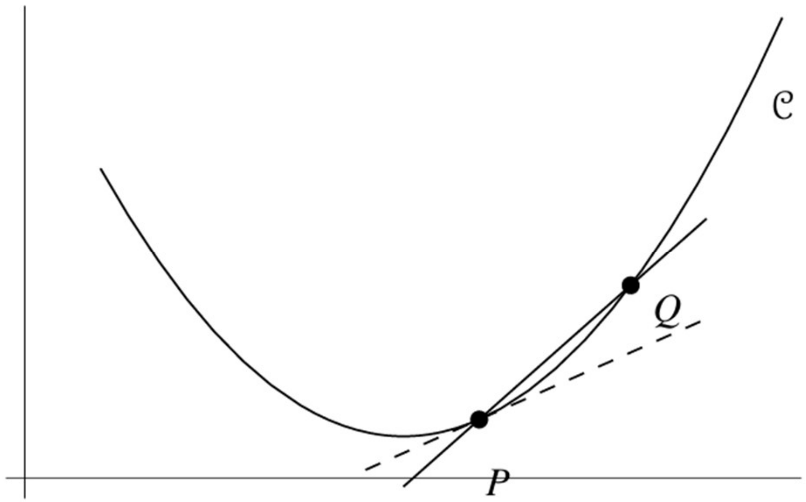
Basics calculus

First we have a curve $\mathcal{C}$ and a point P on the curve. To define the slope of $\mathcal{C}$ at P, take a point Q on the curve different from P. The line PQ is called a *secant line at* P. Its slope, denoted by m_{PQ}, can be found using the coordinates of P and Q. If we let Q move along the curve, the slope m_{PQ} changes.



Formula for Slope Suppose $\mathcal{C}$ is given by $y = f(x)$, where f is a function; and $P(x_0, f(x_0))$ is a point on $\mathcal{C}$. For any point Q on $\mathcal{C}$ with $Q \neq P$, its x-coordinate can be written as $x_0 + h$ where $h \neq 0$ (if $h > 0$, Q is on the right of P; if $h < 0$, Q is on the left of P). Thus, Q can be written as $(x_0 + h, f(x_0 + h))$. The slope m_{PQ} of the secant line PQ is

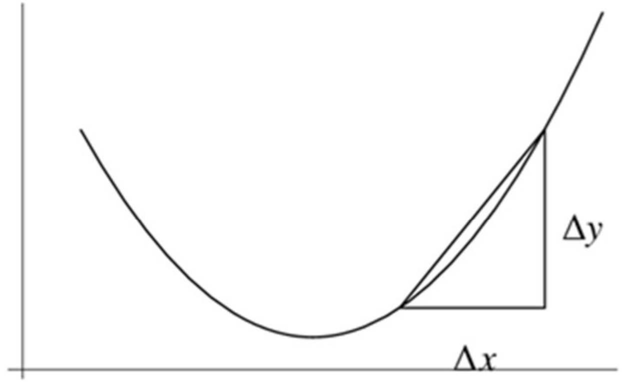
$$\begin{aligned} m_{PQ} &= \frac{f(x_0 + h) - f(x_0)}{(x_0 + h) - x_0} \\ &= \frac{f(x_0 + h) - f(x_0)}{h} \end{aligned}$$

Note that as Q approaches P, the number h approaches 0. From these, we see that the slope of $\mathcal{C}$ at P (denoted by m_P) is

$$m_P = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

provided that the limit exists.

Some readers may wonder why we have the $'$ notation as well as the $\frac{\mathrm{d}}{\mathrm{d}x}$ notation. Calculus was "*invented*" by Newton and Leibniz independently in the late 17th century. Newton used $\dot{x}$ whereas Leibniz used $\frac{\mathrm{d}x}{\mathrm{d}t}$ to denote the derivative of x with respect to t (time). The notation y' is simple whereas $\frac{\mathrm{d}y}{\mathrm{d}x}$ reminds us that it is defined as a limit of *difference quotient*:



$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

where $\Delta x = h = (x + h) - x$ and $\Delta y = f(x + h) - f(x)$ are changes in x and y respectively. $\square$

Example Find the slope of the curve given by $y = x^2$ at the point $P(3, 9)$.

Solution Put $f(x) = x^2$.

$$
\begin{aligned}
m_P &= \lim_{h\to 0} \frac{f(3+h) - f(3)}{h} \\
&= \lim_{h\to 0} \frac{(3+h)^2 - 3^2}{h} \\
&= \lim_{h\to 0} \frac{(9 + 6h + h^2) - 9}{h} \\
&= \lim_{h\to 0} \frac{6h + h^2}{h} \\
&= \lim_{h\to 0} (6 + h) \\
&= 6.
\end{aligned}
$$

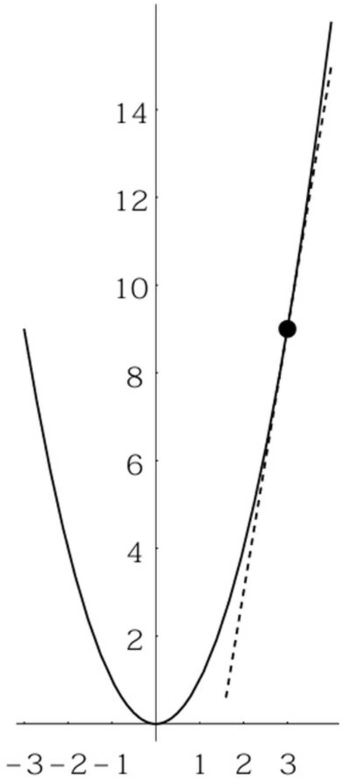
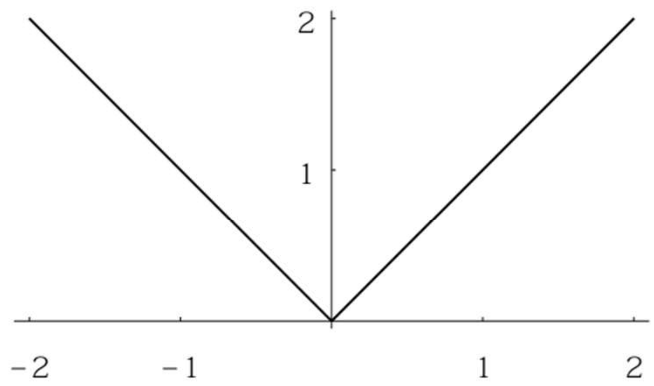


Figure 4.4

Example Let $f(x) = |x|$. The domain of f is $\mathbb{R}$.



The function f is continuous at 0. This is because

- $\lim_{x \to 0-} f(x) = \lim_{x \to 0-} (-x) = 0;$
- $\lim_{x \to 0+} f(x) = \lim_{x \to 0+} x = 0,$

and so $\lim_{x \to 0} f(x) = 0 = f(0)$.

