Advanced Topics in Control: Distributed Systems and Control

Lecture 9: Introduction to Event-Triggered Control

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Summary of Lecture 8 on Stochastic Stability

Consider the discrete-time stochastic dynamical system

$$x(t+1) = f(x(t), w(t)), \quad x(t_0) = x(0)$$

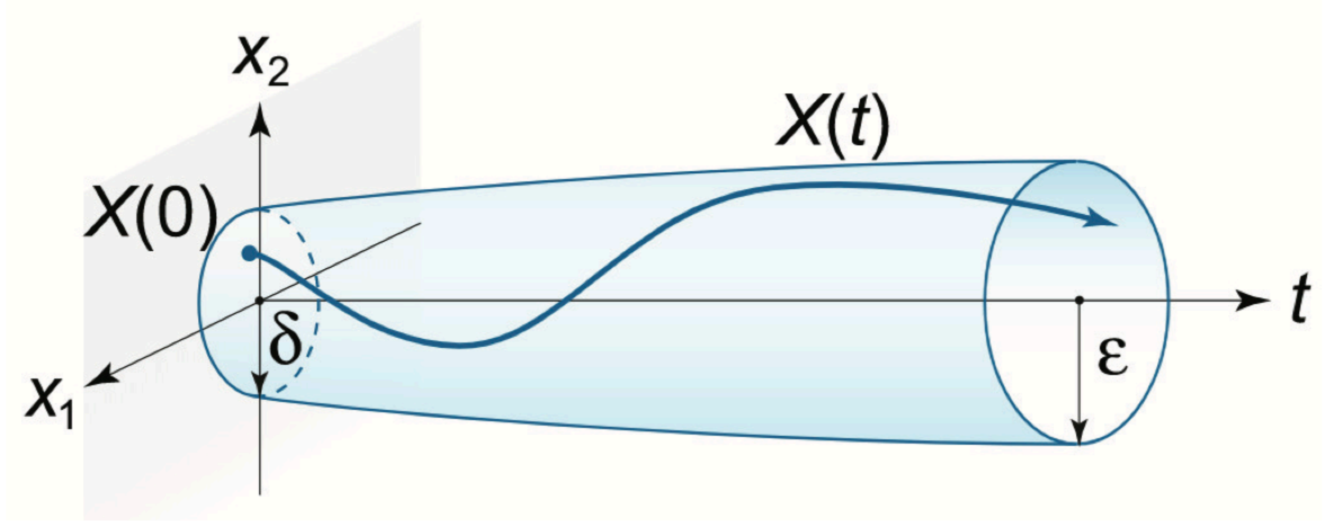
f is a (Lipschitz) continuous function and $w(t), t \in \mathbb{N}_0$ is a sequence of Markov random variables. We are interested in stability of the equilibrium $x(t, x(0)) = 0$

Recall stability notions for deterministic system $x(t+1) = f(x(t))$. $\forall \varepsilon > 0, \ \exists \delta > 0$:

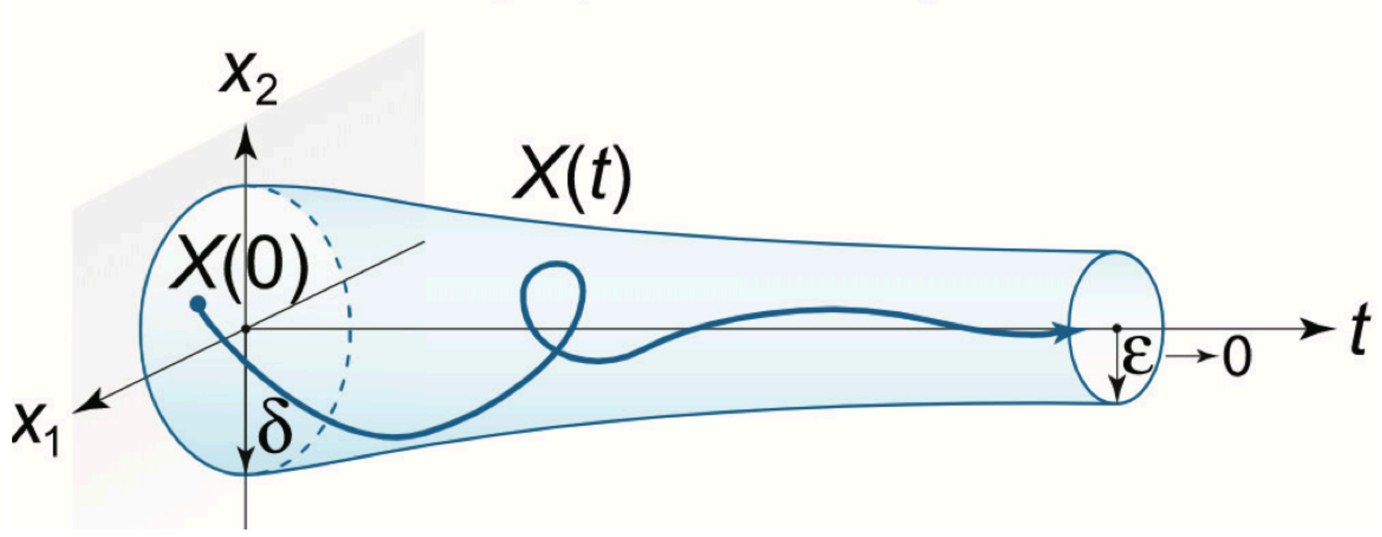
1. **Lyapunov stability:** trajectory starts in δ-vicinity of the equilibrium, always remains in its ε-vicinity

2. **Asymptotic stability:** trajectory starts in δ-vicinity of equilibrium, it stays in its ε-vicinity and eventually converges to the equilibrium

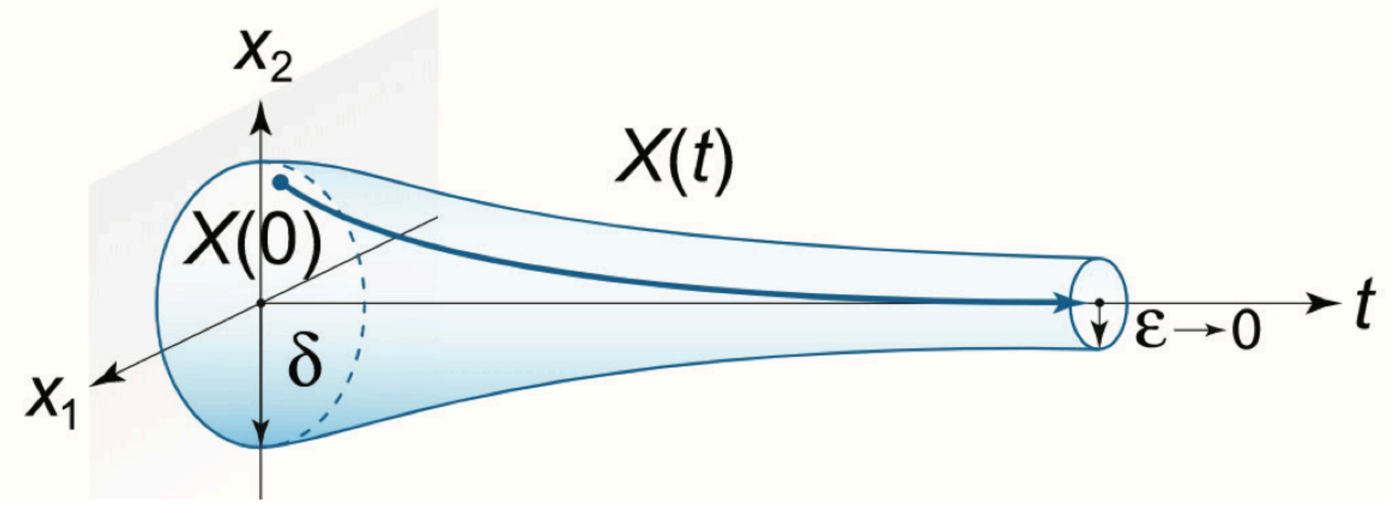
3. **Exponential stability:** trajectory starts in δ-vicinity of equilibrium, it remains in its ε-vicinity and smoothly converges to equilibrium with a particular rate



