Math 231 Calculus - Spring 2016

§2.2 The Limit of a Function

Example. Julia's sushi and salad buffet costs \$10 per pound, but if you get exactly one pound, your meal is free. Let $y = f(x)$ represent the price of your lunch in dollars as a function of its weight x in pounds.

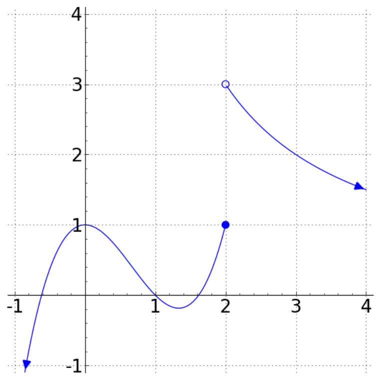
Write an equation $y = f(x)$ as a piecewise defined function.

Draw a graph of $y = f(x)$.

Describe the behavior of $f(x)$ when x is near 1 but not equal to 1.

Definition. (Informal definition) For real numbers a and L, $\lim_{x \to a} f(x) = L$ means

Example. Describe the behavior of $g(x)$ (graphed below) when x is near 2.

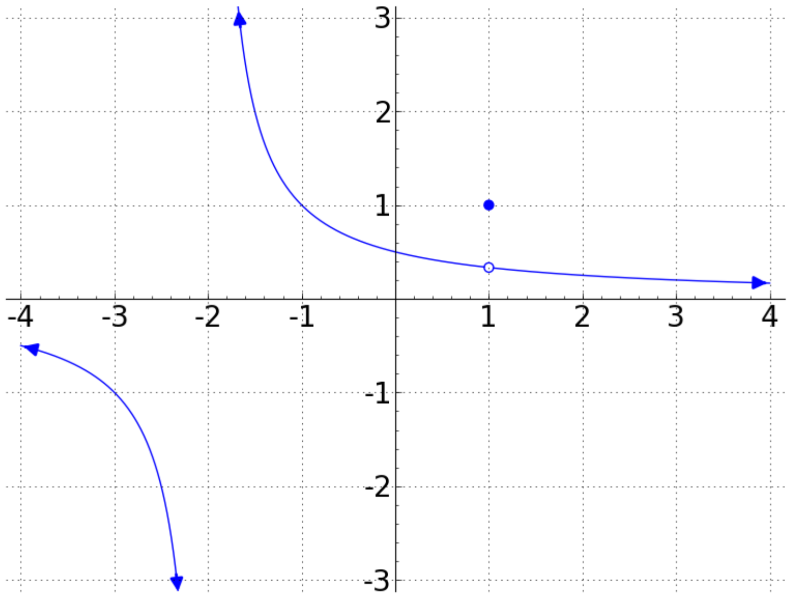


Definition. (Informal definition) For real numbers a and L,

$\lim_{x \to a^-} f(x) = L$ means

$\lim_{x \to a^+} f(x) = L$ means

Example. Describe the behavior of $y = h(x)$ when x is near -2 in terms of limits.



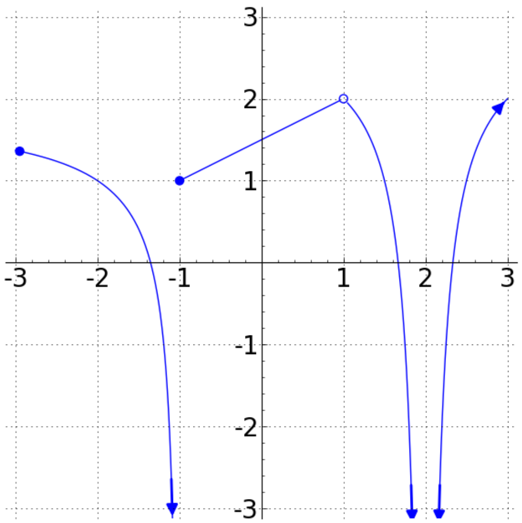
Review. $\lim_{x \to 2} f(x) = 5$ means:

$\lim_{x \to 3^+} f(x) = 4$ means:

$\lim_{x \to 3^-} f(x) = 1$ means:

$\lim_{x \to -1^+} f(x) = \infty$ means:

Example. For the function $f(x)$ graphed below,



Find $\lim_{x \to -1^-} f(x)$ $\quad \lim_{x \to -1^+} f(x)$ $\quad \lim_{x \to -1} f(x)$

$\lim_{x \to 1^-} f(x)$ $\quad \lim_{x \to 1^+} f(x)$ $\quad \lim_{x \to 1} f(x)$

$\lim_{x \to 2^-} f(x)$ $\quad \lim_{x \to 2^+} f(x)$ $\quad \lim_{x \to 2} f(x)$

At what values of a does $\lim_{x \to a} f(x)$ fail to exist as a finite number?

Question. If $\lim\limits_{x\to a^+} f(x)$ and $\lim\limits_{x\to a^-} f(x)$ exist, does $\lim\limits_{x\to a} f(x)$ exist?

Question. What is the relationship between limits and asymptotes?

Note. Ways that limits can fail to exist as a finite number:

Example. What is $\lim_{x \to 0} \sin\left(\frac{\pi}{x}\right)$? Use a table of values for evidence.

True or False: If $\lim_{x\to 2} f(x) = 5$ then $f(2) = 5$.

True or False: If $f(2) = 5$ then $\lim_{x\to 2} f(x) = 5$.

Extra Example. Sketch the graph of a function $f(x)$ defined for $-2 < x < 3$ with the following properties:

1. $\lim_{x\to -2^+} f(x) = 3$
2. $\lim_{x\to 0} f(x) = -2$
3. $f(0) = 4$
4. $\lim_{x\to 1^-} f(x) = \infty$
5. $\lim_{x\to 1^+} f(x) = 4$
6. $f(1) = 4$
7. $\lim_{x\to 3^-} f(x) = 0$

Extra Example. Use a table of values to estimate the value of the limit. Use a graph to confirm your result.

1. $\lim_{x \to 0} \frac{\sin(x)}{\sin(\pi x)}$
2. $\lim_{x \to 3} \frac{|x - 3|}{x - 3}$
3. $\lim_{x \to 1} \frac{x + 1}{x - 1}$

§2.3 Calculating Limits Using the Limit Laws

Example. Suppose $\lim_{x\to 7} f(x) = 30$ and $\lim_{x\to 7} g(x) = 2$. What is $\lim_{x\to 7} \dfrac{f(x)}{-3g(x)}$?

Theorem. *(Limit Laws) Suppose that c is a constant and the limits* $\lim_{x\to a} f(x)$ *and* $\lim_{x\to a} g(x)$ *exist as finite numbers. Then*

- $\lim_{x\to a}[f(x) + g(x)]$

- $\lim_{x\to a}[f(x) - g(x)]$

- $\lim_{x\to a}[cf(x)]$

- $\lim_{x\to a}[f(x)g(x)]$

- $\lim_{x\to a} \dfrac{f(x)}{g(x)}$ *, provided that*

Example. Find $\lim_{x \to 2} \dfrac{x^2 + 3x + 6}{x + 9}$

Review. If $\lim_{x\to 7} f(x) = 4$ and $\lim_{x\to 7} g(x) = -8$, then what is $\lim_{x\to 7} \dfrac{(f(x))^2}{g(x)+10}$?

Review. The limit laws make it easy to evaluate limits like

$$\lim_{x\to -2} \frac{x+7}{x^2+1}$$

What is the easy way to evaluate this and what is the answer?

Question. Which of these limits can be evaluated just by plugging in?

- $\lim_{x \to -3^+} \frac{\sqrt{x+3}}{x^2+1}$
- $\lim_{x \to 3} \frac{-4x}{x-3}$
- $\lim_{x \to 1} \frac{x^3-1}{x^2-1}$

Note. $\lim_{x \to a} \frac{f(x)}{g(x)}$ can usually be evaluated just by plugging in a as long as ....

Question. When $\lim_{x\to a} g(x) = 0$, how can we find $\lim_{x\to a} \dfrac{f(x)}{g(x)}$?

Example. Find $\lim_{x \to 3} \frac{-4x}{x-3}$

Example. Find $\lim\limits_{x \to -4} \dfrac{5x}{|x+4|}$

Example. Use algebra to find the following limits. If the limit DNE as a finite number, decide if it is ∞, $-\infty$, or just DNE.

1. $\lim_{x\to 1} \dfrac{x^3-1}{x^2-1}$
2. $\lim_{z\to 0} \dfrac{(5-z)^2-25}{z}$
3. $\lim_{r\to 0} \dfrac{\frac{1}{r+3}-\frac{1}{3}}{r}$
4. $\lim_{x\to 1} \dfrac{\sqrt{x+3}-2}{x-1}$
5. $\lim_{x\to -5} \dfrac{2x+10}{|x+5|}$

1. $\lim\limits_{x \to 1} \dfrac{x^3 - 1}{x^2 - 1}$

2. $\lim_{z \to 0} \dfrac{(5-z)^2 - 25}{z}$

3. $\lim_{r\to 0} \dfrac{\frac{1}{r+3} - \frac{1}{3}}{r}$

4. $\lim_{x\to 1} \dfrac{\sqrt{x+3}-2}{x-1}$

5. $\lim\limits_{x \to -5} \dfrac{2x + 10}{|x + 5|}$

1. $\lim\limits_{x \to 1} \dfrac{x^3 - 1}{x^2 - 1}$
2. $\lim\limits_{z \to 0} \dfrac{(5 - z)^2 - 25}{z}$
3. $\lim\limits_{r \to 0} \dfrac{\frac{1}{r+3} - \frac{1}{3}}{r}$
4. $\lim\limits_{x \to 1} \dfrac{\sqrt{x + 3} - 2}{x - 1}$
5. $\lim\limits_{x \to -5} \dfrac{2x + 10}{|x + 5|}$

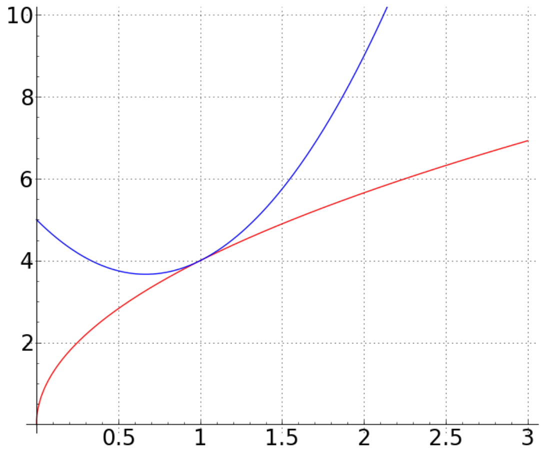
Question. Do all $\frac{0}{0}$ limits exist?

Note. If $\lim\limits_{x \to a} f(x) \neq 0$ and $\lim\limits_{x \to a} g(x) = 0$ then $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)}$ COULD BE ...

Note. If $\lim\limits_{x \to a} f(x) = 0$ and $\lim\limits_{x \to a} g(x) = 0$ then $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)}$ COULD BE ...

Example. For the function $g(x)$ suppose that, for x-values near 1,

$$4\sqrt{x} \leq g(x) \leq 3x^2 - 4x + 5$$



What can we say about $\lim_{x \to 1} g(x)$?

Theorem. *(The Squeeze Theorem)*

Example. Find $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right)$.

Review. True or False: If $f(x) \le g(x) \le h(x)$ near $x = 3$ and $\lim_{x\to 3} f(x)$ exists and $\lim_{x\to 3} h(x)$ exists, then $\lim_{x\to 3} g(x)$ exists.

Review. Suppose

$$\frac{t^2+t-6}{t+3} \le \frac{f(t)}{2t} \le \frac{t^2+3t-1}{t+2}$$

Find $\lim_{t\to -1} f(t)$.

§2.5 Continuity

Definition. (Informal Definition) A function $f(x)$ is continuous if

Question. What are some ways that a function can fail to be continuous?

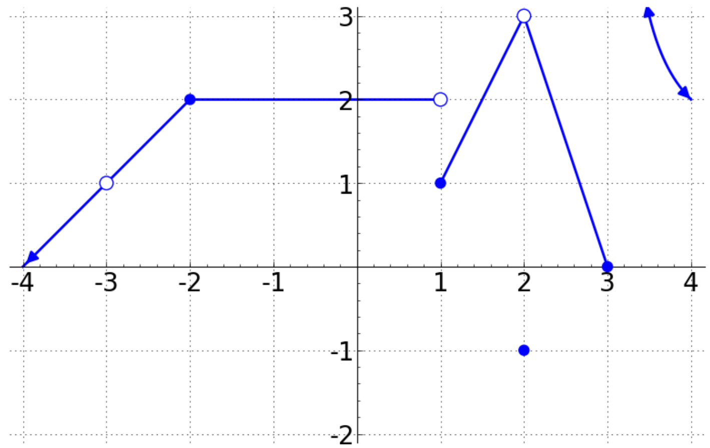
Definition. A function $f(x)$ is continuous at a number a if

1.

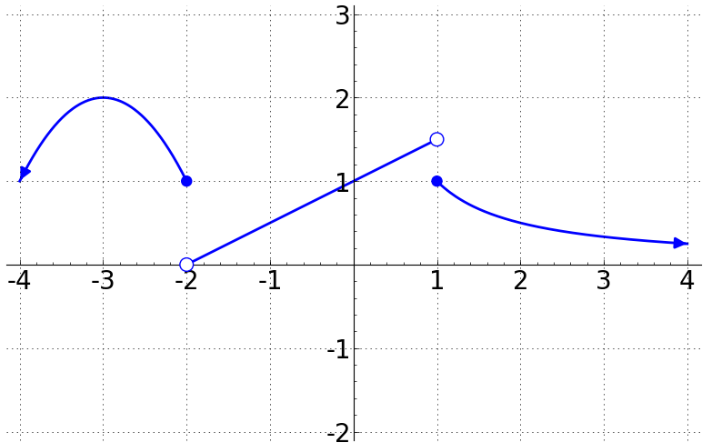
2.

3.

Example. What are all the places where f is not continuous, and why?



Example. Consider the function $f(x)$ at $x = -2$ and at $x = 1$.

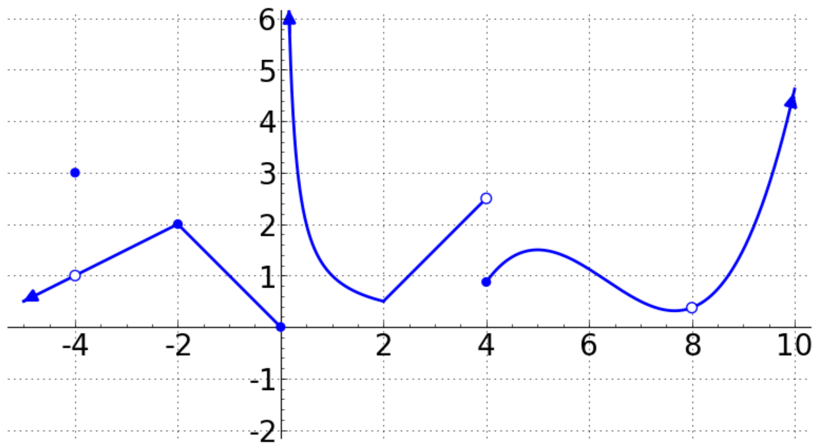


Definition. A function $f(x)$ is **continuous from the left** at $x = a$ if

A function $f(x)$ is **continuous from the right** at $x = a$ if

Review. We say that a function $f(x)$ is continuous at $x = a$ if ...

1. At what points is f discontinuous, and why?



2. At what points is f continuous from the right but not continuous?

3. At what points is f continuous from the left but not continuous?

Question. If $f(x)$ is continuous from the left and from the right at $x = a$, is it continuous at $x = a$?

Example. Consider the function defined as follows, where a and b are unknown constants.

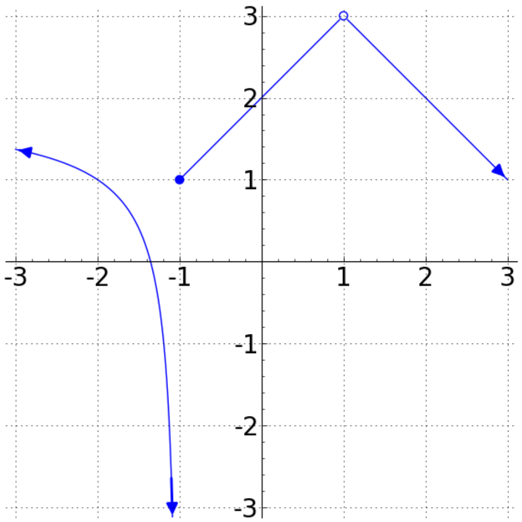
$$f(x) = \begin{cases} ax^2 & \text{if } x < -2 \\ bx + \sqrt{x+3} & \text{if } -2 \leq x \leq 1 \\ 4 + ax & \text{if } 1 < x \end{cases}$$

What values of a and b will make f continuous?

Definition. We say that a function $f(x)$ is continuous on the interval (b, c) if

We say that $f(x)$ is continuous on an interval $[b, c]$ if

Example. On what intervals is $g(x)$ continuous?



Question. What kinds of functions are continuous everywhere?

Question. What kinds of functions are continuous on their domains?

Example. For what real numbers is $f(x) = \ln\left(\frac{\cos^2(x)}{x}\right)$ discontinuous?

Example. Find $\lim_{x\to 0} \cos(x)$.

Example. Find $\lim_{x\to 2} \cos\left(\dfrac{(x^2-4)\pi}{12x-24}\right)$.

Note. If $f(x)$ is continuous, them $\lim_{x\to a} f(g(x)) =$

Question. True or False: If $f(x)$ is continuous on $[0, 4]$ and $f(0) = 5$ and $f(4) = 1$, then $f(a) = \sqrt{2}$ for some number a.

Theorem. *(Intermediate Value Theorem)*

Example. Prove that the polynomial

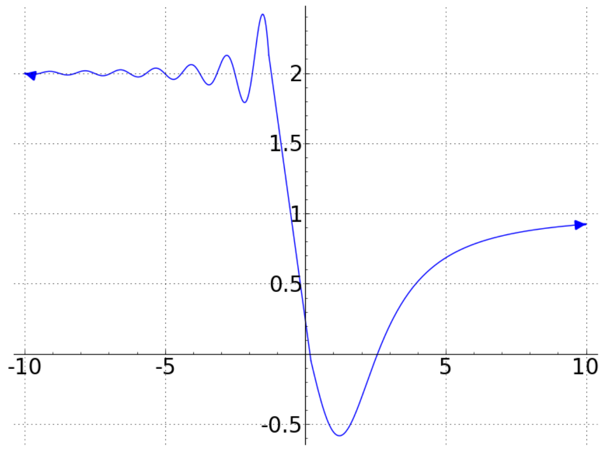
$$p(x) = 5x^4 - 3x^3 - 12x - 25$$

has a real root.

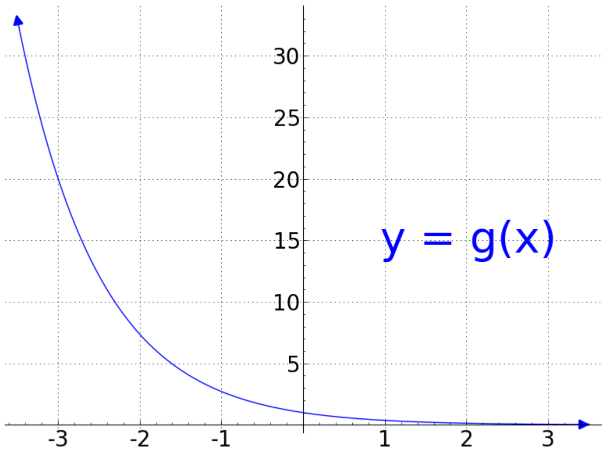
Question. Are there two points on exactly opposite sides of the earth that at this moment have the exact same temperature?