However, f is not differentiable at 0. This is because $\lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$ does not exist as the left-side and right-side limits are unequal:

$$\begin{aligned} \lim_{h \to 0+} \frac{f(0+h) - f(0)}{h} &= \lim_{h \to 0+} \frac{h - 0}{h} \\ &= \lim_{h \to 0+} 1 \\ &= 1 \end{aligned} \qquad \text{and} \qquad \begin{aligned} \lim_{h \to 0-} \frac{f(0+h) - f(0)}{h} &= \lim_{h \to 0-} \frac{-h - 0}{h} \\ &= \lim_{h \to 0-} -1 \\ &= -1. \end{aligned}$$

Power Rule for Differentiation (positive integer version) *Let n be a positive integer. Then the power function x^n is differentiable on $\mathbb{R}$ and we have*

$$\frac{\mathrm{d}}{\mathrm{d}x}x^n = nx^{n-1}.$$

Explanation In the above formula, we use x^n to denote the n-th *power function*, that is, the function f given by $f(x) = x^n$. The domain of f is $\mathbb{R}$. The result means that f is differentiable on $\mathbb{R}$ and $f'(x) = nx^{n-1}$ for all $x \in \mathbb{R}$. When $n = 1$, the formula becomes $\frac{\mathrm{d}}{\mathrm{d}x}x = 1x^0$. In the expression on the right side, x^0 is understood to be the constant function 1 and so the formula reduces to $\frac{\mathrm{d}}{\mathrm{d}x}x = 1$ which is the rule for derivative of the identity function. To prove that the result is true for all positive integers n, we can use mathematical induction. For base step, we know that the result is true when $n = 1$. For the induction step, we can apply product rule (will be discussed later). Below we will give alternative proofs for the power rule (positive integer version).

Proof Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function given by $f(x) = x^n$ for $x \in \mathbb{R}$. By definition, we get

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}. \end{aligned}$$

To find the limit, we "simplify" the numerator to obtain a factor h and then cancel it with the factor h in the denominator. For this, we can use:

- a *factorization formula* $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1})$

By the factorization formula, we get

$$
\begin{aligned}
(x+h)^n - x^n &= (x+h-x)\Big((x+h)^{n-1} + (x+h)^{n-2}x + \cdots + (x+h)x^{n-2} + x^{n-1}\Big) \\
&= h\Big((x+h)^{n-1} + (x+h)^{n-2}x + \cdots + (x+h)x^{n-2} + x^{n-1}\Big)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\Big((x+h)^{n-1} + (x+h)^{n-2}x + \cdots + (x+h)x^{n-2} + x^{n-1}\Big) \\
&= (x+0)^{n-1} + (x+0)^{n-2}\cdot x + \cdots + (x+0)\cdot x^{n-2} + x^{n-1} \\
&= \underbrace{x^{n-1} + x^{n-1} + \cdots + x^{n-1} + x^{n-1}}_{n \text{ terms}} \\
&= nx^{n-1}
\end{aligned}
$$

Example Let $y = x^{123}$. Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

Explanation The notation $y = x^{123}$ represents a power function. To find $\frac{\mathrm{d}y}{\mathrm{d}x}$ means to find the derivative of the function.

Solution

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}x^{123} \\ &= 123x^{123-1} && \text{Power Rule} \\ &= 123x^{122} \end{aligned}$$

□

Constant Multiple Rule for Differentiation *Let f be a function and let k be a constant. Suppose that f is differentiable. Then the function kf is also differentiable. Moreover, we have*

$$\frac{\mathrm{d}}{\mathrm{d}x}(kf)(x) = k \cdot \frac{\mathrm{d}}{\mathrm{d}x} f(x).$$

Explanation The function kf is defined by $(kf)(x) = k \cdot f(x)$ for $x \in \operatorname{dom}(f)$. The result means that if $f'(x)$ exists for all $x \in \operatorname{dom}(f)$, then $(kf)'(x) = k \cdot f'(x)$ for all $x \in \operatorname{dom}(f)$, that is, $(kf)' = k \cdot f'$.

Proof By definition, we have

$$\begin{aligned}
(kf)'(x) &= \lim_{h \to 0} \frac{(kf)(x+h) - (kf)(x)}{h} \\
&= \lim_{h \to 0} \frac{k \cdot f(x+h) - k \cdot f(x)}{h} && \text{Definition of } kf \\
&= \lim_{h \to 0} \left(k \cdot \frac{f(x+h) - f(x)}{h} \right) \\
&= k \cdot \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} && \text{Limit Rule (La5s)} \\
&= k \cdot f'(x)
\end{aligned}$$

$\square$

Example Let $y = 3x^4$. Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

Solution

$$
\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}3x^4 \\
&= 3 \cdot \frac{\mathrm{d}}{\mathrm{d}x}x^4 && \text{Constant Multiple Rule} \\
&= 3 \cdot (4x^{4-1}) && \text{Power Rule} \\
&= 12x^3
\end{aligned}
$$

A particle's position on the x axis of Fig. 2-1 is given by

$$x = 4 - 27t + t^3,$$

with x in meters and t in seconds.

(a) Because position x depends on time t, the particle must be moving. Find the particle's velocity function $v(t)$ and acceleration function $a(t)$.

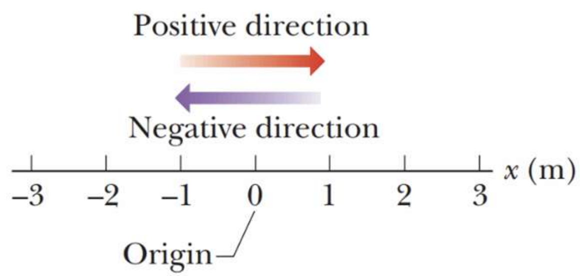


Fig. 2-1 Position is determined on an axis that is marked in units of length (here meters) and that extends indefinitely in opposite directions. The axis name, here x, is always on the positive side of the origin.

Calculations: Differentiating the position function, we find

$$v = -27 + 3t^2, \qquad \text{(Answer)}$$

with v in meters per second. Differentiating the velocity function then gives us

$$a = +6t, \qquad \text{(Answer)}$$

with a in meters per second squared.

(b) Is there ever a time when $v = 0$?

Calculation: Setting $v(t) = 0$ yields

$$0 = -27 + 3t^2,$$

which has the solution

$$t = \pm 3 \text{ s.} \qquad \text{(Answer)}$$

Thus, the velocity is zero both 3 s before and 3 s after the clock reads 0.