Lyapunov stability



Asymptotic stability



Exponential stability

Summary of Lecture 8 on Stochastic Stability

☑ Exponential stability $\Rightarrow$ Asymptotic stability $\Rightarrow$ Lyapunov stability

Generally for stochastic systems, stability is discussed in the sense of expectation or probability, because trajectories vary from one sample to another due to random dynamics, i.e., $w(t)$

$$x(t+1) = f(x(t), w(t)), \quad x(t_0) = x(0)$$

Lyapunov-based stochastic stability:

1. Lyapunov stability in probability: For any $\varepsilon_1 > 0$, $\varepsilon_2 \in [0,1]$, there exists $\delta(\varepsilon_1, \varepsilon_2, t_0)$, s.t. $\|x(0)\| < \delta$ implies

$$\mathrm{P}\{\sup_{t>t_0} \|x(t, x_0)\| > \varepsilon_1\} < \varepsilon_2$$

2. Lyapunov stability in the m^{th} mean: For any $\varepsilon > 0$, there exists $\delta(\varepsilon, t_0)$, s.t. $\|x(0)\|_m < \delta$ implies

$$\mathrm{E}\{\sup_{t>t_0} \|x(t, x(0))\|_m^m\} \leq \varepsilon$$

Summary of Lecture 8 on Stochastic Stability

Asymptotic stochastic stability:

1. Asymptotic stability in probability: if Lyapunov stable in probability and $\exists \delta > 0$ s.t. $\|x(0)\| < \delta$ implies

$$\lim_{\tau \to \infty} \mathrm{P}\{\sup_{t \geq \tau} \|x(t, x(0))\| > \varepsilon\} = 0$$

2. Asymptotic stability in the m^{th} mean: if Lyapunov stable in m^{th} mean and $\exists \delta > 0$ s.t. $\|x(0)\| < \delta$ implies

$$\lim_{\tau \to \infty} \mathrm{E}\{\sup_{t \geq \tau} \|x(t, x(0))\|_m^m\} = 0$$

Exponential stochastic stability:

Exponential stability of the m^{th} mean: if $\exists \delta > 0$ and constants α, β s.t. $\|x(0)\| < \delta$ implies

$$\mathrm{E}\{\|x(t, x_0)\|_m^m\} \leq \beta \|x_0\|_m^m \, e^{-\alpha(t - t_0)}$$

☑ Exponential stability of m^{th} mean $\nLeftarrow$ Asymptotic stability in m^{th} mean $\Rightarrow$ Lyapunov stability in m^{th} mean

☑ Stability in the mean is stronger than stability in probability!

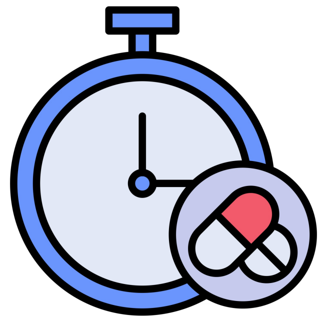
Event-Triggered Scheme

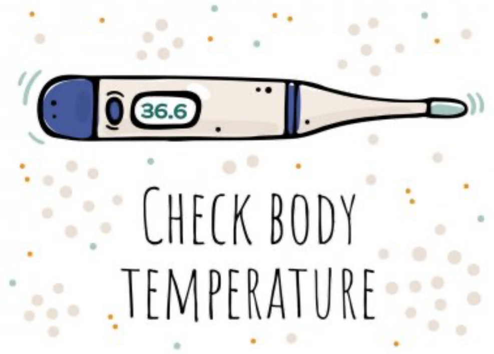
When execution of an action (control, sampling, actuation, communication, etc.) is performed based on the occurrence of a **defined “event”**, we call the action scheme ***event-triggered*** or ***event-based***.

Compared with **Time-Triggered** scheme, where an action is executed at pre-defined and equidistant time instances!

We do things in event-based fashion in our daily lives as well, e.g., when sick:

- Take medication every 8 hours → Time-triggered
- Take medication If body temperature above 38 → Event-triggered





Intuition: Actions often entail costs (implementation/resources), so why not perform an action only when necessary?

Major benefit: reduction of actions and consequently reduction of consuming costly resources

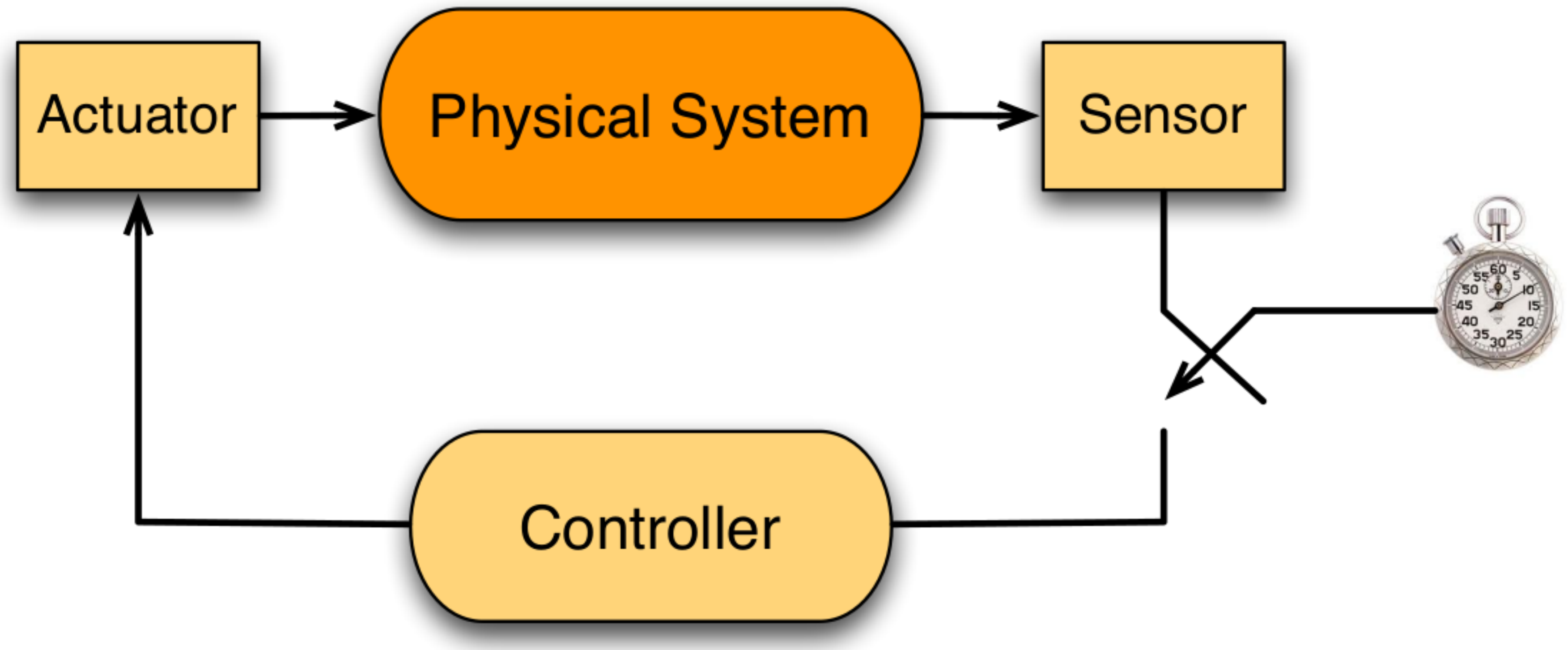
Event-Triggered vs. Time-Triggered Sampling

