Chapter 7

Linear Momentum and Collisions

Intended Learning Outcomes

- To apply the concept of linear momentum in isolated and non-isolated systems
- To solve problems of collisions in one dimension and two dimensions
- To identify the center of mass of different systems

Newton's Laws and Conservation law

- Consider an isolated system of two particles with masses m1 and m2 moving with velocities $\vec{v}_1$ and $\vec{v}_2$ at an instant of time.
- Since the system is isolated, the only force on particle 1 is that from the particle 2 and vice versa (action and reaction forces according to Newton's third law). $\vec{F}_{12} = -\vec{F}_{21}$

$$\vec{F}_{21} + \vec{F}_{12} = 0 \Rightarrow m_1\vec{a}_1 + m_2\vec{a}_2 = 0 \Rightarrow m_1\frac{d\vec{v}_1}{dt} + m_2\frac{d\vec{v}_2}{dt} = 0 \Rightarrow \frac{d(m_1\vec{v}_1)}{dt} + \frac{d(m_2\vec{v}_2)}{dt} = 0$$

- From Newton's second law, $\dfrac{d(m_1\vec{v}_1 + m_2\vec{v}_2)}{dt} = 0$

$(m_1\vec{v}_1 + m_2\vec{v}_2)$ is conserved (a constant).

Conservation of Linear Momentum

- If we define the **linear momentum** (or momentum, $\vec{p}$) of a particle of mass *m* moving with a velocity $\vec{v}$ to be the product of the mass and velocity:
- $$\vec{\mathbf{p}} \equiv m\vec{\mathbf{v}}$$

If there is no net external force acting on a system, the total momentum of the system remains constant.

$\vec{p}$ is a vector and has a unit of $kgms^{-1}$.

Newton's Second Law and Momentum

•Using the definition of momentum, we can rewrite Newton's second law as the following:

$$\vec{F}_{net} = m\vec{a} = m\frac{d\vec{v}}{dt} = \frac{d(m\vec{v})}{dt} = \frac{d\vec{p}}{dt}$$

•i.e. the effect of a net force on a system is to change its linear momentum.

Example 9.1

- A 60kg man stands at rest on frictionless ice and fires a 0.50 kg arrow at 50 ms^{-1}. Determine the speed of the man after firing.

$$m_1\vec{\mathbf{v}}_{1f} + m_2\vec{\mathbf{v}}_{2f} = 0$$

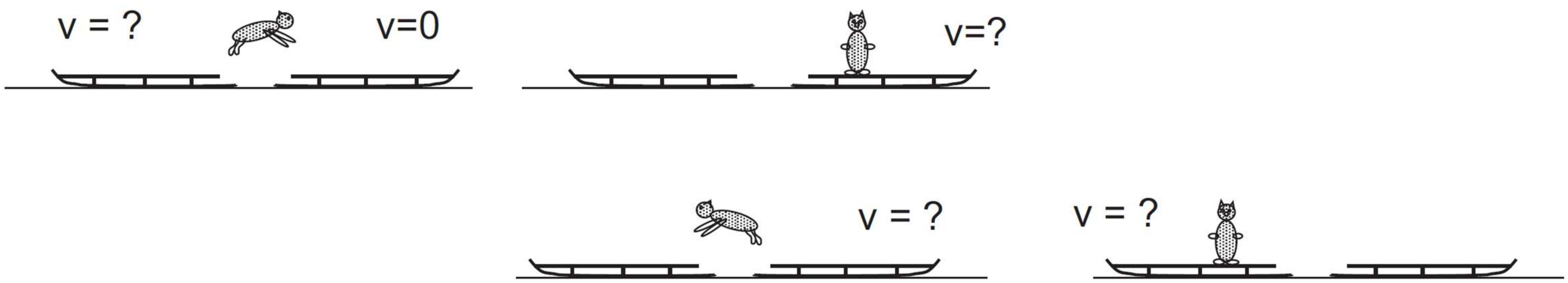
$$\vec{\mathbf{v}}_{1f} = -\frac{m_2}{m_1}\vec{\mathbf{v}}_{2f} = -\left(\frac{0.50\text{ kg}}{60\text{ kg}}\right)(50\hat{\mathbf{i}}\text{ m/s}) = \boxed{-0.42\hat{\mathbf{i}}\text{ m/s}}$$

***Quick Quiz* 9.1** Two objects have equal kinetic energies. How do the magnitudes of their momenta compare? **(a)** $p_1 < p_2$ **(b)** $p_1 = p_2$ **(c)** $p_1 > p_2$ **(d)** not enough information to tell

***Quick Quiz* 9.2** Your physical education teacher throws a baseball to you at a certain speed and you catch it. The teacher is next going to throw you a medicine ball whose mass is ten times the mass of the baseball. You are given the following choices: You can have the medicine ball thrown with **(a)** the same speed as the baseball, **(b)** the same momentum, or **(c)** the same kinetic energy. Rank these choices from easiest to hardest to catch.

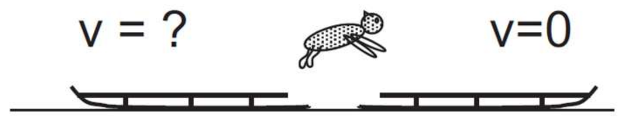
Challenging Question

8. Two $22.7\,\mathrm{kg}$ ice sleds are placed a short distance apart, one directly behind the other, as shown in Fig. 7.10 (a). A $3.63\,\mathrm{kg}$ cat, standing on one sled, jumps across to the other and immediately back to the first. Both jumps are made at a speed on $3.05\,\frac{\mathrm{m}}{\mathrm{s}}$ relative to the ice. Find the final speeds of the two sleds.



We will let the x axis point to the right. In the initial picture (not shown) the cat is sitting on the left sled and both are motionless. Taking our system of interacting "particles" to be the cat and the left sled, the initial momentum of the system is $P = 0$.

After the cat has made its first jump, the velocity of this sled will be $v_{L,x}$, and the (final) total momentum of the system will be



$$P_f = (22.7\,\mathrm{kg})v_{L,x} + (3.63\,\mathrm{kg})(+3.05\,\tfrac{\mathrm{m}}{\mathrm{s}})$$

Note, we are using velocities with respect to the ice, and that is how we were given the velocity of the cat. Now as there are no net external forces, the momentum of this system is conserved. This gives us:

$$0 = (22.7\,\mathrm{kg})v_{L,x} + (3.63\,\mathrm{kg})(+3.05\,\tfrac{\mathrm{m}}{\mathrm{s}})$$

with which we easily solve for $v_{L,x}$:

$$v_{L,x} = -\frac{(3.63\,\mathrm{kg})(+3.05\,\frac{\mathrm{m}}{\mathrm{s}})}{(22.7\,\mathrm{kg})} = -0.488\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

so that the left sled moves at a speed of $0.488\,\frac{\mathrm{m}}{\mathrm{s}}$ *to the left* after the cat's first jump.

The cat lands on the right sled and after landing it moves with the same velocity as that sled; the collision here is completely inelastic. For this part of the problem, the system of "interacting particles" we consider is *the cat and the right sled*. (The left sled does not interact with *this* system.) The initial momentum of this system is just that of the cat,

$$P_i = (3.63\,\mathrm{kg})(+3.05\,\tfrac{\mathrm{m}}{\mathrm{s}}) = 11.1\,\tfrac{\mathrm{kg\cdot m}}{\mathrm{s}}$$

If the final velocity of both cat and sled is $v_{R,x}$ then the final momentum

$$P_f = (22.7\,\mathrm{kg} + 3.63\,\mathrm{kg})v_{R,x} = (26.3\,\mathrm{kg})v_{R,x}$$

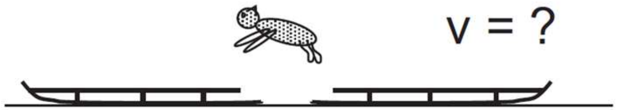
(The cat and sled move as one mass, so we can just add their individual masses.) Conservation of momentum of this system, $P_i = P_f$ gives

$$11.1\,\tfrac{\mathrm{kg\cdot m}}{\mathrm{s}} = (26.3\,\mathrm{kg})v_{R,x}$$

so

$$v_{R,x} = \frac{\left(11.1\,\tfrac{\mathrm{kg\cdot m}}{\mathrm{s}}\right)}{(26.3\,\mathrm{kg})} = 0.422\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

And now the cat makes a jump back to the left sled, as shown in Fig. 7.11 (a). Again, we take the system to be the cat and the right sled. Its initial momentum is



$$P_i = (22.7\,\mathrm{kg} + 3.63\,\mathrm{kg})(0.422\,\tfrac{\mathrm{m}}{\mathrm{s}}) = 11.1\,\tfrac{\mathrm{kg \cdot m}}{\mathrm{s}}$$

Now after the cat leaps, the velocity of the cat (with respect to the ice) is $-3.05\,\frac{\mathrm{m}}{\mathrm{s}}$, as specified in the problem. If the velocity of the right sled after the leap is $v'_{R,x}$ then the final momentum of the system is

$$P_f = (3.63\,\mathrm{kg})(-3.05\,\tfrac{\mathrm{m}}{\mathrm{s}}) + (22.7\,\mathrm{kg})v'_{R,x}$$

Conservation of momentum for the system, $P_i = P_f$, gives

$$11.1\,\tfrac{\mathrm{kg \cdot m}}{\mathrm{s}} = (3.63\,\mathrm{kg})(-3.05\,\tfrac{\mathrm{m}}{\mathrm{s}}) + (22.7\,\mathrm{kg})v'_{R,x}$$

so that we can solve for $v'_{R,x}$:

$$v'_{R,x} = \frac{(11.1\,\frac{\mathrm{kg \cdot m}}{\mathrm{s}} + 11.1\,\frac{\mathrm{kg \cdot m}}{\mathrm{s}})}{(22.7\,\mathrm{kg})} = 0.975\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

so during its second leap the cat makes the right sled go faster!

