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Heston vs Black Scholes stock price modelling

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Abstract

In this thesis the Black Scholes and the Heston stock prices are investigated and the models are compared. The Black Scholes model assumes that the volatility is constant, while the Heston model allows stochastic volatility which is more flexible and can perform better with empirical data. Both models are analysed and simulated, and the parameters are estimated based on empirical data of S&P 500. Results are based on simulations and characteristic functions which are presented with figures of probability density functions.

Acknowledgements

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I would also like to thank my family for the bounteous support throughout my schooling.

Contents

1 Introduction

In a well-functioning market, it is important to have fair trading that does not involve portfolios with sure profits.

The Black Scholes model is a continuous time model of stock prices built on a Brownian motion z which includes a constant parameter volatility σ describing the risk of the stock. The Black Scholes model uses a partial differential equation from which a formula of the option price can be derived, which can be applied to corporate liabilities such as common stocks, as follows from [1]. In order to estimate the parameters we use the normal property of the log returns, from which the characteristic function can be obtained. Finally we simulate the paths of the Black Scholes model.

The CIR model applies a rational asset pricing model to study the term structure of interest rates; it has a simple closed-form solution for bond prices which depends on observable variables, as stated in [2]. We analyse the CIR model since the variance in the Heston model follows the CIR process. We also simulate the process using the Euler method as well as the exact algorithm, and we present a figure comparing the two methods.

In the Heston model, a stock price S_t follows a Black Scholes type stochastic process while its stochastic variance v_t follows a CIR process, as stated in [4]. The model describes stock options and other derivatives and it has a closed-form solution. For the analysis of the Heston model we first explain its properties and present the figures in which we can observe the effect of the parameters. Then we continue to the derivation of the Heston PDE so that we can obtain the characteristic function.We estimate the parameters using the Maximum likelihood estimation followed by Atiya and Wall [5] and simulate the Heston model to show the graph of stock prices vs variance level.

Finally we compare the Black Scholes model to the Heston model by computing the probability density functions using the inverse Fourier transform of the characteristic functions. Another way in which we compare models is by obtaining probability densities form simulated paths and in the case of the Black Scholes model, by plotting the normal density function.

For the empirical data we will use *The Standard and Poor's 500* (S&P 500) which is a stock market index of 500 of the largest companies in the United States.

In §2 we discuss some basic results from stochastic calculus and financial mathematics in order to analyse the models. In §3 we describe the Black Scholes model and its properties and in §3.3 we estimate the parameters. In §4 we summarize some basic properties of the CIR model. In §5 we analyse the Heston model and in §5.3.1 we estimate its parameters, after which we continue with the simulation. Finally in §6 we compare the models and give some ideas for the future investigations.

2 Preliminaries

The most common way to describe the behaviour of a stock price S is to use a geometric Brownian motion described by the stochastic differential equation (SDE)

$$dS = \mu S dt + \sigma S dz. \tag{2.1}$$

Here, μ is the expected rate of the return of the stock (that is, expected *drift rate* divided by the stock price) and σ is the standard deviation of the stock price return per time unit t, which represents the risk, or *volatility*, of the stock price. The process z is a Brownian motion, as defined in [7], that is, a Gaussian process with continuous paths and with independent increments such that $z_0 = 0$ with probability 1, $E[z_t] = 0$, and $\mathrm{Var}[z_t - z_s] = t - s$ for all $0 \leq s \leq t$. In particular, $z_t - z_s \sim N(0, t-s)$ for $0 \leq s \leq t$ and for any two disjoint intervals, for example (t_1, t_2),(t_3, t_4) with $t_1 \leq t_2 \leq t_3 \leq t_4$, the increments $z_{t_2} - z_{t_1}$ and $z_{t_4} - z_{t_3}$ are independent. The Brownian motion z is also referred to as a Wiener process, and it is a type of a Markov process with a mean change of zero and variance rate one per time unit. A Markov process is in turn a stochastic process which only uses the current value of the variable to obtain the future value. In case of a discrete time approximation, which we will use when programming, we have

$$\Delta S = \mu S \Delta t + \sigma S \epsilon \sqrt{\Delta t},$$

where ΔS is the change in the stock price S in the time interval Δt, and ϵ is a random number with standard normal distribution. The smaller the Δt is, the better an approximation of a process is, and as Δt tends to zero, we obtain (2.1). We can also write it as

$$\frac{\Delta S}{S} = \mu \Delta t + \sigma \epsilon \sqrt{\Delta t},$$

where the right hand side of the equation is a discrete approximation of the return provided by the stock in a short period of time Δt, $\mu \Delta t$ is the expected value of the return and $\sigma \epsilon \sqrt{\Delta t}$ is the stochastic component of the return, as stated in [6]. In order to compare the Black Scholes with the Heston models we use returns of stock prices, which show the profit or loss of an investment during a short time.

Another important result in stochastic calculus is Itô's lemma.

Theorem 1 (Itô's lemma) *Suppose that the value of a stochastic process $u = \{u(t)\}_{t\geq 0}$ satisfies the SDE*

$$du = a(u,t)dt + b(u,t)dz$$

where z is a Wiener process and a and b are functions of u and t. Assume G is a function of u and t, twice continuously differentiable in u and once continuously differentiable in t. Then $x = G(t, u(t))$ satisfies

$$dx = \left(\frac{\partial G}{\partial u}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2\right)dt + \frac{\partial G}{\partial x}bdz.$$

This means that x has a drift rate of $\frac{\partial G}{\partial u}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial u^2}b^2$ and a variance rate of $\left(\frac{\partial G}{\partial u}\right)^2 b^2$ which represents uncertainty.

Let S_t be given by (2.1) and $x_t = \ln(S_t)$. Using Itô 's lemma, we obtain

$$dx_t = \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma dz_t. \tag{2.2}$$

This means that $\ln(S_t)$ is a so called linear Brownian motion with uncertainty parameter σ and drift $\mu - \frac{\sigma^2}{2}$. The change in $\ln(S_t)$ from time 0 to T follows a normal distribution with mean $(\mu - \sigma^2/2)T$ and variance $\sigma^2 T$, that is

$$\ln\left(\frac{S_T}{S_0}\right) \sim N\left(\left(\mu - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right). \tag{2.3}$$

The exact formula for the log return during the time interval $[t, t+\Delta t]$ is

$$\ln\left(\frac{S_{t+\Delta t}}{S_t}\right) \sim N\left(\left(\mu - \frac{\sigma^2}{2}\right)\Delta t, \sigma^2 \Delta t\right). \tag{2.4}$$

We have described a stochastic process for a single variable which will help us understand the Black Scholes model, but to understand the Heston model we consider correlated processes. Let

$$dx_1 = a_1 dt + b_1 dz_1 \quad \text{and} \quad dx_2 = a_2 dt + b_2 dz_2$$

where dz_1 and dz_2 are Wiener processes with the correlation ρ. In the discrete approximation we have

$$\Delta x_1 = a_1 \Delta t + b_1 \epsilon_1 \sqrt{\Delta t} \quad \text{and} \quad \Delta x_2 = a_2 \Delta t + b_2 \epsilon_2 \sqrt{\Delta t}.$$

Assume ϵ_1 and ϵ_2 are normally distributed with the correlation ρ. They can have the representation

$$\epsilon_1 = u_1 \quad \text{and} \quad \epsilon_2 = \rho u_1 + \sqrt{1-\rho^2}u_2$$

where u_1 and u_2 are uncorrelated standard normally distributed variables.

We now introduce the definition of arbitrage following [10].

Definition 2.1 *An arbitrage opportunity is a self-financing strategy ϕ which can lead to a positive terminal gain, without any probability of intermediate loss:*

$$\mathbb{P}(\forall t \in [0,T], V_t(\phi) \geq 0) = 1,\ \mathbb{P}(V_T(\phi) > V_0(\phi)) \neq 0.$$

where T is maturity time, $V_t(\phi)$ is a portfolio value at time t and $\mathbb{P}$ is only used to specify whether the profit is possible or impossible.

This means that we are certain of gaining a profit from trading a self-financing strategy ϕ, and its portfolio value at the maturity time will be greater than it is at the present time. Taking advantage of arbitrage implies that profit can be made without any risk.

For log-returns in these models we are assuming an arbitrage-free market, since that is the only well functioning type of market. Absence of arbitrage is also called the law of one price, which means that two self-financing strategies always have the same value independent of time.

There are two main results on arbitrage-free pricing according to [10].

Proposition 2.1 (Risk-neutral pricing) *In a market described by a probability measure $\mathbb{P}$ on scenarios, any arbitrage-free linear pricing rule Π can be represented as*

$$\Pi_t(H) = e^{-r(T-t)}E^{\mathbb{Q}}[H|\mathcal{F}_t]$$

where H is a payoff, $\mathbb{Q}$ is an equivalent martingale measure: a probability measure on the market scenarios such that

$$\mathbb{P} \sim \mathbb{Q} : \forall A \in \mathcal{F}\ \mathbb{Q}(A) = 0 \iff \mathbb{P}(A) = 0$$

and

$$\forall i = 1,..,d,\ E^{\mathbb{Q}}[\hat{S}^i_T|\mathcal{F}] = \hat{S}^i_t.$$

From this proposition we can conclude that arbitrage-free pricing rule is equivalent to prices expressed as $\Pi_t(H)$. Forward probability $\mathbb{Q}$ and subjective probability $\mathbb{P}$ assign zero probability to the same events. Since $\mathbb{P}(\forall t \in [0,T], V_t(\phi) \geq 0) = 1$, $\mathbb{P}(V_T(\phi) > V_0(\phi)) \neq 0$ and by assumption $\mathbb{Q}(\forall t \in [0,T], V_t(\phi) \geq 0) = 1$, $\mathbb{Q}(V_T(\phi) > V_0(\phi)) \neq 0$ it holds that $\Pi_t(H) \neq 0$. Otherwise, $\Pi_t(H) = 0$ would mean that there is an arbitrage opportunity.