A particle moves so that its position (in meters) as a function of time (in seconds) is $\vec{r} = \hat{i} + 4t^2\hat{j} + t\hat{k}$. Write expressions for (a) its velocity and (b) its acceleration as functions of time.
[$8t\hat{j} + \hat{k}$; $8\hat{j}$]

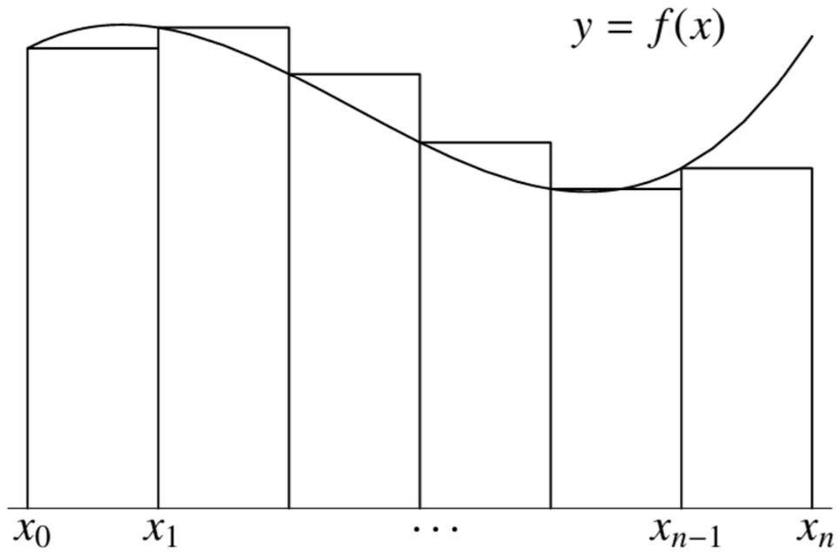
Theorem *Let f be a function that is continuous on a closed and bounded interval $[a, b]$. Then the following limit exists:*

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1}) \cdot \frac{b-a}{n},$$

where $x_i = a + \frac{i}{n}(b-a)$ for $0 \leq i \leq n$.

Explanation

- By the construction of the x_i's, we have $x_0 = a$, $x_n = b$, $x_0 < x_1 < \cdots < x_n$, and for every $i = 1, \ldots, n$, the subinterval $[x_{i-1}, x_i]$ has length $\frac{b-a}{n}$ and x_{i-1} is the left-endpoint of the subinterval.
- If f is non-negative on $[a, b]$, that is, $f(x) \geq 0$ for all $x \in [a, b]$, then $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1}) \cdot \frac{b-a}{n}$ is the area bounded by the graph of f, the x-axis and the vertical lines given by $x = a$ and $x = b$.



Definition Let f be a function that is continuous on a closed and bounded interval $[a,b]$. The number $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1}) \cdot \frac{b-a}{n}$, where $x_i = a + \frac{i}{n}(b-a)$ for $0 \le i \le n$, is called the *definite integral* of f *from* a *to* b and is denoted by $\int_a^b f(x)\,\mathrm{d}x$, that is,

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1}) \cdot \frac{b-a}{n}.$$

More generally, the subintervals $[x_0, x_1], \ldots, [x_{n-1}, x_n]$ need not be of equal lengths. All we need is that the lengths are small enough: if $a = x_0 < x_1 < \cdots < x_n = b$ and $\Delta x_1, \ldots, \Delta x_n$ are small enough, where Δx_i is the length of the ith subinterval $[x_{i-1}, x_i]$, then for every choice of $t_1, \ldots, t_n$ with $t_i \in [x_{i-1}, x_i]$ for $1 \le i \le n$, the sum (called a *Riemann Sum*)

$$\sum_{i=1}^{n} f(t_i)\Delta x_i$$

is close to $\int_a^b f(x)\,\mathrm{d}x$. Many authors use this to define definite integral.

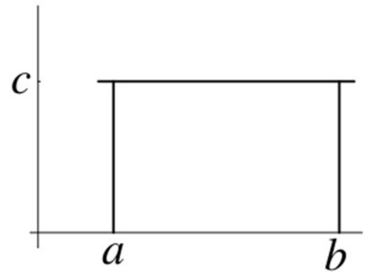
Definite Integral for Constant Functions Let c be a constant and let a and b be real numbers with $a < b$. Then we have $\displaystyle\int_a^b c\,\mathrm{d}x = c \cdot (b-a)$.

Proof Applying (6.1.2) to $f(x) = c$ and $x_i = a + \frac{i}{n}(b-a)$ for $0 \leq i \leq n$, we get

$$\begin{aligned}
\int_a^b c\,\mathrm{d}x &= \lim_{n\to\infty} \sum_{i=1}^{n} c \cdot \frac{b-a}{n} \\
&= \lim_{n\to\infty} c \cdot \frac{b-a}{n} \times n && \text{Sum of Constants} \\
&= \lim_{n\to\infty} c \cdot (b-a) && \text{Rule (L1) for Limit} \\
&= c \cdot (b-a)
\end{aligned}$$

□

Remark If $c > 0$, then $\int_a^b c\,\mathrm{d}x$ is the area of the rectangular region shown in Figure 6.5.

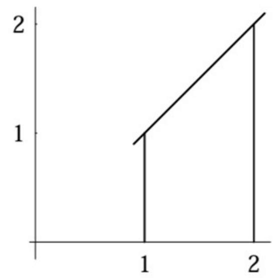


Example Use definition to find $\int^{2} x\,\mathrm{d}x$.

Solution $f(x) = x$, $a = 1$, $b = 2$ and $x_i = 1 + \frac{i}{n}$ for $0 \le i \le n$, we get

$$
\begin{aligned}
\int_1^2 x\,\mathrm{d}x &= \lim_{n\to\infty} \sum_{i=1}^{n} \left(1 + \frac{i-1}{n}\right) \cdot \frac{1}{n} \\
&= \lim_{n\to\infty} \frac{1}{n^2} \sum_{i=1}^{n} (n + i - 1) \\
&= \lim_{n\to\infty} \frac{1}{n^2} \cdot \frac{n(n + 2n - 1)}{2} && \text{Sum of A.P.} \\
&= \lim_{n\to\infty} \frac{3n - 1}{2n} \\
&= \lim_{n\to\infty} \frac{3n}{2n} && \text{Leading Term Rule} \\
&= \frac{3}{2}.
\end{aligned}
$$

Remark The value of the definite integral is the area of the trapezoidal region shown in Figure . Readers can check that the result agrees with that obtained by using formula for area of trapezoid.



Fundamental Theorem of Calculus I

We have seen two types of integrals:

1. **Indefinite:** $\displaystyle\int f(x)\,dx = F(x) + C$

   where $F(x)$ is an antiderivative of $f(x)$.

2. **Definite:** $\displaystyle\int_a^b f(x)\,dx =$ signed area bounded by $f(x)$ over $[a, b]$.

Theorem (Fundamental Theorem of Calculus I)

Let $f(x)$ be a continuous function on $[a, b]$. Then

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b := F(b) - F(a)$$

where $F(x)$ is an antiderivative of $f(x)$.

Note: The result is independent of the chosen antiderivative $F(x)$.

We have **three** ways of evaluating definite integrals:

1. Use of area formulas if they are available.

   (This is what we did last lecture.)

