## *Example* 10.11 Rotating Rod Revisited

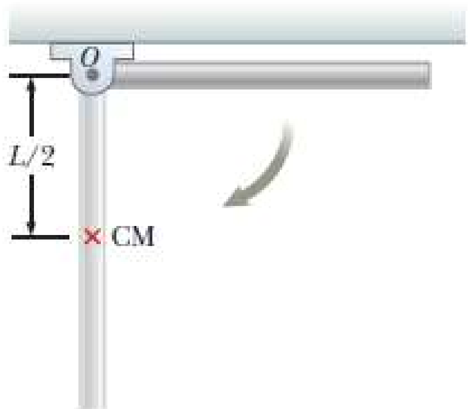
A uniform rod of length $L$ and mass $M$ is free to rotate on a frictionless pin passing through one end (Fig 10.21). The rod is released from rest in the horizontal position.

**(A)** What is its angular speed when the rod reaches its lowest position?

$$K_f + U_f = K_i + U_i$$

$$\tfrac{1}{2}I\omega^2 + 0 = 0 + \tfrac{1}{2}MgL$$

$$\omega = \sqrt{\frac{MgL}{I}} = \sqrt{\frac{MgL}{\frac{1}{3}ML^2}} = \boxed{\sqrt{\frac{3g}{L}}}$$




**(B)** Determine the tangential speed of the center of mass and the tangential speed of the lowest point on the rod when it is in the vertical position.

$$v_{CM} = r\omega = \frac{L}{2}\omega = \boxed{\tfrac{1}{2}\sqrt{3gL}}$$

$$v = 2v_{CM} = \boxed{\sqrt{3gL}}$$

## Example 10.12 Energy and the Atwood Machine

Two blocks having different masses $m_1$ and $m_2$ are connected by a string passing over a pulley as shown in Active Figure 10.22. The pulley has a radius $R$ and moment of inertia $I$ about its axis of rotation. The string does not slip on the pulley, and the system is released from rest. Find the translational speeds of the blocks after block 2 descends through a distance $h$ and find the angular speed of the pulley at this time.

$$K_f + U_f = K_i + U_i$$

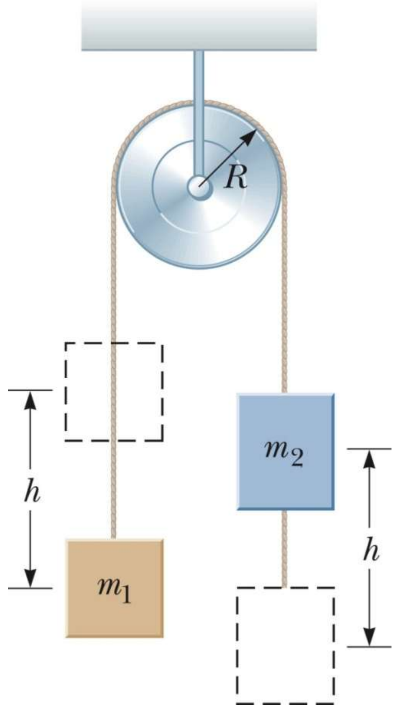
$$\left(\tfrac{1}{2}m_1 v_f^2 + \tfrac{1}{2}m_2 v_f^2 + \tfrac{1}{2}I\omega_f^2\right) + (m_1 gh - m_2 gh) = 0 + 0$$

$$\tfrac{1}{2}m_1 v_f^2 + \tfrac{1}{2}m_2 v_f^2 + \tfrac{1}{2}I\frac{v_f^2}{R^2} = m_2 gh - m_1 gh$$

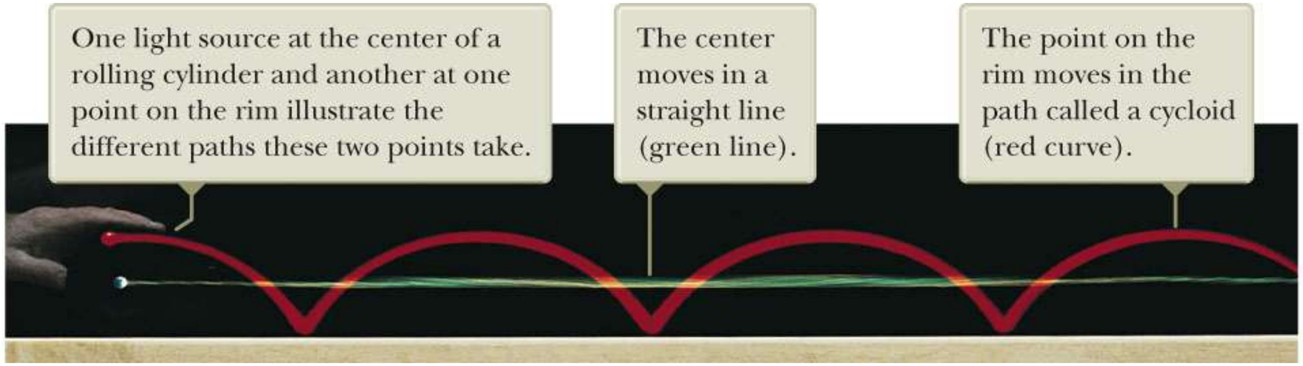
$$\tfrac{1}{2}\left(m_1 + m_2 + \frac{I}{R^2}\right)v_f^2 = m_2 gh - m_1 gh$$

$$(1) \quad v_f = \left[\frac{2(m_2 - m_1)gh}{m_1 + m_2 + I/R^2}\right]^{1/2}$$

$$\omega_f = \frac{v_f}{R} = \frac{1}{R}\left[\frac{2(m_2 - m_1)gh}{m_1 + m_2 + I/R^2}\right]^{1/2}$$



# Rolling Object




- The red curve shows the path moved by a point on the rim of the object.
  - This path is called a ***cycloid.***
- The green line shows the path of the center of mass of the object.
- In pure rolling motion, an object rolls without slipping.
- In such a case, there is a simple relationship between its rotational and translational motions.

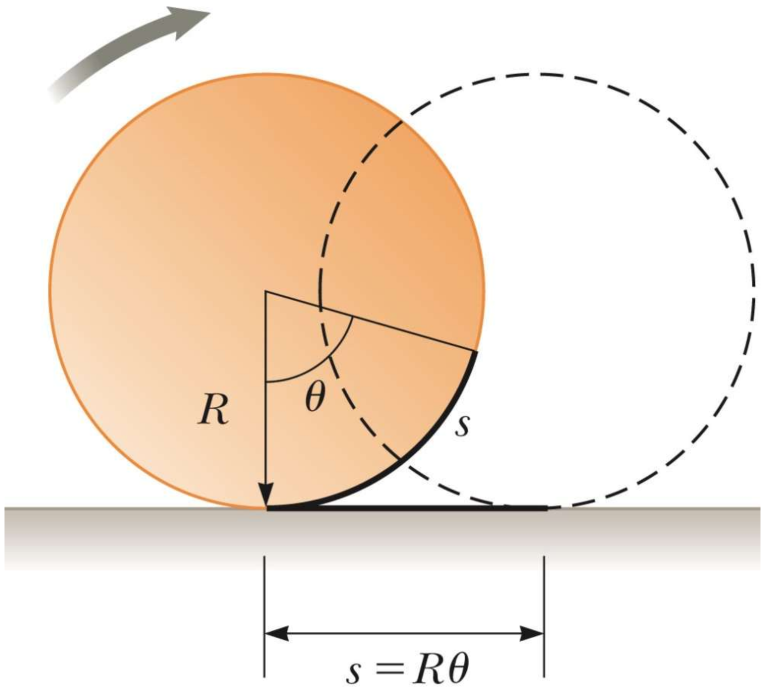
# Pure Rolling Motion, Object's Center of Mass