Finally, for the cat's landing on the left sled we consider the (isolated) system of the cat and the left sled. We already have the velocities of the cat and sled at this time; its initial momentum is

$$P_i = (22.7\,\mathrm{kg})(-0.488\,\tfrac{\mathrm{m}}{\mathrm{s}}) + (3.63\,\mathrm{kg})(-3.05\,\tfrac{\mathrm{m}}{\mathrm{s}}) = -22.1\,\tfrac{\mathrm{kg\cdot m}}{\mathrm{s}}\ .$$

After the cat has landed on the sled, it is moving with the same velocity as the sled, which we will call $v'_{L,x}$. Then the final momentum of the system is

$$P_f = (22.7\,\mathrm{kg} + 3.63\,\mathrm{kg})v'_{L,x} = (26.3\,\mathrm{kg})v'_{L,x}$$

And momentum conservation for *this* collision gives

$$-22.1\,\tfrac{\mathrm{kg\cdot m}}{\mathrm{s}} = (26.3\,\mathrm{kg})v'_{L,x}$$

and then

$$v'_{L,x} = \frac{(-22.1\,\frac{\mathrm{kg\cdot m}}{\mathrm{s}})}{(26.3\,\mathrm{kg})} = -0.842\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

Summing up, the final velocities of the sleds (after the cat is done jumping) are:

$$\text{Left Sled:}\qquad v'_{L,x} = -0.842\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

$$\text{Right Sled:}\qquad v'_{r,x} = +0.975\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

Impulse and Momentum

- The momentum of a system changes if a net force from the environment acts on the system.
- From Newton's Second Law, $\vec{\mathbf{F}} = \dfrac{d\vec{\mathbf{p}}}{dt}$
- Solving for $d\vec{\mathbf{p}}$ gives $d\vec{\mathbf{p}} = \sum \vec{\mathbf{F}}dt$
- Integrating to find the change in momentum over some time interval.

$$\Delta\vec{\mathbf{p}} = \vec{\mathbf{p}}_f - \vec{\mathbf{p}}_i = \int_{t_i}^{t_f} \vec{\mathbf{F}}dt = \vec{\mathbf{I}}$$

- The integral is called the *impulse,* $\vec{\mathbf{I}}$*,* of the force acting on an object over Δt.

More About Impulse

•Impulse is a vector quantity.

•The magnitude of the impulse is equal to the area under the force-time curve.

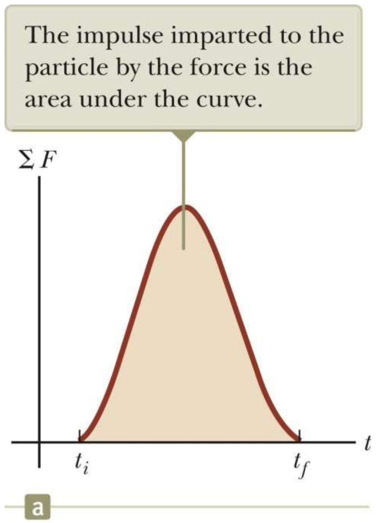
- The force may vary with time.

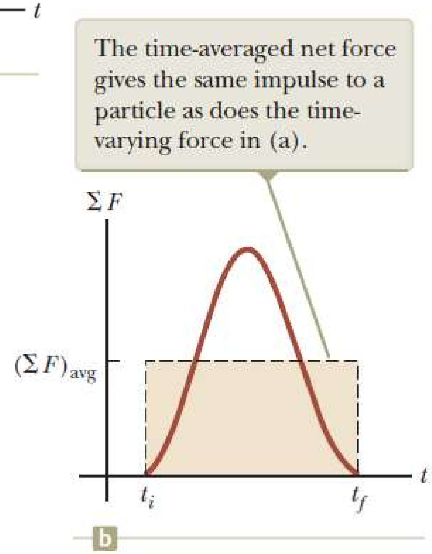
•Impulse is a measure of the change in momentum of the particle.

•The impulse can also be found by using the time averaged (impulsive) force.

$$\vec{\mathbf{I}} = \sum \vec{\mathbf{F}} \Delta t$$

•This would give the same impulse as the time-varying force does.





Example 9.3 How Good Are the Bumpers?

In a particular crash test, a car of mass 1 500 kg collides with a wall as shown in Figure 9.4 (page 242). The initial and final velocities of the car are $\vec{\mathbf{v}}_i = -15.0\hat{\mathbf{i}}$ m/s and $\vec{\mathbf{v}}_f = 2.60\hat{\mathbf{i}}$ m/s, respectively. If the collision lasts 0.150 s, find the impulse caused by the collision and the average net force exerted on the car.

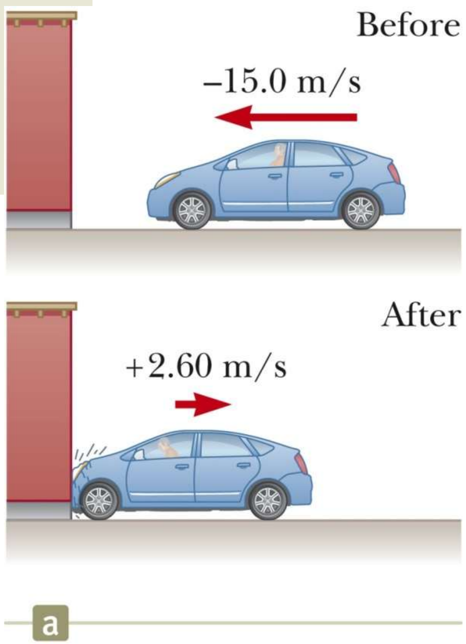
$$\vec{\mathbf{p}}_i = m\vec{\mathbf{v}}_i = (1\,500\text{ kg})(-15.0\hat{\mathbf{i}}\text{ m/s}) = -2.25 \times 10^4\hat{\mathbf{i}}\text{ kg}\cdot\text{m/s}$$

$$\vec{\mathbf{p}}_f = m\vec{\mathbf{v}}_f = (1\,500\text{ kg})(2.60\hat{\mathbf{i}}\text{ m/s}) = 0.39 \times 10^4\hat{\mathbf{i}}\text{ kg}\cdot\text{m/s}$$

$$\vec{\mathbf{I}} = \Delta\vec{\mathbf{p}} = \vec{\mathbf{p}}_f - \vec{\mathbf{p}}_i = 0.39 \times 10^4\hat{\mathbf{i}}\text{ kg}\cdot\text{m/s} - (-2.25 \times 10^4\hat{\mathbf{i}}\text{ kg}\cdot\text{m/s})$$

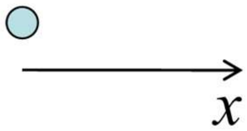
$$= \boxed{2.64 \times 10^4\hat{\mathbf{i}}\text{ kg}\cdot\text{m/s}}$$

$$\left(\sum\vec{\mathbf{F}}\right)_{\text{avg}} = \frac{\vec{\mathbf{I}}}{\Delta t} = \frac{2.64 \times 10^4\hat{\mathbf{i}}\text{ kg}\cdot\text{m/s}}{0.150\text{ s}} = \boxed{1.76 \times 10^5\hat{\mathbf{i}}\text{ N}}$$



For a top player, a tennis ball may leave the racket on the serve with a speed of 55 m/s (about 120 mi/h). If the ball has a mass of 0.060 kg and is in contact with the racket for about 4 ms (4 x 10^{-3} s), estimate the average force on the ball.

[Solution]



$$\vec{J} = \vec{F}_{ave}\Delta t = \Delta\vec{p}$$

$$F_{ave} = \frac{\Delta p}{\Delta t} = \frac{mv - 0}{\Delta t}$$

$$= \frac{0.06 \times 55}{0.004}$$

$$= 800\ (N)$$

***Quick Quiz* 9.3** Two objects are at rest on a frictionless surface. Object 1 has a greater mass than object 2. **(i)** When a constant force is applied to object 1, it accelerates through a distance d in a straight line. The force is removed from object 1 and is applied to object 2. At the moment when object 2 has accelerated through the same distance d, which statements are true? (a) $p_1 < p_2$ (b) $p_1 = p_2$ (c) $p_1 > p_2$ (d) $K_1 < K_2$ (e) $K_1 = K_2$ (f) $K_1 > K_2$ **(ii)** When a force is applied to object 1, it accelerates for a time interval Δt. The force is removed from object 1 and is applied to object 2. From the same list of choices, which statements are true after object 2 has accelerated for the same time interval Δt?

***Quick Quiz* 9.4** Rank an automobile dashboard, seat belt, and air bag in terms of (a) the impulse and (b) the average force each delivers to a front-seat passenger during a collision, from greatest to least.

Collisions – Characteristics

- The term **collision** represents an event during which two particles come close to each other and interact by means of forces.
- Direct physical contact is not necessary
- The interaction forces are assumed to be much greater than any external forces present.
  - This means the impulse approximation can be used.