Time-triggered sampling is an open-loop scheme where sampling is performed using a clock

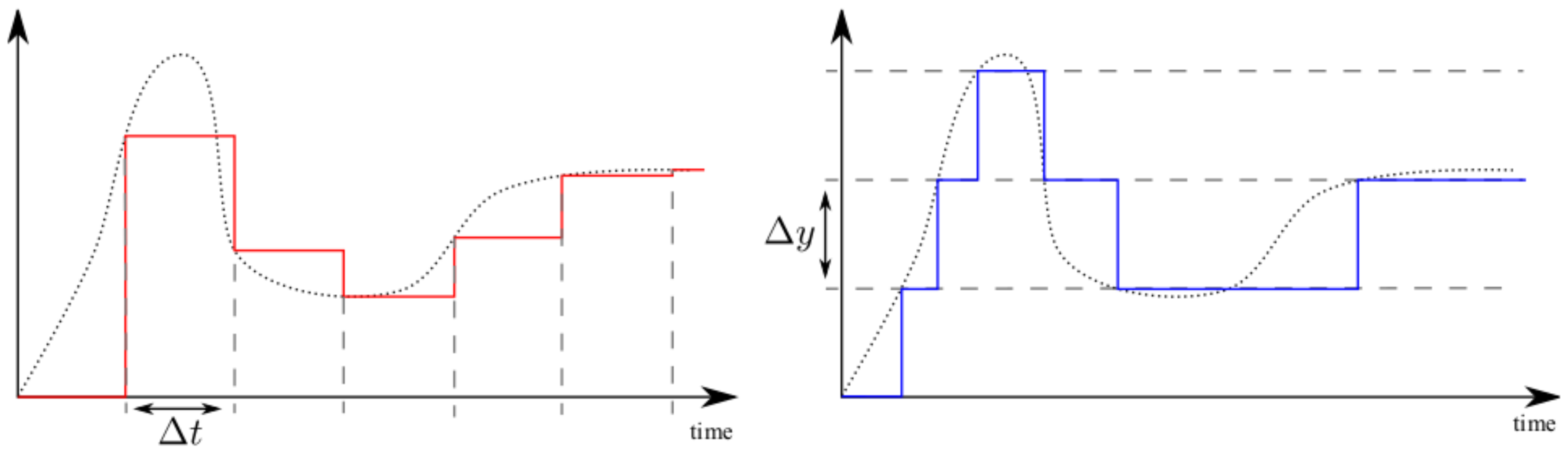
- Pre-specified sampling instances
- Deterministic and synchronous sampling
- Not resource efficient and not flexible

Event-triggered sampling uses feedback in the sampling process

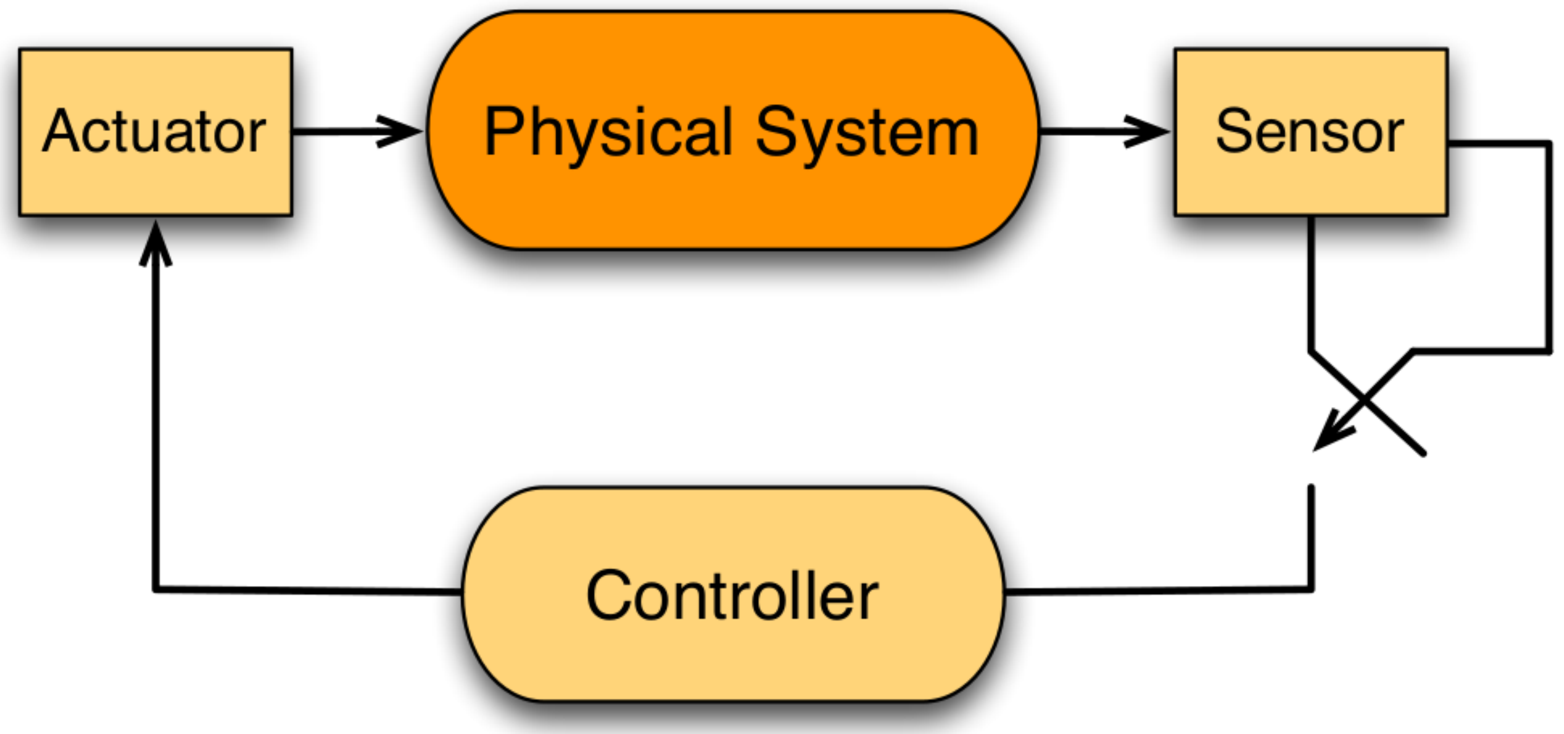
- Real-time sampling instances
- Asynchronous sampling and difficult to predict
- Resource efficient (sample only when performance not satisfactory)



