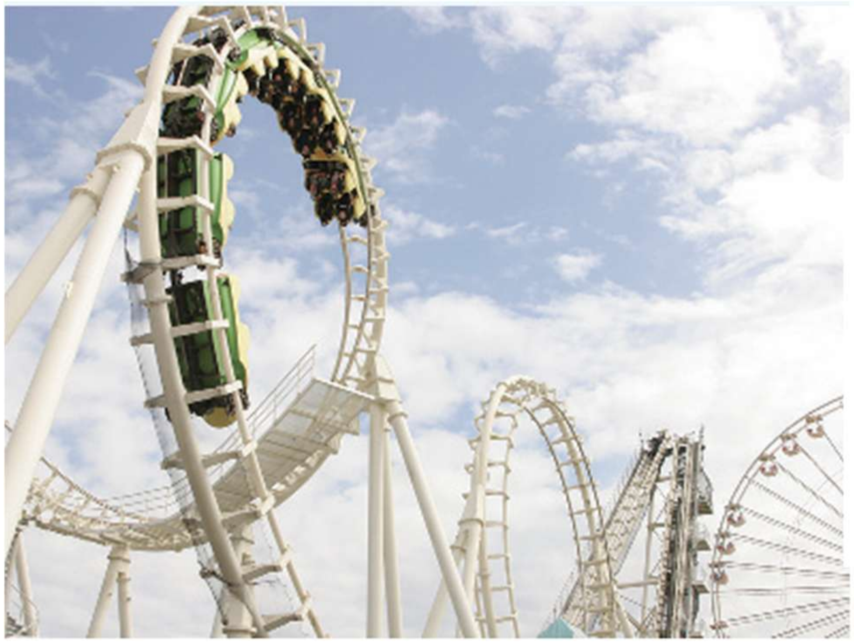
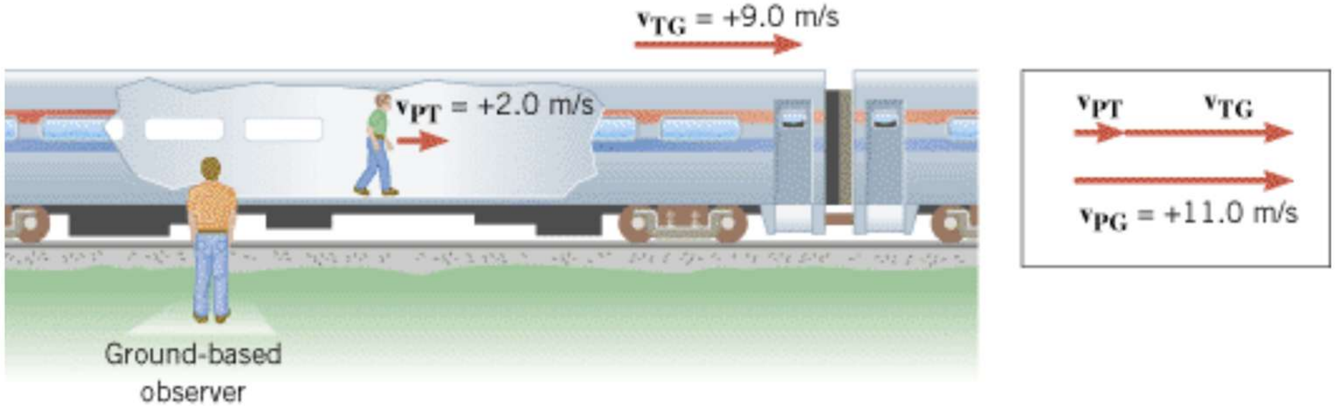
# Centripetal Acceleration and relative velocity





# Uniform Circular Motion

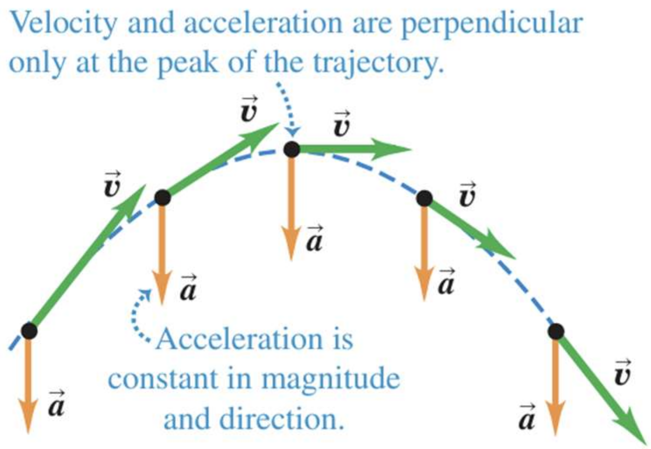
•***Uniform circular motion*** occurs when an object moves in a circular path with a *constant speed*.

•An acceleration exists since the *direction* of the motion is changing .

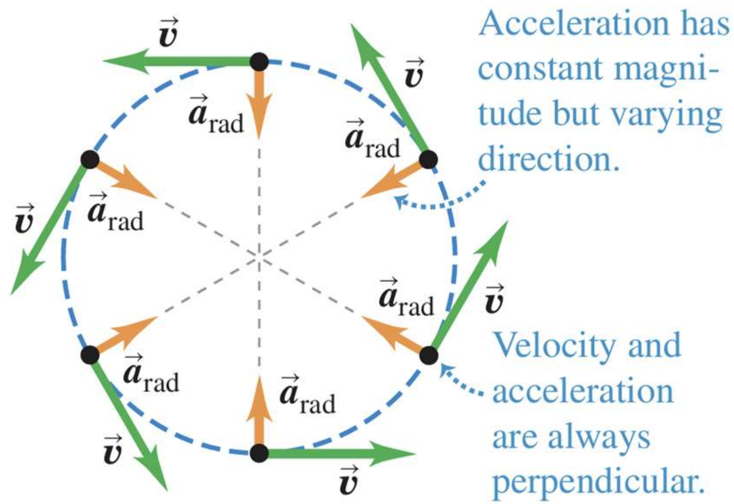
- This change in velocity is related to an acceleration.

•The constant-magnitude velocity vector is always tangent to the path of the object.

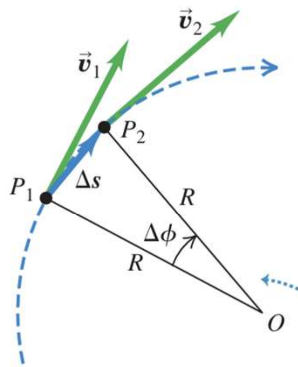
***Projectile motion***



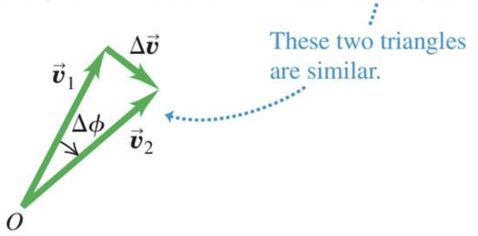

***Uniform circular motion***



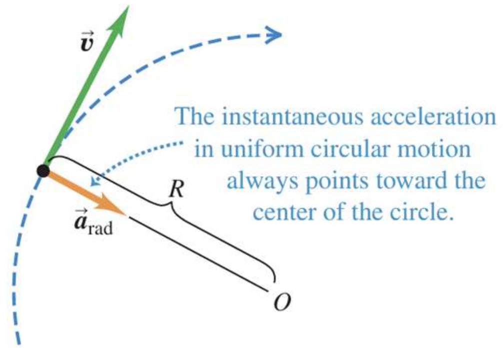
(a) A particle moves a distance $\Delta s$ at constant speed along a circular path.




(b) The corresponding change in velocity $\Delta\vec{v}$. The average acceleration is in the same direction as $\Delta\vec{v}$.