Types of Collisions

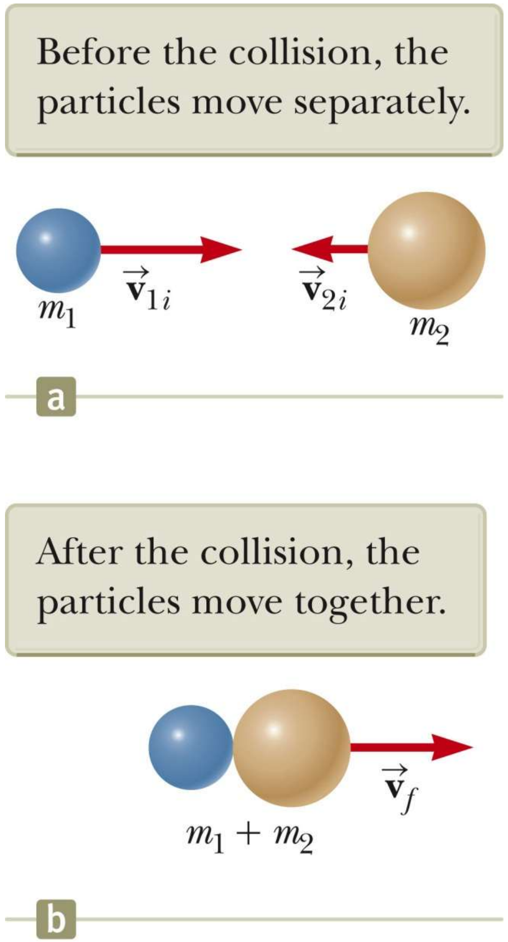
- In an ***elastic*** collision, momentum and kinetic energy are conserved.
  - Perfectly elastic collisions occur on a microscopic level.
  - In macroscopic collisions, only approximately elastic collisions actually occur.
- In an ***inelastic*** collision, kinetic energy is not conserved, although momentum is still conserved.
  - If the <u>objects stick together</u> after the collision, it is a ***perfectly inelastic*** collision.
- Momentum is conserved in all collisions

Perfectly Inelastic Collisions

$$\vec{p}_i = \vec{p}_f$$

$$m_1\vec{v}_{1i} + m_2\vec{v}_{2i} = (m_1 + m_2)\vec{v}_f$$

$$\vec{v}_f = \frac{m_1\vec{v}_{1i} + m_2\vec{v}_{2i}}{m_1 + m_2}$$



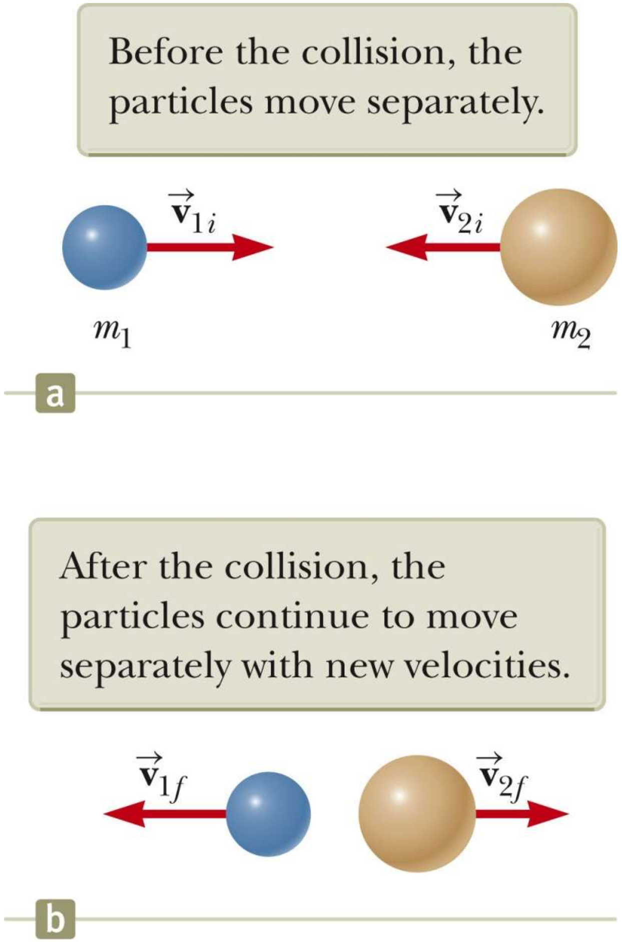
Elastic Collisions

- Both momentum and kinetic energy are conserved.

$$m_1\vec{v}_{1i} + m_2\vec{v}_{2i} = m_1\vec{v}_{1f} + m_2\vec{v}_{2f}$$

$$\frac{1}{2}m_1\vec{v}_{1i}^2 + \frac{1}{2}m_2\vec{v}_{2i}^2 = \frac{1}{2}m_1\vec{v}_{1f}^2 + \frac{1}{2}m_2\vec{v}_{2f}^2$$

- Typically, there are two unknowns to solve for and so you need two equations.



Elastic Collisions

For one dimensional case,

KE: $\frac{1}{2}m_1v_{1i}^2 + \frac{1}{2}m_2v_{2i}^2 = \frac{1}{2}m_1v_{1f}^2 + \frac{1}{2}m_2v_{2f}^2$

$$\frac{m_1}{m_2}\left(v_{1i}^2 - v_{1f}^2\right) = \left(v_{2f}^2 - v_{2i}^2\right)$$

$$\frac{m_1}{m_2}\left(v_{1i} - v_{1f}\right)\left(v_{1i} + v_{1f}\right) = \left(v_{2f} - v_{2i}\right)\left(v_{2f} + v_{2i}\right) \Leftarrow$$

$$\left(v_{1i} + v_{1f}\right) = \left(v_{2f} + v_{2i}\right)$$

$$v_{1i} - v_{2i} = -\left(v_{1f} - v_{2f}\right)$$

Momentum:

$$m_1v_{1i} + m_2v_{2i} = m_1v_{1f} + m_2v_{2f}$$

$$\frac{m_1}{m_2}\left(v_{1i} - v_{1f}\right) = \left(v_{2f} - v_{2i}\right)$$

$$\begin{cases} \frac{m_1}{m_2}v_{1f} + v_{2f} = \frac{m_1}{m_2}v_{1i} + v_{2i} \\ v_{1f} - v_{2f} = -v_{1i} + v_{2i} \end{cases} \Rightarrow \begin{aligned} v_{1f} &= \left(\frac{m_1 - m_2}{m_1 + m_2}\right)v_{1i} + \left(\frac{2m_2}{m_1 + m_2}\right)v_{2i} \\ v_{2f} &= \left(\frac{2m_1}{m_1 + m_2}\right)v_{1i} - \left(\frac{m_1 - m_2}{m_1 + m_2}\right)v_{2i} \end{aligned}$$

Elastic Collisions, Special Cases:

$$v_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)v_{1i} + \left(\frac{2m_2}{m_1 + m_2}\right)v_{2i}$$

$$v_{2f} = \left(\frac{2m_1}{m_1 + m_2}\right)v_{1i} - \left(\frac{m_1 - m_2}{m_1 + m_2}\right)v_{2i}$$

S.C. 1: $m_1 = m_2;$ $\quad v_{1f} = v_{2i}$, $\quad v_{2f} = v_{1i}$ Exchange velocities.

S.C. 2: $v_{2i} = 0;$

$$v_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)v_{1i}$$

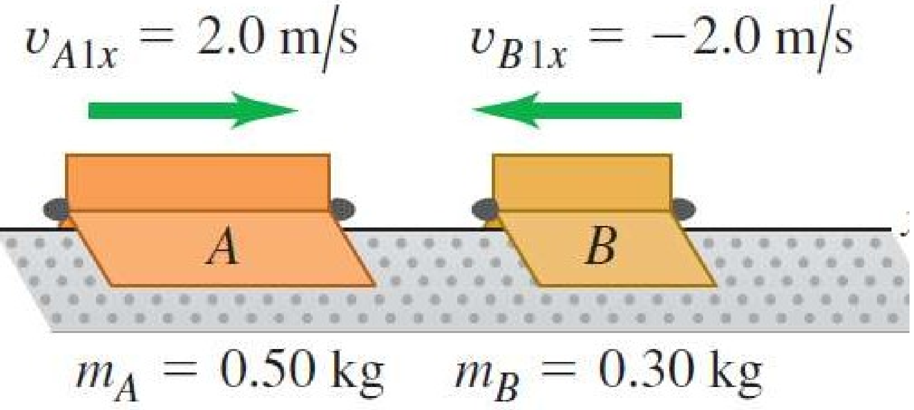
$$v_{2f} = \left(\frac{2m_1}{m_1 + m_2}\right)v_{1i}$$