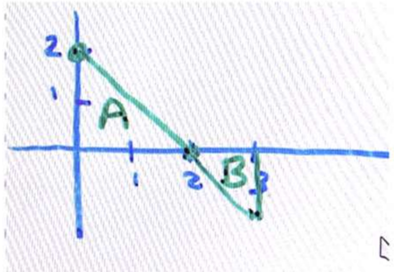
2. Use of the Fundamental Theorem of Calculus (F.T.C.)

3. Use of the Riemann sum $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$

   (This we will not do in this course.)

Example: Evaluate $\int_0^3 (2-x)\,dx$ using the first two methods.

Solution:

1) Areas: 

$$\int_0^3 (2-x)\,dx = A - B = \frac{1}{2}\cdot 2\cdot 2 - \frac{1}{2}\cdot 1\cdot 1 = 2 - \frac{1}{2} = \frac{3}{2}$$

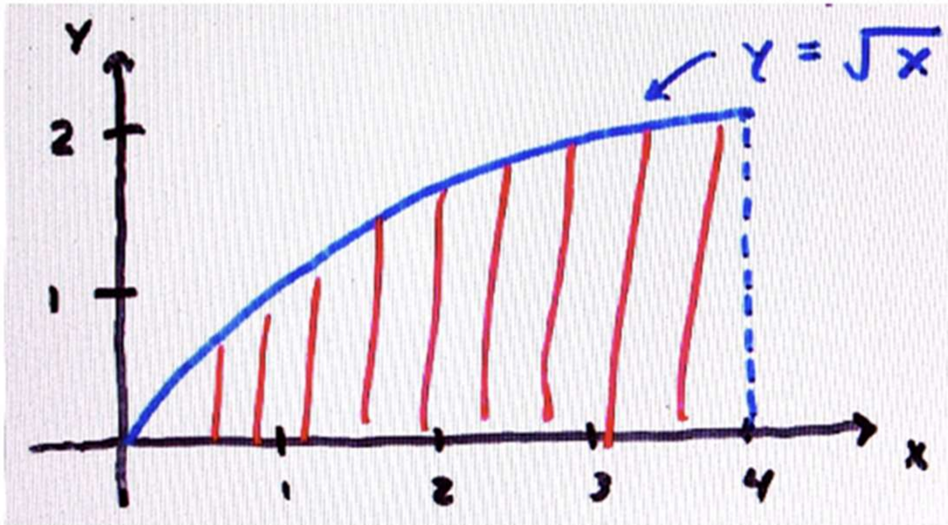
2) F.T.C. (No graph required)

$$\int_0^3 (2-x)\,dx = \underbrace{2x - \frac{x^2}{2}}_{\text{antideriv.}}\Bigg|_0^3$$

$$= (2\cdot 3 - \frac{3^2}{2}) - (2\cdot 0 - \frac{0^2}{2})$$

$$= 6 - \frac{9}{2} = \frac{3}{2}$$

Example: Find the area of the region below.



Solution:

$$A = \int_0^4 \sqrt{x}\, dx = \int_0^4 x^{1/2}\, dx$$

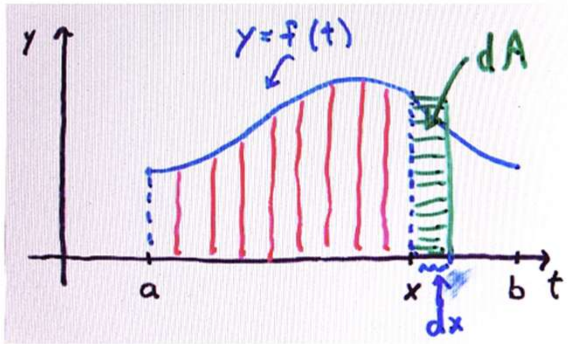
$$= \left. \frac{x^{3/2}}{3/2} \right|_0^4$$

$$= \frac{2}{3} \cdot 4^{3/2} - 0$$

$$= \frac{2}{3} \cdot 8 = \frac{16}{3}$$

Fundamental Theorem of Calculus II

Let $f(t)$ be a continuous function on $[a, b]$.



Let x be a point with $a < x < b$.

Let $A(x) = \displaystyle\int_a^x f(t)\,dt$ = Signed Area bounded by $f(t)$ over $[a, x]$.

Goal: Find the rate that the area $A(x)$ increases or decreases, that is, find $\frac{dA}{dx}$.

Let dx = infinitesimal change in x.

dA = resulting change in the area

dA = height $\times$ base $= f(x) \cdot dx$.

Thus: $\displaystyle\frac{dA}{dx} = \frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x).$

Theorem (Fundamental Theorem of Calculus II)

Let $f(x)$ be a continuous function on $[a, b]$. Then for any x in (a, b) we have

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

where $f(x)$ is the evaluation of $f(t)$ at x.

"The derivative of the integral of a function is the function."

Example: Find $\dfrac{d}{dx}\displaystyle\int_2^x e^t \cdot \cos(5t)\,dt$

Solution: $\dfrac{d}{dx}\displaystyle\int_2^x e^t \cdot \cos(5t)\,dt = e^x \cos(5x)$, by F.T.C. II.

Example Find $\int 2x^7 \, \mathrm{d}x$.

Explanation The question is to find the family of functions that are antiderivatives for the integrand (on some open intervals). The answer should be given in the form "*a function of* x $+C$". Usually, for integration problems, there is no need to mention the underlying open intervals. For the given problem, the function $2x^7$ is continuous on $\mathbb{R}$ and so it has antiderivatives on $\mathbb{R}$.

Solution
$$\begin{aligned}
\int 2x^7 \, \mathrm{d}x &= 2\int x^7 \, \mathrm{d}x && \text{Constant Multiple Rule} \\
&= 2\left(\frac{x^{7+1}}{7+1} + C\right) && \text{Power Rule} \\
&= \frac{1}{4}x^8 + 2C
\end{aligned}$$

□

Remark From the answer, we see that the function $\frac{1}{4}x^8$ is an antiderivative for the function $2x^7$ (on $\mathbb{R}$). Therefore, we can also write

$$\int 2x^7 \, \mathrm{d}x = \frac{1}{4}x^8 + C.$$

Example we have

$$
\begin{aligned}
\int_0^1 x^2 \, \mathrm{d}x &= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^2 \cdot \frac{1}{n} \\
&= \lim_{n\to\infty} \frac{1}{n^3} \sum_{i=1}^{n} i^2 \\
&= \lim_{n\to\infty} \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} && \text{Sum of Squares Formula} \\
&= \lim_{n\to\infty} \frac{2n^3}{6n^3} && \text{Leading Term Rule} \\
&= \frac{1}{3}
\end{aligned}
$$

Try to do it by FTC!