- The translational speed of the center of mass is

$$v_{CM} = \frac{ds}{dt} = R\frac{d\theta}{dt} = R\omega$$

- The linear acceleration of the center of mass is

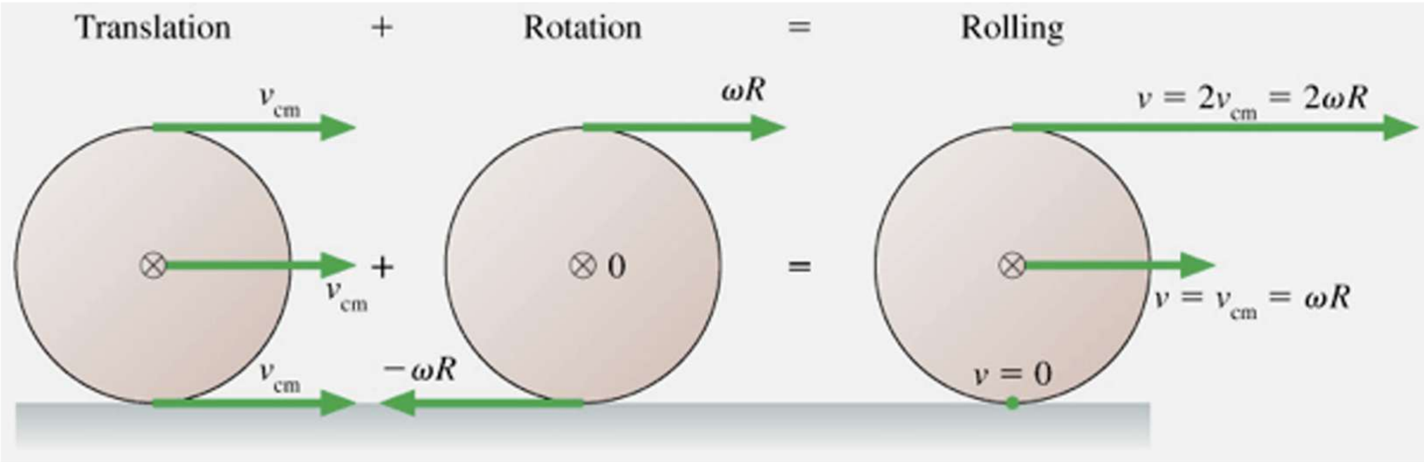
$$a_{CM} = \frac{dv_{CM}}{dt} = R\frac{d\omega}{dt} = R\alpha$$



# Rolling Motion Cont.

•Rolling motion can be modeled as a combination of pure translational motion and pure rotational motion.

•The contact point between the surface and the cylinder has a translational speed of zero (c).




$$\boxed{\vec{v}_i = \vec{v}_{cm} + \vec{v}_{i,rel}}$$

$v_{bottom} = 0$ $\quad v_{axel} = v_{cm} = \omega R$ $\quad v_{top} = 2v_{cm} = 2\omega R$

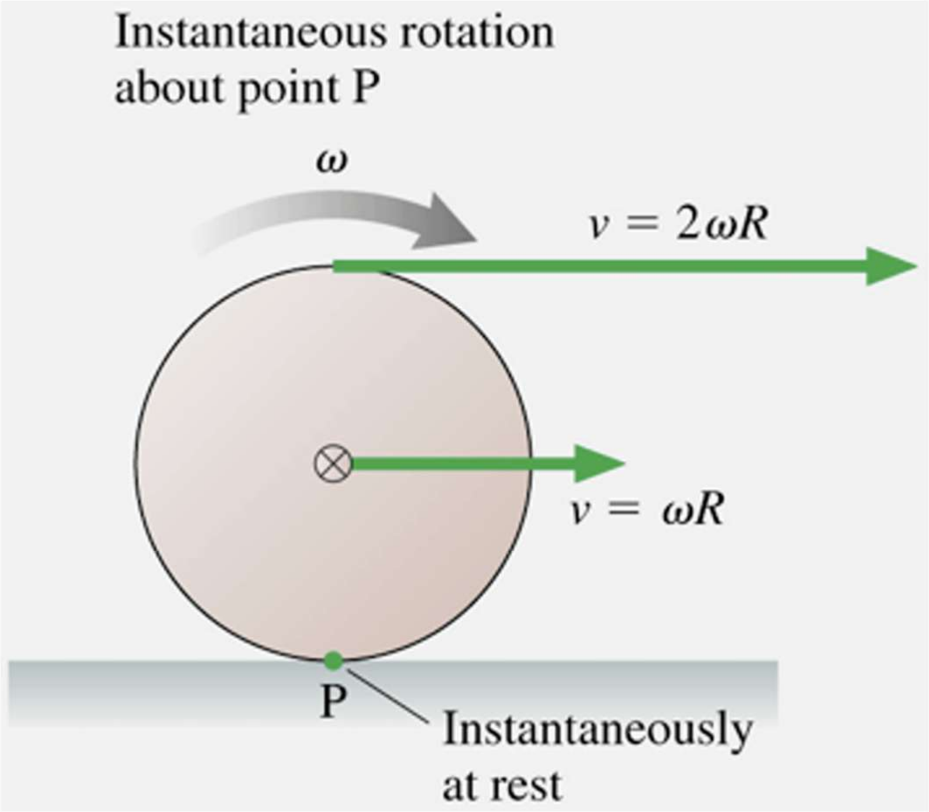
# Total Kinetic Energy of a Rolling Object

- The total kinetic energy of a rolling object is the sum of the translational energy of its center of mass and the rotational kinetic energy about its center of mass.
  - $K = ½ I_{\mathrm{CM}}\, \omega^2 + ½ M v_{\mathrm{CM}}^2$
    - The $½ I_{CM}\omega^2$ represents the rotational kinetic energy of the cylinder about its center of mass.
    - The $½ Mv^2$ represents the translational kinetic energy of the cylinder about its center of mass.

# Kinetic Energy of Rolling

$$E_{\text{mech}} = K_{\text{cm}} + K_{\text{rot}} + U_g$$
$$= \tfrac{1}{2} M v_{\text{cm}}{}^2 + \tfrac{1}{2} I\omega^2 + Mgy_{\text{cm}}$$

**Trick:** Instead of treating the rotation and translation separately, combine them by considering that instantaneously the system is rotating **abut the point of contact**.




$$E_{\text{mech}} = K_{\text{rot,P}} = \tfrac{1}{2} I_{\text{P}}\omega^2$$

$$I_{\text{P}} = \tfrac{1}{2} MR^2 + MR^2 = \tfrac{3}{2} MR^2$$

$$K_{\text{rot,P}} = \tfrac{1}{2}(\tfrac{3}{2} MR^2)\omega^2 = \tfrac{3}{4} MR^2\omega^2 = \tfrac{3}{4} Mv^2$$

# Conservation of Energy

The total kinetic energy of a rolling object is the sum of its linear and rotational kinetic energies:

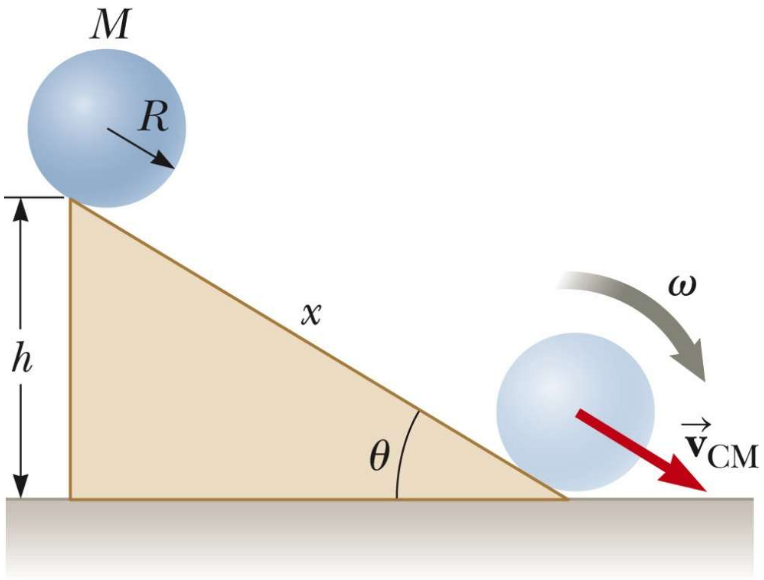
$$K = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

$$= \frac{1}{2}mv^2 + \frac{1}{2}I\left(\frac{v}{r}\right)^2 = \frac{1}{2}mv^2\left(1 + \frac{I}{mr^2}\right)$$

The second equation makes it clear that the kinetic energy of a rolling object is a **multiple** of the kinetic energy of translation.

## *Example* 10.13 Sphere Rolling Down an Incline

- Accelerated rolling motion is possible only if friction is present between the sphere and the incline.
  - The friction produces the net torque required for rotation.
  - No loss of mechanical energy occurs because the contact point is at rest relative to the surface at any instant.
  - In reality, rolling friction causes mechanical energy to transform to internal energy.
    - Rolling friction is due to deformations of the surface and the rolling object.



