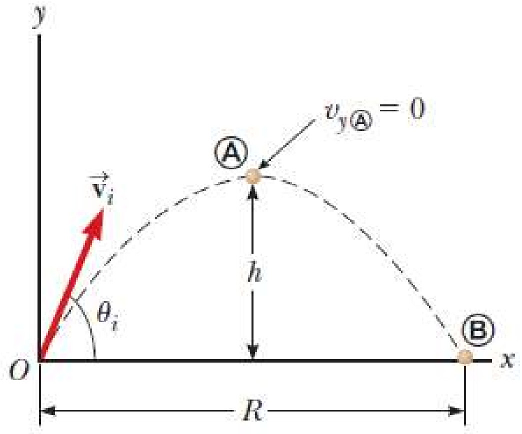
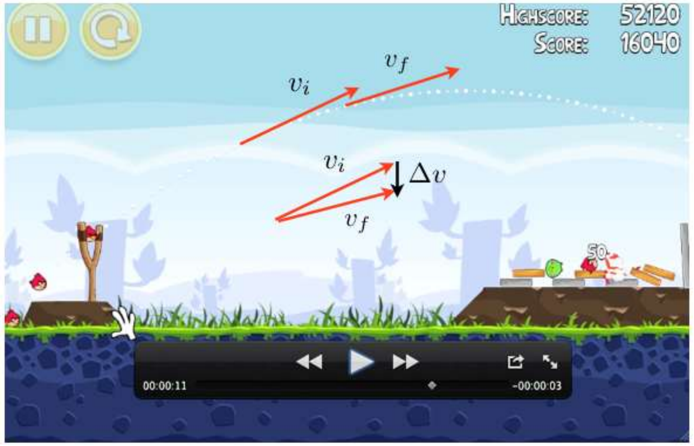
Kinematic Equations for Two-Dimensional Motion

- When the two-dimensional motion has a constant acceleration, a series of equations can be developed that describe the motion.
- These equations will be similar to those of one-dimensional kinematics.
- **Motion in two dimensions can be modeled as two *independent* motions in each of the two perpendicular directions associated with the x and y axes.**
  - Any influence in the y direction does not affect the motion in the x direction and vice versa.