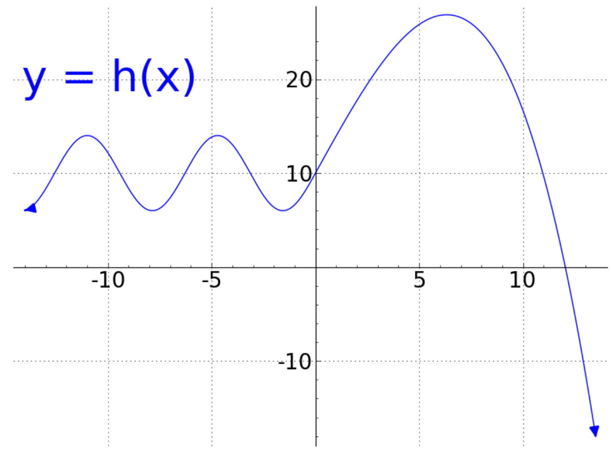
S2.6 Limits at Infinity

Example. What happens to $f(x)$ as x goes through larger and larger positive numbers? Larger and larger negative numbers?



Example. For the functions $g(x)$ and $h(x)$ drawn below, what are the limits at infinity?





Example. What are $\lim_{x\to\infty} \frac{1}{x}$ and $\lim_{x\to-\infty} \frac{1}{x}$?

Example. What are $\lim_{x\to\infty} \frac{1}{x^3}$ and $\lim_{x\to\infty} \frac{1}{\sqrt{x}}$?

Note. $\lim_{x\to\infty} \frac{1}{x^r} = 0$ whenever ....

Example. Evaluate $\lim_{x\to\infty} \dfrac{5x^2 - 4x}{2x^3 - 11x^2 + 12x}$

Example. Find $\lim_{x \to -\infty} \dfrac{3x^3 + 6x^2 + 10x + 2}{2x^3 + x^2 + 5}$

Example. Find $\displaystyle\lim_{x \to -\infty} \frac{x^4 - 3x^2 + 6}{-5x^2 + x + 2}$

Recap:

$$\lim_{x\to\infty} \frac{5x^2 - 4x}{2x^3 - 11x^2 + 12x}$$

$$\lim_{x\to-\infty} \frac{3x^3 + 6x^2 + 10x + 2}{2x^3 + x^2 + 5}$$

$$\lim_{x\to-\infty} \frac{x^4 - 3x^2 + 6}{-5x^2 + x + 2}$$

Review. What is

$$\lim_{x\to-\infty} \frac{-2x^4 + 3x^3 - 4x}{3x^3 + 5x + 6}$$

Review. What is

$$\lim_{x\to\infty} \frac{(5x+1)(3x-2)}{(x-4)(2x+1)}$$

Question. If the graph of $y = f(x)$ has a horizontal asymptote at $y = -2$, then what can we conclude about a limit of $f(x)$?

Question. True or False: A function can have at most one horizontal asymptote.

Question. What are the horizontal and vertical asymptotes of $f(x) = \frac{1-4x^2}{2x^2+x}$?

Example. Find

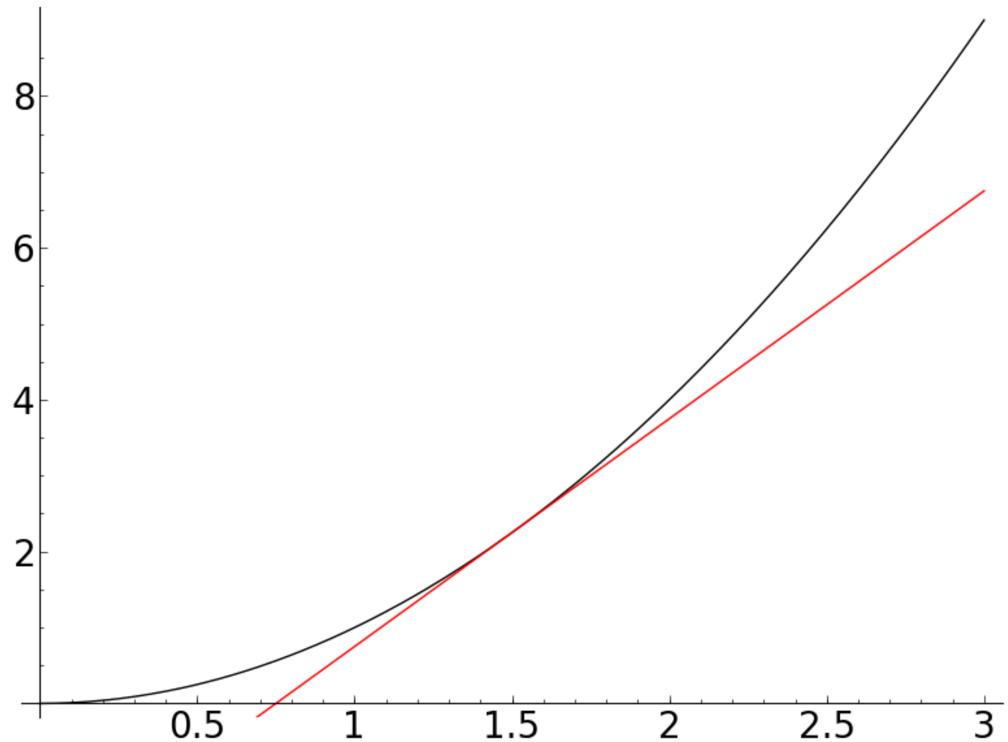
$$\lim_{x \to -\infty} \frac{\sqrt{13x^2 - 3x + 2}}{5x + 6} \qquad \text{and} \qquad \lim_{x \to \infty} \frac{\sqrt{13x^2 - 3x + 2}}{5x + 6}$$

Example. Find $\lim_{x\to\infty} -2x^3 + 11x^2 + 12x$

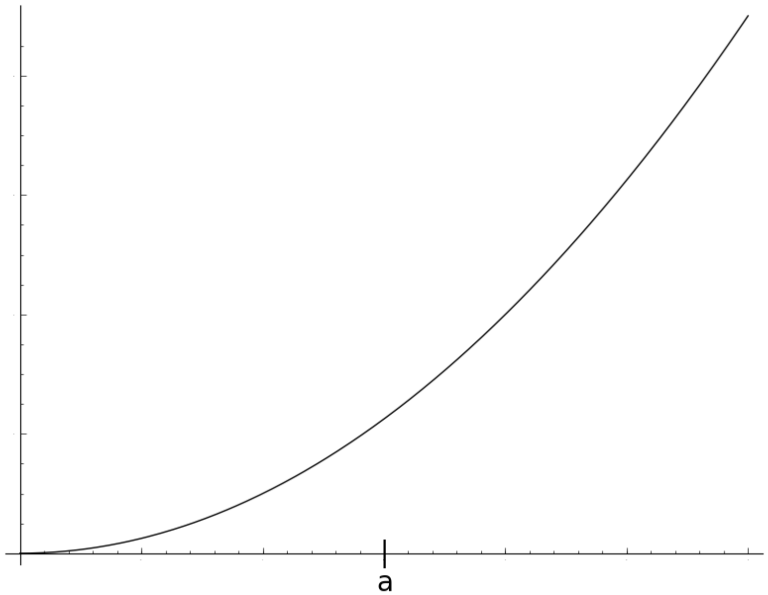
Example. Evaluate $\lim_{x\to\infty} \sqrt{3+2x^2} - 5x$

S2.7 Derivatives

Find the slope of the tangent line.



In general, to find the slope of the tangent line for a function $y = f(x)$ at the point $(a, f(a))$, we can take a limit of slopes of secant lines.



The slope of a secant line through $(a, f(a))$ and $(x, f(x))$ is:

The slope of the tangent line at $(a, f(a))$ is:

An alternate expression for the slope of the tangent line at $(a, f(a))$ is:

Definition. The **derivative** a function $y = f(x)$ at an x-value a is:

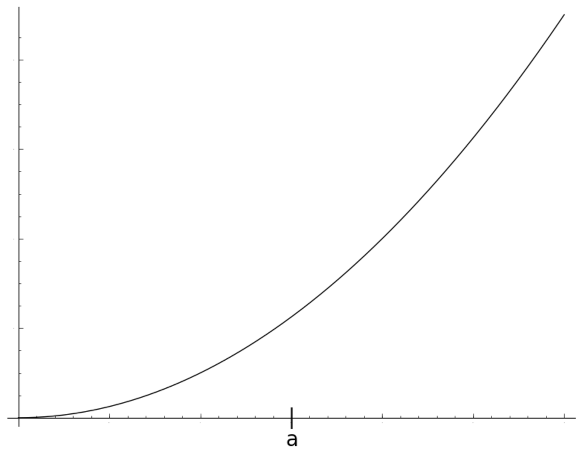
OR

Example. The following expressions represent the derivative of some function at some value a. For each example, find the function and the value of a.

1. $\lim_{x \to -1} \dfrac{(x+5)^2 - 16}{x+1}$

2. $\lim_{h \to 0} \dfrac{3^{2+h} - 9}{h}$

Review. The derivative of a function $f(x)$ at $x = a$, is defined as the slope of the ____________________________ , which is the limit of the slopes of ____________________________ .



The slope of a secant line through $(a, f(a))$ and $(x, f(x))$ is:

An alternate expression for the slope of this secant line is:

The slope of the tangent line at $(a, f(a))$ is:

OR

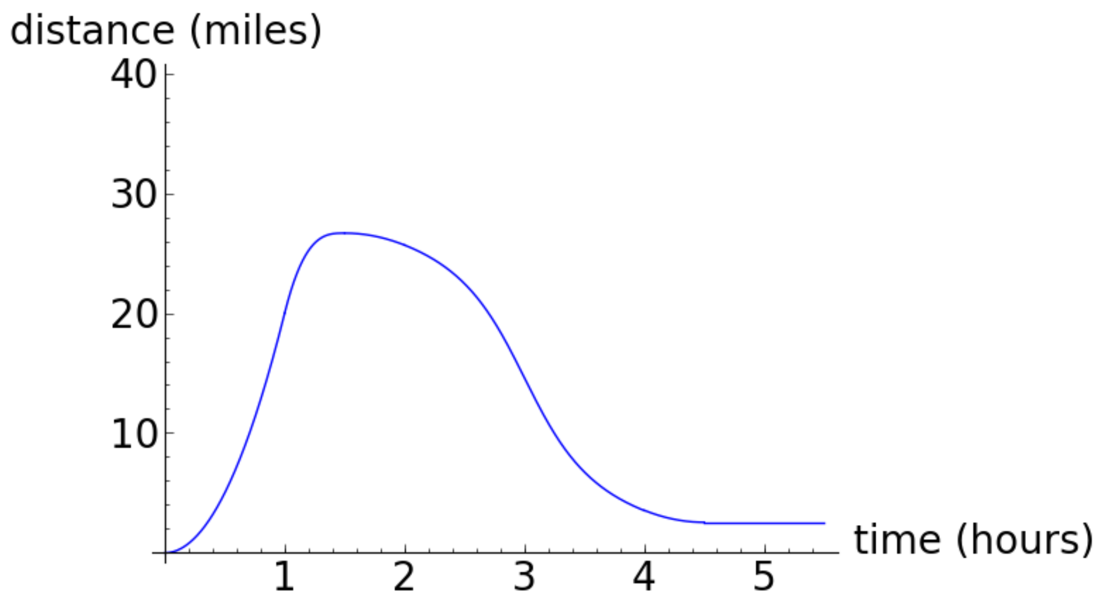
Example. Write out the limit definition of the derivative of $f(x) = 5x + 1/x$ at $x = 1$. You do not need to calculate the limit.

Example. This expression represents the derivative of what function at which value of a?

$$\lim_{x \to 3} \frac{\frac{1}{\sqrt{12x}} - \frac{1}{6}}{x - 3}$$

Derivatives and Rates of Change

Example. The graph of $y = f(x)$ represents my distance from campus on a bike ride heading due north.



- Interpret the slope of secant line through the points $(3, f(3))$ and $(4, f(4))$.
- Interpret the slope of the tangent line at $x = 3$.

Example. Suppose $f(x)$ represents the temperature of your coffee in degrees Fahrenheit as a function of time in minutes x since you've set it on the counter. Interpret the following equations:

1. $f(0) = 140$

2. $f(10) - f(0) = -20$

3. $\dfrac{f(10) - f(0)}{10} = -2$

4. $f'(15) = -0.5$

Extra Example. Suppose $g(x)$ represents the fuel efficiency of a Toyota Prius in miles/gallon as a function of x, the speed in miles/ hour that it is traveling. Interpret the following equations.

1. $g(45) = 52$

2. $g(40) - g(35) = 10$

3. $\dfrac{g(40) - g(35)}{5} = 2$

4. $g'(60) = -2$

Extra Example. The following table represents the amount of snow that has accumulated, $s(t)$, in inches, at t hours after noon.

t	0	2	4	6	8	10
$s(t)$	0	3.2	7.2	10.0	10.8	11.2

Question. If $s(t)$ represents the accumulation of snow in inches at t hours after noon, what does $s'(8)$ represent?

Example. Find the derivative of $f(x) = \dfrac{1}{\sqrt{3-x}}$ at $x = -1$.

Example. Find the equation of the tangent line to $y = x^3 - 3x$ at $x = 2$.

Extra Example. Find the derivative of $f(x) = 5x + 1/x$ at $x = 1$.

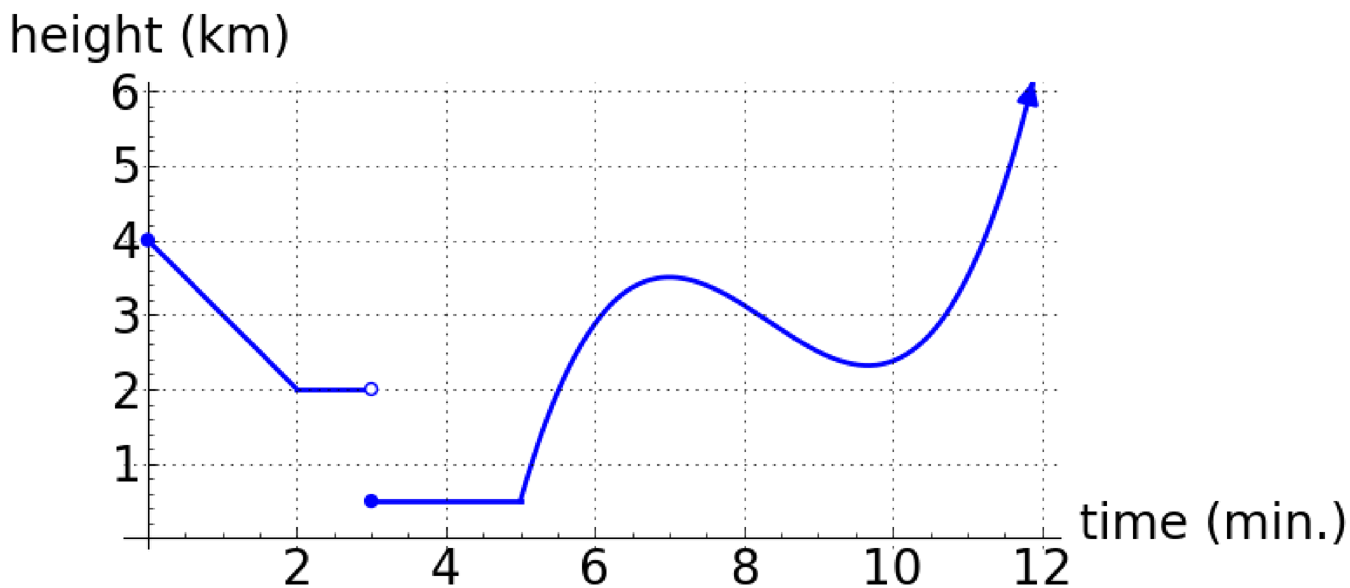
Extra Example. If the equation of the tangent line to the graph of $y = p(x)$ at $x = 5$ is $y = 3x - 2$, find $p(5)$ and $p'(5)$.

§2.8 The Derivative as a Function

Recall: The derivative of $f(x)$ at $x = a$ is given by:

Example. For the function $f(x) = \sqrt[3]{x}$, find the derivative $f'(x)$.

Example. The height of an alien spaceship above the earth's surface is graphed below. Graph the rate of change of the height as a function of time.



Note. Ways that a derivative can fail to exist at $x = a$.

Definition. A function is **differentiable** at $x = a$ if ...

Definition. A function is **differentiable** on an open interval (b, c) if ...

Theorem. *If $f(x)$ is not continuous at $x = a$, then ...*

If $f(x)$ is differentiable at $x = a$, then ...

If $f(x)$ is continuous at $x = a$,

Higher Order Derivatives and Notation

§3.1 Derivatives of Polynomials and Exponential Functions

1. Derivative of a constant

2. Derivative of $f(x) = x$

3. Power Rule

Example. Find the derivatives of these functions:

1. $y = x^{15}$

2. $f(x) = \sqrt[3]{x}$

3. $g(x) = \frac{1}{x^{3.7}}$

4. Derivative of a constant multiple

Example. Find the derivative of $f(x) = 5x^3$.

5. Derivative of a sum

6. Derivative of a difference

Example. Find the derivative of $y = 7x^3 - 5x^2 + 4x - 2$.

Review. Which of the following is NOT a correct derivative rule? (c represents a constant)

A. $\frac{d}{dx}c = 0$

B. $\frac{d}{dx}cf(x) = c\frac{d}{dx}f(x)$

C. $\frac{d}{dx}\left(f(x) + g(x)\right) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$

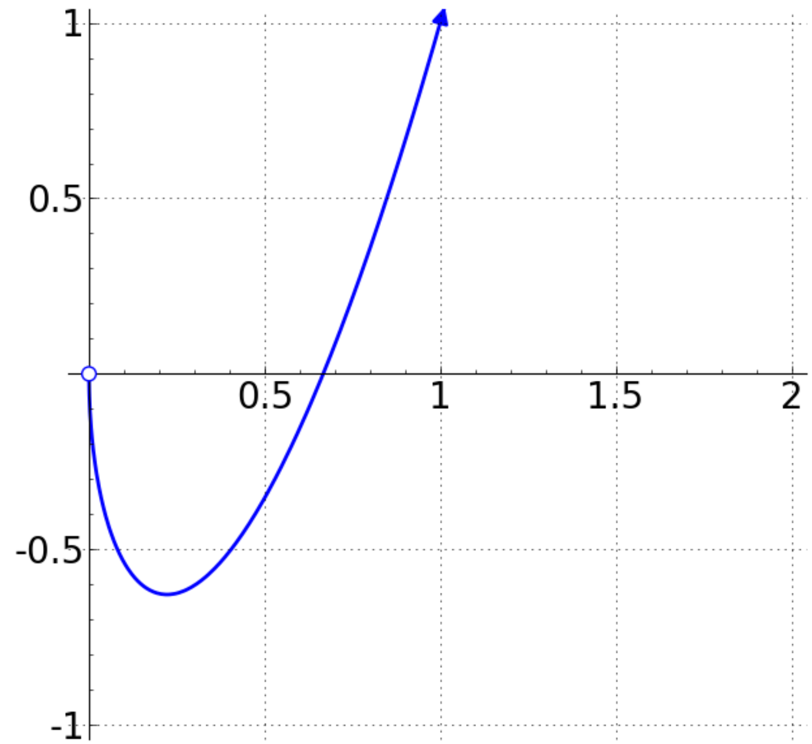
D. $\frac{d}{dx}\left(f(x) \cdot g(x)\right) = \frac{d}{dx}f(x) \cdot \frac{d}{dx}g(x)$

E. $\frac{d}{dx}x^c = cx^{c-1}$

Example. Calculate the derivative: $y = \sqrt{x} + \sqrt{\pi}$

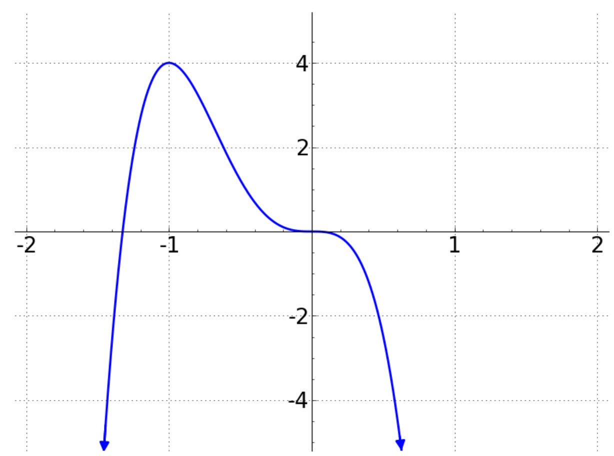
Example. Calculate the derivative: $g(t) = 4t^2 + \frac{1}{4t^2}$

Example. Find the slope of the tangent line and the normal line to $y = \frac{3x^2-2x}{\sqrt{x}}$ at $x = \frac{1}{4}$.