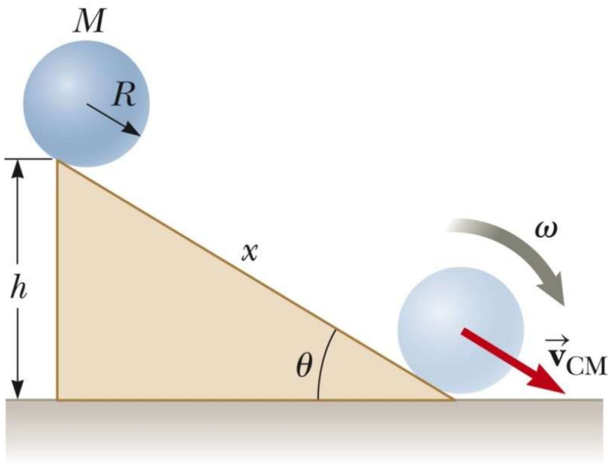
$$K = \tfrac{1}{2} I_{CM}\omega^2 + \tfrac{1}{2} M v_{CM}^2$$

$$K = \tfrac{1}{2} I_{CM}\left(\frac{v_{CM}}{R}\right)^2 + \tfrac{1}{2} M v_{CM}^2 = \tfrac{1}{2}\left(\frac{I_{CM}}{R^2} + M\right) v_{CM}^2$$

$$K_f + U_f = K_i + U_i$$

$$\tfrac{1}{2}\left(\frac{I_{CM}}{R^2} + M\right) v_{CM}^2 + 0 = 0 + Mgh$$

$$v_{CM} = \left[\frac{2gh}{1 + (I_{CM}/MR^2)}\right]^{1/2}$$




For the solid sphere shown in Active Figure 10.26, calculate the translational speed of the center of mass at the bottom of the incline and the magnitude of the translational acceleration of the center of mass.

$$(1) \quad v_{CM} = \left[\frac{2gh}{1 + (\frac{2}{5}MR^2/MR^2)}\right]^{1/2} = (\tfrac{10}{7}gh)^{1/2}$$

$$v_{CM}^2 = \tfrac{10}{7} g x \sin\theta$$

$$v_{CM}^2 = 2a_{CM}x$$

$$a_{CM} = \tfrac{5}{7} g \sin\theta$$

# Summary

**Definitions**

The **angular position** of a rigid object is defined as the angle $\theta$ between a reference line attached to the object and a reference line fixed in space. The **angular displacement** of a particle moving in a circular path or a rigid object rotating about a fixed axis is $\Delta\theta \equiv \theta_f - \theta_i$.

The **instantaneous angular speed** of a particle moving in a circular path or of a rigid object rotating about a fixed axis is

$$\omega \equiv \frac{d\theta}{dt} \qquad (10.3)$$

The **instantaneous angular acceleration** of a particle moving in a circular path or of a rigid object rotating about a fixed axis is

$$\alpha \equiv \frac{d\omega}{dt} \qquad (10.5)$$

When a rigid object rotates about a fixed axis, every part of the object has the same angular speed and the same angular acceleration.

The **moment of inertia of a system of particles** is defined as

$$I \equiv \sum_i m_i r_i^2 \qquad (10.15)$$

where $m_i$ is the mass of the $i$th particle and $r_i$ is its distance from the rotation axis.

The magnitude of the **torque** associated with a force $\vec{\mathbf{F}}$ acting on an object at a distance $r$ from the rotation axis is

$$\tau = rF\sin\phi = Fd \qquad (10.19)$$

where $\phi$ is the angle between the position vector of the point of application of the force and the force vector, and $d$ is the moment arm of the force, which is the perpendicular distance from the rotation axis to the line of action of the force.

# Summary

## Concepts and Principles

When a rigid object rotates about a fixed axis, the angular position, angular speed, and angular acceleration are related to the translational position, translational speed, and translational acceleration through the relationships

$$s = r\theta \quad \text{(10.1a)}$$

$$v = r\omega \quad \text{(10.10)}$$

$$a_t = r\alpha \quad \text{(10.11)}$$

If a rigid object rotates about a fixed axis with angular speed $\omega$, its **rotational kinetic energy** can be written

$$K_R = \tfrac{1}{2} I \omega^2 \quad \text{(10.16)}$$

where $I$ is the moment of inertia of the object about the axis of rotation.

The **moment of inertia of a rigid object** is

$$I = \int r^2 \, dm \quad \text{(10.17)}$$

where $r$ is the distance from the mass element $dm$ to the axis of rotation.

The rate at which work is done by an external force in rotating a rigid object about a fixed axis, or the **power** delivered, is

$$P = \tau\omega \quad \text{(10.23)}$$

If work is done on a rigid object and the only result of the work is rotation about a fixed axis, the net work done by external forces in rotating the object equals the change in the rotational kinetic energy of the object:

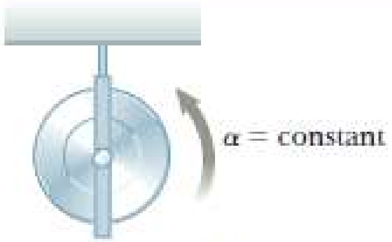
$$\sum W = \tfrac{1}{2} I \omega_f^2 - \tfrac{1}{2} I \omega_i^2 \quad \text{(10.24)}$$

The **total kinetic energy** of a rigid object rolling on a rough surface without slipping equals the rotational kinetic energy about its center of mass plus the translational kinetic energy of the center of mass:

$$K = \tfrac{1}{2} I_{\text{CM}} \omega^2 + \tfrac{1}{2} M v_{\text{CM}}^2 \quad \text{(10.28)}$$

# Summary

**Analysis Models for Problem Solving**



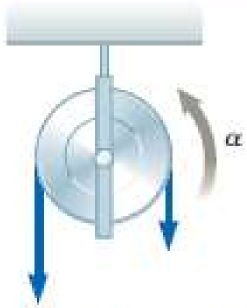

**Rigid Object Under Constant Angular Acceleration.** If a rigid object rotates about a fixed axis under constant angular acceleration, one can apply equations of kinematics that are analogous to those for translational motion of a particle under constant acceleration:

$$\omega_f = \omega_i + \alpha t \quad (10.6)$$

$$\theta_f = \theta_i + \omega_i t + \tfrac{1}{2}\alpha t^2 \quad (10.7)$$

$$\omega_f^2 = \omega_i^2 + 2\alpha(\theta_f - \theta_i) \quad (10.8)$$

$$\theta_f = \theta_i + \tfrac{1}{2}(\omega_i + \omega_f)t \quad (10.9)$$




**Rigid Object Under a Net Torque.** If a rigid object free to rotate about a fixed axis has a net external torque acting on it, the object undergoes an angular acceleration $\alpha$, where

$$\sum \tau_{ext} = I\alpha \quad (10.21)$$

This equation is the rotational analog to Newton's second law in the particle under a net force model.

# Intended Learning Outcomes

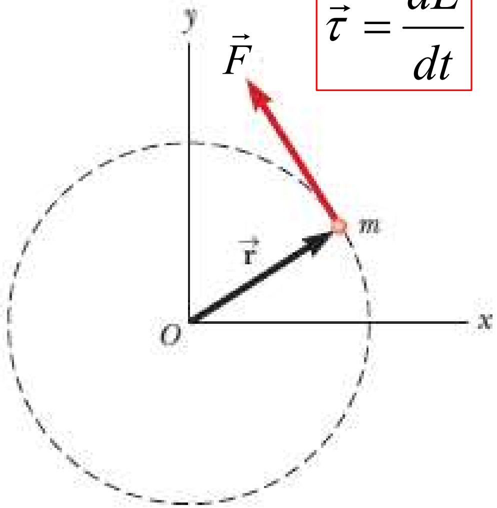
- To apply the concept of angular momentum in isolated and non-isolated systems

# Newton's second law in terms of angular momentum

$$\vec{\tau} = \frac{d\vec{L}}{dt}$$

- Consider a force is applied on a particle of mass *m* located at the vector position for rotational motion (The force actually provides a torque on the mass for the rotation).

$$\vec{\tau} = \vec{r} \times \vec{F} = \vec{r} \times \frac{d\vec{p}}{dt}$$



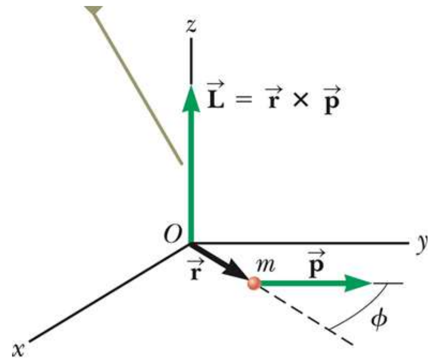

- Adding a term $\frac{d\vec{r}}{dt} \times \vec{p} = 0$ to the above equation,

$$\vec{\tau} = \vec{r} \times \frac{d\vec{p}}{dt} + \frac{d\vec{r}}{dt} \times \vec{p} = \frac{d}{dt}(\vec{r} \times \vec{p}) = \frac{d\vec{L}}{dt}$$

- where $\vec{L} \equiv \vec{r} \times \vec{p}$ is defined as the angular momentum.
- The SI units of angular momentum are (kg·m$^2$)/ s.