Time-triggered sampling



Time-triggered (left) vs. event-triggered (right) sampling of a signal



Event-triggered sampling

Event-Triggered vs. Time-Triggered Sampling: A more complex scenario

Event-triggered actions: events can be designed for various functionalities across a control system

A single control system

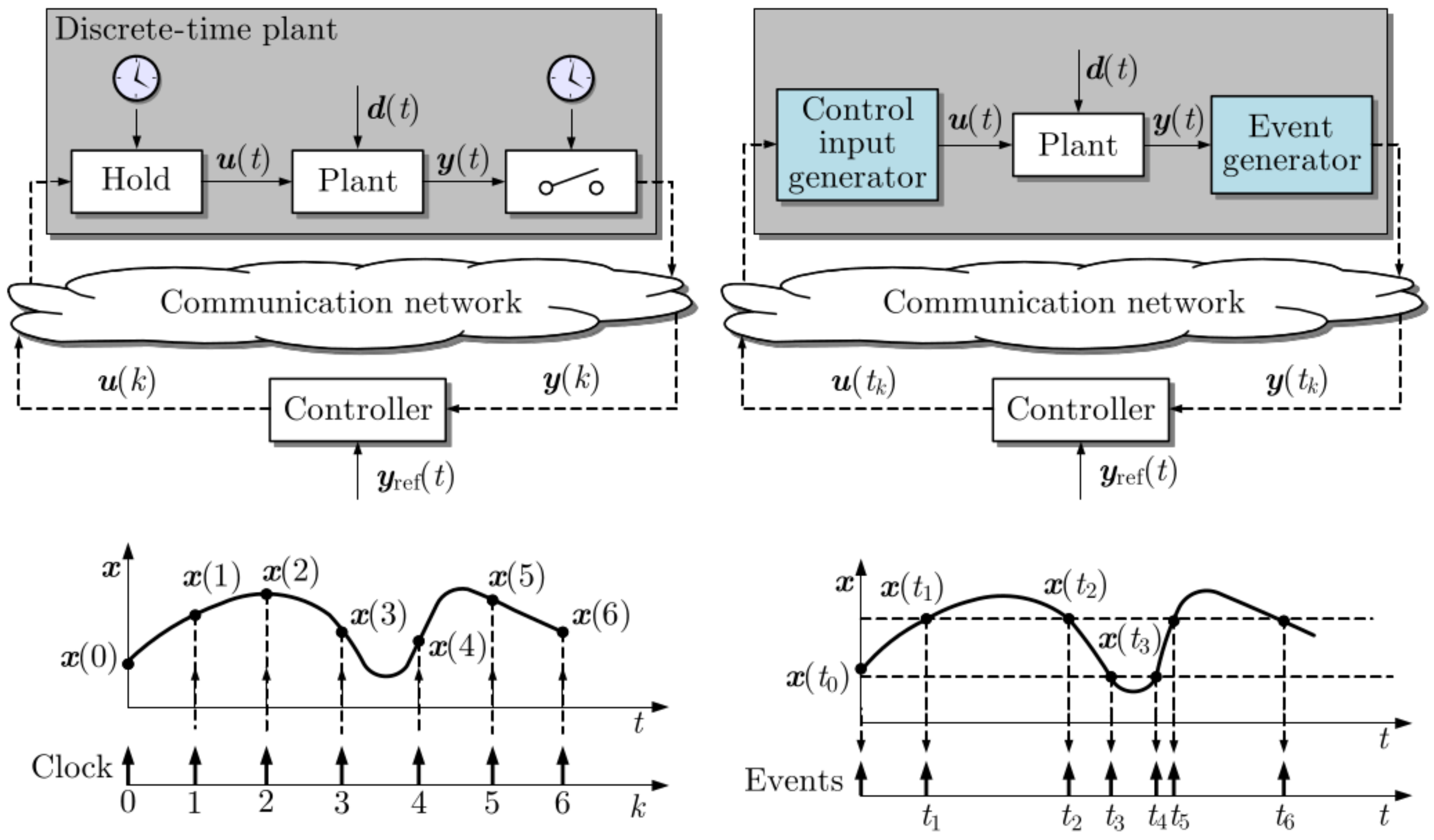
- When sample the system states
- When update the control input/actuate

A networked/multi-agent system

- When to communicate and with whom
- Which loop to be closed, if resources limited

Challenges of event-triggered schemes:

- Complex to design and implement
- Optimal events often difficult to be found
- Requires constant monitoring of the events
- Triggering instances often not predictable