Velocity and Position by Integration

If $t_1 = 0$ and t_2 is any later time t, and if x_0 and v_{0x} are the position and velocity, respectively, at time $t = 0$

$$v_x = v_{0x} + \int_0^t a_x \, dt \qquad (2.17)$$

$$x = x_0 + \int_0^t v_x \, dt \qquad (2.18)$$

Example 2.9 Motion with changing acceleration

Sally is driving along a straight highway in her 1965 Mustang. At $t = 0$ when she is moving at 10 m/s in the positive x-direction, she passes a signpost at $x = 50$ m. Her x-acceleration as a function of time is:

$$a_x = 2.0\ \mathrm{m/s^2} - (0.10\ \mathrm{m/s^3})t$$

(a) Find her x-velocity and position x as functions of time. (b) When is her x-velocity greatest? (c) What is that maximum x-velocity? (d) Where is the car when it reaches that maximum x-velocity?

Analysis: The x-acceleration is a function of time, so we *cannot* use the constant-acceleration formulas of Section 2.4. Instead, we use Eq. (2.17) to obtain an expression for v_x as a function of time, and then use that result in Eq. (2.18) to find an expression for x as a function of t.

Example 2.9 Motion with changing acceleration

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$$a_x = 2.0\ \mathrm{m/s^2} - (0.10\ \mathrm{m/s^3})t$$

(a) Find her x-velocity and position x as functions of time.

$$\begin{aligned} v_x &= 10\ \mathrm{m/s} + \int_0^t [2.0\ \mathrm{m/s^2} - (0.10\ \mathrm{m/s^3})t]\,dt \\ &= 10\ \mathrm{m/s} + (2.0\ \mathrm{m/s^2})t - \tfrac{1}{2}(0.10\ \mathrm{m/s^3})t^2 \end{aligned}$$

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$$a_x = 2.0 \text{ m/s}^2 - (0.10 \text{ m/s}^3)t$$

(a) Find her x-velocity and position x as functions of time.

$$v_x = 10 \text{ m/s} + (2.0 \text{ m/s}^2)t - \tfrac{1}{2}(0.10 \text{ m/s}^3)t^2$$

$$\begin{aligned} x &= 50 \text{ m} + \int_0^t \left[10 \text{ m/s} + (2.0 \text{ m/s}^2)t - \tfrac{1}{2}(0.10 \text{ m/s}^3)t^2\right] dt \\ &= 50 \text{ m} + (10 \text{ m/s})t + \tfrac{1}{2}(2.0 \text{ m/s}^2)t^2 - \tfrac{1}{6}(0.10 \text{ m/s}^3)t^3 \end{aligned}$$

Example 2.9 Motion with changing acceleration

Sally is driving along a straight highway in her 1965 Mustang. At $t = 0$ when she is moving at 10 m/s in the positive x-direction, she passes a signpost at $x = 50$ m. Her x-acceleration as a function of time is:

$$\boxed{a_x} = 2.0 \text{ m/s}^2 - (0.10 \text{ m/s}^3)t$$

(b) When is her x-velocity greatest? *Greatest when acceleration* $\boxed{a_x \textit{ is } 0!}$

$$\boxed{0} = 2.0 \text{ m/s}^2 - (0.10 \text{ m/s}^3)t$$

$$t = \frac{2.0 \text{ m/s}^2}{0.10 \text{ m/s}^3} = 20 \text{ s}$$

Example 2.9 Motion with changing acceleration

Sally is driving along a straight highway in her 1965 Mustang. At $t = 0$ when she is moving at 10 m/s in the positive x-direction, she passes a signpost at $x = 50$ m. Her x-acceleration as a function of time is:

$$a_x = 2.0\ \mathrm{m/s^2} - (0.10\ \mathrm{m/s^3})t$$

(c) What is that maximum x-velocity? Plug in the time we just calculated

$$t = \frac{2.0\ \mathrm{m/s^2}}{0.10\ \mathrm{m/s^3}} = 20\ \mathrm{s}$$

$$v_{\text{max-}x} = 10\ \mathrm{m/s} + (2.0\ \mathrm{m/s^2})(20\ \mathrm{s}) - \tfrac{1}{2}(0.10\ \mathrm{m/s^3})(20\ \mathrm{s})^2$$

$$= 30\ \mathrm{m/s}$$

Example 2.9 Motion with changing acceleration

Sally is driving along a straight highway in her 1965 Mustang. At $t = 0$ when she is moving at 10 m/s in the positive x-direction, she passes a signpost at $x = 50$ m. Her x-acceleration as a function of time is:

$$a_x = 2.0 \text{ m/s}^2 - (0.10 \text{ m/s}^3)t$$

(c) Where is the car when it reaches that maximum x-velocity

Asking about x when v_x reaches maximum

We substitute $t = 20$ s into the expression for x from part (a):

$$x = 50 \text{ m} + (10 \text{ m/s})(20 \text{ s}) + \tfrac{1}{2}(2.0 \text{ m/s}^2)(20 \text{ s})^2$$

$$- \tfrac{1}{6}(0.10 \text{ m/s}^3)(20 \text{ s})^3 \quad = 517 \text{ m}$$

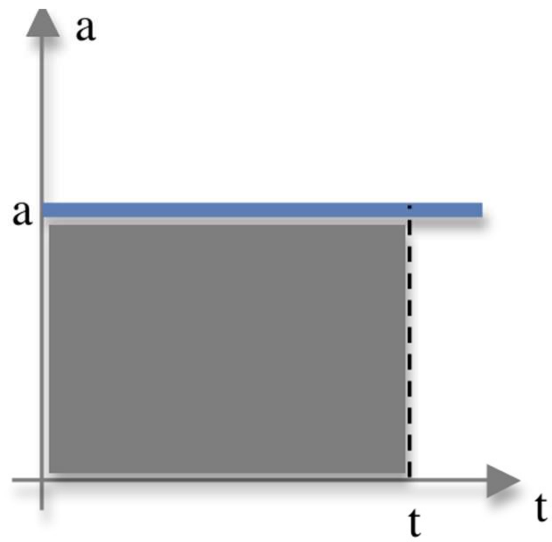
Applications (constant acceleration)

Since the acceleration is constant, it can come through the integral sign. Integrating dt, just gives t evaluated at zero and the final time.

$$v - v_o = \int_0^t a\,dt \Rightarrow v - v_o = a\int_0^t dt \Rightarrow v - v_o = at\Big|_0^t \Rightarrow v - v_o = a(t-0) \Rightarrow v - v_o = at\ .$$