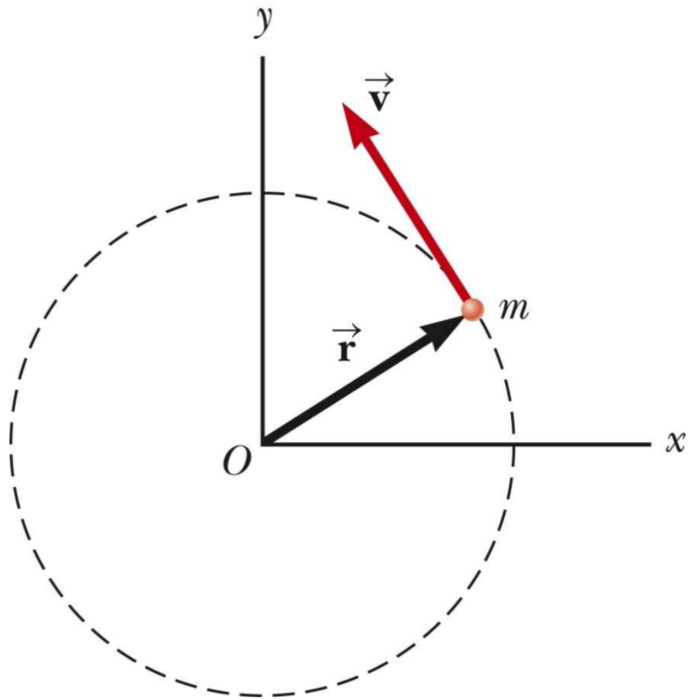
$$L = \left|\vec{L}\right| = rp\sin\phi$$

- Both the magnitude and direction of the angular momentum depend on the choice of origin.



## *Example* 11.3 Angular Momentum of a Particle in Circular Motion

A particle moves in the *xy* plane in a circular path of radius *r* as shown in Figure 11.5. Find the magnitude and direction of its angular momentum relative to an axis through *O* when its velocity is $\vec{\mathbf{v}}$.




- The vector $\vec{L} \equiv \vec{r} \times \vec{p}$ is pointed out of the diagram.
- The magnitude is $L = rp \sin 90^o = mvr$
  - $\sin 90^o$ is used since $v$ is perpendicular to $r$.
- A particle in uniform circular motion has a constant angular momentum about an axis through the center of its path.

**Example 11.4** **A System of Objects**

A sphere of mass $m_1$ and a block of mass $m_2$ are connected by a light cord that passes over a pulley as shown in Figure 11.6. The radius of the pulley is $R$, and the mass of the thin rim is $M$. The spokes of the pulley have negligible mass. The block slides on a frictionless, horizontal surface. Find an expression for the linear acceleration of the two objects, using the concepts of angular momentum and torque.

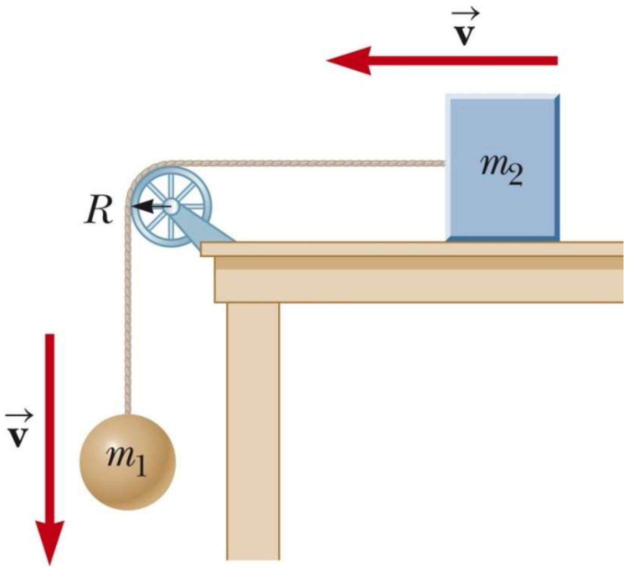
1. Using the torque-angular momentum approach:

$$\tau_{ext} = \frac{dL}{dt}$$

$$m_1 gR = \frac{d}{dt}\left( m_1 vR + m_2 vR + MR^2 \frac{v}{R} \right)$$

$$m_1 gR = \left( m_1 + m_2 + M \right) R \frac{dv}{dt}$$

$$a = \frac{m_1 g}{\left( m_1 + m_2 + M \right)}$$



2. Using the torque-angular acceleration approach:

For $m_1$:

$$m_1 g - T_1 = m_1 a$$

For pulley:

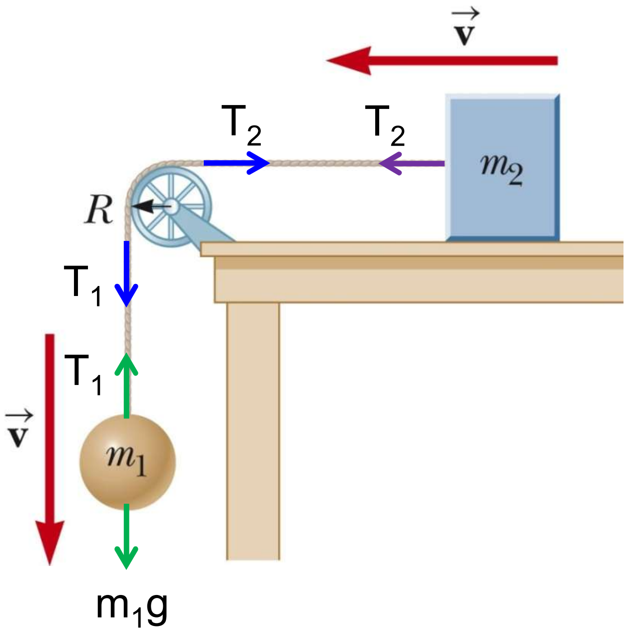
$$T_1 R - T_2 R = \left(MR^2\right)\alpha$$

For $m_2$:

$$T_2 = m_2 a$$

$$\left(m_1 g - m_1 a\right)R - m_2 a R = \left(MR^2\right)\frac{a}{R}$$

$$a = \frac{m_1 g}{\left(m_1 + m_2 + M\right)}$$

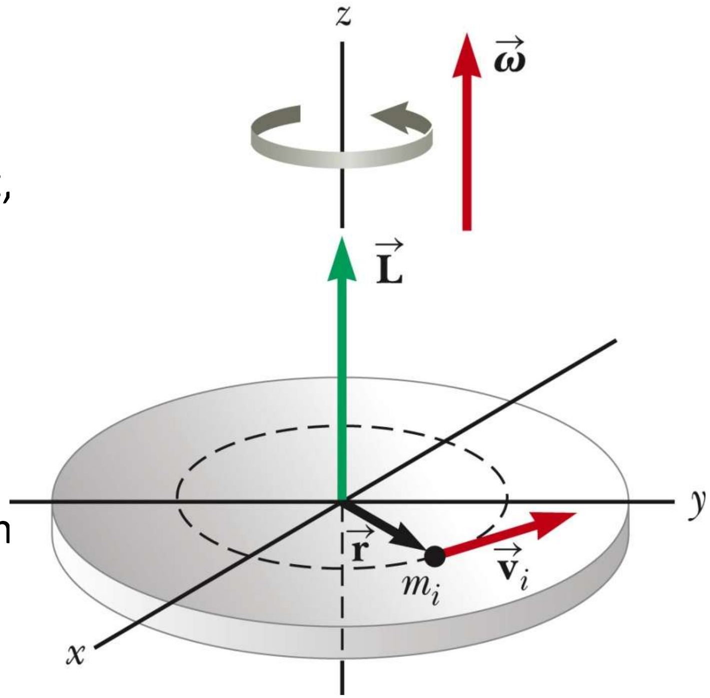


# Angular Momentum of a Rotating Rigid Object

- Consider an rigid object rotating along an axis with angular velocity $\vec{\omega}$.
- To calculate the angular momentum of the rigid object, we divide it into many small particles.
- The angular momentum of an individual particle is

$$L_i = r_i p_i = r_i m_i v_i = r_i m_i r_i \omega_i = m_i r_i^2 \omega$$

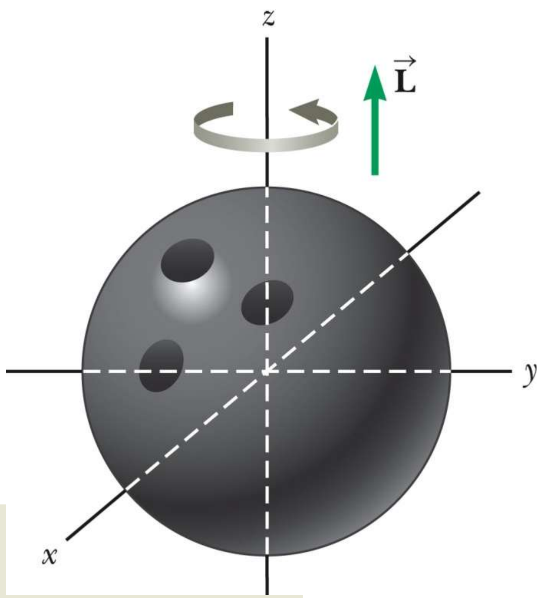
- $\vec{L}_i$ and $\vec{\omega}$ are directed along the *z* axis.
- The total angular momentum of the rigid object is then




$$L_z = \sum_i L_i = \sum_i m_i r_i^2 \omega = \left( \sum_i m_i r_i^2 \right) \omega = I\omega$$