(c) The instantaneous acceleration




$$\frac{|\Delta\vec{v}|}{v_1} = \frac{\Delta s}{R} \qquad \text{or} \qquad |\Delta\vec{v}| = \frac{v_1}{R}\Delta s$$

The magnitude $a_{av}$ of the average acceleration during $\Delta t$ is therefore

$$a_{av} = \frac{|\Delta\vec{v}|}{\Delta t} = \frac{v_1}{R}\frac{\Delta s}{\Delta t}$$

The magnitude $a$ of the *instantaneous* acceleration $\vec{a}$ at point $P_1$ is the limit of this expression as we take point $P_2$ closer and closer to point $P_1$:

$$a = \lim_{\Delta t \to 0} \frac{v_1}{R}\frac{\Delta s}{\Delta t} = \frac{v_1}{R} \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}$$

If the time interval $\Delta t$ is short, $\Delta s$ is the distance the particle moves along its curved path. So the limit of $\Delta s/\Delta t$ is the speed $v_1$ at point $P_1$. Also, $P_1$ can be any point on the path, so we can drop the subscript and let $v$ represent the speed at any point. Then

Magnitude of acceleration of an object in **uniform circular motion**

$$a_{rad} = \frac{v^2}{R}$$

Speed of object

Radius of object's circular path

# Uniform Circular Motion: Centripetal Acceleration

A particle undergoes uniform circular motion (I.e. moving with a constant speed *v* in a circular path of radius *r*) needs an acceleration with magnitude given by

$$a_c = \frac{v^2}{r}$$

which always pointed towards the center of the circular path.

Such acceleration is called the ***Centripetal Acceleration***.

In vector form, it is expressed as:

$$\vec{a}_c = -\frac{v^2}{r}\hat{r}$$

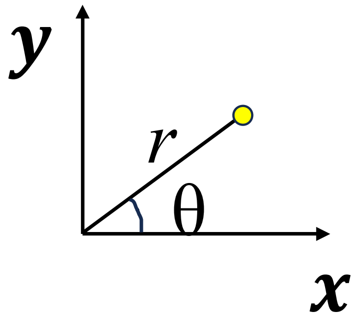
where $\hat{r}$ is a unit vector pointing radially outward.

# Circular Motion: Polar Coordinates

Denote any position A by:

- r: distance to origin
- θ: angle of $r$ from the reference direction ($x$ axis, typically)

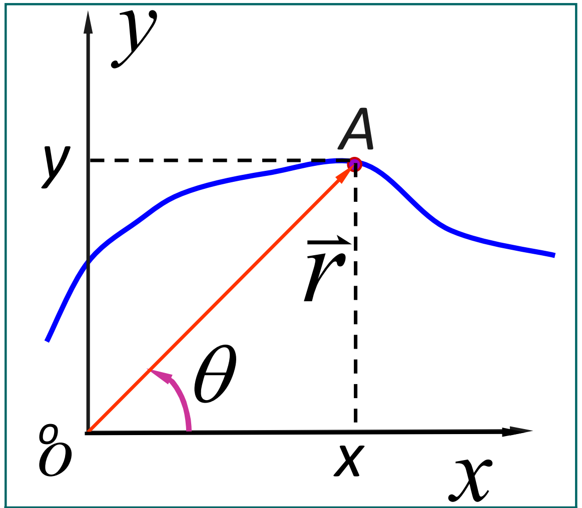
<u>Polar ($r$, θ)</u>

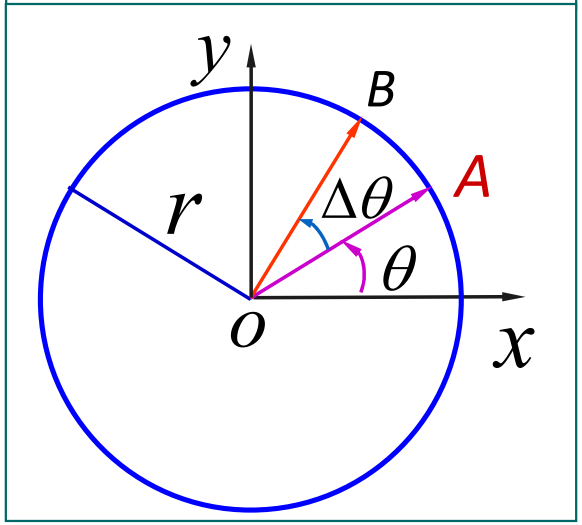



Conversion to Cartesian:

$$\boxed{x = r \cos \theta \qquad y = r \sin \theta}$$

Suitable for problems with rotational symmetry



# Centripetal Acceleration: Derivation Using Calculus

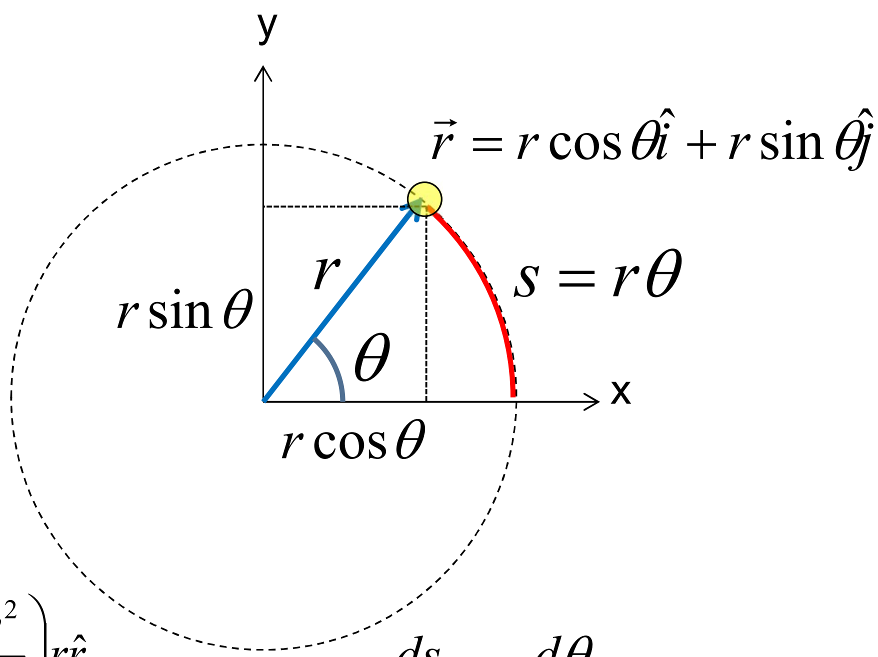
$$\vec{r} = r\cos\theta\hat{i} + r\sin\theta\hat{j}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = -r(\sin\theta)\omega\hat{i} + r(\cos\theta)\omega\hat{j}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = -r(\cos\theta)\omega^2\hat{i} - r(\sin\theta)\omega^2\hat{j}$$

$$= -\omega^2\left(r\cos\theta\hat{i} + r\sin\theta\hat{j}\right) = -\omega^2\vec{r} = -\left(\frac{v^2}{r^2}\right)r\hat{r}$$