So, in fact, we get the same result as we did looking at the graph. Solving for the final speed,

$$v = v_o + at\ .$$



Now that we know the area, let's do the calculus and see that we get the same answer. Recall we had,

$$x - x_o = \int_0^t v\,dt \,.$$

Substituting the expression for the velocity as a function of time from above,

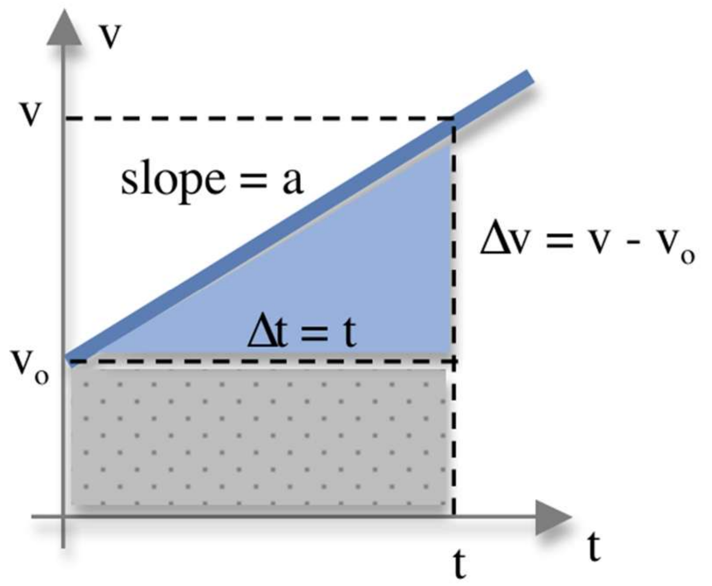
$$x - x_o = \int_0^t (v_o + at)\,dt = \int_0^t v_o\,dt + \int_0^t at\,dt \,.$$

Since the acceleration and initial velocity don't change with time,

$$\int_{x_o}^{x} dx = v_o \int_0^t dt + a \int_0^t t\,dt \Rightarrow x - x_o = v_o t\Big|_0^t + \tfrac{1}{2}at^2\Big|_0^t \Rightarrow x - x_o = v_o t + \tfrac{1}{2}at^2 \,.$$

This agrees with the area we calculated earlier. Solving for the final position,

$$x = x_o + v_o t + \tfrac{1}{2}at^2 \,.$$



$$V = \frac{dx}{dt}$$

$$= \frac{dx}{dv} \cdot \frac{dv}{dt}$$

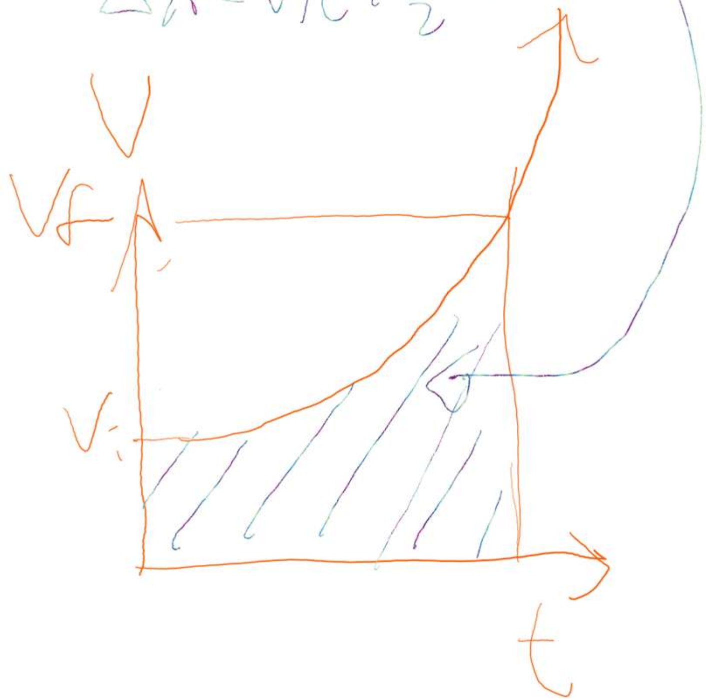
$$= \frac{dx}{dv} \cdot a$$

$$\Rightarrow \int_{v_i}^{v_f} v\,dv = a\int_{x_i}^{x_f} dx$$

$$\left.\frac{v^2}{2}\right|_{v_i}^{v_f} = a\Delta x$$

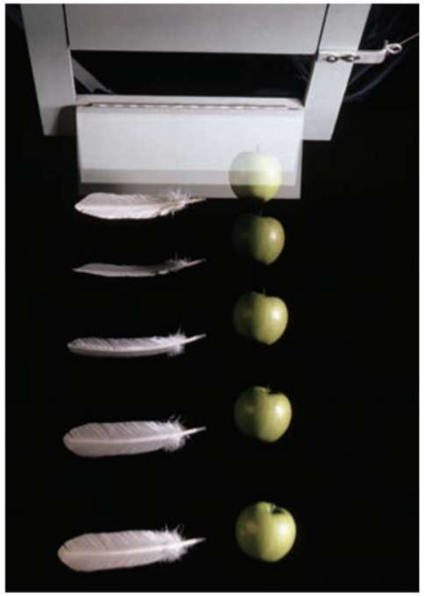
$$\Rightarrow V_f^2 = V_i^2 + 2a\Delta x$$

$$\Delta x = v_i t + \frac{1}{2}at^2$$



Acceleration of Freely Falling Object

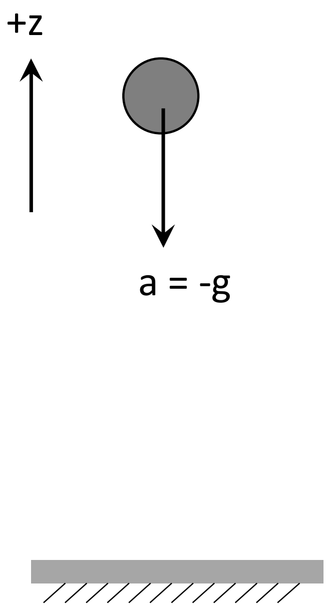
- An application of kinematic equations is a ***freely falling object*** which has a constant acceleration *g* (always pointing towards the center of Earth, i.e. downwards at the Earth's surface).
- The magnitude of free fall acceleration is g = 9.80 m/s^2.
  - *g* decreases with increasing altitude
  - *g* varies with latitude
  - 9.80 m/s^2 is the average at the Earth's surface
  - The italicized *g* will be used for the acceleration due to gravity.
    - Not to be confused with g for grams



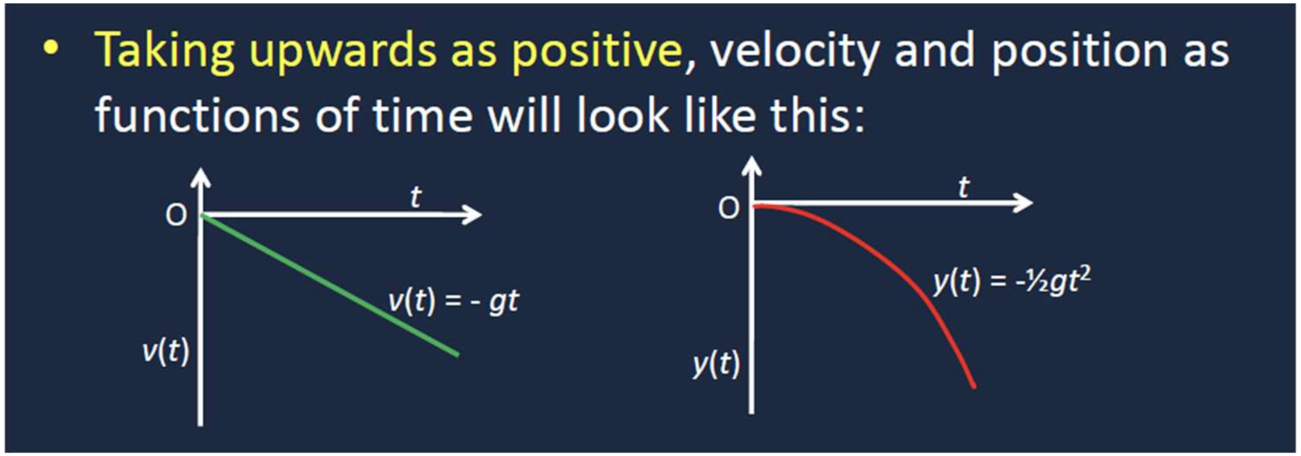
A feather and an apple free fall in vacuum at the same magnitude of acceleration g. The acceleration increases the distance between successive images. In the absence of air, the feather and apple fall together.(Jim Sugar/Corbis Images)