*Example* **11.5** **Bowling Ball**

Estimate the magnitude of the angular momentum of a bowling ball with mass of 7.0 kg and a radius of 12 cm which is spinning at 10 rev/s as shown.




$$I = \frac{2}{5}MR^2 = \frac{2}{5}(7.0\ \mathrm{kg})(0.12\ \mathrm{m})^2 = 0.040\ \mathrm{kg \cdot m^2}$$

$$L_z = I\omega = (0.040\ \mathrm{kg \cdot m^2})(10\ \mathrm{rev/s})(2\pi\ \mathrm{rad/rev}) = 2.53\ \mathrm{kg \cdot m^2/s}$$

# Conservation of Angular Momentum

- Similar to the linear case with $\vec{F}_{ext} = \frac{d\vec{p}}{dt}$ , we have conservation of linear
- momentum ($\vec{p}$) when there is no net force acting on the system.
- In angular case with $\tau_{ext} = \frac{dL}{dt}$ , we have conservation of angular momentum ($\vec{L}$ )
- when there is no net torque acting on the system.

$\boxed{\vec{L} \text{ is conserved if there is no net torque acting on the system.}}$

# Conservation of Angular Momentum, cont

- If the system is deformable such that the mass of the isolated system undergoes redistribution, the moment of inertia changes.
  - The conservation of angular momentum requires a compensating change in the angular velocity.
  - $I_i \omega_i = I_f \omega_f$ = constant
    - This holds for rotation about a fixed axis and for rotation about an axis through the center of mass of a moving system.
    - The net torque must be zero in any case.

When his arms and legs are close to his body, the skater's moment of inertia is small and his angular speed is large.

To slow down for the finish of his spin, the skater moves his arms and legs outward, increasing his moment of inertia.

# Conservation Law Summary

- For an isolated system -

(1) Conservation of Energy:

- $E_i = E_f$
- If there is no energy transfers across the system boundary

(2) Conservation of Linear Momentum:

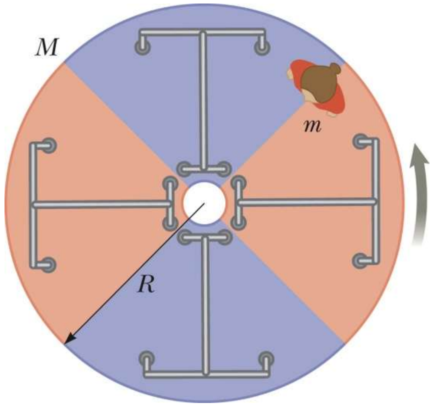
- $\vec{\mathbf{p}}_i = \vec{\mathbf{p}}_f$
- If the net external force on the system is zero

(3) Conservation of Angular Momentum:

- $\vec{\mathbf{L}}_i = \vec{\mathbf{L}}_f$
- If the net external torque on the system is zero

## *Example* 11.8 The Merry-Go-Round

A horizontal platform in the shape of a circular disk rotates freely in a horizontal plane about a frictionless, vertical axle (Fig. 11.11). The platform has a mass $M =$ 100 kg and a radius $R = 2.0$ m. A student whose mass is $m = 60$ kg walks slowly from the rim of the disk toward its center. If the angular speed of the system is 2.0 rad/s when the student is at the rim, what is the angular speed when she reaches a point $r = 0.50$ m from the center?




$$I_i = I_{pi} + I_{si} = \tfrac{1}{2}MR^2 + mR^2 \qquad I_f = I_{pf} + I_{sf} = \tfrac{1}{2}MR^2 + mr^2$$

$$I_i\omega_i = I_f\omega_f$$

$$(\tfrac{1}{2}MR^2 + mR^2)\omega_i = (\tfrac{1}{2}MR^2 + mr^2)\omega_f$$

$$\omega_f = \left(\frac{\tfrac{1}{2}MR^2 + mR^2}{\tfrac{1}{2}MR^2 + mr^2}\right)\omega_i$$

$$\omega_f = \left[\frac{\tfrac{1}{2}(100\text{ kg})(2.0\text{ m})^2 + (60\text{ kg})(2.0\text{ m})^2}{\tfrac{1}{2}(100\text{ kg})(2.0\text{ m})^2 + (60\text{ kg})(0.50\text{ m})^2}\right](2.0\text{ rad/s}) = \boxed{4.1\text{ rad/s}}$$

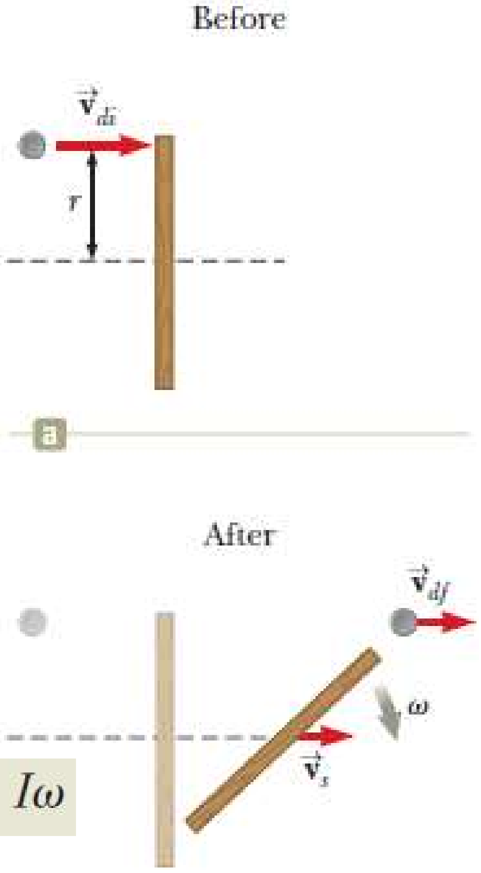
**WHAT IF?** What if you measured the kinetic energy of the system before and after the student walks inward? Are the initial kinetic energy and the final kinetic energy the same?

$$K_i = \tfrac{1}{2}I_i\omega_i^2 = \tfrac{1}{2}(440\text{ kg}\cdot\text{m}^2)(2.0\text{ rad/s})^2 = 880\text{ J}$$

$$K_f = \tfrac{1}{2}I_f\omega_f^2 = \tfrac{1}{2}(215\text{ kg}\cdot\text{m}^2)(4.1\text{ rad/s})^2 = 1.80 \times 10^3\text{ J}$$

## *Example* 11.9 Disk and Stick Collision

A 2.0-kg disk traveling at 3.0 m/s strikes a 1.0-kg stick of length 4.0 m that is lying flat on nearly frictionless ice as shown in the overhead view of Figure 11.12a. The disk strikes at the endpoint of the stick, at a distance $r$ = 2.0 m from the stick's center. Assume the collision is elastic and the disk does not deviate from its original line of motion. Find the translational speed of the disk, the translational speed of the stick, and the angular speed of the stick after the collision. The moment of inertia of the stick about its center of mass is 1.33 kg · m².