$$= -\frac{v^2}{r}\hat{r}$$

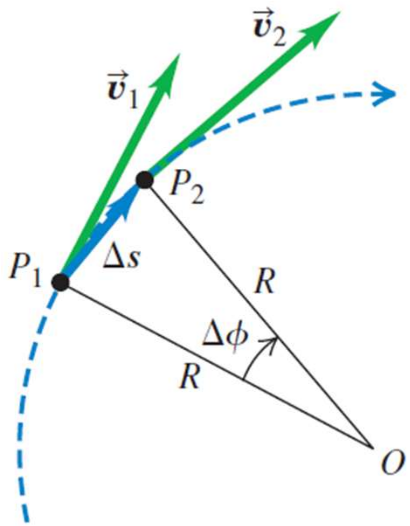



$$v = \frac{ds}{dt} = r\frac{d\theta}{dt} = r\omega$$

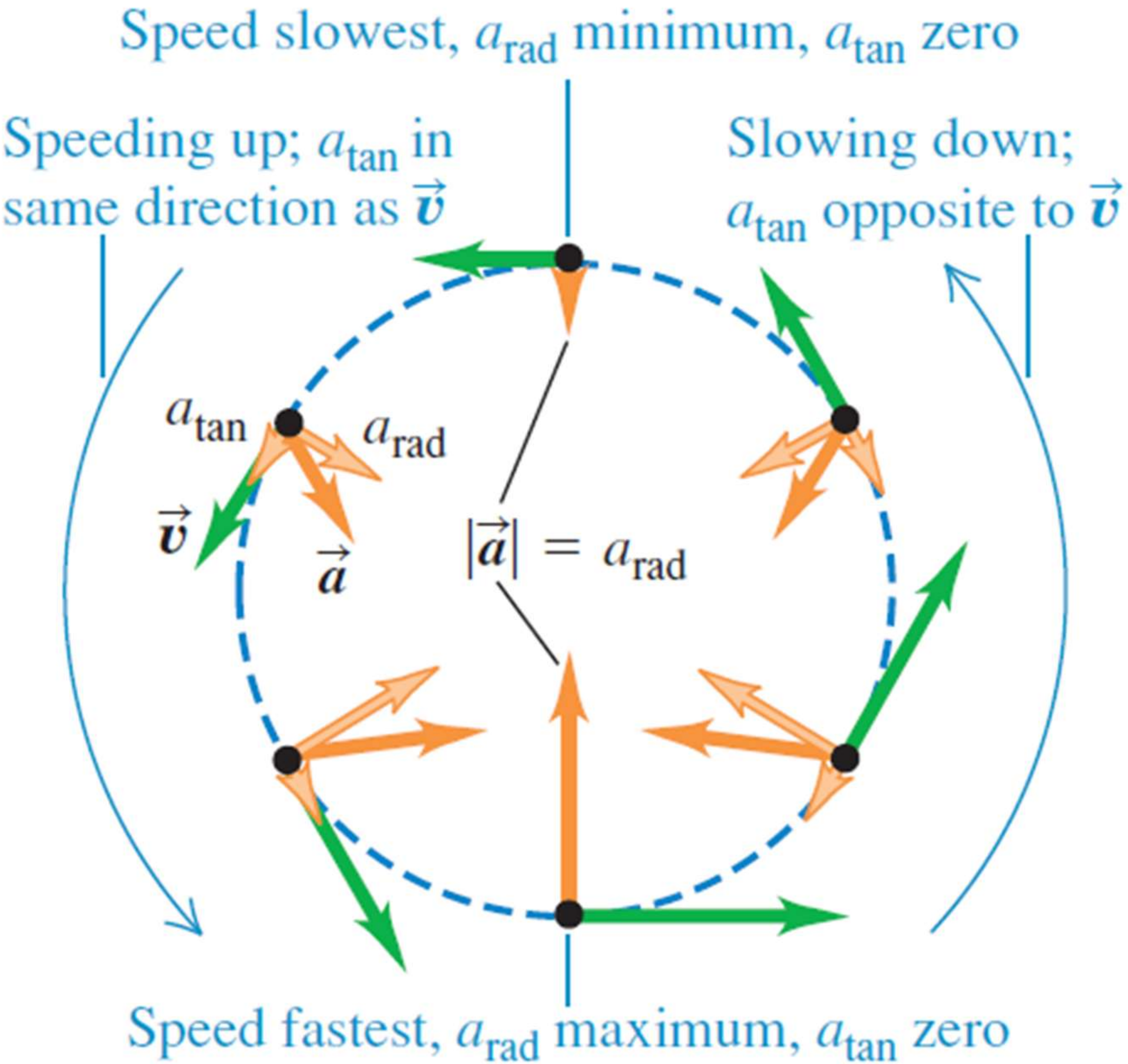
$\omega : angular\ velocity$

- In nonuniform circular motion there is also a component of acceleration that is parallel to the instantaneous velocity. This is the component $a_{\parallel}$ that we discussed before; here we call this component $a_{tan}$ to emphasize that it is tangent to the circle. This component, called the tangential acceleration $a_{tan}$, is equal to the rate of change of speed. Thus

$$a_{\text{rad}} = \frac{v^2}{R} \qquad \text{and} \qquad a_{\text{tan}} = \frac{d|\vec{v}|}{dt} \quad \text{(nonuniform circular motion)}$$



# Nonuniform Circular Motion




$$|d\vec{v}/dt| = \sqrt{a_{rad}^2 + a_{tan}^2}$$

We can also express the magnitude of the acceleration in uniform circular motion in terms of the **period** $T$ of the motion, the time for one revolution (one complete trip around the circle). In a time $T$ the particle travels a distance equal to the circumference $2\pi R$ of the circle, so its speed is

Magnitude of acceleration of an object in **uniform circular motion**

$$a_{\text{rad}} = \frac{4\pi^2 R}{T^2}$$

Radius of object's circular path

Period of motion

# EXAMPLE Centripetal acceleration on a fairground ride

Passengers on a fairground ride move at constant speed in a horizontal circle of radius 5.0 m, making a complete circle in 4.0 s. What is their acceleration?

**IDENTIFY and SET UP** The speed is constant, so this is uniform circular motion. We are given the radius $R = 5.0\ \text{m}$ and the period $T = 4.0\ \text{s}$, so we can use Eq. (3.29) to calculate the acceleration directly, or we can calculate the speed $v$ by using Eq. (3.28) and then find the acceleration by using Eq. (3.27).

**EXECUTE** From Eq. (3.29),

$$a_{\text{rad}} = \frac{4\pi^2(5.0\ \text{m})}{(4.0\ \text{s})^2} = 12\ \text{m/s}^2 = 1.3g$$

**EVALUATE** We can check this answer by using the second, roundabout approach. From Eq. (3.28), the speed is

$$v = \frac{2\pi R}{T} = \frac{2\pi(5.0\ \text{m})}{4.0\ \text{s}} = 7.9\ \text{m/s}$$

The centripetal acceleration is then

$$a_{\text{rad}} = \frac{v^2}{R} = \frac{(7.9\ \text{m/s})^2}{5.0\ \text{m}} = 12\ \text{m/s}^2$$

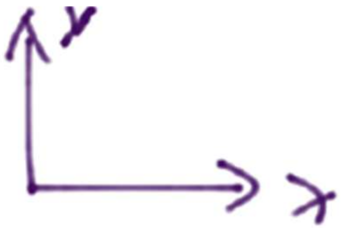
**3.29** • A Ferris wheel with radius 14.0 m is turning about a horizontal axis through its center (Fig. E3.29). The linear speed of a passenger on the rim is constant and equal to 7.00 m/s. What are the magnitude and direction of the passenger's acceleration as she passes through (a) the lowest point in her circular motion? (b) The highest point in her circular motion? (c) How much time does it take the Ferris wheel to make one revolution?