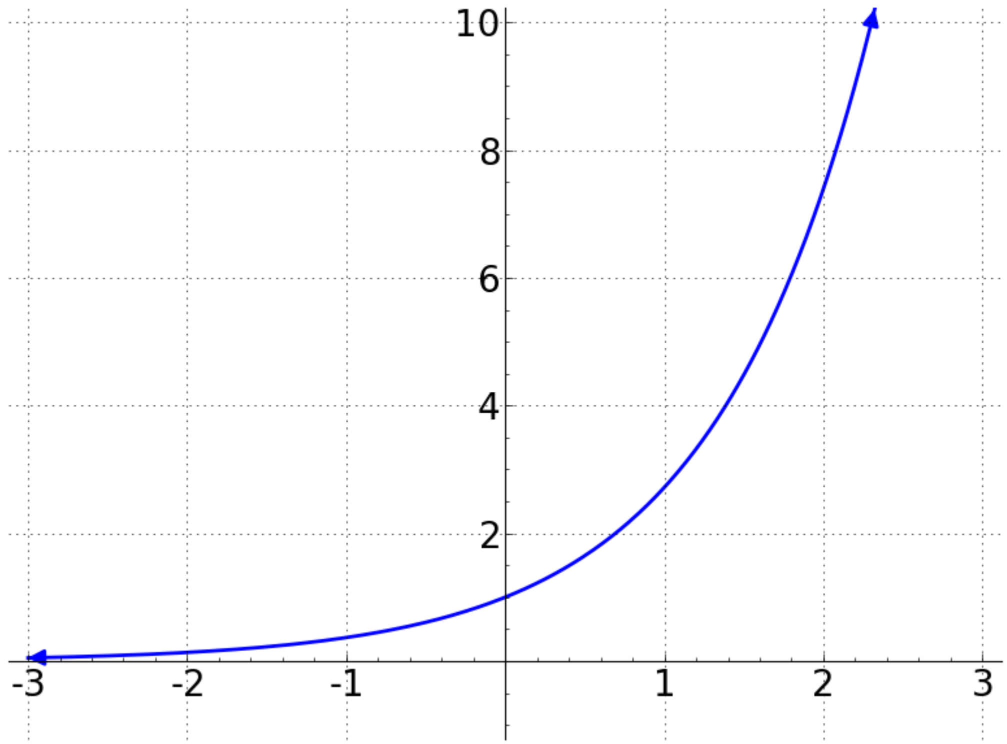
Example. At what x-values is the tangent line of this graph horizontal?

$$y = x^3(x^2 - 10x - 15)$$



Extra Example. Find $f^{(100)}(1)$ if $f(x) = (x^2 + 7x + 9)^5$.

The Marvelous Number e



$y = e^x$

Three facts about e.

1.

2.

3.

Example. Find the derivative of $g(x) = ex^2 + 2e^x + xe^2 + x^{e^2}$.

Note. For a constant c, $\dfrac{d}{dx}(c) = 0$.

Proof. .

Note. $\dfrac{d}{dx}(x) = 1$.

Proof. .

Theorem. *(The Power Rule)* $\dfrac{d}{dx}(x^n) = nx^{n-1}$

Proof. (Proof for n a positive integer only).

Theorem. *(Constant Multiple Rule) If c is a constant, and f is a differentiable function, then*

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x)$$

Proof. .

Theorem. *(Sum Rule) If f and g are differentiable functions, then*

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

Proof. .

Theorem. *(Difference Rule) If f and g are differentiable functions, then*

$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$$

Proof. .

§3.2 The Product Rule and the Quotient Rule

Recall Sum and Difference Rules: If f and g are differentiable functions, then

The Product Rule If f and g are differentiable functions, then

$$\frac{d}{dx}\left[f(x)g(x)\right] =$$

Example. Find $\left(\pi t\sqrt{t}\, e^{t}\right)'$

The Quotient Rule If f and g are differentiable functions, then

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) =$$

Example. Find $\dfrac{d}{dz}\left(\dfrac{z^2}{z^3+1}\right)$

Review. Product Rule:

$$(fg)' = f\,g' + f'\,g$$

Quotient Rule:

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

Example. (#40) Find $f'(t)$ and $f''(t)$ if $f(t) = (t^2 - 1)e^t$.

Example. If $f(9) = 6$ and $f'(9) = 2$, find $\dfrac{d}{dx}\left(\dfrac{4\sqrt{x}}{f(x)}\right)$ at $x = 9$.

Example. Find $f'(z)$ for $f(z) = \left(\dfrac{az+b}{cz+d}\right)$

Example. Find $\left(\dfrac{xe^x}{x^2 + \pi e^x}\right)'$

Example. In 2010, UNC's out-of-state undergraduate population was growing while its in-state population was shrinking, at the rates shown below. Tuition and fees were growing for both populations. At what rate was the *total income* from undergraduate tuition and fees changing in 2010?

Population	Students	Rate of change (# students / year)	Tuition and fees ($)	Rate of change ($ / year)
In-state undergrads	14,550	-90	6,670	690
Out-of-state undergrads	3,390	210	25,280	1660

Data from UNC's Data Dashboard http://northcarolina.edu/content/unc-data-dashboard and from collegefactual.com .

Proofs

The Product Rule If f and g are differentiable functions, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + \frac{d}{dx}[f(x)]\,g(x)$$

The Reciprocal Rule If f is a differentiable function, then

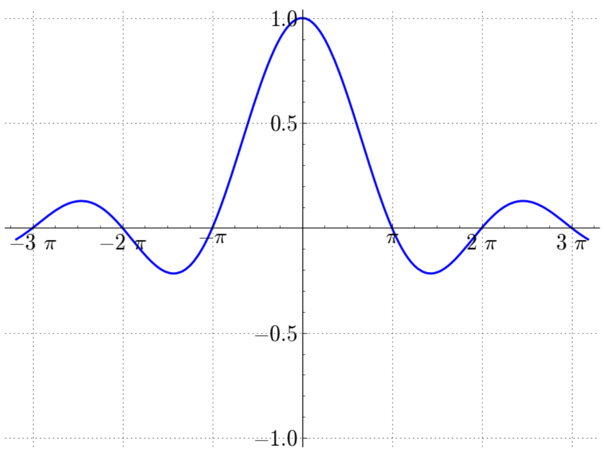
$$\frac{d}{dx}\left(\frac{1}{f(x)}\right) = -\frac{\frac{d}{dx}\left(f(x)\right)}{\left(f(x)\right)^2}$$

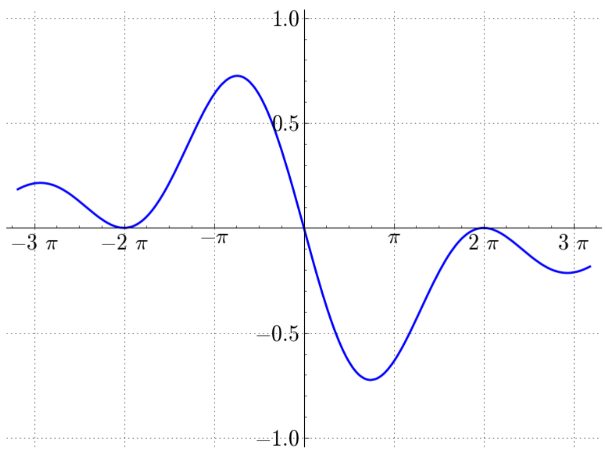
The Quotient Rule If f and g are differentiable functions, then

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{d}{dx}\left[f(x)\right] - f(x)\frac{d}{dx}\left[g(x)\right]}{\left(g(x)\right)^2}$$

§3.3 Limits and Derivatives of Trig Functions

Special Limit #1: $\lim\limits_{\theta\to 0} \dfrac{\sin\theta}{\theta}$ **Special Limit #2:** $\lim\limits_{\theta\to 0} \dfrac{\cos\theta - 1}{\theta}$

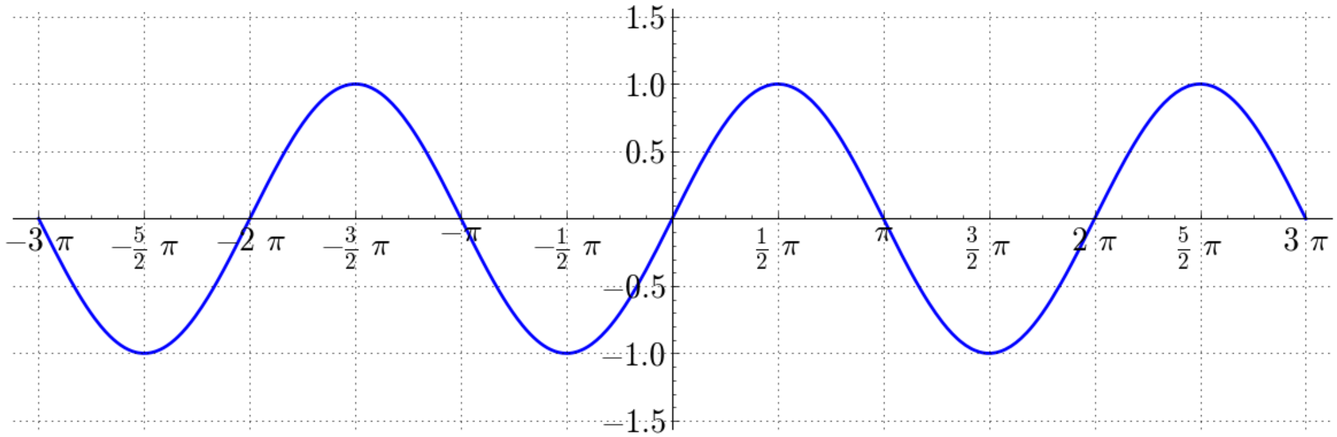


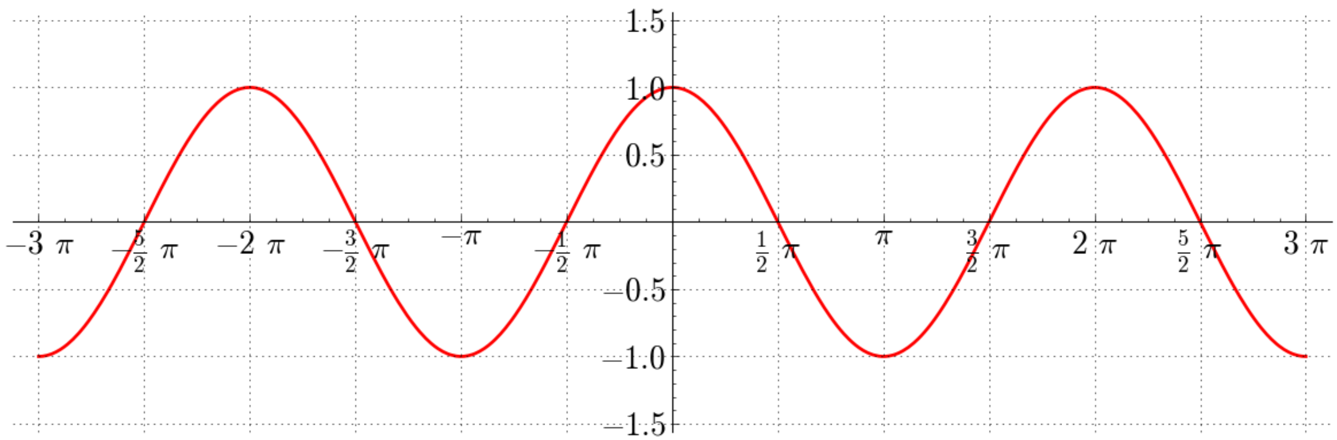


Example. Estimate $\sin 0.01769$ without a calculator.

Example. Find $\lim\limits_{x \to 0} \dfrac{\sin(2x)}{2x}$.

Example. Find $\lim\limits_{x \to 0} \dfrac{\tan 7x}{\sin 4x}$.





$$\frac{d}{dx}(\sin x) =$$

$$\frac{d}{dx}(\cos x) =$$

Example. Find $\dfrac{d}{dx}(\tan x)$.

Example. Find $\dfrac{d}{dx}(\sec x)$.

Example. Find Find $\dfrac{d}{dx}(\cot x)$ and $\dfrac{d}{dx}(\csc x)$.

Summary

$\dfrac{d}{dx}\sin x =$	$\dfrac{d}{dx}\cos x =$
$\dfrac{d}{dx}\tan x =$	$\dfrac{d}{dx}\cot x =$
$\dfrac{d}{dx}\sec x =$	$\dfrac{d}{dx}\csc x =$

Example. For $g(x) = \left(\dfrac{x\cos(x)}{M + \cot(x)}\right)$, find $g'(x)$.

Proofs

Proof that $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$

Proof that $\lim_{\theta \to 0} \dfrac{\cos\theta - 1}{\theta} = 0$

Proof that $\frac{d}{dx}(\sin x) = \cos x$

Proof that $\frac{d}{dx}(\cos x) = -\sin x$

§3.4 The Chain Rule

Compositions of Functions

$$f \circ g(x) =$$

Example. Write the following functions as a composition of functions:

1. $h(x) = \sqrt{\sin(x)}$

2. $k(x) = 5(\tan x + \sec x)^3$

3. $r(x) = e^{\sin(x^2)}$

The Chain Rule If g is differentiable at x and f is differentiable at $g(x)$, then $f \circ g$ is differentiable at x and

$$(f \circ g)'(x) =$$

Example. Find the derivative of $h(x) = \sqrt{\sin(x)}$.

Example. Find the derivative of $k(x) = 5(\tan x + \sec x)^3$

Example. Find the derivative of $r(x) = e^{\sin(x^2)}$

Review. Chain Rule:

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Example. Using the following table of values, find $\left.\dfrac{d}{dx}(f \circ g)\right|_{x=1}$

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	3	2	9	-5
2	4	5	10	-3
3	1	3	2	-1
4	0	2	6	0

Example. Find y' for $y = (1 + 3x^4)^{10}$. Also find y'' and $y^{(100)}$.

Example. Find $\frac{dy}{dt}$ for $y = \cos^4(3t)$.

Example. (#45) Find the derivative of $f(x) = \cos\left(\sqrt{\sin(\tan(\pi x))}\right)$

Example. Suppose that at 10 AM one day, shirts are being produced in your factory at the rate of 36 shirts per hour, and they can be sold to distributors for $4 per shirt. If M represents the total amount of money earned from shirt sales to distributors, at what rate is M increasing with respect to time at 10AM?

Example. Find $\dfrac{d}{dx}(5^x)$. Hint: rewrite 5 as e to a power.

Note. .

$\frac{d}{dx}(a^x) =$

$\frac{d}{dx}(x^a) =$

Example. (#43) $g(x) = (2ra^{rx} + n)^p$. Find $\frac{dg}{dx}$.

Extra Example. The temperature in a certain wire is not the same at all points. At x cm from the left end, the temperature is x^2 degrees Celsius. A bug crawls along the wire at a rate of 2 cm per minute. How fast is the temperature increasing per minute for the bug when the bug is 3 cm from the cool end?

Example. If $f(3) = 5$, $f'(3) = -2$, $f(9) = 4$, $f'(9) = 7$, find

1. $g'(3)$, where $g(x) = (f(x))^2$.
2. $h'(3)$, where $h(x) = f(x^2)$.

Extra Example. In special relativity, the momentum p of a particle traveling at velocity v is given by

$$\rho = \frac{m_0 v}{\sqrt{1 - \frac{v^2}{c^2}}}$$

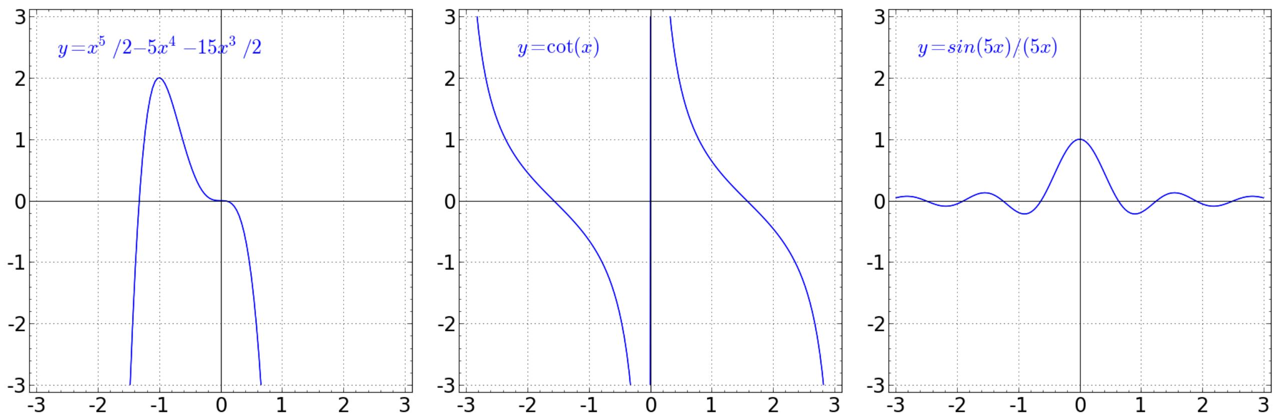
where m_0 is the mass at rest and c is the speed of light. Find an expression for the rate at which momentum is changing over time. Hint: $\frac{d\rho}{dt} = \frac{d\rho}{dv} \cdot \frac{dv}{dt}$.