Proposition 2.2 (Fundamental theorem of asset pricing) *The market model defined by $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ and asset prices $(S_t)_{t\in[0,T]}$ is arbitrage-free if and only if there exists a probability measure $\mathbb{Q} \sim \mathbb{P}$ such that the discounted assets $(\hat{S}_t)_{t\in[0,T]}$, where $\hat{S}_t = e^{-rt}S_t$, form a martingale with respect to $\mathbb{Q}$.*

This proposition provides us with a market model including no arbitrage opportunities. Here, $\mathbb{Q}$ is a risk neutral probability measure, and discounted assets $\hat{S}_t$ are martingales which can be interpreted as fair trading. This means that we will have a fair price at the beginning which will not lead to an arbitrage opportunity, assuming that the price is S_t at the time t.

In order to compare the results of the models, we will use characteristic functions, as defined in [10].

Definition 2.2 *The characteristic function of the $\mathbb{R}^d$-valued random variable X is the function $\varphi_X : \mathbb{R}^d \mapsto \mathbb{R}$ defined by*

$$\forall h \in \mathbb{R}^d,\ \varphi_X(h) = E[e^{ihX}] = \int_{\mathbb{R}^d} e^{ihx} dF_X(x) = \int_{\mathbb{R}^d} e^{ihx} f_X(x)dx,$$

where F_X is the cumulative distribution function and f_X is probability density function of X.

The characteristic function of a random variable is the Fourier transform of its distribution.

For the transformation of both Black Scholes and Heston model from the historical measure $\mathbb{P}$ to the risk neutral measure $\mathbb{Q}$ we use Girsanov's theorem, following [9].

Theorem 2 (Girsanov's theorem) *Let $y_t \in \mathbb{R}^n$ be an Itô process of the form*

$$dy_t = a(t,w)dt + dz_t, \quad t \leq T,\ Y_0 = 0,$$

where $t \leq \infty$ is a given constant and z_t is n-dimensional Brownian motion. Put

$$M_t = \exp\left(-\int_0^t (a(s,w)dz_s - \frac{1}{2}\int_0^t a^2(s,w)ds\right), \quad t \leq T.$$

Assume that $a(s,w)$ satisfies Novikov's condition

$$E\left[\exp\left(\frac{1}{2}\int_0^T a^2(s,w)ds\right)\right] < \infty,$$

where $E = E_{\mathbb{P}}$ is the expectation w.r.t. $\mathbb{P}$. define the measure $\mathbb{Q}$ on $(\Omega, \mathcal{F}_T)$ by

$$d\mathbb{Q}(w) = M_T(w)d\mathbb{P}(w).$$

Then $Y(t)$ is an n-dimensional Brownian motion w.r.t. the probability law $\mathbb{Q}$ for $t \leq T$.

3 The Black Scholes Model

Fischer Black and Myron Scholes proposed a model for option pricing in the article [1] in 1973. It was one of the first times that such an approach had been used in option pricing.

The stock prices follow the equation

$$dS_t = \mu S_t dt + \sigma S_t dz_t. \tag{3.1}$$

where μ is the expected return in a short period of time and σ is the volatility of the stock price. It can be written in integral form as

$$S_t = S_0 + \mu \int_0^t S_u du + \sigma \int_0^t S_u dz_u. \tag{3.2}$$

A discrete approximation is given by

$$S(t + \Delta t) = S(t) + \mu S(t)\Delta t + \sigma S(t)(z(t + \Delta t) - z(t))$$

which can be shortly expressed as

$$\Delta S = \mu S \Delta t + \sigma S \Delta z. \tag{3.3}$$

From (2.3) we obtain the solution for the stock price

$$S_t = S_0 e^{(\mu - \sigma^2/2)t + \sigma z_t}.$$

The stock price and variance in (3.1) are under the historical or empirical measure $\mathbb{P}$, which is also referred to as the physical measure. We would like to obtain a risk-neutral process for pricing purposes which will be under the risk-neutral measure $\mathbb{Q}$. By using Girsanov's theorem we express the evolution of S_t in the form

$$dS_t = rS_t dt + \sigma S_t d\tilde{z}_t,$$

where

$$\tilde{z}_t = \left(z_t + \frac{\mu - r}{\sigma} t \right)$$

is a Brownian motion under $\mathbb{Q}$. For the comparison with the Heston model, we will consider the Black Scholes model under the risk neutral measure where μ is replaced by the risk-free interest rate r. We will omit the tilde in $\tilde{z}$ for simplicity.

For the Black Scholes model we have normally distributed values of log returns, as in (2.4) that is

$$\ln\left(\frac{S_{t+\Delta t}}{S_t}\right) \sim N\left(\left(r-\frac{\sigma^2}{2}\right)\Delta t, \sigma^2 \Delta t\right).$$

Therefore, since the characteristic function for the normal distribution is given as a Fourier transform of the normal density function f as

$$\varphi_X(\phi) = E(e^{i\phi X}) = \int_{-\infty}^{\infty} f(x)e^{i\phi x}dx = e^{i\mu\phi - \frac{1}{2}(\sigma\phi)^2},$$

the characteristic function of $\ln(S_T)$ in the Black Scholes model, using (2.4) and following [4], is given by

$$E[e^{i\phi \ln S_T}] = e^{i\phi\left[\ln S_t + \left(r - \frac{1}{2}\sigma^2\right)\tau\right] - \frac{1}{2}\phi^2\sigma^2\tau}, \tag{3.4}$$

where $\tau = T - t$.

We assume the following conditions for the stock and the option, as stated in [1] and [6]:

(i) The risk-free interest rate is known and constant through time.

(ii) The stock price follows the process $dS_t = \mu S_t dt + \sigma S_t dz_t$ where μ and σ are constant.

(iii) The stock pays no dividends.

(iv) An option can only be exercised at maturity, i.e. it is a European option.

(v) There are no transaction costs, all securities are divisible.

(vi) It is possible to borrow any fraction of the price of a security to buy it or to hold it, at the short-term interest rate.

(vii) The short selling of securities is permitted.

3.1 The Black Scholes PDE

Options under the Black Scholes model are obtained by well known Black Scholes partial differential equations (PDE). Here we give a short derivation. Using Itô's lemma, we obtain an equation for a call option f,

$$df = \left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)dt + \frac{\partial f}{\partial S}\sigma S dz.$$

The discrete approximation is given by

$$\Delta f = \left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)\Delta t + \frac{\partial f}{\partial S}\sigma S \Delta z. \tag{3.5}$$

If we assume that a holder of a portfolio is f short and $\dfrac{\partial f}{\partial S}$ shares long, then we can describe the portfolio Π with the equation

$$\Pi = -f + \frac{\partial f}{\partial S}S. \tag{3.6}$$

During the time interval Δt we observe the change in portfolio

$$\Delta\Pi = -\Delta f + \frac{\partial f}{\partial S}\Delta S.$$

Using equations (3.3) and (3.5) we obtain

$$\Delta\Pi = \left(-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)\Delta t. \tag{3.7}$$

This implies that the return from the risk-less portfolio is the same as a risk-free interest rate, otherwise there would be an opportunity for arbitrage. Hence we obtain equation

$$\Delta\Pi = r\Pi\Delta t. \tag{3.8}$$

Now using equations (3.6),(3.7) and (3.8) we obtain

$$\left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)\Delta t = r\left(f - \frac{\partial f}{\partial S}S\right)\Delta t.$$

This leads to the differential equation

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 = rf.$$

The price of a derivative, such as an option, that depends on a stock must satisfy the Black Scholes differential equation.

The boundary condition for the European call option is given by

$$f = \max(S - K, 0) \text{ when } t = T.$$

We can see that the Black Scholes PDE does not involve the parameter μ so it is independent of risk, since μ increases with the risk level. Therefore, the expected return will be a risk-free rate r. Calculating the solutions to the PDE with the risk neutral valuation can be used in the risk averse markets too. The expected payoff from the derivative changes and the discount rate that is used for the payoff changes, and those changes offset each other. The concepts are explained further in [6].

3.2 The Black Scholes formula

A European call gives the owner the right to buy a stock for a pre-determined price at a future time. European call option prices under the Black Scholes model can be described by the Black Scoles formula, as in [1],

$$C(S_t, t) = S_t N(d_1) - K e^{r\tau} N(d_2),$$

where $C(S_t, t)$ is the value of the option as a function of the stock price S_t and time t. Furthermore, N is a cumulative normal density function, $\tau = T - t$ is the time to maturity where T is the time at which an option can be exercised, r is the interest rate, K is the exercise price,

$$d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(\tau)}{\sigma\sqrt{\tau}},$$

and

$$d_2 = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)(\tau)}{\sigma\sqrt{\tau}}.$$

3.3 Parameter estimation

We can estimate the expected return μ and volatility σ of a stock price by using empirical data. We will use Matlab to estimate the parameters based on discrete approximation and logarithmic return of a stock price. We estimate μ by using the method of moments introduced by Chebyshev and σ following [10]. The logarithmic return during the time interval $[t, t + \Delta t]$ is, with some ambiguous notation,

$$x_{\Delta t} = \ln\left(\frac{S_{t+\Delta t}}{S_t}\right)$$

so $x_{\Delta t}$ is normally distributed

$$x_{\Delta t} \sim N\left[\left(\mu - \frac{\sigma^2}{2}\right)\Delta t, \sigma^2 \Delta t\right]$$

with expected return

$$E[x_{\Delta t}] = \left(\mu - \frac{\sigma^2}{2}\right)\Delta t$$

and variance

$$\mathrm{Var}[x_{\Delta t}] = \sigma^2 \Delta t.$$

From this we determine that reasonable estimates for μ and σ are

$$\hat{\mu} = \frac{2\hat{E}[x_{\Delta t}] + \widehat{\mathrm{Var}}[x_{\Delta t}]\Delta t}{2\Delta t}$$

and

$$\hat{\sigma} = \frac{\widehat{\mathrm{Var}}[x_{\Delta t}]}{\Delta t}.$$

The estimated parameters for empirical S&P 500 data are given in Table 1. Note that during this time period the estimated drift parameter $\hat{\mu}$ is actually negative, which is by Figure 1 below.