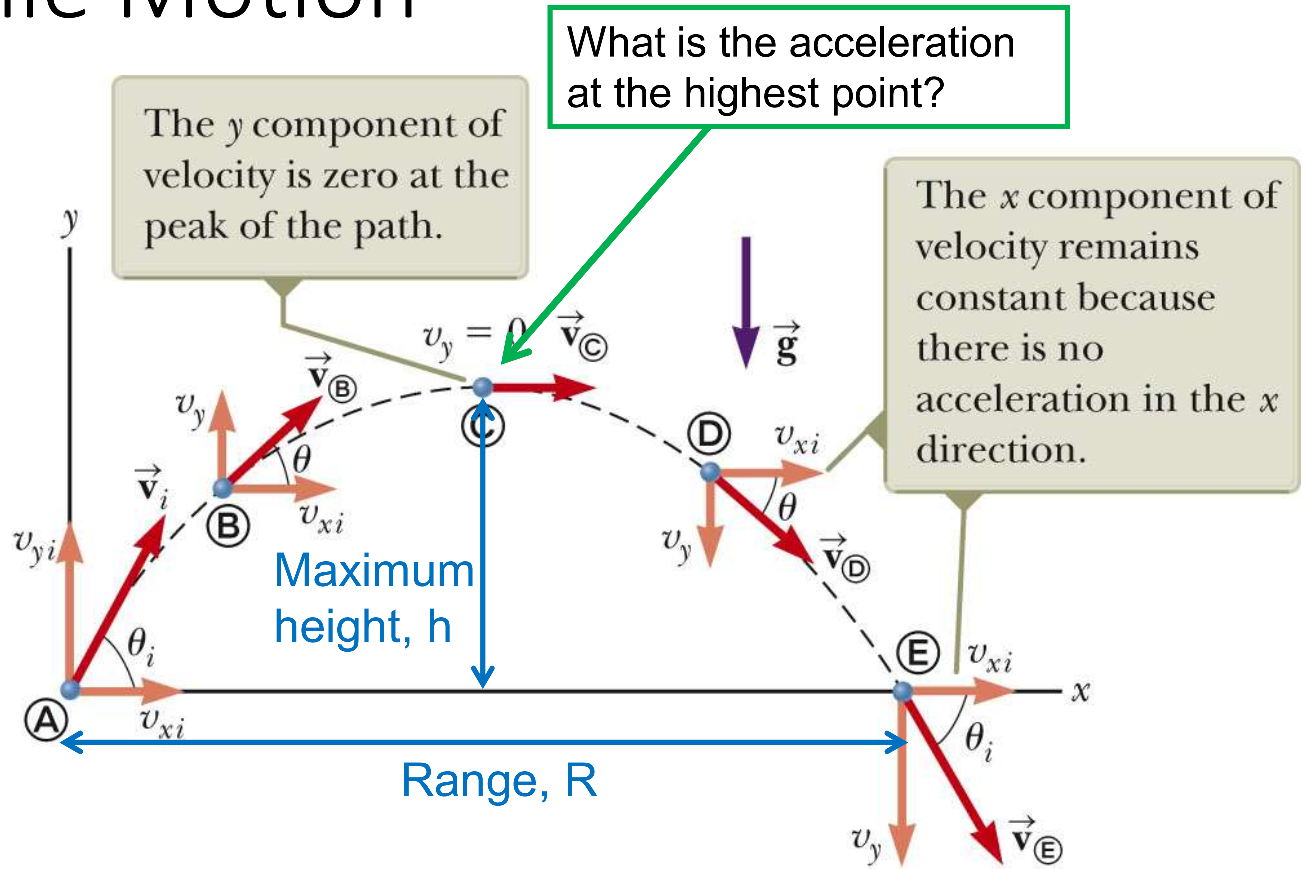
Projectile Motion





Projectile motion is the trajectory obtained when an object is thrown with some initial velocity at some angle and then moves under the influence of gravity alone.

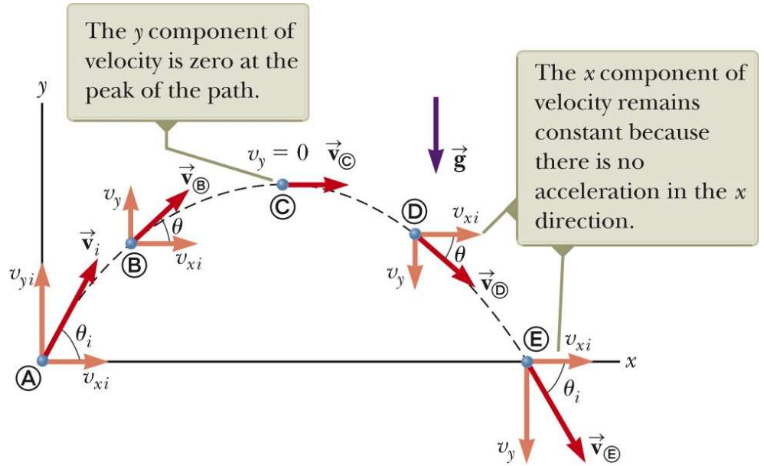
Projectile Motion



We are most interested in calculating h & R.

Projectile Motion Diagram

As mentioned before, motion in two dimensions can be modeled as two *independent* motions in each of the two perpendicular directions associated with the x and y axes, but the total times for the two motions are the same.



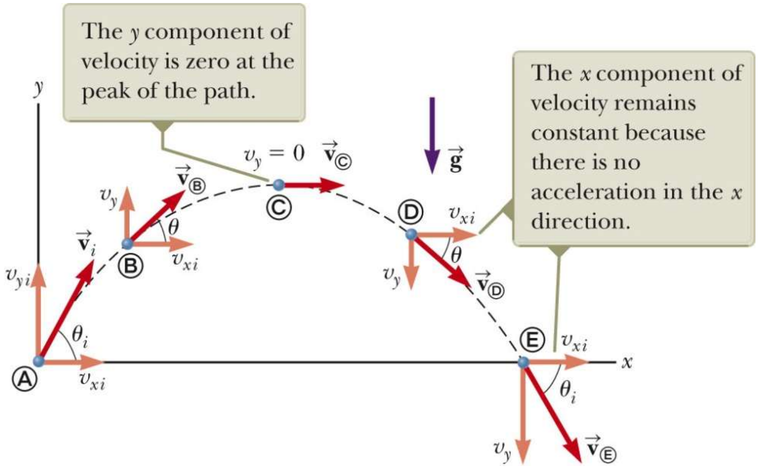
Consider the motion in y-direction from point A to point C:

$$v_{yi} = v_i \sin\theta_i \qquad v_{yf} = 0 \qquad a_y = -g \qquad s_y = h$$

$$v_{yf}^2 - v_{yi}^2 = 2a_y s_y \Rightarrow s_y = \frac{v_{yf}^2 - v_{yi}^2}{2a_y} \Rightarrow h = \frac{0 - (v_i \sin\theta_i)^2}{-2g} = \frac{v_i^2 \sin^2\theta_i}{2g}$$

$$t_C = \frac{v_{yf} - v_{yi}}{a_y} = \frac{0 - v_i \sin\theta_i}{-g} = \frac{v_i \sin\theta_i}{g}$$

Projectile Motion Diagram



Consider the motion in x-direction from point a to point E:

$$v_{xi} = v_i \cos\theta_i \qquad a_x = 0 \qquad s_x = R$$

$$t_E = 2t_C = \frac{2v_i \sin\theta_i}{g}$$

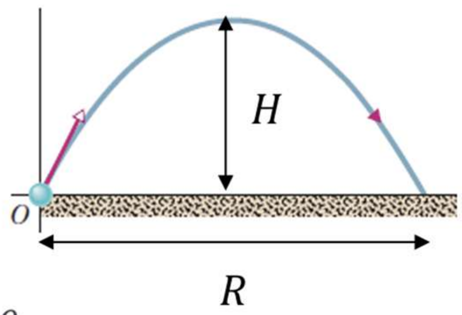
$$s_x = v_{xi}t + \frac{1}{2}a_x t^2 \Rightarrow$$

$$R = v_i \cos\theta_i \frac{2v_i \sin\theta_i}{g} = \frac{v_i^2 2\sin\theta_i \cos\theta_i}{g} = \frac{v_i^2 \sin 2\theta_i}{g}$$

Projectile Motion

- Equation of the path (trajectory (軌跡))
  - $\begin{cases} x = u\cos\theta\, t \\ y = u\sin\theta\, t - \frac{1}{2}gt^2 \end{cases} \xrightarrow{eliminating\ t} y = x\tan\theta - \frac{gx^2}{2u^2\cos^2\theta}$

- Maximum height, *H*
  - $v_y = 0 \quad \rightarrow \quad u\sin\theta - gt = 0 \quad \rightarrow \quad t = \frac{u\sin\theta}{g}$
  - $H = u\sin\theta\left(\frac{u\sin\theta}{g}\right) - \frac{1}{2}g\left(\frac{u\sin\theta}{g}\right)^2 = \frac{u^2\sin^2\theta}{2g}$



- Horizontal range, *R*
  - $\begin{cases} R = u\cos\theta\, t \\ 0 = u\sin\theta\, t - \frac{1}{2}gt^2 \end{cases} \xrightarrow{eliminating\ t} R = \frac{u^2\sin 2\theta}{g}$