Justification of the Chain Rule If g is differentiable at x and f is differentiable at $g(x)$, then $f \circ g$ is differentiable at x and

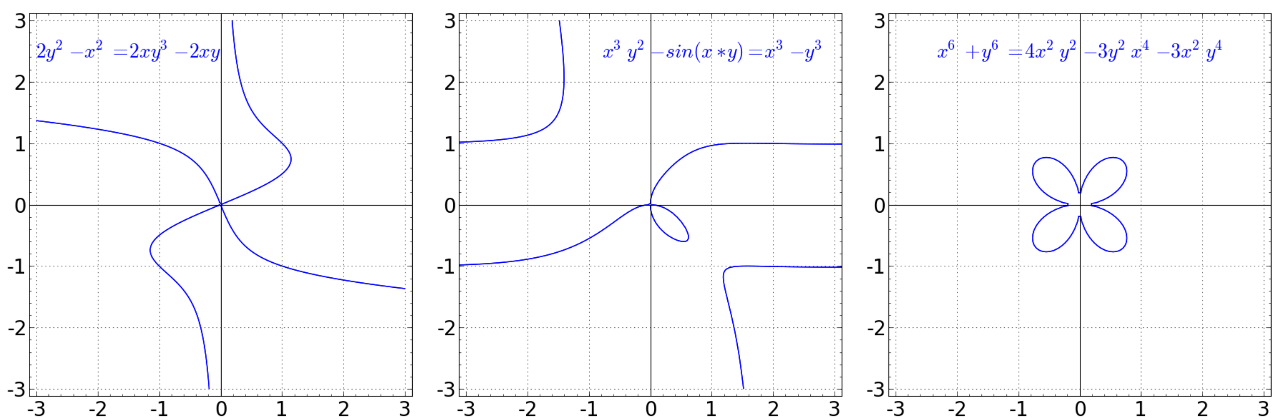
$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

§3.5 Implicit Differentiation

Explicitly defined functions



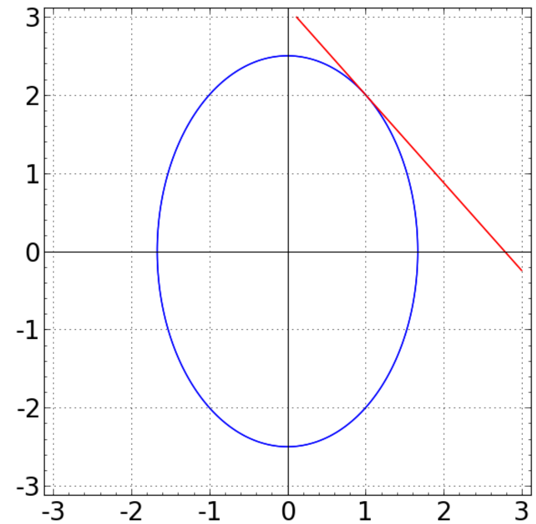
Implicitly defined curves



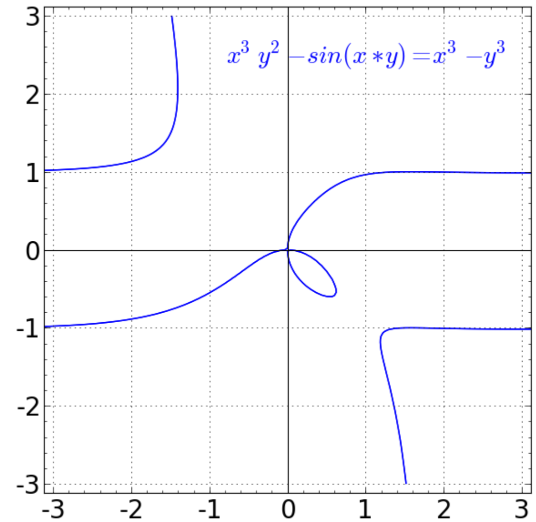
Example. Find the equation of the tangent line for

$$9x^2 + 4y^2 = 25$$

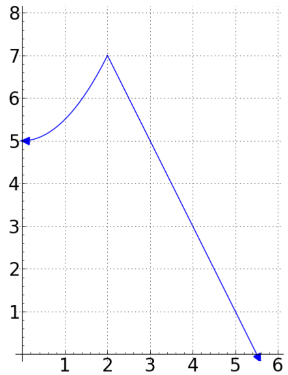
at the point $(1, 2)$.



Example. Find y' if $x^3y^2 - \sin(xy) = x^3 - y^3$



Review. If y is the function of x graphed below, then find $\dfrac{d}{dx}\left(y^2\right)$ at $x = 3$.



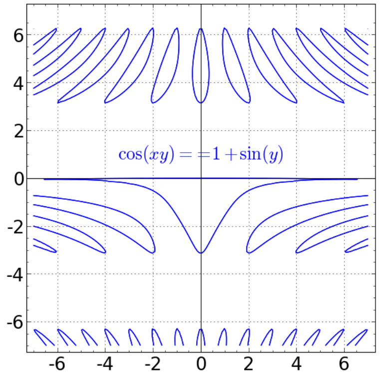
Example. Find the derivative of $x\cos(y) + x^3 + y^3$ (assume y is a function of x).

Example. Find $\dfrac{dy}{dx}$ for the curve defined by

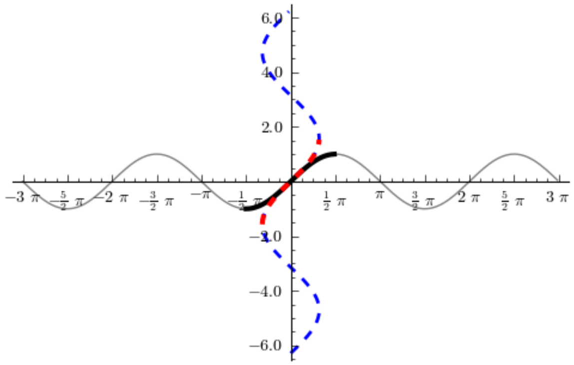
$$\cos(xy) = 1 + \sin(y)$$

Find the equation to the tangent line for this curve at the point $(1, -\frac{\pi}{2})$.

Find the equation to the tangent line for this curve at the point $(0, \pi)$.



Inverse Sine Function



Restricted $\sin(x)$ **has** **Domain:** **Range:**

arcsin(x) has Domain: Range:

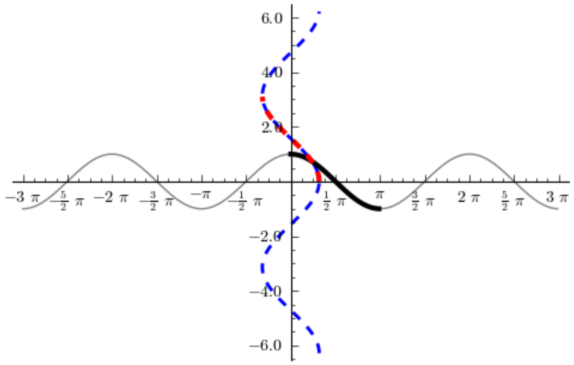
arcsin(x) is the (circle one) angle / number between and whose ...

$y = \arcsin(x)$ means:

Alternative Notation:

Warning:

Inverse Cosine Function



Restricted $\cos(x)$ **has** **Domain:** **Range:**

arccos(x) has Domain: Range:

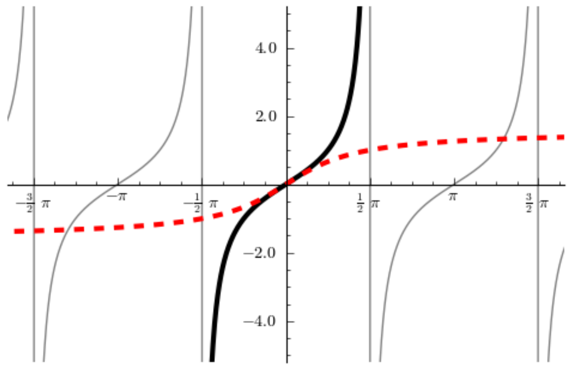
arccos(x) is the (circle one) angle / number between and whose ...

$y = \arccos(x)$ means:

Alternative Notation:

Warning:

Inverse Tangent Function



Restricted $\tan(x)$ **has** **Domain:** **Range:**

arctan(x) has Domain: Range:

arctan(x) is the (circle one) angle / number between and whose ...

$y = \arctan(x)$ means:

Alternative Notation:

Warning:

Review. $y = \arcsin(x)$ means:

Find $\dfrac{d}{dx}\arcsin(x)$

Review. $y = \arccos(x)$ means:

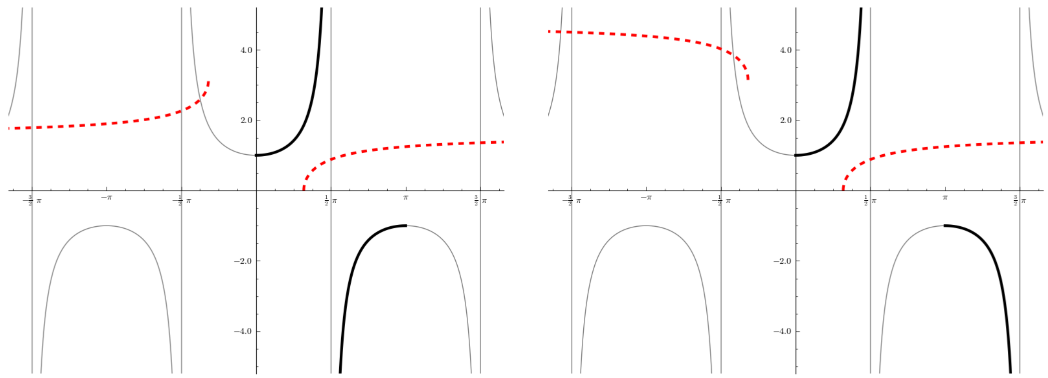
Find $\dfrac{d}{dx}\arccos(x)$

Review. $y = \arctan(x)$ means:

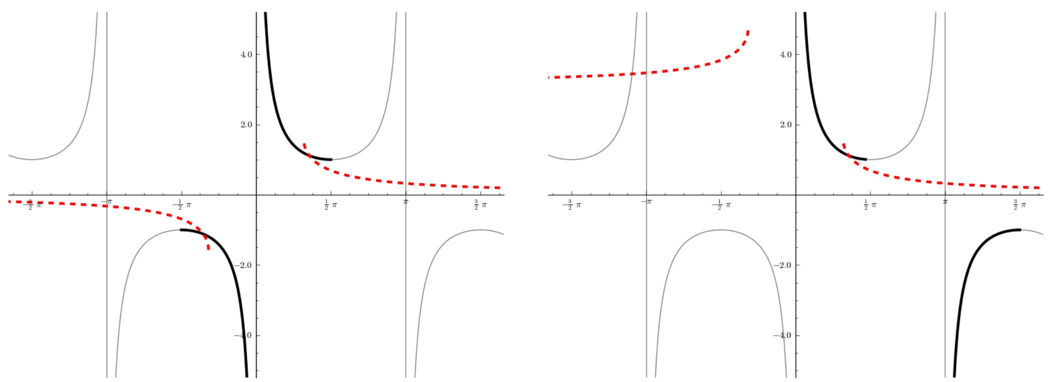
Find $\dfrac{d}{dx}\arctan(x)$

Summary

$\dfrac{d}{dx}\sin^{-1}x =$	$\dfrac{d}{dx}\cos^{-1}x =$
$\dfrac{d}{dx}\tan^{-1}x =$	$\dfrac{d}{dx}\cot^{-1}x =$
$\dfrac{d}{dx}\sec^{-1}x =$	$\dfrac{d}{dx}\csc^{-1}x =$



$\sec(x)$ and $\sec^{-1}(x)$



$\csc(x)$ and $\csc^{-1}(x)$

Example. Find the derivative of $y = \tan^{-1}\left(\dfrac{a+x}{a-x}\right)$.

Example. Find y' for $y = (\cos^{-1}(x))^{-1}$.

§3.6 Derivatives of Log Functions

Find $\dfrac{d}{dx}\log_a(x)$

Find $\dfrac{d}{dx} \ln |x|$

Review. Which of the following is true?

A. $\dfrac{d}{dx}\log_5(x) = \dfrac{1}{x}$

B. $\dfrac{d}{dx}\log_5(x) = \dfrac{\ln(5)}{x}$

C. $\dfrac{d}{dx}\log_5(x) = \dfrac{1}{x\ln(5)}$

D. $\dfrac{d}{dx}\log_5(x) = \dfrac{1}{\log_5(x)}$

Example. Find the derivatives of the following functions.

1. $y = x^{\ln 5}$

2. $y = 5^{\ln x}$

3. $y = (\ln 5)^x$

4. $y = (\ln x)^5$

Example. Match the functions to their derivatives.

1. $y = x^{\ln 5}$

A. $y' = (\ln 5) \cdot 5^{\ln x} \cdot \dfrac{1}{x}$

2. $y = 5^{\ln x}$

B. $y' = 5(\ln x)^4 \cdot \dfrac{1}{x}$

3. $y = (\ln 5)^x$

C. $y' = \ln(\ln 5) \cdot (\ln 5)^x$

4. $y = (\ln x)^5$

D. $y' = (\ln 5) \cdot x^{(\ln 5)-1}$

Example. One version of Fitt's Law says that the "movement time" it takes a person to point to an object (MT) depends on the distance to the object (D) and the width of the object (W), according to the formula:

$$MT = a + b\log_2\left(\frac{2D}{W}\right)$$

At what rate does the movement time decrease as the width increases, if all other variables are held constant? (What are the units for this rate, if MT is measured in seconds and D and W are measured in cm?)

Logarithmic Differentiation

Recall:

$$\frac{d}{dx}x^a =$$

$$\frac{d}{dx}a^x =$$

Example. Which do you think is true about the derivative of x^x?

A. $\frac{d}{dx}x^x = x \cdot x^{x-1}$

B. $\frac{d}{dx}x^x = \ln|x| \cdot x^x$

C. None of the above.

Example. Find $\dfrac{d}{dx}(x^x)$.

Example. Find $\dfrac{d}{dx}(\tan x)^{1/x}$.

Example. Find the derivative of $y = \dfrac{x\cos(x)}{(x^2+x)^5}$

Example. Which of the following should be differentiated using logarithmic differentiation?

1. $y = [f(x)]^{g(x)}$
2. $y = [f(x)]^{e}$
3. $y = e^{f(x)}$
4. $y = e^{5}$
5. $y = \ln(x^{g(x)})$
6. $y = [\ln(x)]^{g(x)}$
7. $y = \ln(g(x))$

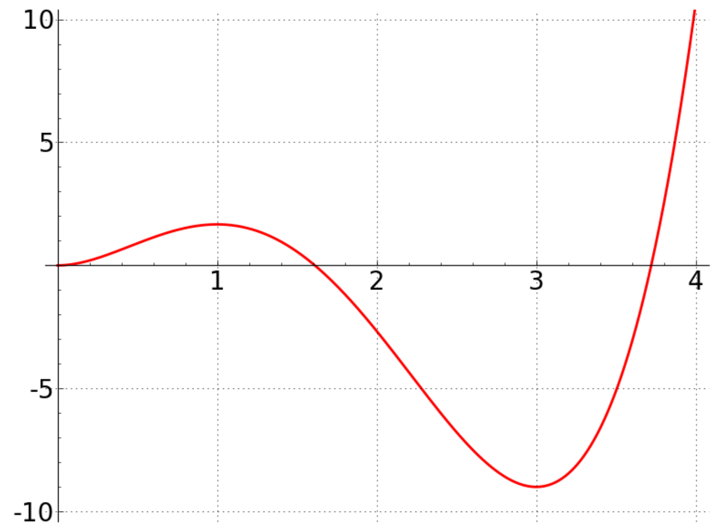
§3.7 Rates of Change in the Natural and Social Sciences

Example. A particle moves up and down along a straight line. Its position in mm at time t seconds is given by the equation $s(t) = t^4 - \frac{16}{3}t^3 + 6t^2$.

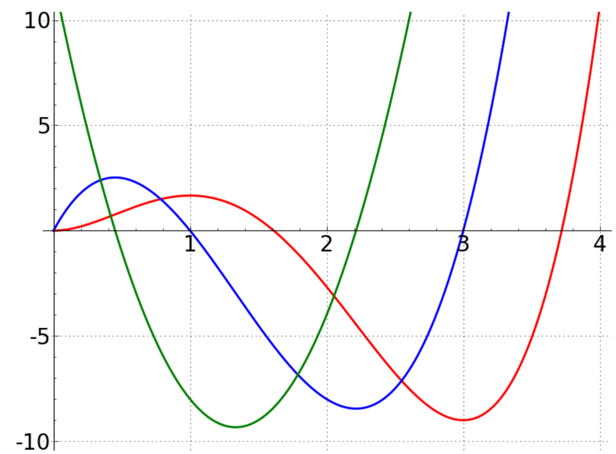
Find $s'(t)$ and $s''(t)$. What do they represent?

Example. Use the table of values to describe the particle's motion at time 1.5 seconds and at time 2.5 seconds.

t	$s(t)$	$s'(t)$	$s''(t)$
1.5	0.5625	−4.5	−9
2.5	−6.77	−7.5	7



Example. A particle moves up and down along a straight line. Its position in mm at time t seconds is given by the equation $s(t) = t^4 - \frac{16}{3}t^3 + 6t^2$.



1. When is the particle at rest?

2. When is the particle moving up? moving down?

3. When is the particle speeding up? slowing down?

Example. A particle moves up and down along a straight line. Its position in mm at time t seconds is given by the equation $s(t) = t^4 - \frac{16}{3}t^3 + 6t^2$.

1. What is the net change in position for the particle between 1 and 4 seconds?

2. What is the total distance traveled by the particle between 1 and 4 seconds?

Review. Suppose $s(t)$ represents the height of a mosquito above your head in cm. Suppose $s'(t) < 0$. Which of the following must be true?

A. The mosquito is below the top of your head.

B. The velocity of the mosquito is negative.

C. The mosquito is moving down.

D. The height of the feather is decreasing.

E. The acceleration of the mosquito is negative.

F. The velocity of the mosquito is decreasing.

G. The mosquito is slowing down.

H. The force on the mosquito is in the downwards direction.

Review. Suppose $s(t)$ represents the height of a mosquito above your head in cm. Suppose $s''(t) < 0$. Which of the following must be true?

A. The mosquito is below the top of your head.

B. The velocity of the mosquito is negative.

C. The mosquito is moving down.

D. The height of the feather is decreasing.

E. The acceleration of the mosquito is negative.

F. The velocity of the mosquito is decreasing.

G. The mosquito is slowing down.

H. The force on the mosquito is in the downwards direction.

Example. $s(t)$ represents the position of a squirrel that is running left and right along a telephone wire that crosses over a street. $s(t)$ is the number of feet to the right of the center of the street and t is time in seconds. Which of the following are true about the motion of the squirrel at time $t = 1$? $t = 2$? $t = 3$?

t	$s(t)$	$s'(t)$	$s''(t)$
1	-2.5	0.2	4.2
2	1.5	0.5	-5.6
3	-2.2	-1.0	35.6

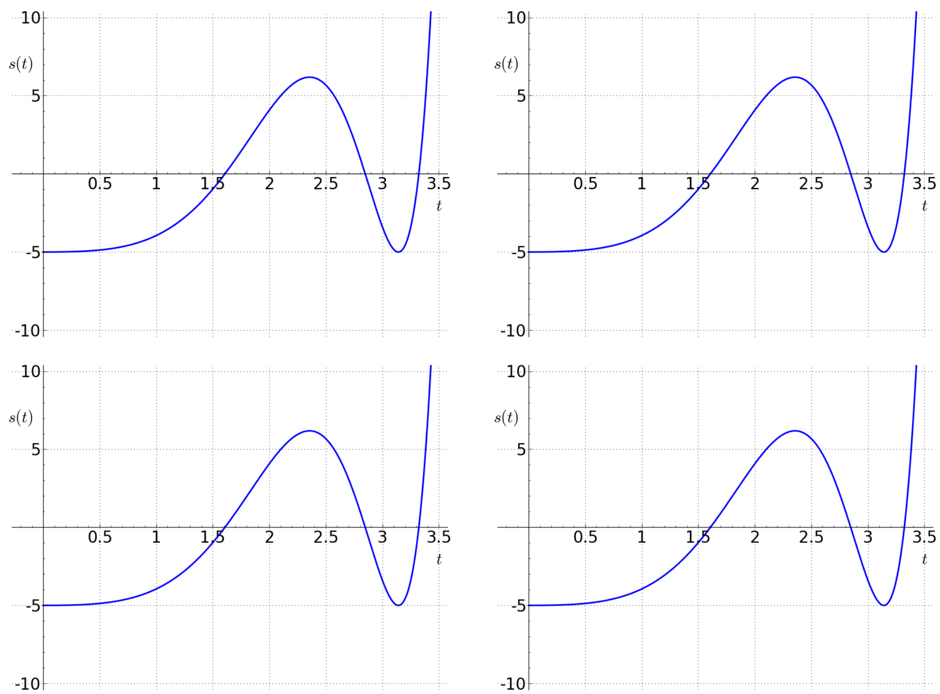
A. The squirrel is to the right of the center of the street.

B. The squirrel is moving right.

C. The squirrel's velocity is increasing.

D. The squirrel is speeding up.

Example. Below is a graph of the squirrel's position $s(t)$. As before, $s(t)$ is the number of feet to the right of the center of the street and t is time in seconds. When is the squirrel to the right of center? When is the squirrel's velocity positive? When is the squirrel's velocity increasing? When is the squirrel speeding up?