Estimates	$\hat{\mu}$	$\hat{\sigma}$
	-0.1054	0.2042

Table 1: Estimated parameters for the Black Scholes model.

3.4 Simulation

Instead of using a discretization based on (3.3), an exact discretization (i.e. it has no errors at the grid points) is given by

$$x(t + \Delta t) = x(t) + (\mu - \frac{1}{2}\sigma^2)\Delta t + \sigma(z(t + \Delta t) - z(t)),$$

referring to (2.2). Additionally, let

$$S_t = S_0 e^{x_t}$$

be the simulated stock price. Having obtained values for μ and σ using parameter estimation (see previous section), we now simulate a few paths of the stock prices following the Black Scholes model. Every simulated path is different so here we provided one example of such an experiment. Figure 1 shows the simulation following the Black Scholes model using the data from Table 1 and empirical S&P 500 data path. We can observe that the stock price index went down which is verified by the empirical stock price index path itself.

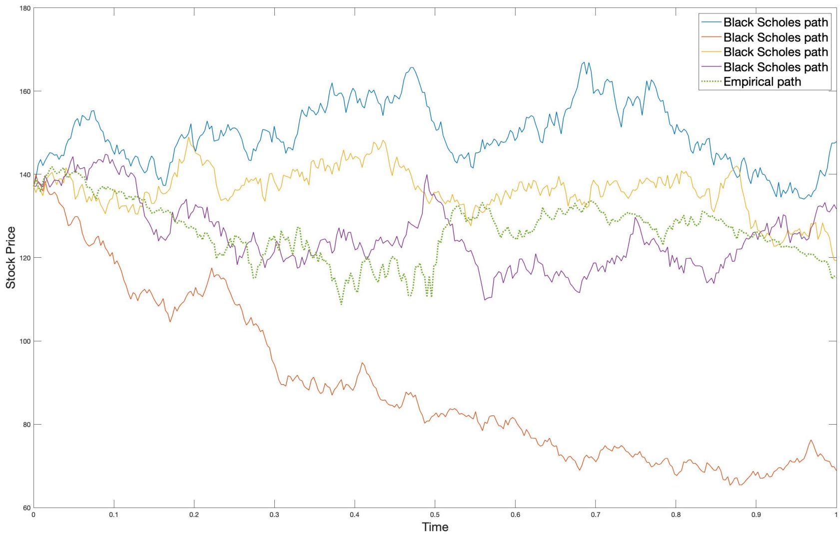

Figure 1: Simulation of paths of stock prices following the Black Scholes model and empirical data.

Using the empirical data from S&P 500 we obtained the empirical estimates as in Table 1. After that we simulated the data given the estimated parameters and then estimated the parameters from that simulated path to check for the validity of the estimates.

4 The CIR Model

In 1985 John C. Cox, Jonathan E. Ingersoll and Stephen A. Ross proposed a process that includes the square root diffusion in the article [2]. It models short-term interest rates using an intertemporal general equilibrium asset pricing model. In order to determine bond prices it anticipates future events, risk preferences as well as timing of the consumption and habitat.

The CIR process is the solution to the stochastic differential equation

$$dr_t = \kappa(\theta - r_t)dt + \sigma\sqrt{r_t}dz_t, \tag{4.1}$$

which describes a continuous time first-order autoregressive process where the randomly moving interest rate r_t is elastically pulled toward a central location of the long-term value $\theta > 0$, z_t is a Wiener process, and κ is the speed of adjustment, as stated in [2]. In this context r_t is the standard notation for CIR process, not to be confused with the risk-free interest rate r.

We can write (4.1) in the integral form as

$$r_t = (r_0 - \theta)e^{\kappa t} + \sigma e^{\kappa t} \int_0^t e^{\kappa u} \sqrt{r_u} dz_u,$$

as described in [7].

There are three assumptions in the model, as in [2]:

(i) The change in the interest rate over time is described by a single state variable r.

(ii) The means and variances of the rates of return in the processes are proportional to r, which means that they won't dominate the portfolio decision for large values of r.

(iii) The development of the state variable r follows the equation (4.1).

The CIR model has the property that r reaches zero if $\sigma^2 > 2\kappa\theta$, and if $\sigma^2 \leq 2\kappa\theta$ the upward drift is sufficiently large to make the origin inaccessible.

The CIR model has some more interesting properties [2]:

(i) There are no negative interest rates, since the diffusion term $\sigma\sqrt{r_t}$ decreases to zero as r_t approaches the origin, as stated by [8].

(ii) If the interest rate reaches zero, it can become positive.

(iii) The absolute variance of the interest rate increases when the interest rate increases.

(iv) There is a steady state distribution for the interest rate.

This means that the CIR model has a limiting non-degenerate distribution centered at θ. The process does not converge to a point but to a distribution. Following [8], the distribution of r_t given r_u for some $u < t$ is, up to a scale factor, a noncentral chi-square distribution which is

$$P(\chi_\nu'^2(\lambda) \leq y) = e^{-\lambda/2} \sum_{j=0}^{\infty} \frac{(1/2\lambda)^j/j!}{2^{(\nu/2)+j}\Gamma(\nu/2+j)} \int_0^y u^{(\nu/2)+j-1} e^{-u/2} du,$$

for some parameter values λ and ν.

4.1 Simulation

For the simulation of the CIR process we will use Euler approximation but also the exact algorithm.

4.1.1 Euler method

The Euler method is an approximation scheme used to generate solutions to deterministic differential equations as described in [7]. Given an Itô process solution of the stochastic differential equation

$$dr_t = \mu dt + \sigma dz_t$$

with initial deterministic value $r_{t_0} = r_0$ and the discretization $\Pi_N = \Pi_N([0,T])$ of the interval $[0,T]$, $0 = t_0 < t_1 < ... < t_n = T$, the Euler approximation of r is a continuous stochastic process y satisfying the iterative scheme

$$y_{i+1} = y_i + \mu(t_{i+1} - t_i) + \sigma(z_{i+1} - z_i),$$

for $i = 0, 1, ...N-1$ and $y_0 = x_0$. The time increment $\Delta t = t_{i+1} - t_i$ is constantly equal to $\Delta t = \frac{1}{N}$. We consider linear interpolation

$$y_t = y_i + \frac{t - t_i}{t_{i+1} - t_i}(y_{i+1} - y_i) \ , \ t \in [t_i, t_{i+1}).$$

We consider the particular case of CIR model and using the transformation $y_t = \sqrt{r_t}$ we obtain the transformed stochastic differential equation

$$dy_t = \frac{1}{2y_t}\left(\kappa(\sigma - (y_t)^2) - \frac{1}{4}\sigma^2\right) dt + \frac{1}{2}\sigma dz_t,$$

and the Euler scheme is

$$\Delta y = \frac{1}{2y_i}\left(\kappa(\sigma - (y_i)^2) - \frac{1}{4}\sigma^2\right) \Delta t + \frac{1}{2}\sigma\epsilon\sqrt{\Delta t},$$

where $\epsilon \sim N(0,1)$.

4.1.2 Exact algorithm

In the exact algorithm we can obtain the exact distribution with no distributional error at the grid points as opposed to the Euler approximation, where at grid points we obtain an approximative distribution. We follow an algorithm from [8]. We sample the noncentral chi-square distribution $\chi'^2_\nu(\lambda)$ by generating a Poisson random variable N and, conditional on N, we sample a chi-square random variable with $\nu + 2N$ degrees of freedom. To simulate

$$dr_t = \kappa(\theta - r_t)dt + \sigma\sqrt{r_t}dz_t$$

on time grid $0 = t_0 < t_1 < ... < t_n$ with $d = \dfrac{4\theta\kappa}{\sigma^2}$, we have two cases

(i) $d > 1$
for $i = 0, ..., n-1$
$c = \sigma^2 \dfrac{(1 - e^{-\kappa(t_{i+1}-t_i)})}{4\kappa}$
$\lambda = r(t_i)\dfrac{e^{-\kappa(t_{i+1}-t_i)}}{c}$
We generate $Z \sim N(0,1)$
We generate $X \sim \chi^2_{d-1}$
$r(t_{i+1}) = c[(Z + \sqrt{\lambda})^2 + X]$

(ii) $d \leq 1$
for $i = 0, ..., n-1$
$c = \sigma^2 \dfrac{(1 - e^{-\kappa(t_{i+1}-t_i)})}{4\kappa}$
$\lambda = r(t_i)\dfrac{e^{-\kappa(t_{i+1}-t_i)}}{c}$
We generate $N \sim \text{Poisson}(\lambda/2)$
We generate $X \sim \chi^2_{d+2N}$
$r(t_{i+1}) = cX$

In Figure 2 we simulate paths of the CIR process comparing the Euler method with the exact algorithm.