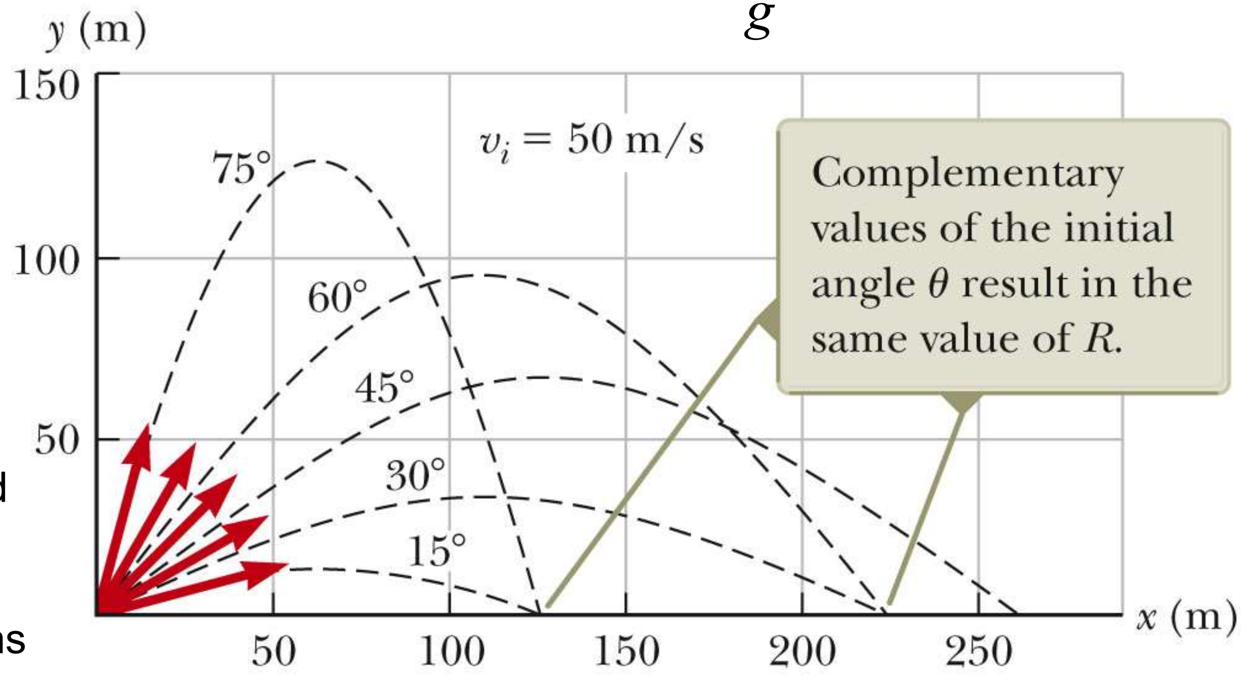
More About the Range of a Projectile

$$h = \frac{v_i^2 \sin^2 \theta_i}{2g}$$

$$R = \frac{v_i^2 \sin 2\theta_i}{g}$$

These two equations are valid only for symmetric motion. Use kinematic equations for other cases.

$$R_{\max} = \frac{v_i^2}{g} \; when \; \theta_i = 45^\circ$$



Example 4.4 That's Quite an Arm!

A stone is thrown from the top of a building upward at an angle of 30.0° to the horizontal

(A) How long does it take the stone to reach the ground?

$$v_{xi} = v_i \cos\theta_i = (20.0 \text{ m/s}) \cos 30.0° = 17.3 \text{ m/s}$$

$$v_{yi} = v_i \sin\theta_i = (20.0 \text{ m/s}) \sin 30.0° = 10.0 \text{ m/s}$$

$$y_f = y_i + v_{yi}t + \tfrac{1}{2}a_y t^2$$

$$-45.0 \text{ m} = 0 + (10.0 \text{ m/s})t + \tfrac{1}{2}(-9.80 \text{ m/s}^2)t^2$$

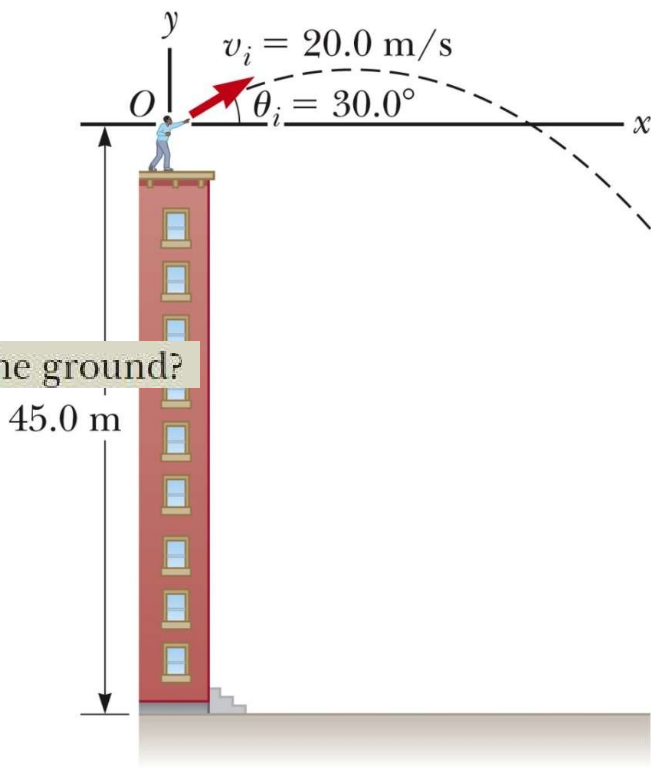
$$t = \boxed{4.22 \text{ s}}$$

(B) What is the speed of the stone just before it strikes the ground?