Conservation of linear momentum:

$$m_d v_{di} = m_d v_{df} + m_s v_s \Rightarrow (1) \quad m_d(v_{di} - v_{df}) = m_s v_s$$

Conservation of angular momentum:

$$-r m_d v_{di} = -r m_d v_{df} + I\omega \Rightarrow (2) \quad -r m_d(v_{di} - v_{df}) = I\omega$$

Conservation of energy:

$$\tfrac{1}{2} m_d v_{di}^2 = \tfrac{1}{2} m_d v_{df}^2 + \tfrac{1}{2} m_s v_s^2 + \tfrac{1}{2} I\omega^2$$

$$\Rightarrow (3) \quad m_d(v_{di} - v_{df})(v_{di} + v_{df}) = m_s v_s^2 + I\omega^2$$

$$(1) \quad m_d(v_{di} - v_{df}) = m_s v_s \qquad (2) \quad -r m_d(v_{di} - v_{df}) = I\omega$$

$$(3) \quad m_d(v_{di} - v_{df})(v_{di} + v_{df}) = m_s v_s^2 + I\omega^2$$

Multiply Equation (1) by $r$ and add to Equation (2):

$$0 = r m_s v_s + I\omega$$

$$(4) \quad \omega = -\frac{r m_s v_s}{I}$$

Divide Equation (3) by Equation (1):

$$\frac{m_d(v_{di} - v_{df})(v_{di} + v_{df})}{m_d(v_{di} - v_{df})} = \frac{m_s v_s^2 + I\omega^2}{m_s v_s}$$

$$(6) \quad v_{di} + v_{df} = v_s\left(1 + \frac{r^2 m_s}{I}\right)$$

Substitute $v_{df}$ from Equation (1) into Equation (6):

$$v_{di} + \left(v_{di} - \frac{m_s}{m_d} v_s\right) = v_s\left(1 + \frac{r^2 m_s}{I}\right)$$

$$v_s = \frac{2v_{di}}{1 + (m_s/m_d) + (r^2 m_s/I)}$$

$$= \frac{2(3.0\ \text{m/s})}{1 + (1.0\ \text{kg}/2.0\ \text{kg}) + [(2.0\ \text{m})^2(1.0\ \text{kg})/1.33\ \text{kg}\cdot\text{m}^2]} = \boxed{1.3\ \text{m/s}}$$

$$\omega = -\frac{(2.0\ \text{m})(1.0\ \text{kg})(1.3\ \text{m/s})}{1.33\ \text{kg}\cdot\text{m}^2} = \boxed{-2.0\ \text{rad/s}}$$

$$v_{df} = v_{di} - \frac{m_s}{m_d} v_s = 3.0\ \text{m/s} - \frac{1.0\ \text{kg}}{2.0\ \text{kg}}(1.3\ \text{m/s}) = \boxed{2.3\ \text{m/s}}$$

**TABLE 11.1** *Comparison of Values in Example 11.9 Before and After the Collision*

| | $v$ (m/s) | $\omega$ (rad/s) | $p$ (kg · m/s) | $L$ (kg · $m^2$/s) | $K_{trans}$ (J) | $K_{rot}$ (J) |
|---|---|---|---|---|---|---|
| **Before** | | | | | | |
| Disk | 3.0 | — | 6.0 | −12 | 9.0 | — |
| Stick | 0 | 0 | 0 | 0 | 0 | 0 |
| Total for system | — | — | 6.0 | −12 | 9.0 | 0 |
| **After** | | | | | | |
| Disk | 2.3 | — | 4.7 | −9.3 | 5.4 | — |
| Stick | 1.3 | −2.0 | 1.3 | −2.7 | 0.9 | 2.7 |
| Total for system | — | — | 6.0 | −12 | 6.3 | 2.7 |

*Note:* Linear momentum, angular momentum, and total kinetic energy of the system are all conserved.

# Summary

## Definitions

Given two vectors $\vec{A}$ and $\vec{B}$, the **vector product** $\vec{A} \times \vec{B}$ is a vector $\vec{C}$ having a magnitude

$$C = AB\sin\theta \qquad (11.3)$$

where $\theta$ is the angle between $\vec{A}$ and $\vec{B}$. The direction of the vector $\vec{C} = \vec{A} \times \vec{B}$ is perpendicular to the plane formed by $\vec{A}$ and $\vec{B}$, and this direction is determined by the right-hand rule.

The **torque** $\vec{\tau}$ due to a force $\vec{F}$ about an axis through the origin in an inertial frame is defined to be

$$\vec{\tau} \equiv \vec{r} \times \vec{F} \qquad (11.1)$$

The **angular momentum** $\vec{L}$ about an axis through the origin of a particle having linear momentum $\vec{p} = m\vec{v}$ is

$$\vec{L} \equiv \vec{r} \times \vec{p} \qquad (11.10)$$

where $\vec{r}$ is the vector position of the particle relative to the origin.

## Concepts and Principles

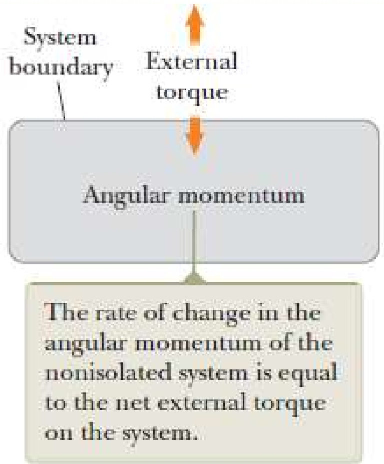
The $z$ component of angular momentum of a rigid object rotating about a fixed $z$ axis is

$$L_z = I\omega \qquad (11.14)$$

where $I$ is the moment of inertia of the object about the axis of rotation and $\omega$ is its angular speed.

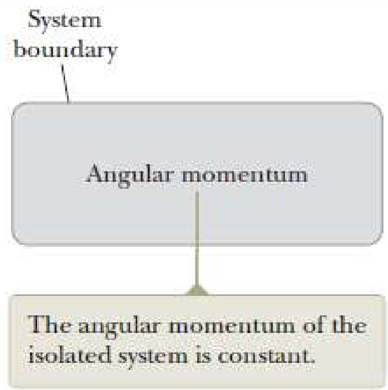
# Summary

## Analysis Models for Problem Solving




**Nonisolated System (Angular Momentum).** If a system interacts with its environment in the sense that there is an external torque on the system, the net external torque acting on a system is equal to the time rate of change of its angular momentum:

$$\sum \vec{\boldsymbol{\tau}}_{\text{ext}} = \frac{d\vec{\mathbf{L}}_{\text{tot}}}{dt} \qquad (11.13)$$




**Isolated System (Angular Momentum).** If a system experiences no external torque from the environment, the total angular momentum of the system is conserved:

$$\vec{\mathbf{L}}_i = \vec{\mathbf{L}}_f \qquad (11.18)$$

Applying this law of conservation of angular momentum to a system whose moment of inertia changes gives

$$I_i\omega_i = I_f\omega_f = \text{constant} \qquad (11.19)$$

# Static Equilibrium

# Intended Learning Outcomes

- To solve problems for a rigid body in static equilibrium

# Static Equilibrium

- Equilibrium implies that the object moves with both constant velocity and constant angular velocity relative to an observer in an inertial reference frame.
- In this course, we will deal with the special case in which both of these velocities are equal to zero
  - This is called *static equilibrium.*
- Static equilibrium is a common situation in engineering.
- The principles involved are of particular interest to civil engineers, architects, and mechanical engineers.

# Conditions for Equilibrium

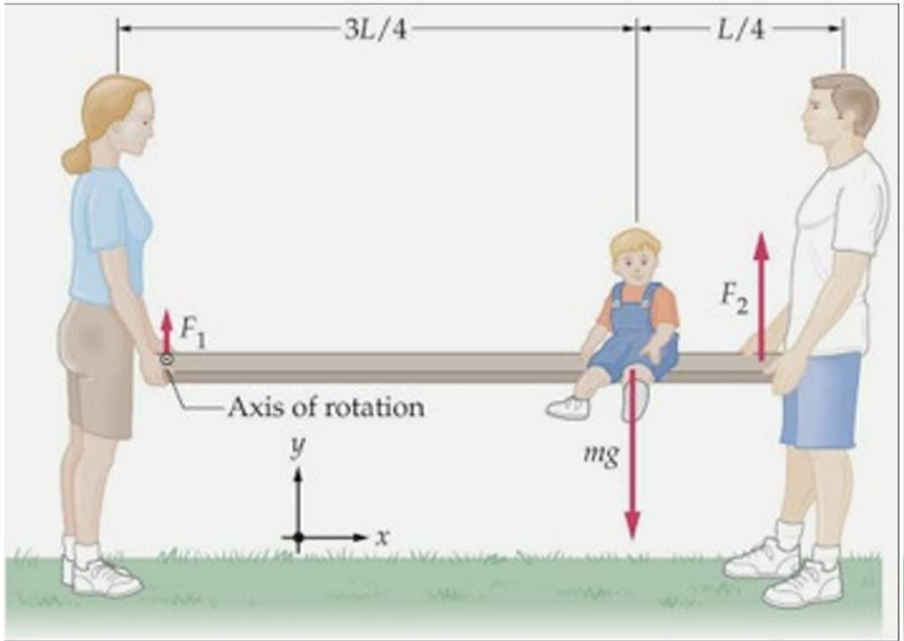
- The net external force on the object must equal zero.
  - $$\sum \vec{\mathbf{F}}_{ext} = 0$$
- The net external torque on the object about any axis must be zero.
  - $$\sum \vec{\tau}_{ext} = 0$$
- These conditions describe the **rigid object in equilibrium analysis model.**

