Fundamental Theorem of Calculus (FTC) and Its Relationship with Density Functions and Cumulative Distribution Functions (CDFs)

1. The Fundamental Theorem of Calculus (FTC)

The Fundamental Theorem of Calculus (FTC) establishes the essential connection between **differentiation** and **integration**. It is usually divided into two complementary parts:

FTC Part 1 – Differentiation of an Integral

If f is continuous on an interval [a,b], and we define

$$F(x) = \int_a^x f(t)\,dt,$$

then F is differentiable on (a,b), and its derivative is

$$F'(x) = f(x).$$

Interpretation: Differentiation and integration are inverse processes. The derivative of the accumulated area under the curve of f(t)f(t)f(t) is simply the function itself.

FTC Part 2 – Evaluation of a Definite Integral

If F is any antiderivative of f (i.e., $F'(x)=f(x)$), then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Interpretation: A definite integral can be computed by evaluating an antiderivative of the integrand at the endpoints of the interval.

2. Connection to Probability Theory

In probability theory, the FTC provides the mathematical foundation for linking **probability density functions (PDFs)** and **cumulative distribution functions (CDFs)** for continuous random variables.

2.1 Probability Density Function (PDF)

A **probability density function** f(x) describes how probability is distributed over the real line for a continuous random variable X.

Key properties:

- $f(x) \geq 0$ for all x.
- The total probability is 1:

$$\int_{-\infty}^{\infty} f(x)\,dx = 1.$$

Important Note: For continuous random variables, f(x) itself is *not* the probability that X takes the value x. Instead, probabilities are obtained as **areas under the density curve**. For example, the probability that X lies in an interval [a,b] is:

$$P(a \leq X \leq b) = \int_{a}^{b} f(t)\,dt.$$

2.2 Cumulative Distribution Function (CDF)

The **cumulative distribution function** (CDF) of X is defined as:

$$F(x) = P(X \leq x).$$

Using the PDF, this can be expressed as:

$$F(x) = \int_{-\infty}^{x} f(t)\,dt.$$

This definition is a direct application of FTC Part 2: the CDF accumulates the probability density up to the point x.

2.3 Connecting PDFs and CDFs via the FTC

The relationship between PDFs and CDFs follows directly from the two parts of the FTC:

1. **From CDF to PDF (Differentiation):**
   If F(x) is differentiable, then by FTC Part 1:

$$f(x) = F'(x).$$

The PDF is the derivative of the CDF.

2. **From PDF to CDF (Integration):**
   Since the CDF is defined as an integral of the PDF, by FTC Part 2:

$$F(x) = \int_{-\infty}^{x} f(t)\,dt.$$

The CDF is the accumulated area under the PDF curve.

3. **Probability over an interval:**
   The probability that X falls between two values a and b can be written either as an integral of the PDF or as a difference of CDF values:

$$P(a \le X \le b) = \int_{a}^{b} f(t)\,dt = F(b) - F(a).$$