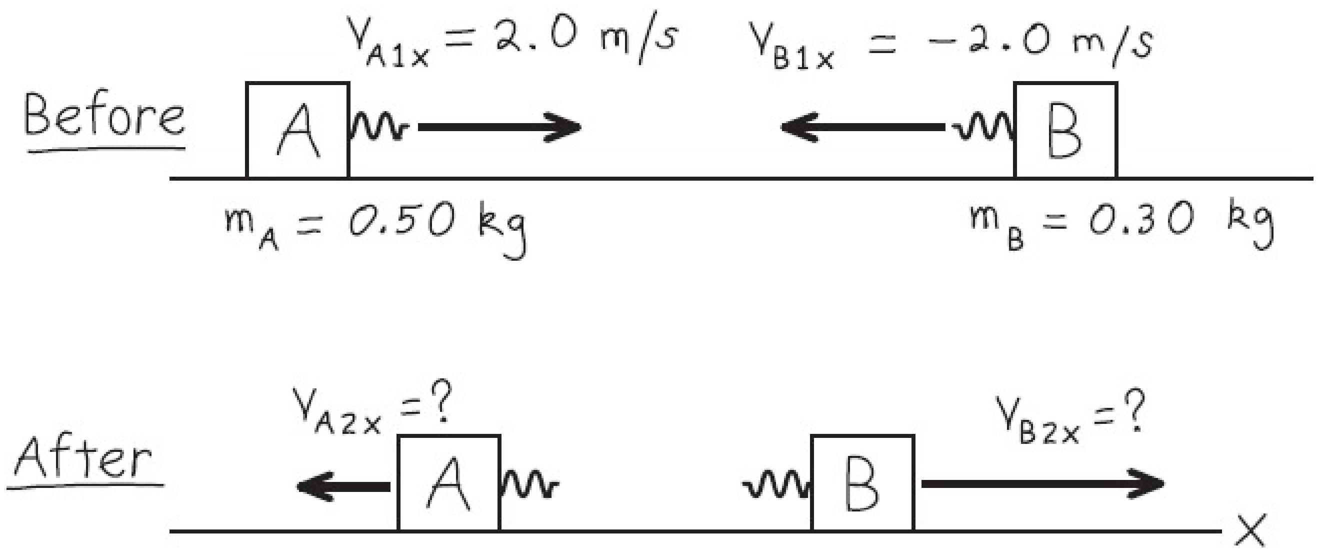
If $m_1 >> m_2$, $\quad v_{1f} \approx v_{1i}$, $\quad v_{2f} \approx 2v_{1i}$

If $m_2 >> m_1$, $\quad v_{1f} \approx -v_{1i}$, $\quad v_{2f} \approx 0$

Example : Elastic straight-line collision

We repeat the air-track collision of Example 8.5 (Section 8.2), but now we add ideal spring bumpers to the gliders so that the collision is elastic. What are the final velocities of the gliders?





Example : Elastic straight-line collision

We repeat the air-track collision of Example 8.5 (Section 8.2), but now we add ideal spring bumpers to the gliders so that the collision is elastic. What are the final velocities of the gliders?

From conservation of momentum,

$$m_A v_{A1x} + m_B v_{B1x} = m_A v_{A2x} + m_B v_{B2x}$$

Plug in numbers:

$$(0.50\ \text{kg})(2.0\ \text{m/s}) + (0.30\ \text{kg})(-2.0\ \text{m/s}) = (0.50\ \text{kg})v_{A2x} + (0.30\ \text{kg})v_{B2x}$$

Simplify to:

$$0.50 v_{A2x} + 0.30 v_{B2x} = 0.40\ \text{m/s}$$

Example : Elastic straight-line collision

We repeat the air-track collision of Example 8.5 (Section 8.2), but now we add ideal spring bumpers to the gliders so that the collision is elastic. What are the final velocities of the gliders?

$$0.50v_{A2x} + 0.30v_{B2x} = 0.40 \text{ m/s}$$

$$\begin{aligned} v_{B2x} - v_{A2x} &= -(v_{B1x} - v_{A1x}) \\ &= -(-2.0 \text{ m/s} - 2.0 \text{ m/s}) = 4.0 \text{ m/s} \end{aligned}$$

Solving these equations simultaneously, we find

$$v_{A2x} = -1.0 \text{ m/s} \qquad v_{B2x} = 3.0 \text{ m/s}$$

Example : Elastic straight-line collision

The total kinetic energy before the collision (which we calculated in Example 8.7) is 1.6 J. The total kinetic energy after the collision is

$$\tfrac{1}{2}(0.50\ \text{kg})(-1.0\ \text{m/s})^2 + \tfrac{1}{2}(0.30\ \text{kg})(3.0\ \text{m/s})^2 = 1.6\ \text{J}$$

***Quick Quiz* 9.5** In a perfectly inelastic one-dimensional collision between two moving objects, what condition alone is necessary so that the final kinetic energy of the system is zero after the collision? **(a)** The objects must have initial momenta with the same magnitude but opposite directions. **(b)** The objects must have the same mass. **(c)** The objects must have the same initial velocity. **(d)** The objects must have the same initial speed, with velocity vectors in opposite directions.

***Quick Quiz* 9.6** A table-tennis ball is thrown at a stationary bowling ball. The table-tennis ball makes a one-dimensional elastic collision and bounces back along the same line. Compared with the bowling ball after the collision, does the table-tennis ball have **(a)** a larger magnitude of momentum and more kinetic energy, **(b)** a smaller magnitude of momentum and more kinetic energy, **(c)** a larger magnitude of momentum and less kinetic energy, **(d)** a smaller magnitude of momentum and less kinetic energy, or **(e)** the same magnitude of momentum and the same kinetic energy?

Example 9.6 The Ballistic Pendulum

The ballistic pendulum (Fig. 9.9, page 248) is an apparatus used to measure the speed of a fast-moving projectile such as a bullet. A projectile of mass m_1 is fired into a large block of wood of mass m_2 suspended from some light wires. The projectile embeds in the block, and the entire system swings through a height h. How can we determine the speed of the projectile from a measurement of h?

Conservation of momentum: (1) $v_B = \dfrac{m_1 v_{1A}}{m_1 + m_2}$

KE after collision: (2) $K_B = \frac{1}{2}(m_1 + m_2)v_B^{\,2}$

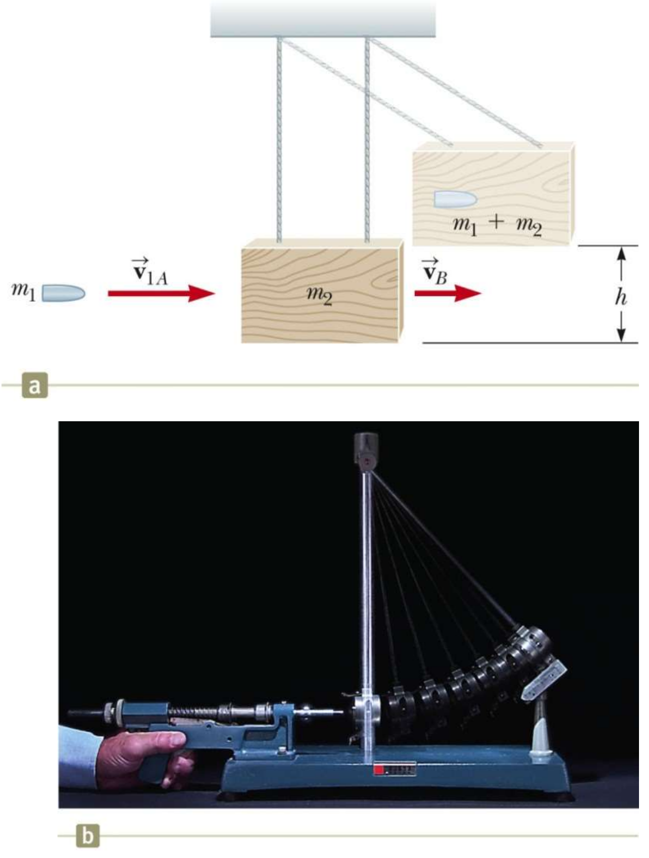
$$K_B = \frac{m_1^{\,2} v_{1A}^{\,2}}{2(m_1 + m_2)}$$

Conservation of energy in stage of BC:

$$K_B + U_B = K_C + U_C$$

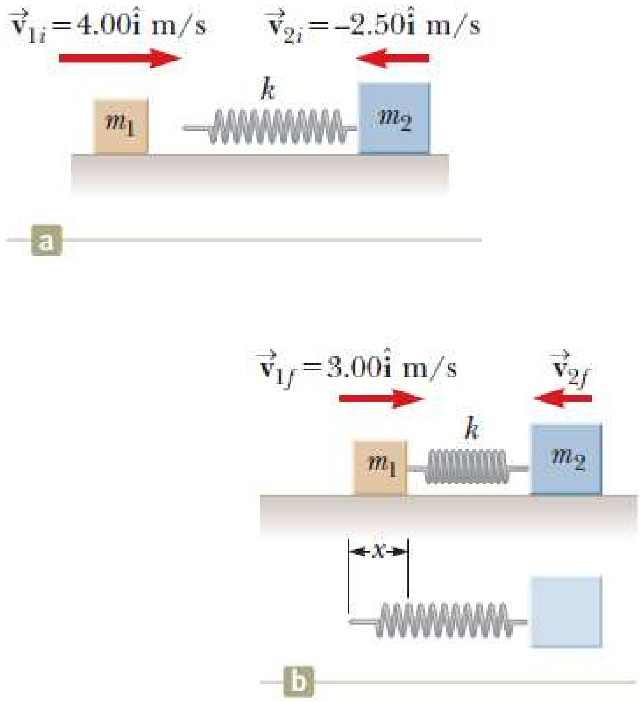
$$\frac{m_1^{\,2} v_{1A}^{\,2}}{2(m_1 + m_2)} + 0 = 0 + (m_1 + m_2)gh$$

$$v_{1A} = \left(\frac{m_1 + m_2}{m_1}\right)\sqrt{2gh}$$



Example 9.7 A Two-Body Collision with a Spring

A block of mass $m_1 = 1.60$ kg initially moving to the right with a speed of 4.00 m/s on a frictionless, horizontal track collides with a light spring attached to a second block of mass $m_2 = 2.10$ kg initially moving to the left with a speed of 2.50 m/s as shown in Figure 9.10a. The spring constant is 600 N/m.