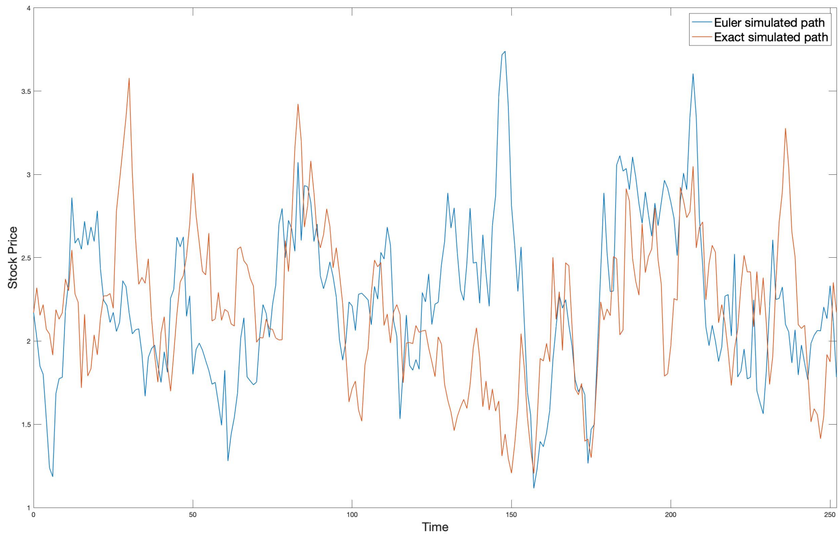

Figure 2: Simulation of paths of stock prices following the CIR model

5 The Heston Model

Steven Heston introduced a new model in 1993 in the article [3]. It is a generalization of the Black Scholes model which relates the distribution of the spot returns to the cross-sectional properties of option prices. This is a model of stochastic volatility with a closed-form solution for the price of a European call option when the spot asset is correlated with volatility, which also incorporates stochastic interest rates. In the Heston model, volatility is a *mean reverting* process. Mean reversion means that the process does not wander off to infinity but oscillates around a well-defined long term average value, as defined in [10]. We assume that the asset price S at time t follows the diffusion equation

$$dS_t = \mu S_t dt + \sqrt{v_t} S_t dz_{1,t}, \tag{5.1}$$

where μ is the drift of the process for the stock, v_t is the stochastic variance, v_0 is the initial level of variance, and $z_{1,t}$ is the Wiener process. The variance v follows the square-root CIR process

$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t} dz_{2,t}, \tag{5.2}$$

where κ is the mean reversion speed for the variance, θ is the mean reversion level for the variance, and $z_{2,t}$ is the Wiener process.

The stock price and variance in (5.1) and (5.2) are under the historical measure $\mathbb{P}$. By applying the Girsanov theorem one can find a probability measure $\mathbb{Q}$ such that we can represent the stock price in the form

$$dS_t = rS_t dt + \sqrt{v_t} S_t d\tilde{z}_{1,t},$$

where

$$\tilde{z}_{1,t} = \left(z_{1,t} + \frac{\mu - r}{\sqrt{v_t}} t \right)$$

is a Brownian motion under $\mathbb{Q}$. The variance process still has the same form under $\mathbb{Q}$ as in (5.2). There is a possibility to make some additional change of measure under which the risk premium is included for the volatility which is useful for volatility options following [3] and [4], but that will not be investigated in this thesis. The variance drifts toward a long-run mean θ with the mean reversion speed κ. Therefore, an increase in the average variance θ increases the prices of options. When mean reversion is positive, the variance has a steady state distribution with mean θ. By the Central limit theorem log-returns over long periods will have asymptotically normal distributions, with variance per unit of time given by θ, following [3] and [2].

We can write the process in (5.1) in terms of the log price, for $x_t = \ln S_t$ the risk neutral log price process is

$$x_t = \left(r - \frac{1}{2} \right) dt + \sqrt{v_t} d\tilde{z}_{1,t}.$$

From now on, if nothing else is said, we will assume that the stock and variance processes are risk-neutral, so we will replace μ by r. We will omit the tilde in $\tilde{z}$ for simplicity. For the analysis of the Heston model, we use the book [4] and its Matlab codes.

Equations (5.1) and (5.2) can be written in integral form as

$$S_t = S_0 + r \int_0^t S_u du + \int_0^t \sqrt{v_u} S_u dz_u$$

$$v_t = v_0 + \kappa \int_0^t (\theta - v_u) du + \sigma \int_0^t \sqrt{v_u} dz_u.$$

The volatility of the variance parameter σ controls the kurtosis, which is a measure describing the tails of the probability distribution. It is expressed with the formula

$$\mathrm{Kurt}[X] = E\left[\left(\frac{X - \mu}{\sigma} \right)^4 \right],$$

for random variable X with mean μ and variance σ^2. When σ is large the variance process v_t is highly dispersed and the distribution of returns has higher kurtosis and fatter tails. Kurtosis is the effect of the volatility clustering.

In Figure 3 we compare the probability density functions for different values of σ, that is $\sigma = 0$, $\sigma = 0.3$ and $\sigma = 0.5$. The other parameters are $\kappa = 2$, $\theta = 0.01$,$v_0 = 0.01$ and $\rho = 0$.

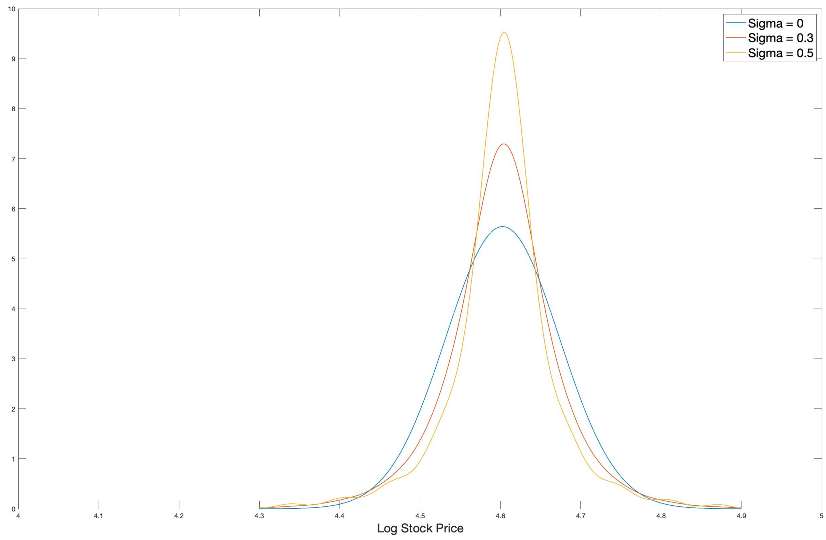

Figure 3: Comparison of different values of σ

The correlation between $z_{2,t}$ and $z_{1,t}$ is ρ, that is expressed somewhat heuristically as $E[dz_{1,t}dz_{2,t}] = \rho dt$. Assuming we are given the time interval $[0, T]$ where T is maturity time, we can describe ρ as a parameter which controls the skewness of the density of $\ln(S_T)$ and of the continuously compounded return $\ln\left(\frac{S_T}{S_0}\right)$, where we recall that the skewness of a random variable X is

$$\text{Skew}[X] = E\left[\left(\frac{X-\mu}{\sigma}\right)^3\right].$$

If $\rho > 0$ then the probability densities are positively skewed and it implies a rise in variance when the stock price rises which has the effect of fattening the right tail of the distribution and thinning the left tail. On the other hand, if $\rho < 0$ then the densities are negatively skewed as stated in [3] and [4].

In Figure 4 we compare the probability density functions for different values of ρ, that is $\rho = -0.9$, $\rho = 0$, and $\rho = 0.9$. Other parameters are $\kappa = 2$, $\theta = 0.01$, $\sigma = 0.1$ and $v_0 = 0.01$.

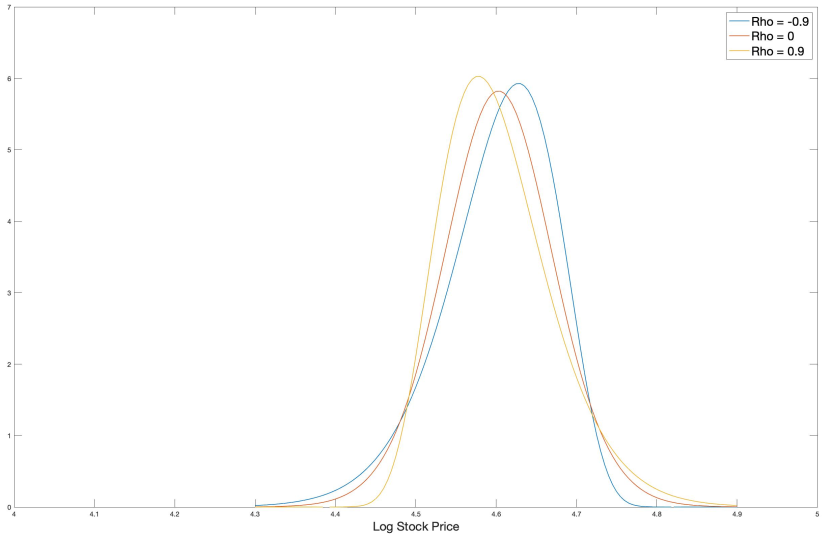

Figure 4: Comparison of different values of ρ

5.1 The Heston PDE

Options under the Heston model can be obtained by a certain PDE. The derivation of the Heston PDE is similar to the Black Scholes PDE. In the Heston model, following [4], we need another derivative in the portfolio, in order to hedge the volatility. We have the portfolio

$$\Pi = V + \delta S + \varphi U,$$

consisting of one option $V(S, v, t)$, δ units of the stock S and φ units of the option $U(S, v, t)$. The change in the portfolio value is

$$d\Pi = dV + \delta dS + \varphi dU.$$

Now we apply Itô's lemma to $V(S, v, t)$ and differentiate V with respect to S,v and t and obtain a Taylor series