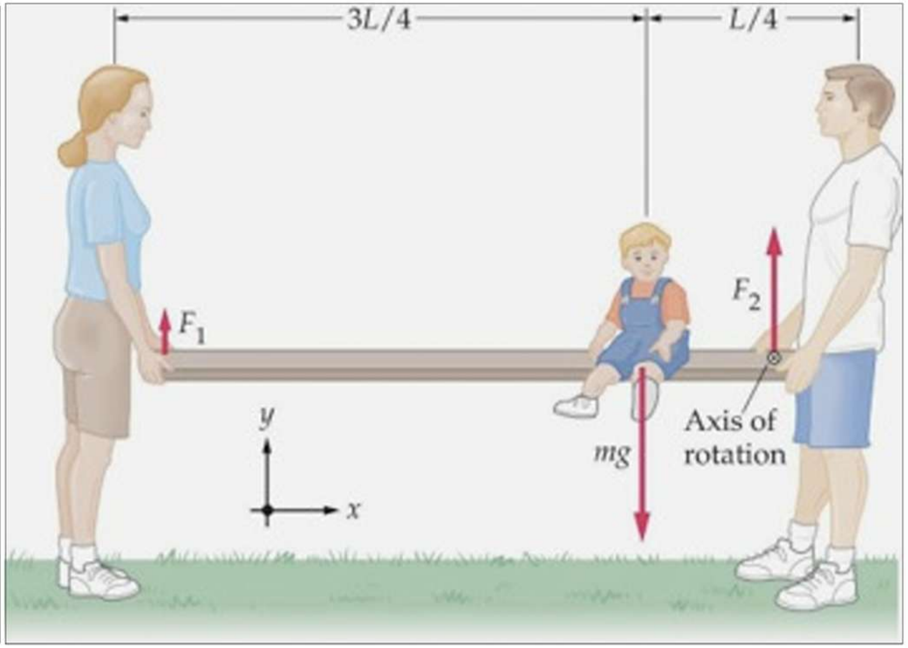
# Equilibrium Equations

- We will restrict the applications to situations in which all the forces lie in the xy plane.
  - These are called coplanar forces since they lie in the same plane.
  - This restriction results in three scalar equations.
- There are three resulting equations:
  - $\Sigma F_x = 0$
  - $\Sigma F_y = 0$
  - $\Sigma \tau_z = 0$
    - The location of the axis for the torque equation is arbitrary.

# Zero Torque & Static Equilibrium

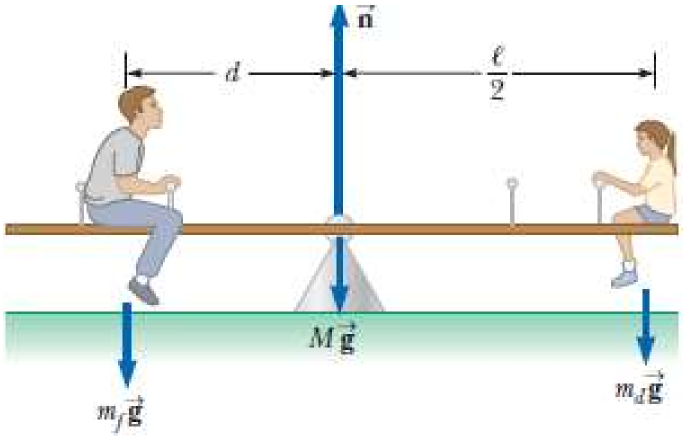
If the net torque is zero, it doesn't matter which axis we consider rotation to be around; we are free to choose the one that makes our calculations easiest.



**Example 12.1** The Seesaw Revisited

A seesaw consisting of a uniform board of mass $M$ and length $\ell$ supports at rest a father and daughter with masses $m_f$ and $m_d$, respectively, as shown in Figure 12.7. The support (called the *fulcrum*) is under the center of gravity of the board, the father is a distance $d$ from the center, and the daughter is a distance $\ell/2$ from the center.




**(A)** Determine the magnitude of the upward force $\vec{n}$ exerted by the support on the board.

$$n - m_f g - m_d g - Mg = 0$$
$$n = m_f g + m_d g + Mg = (m_f + m_d + M)g$$

**(B)** Determine where the father should sit to balance the system at rest.

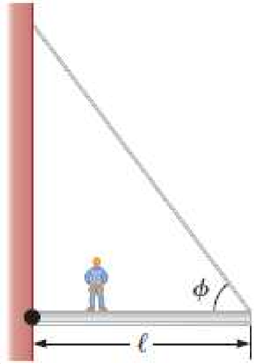
$$(m_f g)(d) - (m_d g)\frac{\ell}{2} = 0 \quad \Rightarrow \quad d = \left(\frac{m_d}{m_f}\right)\frac{\ell}{2}$$

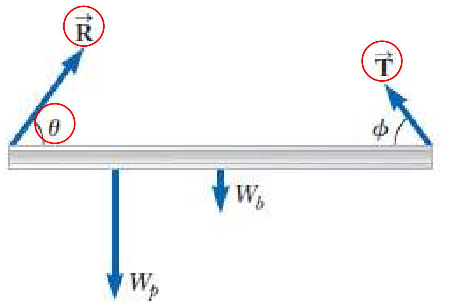
**WHAT IF?** Suppose we had chosen another point through which the rotation axis were to pass. For example, suppose the axis is perpendicular to the page and passes through the location of the father. Does that change the results to parts (A) and (B)?

$$n(d) - (Mg)(d) - (m_d g)\left(d + \frac{l}{2}\right) = (m_f + m_d + M)gd - Mgd - m_d g\left(d + \frac{l}{2}\right)$$
$$= m_f gd - m_d g\left(\frac{l}{2}\right) = m_f g\left(\frac{m_d}{m_f}\right)\left(\frac{l}{2}\right) - m_d g\left(\frac{l}{2}\right) = 0$$

## Example 12.2 Standing on a Horizontal Beam

A uniform horizontal beam with a length of $\ell$ = 8.00 m and a weight of $W_b$ = 200 N is attached to a wall by a pin connection. Its far end is supported by a cable that makes an angle of $\phi$ = 53.0° with the beam (Fig. 12.8a). A person of weight $W_p$ = 600 N stands a distance $d$ = 2.00 m from the wall. Find the tension in the cable as well as the magnitude and direction of the force exerted by the wall on the beam.



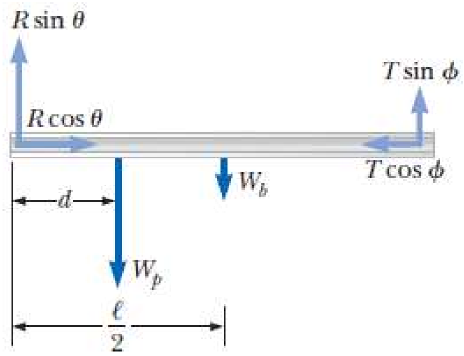

We have 3 unknowns, so we need 3 equations to solve them.

$$(1)\ \sum F_x = R\cos\theta - T\cos\phi = 0$$

$$(2)\ \sum F_y = R\sin\theta + T\sin\phi - W_p - W_b = 0$$

$$(3)\ \sum \tau_z = (T\sin\phi)(\ell) - W_p d - W_b\left(\frac{\ell}{2}\right) = 0$$

$$T = \frac{W_p d + W_b(\ell/2)}{\ell \sin\phi} = \frac{(600\text{ N})(2.00\text{ m}) + (200\text{ N})(4.00\text{ m})}{(8.00\text{ m})\sin 53.0°} = \boxed{313\text{ N}}$$

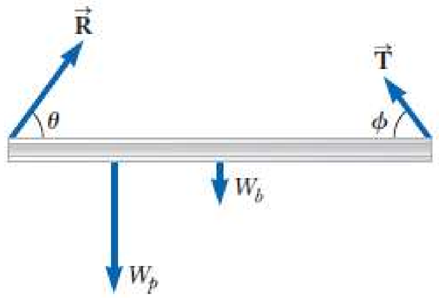


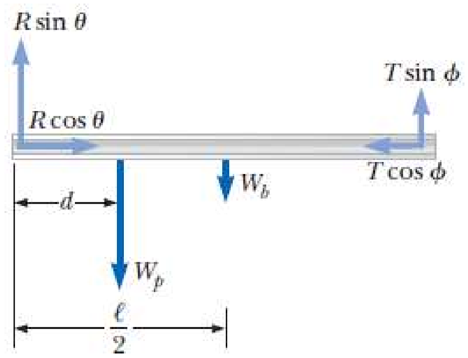
$$\frac{R\sin\theta}{R\cos\theta} = \tan\theta = \frac{W_p + W_b - T\sin\phi}{T\cos\phi}$$

$$\theta = \tan^{-1}\left(\frac{W_p + W_b - T\sin\phi}{T\cos\phi}\right)$$

$$= \tan^{-1}\left[\frac{600\text{ N} + 200\text{ N} - (313\text{ N})\sin 53.0^\circ}{(313\text{ N})\cos 53.0^\circ}\right] = \boxed{71.1^\circ}$$

$$R = \frac{T\cos\phi}{\cos\theta} = \frac{(313\text{ N})\cos 53.0^\circ}{\cos 71.1^\circ} = \boxed{581\text{ N}}$$