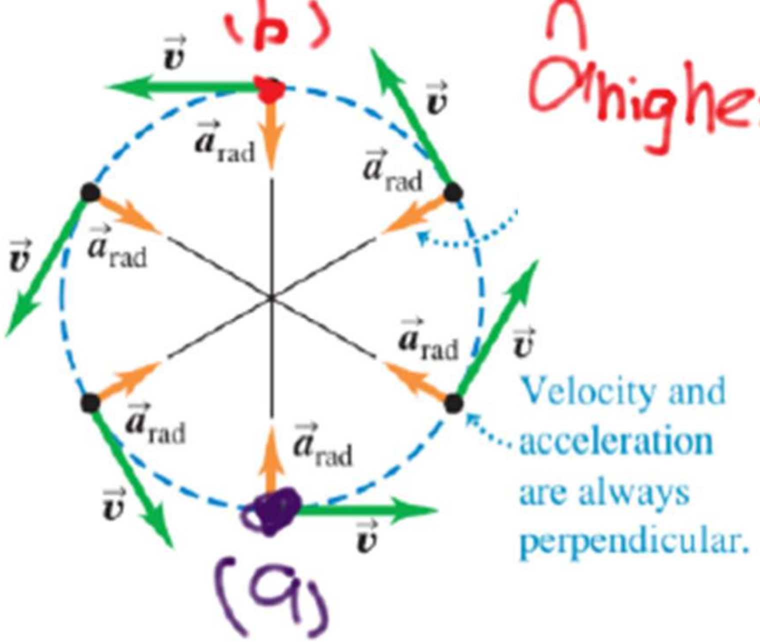
Figure **E3.29**

# Ferris Wheel

For both (a) and (b).

$$|\vec{a}_{rad}| = \frac{v^2}{R} = \frac{(7.0\,m/s)^2}{14\,m} = 3.5\,m/s^2$$




$\hat{a}_{highest}$ ↓ in $-y$ direction

↑ $\hat{a}_{lowest}$ in $+y$ direction

# Ferris Wheel

For both (a) and (b).

$$|\vec{a}_{rad}| = \frac{v^2}{R} = \frac{(7.0\,m/s)^2}{14\,m} = 3.5\,m/s^2$$

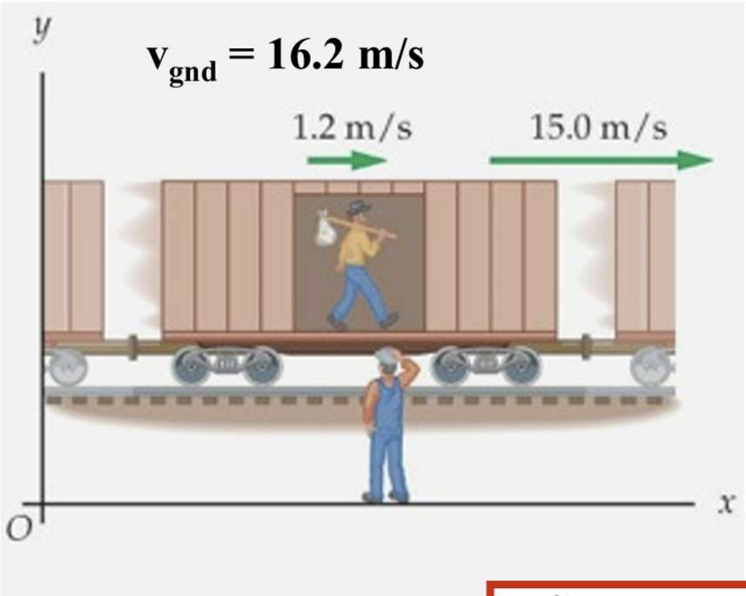
$$t = \frac{S}{v}$$ ← distance to travel in One round

$$S = 2\pi R$$

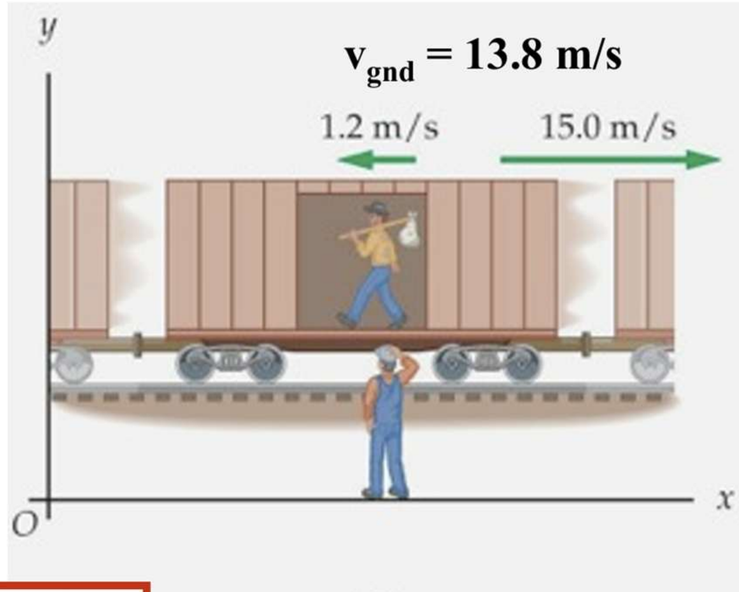
$$t = \frac{2 \cdot 3.14159 \cdot 14\,m}{7\,m/s} \simeq 12.57\,s$$

# Relative Motion

The speed of the passenger with respect to the ground depends on the relative directions of the passenger's and train's speeds:




(a)

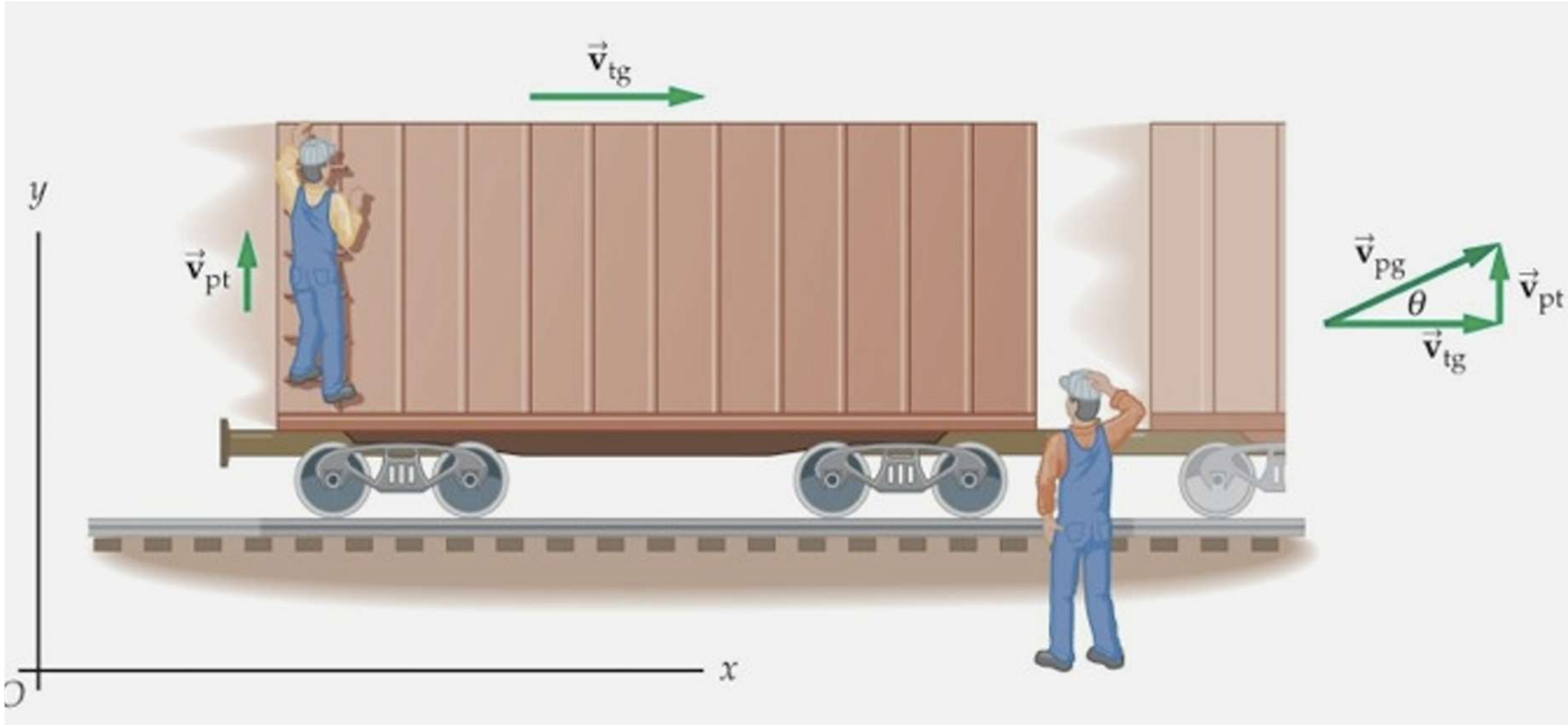



(b)

$$\boxed{\vec{\mathbf{v}}_{pg} = \vec{\mathbf{v}}_{pt} + \vec{\mathbf{v}}_{tg}}$$

