# Mobile Robots

## Assembly and Control

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Advanced Topics in Control
May 16, 2022

# Announcements

1. Homework 3 is extended, Vahid is coming at 17:45 to answer questions

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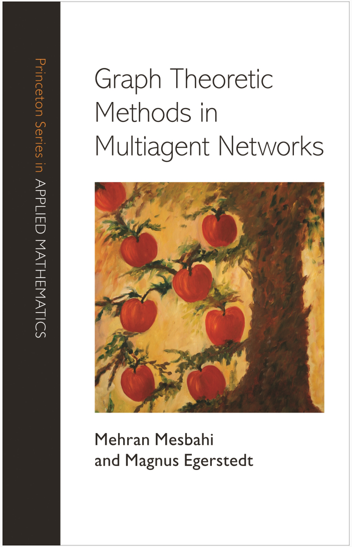
2. Project ideas should be finalized

# Recap from last week

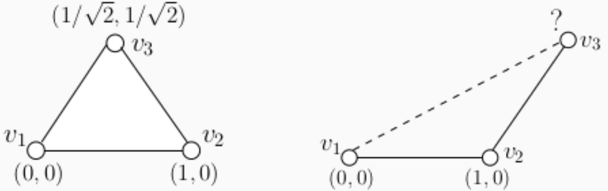

Princeton Series in APPLIED MATHEMATICS
Graph Theoretic Methods in Multiagent Networks
Mehran Mesbahi
and Magnus Egerstedt

# Recap of Rigidity

**Desired Interagent distances**

$$D = \{d_{ij} \in \mathbb{R} \ : \ d_{ij} > 0; i, j = 1, \ldots, n, i \neq j\}$$

$D$ must be **feasible:** there exist points $\Xi := \{\xi_1, \ldots, \xi_n \mid \xi_i \in \mathbb{R}^p\}$ such that $\|\xi_i - \xi_j\| = d_{ij}$,




$D = \{d_{12} = d_{13} = d_{23} = 1\}$

(a) Feasible formation

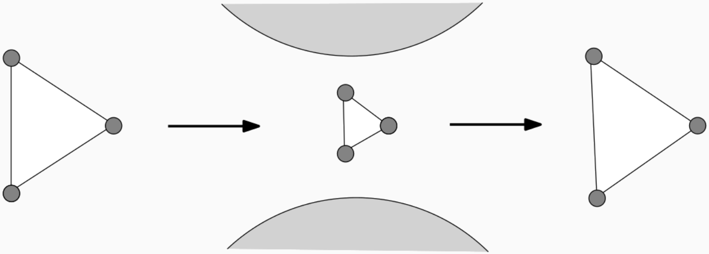
$D = \{d_{12} = d_{23} = 1, \ d_{13} = 3\}$

(b) Infeasible formation

# Recap of Rigidity

We refer to the **scale invariant** formation $D$ as any set of distances $D'$ such that $D' = \alpha D$ for any $\alpha \in \mathbb{R}_+$.

Used in 'cluttered' environments where strict interagent distances are not required.



# Recap of Rigidity

Interagent distances

$$D = \{d_{ij} = d_{ji} \geq 0,\ i, j = 1, \ldots, n,\ i \neq j\}$$

| formation | specification | interpretation |
|---|---|---|
| scale invariant | $D$ | $\|x_i - x_j\| = \alpha d_{ij}$ for some $\alpha > 0$ |
| rigid | $D$ | $\|x_i - x_j\| = d_{ij}$ |
| translational invariant | $\Xi$ | $x_i = \xi_i + \tau$ for some $\tau \in \mathbf{R}^p$ |

# Rigidity

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- is called **edge consistent** if $\|x_i(t) - x_j(t)\|$ is constant for all $ij \in \mathcal{E}, t \in \mathbb{R}_+$, or
- **rigid**, if $\|x_i(t) - x_j(t)\|$ is constant for all $i \neq j$.

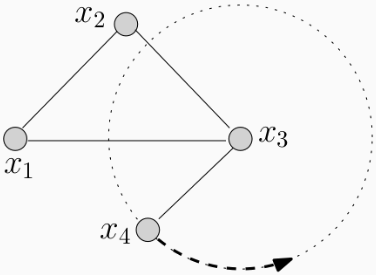
# Rigidity

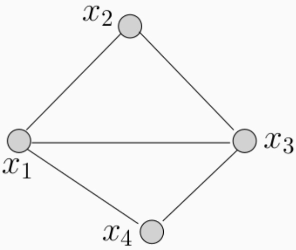
**Definition**

A framework $(\Xi, \mathcal{G}_f)$ is called **rigid** if and only if all edge-consistent trajectories of the framework are rigid trajectories.

Otherwise, the framework is called flexible.