(A) Find the velocities of the two blocks after the collision.

$$(1)\quad m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$

$$(2)\quad v_{1i} - v_{2i} = -(v_{1f} - v_{2f})$$

$$(3)\quad m_1 v_{1i} - m_1 v_{2i} = -m_1 v_{1f} + m_1 v_{2f}$$

$$2m_1 v_{1i} + (m_2 - m_1) v_{2i} = (m_1 + m_2) v_{2f}$$

$$v_{2f} = \frac{2m_1 v_{1i} + (m_2 - m_1) v_{2i}}{m_1 + m_2}$$

$$v_{2f} = \frac{2(1.60\text{ kg})(4.00\text{ m/s}) + (2.10\text{ kg} - 1.60\text{ kg})(-2.50\text{ m/s})}{1.60\text{ kg} + 2.10\text{ kg}} = \boxed{3.12\text{ m/s}}$$

$$v_{1f} = v_{2f} - v_{1i} + v_{2i} = 3.12\text{ m/s} - 4.00\text{ m/s} + (-2.50\text{ m/s}) = \boxed{-3.38\text{ m/s}}$$

(B) Determine the velocity of block 2 during the collision, at the instant block 1 is moving to the right with a velocity of +3.00 m/s as in Figure 9.10b.

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$

$$v_{2f} = \frac{m_1 v_{1i} + m_2 v_{2i} - m_1 v_{1f}}{m_2}$$

$$v_{2f} = \frac{(1.60\ \text{kg})(4.00\ \text{m/s}) + (2.10\ \text{kg})(-2.50\ \text{m/s}) - (1.60\ \text{kg})(3.00\ \text{m/s})}{2.10\ \text{kg}} = \boxed{-1.74\ \text{m/s}}$$

(C) Determine the distance the spring is compressed at that instant.

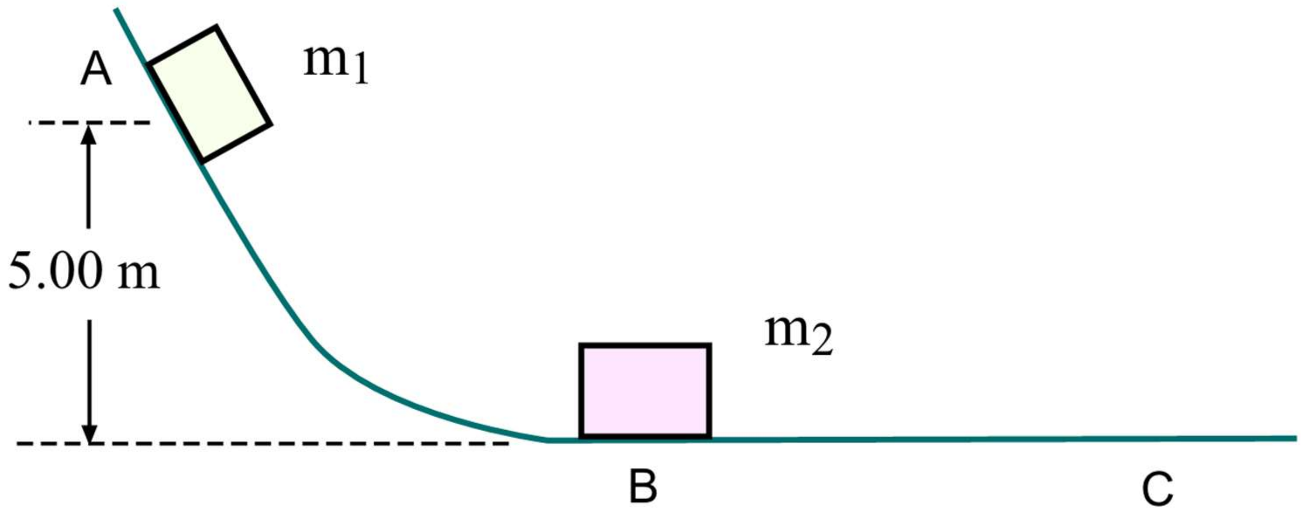
$$K_i + U_i = K_f + U_f$$

$$\tfrac{1}{2}m_1 {v_{1i}}^2 + \tfrac{1}{2}m_2 {v_{2i}}^2 + 0 = \tfrac{1}{2}m_1 {v_{1f}}^2 + \tfrac{1}{2}m_2 {v_{2f}}^2 + \tfrac{1}{2}kx^2$$

$$\tfrac{1}{2}(1.60\ \text{kg})(4.00\ \text{m/s})^2 + \tfrac{1}{2}(2.10\ \text{kg})(2.50\ \text{m/s})^2 + 0$$
$$= \tfrac{1}{2}(1.60\ \text{kg})(3.00\ \text{m/s})^2 + \tfrac{1}{2}(2.10\ \text{kg})(1.74\ \text{m/s})^2 + \tfrac{1}{2}(600\ \text{N/m})x^2$$

$$x = \boxed{0.173\ \text{m}}$$

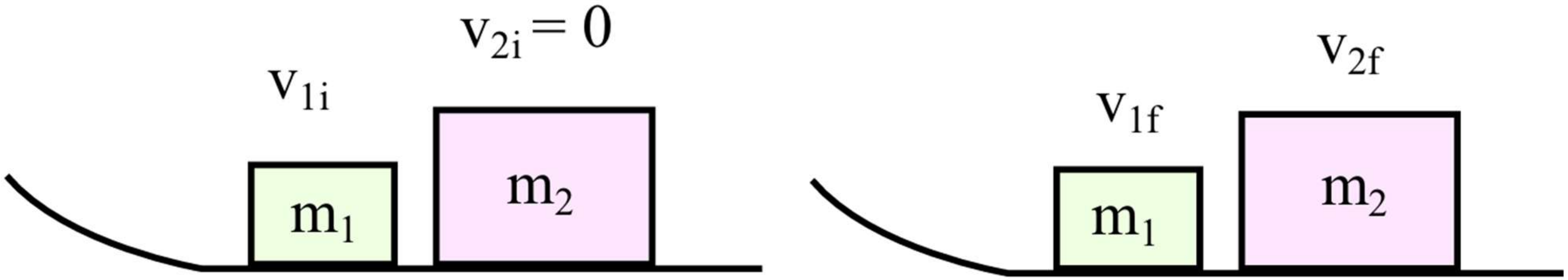
Consider a frictionless track ABC as shown in Fig. 7.7. A block of mass $m_1 = 5.001\,\mathrm{kg}$ is released from A. It makes a head–on elastic collision with a block of mass $m_2 = 10.0\,\mathrm{kg}$ at B, initially at rest. Calculate the maximum height to which m_1 rises after the collision.



$$m_1 gh = \tfrac{1}{2} m v_{1i}^2 \qquad \Longrightarrow \qquad v_{1i}^2 = 2gh = 2(9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2})(5.00\,\mathrm{m}) = 98.0\,\tfrac{\mathrm{m}^2}{\mathrm{s}^2}$$

so that

$$v_{1i} = +9.90\,\tfrac{\mathrm{m}}{\mathrm{s}}$$



$$v_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) v_{1i} = \left(\frac{5.001\,\mathrm{kg} - 10.0\,\mathrm{kg}}{5.001\,\mathrm{kg} + 10.0\,\mathrm{kg}}\right)(+9.90\,\tfrac{\mathrm{m}}{\mathrm{s}}) = -3.30\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

$$v_{2f} = \left(\frac{2m_1}{m_1 + m_2}\right) v_{1i} = \left(\frac{2(5.001\,\mathrm{kg})}{5.001\,\mathrm{kg} + 10.0\,\mathrm{kg}}\right)(+9.90\,\tfrac{\mathrm{m}}{\mathrm{s}}) = +6.60\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

So after the collision, m_1 has a *velocity* of $-3.30\,\frac{\mathrm{m}}{\mathrm{s}}$; that is, it has *speed* $3.30\,\frac{\mathrm{m}}{\mathrm{s}}$ and it is now moving *to the left*. After the collision, m_2 has velocity $+6.60\,\frac{\mathrm{m}}{\mathrm{s}}$, so that it is moving to the right with speed $6.60\,\frac{\mathrm{m}}{\mathrm{s}}$.

Since m_1 is now moving to the left, it will head back up the slope. (See Fig. 7.9.) How high will it go? Once again, we can use energy conservation to give us the answer. For the trip back up the slope, the initial energy (all kinetic) is

$$E_i = K_i = \tfrac{1}{2}m(3.30\,\tfrac{\mathrm{m}}{\mathrm{s}})^2$$

and when it reaches maximum height (h) its speed is zero, so its energy is the potential energy,

$$E_f = U_f = mgh$$

Conservation of energy, $E_i = E_f$ gives us:

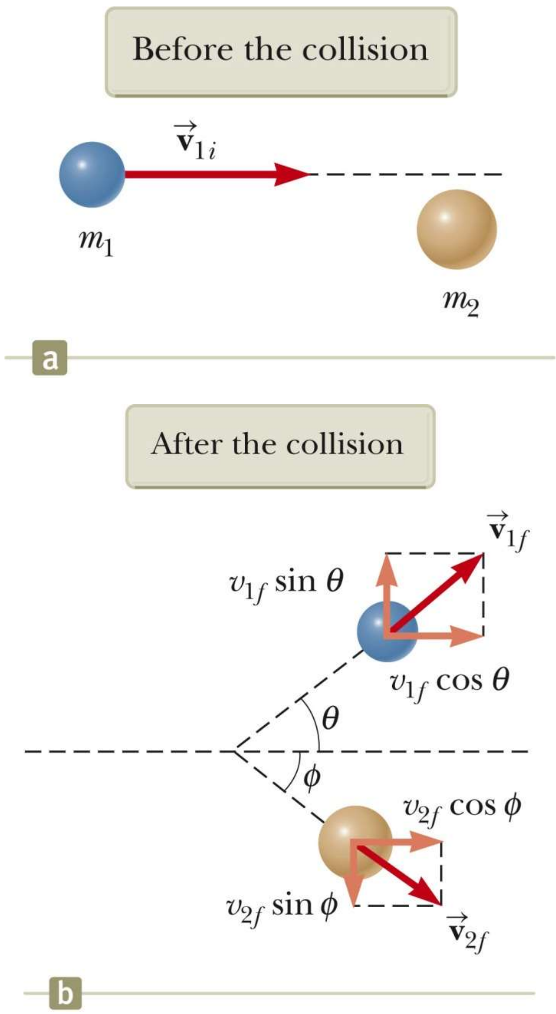
$$\tfrac{1}{2}m(3.30\,\tfrac{\mathrm{m}}{\mathrm{s}})^2 = mgh \qquad \Longrightarrow \qquad h = \frac{(3.30\,\frac{\mathrm{m}}{\mathrm{s}})^2}{2g} = 0.556\,\mathrm{m}$$

Mass m_1 will travel back up the slope to a height of $0.556\,\mathrm{m}$.