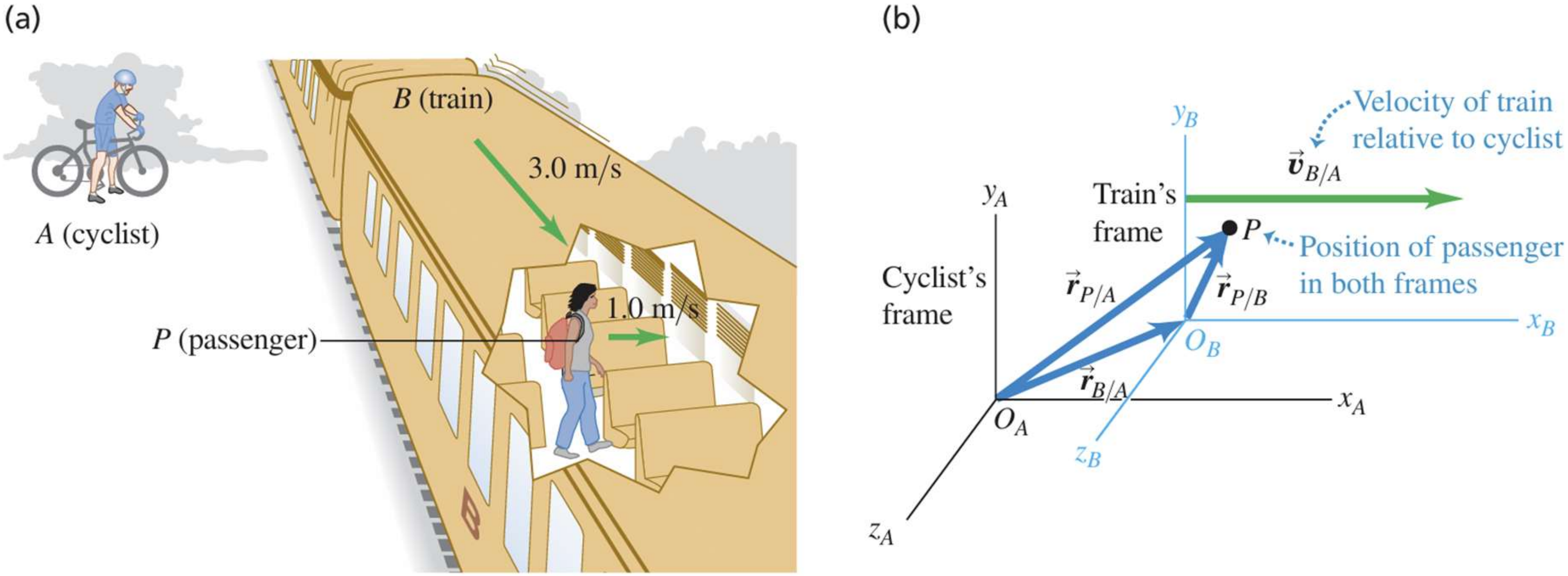
# Relative Motion

This also works in two dimensions:




$$\boxed{\vec{\mathbf{v}}_{pg} = \vec{\mathbf{v}}_{pt} + \vec{\mathbf{v}}_{tg}}$$

- Observer on ground: sees the sum of two motions one horizontal and one vertical




vectors $\vec{r}$ because the problem is now two-dimensional.

$$\vec{r}_{P/A} = \vec{r}_{P/B} + \vec{r}_{B/A}$$

Just as we did before, we take the time derivative of this equation to get a relationship among the various velocities; the velocity of $P$ relative to $A$ is $\vec{v}_{P/A} = d\vec{r}_{P/A}/dt$ and so on for the other velocities. We get

**Relative velocity in space:**

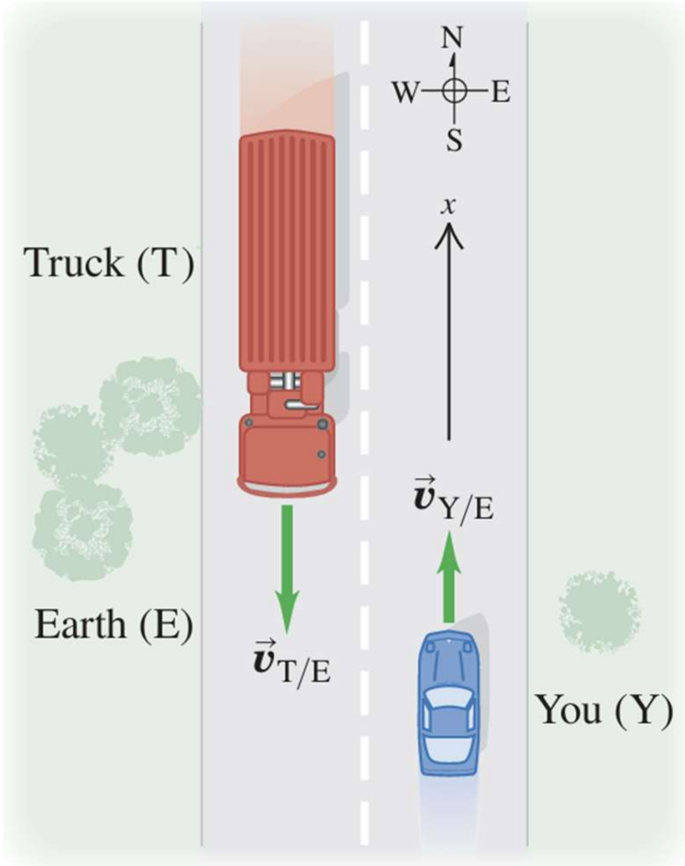
$$\vec{v}_{P/A} = \vec{v}_{P/B} + \vec{v}_{B/A}$$

Velocity of $P$ relative to $A$

Velocity of $P$ relative to $B$

Velocity of $B$ relative to $A$

You drive north on a straight two-lane road at a constant 88 km/h. A truck in the other lane approaches you at a constant 104 km/h . Find (a) the truck's velocity relative to you and (b) your velocity relative to the truck. (c) How do the relative velocities change after you and the truck pass each other? Treat this as a one-dimensional problem.



**EXECUTE** (a) To find $v_{T/Y\text{-}x}$, we write Eq. (3.32) for the known $v_{T/E\text{-}x}$ and rearrange:

$$v_{T/E\text{-}x} = v_{T/Y\text{-}x} + v_{Y/E\text{-}x}$$
$$v_{T/Y\text{-}x} = v_{T/E\text{-}x} - v_{Y/E\text{-}x} = -104 \text{ km/h} - 88 \text{ km/h} = -192 \text{ km/h}$$

The truck is moving at 192 km/h in the negative $x$-direction (south) relative to you.