Gravitational Acceleration (Graphic View)

- Acceleration $a = -g$, where g ~= 9.8 m/s^2 around sea level



- Notice the minus sign, because we usually take the direction in which height (h) increases as +z direction



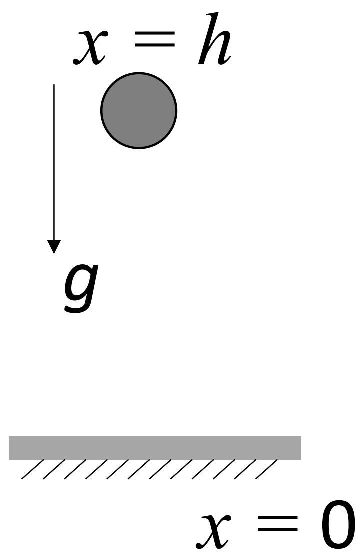
Case 1: Falling from height h

Q: how long does it take for a particle to fall back from a height *h*? Derive an analytic equation.

Assume: air resistance is negligible and initial velocity is 0

g ~= 9.8 m/s^2

$$v = v_0 + at$$
$$x = x_0 + v_0 t + \tfrac{1}{2} a t^2$$
$$\bar{v} = \frac{v_0 + v_1}{2}$$
$$v^2 = v_0^2 + 2a(x - x_0)$$



Answer:

$$\underset{0}{\boldsymbol{x}} = \underset{h}{\boldsymbol{x_0}} + \underset{0}{\boldsymbol{v_0}}\boldsymbol{t} + \frac{\mathbf{1}}{\mathbf{2}}\underset{-g}{\boldsymbol{a}}\boldsymbol{t^2}$$

$$\mathbf{0} = \boldsymbol{h} - \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{g t^2}$$

Rearrange

$$\boldsymbol{t} = \sqrt{\frac{\mathbf{2}\boldsymbol{h}}{\boldsymbol{g}}}$$

Case 2: Throwing a ball upward

Question: how long does it take for a particle to fall back if we throw it upward from $z = 0$ with an initial velocity $v = v_0$

Derive an analytic equation.

Approach 1:

$$\boxed{x_1 = x_0 = 0}$$

$$\boxed{v^2 - {v_0}^2 = 2a(x_1 - x_0)}$$

$$v^2 = {v_0}^2$$

$$v = -v_0$$

$$\boxed{-v_0 = v_0 - gt}$$

$$t = 2v_0/g$$

Approach 2: still use:

$$x = x_0 + v_0 t + \frac{1}{2}at^2$$

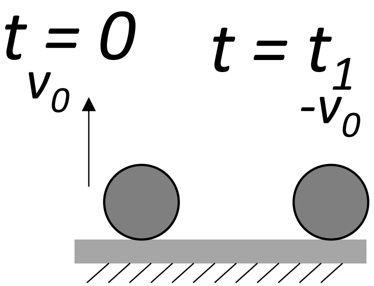
g ~= 9.8 m/s^2

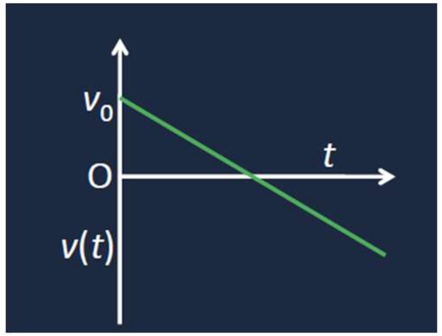
$$\boxed{v = v_0 + at}$$

$$\boxed{x = x_0 + v_0 t + \tfrac{1}{2}at^2}$$

$$\bar{v} = \frac{v_0 + v_1}{2}$$

$$\boxed{v^2 = v_0^2 + 2a(x - x_0)}$$



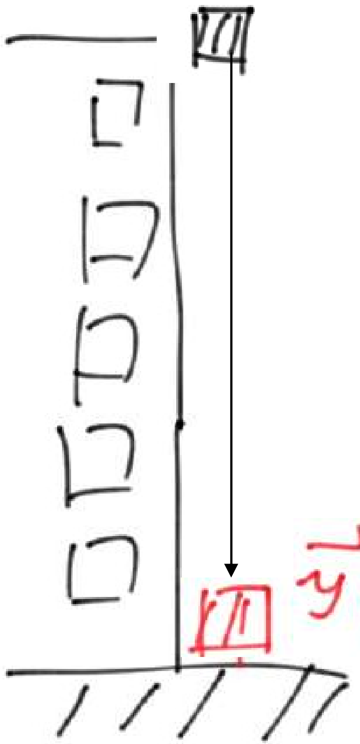


Exercise 2: Drop Brick (Don't do it in real life)

$\vec{y}(t=0) = \vec{y}_0$

$\vec{V}(t=0) = 0$

2.42 •• A brick is dropped (zero initial speed) from the roof of a building. The brick strikes the ground in 2.50 s. You may ignore air resistance, so the brick is in free fall. (a) How tall, in meters, is the building? (b) What is the magnitude of the brick's velocity just before it reaches the ground? (c) Sketch a_y-t, v_y-t, and y-t graphs for the motion of the brick.