Two-Dimensional Collision

$$m_1 v_{1ix} + m_2 v_{2ix} = m_1 v_{1fx} + m_2 v_{2fx}$$

$$m_1 v_{1iy} + m_2 v_{2iy} = m_1 v_{1fy} + m_2 v_{2fy}$$



Example 9.8 Collision at an Intersection

A 1 500-kg car traveling east with a speed of 25.0 m/s collides at an intersection with a 2 500-kg truck traveling north at a speed of 20.0 m/s as shown in Figure 9.12. Find the direction and magnitude of the velocity of the wreckage after the collision, assuming the vehicles stick together after the collision.

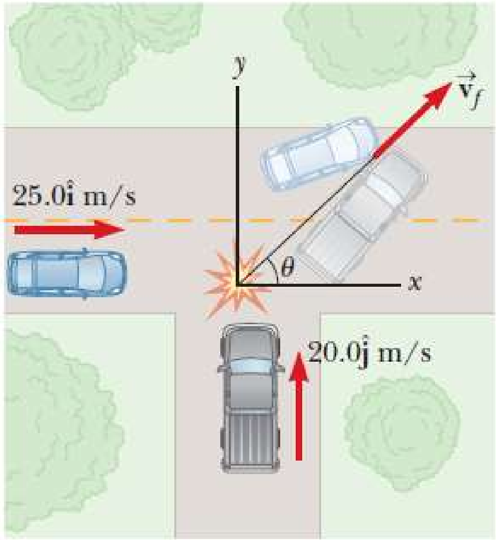
$$\sum p_{xi} = \sum p_{xf} \quad \rightarrow \quad (1)\ m_1 v_{1i} = (m_1 + m_2) v_f \cos\theta$$

$$\sum p_{yi} = \sum p_{yf} \quad \rightarrow \quad (2)\ m_2 v_{2i} = (m_1 + m_2) v_f \sin\theta$$

$$\frac{m_2 v_{2i}}{m_1 v_{1i}} = \frac{\sin\theta}{\cos\theta} = \tan\theta$$

$$\theta = \tan^{-1}\left(\frac{m_2 v_{2i}}{m_1 v_{1i}}\right) = \tan^{-1}\left[\frac{(2\,500\text{ kg})(20.0\text{ m/s})}{(1\,500\text{ kg})(25.0\text{ m/s})}\right] = \boxed{53.1^\circ}$$

$$v_f = \frac{m_2 v_{2i}}{(m_1 + m_2)\sin\theta} = \frac{(2\,500\text{ kg})(20.0\text{ m/s})}{(1\,500\text{ kg} + 2\,500\text{ kg})\sin 53.1^\circ} = \boxed{15.6\text{ m/s}}$$



The Center of Mass

•There is a special point in a system or object, called the ***center of mass***, that moves as if all of the mass of the system is concentrated at that point.

•The system will move as if an external force were applied to a single particle of mass *M* located at the center of mass.

- *M* is the total mass of the system.

This behavior is independent of other motion, such as rotation or vibration, or deformation of the system.

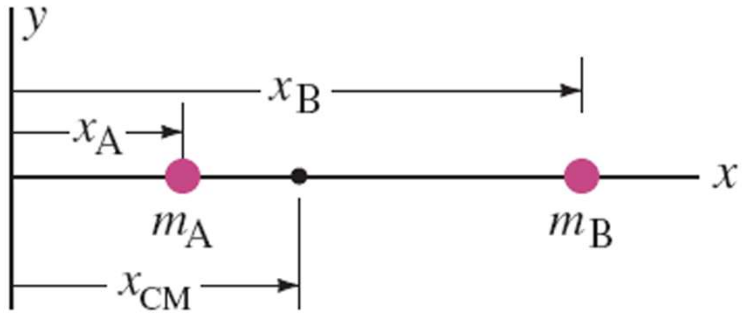
- This is the particle model.

http://blogs.exploratorium.edu/passionately-curious/about/

For two particles, the center of mass lies closer to the one with the most mass:

$$x_{\mathrm{CM}} = \frac{m_{\mathrm{A}} x_{\mathrm{A}} + m_{\mathrm{B}} x_{\mathrm{B}}}{m_{\mathrm{A}} + m_{\mathrm{B}}} = \frac{m_{\mathrm{A}} x_{\mathrm{A}} + m_{\mathrm{B}} x_{\mathrm{B}}}{M},$$

where *M* is the total mass.



For example, if x_A=1m, x_B=2m, m_A=10kg, m_B=10kg; then x_{CM}=1.5m.

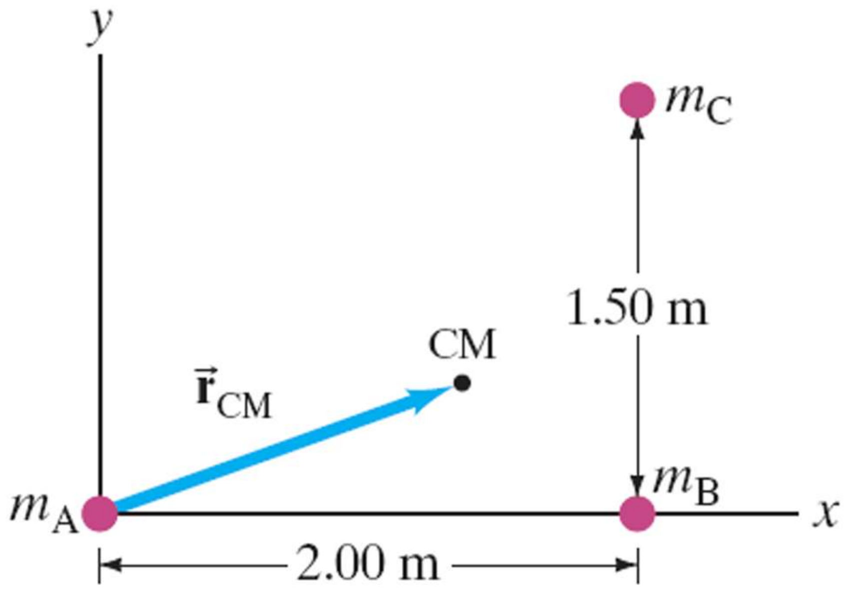
For example, if x_A=1m, x_B=2m, m_A=10kg, m_B=40kg; then x_{CM}=1.8m.

For example, if x_A=1m, x_B=2m, m_A=40kg, m_B=10kg; then x_{CM}=1.2m.

In general, $$x_{CM} = \frac{m_A x_A + m_B x_B}{m_A + m_B}, \quad y_{CM} = \frac{m_A y_A + m_B y_B}{m_A + m_B},$$

Exercise: Three particles in 2-D.

Three particles, each of mass 2.50 kg, are located at the corners of a right triangle whose sides are 2.00 m and 1.50 m long, as shown. Locate the center of mass.



[Solution]

$$x_{CM} = \frac{m_A x_A + m_B x_B + m_C x_C}{m_A + m_B + m_C}$$

$$= \frac{1}{3}\left(x_A + x_B + x_C\right) \quad (\because m_A = m_B = m_C)$$

$$= \frac{1}{3}\left(0 + 2.0 + 2.0\right)$$

$$= 1.33\,(m)$$

$$y_{CM} = \frac{m_A y_A + m_B y_B + m_C y_C}{m_A + m_B + m_C}$$

$$= \frac{1}{3}\left(y_A + y_B + y_C\right)$$

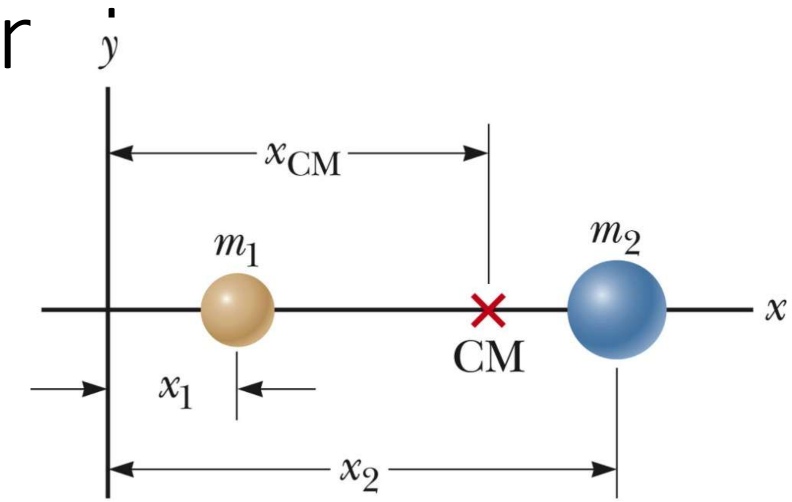
$$= \frac{1}{3}\left(0 + 0 + 1.5\right)$$

$$= 0.50\,(m)$$

Center of Mass, Coordir

•The coordinates of the center of mass are

$$x_{CM} = \frac{\sum_i m_i x_i}{M}; y_{CM} = \frac{\sum_i m_i y_i}{M}; z_{CM} = \frac{\sum_i m_i z_i}{M}$$



- M is the total mass of the system.