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{\partial V}{\partial v}dv + \frac{1}{2}vS^2\frac{\partial^2 V}{\partial S^2}dt + \frac{1}{2}v\sigma^2\frac{\partial^2 V}{\partial v^2}dt + \sigma\rho v S\frac{\partial^2 V}{\partial S\partial v}dt,$$

with the assumption that $(dS)^2 = vS^2(dz_1)^2 = vS^2dt$, $(dv)^2 = \sigma^2 vdt$ and $dSdv = \sigma vSdz_1dz_2 = \sigma\rho vSdt$, with $(dt)^2 = 0$, $dz_1dt = dz_2dt = 0$. Using the same arguments we can derive the expression for $U(S, v, t)$. Combining those expressions we now obtain

$$\begin{aligned} d\Pi = & \left(\frac{\partial V}{\partial t} + \frac{1}{2}vS^2\frac{\partial^2 V}{\partial S^2} + \sigma\rho vS\frac{\partial^2 V}{\partial S\partial v} + \frac{1}{2}v\sigma^2\frac{\partial^2 V}{\partial v^2}\right)dt \\ & + \varphi\left(\frac{\partial U}{\partial t} + \frac{1}{2}vS^2\frac{\partial^2 U}{\partial S^2} + \sigma\rho vS\frac{\partial^2 U}{\partial S\partial v} + \frac{1}{2}v\sigma^2\frac{\partial^2 U}{\partial v^2}\right)dt \\ & + \left(\frac{\partial V}{\partial S} + \varphi\frac{\partial U}{\partial S} + \delta\right)dS + \left(\frac{\partial V}{\partial v} + \varphi\frac{\partial U}{\partial v}\right)dv. \end{aligned} \tag{5.3}$$

If the portfolio is to be hedged against movements in stock and volatility, then the last two terms in the latter equation are zero, therefore

$$\varphi = \frac{\frac{\partial V}{\partial v}}{\frac{\partial U}{\partial v}} \quad \text{and} \quad \delta = -\varphi\frac{\partial U}{\partial S} - \frac{\partial V}{\partial S}. \tag{5.4}$$

Using the same argumnet as in the Black Scholes case, we obtain the equation for the change in the portfolio value as in (3.8) and hence

$$d\Pi = r(V + \delta S + \varphi U)dt.$$

Using equations (5.3) and (5.4), the equation for the change in the portfolio value now becomes

$$\begin{aligned} & \frac{\left(\frac{\partial V}{\partial t} + \frac{1}{2}vS^2\frac{\partial^2 V}{\partial S^2} + \sigma\rho vS\frac{\partial^2 V}{\partial S\partial v} + \frac{1}{2}v\sigma^2\frac{\partial^2 V}{\partial v^2}\right) - rV + rs\frac{\partial V}{\partial S}}{\frac{\partial V}{\partial v}} \\ = & \frac{\left(\frac{\partial U}{\partial t} + \frac{1}{2}vS^2\frac{\partial^2 U}{\partial S^2} + \sigma\rho vS\frac{\partial^2 U}{\partial S\partial v} + \frac{1}{2}v\sigma^2\frac{\partial^2 U}{\partial v^2}\right) - rU + rS\frac{\partial U}{\partial S}}{\frac{\partial U}{\partial v}}. \end{aligned} \tag{5.5}$$

Following [3], both sides of the equation (5.5) can be written as

$$f(S, v, t) = -\kappa(\theta - v) + \lambda(S, v, t)$$

where $\lambda(S, v, t)$ is a measure of risk premium, allowed to be zero. If we now use the left hand side of Equation (5.5) to substitute for $f(S, v, t)$ we obtain

$$-\kappa(\theta - v) + \lambda(S, v, t) = \frac{\left(\frac{\partial U}{\partial t} + \frac{1}{2}vS^2\frac{\partial^2 U}{\partial S^2} + \sigma\rho vS\frac{\partial^2 U}{\partial S\partial v} + \frac{1}{2}v\sigma^2\frac{\partial^2 U}{\partial v^2}\right) - rU + rS\frac{\partial U}{\partial S}}{\frac{\partial U}{\partial v}},$$

which leads to the Heston partial differential equation (PDE)

$$\left(\frac{\partial U}{\partial t}+\frac{1}{2}vS^2\frac{\partial^2 U}{\partial S^2}+\sigma\rho vS\frac{\partial^2 U}{\partial S\partial v}+\frac{1}{2}v\sigma^2\frac{\partial^2 U}{\partial v^2}\right)-rU+rS\frac{\partial U}{\partial S}+(\kappa(\theta-v)-\lambda(S,v,t))\frac{\partial U}{\partial v}=0.$$

For the European call option with maturity time T and strike price K, the call is worth its intrinsic value, as stated it [4]

$$U(S,v,T)=\max(0,S-K)$$

and we have the boundary conditions

$$U(0,v,t)=0\ ,\ \frac{\partial U}{\partial S}(\infty,v,t)=1 \text{ and } U(S,\infty,t)=S.$$

5.2 The Heston formula and the characteristic functions

We can express the Heston call price in a similar way to how it is done in the Black Scholes model. Following [4] we have

$$\begin{aligned} C(S_t,t) &= e^{-r(T-t)}E^{\mathbb{Q}}[(S_T-K)^+|S_t] \\ &= e^{-r(T-t)}E^{\mathbb{Q}}[(S_T-K)\mathbb{1}_{S_T>K}|S_t] \\ &= e^{-r(T-t)}E^{\mathbb{Q}}[S_TS_t^{-1}\mathbb{1}_{S_T>K}|S_t]-Ke^{-r(T-t)}E^{\mathbb{Q}}[\mathbb{1}_{S_T>K}|S_t] \\ &= S_tP_1-Ke^{-r(T-t)}P_2 \end{aligned}$$

where S_t is the spot or current price, K is the exercise price, T is the maturity time and $P_1=E^{\mathbb{Q}}[S_TS_t^{-1}\mathbb{1}_{S_T>K}|S_t]$ and $P_2=E^{\mathbb{Q}}[\mathbb{1}_{S_T>K}|S_t]$. The PDE for P_1 and P_2 is

$$\frac{\partial P_j}{\partial t}+\rho\sigma v\frac{\partial^2 P_j}{\partial v\partial x}+\frac{1}{2}v\frac{\partial^2 P_j}{\partial x^2}+\frac{1}{2}\sigma^2 v\frac{\partial^2 P_j}{\partial v^2}+(r+u_jv)\frac{\partial P_j}{\partial x}+(a-b_jv)\frac{\partial P_j}{\partial v}=0, \tag{5.6}$$

where $j=1,2$, $u_1=\frac{1}{2}$, $u_2=-\frac{1}{2}$, $a=\kappa\theta$, $b_1=\kappa+\lambda-\rho\sigma$ and $b_2=\kappa+\lambda$.

One can express P_1 in the Gil-Pelaez inversion form

$$P_1=\frac{1}{2}+\frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-i\phi\ln K}\varphi_1(\phi;x,v)}{i\phi}\right]d\phi,$$

where

$$\varphi_1(\phi;x_t,v_t)=E^{\mathbb{Q}}[S_TS_t^{-1}e^{i\phi x_T}|x_t]=e^{(C_1(\tau,\phi)+D_1(\tau,\phi)v_t+i\phi x_t)} \tag{5.7}$$

where $\tau = T - t$. Similarly one can express P_2 as

$$P_2 = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-i\phi \ln K}\varphi_2(\phi; x, v)}{i\phi}\right] d\phi,$$

where

$$\varphi_2(\phi; x_t, v_t) = E^{\mathbb{Q}}[e^{i\phi x_T}|x_t] = e^{(C_2(\tau,\phi)+D_2(\tau,\phi)v_t+i\phi x_t)}. \tag{5.8}$$

At maturity, that is when $\tau = 0$, $x_T = \ln S_T$ is known, the conditional expectations degenerate into $e^{i\phi x_T}$ so the initial conditions are $D_j(0,\phi) = 0$ and $C_j(0,\phi) = 0$, while v_0 must be estimated. Here

$$C_j(\tau,\phi) = ri\phi\tau + \frac{a}{\sigma^2}\left[(b_j - \rho\sigma i\phi + d_j)\tau - 2\ln\left(\frac{1 - g_j e^{d_j\tau}}{1 - g_j}\right)\right],$$

$$D_j(\tau,\phi) = \frac{b_j - \rho\sigma i\phi + d_j}{\sigma^2}\left(\frac{1 - e^{d_j\tau}}{1 - g_j e^{d_j\tau}}\right),$$

and

$$g_j = \frac{b_j - \rho\sigma i\phi + d_j}{b_j - \rho\sigma i\phi - d_j},$$

$$d_j = \sqrt{(\rho\sigma i\phi - b_j)^2 - \sigma^2(2u_j i\phi - \phi^2)}.$$

From the characteristic functions of log returns one can obtain probability density functions by taking inverse Fourier transform.

5.3 Parameter estimation

So far, the parameters for the Heston model fit of data are unknown. Now by using the empirical data we will estimate those parameters.

