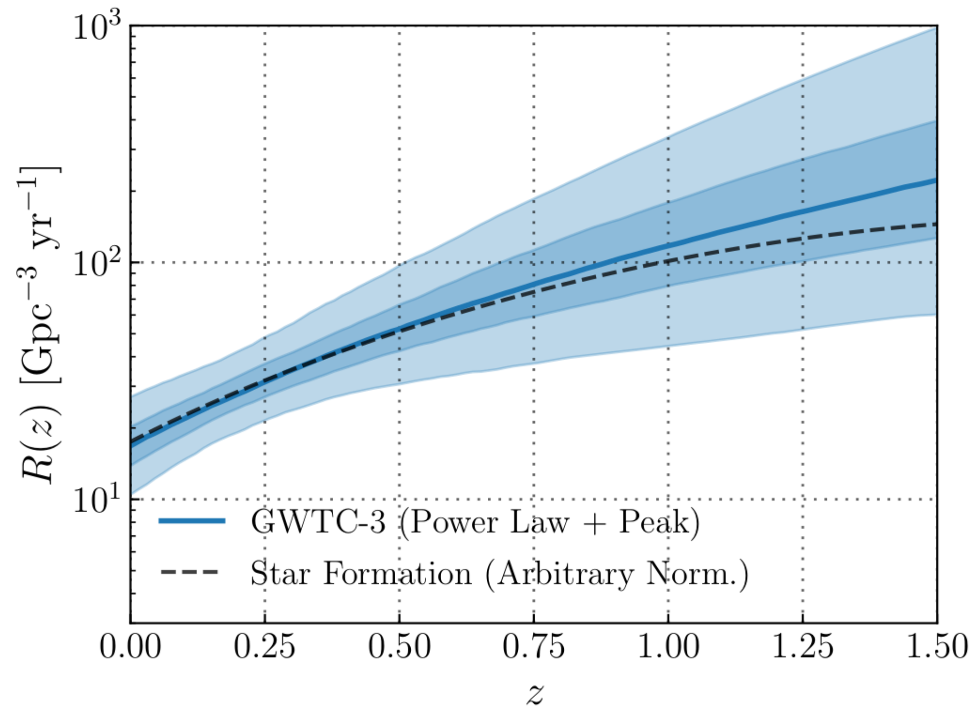
Abbott et al. 2020, https://ui.adsabs.harvard.edu/abs/2020arXiv201014533T/abstract

Note: every rate estimation requires assumptions about the population!

Rates from GWTC-3:

BNS Rate ~ 10 – 1700 Gpc^{-3} yr^{-1}

BHNS Rate ~ 8 – 140 Gpc^{-3} yr^{-1}



BBH Rate ~ 18 – 44 Gpc^{-3} yr^{-1}

if redshift evolution allowed

BBH Rate ~ 16 – 61 Gpc^{-3} yr^{-1}

otherwise

Abbott et al. 2022, population

Note: every rate estimation requires assumptions about the population!

References:

* Abbott et al. 2016,https://ui.adsabs.harvard.edu/abs/2016PhRvL.116f1102A/

* Abbott et al. 2016, https://ui.adsabs.harvard.edu/abs/2016ApJ...818L..22A/

* Abbott et al. 2019, GWTC1, https://ui.adsabs.harvard.edu/abs/2019PhRvX...9c1040A/

* Abbott et al. 2019, population from GWTC-1, https://ui.adsabs.harvard.edu/abs/2019ApJ...882L..24A/

* Abbott et al. 2019, Observing scenarios paper, https://dcc.ligo.org/LIGO-P1200087-V57/public

* Abbott et al. 2021, GWTC-2, https://arxiv.org/abs/2010.14527

* Abbott et al. 2021, population from GWTC-2, https://ui.adsabs.harvard.edu/abs/2020arXiv201014533T/abstract

* Abbott et al. 2022, GWTC-2.1

* Abbott et al. 2022, GWTC-3

* Abbott et al. 2022, populations of GWTC-3

* Public alert webpage (GRACEDB): https://gracedb.ligo.org/superevents/public/O3/