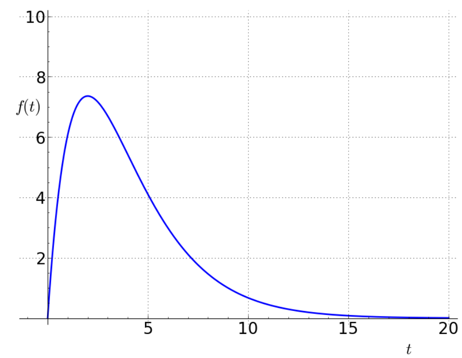


Approximately what distance does the squirrel travel between $t = 0$ and $t = 3.4$?

What is the squirrel's displacement between between $t = 0$ and $t = 3.4$?

Example. A feather is moving up and down. Its height in meters above the ground is given by the equation

$$f(t) = 10te^{-t/2}$$



where t represents seconds and $0 \leq t \leq 8$.

1. When is the feather at rest?

2. When is the feather moving upwards?

3. What is the maximum height reached by the feather?

4. What is the total distance traveled during the first 8 seconds?

5. When does the feather have positive acceleration?

6. When is the feather speeding up?

Example. In special relativity, the momentum p of a particle traveling at velocity v is given by

$$p = \frac{m_0 v}{\sqrt{1 - \frac{v^2}{c^2}}}$$

where m_0 is the mass at rest and c is the speed of light. Find an expression for the rate at which momentum is changing over time.

Example. Suppose that the total cost of producing x tie-dyed T-shirts is $C(x)$.

Is $C(x)$ an increasing or decreasing function?

Is $C'(x)$ increasing or decreasing?

Interpret the following:

1. $C(204) - C(200)$

2. $\dfrac{C(204) - C(200)}{4}$

3. $C'(200)$

$C(x)$ is called:

$C'(x)$ is called:

Example. Suppose the cost for producing x iPads is given by $C(x) = 500 + 300\sqrt{x}$.

1. Find and interpret $C(401) - C(400)$.

2. Find and interpret $C'(400)$.

§3.9 Related Rates

Example. A tornado is 20 miles west of us, heading due east towards Phillips Hall at a rate of 40 mph. You hop on your bike and ride due south at a speed of 12 mph. How fast is the distance between you and the tornado changing after 15 minutes?

Example. Water flows into a tank at a rate of 3 cubic meters per minute. The tank is shaped like a cone with a height of 4 meters and a radius of 5 meters at the top. Find the rate at which the water level is rising in the tank when the water height is 2 meters.

Example. A lighthouse that is half a mile west of shore has a rotating light that makes 2 revolutions per minute in the counterclockwise direction. The shoreline runs north-south, and there is a cave on the shore directly east of the lighthouse. How fast is the beam of light moving along the shore at a point 1 mile north of the cave?

Example. A girl 1 meter tall stands 3 meters away from a wall and shines a headlamp on a spider that is crawling up the wall. At what rate is the angle of her headlamp changing when the spider is 2 meters high?

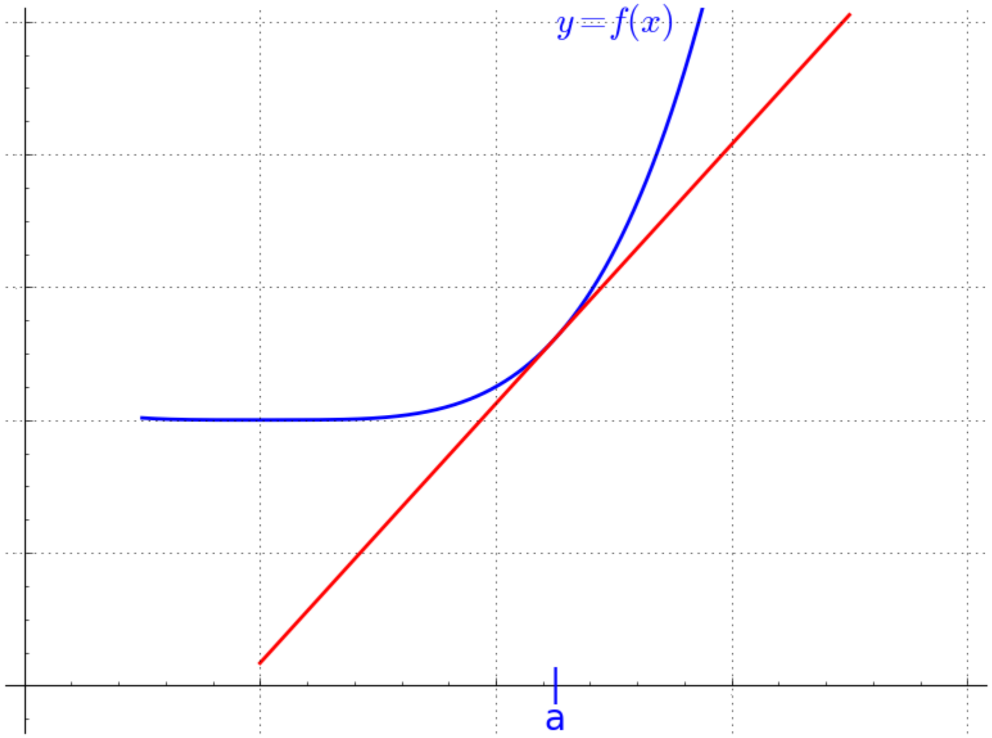
Example. A woman 1.5 meters tall walks away from a lamp that is 15 meters high at a rate of 3 km/hr. At what rate is her shadow growing when she is 3 meters from the lamp? At what rate is the tip of her shadow moving when she is 3 meters from the lamp?

Example. A spherical snowball evaporates at a rate proportional to its surface area. Show that its radius shrinks at a constant rate.

§3.10 Linear Approximation and Differentials

Example. Suppose $f(t)$ is the temperature in degrees Fahrenheit at time t (measured in hours), where $t = 0$ represents midnight. Suppose that $f(6) = 60^\circ$ and $f'(6) = 3^\circ/\text{hr}$. What is your best estimate for the temperature at 7:00 am? 8:00 am?

Linear Approximation



Approximation Principle:

Linear Approximation:

Example. Use the approximation principle to estimate $\sqrt{59}$ without a calculator.

Example. Use a linearization of $y = \sin(x)$ to estimate $\sin(33°)$ without a calculator.

Review. .
Idea of Linearization: Functions can be approximated by lines.

Intuition: Zoom in on your favorite function.

Question. Can all functions be approximated by lines?

Review. .

True or False: The linear approximation (linearization) of a differentiable function f at a, given by

$$L(x) = f(a) + f'(a)(x - a)$$

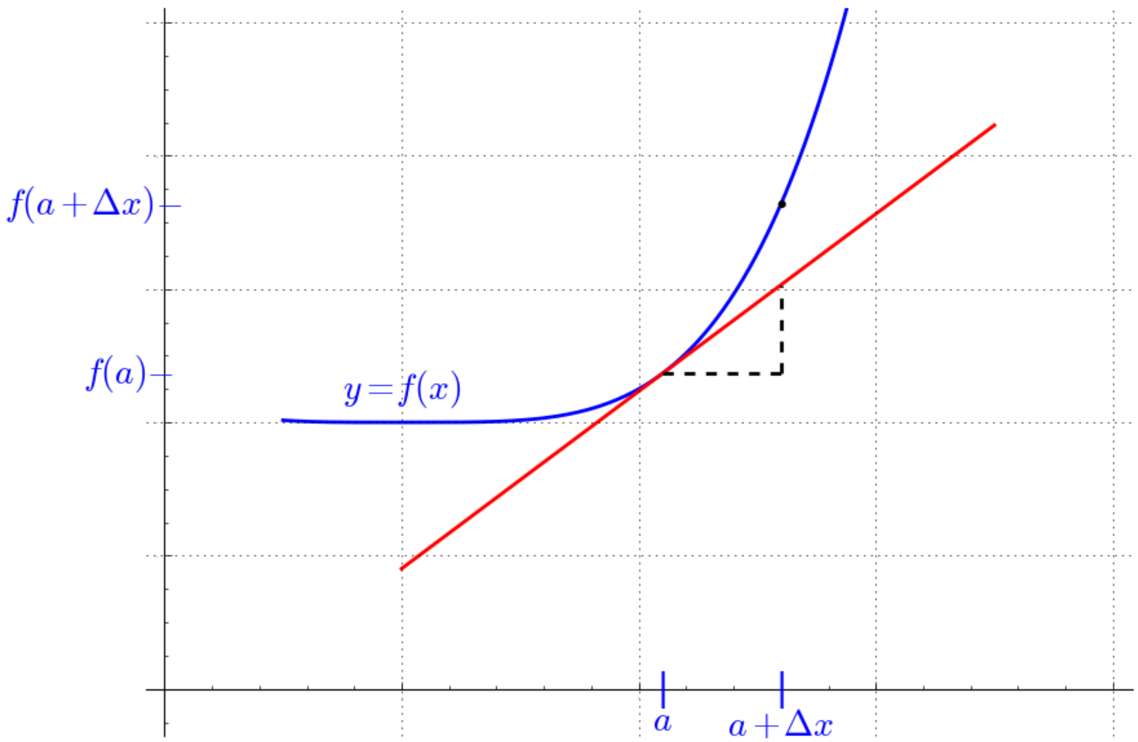
is the same curve as the tangent line to the graph of f at a.

True/False: If f is a differentiable function and a is a number in its domain, then $f(x) = f(a) + f'(a) \cdot (x - a)$ for all x-values in the function's domain.

Example. Find the linearization of $f(x) = \ln(\sqrt{x+1})$ at $x = 0$ and use it to approximate $\ln(\sqrt{1.5})$.

Example. Use a linear approximation to estimate $\sqrt[3]{10}$ without a calculator.

Definitions and Terminology



Example. For $f(x) = x \ln x$, find:

1. df

2. df when $a = 2$ and $\Delta x = -0.3$

3. Δf when $a = 2$ and $\Delta x = -0.3$

Example. You've been hired to paint the inside of the dome on the Morehead Planetarium and you need to calculate the surface area involved. The dome is a perfect hemisphere and you've measured the diameter to be 18 meters, which you believe to be accurate to within 40 cm. Use the differential to estimate the possible error in computing the surface area based on your measurement.

Estimate the relative error for the surface area.

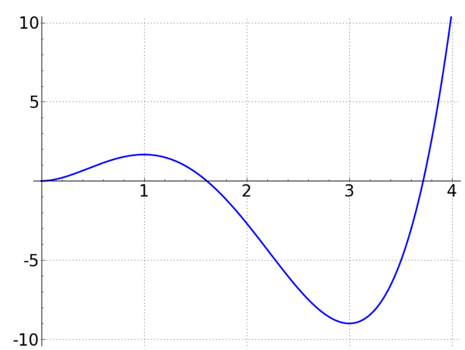
§4.1.1 Maximum and Minimum Values

Definition. A function $f(x)$ has an **absolute maximum** at $x = c$ if

The y-value $f(c)$ is called the ______________________________
and the point $(c, f(c))$ is called ______________________________

Definition. A function $f(x)$ has an **absolute minimum** at $x = c$ if

The y-value $f(c)$ is called the ______________________________
and the point $(c, f(c))$ is called ______________________________



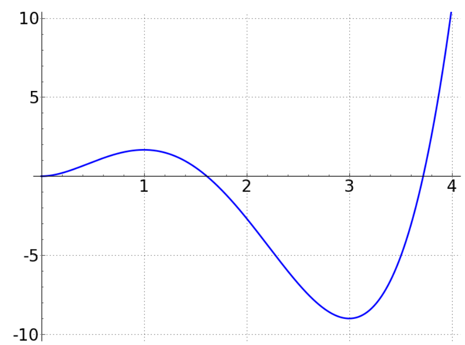
Definition. Absolute maximum and minimum values can also be called

Definition. A function $f(x)$ has an **local maximum** at $x = c$ if

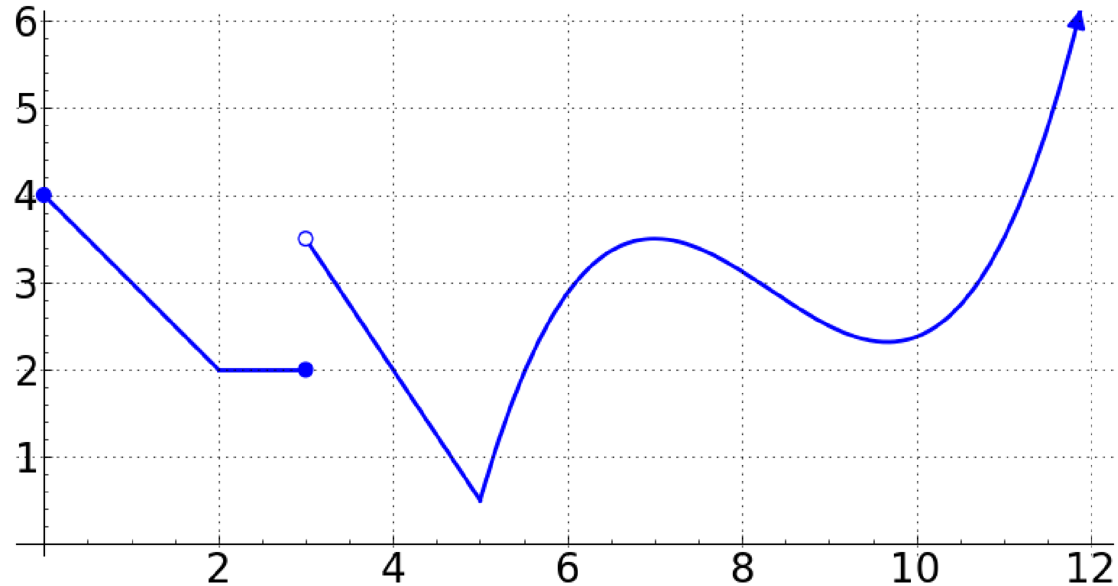
The y-value $f(c)$ is called the ____________________________
and the point $(c, f(c))$ is called ____________________________

Definition. A function $f(x)$ has an **local minimum** at $x = c$ if

The y-value $f(c)$ is called the ____________________________
and the point $(c, f(c))$ is called ____________________________



Definition. Local maximum and minimum values can also be called

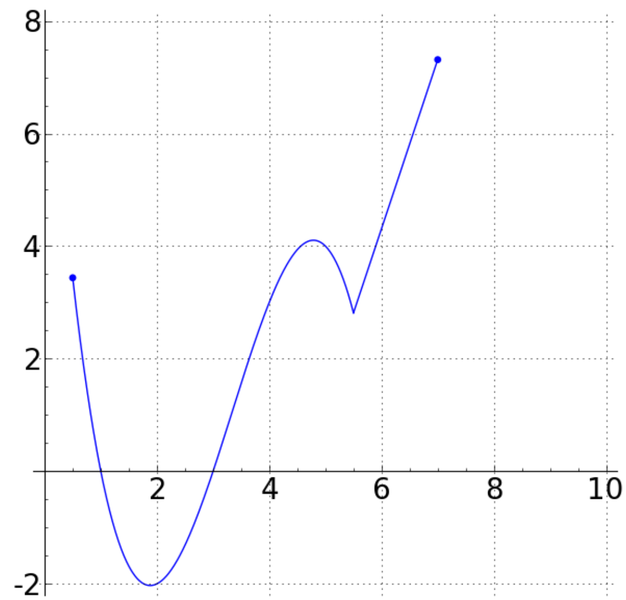


Example. 1. Mark all local maximum and minimum points.

2. Mark all global maximum and and minimum points.

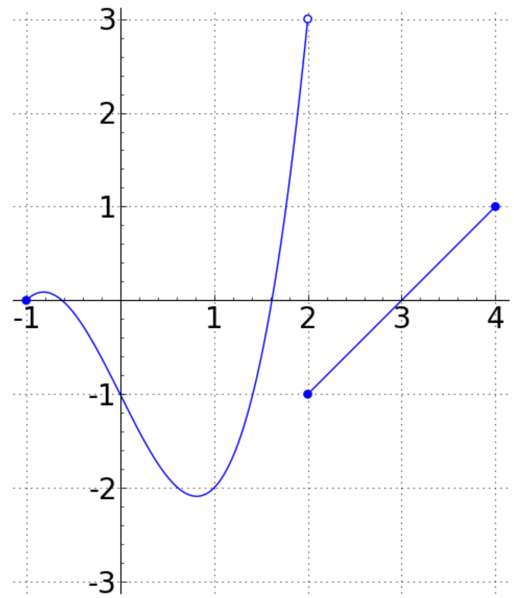
3. What is the absolute maximum value of the function? What is the absolute minimum value?

Question. What do you notice about the derivative of a function at its local maximum and minimum points?



Definition. A number c is **critical number** for a function c if:

Review. Find the local maximum points, the local minimum points, the absolute maximum points, and the absolute minimum points.



Review. A critical number for a function $f(x)$ is:

True or False: If f has a local maximum or local minimum at $x = c$, then c is a critical number for a function f.

True or False: If f has a critical number at c, then f has a local maximum or minimum at $x = c$.

Question. What is the relationship between absolute maximum points and local maximum points?

- True or False: A local maximum point is automatically also an absolute maximum point.

- True or False: An absolute maximum point is automatically also a local maximum point.

Question. Does a function always have at least one local maximum point?

Question. Does a function always have at least one absolute maximum point?

Theorem. *(Extreme Value Theorem) A ________________ function, defined on a domain that is ________________________ must achieve an absolute maximum value and an absolute minimum value on its domain.*

.

Strategy for finding absolute extreme values for a continuous function f on a closed interval $[a, b]$.

1.

2.

3.

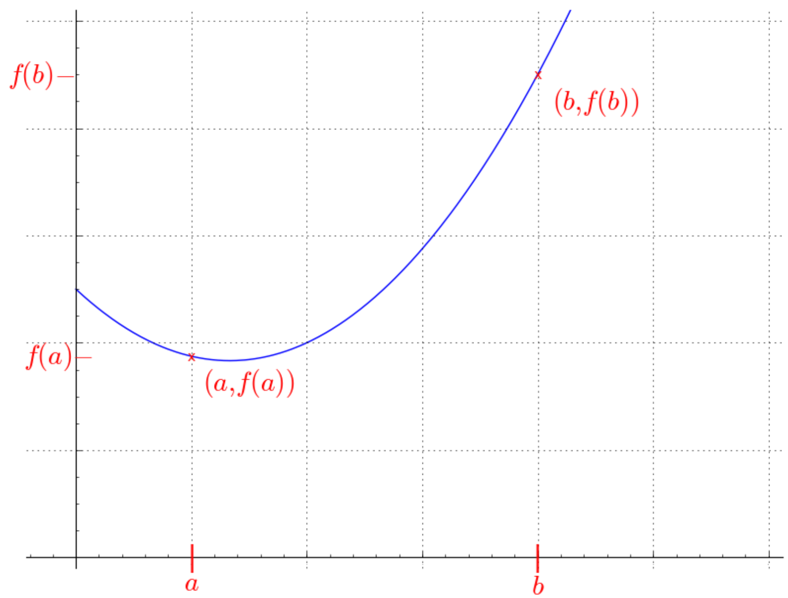
4.

Example. Find the absolute extreme values for $f(\theta) = \cos(2\theta) + 2\sin(\theta)$ on the interval $[0, 2\pi]$.

Extra Example. Find the absolute maximum and minimum values for $g(x) = \dfrac{x-1}{x^2+x+2}$ on the interval $[0, 4]$.

Extra Example. Find the local and absolute extreme values for $f(x) = |3x| - x^2$ on the interval $[-2, 2]$.