**WHAT IF?** What if the person walks farther out on the beam? Does $T$ change? Does $R$ change? Does $\theta$ change?

**Answer** $T$ must increase because the weight of the person exerts a larger torque about the pin connection, which must be countered by a larger torque in the opposite direction due to an increased value of $T$. If $T$ increases, the vertical component of $\vec{R}$ decreases to maintain force equilibrium in the vertical direction. Force equilibrium in the horizontal direction, however, requires an increased horizontal component of $\vec{R}$ to balance the horizontal component of the increased $\vec{T}$. This fact suggests that $\theta$ becomes smaller, but it is hard to predict what happens to $R$. Problem 66 asks you to explore the behavior of $R$.

**Example 12.3** **The Leaning Ladder**

A uniform ladder of length $\ell$ rests against a smooth, vertical wall (Fig. 12.9a). The mass of the ladder is $m$, and the coefficient of static friction between the ladder and the ground is $\mu_s = 0.40$. Find the minimum angle $\theta_{\min}$ at which the ladder does not slip.


(1) $\sum F_x = f_s - P = 0$

(2) $\sum F_y = n - mg = 0$

(3) $P = f_s$

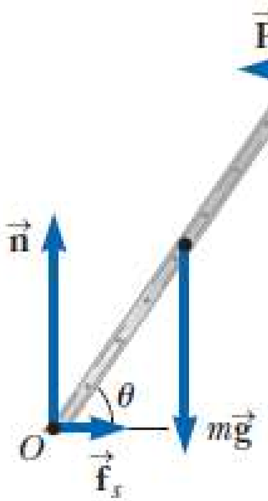
(4) $n = mg$

(5) $P = f_{s,\max} = \mu_s n = \mu_s mg$

$$\sum \tau_O = P\ell \sin\theta_{\min} - mg\frac{\ell}{2}\cos\theta_{\min} = 0$$

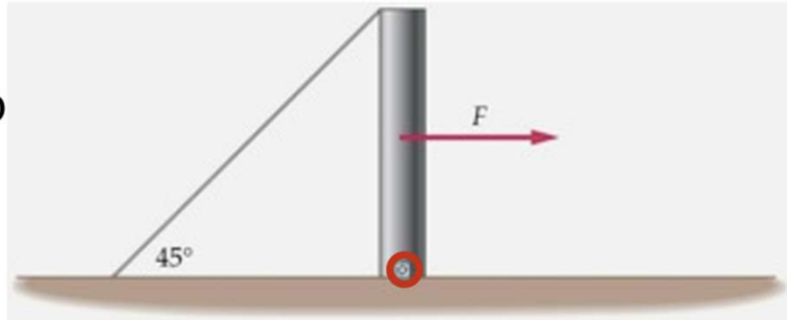
$$\frac{\sin\theta_{\min}}{\cos\theta_{\min}} = \tan\theta_{\min} = \frac{mg}{2P} = \frac{mg}{2\mu_s mg} = \frac{1}{2\mu_s}$$

$$\theta_{\min} = \tan^{-1}\left(\frac{1}{2\mu_s}\right) = \tan^{-1}\left[\frac{1}{2(0.40)}\right] = 51^\circ$$



# Example: Rod and Wire

A rigid vertical rod of length $L$ and negligible mass is connected to the floor by a bolt through its lower end. The rod also has a 45° wire connecting its top end to the floor.




If a horizontal force $F$ is applied to the center of the rod, find the tension in the wire.

Find the horizontal and vertical components of the force exerted by the bold on the rod.

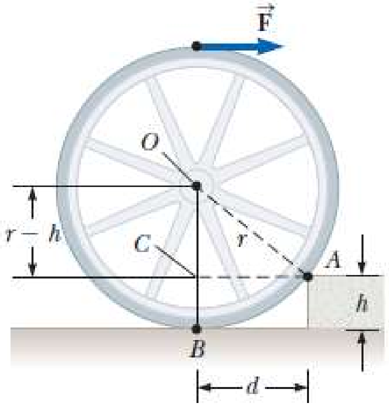
$$\tau = 0 = F(L/2) - T\,\mathrm{Cos}\theta\, L; \quad T = \frac{F}{2\,\mathrm{Cos}\theta} = F/\sqrt{2}$$

$$\sum F_x = 0 = F + f_x - T\,\mathrm{Cos}\theta; \quad f_x = -F/2$$

$$\sum F_y = 0 = f_y - T\,\mathrm{Sin}\theta; \quad f_y = F/2$$

## *Example* 12.4 Negotiating a Curb

(A) Estimate the magnitude of the force $\boldsymbol{F}$ a person must apply to a wheelchair's main wheel to roll up over a sidewalk curb. This main wheel that comes in contact with the curb has a radius $r = 30$ cm, and the height of the curb is $h = 10$ cm. Assume that the combined weight for the person and the wheelchair is $mg = 1400$ N.




Taking torques about an axis through A:

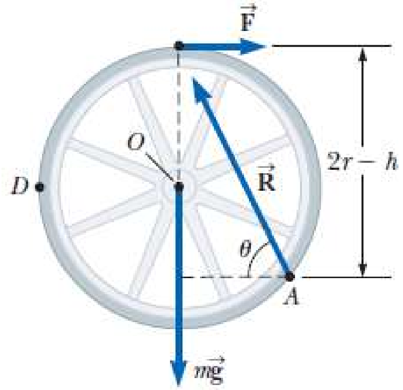
$$\sum \tau_A = mgd - F(2r - h) = 0$$

$$F = \frac{mg\sqrt{2rh - h^2}}{2r - h}$$

$$F = \frac{(700\text{ N})\sqrt{2(0.3\text{ m})(0.1\text{ m}) - (0.1\text{ m})^2}}{2(0.3\text{ m}) - 0.1\text{ m}}$$

$$= 3 \times 10^2\text{ N}$$

How large is this force?



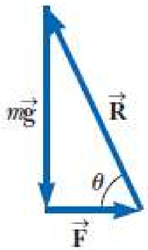

$$d = \sqrt{r^2 - (r - h)^2} = \sqrt{2rh - h^2}$$

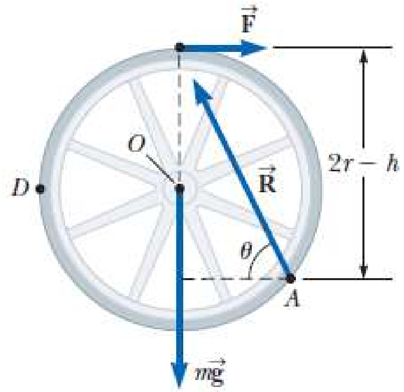
**(B)** Determine the magnitude and direction of $\vec{\mathbf{R}}$.

Net force = 0

$$\sum F_x = F - R\cos\theta = 0$$
$$\sum F_y = R\sin\theta - mg = 0$$

$$\frac{R\sin\theta}{R\cos\theta} = \tan\theta = \frac{mg}{F}$$




$$\theta = \tan^{-1}\left(\frac{mg}{F}\right) = \tan^{-1}\left(\frac{700\text{ N}}{300\text{ N}}\right) = \boxed{70^\circ}$$
$$R = \frac{mg}{\sin\theta} = \frac{700\text{ N}}{\sin 70^\circ} = \boxed{8 \times 10^2\text{ N}}$$

# Summary

## Analysis Models for Problem Solving

**Rigid Object in Equilibrium** A rigid object in equilibrium exhibits no translational or angular acceleration. The net external force acting on it is zero, and the net external torque on it is zero about any axis:

$$\sum \vec{\mathbf{F}}_{\text{ext}} = 0 \qquad (12.1)$$

$$\sum \vec{\boldsymbol{\tau}}_{\text{ext}} = 0 \qquad (12.2)$$

The first condition is the condition for translational equilibrium, and the second is the condition for rotational equilibrium.