# Rigidity




**Left:** Flexible framework with edge-consistent trajectories.

**Right:** Rigid framework

# Rigidity

**Definition**

A framework $\mathcal{G}(\Xi)$ is **infinitesimally rigid** if $R(\mathcal{G}(\Xi))u = 0$ for all $u$ satisfying

$$(u_i(t) - u_j(t))^T(x_i(t) - x_j(t)) = 0 \text{ for all } ij \in \mathcal{E}$$

Some properties of $R(\mathcal{G}(\Xi))$:

- Has $|\mathcal{E}|$ rows
- Has $pn$ columns ($p$ is the state dimension)
- Is an algebraic representation of a graph plus the points $\Xi$

**Theorem**

If $\mathcal{G}$ is rigid (i.e., there exists at least one $\Xi$ such that $\mathcal{G}(\Xi)$ is infinitesimally rigid), then the set of all $\Xi$ such that $\mathcal{G}(\Xi)$ is infinitesimally rigid is a dense, open subset of $\mathbb{R}^{pn}$.

1. Rigid graphs have a lot of feasible configurations!

# Rigid Graphs

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1. Rigid graphs have a lot of feasible configurations!
2. Feasible configurations can be approximated arbitrarily well (remember what dense + open means)

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2. Shape-based control (controlling to target $\Xi$)
3. Relative state-based control (controlling to $\|x_i - x_j\| = d_{ij}$)

# Formation Specification via Relative States

$$z(t) = \begin{bmatrix} x_1(t) - x_2(t) \\ x_2(t) - x_3(t) \end{bmatrix} \in \mathbb{R}^4$$

This specifies **all** relative formation states, including

$$x_1(t) - x(3) = [x_1(t) - x_2(t)] + [x_2(t) - x_3(t)]$$

# Formation Specification via Relative States

We can also set a formation center $x_o(t)$ and control the states,

$$z(t) = \begin{bmatrix} x_o(t) - x_1(t) \\ x_1(t) - x_2(t) \\ x_2(t) - x_3(t) \end{bmatrix} \in \mathbb{R}^6$$

# Formation Specification via Relative States – Incidence Matrix

$$z(t) = \begin{bmatrix} x_o(t) - x_1(t) \\ x_1(t) - x_2(t) \\ x_2(t) - x_3(t) \end{bmatrix} \in \mathbb{R}^6$$

$$z(t) = D(\mathcal{D})^T x(t)$$

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

$$D(\mathcal{D}) = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} \otimes I_2$$

# Formation Specification via Relative States – RSS

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This is called the **relative state specification (RSS)**.

# Formation Specification via Relative States – RSS Example

# Equivalence/Converting between RSS's

Suppose we have two spanning digraphs $\mathcal{D}_j$ and $\mathcal{D}_d$. Are these equivalent representations of an RSS?

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**Proposition**

Let $\mathcal{D}_j$ be a directed spanning tree RSS and let $\mathcal{D}_d$ be an arbitrary RSS. Then,

$$T_{dj} = D(\mathcal{D}_d)^T D(\mathcal{D}_j) \left[ D(\mathcal{D}_j)^T D(\mathcal{D}_j) \right]^{-1}$$

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$$T_{dj} = D(\mathcal{D}_d)^T D(\mathcal{D}_j) \left[ D(\mathcal{D}_j)^T D(\mathcal{D}_j) \right]^{-1}$$

This is the **projection** of the edge states of $\mathcal{D}_j$ onto those of $\mathcal{D}_d$.

# shape-based control

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The goal of formation control:

(R1): $\|x_i(t) - x_j(t)\| \to d_{ij}$ for all $ij \in E_f$

(R2): If the interaction graph $\mathcal{G}(t)$ is dynamic, then it should converge to $E_f \subseteq \mathcal{E}$ for all $t \geq T$, for finite $T$.

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By reaching agreement over the $\boldsymbol{\tau}_i$'s, each agent would satisfy $x_i - \xi_i = \boldsymbol{\tau}$, and so $\|x_i - x_j\| = d_{ij}$ as required by (R1)

# Shape-based control – The static graph case & consensus

We can let each agent $i$ perform consensus on $\boldsymbol{\tau}_i$ with its neighbours in $E_f$:

$$\dot{\boldsymbol{\tau}}_i(t) = -\sum_{j \in N_f(i)} \left(\boldsymbol{\tau}_i(t) - \boldsymbol{\tau}_j(t)\right)$$

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However, we want this consensus to be over the *formation graph* $\mathcal{G}_f$, since this encodes our target set $\Xi$:

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Since we assume that (R2) is satisfied, $E_f \subseteq \mathcal{E}$, and so this is a feasible communication strategy.