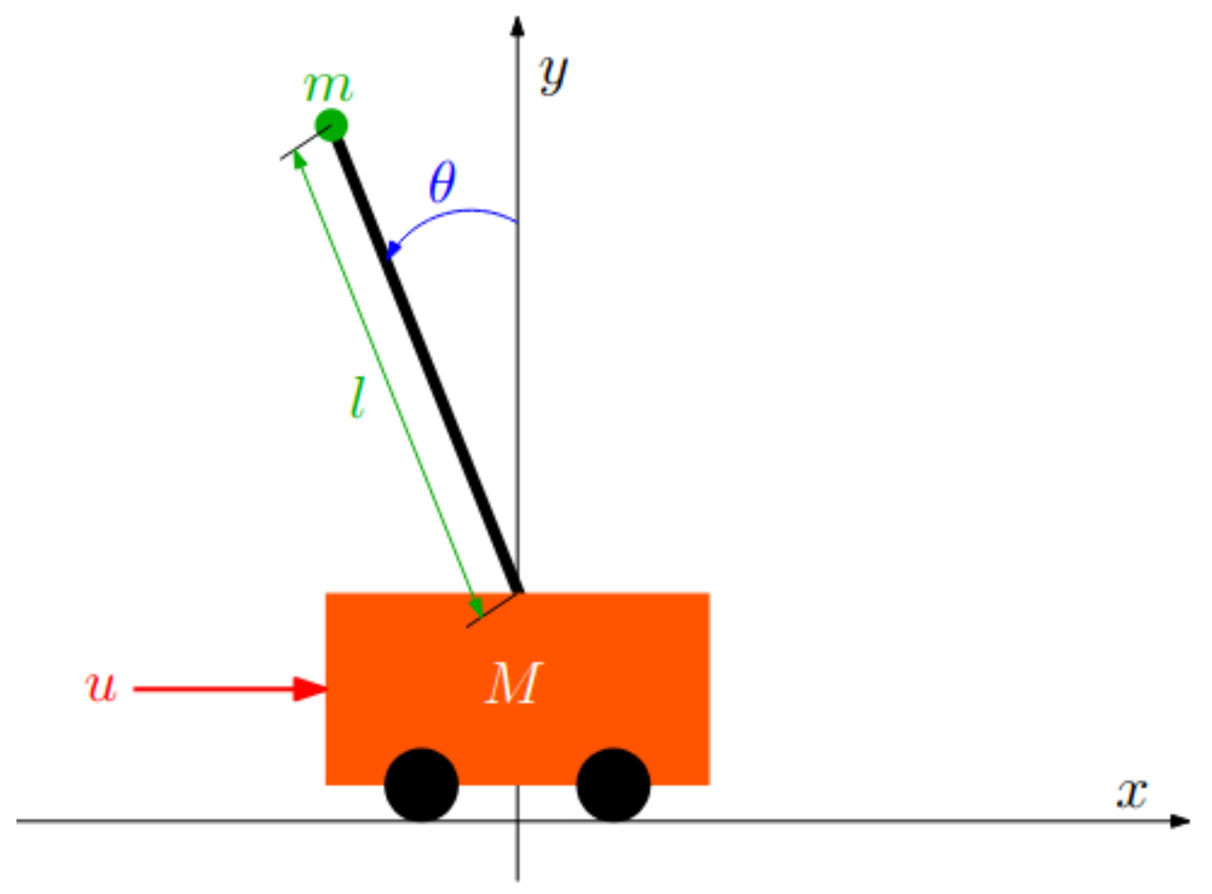


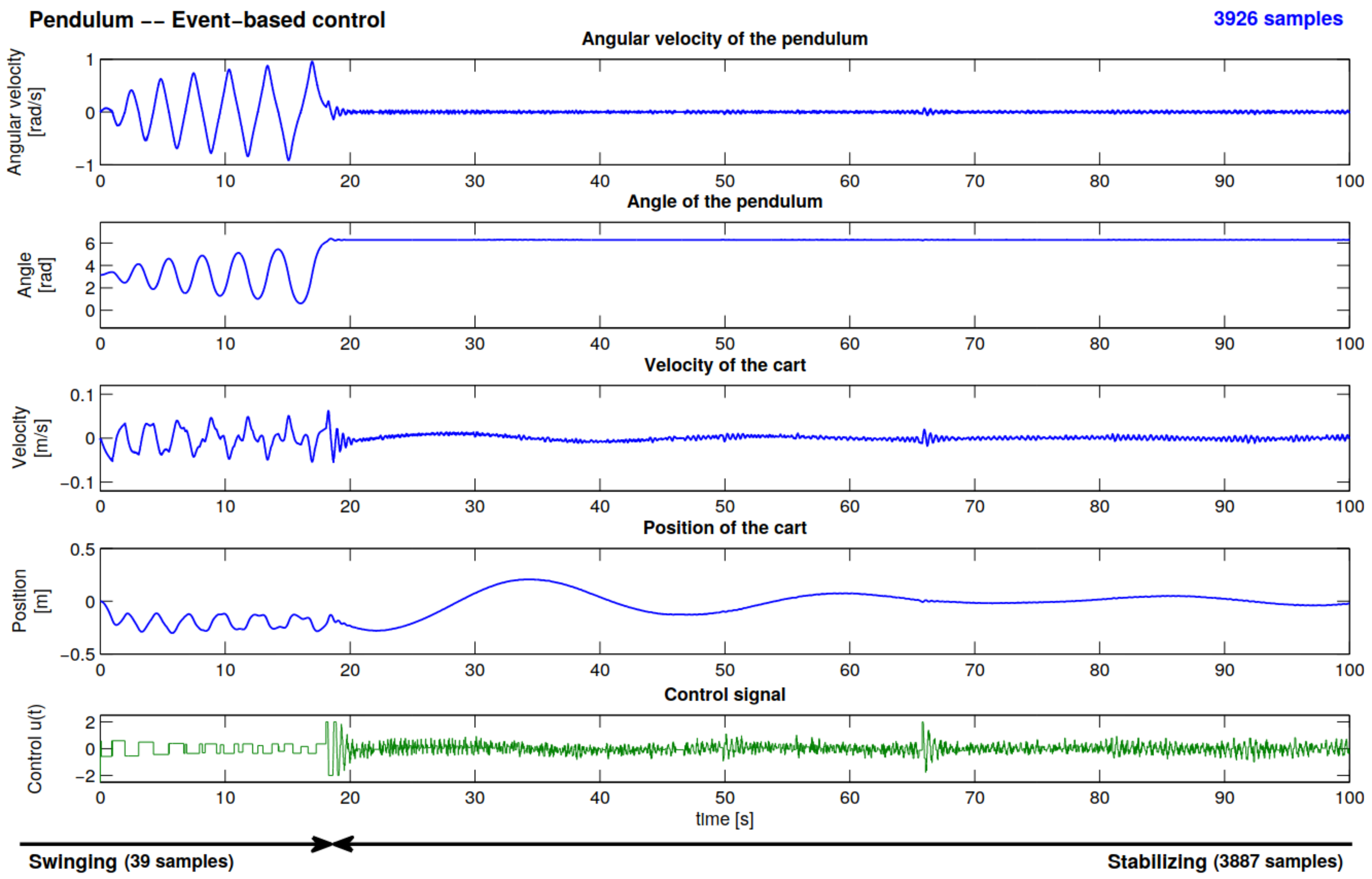
Time-triggered vs. Event-triggered schemes in a networked control system

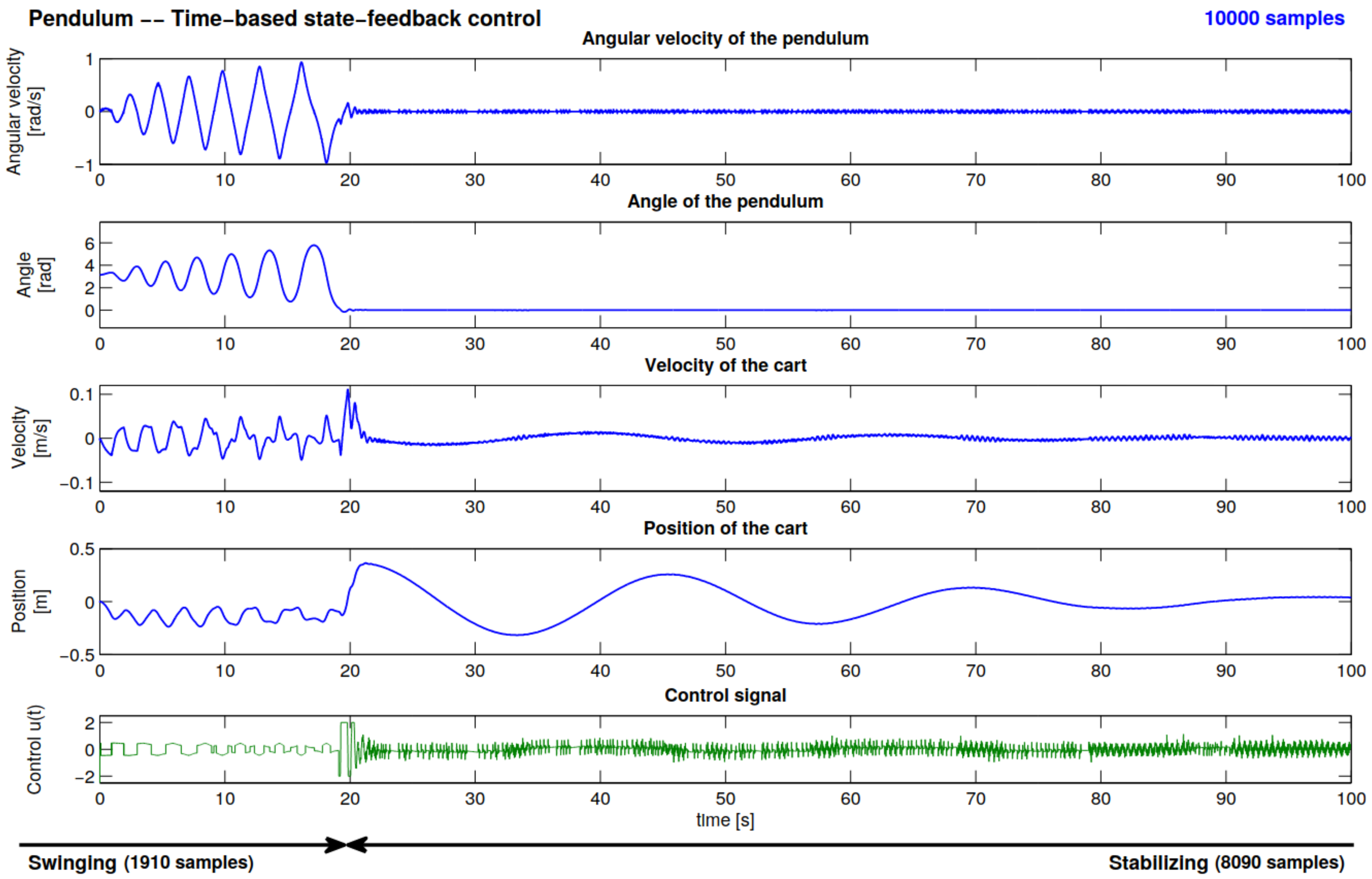
Event-Triggered vs. Time-Triggered Scheme: Inverted Pendulum