§4.2 The Mean Value Theorem



Theorem. *The Mean Value Theorem. Let $f(x)$ be a function defined on $[a, b]$ such that*

1. *$f(x)$ is continuous on $[a, b]$*
2. *$f(x)$ is differentiable on (a, b)*

then there is a number c in $[a, b]$ such that ...

Question. Is the number c unique?

Example. Verify the mean value theorem for $f(x) = 2x^3 - 8x + 1$ on the interval $[1, 3]$.

Example. If f is a differentiable function and $f(1) = 7$ and $-3 \leq f'(x) \leq -2$ on the interval $[1, 6]$, then what is the biggest and smallest values that are possible for $f(6)$?

Theorem. *Rolle's Theorem. If $f(x)$ is a function defined on $[a, b]$ such that*

1. *$f(x)$ is continuous on $[a, b]$*
2. *$f(x)$ is differentiable on (a, b)*
3. *and*

then there is a number c in $[a, b]$ such that ...

Review. For a continuous, differentiable function f defined on the interval $[a,b]$, the Mean Value Theorem says that there is some number c in the interval (a,b), such that:

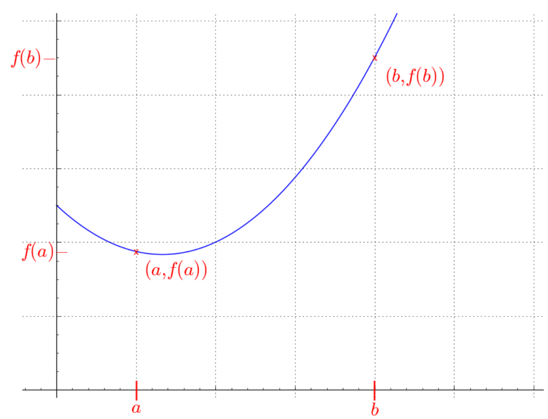
A. the slope of the tangent line at $x = c$ equals the slope of the secant line between $x = a$ and $x = b$

B. f′(c) equals the average value of f on the interval

C. f′(c) equals the average rate of change of f on the interval

D. $f'(c) = \dfrac{f(b) + f(a)}{2}$

E. $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.



Example. A police officer notices (based on timed photos) that a car has travelled 162 miles on I-40 from Chapel Hill, NC to Wilmington, NC in exactly 2 hours. Which box should the officer check on the speeding ticket?

A. 75 - 84 mph speeding violation

B. 85 - 94 mph speeding violation

C. 95 - 104 mph speeding violation

D. 105+ mph speeding violation

Example. If $g(x) = x^3 - 4x$, does $g(x)$ have a tangent line of slope -1 for some x-value between -1 and 2?

Example.

$$f(x) = \begin{cases} 1 + x & \text{if } x < 0 \\ 1 - x & \text{if } x >= 0 \end{cases}$$

Is there a c in $(-2, 2)$ such that $f'(c) = 0$?

Extra Example. Suppose that f is differentiable and $f(5) = 4$ and $f'(x) \geq -3$ for all x . How big can $f(1)$ possibly be?

Proofs

Theorem. *Rolle's Theorem. If $f(x)$ is a function defined on $[a, b]$ such that*

1. *$f(x)$ is continuous on $[a, b]$*
2. *$f(x)$ is differentiable on (a, b)*
3. *and $f(a) = f(b)$*

then there is a number c in $[a, b]$ such that $f'(c) = 0$.

Theorem. *The Mean Value Theorem. Let $f(x)$ be a function defined on $[a, b]$ such that*

1. *$f(x)$ is continuous on $[a, b]$*
2. *$f(x)$ is differentiable on (a, b)*

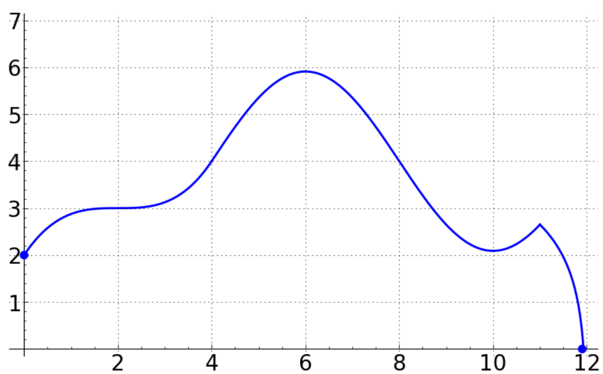
then there is a number c in $[a, b]$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

Proof:

§4.3 Derivatives and the Shape of a Graph

Definition. The function $f(x)$ is *increasing* if:

Definition. The function $f(x)$ is *decreasing* if:



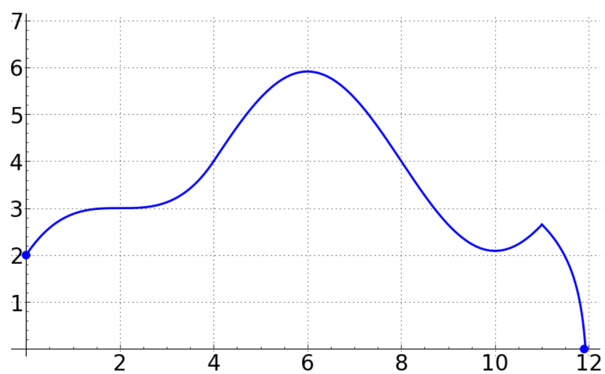
Increasing / Decreasing Test:

If $f'(x) > 0$ for all x on an interval, then:

If $f'(x) < 0$ for all x on an interval, then:

Definition. $f(x)$ is concave up on an interval (a, b) if:

Definition. $f(x)$ is concave down on an interval (a, b) if:

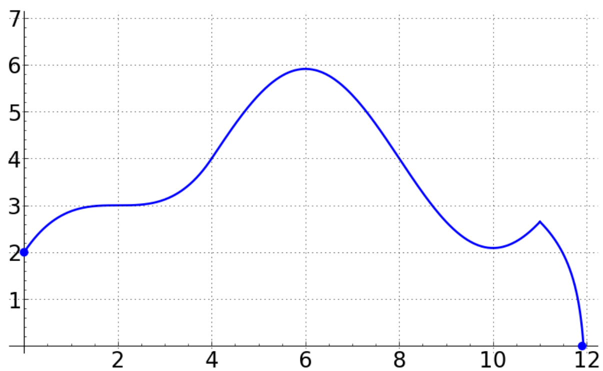


Concavity Test:

If $f''(x) > 0$ for all x on an interval, then:

If $f''(x) < 0$ for all x on an interval, then:

Definition. $f(x)$ has an *inflection point* at $x = c$ if:



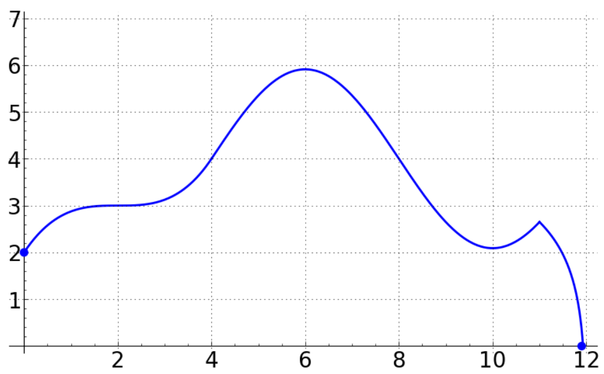
Inflection Point Test:

If $f''(x) > 0$ for all x on an interval, then:

If $f''(x) < 0$ for all x on an interval, then:

Definition. $f(x)$ has a local maximum at $x = c$ if:

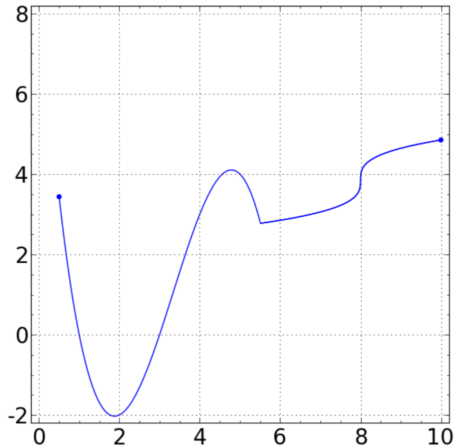
Definition. $f(x)$ has a local minimum at $x = c$ if:



First Derivative Test:

Second Derivative Test:

Review. A graph of $y = f(x)$ shown.



Find where

- f is increasing
- f is decreasing
- f has an inflection point
- f has a local max
- f has a local min
- f is concave up
- f is concave down

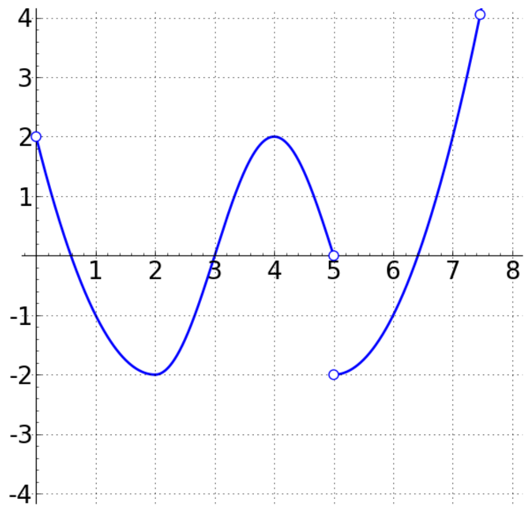
These conditions on $f'(x)$ ...	and / or these conditions on $f''(x)$...	guarantee this feature on the graph of f
		Increasing on the interval (a, b)
		Decreasing on the interval (a, b)
		Concave up on the interval (a, b)
		Concave down on the interval (a, b)
		Local max point at $(c, f(c))$
		Local min point at $(c, f(c))$
		Inflection point at $(c, f(c))$

First Derivative Test:

Second Derivative Test:

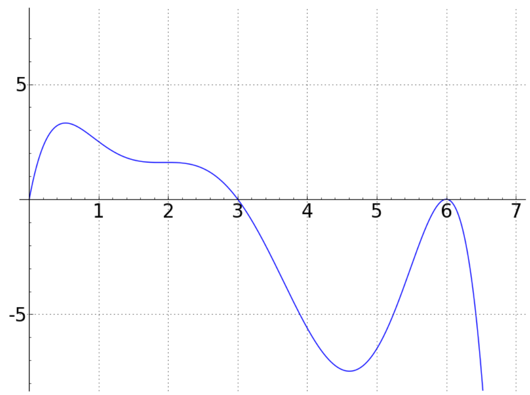
Question. If $f'(c) = 0$, must f have a local max or min at $x = c$?

Question. If $f''(c) = 0$, must f have an inflection point at $x = c$?



Example. Assume $g(x)$ is a continuous function. A graph of THE DERIVATIVE $g'(x)$ is shown. Find the x-values where:

- g is increasing
- g is decreasing
- g has an inflection point
- g has a local max
- g has a local min
- g is concave up
- g is concave down



Example. A graph of $h'(x)$ is shown. Find the x-values where:

- h has local extreme points

- h has inflection points

Example. $g(x) = x^3 - 6x^2 - 15x + 1$

Find where $g(x)$ is increasing and decreasing, has local max and min values, is concave up and down, and has inflection points.

Example. $f(\theta) = \sin\theta\cos\theta + \cos\theta$ for $0 \leq \theta \leq 2\pi$

Find where $f(\theta)$ is increasing and decreasing, has local max and min values, is concave up and down, and has inflection points.

Example. $h(x) = x^{1/3} + x^{4/3}$

Find where $h(x)$ is increasing and decreasing, has local max and min values, is concave up and down, and has inflection points.

§4.4 L'Hospital's Rule

Definition. A limit of the form $\lim_{x\to a} \dfrac{f(x)}{g(x)}$ is called a $\frac{0}{0}$ indeterminate form if:

Definition. A limit of the form $\lim_{x\to a} \dfrac{f(x)}{g(x)}$ is called a $\frac{\infty}{\infty}$ indeterminate form if:

Theorem. *L'Hospital's Rule* Suppose f and g are differentiable and $g'(x) \neq 0$ in an open interval around a (except possibly at a). If $\lim_{x \to a} \dfrac{f(x)}{g(x)}$ is a $\frac{0}{0}$ or $\frac{\infty}{\infty}$ indeterminate form, then

Example. $\lim_{x\to\infty} \frac{x}{3^x}$

Example. $\lim_{x \to 0} \dfrac{\sin(x) - x}{(\sin x)^3}$

Review. If $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$, then $\lim_{x \to a} \frac{f(x)}{g(x)}$ is

A. 0

B. 1

C. DNE

D. Cannot be determined from this information.

Example. $\lim_{x\to\infty} \frac{\sqrt[3]{x}}{\ln(x)}$

Example. $\lim_{x\to 0^+} \sin x \ln x$

Example. $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$

Example. $\lim_{x\to\infty} \ln(x^2 - 1) - \ln(x^5 - 1)$

Tips for using L'Hopital's Rule:

Form	Example	What to do

Question. Which of the following limits would NOT be an appropriate candidate for L'Hospital's Rule?

A. $\lim\limits_{x \to 5} \dfrac{x-5}{x^2-25}$

B. $\lim\limits_{x \to 1^+} \dfrac{e^x}{x-1}$

C. $\lim\limits_{x \to \infty} x \sin(1/x)$

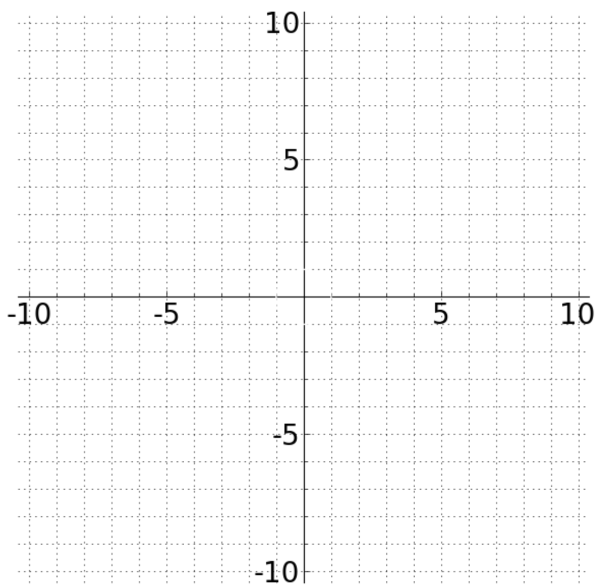
D. $\lim\limits_{x \to 0^+} x^x$

L'Hospital's Rule cannot be used to find ALL limits.

§4.5 Curve Sketching

Example 0.1. Sketch the graph of a function $f(x)$ with the properties:

1. $f'(x) > 0$ when $x < 3$ and $f'(x) < 0$ when $x > 3$
2. $f(x)$ has an absolute maximum value of 4
3. $f''(x) > 0$ when $x < 0$ and when $x > 5$ and $f''(x) < 0$ when $0 < x < 5$
4. $\lim_{x \to -\infty} f(x) = -2$ and $\lim_{x \to \infty} f(x) = 0$



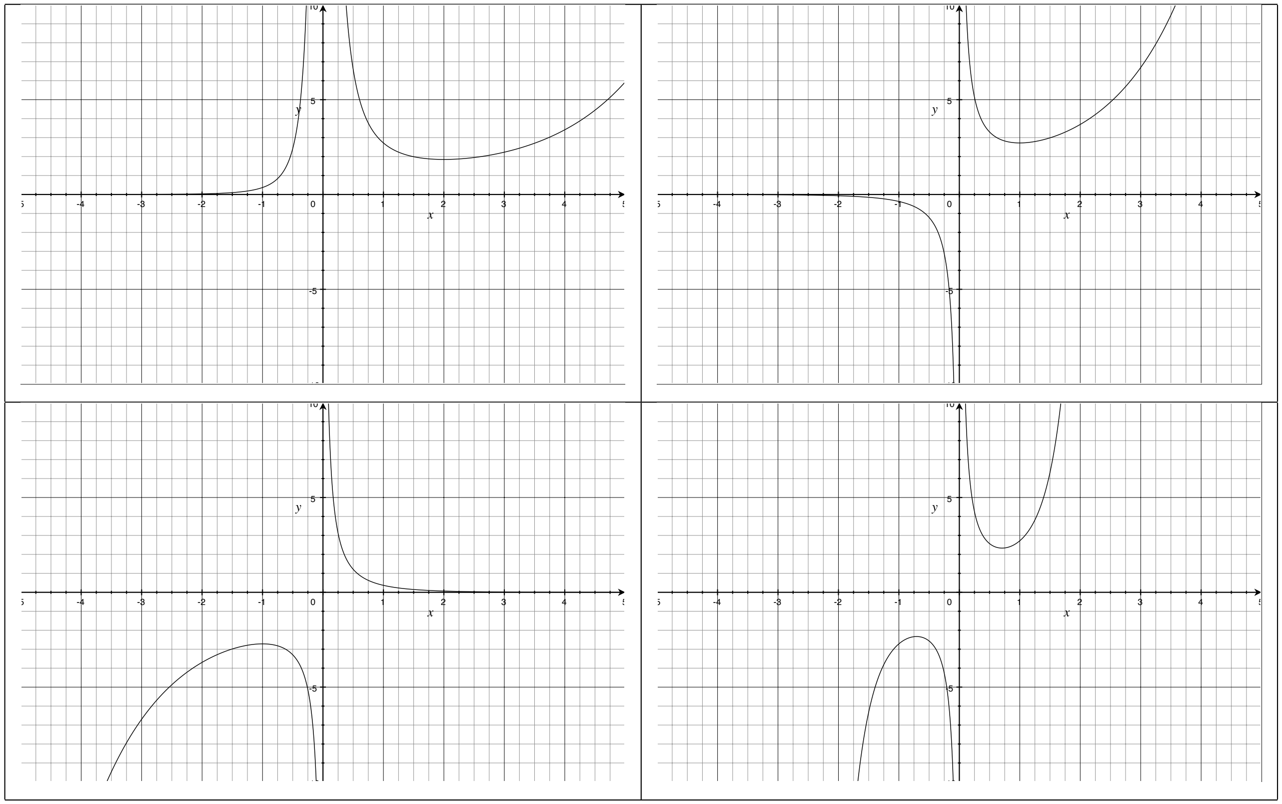
Tricks for graphing functions or recognizing graphs.

Locate the following features:

- Increasing / decreasing
- Local max and mins
- Concave up and concave down
- Inflection points

- Horizontal asymptotes
- Vertical asymptotes
- Intercepts
- Domain
- Symmetry

Example. Graph $y = \dfrac{e^x}{x}$



§4.7 Optimization

Example. • A can is to hold 443 mL (15 oz) of soup.

- Metal for the sides of the can costs 3 cents per cm^2.
- Metal for the top and bottom of the can costs 5 cents per cm^2.

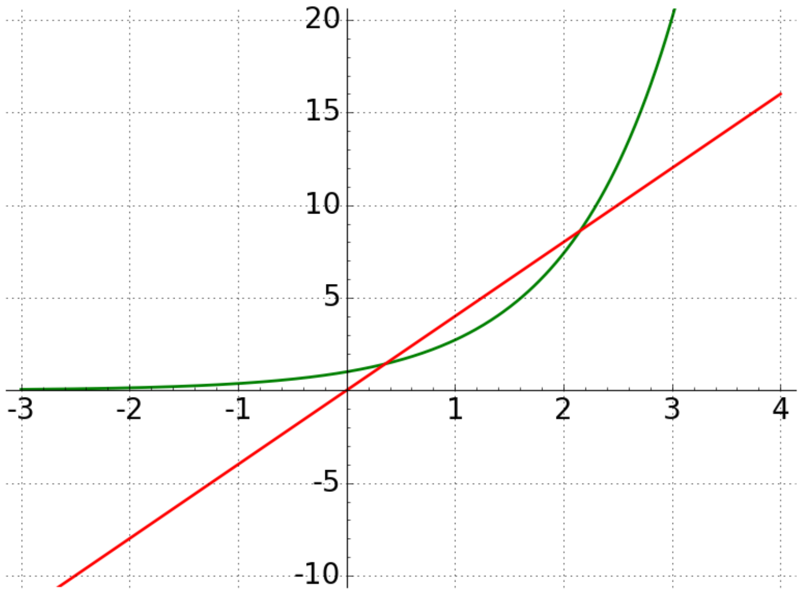
What dimensions should the can have to minimize the cost of metal?