Note that $\dot{\tau}_i = \dot{x}_i$, and

$$\tau_i(t) - \tau_j(t) = x_i(t) - x_j(t) - (\xi_i - \xi_j).$$

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Therefore, we can write

$$\dot{x}_i(t) = \dot{\tau}_i(t) = -\sum_{j \in N_f(i)} (\tau_i(t) - \tau_j(t)) = -\sum_{j \in N_f(t)} \big(x_i(t) - x_j(t)\big) - \big(\xi_i - \xi_j\big)$$

## Shape-based control – The static graph case & consensus

**Theorem**

Consider the protocol,

$$\dot{x}_i(t) = -\sum_{j \in N_f(t)} \big(x_i(t) - x_j(t)\big) - (\xi_i - \xi_j)$$

on a connected formation graph $\mathcal{G}_f = (\mathcal{V}, E_f)$ encoding a set of target locations $\Xi$.

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$$x_i(t) - \xi_i \to \tau, \ \forall i \in \mathcal{N}.$$

as $t \to \infty$.

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(Way harder!)

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**Corollary**

Given a connected, target formation graph $\mathcal{G}_f = (\mathcal{V}, E_f)$, the above protocol will asymptotically drive all agents to a constant displace- ment of the target positions if for all $t \geq 0$, $E_f \subseteq \mathcal{E}(t)$.

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There are many results on graphs that vary randomly over time, but requires complicated machinerty to analyze

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The formation error at time $t$ is defined as,

$$e(t) = z_{\text{ref}} - z(t).$$

# Relative state-based control

We can write the dynamics of the error as,

$$\dot{e}(t) = -D(\mathcal{D})^T u(t).$$

*Proof:*

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A common feedback design is to integrate proportional to the error:

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Then, we can write,

$$\dot{e} = -kL_e(\mathcal{D})e(t)$$

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The edge Laplacian is positive-definite for a spanning tree, and so

$$\lim_{t \to \infty} e(t) = 0,$$

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# Relative state-based control

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and so the $n$ single integrators achieve $z_{\text{ref}}$ if $k > 0$ as $t \to \infty$.

# Relative state-based control – Dynamics

What do the dynamics of $x$ look like under this control?

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**Lemma**

Under the control law $u(t) = kD(\mathcal{D})e(t)$, the state dynamics are,

$$\dot{x}(t) = -kL(\tilde{\mathcal{D}})x(t) + kD(\mathcal{D})z_{\text{ref}},$$

where $\tilde{\mathcal{D}}$ is the undirected version of the spanning tree.

*Proof:*

# Why does the Laplacian use the undirected version of $\mathcal{D}$?

# relative state-based control – double-integrator case

# Relative state-based control – Double Integrator

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We further assume that $\ddot{z}_{\text{ref}} = 0$, i.e. the reference velocity does not change with time.

# Relative state-based control – Double Integrator

Again, we set the error as $e(t) = z_{\text{ref}} - D(\mathcal{D})^T x(t)$.

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We can show that,

$$\ddot{e} = -D(\mathcal{D})^T u(t).$$

*Proof:*

# Relative state-based control – Double Integrator

Define the (proportional-derivative) feedback controller,

$$u(t) = k \begin{bmatrix} D(\mathcal{D}) & D(\mathcal{D}) \end{bmatrix} \begin{bmatrix} e(t) \\ \dot{e}(t) \end{bmatrix}.$$

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Then, the closed-loop system is

$$\begin{bmatrix} \dot{e}(t) \\ \ddot{e}(t) \end{bmatrix} = \begin{bmatrix} 0 & I \\ -kL_e(\mathcal{D}) & -kL_e(\mathcal{D}) \end{bmatrix} \begin{bmatrix} e(t) \\ \dot{e}(t) \end{bmatrix}$$

*Proof:*

# Stability of the closed-loop system

# Stability of the closed-loop system

*continued:*

# Dynamics of the closed-loop system

**Theorem**

$$\begin{bmatrix} \dot{x}(t) \\ \ddot{x}(t) \end{bmatrix} = \begin{bmatrix} 0 & I \\ -kL(\tilde{\mathcal{D}}) & -kL(\tilde{\mathcal{D}}) \end{bmatrix} \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ kD(\mathcal{D}) & kD(\mathcal{D}) \end{bmatrix} \begin{bmatrix} z_{\text{ref}}(t) \\ \dot{z}_{\text{ref}} \end{bmatrix}$$

# Conclusion

1. Shape-based control (Take points $\Xi$ and control towards them)

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2. Relative state-based control (Define relative states $x_i - x_j$ and control them so $\|x_i - x_j\| = d_{ij}$.
3. Single/double integrator versions

# Homework: Extensions of these ideas

1. Internal dynamics: $\dot{x}_i = ax(t) + bu(t)$

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1. Internal dynamics: $\dot{x}_i = ax(t) + bu(t)$
2. Unicycle control: $r_i(t) = x_i(t) + jy_i(t)$

# Notifications

Reading Assignment

- Mesbahi & Egerstedt 6.2-6.6

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- Mesbahi & Egerstedt 6.2-6.6
- Mesbahi & Egerstedt 7.1-7.4

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## Exercise Session

Coverage Control with **Alberto**