Compare the sampling instances:

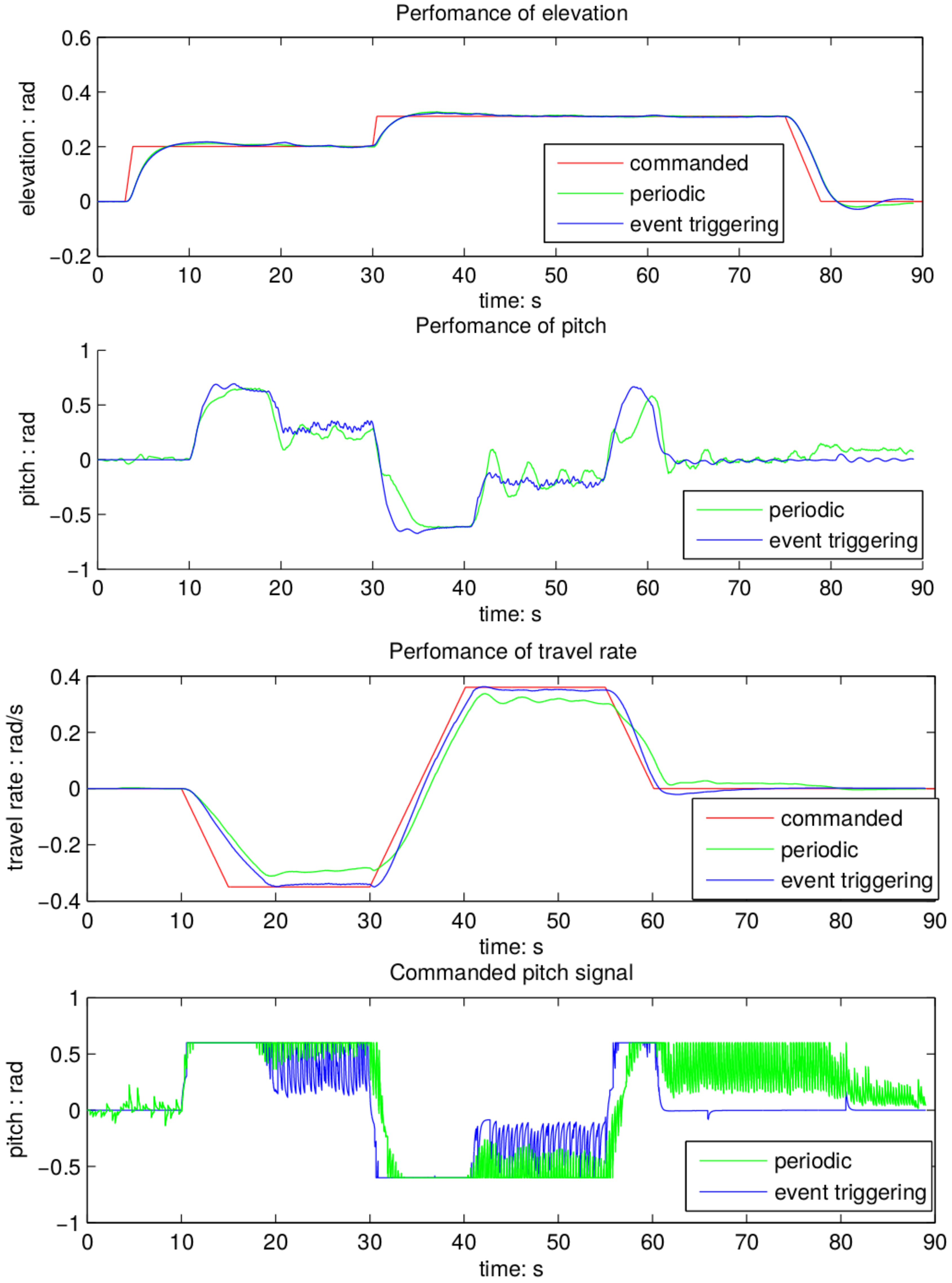
- Time-triggered: 10000 samples
- Event-triggered: 3926 samples
- ~60% reduction in sampling for similar performance