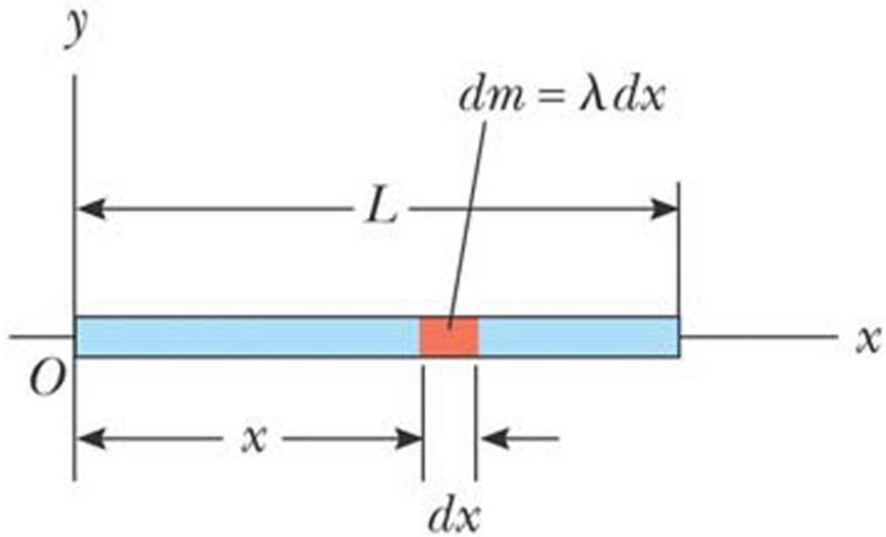
$$\vec{\mathbf{r}}_{CM} = x_{CM}\hat{\mathbf{i}} + y_{CM}\hat{\mathbf{j}} + z_{CM}\hat{\mathbf{k}} = \frac{1}{M}\sum_i m_i x_i \hat{\mathbf{i}} + \frac{1}{M}\sum_i m_i y_i \hat{\mathbf{j}} + \frac{1}{M}\sum_i m_i z_i \hat{\mathbf{k}}$$

For continuous mass distributions,

$$x_{CM} = \lim_{\Delta m_i \to 0} \frac{1}{M}\sum_i x_i \Delta m_i = \frac{1}{M}\int x\,dm \qquad y_{CM} = \frac{1}{M}\int y\,dm \quad \text{and} \quad z_{CM} = \frac{1}{M}\int z\,dm$$

$$\boxed{\vec{\mathbf{r}}_{CM} = \frac{1}{M}\int \vec{\mathbf{r}}\,dm}$$

Example. Find the **center of mass** of a rod of mass *M* and length *L*.



Answer:

Let λ (read as "lambda") denote the linear mass density, or the mass per unit length, then $\lambda = \frac{M}{L}$.

If we divide the rod into elements of length *dx*, then

$\lambda = \frac{dm}{dx}$, or $dm = \lambda\, dx$.

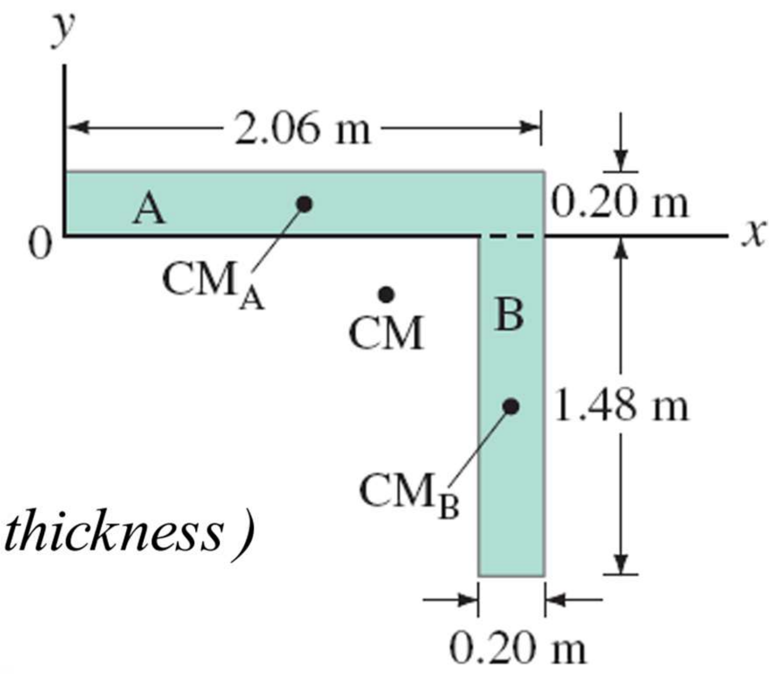
$$X_{cm} = \frac{1}{M}\int_{o}^{L} x\,dm = \frac{1}{M}\int_{o}^{L} x\lambda\,dx$$

$$= \frac{\lambda}{M}\left[\frac{x^2}{2}\right]_{o}^{L} = \frac{\lambda L^2}{2M} = \frac{L}{2}$$

Example: CM of L-shaped flat object.

Determine the CM of the uniform thin L-shaped object shown.

[Solution] The object consists of two rectangular parts: A and B, whose centres of mass are (x_A, y_A) and (x_B, y_B).



$$x_A = 1.03m, \quad y_A = 0.10m$$

$$x_B = 1.96m, \quad y_B = -0.74m$$

$$m_A = \rho t(2.06 \times 0.20)$$

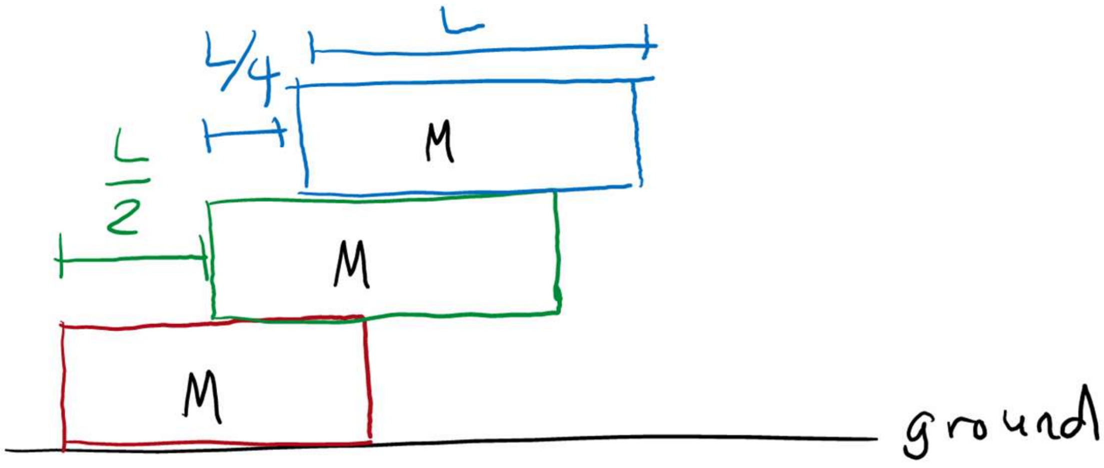
$$= 0.412\rho t \qquad (\rho = density, \quad t = thickness)$$

$$m_B = \rho t(1.48 \times 0.20) = 0.296\rho t$$

$$x_{CM} = \frac{m_A x_A + m_B x_B}{m_A + m_B} = \frac{0.412 \times 1.03 + 0.296 \times 1.96}{0.412 + 0.296} = 1.42\,(m)$$

$$y_{CM} = \frac{m_A y_A + m_B y_B}{m_A + m_B} = \frac{0.412 \times 0.10 + 0.296 \times (-0.74)}{0.412 + 0.296} = -0.25\,(m)$$

Three bricks of length L and mass M are stacked on top of each other as shown in the figure below. Find the x-coordinate of the center of mass of the bricks.



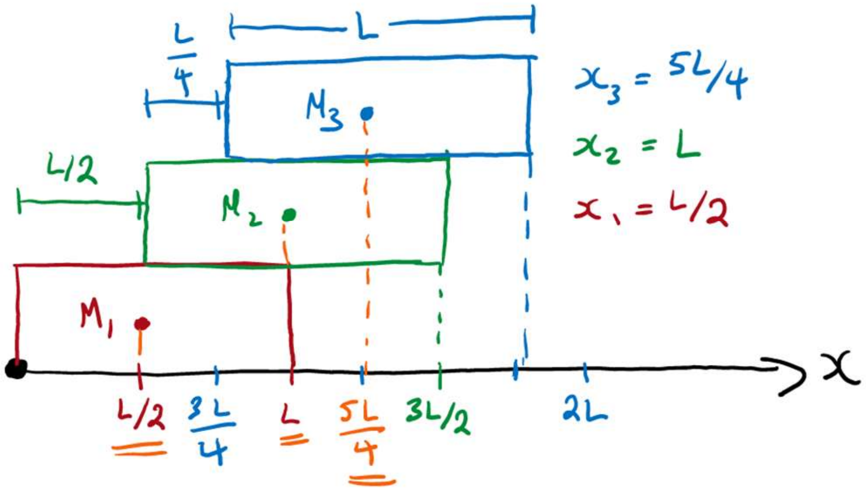
Step #1: Define Origin and Use CM Equation

The easiest place to put the origin is at the left edge of the bottom brick.