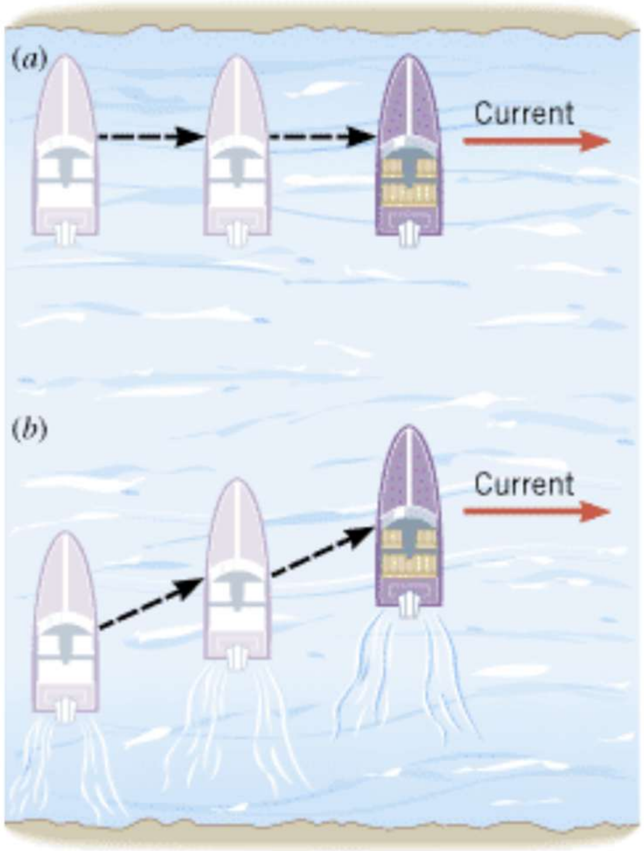
(b) From Eq. (3.33),

$$v_{Y/T\text{-}x} = -v_{T/Y\text{-}x} = -(-192 \text{ km/h}) = +192 \text{ km/h}$$

You are moving at 192 km/h in the positive $x$-direction (north) relative to the truck.

(c) The relative velocities do *not* change after you and the truck pass each other. The relative *positions* of the objects don't matter. After it passes you, the truck is still moving at 192 km/h toward the south relative to you, even though it is now moving away from you instead of toward you.

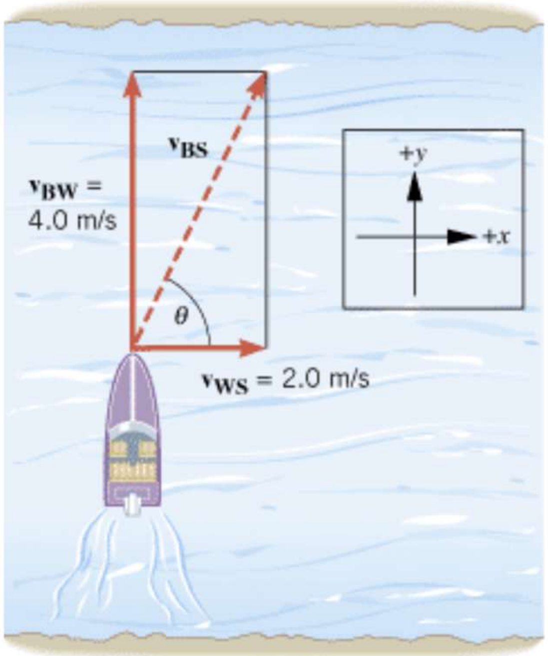
**EVALUATE** To check your answer in part (b), use Eq. (3.32) directly in the form $v_{Y/T\text{-}x} = v_{Y/E\text{-}x} + v_{E/T\text{-}x}$. (The $x$-velocity of the earth with respect to the truck is the opposite of the $x$-velocity of the truck with respect to the earth: $v_{E/T\text{-}x} = -v_{T/E\text{-}x}$.) Do you get the same result?




*(a)* A boat with its engine turned off is carried along by the current. *(b)* With the engine turned on, the boat moves across the river in a diagonal fashion.

## *Example* **Crossing a River**

The engine of a boat drives it across a river that is 1800 m wide. The velocity $\mathbf{v}_{BW}$ of the boat relative to the water is 4.0 m/s, directed perpendicular to the current, as in Figure. The velocity $\mathbf{v}_{WS}$ of the water relative to the shore is 2.0 m/s. (a) What is the velocity $\mathbf{v}_{BS}$ of the boat relative to the shore? (b) How long does it take for the boat to cross the river?




The velocity of the boat relative to the shore is $\mathbf{v}_{BS}$. It is the vector sum of the velocity $\mathbf{v}_{BW}$ of the boat relative to the water and the velocity $\mathbf{v}_{WS}$ of the water relative to the shore: $\mathbf{v}_{BS}=\mathbf{v}_{BW}+\mathbf{v}_{WS}$.

## Solution

(a) Since the vectors $\mathbf{v}_{\mathbf{BW}}$ and $\mathbf{v}_{\mathbf{WS}}$ are perpendicular (see Figure 3.18), the magnitude of $\mathbf{v}_{\mathbf{BS}}$ can be determined by using the Pythagorean theorem:

$$v_{\mathrm{BS}} = \sqrt{v_{\mathrm{BW}}^2 + v_{\mathrm{WS}}^2} = \sqrt{(4.0\,\mathrm{m/s})^2 + (2.0\,\mathrm{m/s})^2} = \boxed{4.5\,\mathrm{m/s}}$$

Thus, the boat moves at a speed of 4.5 m/s with respect to an observer on shore. The direction of the boat relative to the shore is given by the angle $\theta$ in the drawing:

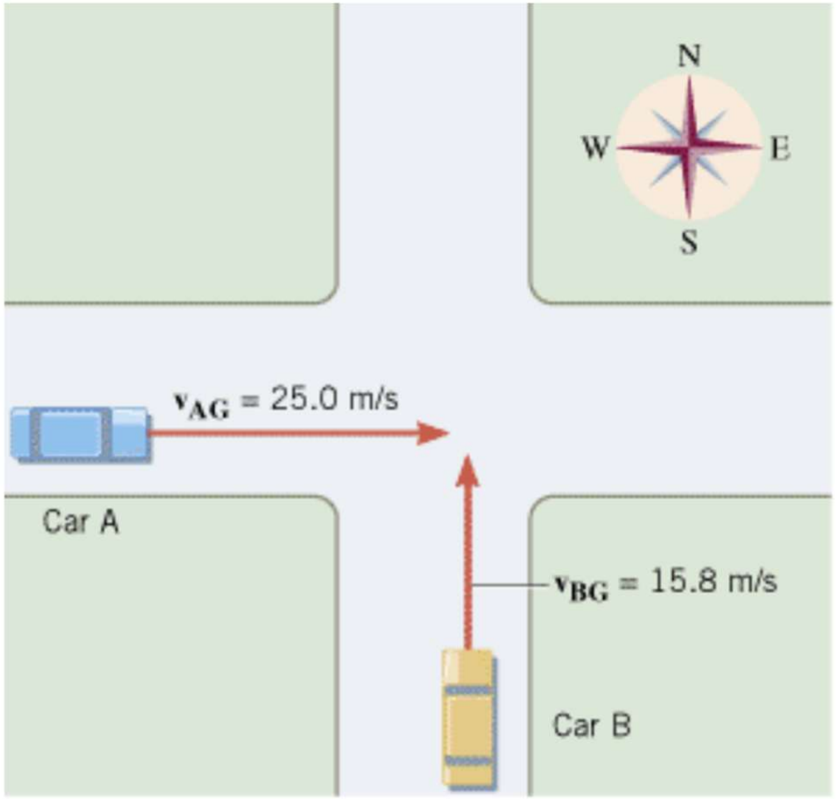
$$\tan\theta = \frac{v_{\mathrm{BW}}}{v_{\mathrm{WS}}} \quad \text{or} \quad \theta = \tan^{-1}\left(\frac{v_{\mathrm{BW}}}{v_{\mathrm{WS}}}\right) = \tan^{-1}\left(\frac{4.0\,\mathrm{m/s}}{2.0\,\mathrm{m/s}}\right) = \boxed{63^\circ}$$