$\vec{y}(t_1 = 2.5s) = 0$

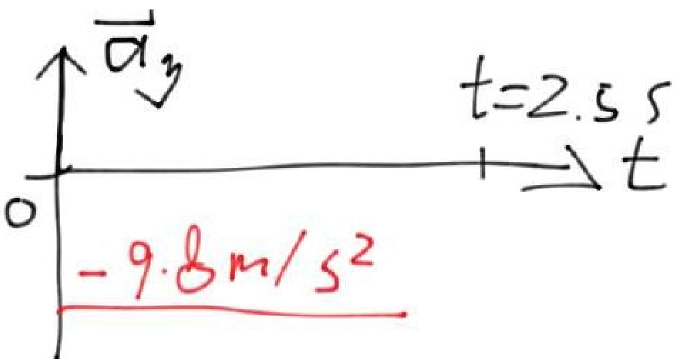
$\vec{V}(t_1 = 2.5s) = ?$

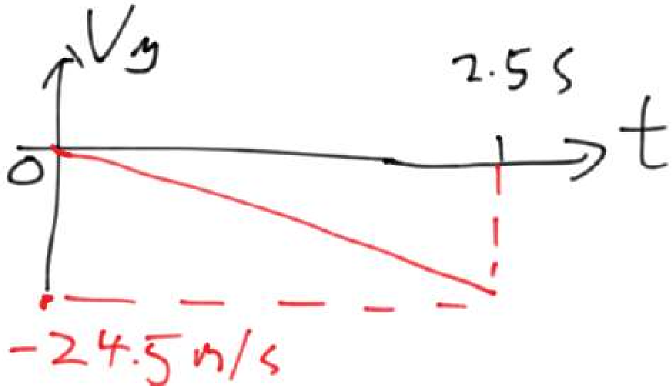
$$\vec{V} = \underbrace{\vec{V}_0}_{0} - gt\,\hat{j}$$

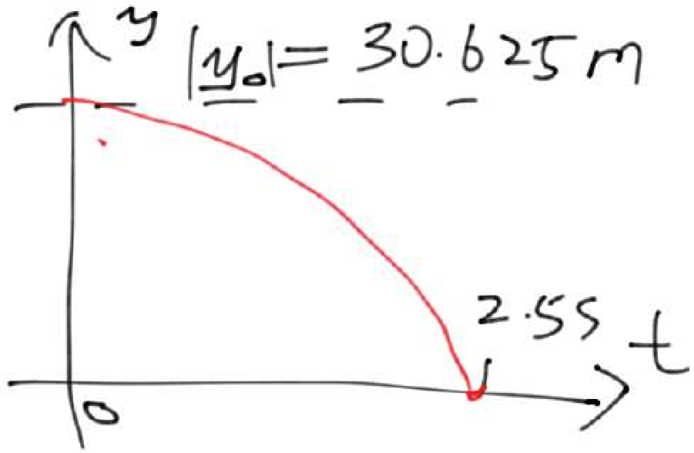
$$\Rightarrow \vec{V} = -9.8\,m/s^2 \cdot 2.5\,s\,\hat{j}$$

Exercise 2: Drop Brick (Don't do it in real life)

2.42 •• A brick is dropped (zero initial speed) from the roof of a building. The brick strikes the ground in 2.50 s. You may ignore air resistance, so the brick is in free fall. (a) How tall, in meters, is the building? (b) What is the magnitude of the brick's velocity just before it reaches the ground? (c) Sketch a_y-t, v_y-t, and y-t graphs for the motion of the brick.

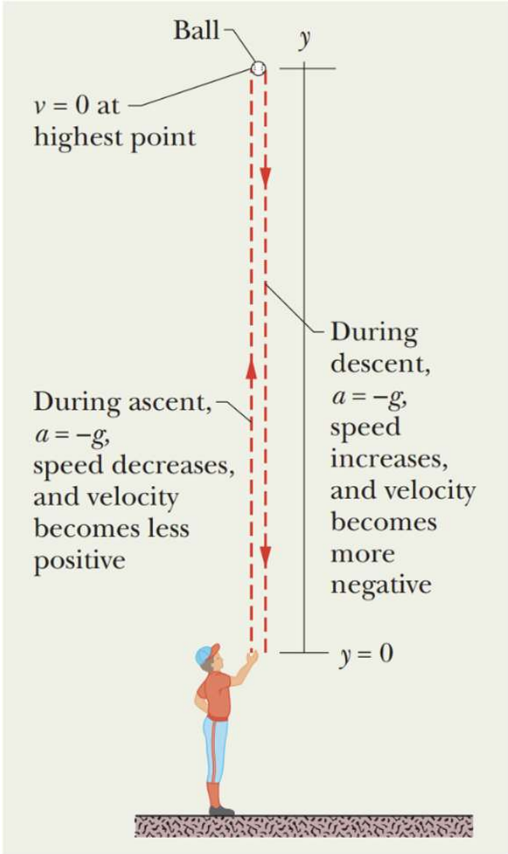






a pitcher tosses a baseball up along a y axis, with an initial speed of 12 m/s.

(a) How long does the ball take to reach its maximum height?



a) those four variables. This yields

$$t = \frac{v - v_0}{a} = \frac{0 - 12 \text{ m/s}}{-9.8 \text{ m/s}^2} = 1.2 \text{ s.} \qquad \text{(Answer)}$$

b) What is the ball's maximum height above its release point

Calculation: We can take the ball's release point to be $y_0 = 0$. We can then write Eq. 2-16 in y notation, set $y - y_0 = y$ and $v = 0$ (at the maximum height), and solve for y. We get

$$y = \frac{v^2 - v_0^2}{2a} = \frac{0 - (12\text{ m/s})^2}{2(-9.8\text{ m/s}^2)} = 7.3\text{ m}. \qquad \text{(Answer)}$$

(c) How long does the ball take to reach a point 5.0 m above its release point?

c) How long does the ball take to reach a point 5.0 m above its release point?

Calculations: We know v_0, $a = -g$, and displacement $y - y_0 = 5.0$ m, and we want t, so we choose Eq. 2-15. Rewriting it for y and setting $y_0 = 0$ give us

$$y = v_0 t - \tfrac{1}{2} g t^2,$$

or $$5.0 \text{ m} = (12 \text{ m/s})t - (\tfrac{1}{2})(9.8 \text{ m/s}^2)t^2.$$

If we temporarily omit the units (having noted that they are consistent), we can rewrite this as

$$4.9t^2 - 12t + 5.0 = 0.$$

Solving this quadratic equation for t yields

$$t = 0.53 \text{ s} \quad \text{and} \quad t = 1.9 \text{ s}. \qquad \text{(Answer)}$$

There are two such times! This is not really surprising because the ball passes twice through $y = 5.0$ m, once on the way up and once on the way down.

Summary: Kinematics in 1D

$$v_{\text{av-}x} = \frac{x_2 - x_1}{t_2 - t_1} = \frac{\Delta x}{\Delta t}$$

$$v_x = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$$

$$a_{\text{av-}x} = \frac{v_{2x} - v_{1x}}{t_2 - t_1} = \frac{\Delta v_x}{\Delta t}$$

$$a_x = \lim_{\Delta t \to 0} \frac{\Delta v_x}{\Delta t} = \frac{dv_x}{dt}$$

$$v_x = v_{0x} + \int_0^t a_x \, dt$$

$$x = x_0 + \int_0^t v_x \, dt$$

Constant x-acceleration only:

$$v_x = v_{0x} + a_x t \quad (2.8)$$

$$x = x_0 + v_{0x} t + \tfrac{1}{2} a_x t^2 \quad (2.12)$$

$$v_x{}^2 = v_{0x}{}^2 + 2a_x(x - x_0) \quad (2.13)$$

$$x - x_0 = \left(\frac{v_{0x} + v_x}{2}\right) t$$