5.3.1 Maximum likelihood estimation

The maximum likelihood method is the most used method in estimating parameters from historical data. We obtain parameters that maximize the likelihood of the observed data. Following [10], starting with a general discussion, let $f(x,\xi)$ be a functional form of the density of the log-returns with a multidimensional parameter ξ and observations x_t, $t = 1, ...n$, where

$$\hat{\xi} = \max_{\xi}\prod_{t=1}^{n} f(x_t, \xi).$$

This is equivalent to maximizing the log-likelihood function

$$l(\xi) = \sum_{t=1}^{n} \ln f(x_t, \xi).$$

We will use a maximum likelihood estimation inspired by the article [5] due to Atiya and Wall. There, an approximation is used that is valid when the time step is short and a semi-analytic approximation for the volatility likelihood function is obtained. For the empirical data we will use S&P 500 historical stock prices.

As in [4], we start by transforming the equation (5.1) to the equation for the log-stock price $x_t = \ln S_t$. In this subsection we initially consider the log return under the empirical probability measure. The volatility equation (5.2) remains the same. After using Itô's lemma we have

$$dx_t = \left(\mu - \frac{1}{2}v_t\right) dt + \sqrt{v_t} dz_{1,t}, \tag{5.9}$$

$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t} dz_{2,t}. \tag{5.10}$$

Note that only the log returns are observed and not the volatilities, which makes the parameter estimation a non-trivial problem. The transition probability density is given by [1]

$$p_{V_{t_{n+1}}|X_{t_0}=x_{t_0},...,X_{t_{n+1}}=x_{t_{n+1}}}(v_{t_{n+1}}|x_{t_0},...,x_{t_{n+1}})$$
$$\propto p_{V_{t_{n+1}}|X_{t_{n+1}},...,X_{t_0}}(v_{t_{n+1}}, x_{t_{n+1}}, x_{t_n}, ..., x_{t_0})$$

$$= \int_{v_{t_n}} p_{V_{t_{n+1}}X_{t_{n+1}}|V_{t_n}=v_{t_n},X_{t_n}=x_{t_n}}(x_{t_{n+1}}, v_{t_{n+1}}|x_{t_n}, v_{t_n})$$
$$\times p_{V_{t_n}|X_{t_n}=x_{t_n},...,X_{t_0}=x_{t_0}}(v_{t_n}|x_{t_n}, ..., x_{t_0})dv_{t_n}$$

where

$$p_{X_{t_{n+1}},V_{t_{n+1}}|X_{t_n}=x_{t_n},V_{t_n}=v_{t_n}}(x_{t_{n+1}}, v_{t_{n+1}}|x_{t_n}, v_{t_n})$$
$$\approx \frac{1}{(2\pi)^{n/2}|\Sigma_{t_{n+1}}|^{n/2}} e^{-\frac{1}{2}\left[\begin{pmatrix}x_{t_{n+1}}\\ v_{t_{n+1}}\end{pmatrix} - \mu_{t_{n+1}}\right]^T \Sigma^{-1}_{t_{n+1}} \left[\begin{pmatrix}x_{t_{n+1}}\\ v_{t_{n+1}}\end{pmatrix} - \mu_{t_{n+1}}\right]}$$

that is

$$p(x_{t+1}, v_{t+1}|x_t, v_t) \approx N(\mu_{t+1}, \Sigma_{t+1}), \tag{5.11}$$

[1]The proportionality symbol can be used here since the likelihood function equals the transition probability density times some factor that does not depend on the parameters.

the transition probability density which is approximately normal. Thinking somewhat heuristically, for small dt, the transition probability densite has the mean vector

$$\mu_{t+1} = \begin{pmatrix} x_t + \left(\mu - \frac{v_t}{2}\right) dt \\ v_t + \kappa(\theta - v_t)dt \end{pmatrix}$$

and covariance matrix

$$\Sigma_{t+1} = v_t dt \begin{pmatrix} 1 & \rho\sigma \\ \rho\sigma & \sigma^2 \end{pmatrix}.$$

Alternatively, in a more compact form following [5], let $x_{1:T}$ denote the past asset price observables from time 1 to T and $v_{0:T}$ past volatilities from time 0 to T. The filtering problem can be formulated as that of obtaining the likelihood of the volatility given the past observations

$$\begin{aligned} L_T(v_T) &= p(v_T|x_{0:T}) \\ &\propto p(v_T, x_{1:T}|x_0) \\ &= \int p(x_{1:T}, v_{0:T}|x_0)dv_{0:T-1}, \end{aligned}$$

where L_T denotes the likelihood function. The maximally likely estimate of the volatility is the maximum of the function $p(v_T, x_{1:T}|x_0)$. Using the Markov property we obtain

$$L_T(v_T) \propto \int_{v_{T-1}} \cdots \int_{v_0} \prod_{t=1}^{T} [p(x_t, v_t|x_{t-1}, v_{t-1})] \times p(v_0)dv_0...dv_{T-1},$$

where $p(v_0)$ represents a priori probability density for the volatility at time $t = 0$. Likelihood at time $t + 1$ can be recursively obtained given the time t as

$$L_{t+1}(v_{t+1}) \propto \int_{v_t} p(x_{t+1}, v_{t+1}|x_t, v_t)L_t(v_t)dv_t,$$

with the starting value $L_0(v_0) = p(v_0)$. Using (5.11) we obtain

$$L_{t+1}(v_{t+1}) \propto dt \int_0^\infty \left[\frac{e^{-av_t} - (b_t/v_t)}{v_t}\right] L_t(v_t)dv_t, \tag{5.12}$$

where

$$a = \frac{(\kappa')^2 + \rho\sigma\kappa' dt + \sigma^2 dt^2/4}{2\sigma^2(1-\rho^2)dt},$$

$$b_t = \frac{(v_{t+1} - \alpha dt)^2 - 2\rho\sigma(v_{t+1} - \alpha dt)(\Delta x_{t+1} - \mu dt) + \sigma^2(\Delta x_{t+1} - \mu dt)^2}{2\sigma^2(1-\rho^2)dt}$$

and

$$d_t = \frac{1}{D} \exp\left(\frac{(2\kappa' + \rho\sigma dt)(v_{t+1} - \alpha dt) - (2\rho\sigma\kappa' + \sigma^2 dt)(\Delta x_{t+1} - \mu dt)}{2\sigma^2(1-\rho^2)dt} \right)$$

where $\mu = r$ is the drift, $\Delta x_{t+1} - x_t$ is the increment between the log stock prices, $\kappa' = 1 - \kappa dt$, $\alpha = \kappa\theta$ and $D = 2\pi\sigma\sqrt{1-\rho^2 dt}$. We obtain v_{t+1} through $v_t = \sqrt{\frac{b_t}{a}}$ so

$$v_{t+1} = \sqrt{B^2 - C} - B \tag{5.13}$$

where

$$B = -\alpha dt - \rho\sigma(\Delta x_{t+1} - \mu dt)$$

and

$$C = (\alpha dt)^2 + 2\rho\sigma\alpha dt(\Delta x_{t+1} - \mu dt) + \sigma^2(\Delta x_{t+1} - \mu dt)^2 - 2{v_t}^2 a\sigma^2(1-\rho^2)dt.$$

The function $e^{-av_t - (b_t/v_t)} v_t$ in (5.12) has a maximum at

$$v_{\text{peak}} = \frac{-1 + \sqrt{1 + 4ab_t}}{2a}.$$

By approximating the likelihood function $L_t(v_t)$ around v_{peak} we obtain

$$L_{t+1}(v_{t+1}) \propto L_t(v_{\text{peak}}) \int_0^\infty \left[\frac{e^{-av_t} - (b_t/v_t)}{v_t} \right] dv_t.$$

According to [11], the integral above is equal to

$$\int_0^\infty v_t^{-1} e^{-av_t - (b_t/v_t)} dv_t = 2K_0(2\sqrt{ab_t})$$

where $K_0(q)$ is the modified Bessel function of the second kind. For large q,

$$K_0(q) = \sqrt{\frac{\pi}{2q}} e^{-q} \sum_{k=0}^{n-1} \frac{1}{(2q)^k} \frac{\Gamma\left(k + \frac{1}{2}\right)}{k!\Gamma\left(-k + \frac{1}{2}\right)} + O(q^{-n})$$

is a good approximation of $K_0(q)$ according to [11], where $q = 2\sqrt{ab_t}$ which is large because of the Δt in the denominator. Hence

$$L_{t+1}(v_{t+1}) \approx L_t(v_{\text{peak}}) \sqrt{\frac{\pi}{\sqrt{ab_t}}} e^{-2\sqrt{ab_t}}.$$

Approximating $v_{\text{peak}} \approx \sqrt{\frac{b_t}{a}}$, the final expression for the likelihood is

$$L_{t+1}(v_{t+1}) \propto d_t (ab_t)^{-1/4} e^{-2\sqrt{ab_t}} L_t \left(\sqrt{\frac{b_t}{a}} \right). \tag{5.14}$$

The log-likelihood is

$$l_{t+1}(v_{t+1}) \propto \ln(d_t) - \frac{1}{4}\ln(ab_t) - 2\sqrt{ab_t} + l_t \left(\sqrt{\frac{b_t}{a}} \right).$$

To obtain the likelihood we use the following algorithm:

(i) We start at $t = 0$, we choose uniform grid for v_0 and we set $L(v_0) = p(v_0)$

(ii) For $t = 0$ to $T - 1$

  (a) We obtain grid points for v_{t+1} following equation (5.13)

  (b) We compute the likelihood L_{t+1} for the grid points following the equation (5.14)

(iii) We calculate Δv and from there ΔS.