$$v_{yf} = v_{yi} + a_y t$$

$$v_{yf} = 10.0 \text{ m/s} + (-9.80 \text{ m/s}^2)(4.22 \text{ s}) = -31.3 \text{ m/s}$$

$$v_f = \sqrt{v_{xf}^2 + v_{yf}^2} = \sqrt{(17.3 \text{ m/s})^2 + (-31.3 \text{ m/s})^2} = \boxed{35.8 \text{ m/s}}$$



Example 4.2 The Long Jump

A long jumper (Fig. 4.11) leaves the ground at an angle of 20.0° above the horizontal and at a speed of 11.0 m/s.

(A) How far does he jump in the horizontal direction?

$$R = \frac{v_i^2 \sin 2\theta_i}{g} = \frac{(11.0 \text{ m/s})^2 \sin 2(20.0°)}{9.80 \text{ m/s}^2} = \boxed{7.94 \text{ m}}$$

(B) What is the maximum height reached?

$$h = \frac{v_i^2 \sin^2\theta_i}{2g} = \frac{(11.0 \text{ m/s})^2 (\sin 20.0°)^2}{2(9.80 \text{ m/s}^2)} = \boxed{0.722 \text{ m}}$$

AP Images

Figure 4.11 (Example 4.2) Romain Barras of France competes in the men's decathlon long jump at the 2008 Beijing Olympic Games.

Example 4.5 The End of the Ski Jump

A ski jumper leaves the ski track moving in the horizontal direction with a speed of 25.0 m/s as shown in Figure 4.14. The landing incline below her falls off with a slope of 35.0°. Where does she land on the incline?

(1) $x_f = v_{xi}t$

(2) $y_f = v_{yi}t + \frac{1}{2}a_y t^2 = -\frac{1}{2}gt^2$

(3) $d\cos\phi = v_{xi}t$

(4) $-d\sin\phi = -\frac{1}{2}gt^2$

$$-d\sin\phi = -\frac{1}{2}g\left(\frac{d\cos\phi}{v_{xi}}\right)^2$$

$$d = \frac{2v_{xi}^2\sin\phi}{g\cos^2\phi} = \frac{2(25.0\text{ m/s})^2\sin 35.0°}{(9.80\text{ m/s}^2)\cos^2 35.0°} = 109\text{ m}$$