We also know that the center of mass of each brick is at its center.

Let's label these and get their values along the x-axis.

Now, we can put them all together into the X_{CM} equation.



$$X_{CM} = \sum_i \frac{m_i x_i}{m_{tot}} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3} = \frac{M\left(\frac{L}{2}\right) + M(L) + M\left(\frac{5L}{4}\right)}{M + M + M} = \frac{\frac{2ML}{4} + \frac{4ML}{4} + \frac{5ML}{4}}{3M} = \frac{\frac{11ML}{4}}{3M} =$$

$$X_{CM} = \frac{11L}{12}$$

Motion of a System of Particles

- Assume the total mass, M, of the system remains constant.
- We can describe the motion of the system in terms of the velocity and acceleration of the center of mass of the system.

$$\vec{\mathbf{v}}_{CM} = \frac{d\vec{\mathbf{r}}_{CM}}{dt} = \frac{1}{M}\sum_i m_i \frac{d\vec{\mathbf{r}}_i}{dt} = \frac{1}{M}\sum_i m_i\vec{\mathbf{v}}_i \qquad \vec{\mathbf{a}}_{CM} = \frac{d\vec{\mathbf{v}}_{CM}}{dt} = \frac{1}{M}\sum_i m_i \frac{d\vec{\mathbf{v}}_i}{dt} = \frac{1}{M}\sum_i m_i\vec{\mathbf{a}}_i$$

- We can also describe the momentum of the system and Newton's Second Law for the system.

$$M\vec{\mathbf{v}}_{CM} = \sum_i m_i\vec{\mathbf{v}}_i = \sum_i \vec{\mathbf{p}}_i = \vec{\mathbf{p}}_{tot} \qquad M\vec{\mathbf{a}}_{CM} = \sum_i m_i\vec{\mathbf{a}}_i = \sum_i \vec{\mathbf{F}}_i$$

The center of mass of a system of particles having combined mass *M* moves like an equivalent particle of mass *M* would move under the influence of the net external force on the system.

Center of Gravity

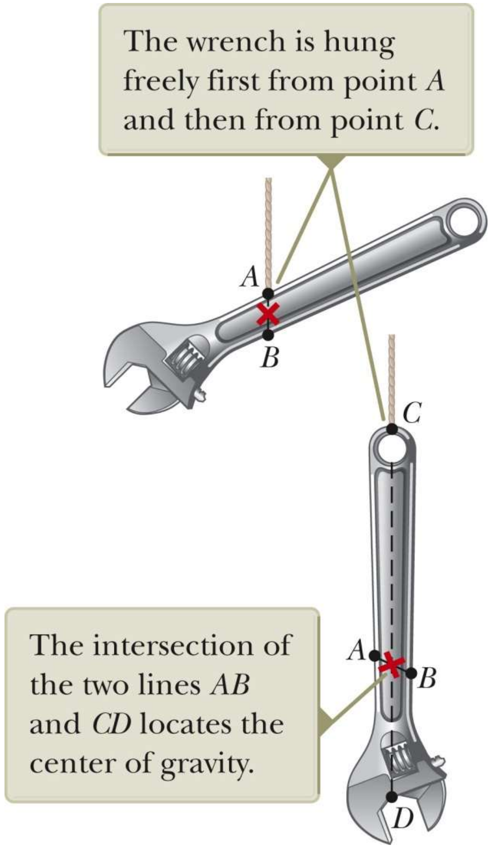
•Each small mass element of an extended object is acted upon by the gravitational force.

•The net effect of all these forces is equivalent to the effect of a single force $M\vec{\mathbf{g}}$ acting through a point called the **center of gravity**

•**I**f $\vec{\mathbf{g}}$ is constant over the mass distribution, the center of gravity coincides with the center of mass.

Finding Center of Gravity, Irregularly Shaped Object

- Suspend the object from one point.
- Then, suspend from another point.
- The intersection of the resulting lines is the center of gravity and half way through the thickness of the wrench.



***Quick Quiz* 9.7** A baseball bat of uniform density is cut at the location of its center of mass as shown in Figure 9.17. Which piece has the smaller mass? **(a)** the piece on the right **(b)** the piece on the left **(c)** both pieces have the same mass **(d)** impossible to determine

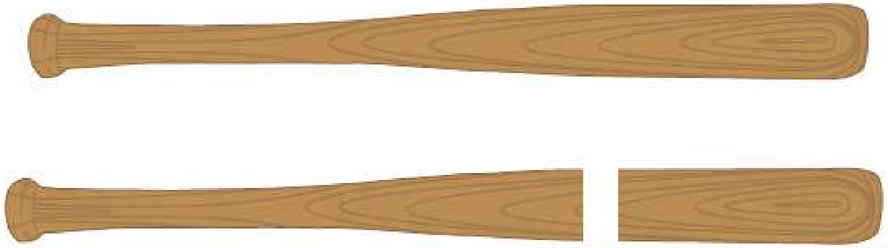


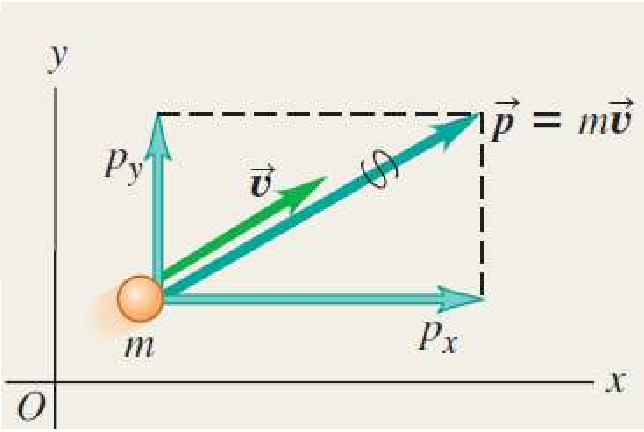
Figure 9.17 (Quick Quiz 9.7) A baseball bat cut at the location of its center of mass.

***Quick Quiz* 9.8** A cruise ship is moving at constant speed through the water. The vacationers on the ship are eager to arrive at their next destination. They decide to try to speed up the cruise ship by gathering at the bow (the front) and running together toward the stern (the back) of the ship. **(i)** While they are running toward the stern, is the speed of the ship (a) higher than it was before, (b) unchanged, (c) lower than it was before, or (d) impossible to determine? **(ii)** The vacationers stop running when they reach the stern of the ship. After they have all stopped running, is the speed of the ship (a) higher than it was before they started running, (b) unchanged from what it was before they started running, (c) lower than it was before they started running, or (d) impossible to determine?

Summary

$$\vec{p} = m\vec{v}$$

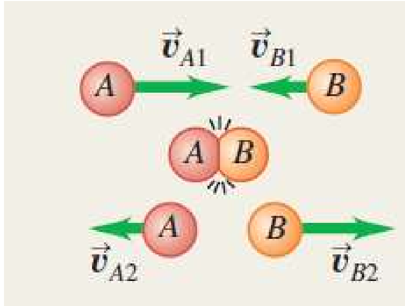
$$\sum \vec{F} = \frac{d\vec{p}}{dt}$$



$$\vec{J} = \sum \vec{F}(t_2 - t_1) = \sum \vec{F}\,\Delta t$$

$$\vec{J} = \int_{t_1}^{t_2} \sum \vec{F}\,dt$$

$$\vec{J} = \vec{p}_2 - \vec{p}_1$$



$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + \cdots}{m_1 + m_2 + m_3 + \cdots}$$

$$= \frac{\sum_i m_i\vec{r}_i}{\sum_i m_i} \qquad (8.29)$$

$$\vec{P} = m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + \cdots$$

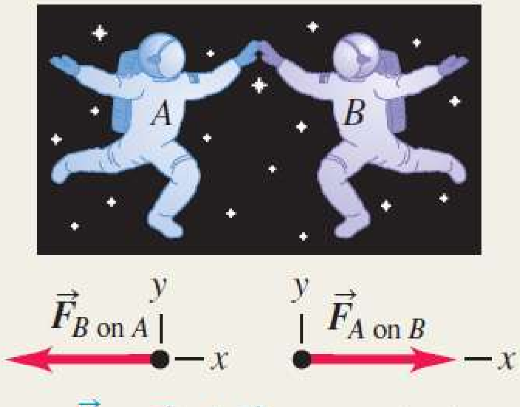
$$= M\vec{v}_{cm} \qquad (8.32)$$

$$\sum \vec{F}_{ext} = M\vec{a}_{cm} \qquad (8.34)$$

$$\vec{P} = \vec{p}_A + \vec{p}_B + \cdots$$

$$= m_A\vec{v}_A + m_B\vec{v}_B + \cdots \qquad (8.14)$$

If $\sum \vec{F} = \mathbf{0}$, then $\vec{P} =$ constant.



$\vec{P} = \vec{p}_A + \vec{p}_B =$ constant