(b) The time $t$ for the boat to cross the river is

$$t = \frac{\text{Width}}{v_{\mathrm{BS}}\sin\theta} = \frac{1800\,\mathrm{m}}{4.0\,\mathrm{m/s}} = \boxed{450\,\mathrm{s}}$$

## *Example* Approaching an Intersection

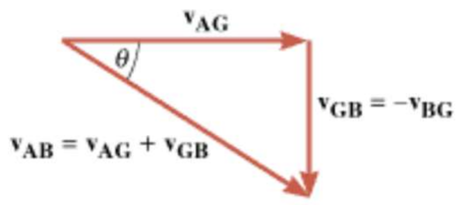
Find the magnitude and direction of $\mathbf{v}_{AB}$ (velocity of car A relative to the observers in car B), where



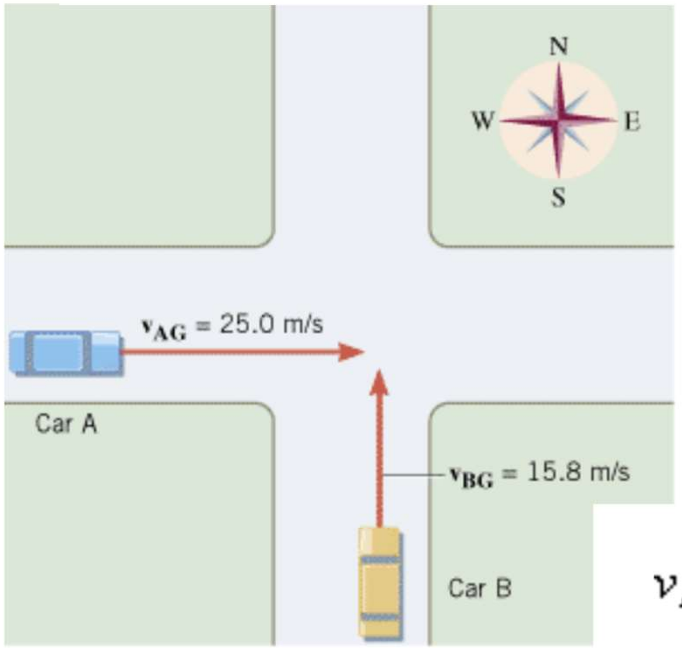
$\mathbf{v}_{\boxed{\mathbf{AG}}}$ = velocity of $\boxed{\text{car A}}$ relative to the $\boxed{\text{Ground}}$ = 25.0 m / s, eastward

$\mathbf{v}_{\boxed{\mathbf{BG}}}$ = velocity of $\boxed{\text{car B}}$ relative to the $\boxed{\text{Ground}}$ = 15.8 m / s, northward

$\mathbf{v}_{\mathbf{AB}}$ = velocity of $\boxed{\text{car A}}$ as measured by a passenger in $\boxed{\text{car B}}$




(b)



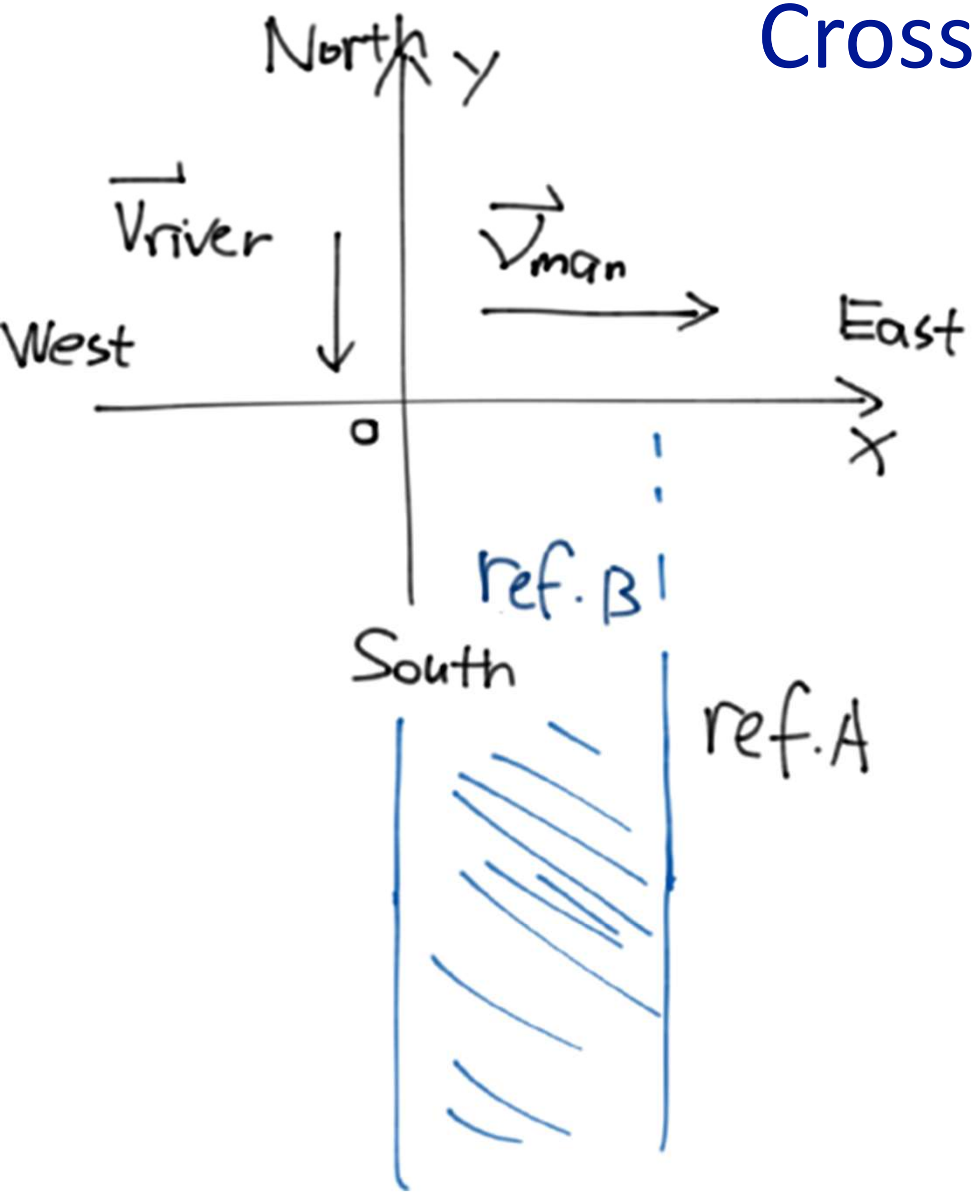

$$v_{AB} = \sqrt{v_{AG}^2 + v_{GB}^2} = \sqrt{(25.0\,\mathrm{m/s})^2 + (-15.8\,\mathrm{m/s})^2} = \boxed{29.6\,\mathrm{m/s}}$$

$$\cos\theta = \frac{v_{AG}}{v_{AB}} \quad \text{or} \quad \theta = \cos^{-1}\left(\frac{v_{AG}}{v_{AB}}\right) = \cos^{-1}\left(\frac{25.0\,\mathrm{m/s}}{29.6\,\mathrm{m/s}}\right) = \boxed{32.4^\circ}$$

**3.35** • **Crossing the River I.** A river flows due south with a speed of 2.0 m/s. A man steers a motorboat across the river; his velocity relative to the water is 4.2 m/s due east. The river is 800 m wide. (a) What is his velocity (magnitude and direction) relative to the earth? (b) How much time is required to cross the river? (c) How far south of his starting point will he reach the opposite bank?

Again, if you don't know what to do, draw things out!