One of the best methods to optimize the approximative likelihood is by using Adaptive Simulated Annealing (ASA) optimization procedure by Ingber, as explained in [5]. ASA is developed to statistically find the best global fit of a non-linear constrained non-convex cost-function over a D-dimensional space which will not be further investigated in this thesis.

Since we consider the Heston model under the risk neutral measure $\mathbb{Q}$, we replace the parameter μ by r, and set $r = 0.001$, as used in [4]. The estimated parameters using Matlab optimizers are given in Table 2.

Estimates	$\hat{\kappa}$	$\hat{\theta}$	$\hat{\sigma}$	$\hat{v_0}$	$\hat{\rho}$
	8.9213	0.0575	2.0000	0.0205	-0.7890

Table 2: Estimated parameters for the Heston model.

5.4 Simulation

The Heston model is represented by the bivariate system of stochastic differential equations and in order to derive option prices we have to use either an analytical approach or a simulation. Using a simulation, we can see how the process develops over time and what values we can expect for the spot asset and the volatility. Since it is a random process we generate random variables which will give us a different path in every simulation while the parameters remain the same. By changing the parameters we can describe the behaviours of paths and get a better understanding of their role.

5.4.1 Euler method

We will use the Euler approximation method similarly to how we used it for the CIR model. Following [4], we first integrate equation (5.1) and (5.2) and we heuristically obtain, under a risk-neutral probability measure for which μ is replaced by r given by a central bank,

$$S_{t+dt} = S_t + r\int_t^{t+dt} S_u\,du + \int_t^{t+dt} \sqrt{v_u} S_u\,dz_u, \tag{5.15}$$

$$v_{t+dt} = v_t + \int_t^{t+dt} \kappa(\theta - v_u)\,du + \int_t^{t+dt} \sigma\sqrt{v_u}\,dz_u. \tag{5.16}$$

Now we can use the Euler discretization and rewrite the equations as

$$S_{t+\Delta t} = S_t + rS_t\Delta t + \sqrt{v_t}S_t\epsilon_S\sqrt{\Delta t}, \tag{5.17}$$

$$v_{t+\Delta t} = v_t + \kappa(\theta - v_t)\Delta t + \sigma\sqrt{v_t}\epsilon_V\sqrt{\Delta t}. \tag{5.18}$$

From here we can simulate the processes as follows

(i) We set S_0 as the spot price and v_0 as the variance.

(ii) We generate independent $N(0,1)$ random variables ϵ_1 and ϵ_2 and set $\epsilon_V = \epsilon_1$ and $\epsilon_S = \rho\epsilon_V + \sqrt{1-\rho^2}\epsilon_2$.

(iii) We calculate Δv and from there ΔS.

There are two main problems regarding Euler approximation of the Heston model. The first problem is the slow speed of convergence and the second problem is the possibility of a negative value of simulated v_t, which is possible even if the Feller condition $2\kappa\theta > \sigma^2$ is satisfied. We can use two methods to solve the problem of negative variances.

(i) Full truncation scheme which replaces v_t with $v_t^+ = \max(0, v_t)$.

(ii) Reflection scheme which replaces v_t with $|v_t|$.

In our analysis, we use the reflection scheme because then the Euler approximated v_t does not have the tendency of staying too long time at the zero level.

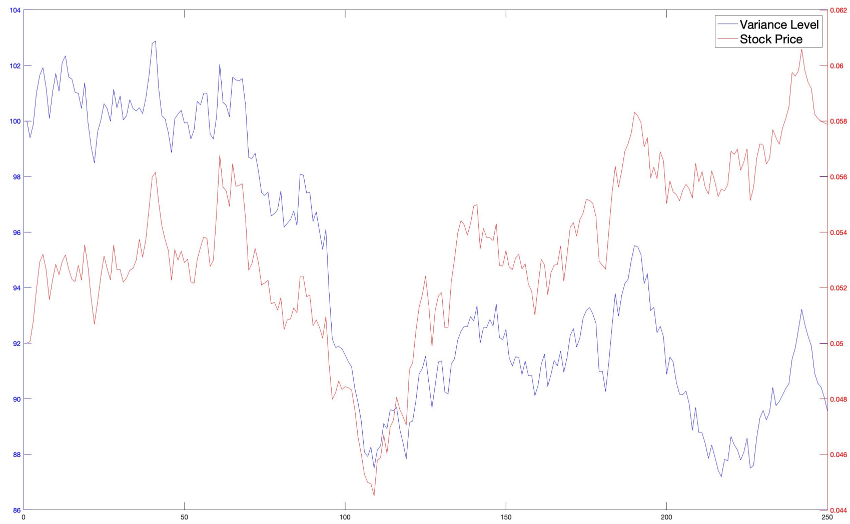

Figure 5: Euler simulation of the Heston model

With the empirical data from S&P 500 we obtained the empirical estimates using the method by Atiya and Wall. After that we simulated the data given the estimated parameters and then estimated the parameters from that simulated path to check the validity of the estimates. Every simulated path is different so here we provided one example of such an experiment in Figure 5. All the data is given in Table 2.

6 Discussion

6.1 Comparison of the models

The Black Scholes model is very useful and widely applicable in option and other corporate liability pricing, such as stocks and bonds. However, it makes an assumption that stock returns are normally distributed with known mean and variance. It doesn't depend on the mean spot return and it can not be generalized by variation of the mean. European call option prices can, under the Black Scholes model, be seen as a strictly increasing functions of implied volatility σ. This means that for a given European call price there is a unique value of σ. One problem with the Black Scholes model is that it predicts a flat profile for the implied volatility surface, while empirical facts indicate that it is not constant as a function of exercise price, nor as a function of time to maturity. In fact, it is in practice often found to be curved in a way that resembles a smile. Another problem arises with the asymmetric leptokurtic features analyzed by kurtosis.

On the other hand, the Heston model provides a model of stochastic volatility, which results in a graphical volatility smile. It assumes that the spot asset is correlated with volatility and has a closed-form solution for the price of a European call option. It also incorporates stochastic interest rates, as explained in [3].

In order to compare the Black Scholes and the Heston model together with the empirical data we use characteristic functions (3.4), (5.7) and (5.8).

In Figure 6 we compare the probability density functions of the Heston and the Black Scholes model. We first estimated the parameters from the empirical data and then we obtained the characteristic functions under a risk-neutral probability measure for the Heston model for which μ is replaced by r and the risk-neutral probability measure under the Black Scholes model for which μ is also replaced by r. From the characteristic functions we computed the probability density functions by taking the inverse Fourier transform. The empirical estimates of the parameters used in Figure 6 are listed in Table 1 for the Black Scholes model and in Table 2 for the Heston model. The figure is showing a return for 348 days as opposed to Figure 7 which presents daily returns.

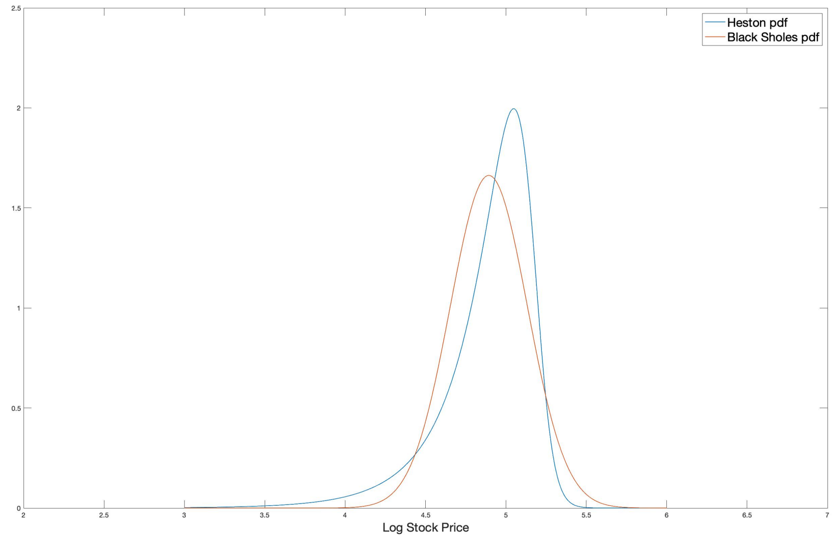

Figure 6: Comparison of the Black Scholes model and Heston model

To create the Figure 7 we first used the empirical estimates of the parameters with μ replaced by r and used kernel density estimate for the empirical log-returns. For the Black Scholes model, log return of the normal density function centered at r with the estimated σ was plotted. For the Heston log return density a long series of log returns were simulated with μ replaced by r and with applied kernel density. An alternative would be to use the characteristic function of the log return density and applying inverse Fourier transform but that approach turned out to perform poorly for the S&P 500 daily return data. We used the Euler approximation method for simulation and we had to take into consideration the possible errors that can occur. This is especially the case with the Heston model since we are using the reflection scheme to avoid the negative variances, which means that we are replacing v_t by $|v_t|$.

After that we used those paths to estimate parameters in order to check the validity of the estimate. Finally we compared the probability density functions of the simulated estimates of the Black Scholes and the Heston model with the probability density function of the empirical data. We obtained the figure of the empirical data but adapted to work under the risk neutral measure by using the formula $x^* = x - \bar{x} + r\Delta t - \dfrac{1}{2}\hat{\theta}\Delta t$.

From Figure 7 we can conclude that the Heston model performs better than the Black Scholes model since the density function of the simulated path is much closer to the density function of the empirical log returns. See also Figure 8 with empirical log returns which indicates that the volatility is indeed not constant which justifies a stochastic volatility modelling.