Example. (Fill in the blanks with something reasonable.) In the summer, Yopo sells an average of ________ cones of yogurt per day at a price of \$2 per cone. For every ________ cents more they charge, they sell ________ fewer cones. If it costs them \$________ per cone in ingredients, how much should they charge to maximize profit?

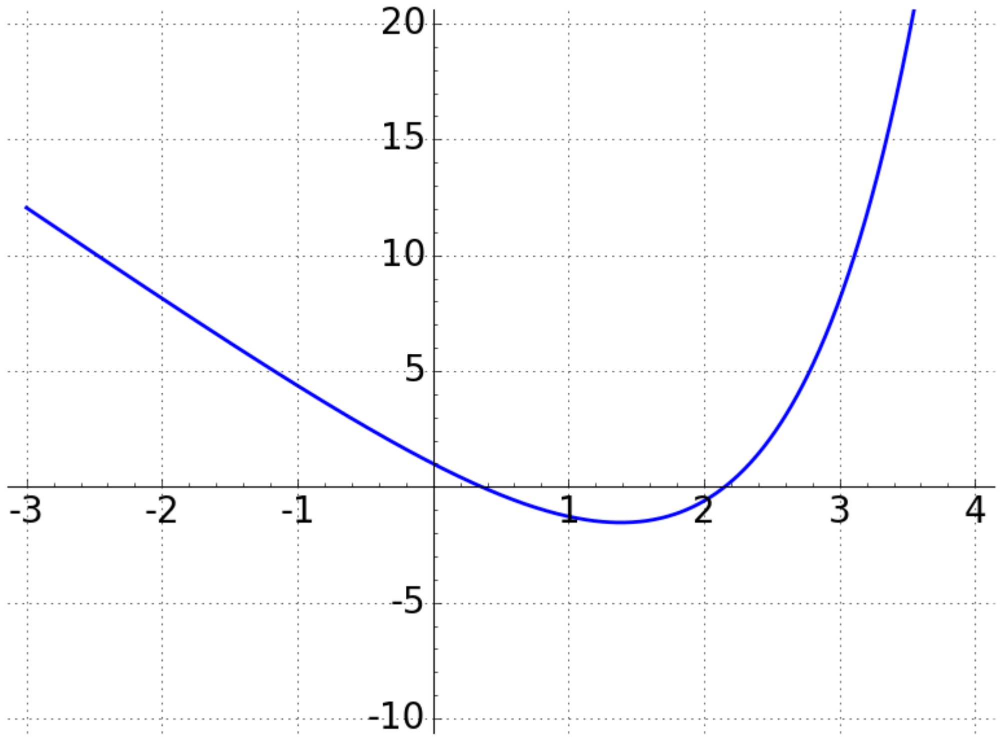
Example. A corridor that is 8 feet wide meets a corridor that is 27 feet wide, at an L-shaped intersection. What is the longest pipe that you can carry around the corridor horizontally?

§4.8 Newton's Method

Example. Find a solution to the equation $e^x = 4x$.



Example. Find a zero for the function $f(x) = e^x - 4x$.



Review. Newton's Method is an approximation method for

1. finding zeros of a function
2. finding solutions to an equation
3. finding x-intercepts for a graph
4. finding maxes and mins

Review. Newton's Method uses successive approximations, where each approximation is obtained from the previous one by:

A. computing a limit

B. using the bisection method

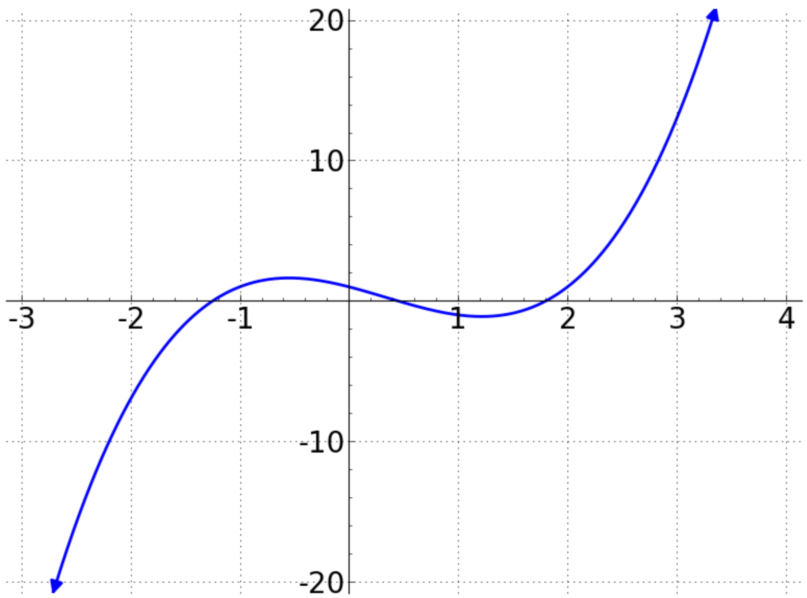
C. find the zero of the tangent line

D. using concavity and the second derivative

Review. The key equation to get from one approximation to the next is:

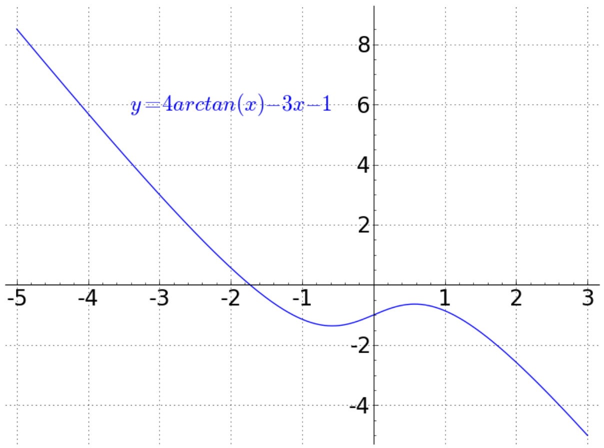
Example. Use Newton's Method to find an approximate solution to

$$x^3 - x^2 - 1 = 2x - 2$$



Example. Use Newton's Method to approximate $\sqrt[3]{2}$ to 4 decimal places.

Question. If you want Newtons Method to converge quickly, which of these guesses is a better first guess?



A. -5

B. 0

Question. Does Newton's Method always work? (How can we break Newton's Method?)

§4.9 Antiderivatives

Example. If $g'(x) = 3x^2$, what could $g(x)$ be?

Definition. A function $F(x)$ is called an antiderivative of $f(x)$ on an interval (a, b) if

Question. What are all the antiderivatives of $f(x) = 3x^2$?

Table of Antiderivatives:

Function $f(x)$	Antiderivative $F(x)$	Function $f(x)$	Antiderivative $F(x)$
1		$\sec^2(x)$	
x		$\sec(x)\tan(x)$	
x^n $(n \neq -1)$		$\frac{1}{1+x^2}$	
x^{-1}		$\dfrac{1}{\sqrt{1-x^2}}$	
e^x		$c \cdot f(x)$	
$\sin(x)$		$f(x) + g(x)$	
$\cos(x)$		$f(x) - g(x)$	

Example. Find the general antiderivative for

$$f(x) = \frac{5}{1+x^2} - \frac{1}{2\sqrt{x}}$$

Review. Which of these is an antiderivative for $\cos(3x)$?

A. $\sin(3x)$

B. $-\sin(3x)$

C. $3\sin(3x)$

D. $-3\sin(3x)$

E. $\frac{1}{3}\sin(3x)$

F. $-\frac{1}{3}\sin(3x)$

Review. Find a general antiderivative for $e^x - 3\sin(x)$.

Example. If $g'(x) = e^x - 3\sin(x)$, and $g(2\pi) = 5$, find $g(x)$.

Definition. An equation like $g'(x) = e^x - 3\sin(x)$ that involves derivatives is called a ____________________ .

An equation like $g(2\pi) = 5$ that can be used to pin down the constant in the antiderivative is called an ________ ________________ .

Example. Find $f(x)$ if $f'(x) = 5e^x + 2x^2$ and $f(0) = 7$.

Example. $f''(x) = \sqrt{x}\left(3 - \frac{1}{x}\right)$. Find an equation for $f(x)$, if $f(1) = 0$ and $f(0) = 2$.

Example. Suppose a particle is moving in along a straight line with acceleration given by $a(t) = \sin t + 3\cos(t) + 5$ and initial velocity $v(0) = -3$ and initial position $s(0) = 1$. Find an equation for the position function $s(t)$.

Example. You stand at the edge of a cliff at height 30 meters. You throw a tomato straight up in the air at a speed of 20 meters per second. How long does it take the tomato to reach the ground? What is its velocity at impact?

Proof that Two Antiderivatives Differ by a Constant

Fact: *If $F(x)$ is one antiderivative of $f(x)$, then any other antiderivative of $f(x)$ can be written in the form $F(x) + C$ for some constant C.*

Note. If $g'(x) = 0$ on the interval (a, b), then $g(x) =$

Note. If $g_1(x)$ and $g_2(x)$ are two functions defined on (a, b) and $g_1'(x) = g_2'(x)$ on (a, b), then

§5.1 Areas and Distances

Background: Review of Sigma Notation

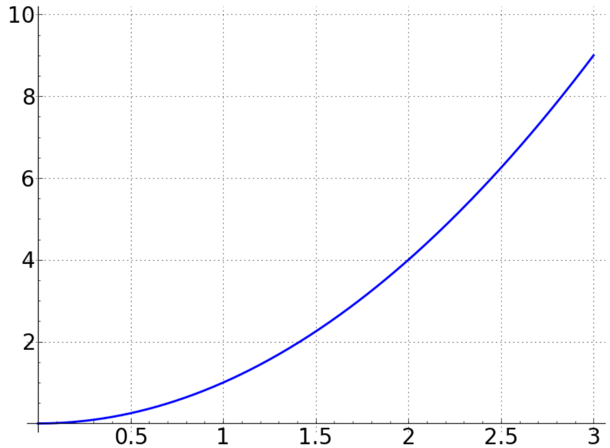
Example. $\displaystyle\sum_{i=1}^{5} 2^i =$

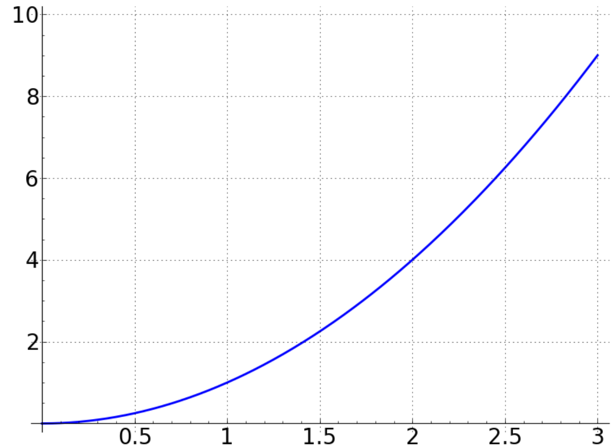
Example. $\displaystyle\sum_{j=3}^{7} \frac{1}{j} =$

Example. Write in Σ notation: $6 + 9 + 12 + 15 + 18$

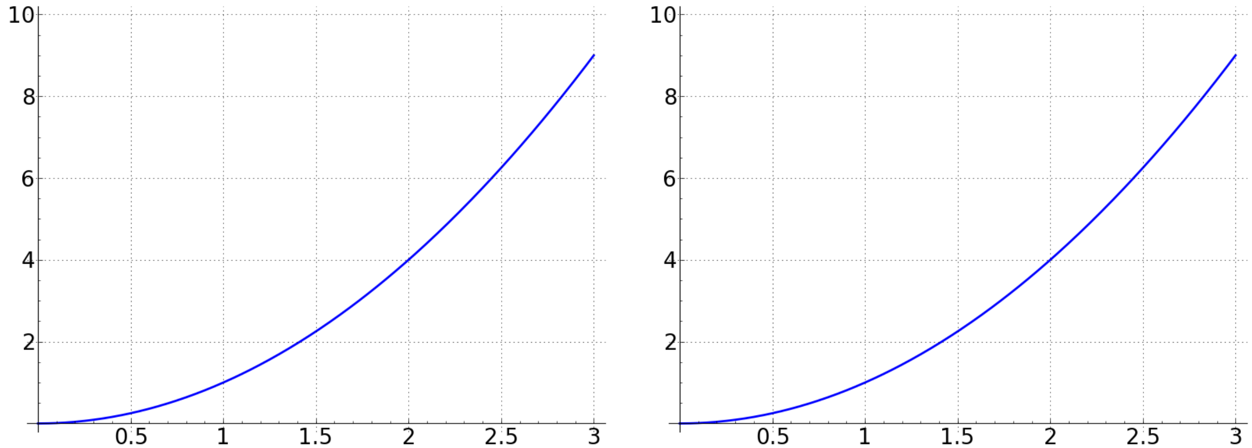
Example. Write in Σ notation: $\dfrac{3}{4} + \dfrac{7}{8} + \dfrac{15}{16} + \dfrac{31}{32}$

Example. Estimate the area under the curve $y = x^2$ between $x = 0$ and $x = 3$ by approximating it with 6 rectangles.

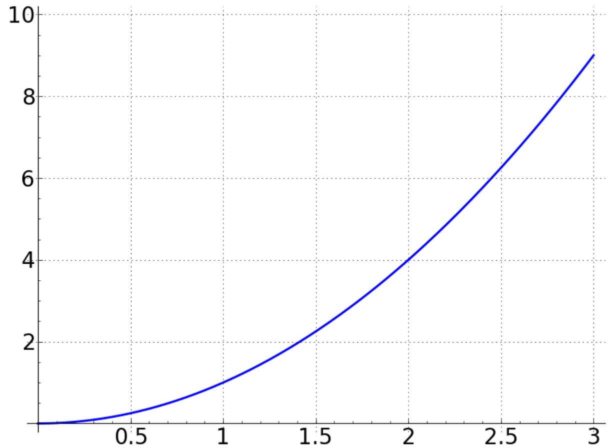


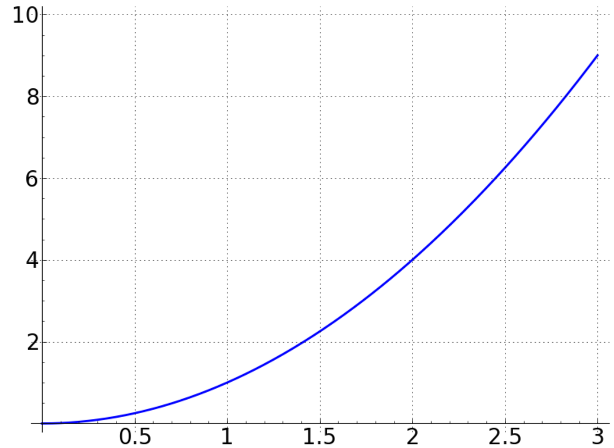


Example. Estimate the area under the curve $y = x^2$ between $x = 0$ and $x = 3$ by approximating it with 12 rectangles.



Example. Estimate the area under the curve $y = x^2$ between $x = 0$ and $x = 3$ by approximating it with n rectangles.





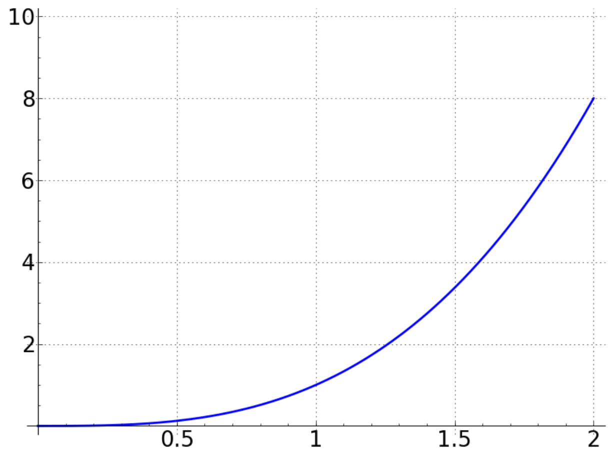
Estimate using n rectangles:

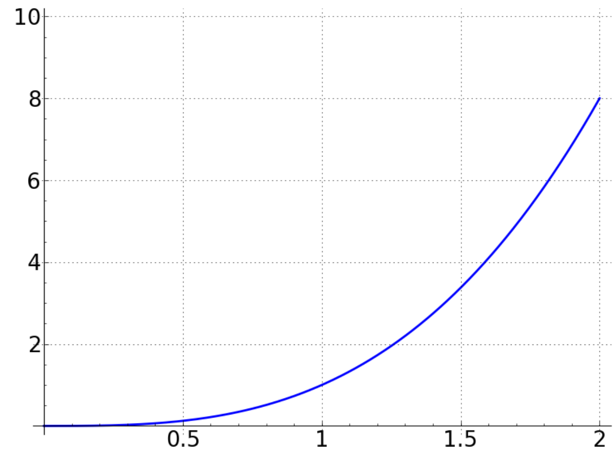
The exact area is given by the limit:

Example. Compute the exact area under the curve $y = x^2$ between $x = 0$ and $x = 3$.

Review. Evaluate the expression $\displaystyle\sum_{i=1}^{3} \frac{5i}{6}$.

Review. Suppose we estimate the area under the curve $y = x^3$ between $x = 0$ and $x = 2$ using 4 rectangles, and using "right endpoints". What is the area of the first rectangle? Second rectangle? Third rectangle? Fourth rectangle?

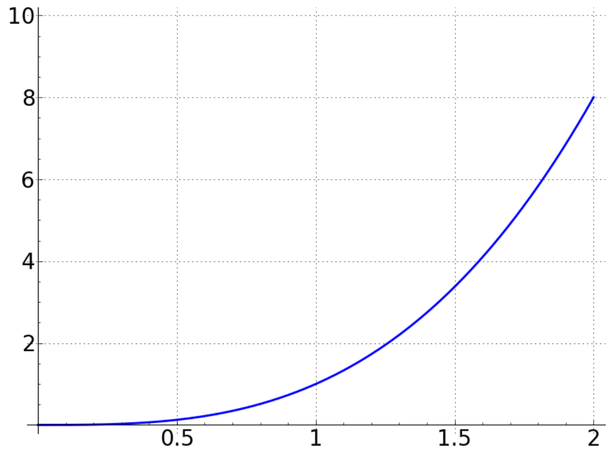


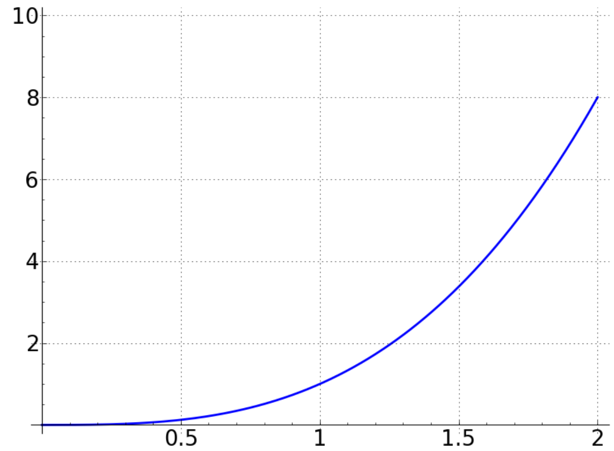


What is the area estimate, using 4 rectangles and right endpoints?