# Crossing the River




$$\vec{V}_{river} = \vec{V}_r = -2.0\,m/s\cdot\hat{j}$$

$\swarrow$ ground

$$= \vec{V}_{B/A}$$

velocity of B (river) relative to A (ground)

$$\vec{V}_{man} = \vec{V}_{P/B} = 4.2\,m/s\cdot\hat{i}$$

$\swarrow$ river

$$\vec{V}_{P/A} = \vec{V}_{P/B} + \vec{V}_{B/A}$$

$$= 4.2\,m/s\,\hat{i} - 2.0\,m/s\cdot\hat{j}$$

# Crossing the River

$$\vec{v}_{P/A} = \vec{v}_{P/B} + \vec{v}_{B/A}$$

$$= 4.2\,\mathrm{m/s}\,\hat{i} - 2.0\,\mathrm{m/s}\cdot\hat{j}$$

(a) To get to the other side, $\Delta x = 800\,\mathrm{m}$

or $\vec{r} = 800\,\mathrm{m}\cdot\hat{i} + y\cdot\hat{j}$

don't need in (a)

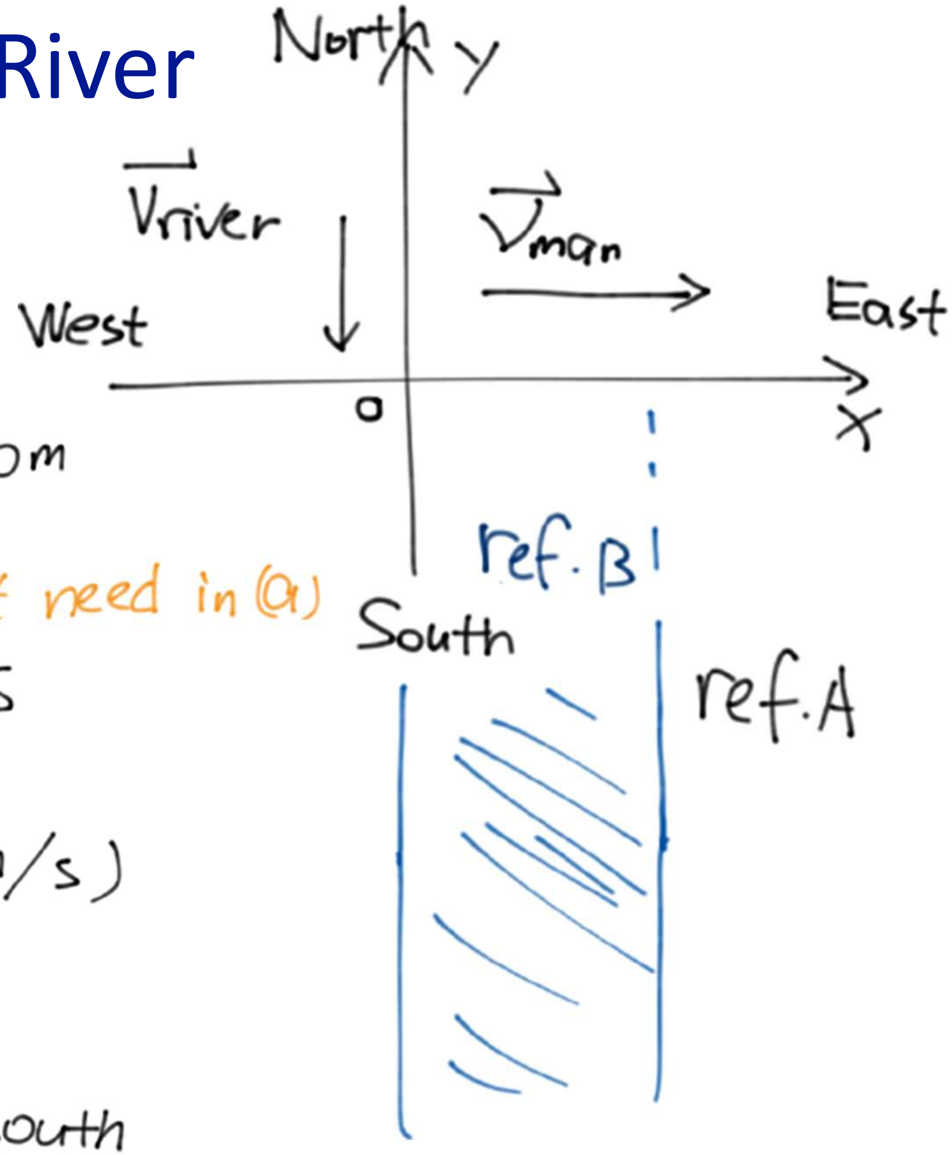
$$\Delta t = \frac{\Delta x}{v_x} = \frac{800\,\mathrm{m}}{4.2\,\mathrm{m/s}} = 190.5\,\mathrm{s}$$

(b) $\Delta y = v_y\cdot\Delta t = 190.5\,\mathrm{s} \times (-2.0\,\mathrm{m/s})$

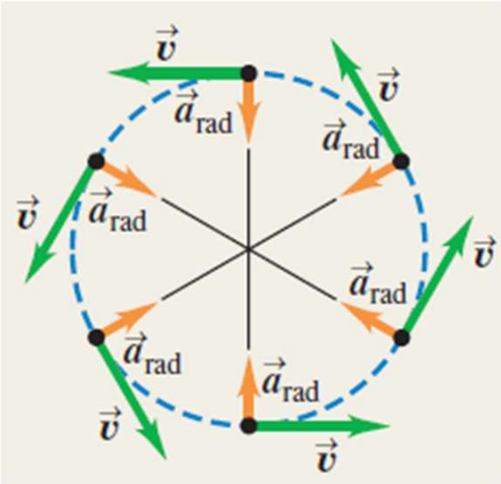
$\simeq -381.0\,\mathrm{m}$

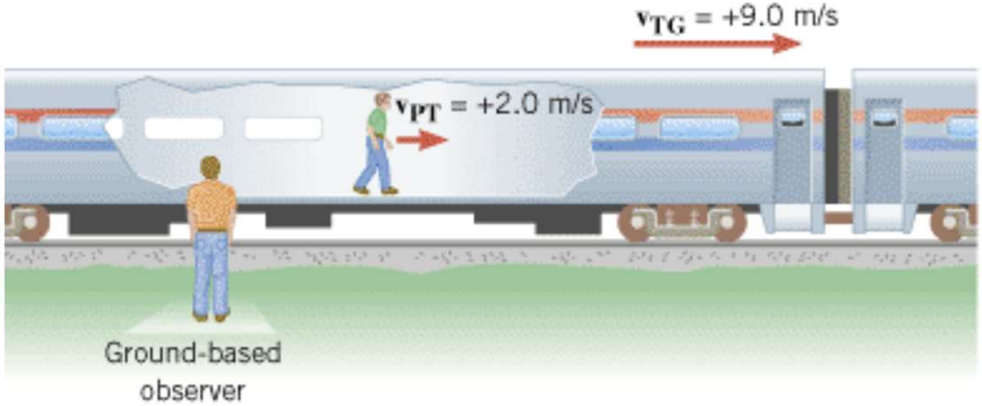
We took north as $+y$

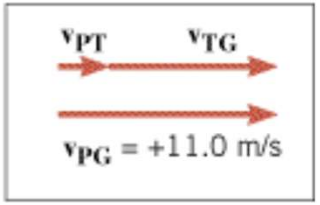
so the guy went 381.0 m south



# Summary




$$a_{rad} = \frac{v^2}{R}$$

$$a_{rad} = \frac{4\pi^2 R}{T^2}$$

$$v_{P/A\text{-}x} = v_{P/B\text{-}x} + v_{B/A\text{-}x}$$

(relative velocity along a line)

$$\vec{v}_{P/A} = \vec{v}_{P/B} + \vec{v}_{B/A}$$

(relative velocity in space)