Heston model: shifting on the volatility surface

Fitting the implied volatility surface is generally a complicated affair. Here, Claudio Pacati, Roberto Renò and Manola Santilli propose a simple extension of the Heston model that allows fast and arbitrage-free interpolation of the volatility surface with just one time-dependent parameter

Ideally, an option pricing model would have the following properties: it would be guaranteed to be free of arbitrage opportunities; it would provide simple (and fast) pricing and hedging formulas; it would readily fit the quoted volatility surfaces across both maturities and strikes; it would avoid overfitting, ie, keep the number of parameters parsimoniously small; and it would adequately describe price and volatility risk.

Arbitrage-free volatility interpolation can be achieved with local volatility models (Derman & Kani 1994; Dupire 1994). However, local volatility models do not accommodate volatility risk. The Heston (1993) and SABR (Hagan *et al* 2002) models include volatility risk in a mathematically convenient way, but they are not flexible enough to fit the whole volatility surface, unless parameters are made time-dependent. The class of stochastic local volatility (SLV) models (Guyon & Henry-Labordère 2012; Ren, Madan & Qian 2007) mixes a local factor with a stochastic factor to overcome these issues. These models can effectively be used to calibrate to exotics, such as barrier options. However, a calibration of the volatility surface using a stochastic volatility model is anyway often required as a preliminary step (see, for example, Tian *et al* 2014).

This article studies affine stochastic volatility models enriched with a time-dependent displacement meant to accommodate the term structure of implied volatilities. The models considered here can be seen as special cases of SLV models, and in particular of a displaced diffusion. Our specifications are of note since only one time-dependent parameter tends to fit the volatility surface reasonably well. We label this class of models Heston++, since the idea is inspired by the deterministic shift extension introduced by Brigo & Mercurio (2001) to fit the term structure of interest rates.

We show that, by adding flexibility on the term structure, the Heston++ models can dramatically improve the fitting of the volatility surface, while, at the same time, preserving the affine structure of the model (and its mathematical tractability) and keeping the number of parameters reasonable.

To illustrate the gain, in terms of pricing, of the proposed model we calibrate it on all daily option panels on the foreign exchange rate €/US\$ from 2005 to 2012, for strikes up to 10Δ and 10 maturities ranging from one week to two years. The Heston++ models are readily fitted with virtually no additional computational cost with respect to the standard versions (MATLAB code is available upon request) and obtain, for the best model considered here, an average daily root mean squared error on implied volatilities of 0.167%, corresponding to a root mean squared relative error of 1.26%.

The Heston++ model

We start with a one-factor specification. In a market in which there is a constant risk-free interest rate r, we assume the price process S_t of a non-dividend-paying underlying follows the risk-neutral dynamics:

$$dS_t = rS_t\,dt + S_t\sqrt{V_t + \phi_t}\,dW_t^S \tag{1}$$

$$dV_t = \alpha(\beta - V_t)\,dt + \Lambda\sqrt{V_t}\,dW_t^V \tag{2}$$

where r, α, β and Λ are non-negative constants with $2\alpha\beta \geqslant \Lambda^2$, $\phi_t \geqslant 0$ is a sufficiently smooth non-negative displacement function (see, for example, Antonov, Arneguy & Audet 2008) with $\phi_0 = 0$, and W_t^S and W_t^V are correlated Wiener processes with time-dependent instantaneous risk-neutral correlation:

$$\operatorname{corr}(dW_t^S, dW_t^V) = \rho\sqrt{\frac{V_t}{V_t + \phi_t}} \tag{3}$$

where $\rho \in [-1, 1]$ is an additional constant. The form of the correlation in (3) guarantees the linearity of the pricing partial differential equation (PDE) associated with the model. The required conditions $|\rho| \leqslant 1$ and $\phi_t \geqslant 0$ guarantee a correlation not exceeding 1 in absolute value. In the extended model, ρ can thus be interpreted as an upper bound on the correlation. The model is then specified by the risk-free rate, four parameters $(\alpha, \beta, \Lambda, \rho)$ and the deterministic function ϕ_t. The risk-free rate is constant for ease of exposition only, and can be made stochastic with an affine diffusion with straightforward modifications of the pricing formulas.

The classical Heston (1993) model is recovered when $\phi_t = 0$. However, the addition of the deterministic factor ϕ_t allows automatic fitting of the model-free implied volatilities, that is of the risk-neutral integrated variance, which is given for $T \geqslant 0$ by:

$$\mathrm{IV}(0, T) = \frac{1}{T}\mathbb{E}^{\mathbb{Q}}\biggl[\int_0^T V_t\,dt\biggr] + \frac{1}{T}\int_0^T \phi_t\,dt$$

that is:

$$\mathrm{IV}(0, T) = \mathrm{IV}^{\mathrm{H}}(0, T) + \frac{1}{T}I_\phi(0, T) \tag{4}$$

where it can be easily shown that:

$$\mathrm{IV}^{\mathrm{H}}(0, T) = \frac{V_0 - \beta}{\alpha T}(1 - e^{-\alpha T}) + \beta \tag{5}$$

and where:

$$I_\phi(t, T) = \int_t^T \phi_u\,du \quad \text{for } 0 \leqslant t \leqslant T \tag{6}$$