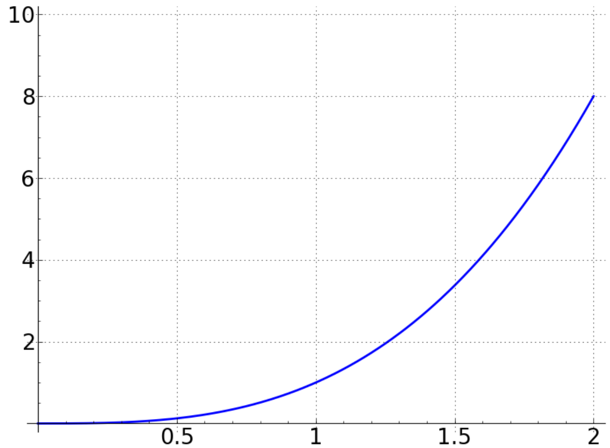
What is the area estimate, using 4 rectangles and left endpoints?

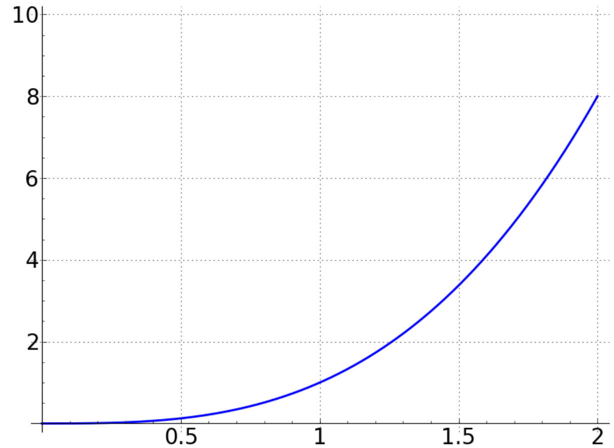
Example. Estimate the area under the curve $y = x^3$ between $x = 0$ and $x = 2$ using 100 rectangles, and using right endpoints. (Then try left endpoints.)





Example. Estimate the area under the curve $y = x^3$ between $x = 0$ and $x = 2$ using n rectangles, and using right endpoints. (or left endpoints)





Example. Write an expression for the exact area under the curve $y = x^3$ between $x = 0$ and $x = b$ using n rectangles, and using right endpoints. Then compute the value of this expression.

Example. Write an expression for an estimate of the area under the curve $y = f(x)$ between $x = a$ and $x = b$ using n rectangles, and using right endpoints. (or left endpoints)

Definition. A Riemann sum for a function $f(x)$ on an interval $[a, b]$ is the sum of areas of rectangles and is given by the formula:

... if we use right endpoints, and

... if we use left endpoints.

The formula for a Riemann sum can also be written as

where the x_i^* are called *sample points* and represent ...
and Δx represents ....

Definition. The *definite integral* of $f(x)$ on the interval $[a, b]$ is written $\displaystyle\int_a^b f(x)\,dx$ and represents a limit of Riemann sums:

$$\int_a^b f(x)\,dx =$$

Definition. A function $f(x)$ on $[a, b]$ is called *integrable* if

Theorem. *If $f(x)$ is ______________________________*
(or ____)
then $f(x)$ is integrable, i.e. $\displaystyle\int_a^b f(x)\,dx$ exists.

Example. Write an expression for the area between the curve $y = \sin(x)$, the x-axis, and the lines $x = 1$ and $x = 2$. Use right endpoints.

Extra Example. Write out the definition of $\int_3^{10} 5x + x^2 \, dx$ on the interval $[3, 10]$.

The Distance Problem

Extra Example. Suppose you bike at velocities given by the following chart:

time (seconds)	0	30	60	90	120
velocity (km/hr)	3.6	21.6	36	28.8	25.2
velocity (m/sec)	1	6	10	8	7

Approximately how far do you go in this period of 120 seconds?

Represent this problem graphically.

§5.2 The Definite Integral

Review. In the expression

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

- $n =$
- $\Delta x =$
- $i =$
- $x_i^* =$
- The area of the first rectangle is:
- The area of the fifth rectangle is:
- The total area of all the rectangles is:

Review. Which of the following expressions does NOT represent $\int_a^b f(x)\,dx$?

A. $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$

B. $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n} f(a+i\Delta x)\Delta x$

C. $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\frac{(b-a)}{n}$

D. $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n} \frac{(b-a)}{n} f\left(\frac{(b-a)}{n}i\right)$

E. $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n} \frac{(b-a)}{n} f\left(a+\frac{(b-a)}{n}i\right)$

Example. Write an expression for the area under the curve $y = 5x + x^2$ on the interval $[3, 10]$.

Example. The expression

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{5}{n} \sqrt{2 + \frac{5}{n} i}$$

represents the integral of a function on an interval. What is the function $f(x)$ and what is the interval?

1. $f(x) = \sqrt{x}$, interval is $[0, 5]$
2. $f(x) = \sqrt{x}$, interval is $[2, 5]$
3. $f(x) = \sqrt{x}$, interval is $[2, 7]$
4. $f(x) = \sqrt{2 + x}$, interval is $[0, 5]$
5. $f(x) = \sqrt{2 + x}$, interval is $[2, 7]$

Example. The expression $\lim_{n\to\infty}\sum_{i=1}^{n}\dfrac{1}{4+\frac{3i}{n}}\cdot\dfrac{3}{n}$ represents the integral of a function on an interval. What is the function and what is the interval?

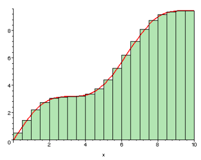
A. $f(x)=\frac{1}{4+x}$, $[0,3]$

B. $f(x)=\frac{1}{x}$, $[4,7]$

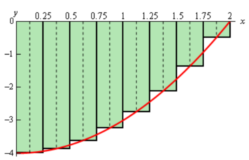
C. $f(x)=\frac{1}{4+3x}$, $[0,1]$

D. $f(x)=\frac{x}{4+x}$, $[0,3]$

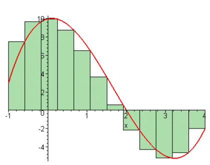
Note. If $f(x)$ is a *positive function* on (a, b) (i.e. $f(x) > 0$ on (a, b)), then $\displaystyle\int_a^b f(x)\,dx$ represents:



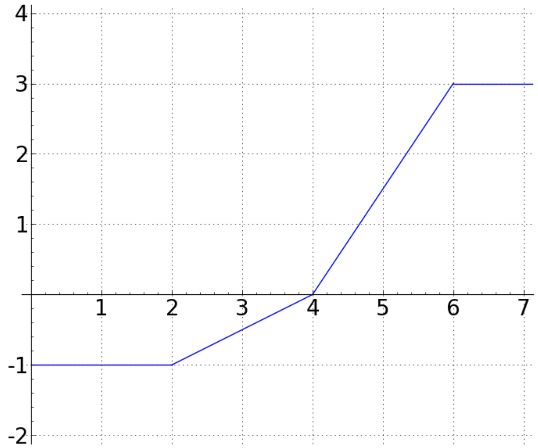
Note. If $f(x)$ is a *negative function* on (a, b) (i.e. $f(x) < 0$ on (a, b)), then $\displaystyle\int_a^b f(x)\,dx$ represents:



Note. If $f(x)$ is both positive and negative on (a, b), then $\displaystyle\int_a^b f(x)\,dx$ represents:



Example. Use the graph to find $\displaystyle\int_2^7 f(x)\,dx$.



Properties of Integrals

Explain the following properties in terms of *area*.

1. $\int_a^a f(x)\,dx = 0$

2. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

3. * $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

4. $\int_a^b c\,dx = c(b-a)$

5. * $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$

6. * $\int_a^b f(x) + g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

7. * $\int_a^b f(x) - g(x)\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

8. If $f(x) \le g(x)$ on $[a, b]$, then $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$

9. If $m \le f(x) \le M$ for $a \le x \le b$ then

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$$

Example. If $\displaystyle\int_1^5 f(x)\,dx = 7$ and $\displaystyle\int_3^5 f(x)\,dx = 4$, what is $\displaystyle\int_1^3 2f(x)\,dx$?

Example. Which inequality is true?

A. $\int_1^4 e^{-x^2}\,dx < \frac{3}{2}$

B. $\int_1^4 e^{-x^2}\,dx = \frac{3}{2}$

C. $\int_1^4 e^{-x^2}\,dx > \frac{3}{2}$

Question. Suppose that you are using rectangles to approximate the area under a the graph of an increasing function between $x = a$ and $x = b$. Which of the following is likely to increase the accuracy of your estimate?

A. Using right endpoints instead of left endpoints.

B. Using midpoints instead of left or right endpoints.

C. Increasing the number of rectangles.

Question. True or False: A Riemann sum using left endpoints gives an underestimate of area and a Riemann sum with right endpoints gives an overestimate.

Question. Suppose you are using rectangles to estimate the area under the graph of a *decreasing* function $f(x)$. Which of the following is guaranteed to get you an underestimate of the area?

A. Using left endpoints

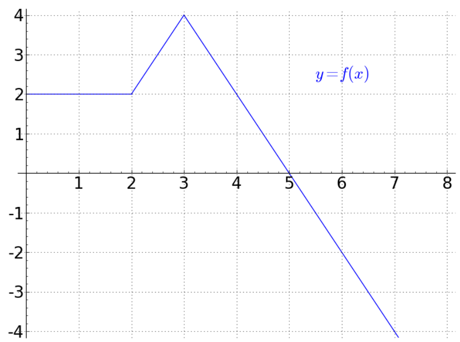
B. Using right endpoints

C. Using midpoints

Extra Example. Evaluate $\int_a^b 2x \, dx$ using the definition of integral.

§5.3 The Fundamental Theorem of Calculus

Example. Suppose $f(x)$ has the graph shown, and let $g(x) = \int_1^x f(t)\,dt$.



Find:

$g(1) =$

$g(2) =$

$g(3) =$

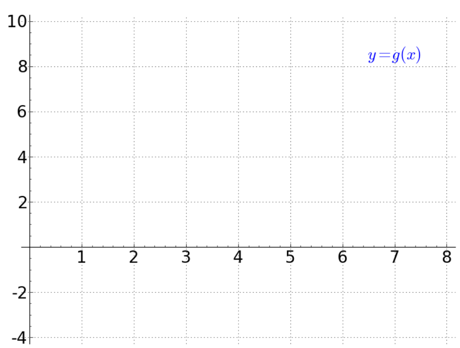
$g(4) =$

$g(5) =$

$g(6) =$

$g(7) =$

$g(0) =$



Theorem. *(Fundamental Theorem of Calculus, Part 1) If $f(x)$ is continuous on $[a, b]$ then for $a \leq x \leq b$ the function*

$$g(x) = \int_a^x f(t)\, dt$$

is continuous on $[a, b]$ and differentiable on (a, b) and

$$g'(x) =$$

Example. Find

1. $\frac{d}{dx}\int_5^x \sqrt{t^2+3}\,dt$

2. $\frac{d}{dx}\int_4^x \sqrt{t^2+3}\,dt$

3. $\frac{d}{dx}\int_x^4 \sqrt{t^2+3}\,dt$

4. $\frac{d}{dx}\int_4^{\sin(x)} \sqrt{t^2+3}\,dt$

Theorem. *(Fundamental Theorem of Calculus, Part 2) If f is continuous on $[a,b]$, then*

$$\int_a^b f(x)\,dx =$$

Example. Find $\int_{-1}^{-5} 3x^2 - \frac{4}{x}\, dx$

Example. Find $\int_1^4 \frac{y^2-y+1}{\sqrt{y}}\,dy$

Review. In your own words (no equations) what does the Fundamental Theorem of Calculus say?

Theorem. *(Fundamental Theorem of Calculus, Part 1)*

Theorem. *(Fundamental Theorem of Calculus, Part 2)*

Fundamental Theorem of Calculus Part 2

Example. Compute $\int_0^2 x^3 \, dx$.

Example. Compute $\int_{-2}^{-1} \frac{y^3+3y-2}{y^2} \, dy$

Example. Find $\int_0^1 e^x + x^e \, dx$

Fundamental Theorem of Calculus Part 1

Example. Find $\frac{d}{dx}\int_1^x t^3\,dt$

Example. Find $\frac{d}{dx}\int_x^5 t^3\,dt$

Example. Find $\frac{d}{dx} \int_1^{\ln(x)} t^3 \, dt$

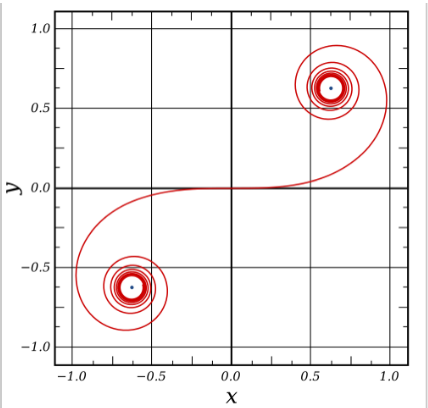
Example.

$$C(x) = \int_0^x \cos\left(t^2\right)\, dt$$

and

$$S(x) = \int_0^x \sin\left(t^2\right)\, dt$$

are the Fresnel integrals, used in optics and in highway and roller coaster design.



Euler spiral $(x, y) = (C(t), S(t))$.

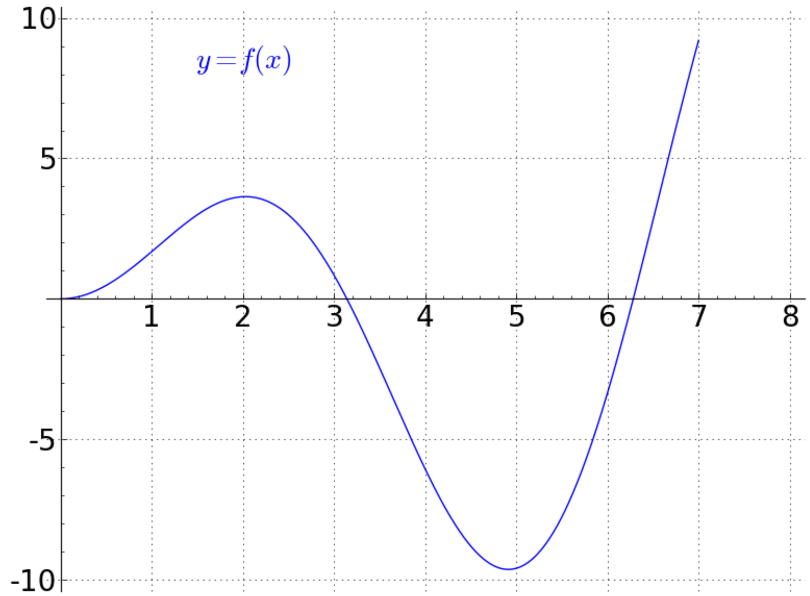
Example. Find

1. $\frac{d}{dx}\int_0^x \sin(t^2)\,dt$

2. $\frac{d}{dx}\int_0^{x^3} \sin(t^2)\,dt$

3. $\frac{d}{dx}\int_x^{x^3} \sin(t^2)\,dt$

Example. The function $f(x)$ is graphed below. Let $g(x) = \int_2^x f(t)\, dt$. Where will $g(x)$ be increasing? Concave up? Have a local max?



Question. Does every continuous function have an antiderivative?

§5.4 Indefinite Integrals and the Net Change Theorem

Recall: A *definite integral* is an integral of the form

Definition. An *indefinite integral* is written:

and means

Example. 1. $\int_{\pi}^{2\pi} \sin(x)\, dx =$

2. $\int \sin(x)\, dx =$

Question. What is $\dfrac{d}{dx}\displaystyle\int_0^1 \frac{1}{1+t^2}\,dt$?

Interpreting Integrals

Example. If $g(t)$ is the rate of change at which water flows into a reservoir in cubic feet per minute, and x is time in minutes, then give the units and interpret the following.

$\Delta x = 2$

$g(x_3^*) = 10$

$g(x_5^*) = -20$

$g(x_5^*)\Delta x = -40$

$\sum_{i=1}^{n} g(x_i^*)\Delta x = -30$

$\int_{10}^{20} g(x)\,dx = -30$

Theorem. *(The Net Change Theorem) The integral of a rate of change represents:*

Example. If $v(t)$ is the velocity of a bicycle at time t in miles per hour, and t is time in hours, then give units and interpret the following.

$\Delta t = 0.5$

$v(t_i^*) = 8$

$v(t_i^*)\Delta t = 4$

$\int_3^7 v(t)\,dt = 1$

$\int_3^7 |v(t)|\,dt = 1$

Example. Let

$$v(t) = \begin{cases} t & \text{if } 0 \le t \le 2 \\ 4 - t & \text{if } 2 < t \le 6 \\ -2 & \text{if } 6 < t \le 8 \end{cases}$$

be the velocity function (in mm/sec) for an ant crawling along the edge of a ruler placed horizontally. Treat crawling to the right as the positive direction of travel.

1. Sketch the graph of v.

2. At what time does the bug turn around to begin crawling in the negative direction of travel?

3. At time 4, the bug has crawled a distance of $D_4 =$________.

4. At time 8, the bug has crawled a distance of $D_8 =$________.

5. At time 8, the bug is ________ units to the (choose one) RIGHT LEFT of its starting point (the location at time 0).

Example. The acceleration function for a particle is given by $a(t) = 2t - 2$, with initial velocity given by $v(0) = -3$.

1. Find the displacement of the particle between time $t = 1$ and $t = 4$.

2. Find the total distance traveled between $t = 1$ and $t = 4$.

§5.5 The Substitution Method

Example. Find $\int 2x \sin(x^2)\, dx$

Example. Evaluate $\int \frac{x}{1+3x^2}\,dx$

Example. $\int e^{7x}\,dx$

Substitution with Definite Integrals

Example. $\int_e^{e^2} \frac{\ln(x)}{x}\, dx$

Why does the substitution method work?

Review. Use substitution to find: $\int \sin^2(5x)\cos(5x)\,dx$

Example. $\int x^4 \sqrt{1 + x^5}\, dx$

Example. $\int \tan^3 t \sec^2 t\, dt$

Example. $\int e^{5-3x}\, dx$

Substitution with Definite Integrals

Example. $\int_0^{1/4} \cos(-2\pi t)\, dt$

Example. $\int_5^8 \frac{x}{4-x^2}\, dx$

Example. $\int \frac{\arctan(x)}{1+x^2}\,dx$

Example. $\int \tan(7x)\,dx$

Example. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

Tricky problems:

Example. $\int x^3 \sqrt{1-x^2}\, dx$

Example. $\int_0^{\sqrt{2}/2} \frac{x}{\sqrt{1-x^4}}\, dx$

More tricky problems:

Example. $\int (1 + x^2)x^3 \, dx$

Example. Find:

1. $\int_{-1}^{1} x^2 \sin(4x^3)\, dx$

2. $\int_{-1}^{1} x^2 \cos(4x^3)\, dx$

Final Exam Review

Question. True or False: If you write the answer to $\int \frac{1}{x}\, dx$ as $\ln(x)$, you will get full credit on the exam.

Question. True or False: You need to show work on multiple choice questions.

Question. True or False: If $v(t) = 4-6x$, then the distance traveled between time 0 and time 1 is $\int_0^1 4 - 6t\, dt$.

Question. True or False: If $f'(x) = \frac{2-3x}{x^3}$, then the only place where f can change from increasing to decreasing or vice versa is at $x = \frac{2}{3}$.

Question. True or False: The Mean Value Theorem will not be on the final exam this semester.

Question. True or False: $\displaystyle\int_0^{x^2} \sqrt{\cos(t+1)}\, dt = \sqrt{\cos(x^2+1)}$

Question. True or False: The graph of $f(x) = \dfrac{x^2 - 16}{x^2 - 8x + 16}$ has a hole at $x = 4$.

Question. True or False: You need to memorize all your basic derivatives and antiderivatives, including the derivative of $\cos^{-1}(x)$ and $\tan^{-1}(x)$.

Question. True or False:

$$\frac{d}{dx}\tan^{-1}(x) = \frac{d}{dx}\left(\frac{\cos(x)}{\sin(x)}\right)$$

Question. True or False: $\arccos\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$