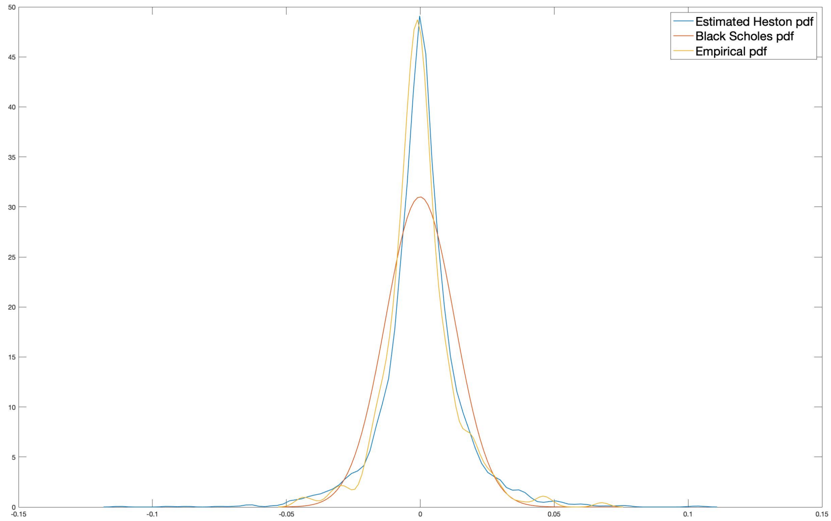

Figure 7: Comparison of the Black Scholes model, Heston model and empirical data

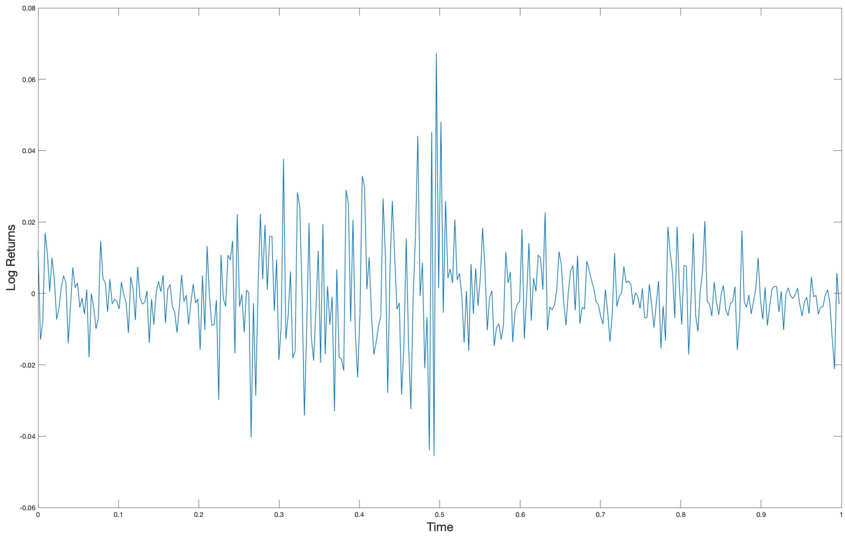

Figure 8: Empirical log returns

6.2 Future investigations

In future investigations it may be interesting to estimate the parameters under the empirical probability which are based on observations. That way we could have a more natural comparison between the modelled and empirical paths. Another extension of this work is to explore in more detail option pricing under the Heston model and estimate the parameters by implied volatilities. Further analysis can compare the estimated parameters using the discrete-time GARCH(1,1), which stands for generalized autoregressive conditional heteroscedasticity. This was discussed already in Heston's original article [12].

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A Matlab code

For Figure 1 and estimation of the Black Scholes parameters, along Figure 8 we use

```
%% Estimate the parameters for the Black Scholes model

S0 = readmatrix('SPY Historical Prices.xls','Range','G2:G2');
S=readmatrix('SPY Historical Prices.xls','Range','G2:G349');
```

```
dt=1/252;
R=diff(log(S),1);
est_mu=mean(R)/dt+var(R)/(2*dt)
est_sigma=sqrt (var(R)/dt)

%% Simulation of the Black Scholes model
n=4;
t=linspace(0,1,348)';
z =cumsum([zeros(1,n); sqrt(dt).*normrnd(0,1,length(t)
    -1,n)]) ; % Brownian motion
SS = S0*exp((est_mu-est_sigma^2 /2)*t*ones(1,n)+
    est_sigma*z) ; %Simulation of the Black Scholes path

plot (t,SS,'LineWidth',1)
hold on
plot (t,S,':','LineWidth',2)
%plot (t(1:end-1),R,'LineWidth',1) %Log-returns
xlabel('Time','FontSize',18)
ylabel('Stock Price','FontSize',18)
l=legend('Black Scholes path','Black Scholes path','
    Black Scholes path','Black Scholes path','Empirical
    path');
l.FontSize=18;
```

For the Figure 2 we use www.mathworks.com/matlabcentral/fileexchange/16670-simulate-a-cox-ingersoll-ross-process for the code below

```
speed=0.13; %alpha
level=2.17; %mu
sigma=0.16;
t = 0:1:252;
r0 = 2.17;

n = 2502;    % number of simulated observations
dt = 1;      % time increment = 1 day

r= cirpath(t,speed,level,sigma,r0);

CIR = cir(speed,level,sigma,'StartState',r0)
[Paths,Times] = simByEuler(CIR,n,'DeltaTime', dt);

plot(Times, Paths,'LineWidth',1)
```

```
hold on
plot(t,r,'LineWidth',1)
xlim([0,252])
xlabel('Time','FontSize',18), ylabel('Stock Price','
    FontSize',18)
l=legend('Euler simulated path','Exact simulated path')
l.FontSize=18;
```

For Figure 3, Figure 4 and Figure 5 we use the code from [4].
Modifying the code from [4] we obtain Figure 6.

```
% Illustration of the log-returns density
% Effect of vol of variance (sigma)
clc; clear;

% Option settings
S = readmatrix('SPY Historical Prices.xls','Range','G2:
    G2');
T = 348/252;    % Maturity in years
rf = 0.001;     % Risk free rate
q = 0;

% Heston parameters
param.kappa =  8.9213;
param.theta =  0.0575;
param.sigma =  2.0000;
param.v0    =  0.0205;
param.rho   =  -0.7890;
trap = 1;
lambda = 0;

sigma=0.2042;
mu=0.001;

%% Obtain the density of log(S(T)) by inverting the
    characteristic function
dphi = 0.01;
phi = [1e-10:dphi:100];
dx = 0.001;
xT = [3:dx:6];
for x=1:length(xT)
    intlo  = real(exp(-i.*phi.*xT(x)).*HestonCF(phi,
```

```
        param ,T,S,rf,q,trap));
    intbs= real(exp(-i.*phi.*xT(x)).*exp(i.*phi.*(log(S)+(mu-0.5.*sigma.*sigma).*T) -0.5.*(phi.*phi).*sigma.*sigma.*T));
    flo(x)  = trapz(intlo) /pi*dphi;
    fbs(x) = trapz(intbs)/pi*dphi;
end

%% Plot the densities

plot(xT,flo,xT,fbs,'LineWidth',1)
axis([2 7 0 2.5])
l=legend('Heston pdf','Black Sholes pdf')
l.FontSize = 18;
xlabel('Log Stock Price',FontSize=18);
```

Modifying the code from [4] we obtain Figure 7.

```
clc; clear;
%% Heston simulation using estimated parameters based on empirical data
% Option features

r = 0.001;              % Risk free rate
q = 0;              % Dividend yield
Mat = 3000/252;              % Maturity in years
S0 = readmatrix('SPY Historical Prices.xls','Range','G2:G2');      % Spot price
K  = 90;              % Strike price
PutCall = 'C';        % 'P'ut or 'C'all

% Heston parameters
kappa =  8.9213;        % Variance reversion speed
theta =  0.0575;     % Variance reversion level
sigma =  2.0000;      % Volatility of Variance
v0    =  0.0205;     % Initial variance
lambda = 0;        % Risk parameter

% Simulation features
N = 1;               % Number of stock price paths
T = 3000;              % Number of time steps per path
alpha = 0.5;         % Weight for explicit-implicit
```

```
scheme
negvar = 'R';        % Use the reflection scheme for negative variances
rho    =   -0.7890;        % Correlation between Brownian motions
params = [kappa theta sigma v0 rho lambda];

%% Simulate the processes and obtain the option prices
schemeV = 'E';
[S V F Price] = EulerMilsteinPrice(schemeV,negvar,params,PutCall,S0,K,Mat,r,q,T,N,alpha);

writematrix(S,'data.xls');

%% Characteristic functions

S0 = readmatrix('SPY Historical Prices.xls','Range','G2:G2');
S=readmatrix('SPY Historical Prices.xls','Range','G2:G349');
dt =1/252;
RE=diff(log(S),1); %Empirical log returns
R=RE-mean(RE)+r*dt-0.5*theta*dt; %Empirical log returns adapted under risk neutral measure

est_sigma=sqrt(var(RE)/dt); % sigma for the Black Scholes model

D=readmatrix('data.xls','Range','A1:A3000');
RD=diff(log(D),1);
[ RD1 , RD2]=ksdensity (RD) ; %Heston simulated path
[ R1 , R2]=ksdensity (R) ; %Empirical data

plot(RD2,RD1,'LineWidth',1);
hold on
plot (R2,normpdf(R2,(r-est_sigma^2/2)*dt,est_sigma*sqrt(dt)),'LineWidth',1);
hold on
plot(R2,R1,'LineWidth',1);

l=legend('Estimated Heston pdf','Black Scholes pdf','
```

```
    Empirical pdf');
l.FontSize=18;
```

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