$$x_f = d\cos\phi = (109\text{ m})\cos 35.0° = 89.3\text{ m}$$

$$y_f = -d\sin\phi = -(109\text{ m})\sin 35.0° = -62.5\text{ m}$$

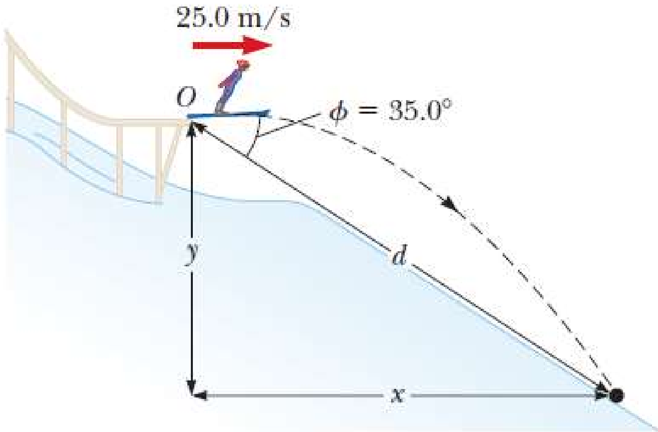
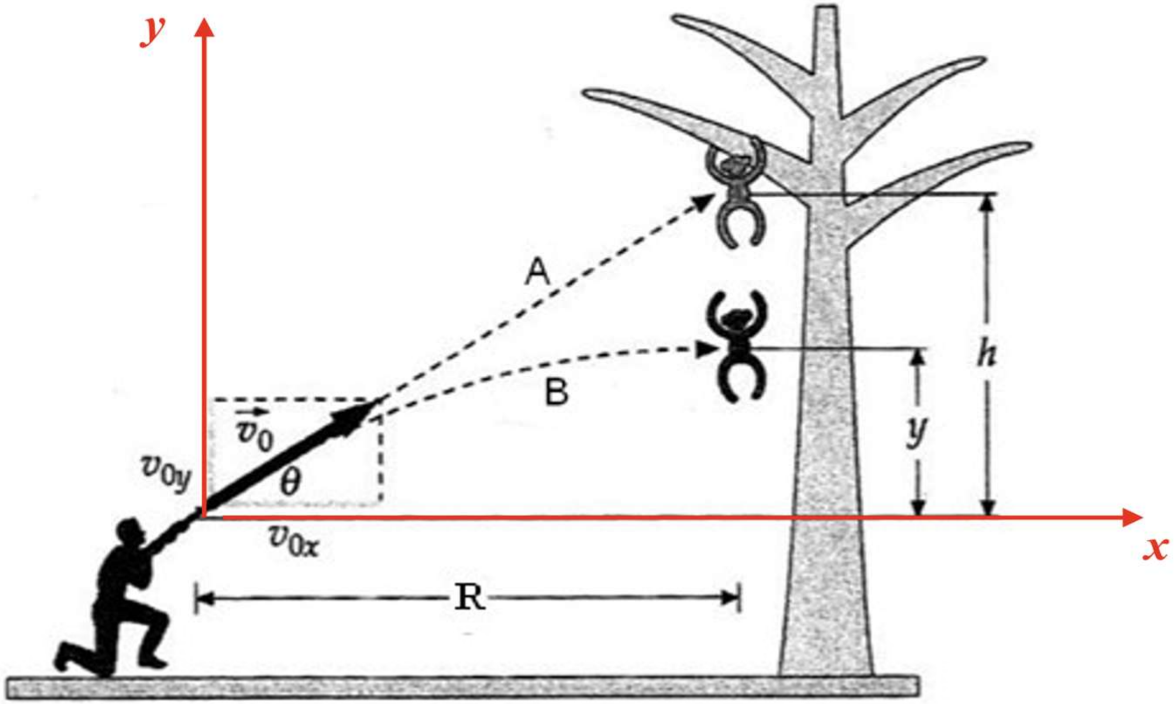


Figure 4.14 (Example 4.5) A ski jumper leaves the track moving in a horizontal direction.

The Monkey and the Hunter Problem



A student fires a dart at a stuff monkey held by an electromagnet a distance **h** vertically above the dart gun and a distance **R** horizontally away from the dart gun. The student aims directly at the monkey and fires, but as the student fires, the power of the electromagnet is turned off causing the monkey to drop simultaneously. Will the dart miss the monkey?

[Shoot the Monkey]

It takes time t the dart to travel a horizontal distance R: $t = \dfrac{R}{v_0 \cos\theta}$

At time t: $y_d = v_0 \sin\theta\, t - \frac{1}{2}gt^2; \quad y_m = h - \frac{1}{2}gt^2$

The difference in y-position at time t:

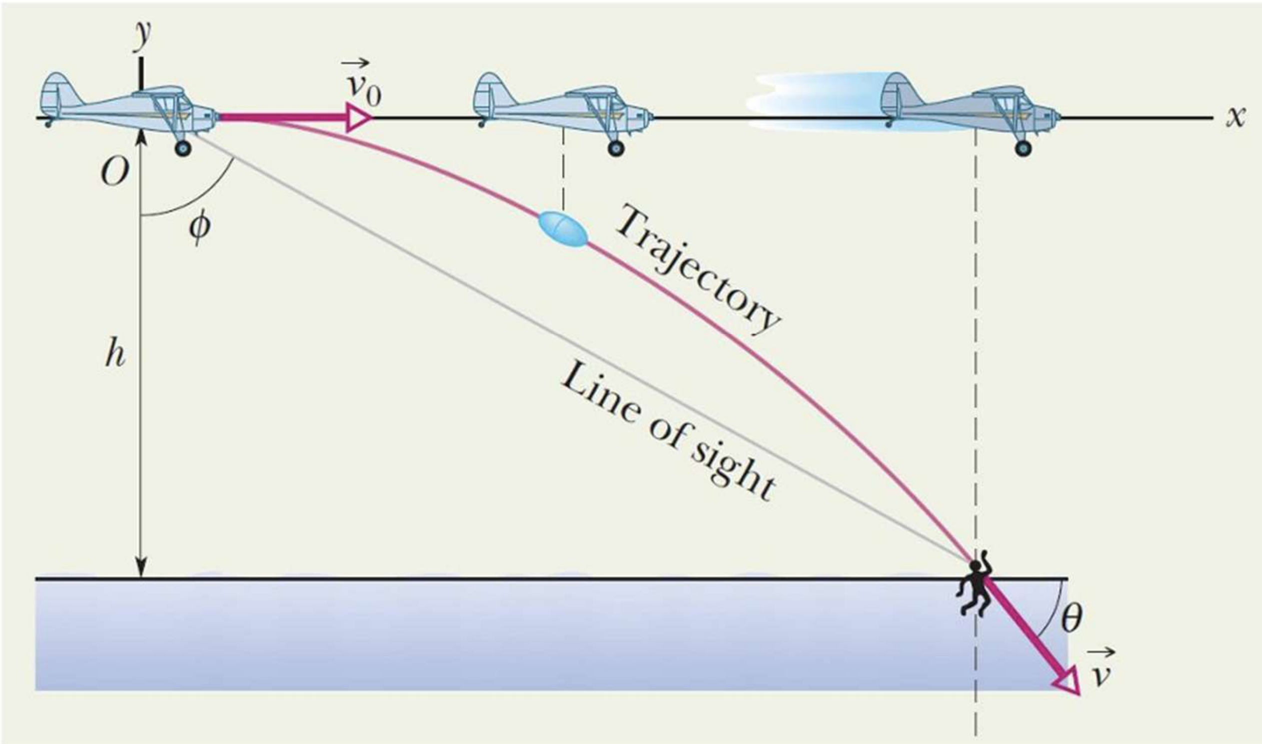
$$
\begin{aligned}
y_d - y_m &= v_0 \sin\theta\, t - \frac{1}{2}gt^2 - \left(h - \frac{1}{2}gt^2\right) \\
&= v_0 \sin\theta\, t - h \\
&= v_0 \sin\theta \frac{R}{v_0 \cos\theta} - h \\
&= R \tan\theta - h \\
&= R\frac{h}{R} - h \\
&= 0
\end{aligned}
$$

Therefore, the dart will hit the monkey.

Sample Problem

- A rescue plane flies at 198 km/h (= 55.0 m/s) and constant height h = 500 m toward a point directly over a victim, where a rescue capsule is to land.

(a) What should be the angle ϕ of the pilot's line of sight to the victim when the capsule release is made?



$$\phi = \tan^{-1}\frac{x}{h}$$

Sample Problem

$$y - y_0 = (v_0 \sin \theta_0)t - \tfrac{1}{2}gt^2.$$

$$-500 \text{ m} = (55.0 \text{ m/s})(\sin 0°)t - \tfrac{1}{2}(9.8 \text{ m/s}^2)t^2$$

$$t = 10.1 \text{ s}.$$

$$x - x_0 = (v_0 \cos \theta_0)t.$$

$$x - 0 = (55.0 \text{ m/s})(\cos 0°)(10.1 \text{ s})$$

$$x = 555.5 \text{ m}.$$

$$\phi = \tan^{-1} \frac{555.5 \text{ m}}{500 \text{ m}} = 48.0°. \qquad \text{(Answer)}$$

Summary

Motion in 1-D

Particle Under Constant Acceleration. If a particle moves in a straight line with a constant acceleration a_x, its motion is described by the kinematic equations:

$$v_{xf} = v_{xi} + a_x t \qquad \textbf{(2.13)}$$

$$x_f = x_i + \tfrac{1}{2}(v_{xi} + v_{xf})t \qquad \textbf{(2.15)}$$

$$x_f = x_i + v_{xi}t + \tfrac{1}{2}a_x t^2 \qquad \textbf{(2.16)}$$

$$v_{xf}^2 = v_{xi}^2 + 2a_x(x_f - x_i) \qquad \textbf{(2.17)}$$

Motion in 2-D

Projectile motion is one type of two-dimensional motion, exhibited by an object launched into the air near the Earth's surface and experiencing free fall. This common motion can be analyzed by applying the particle-under-constant-velocity model to the motion of the projectile in the x direction and the particle-under-constant-acceleration model ($a_y = -g$) in the y direction.