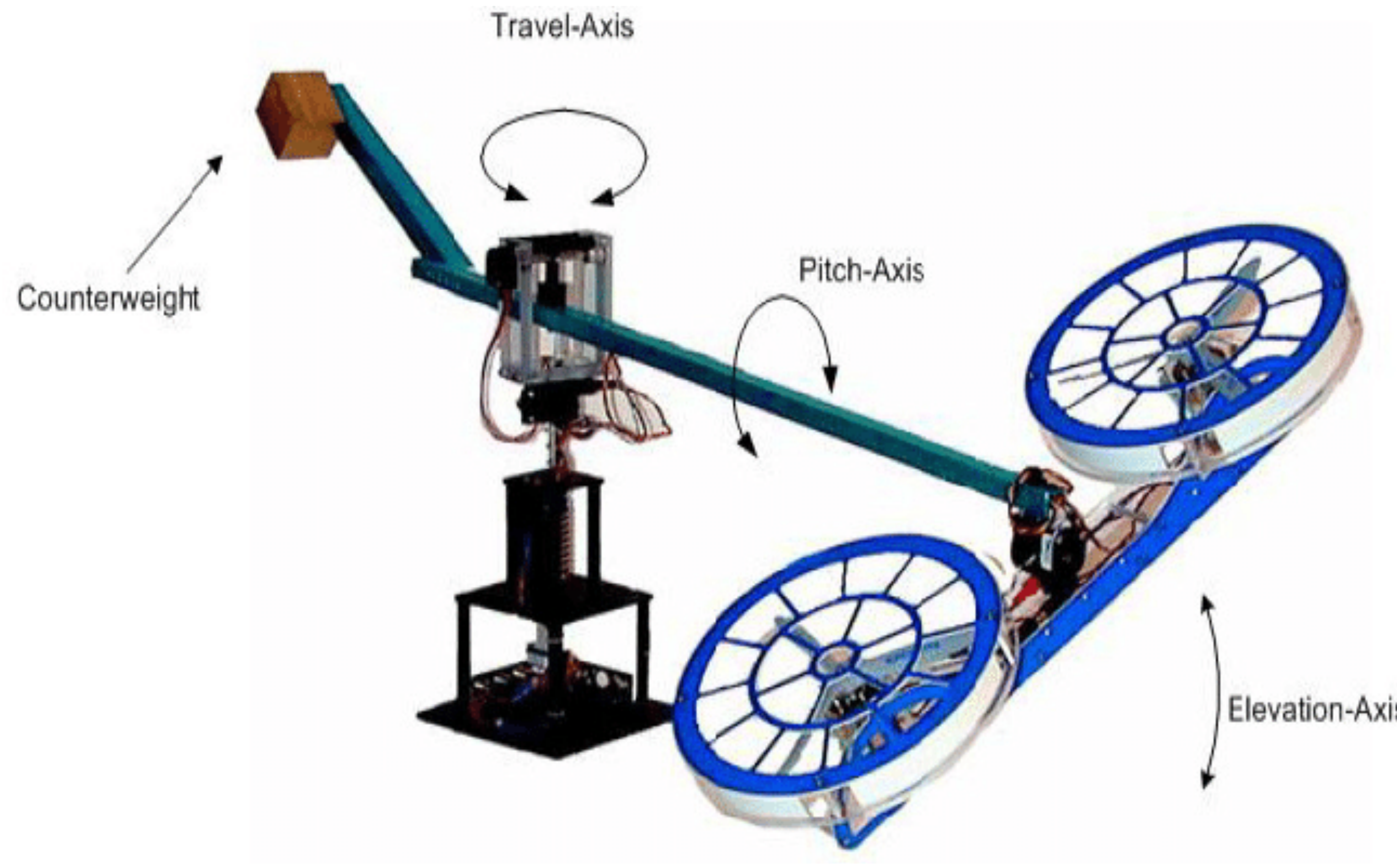


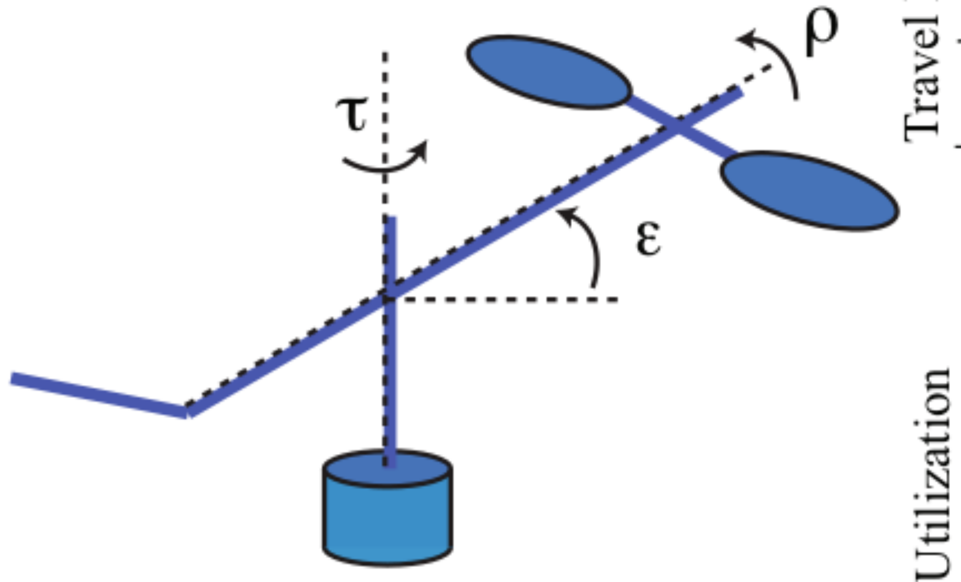


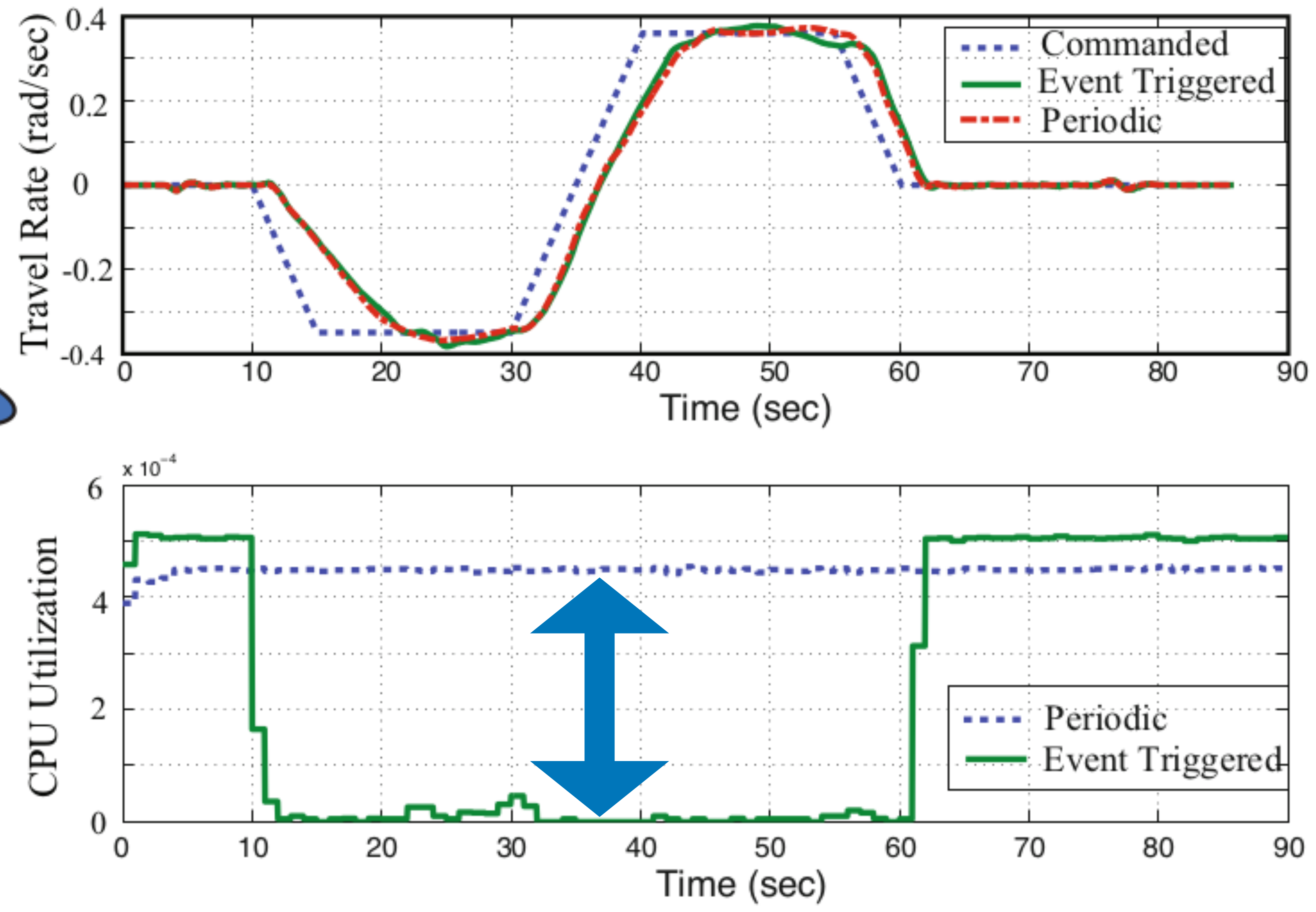


Event-Triggered vs. Time-Triggered Scheme: 3DOF Helicopter









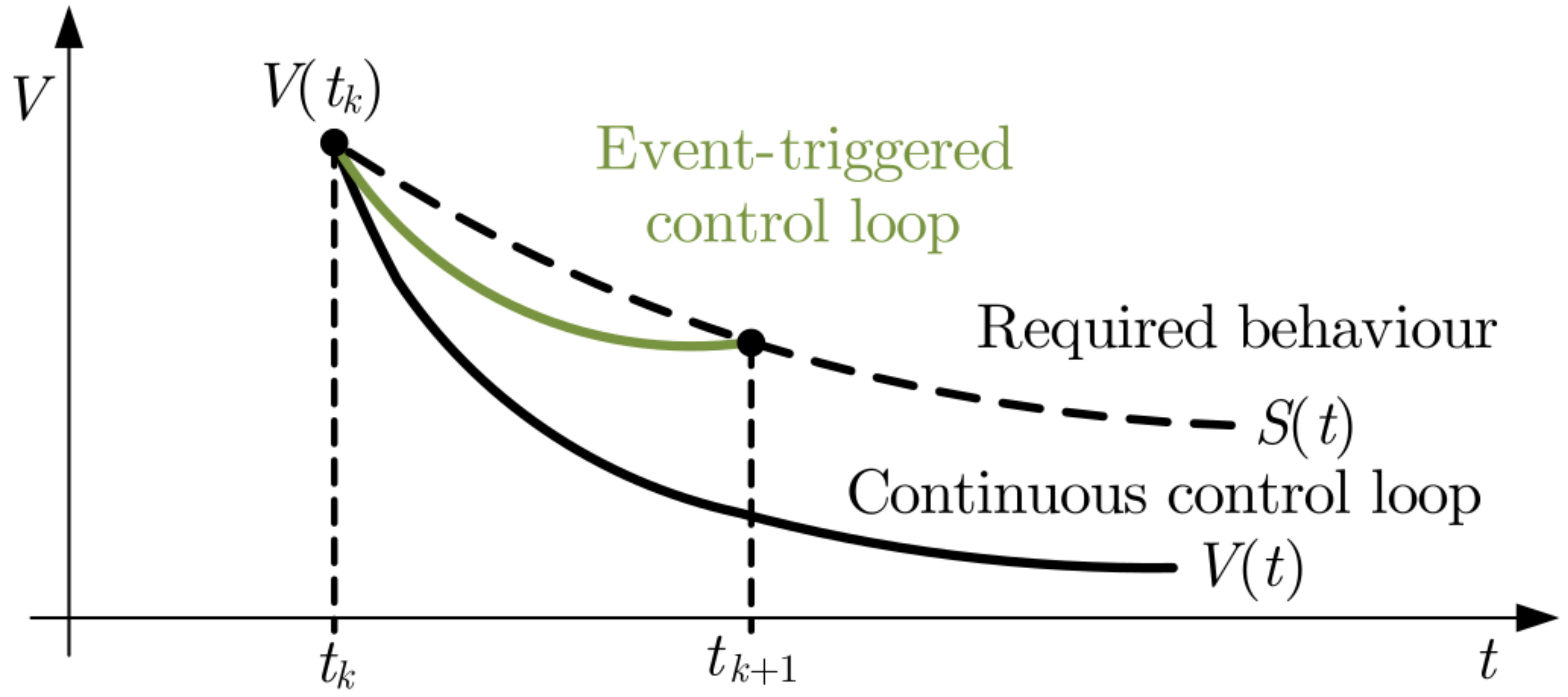
Asynchronous Triggering Schemes

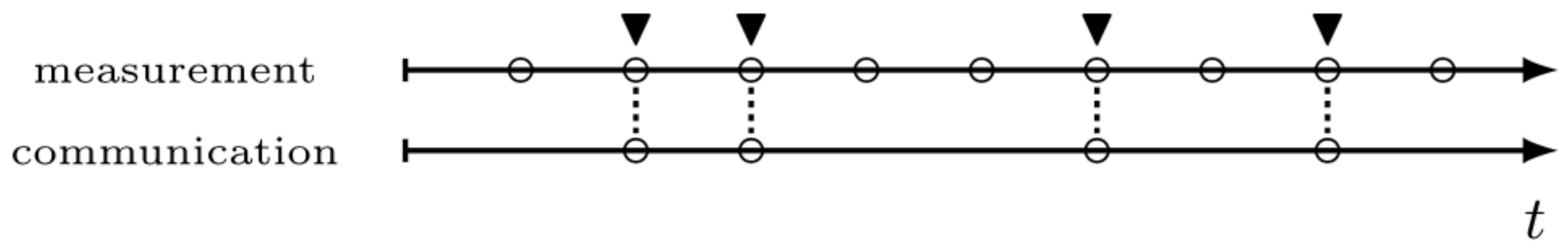
Event-triggered:

- Deterministic triggering functions: events are often deterministically defined with thresholds on system variables, e.g., the states of the system, estimation error, cost value, etc.
- Stochastic triggering functions: if events have stochastic nature, the triggering times are defined in probabilisitic fashion
- Mixed triggering functions: events are triggered partly deterministically and partly stochastically

Periodic event-triggered:

- Continuous monitoring of system states to check when an event is triggered might not always be possible!
- Events are not continuously monitored but periodically, and if triggering condition is met, the action will be performed
- Often accompanied by maximum inter-update deadline, i.e., at least one update over a given interval even if event is not triggered





(a) Event-triggered control.

Event-triggered Control: Continuous-Time Linear Systems

Consider a continuous LTI system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad u(t) = Lx(t), \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m$$

- $L \in \mathbb{R}^{m \times n}$ is a stabilising gain making the closed-loop system asymptotically stable!

Let $\{\tau_1, \tau_2, \tau_3, \dots\}$ be the sequence of executing the control inputs, i.e.,

$$u(t) = Lx(\tau_i), \quad \forall t \in [\tau_i, \tau_{i+1})$$

Define the error measure:

$$e(t) = x(\tau_i) - x(t), \quad \forall t \in [\tau_i, \tau_{i+1})$$

$\forall t \in [\tau_i, \tau_{i+1})$ the closed-loop system:

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + BLx(\tau_i) \\
&= Ax(t) + BLx(t) - BLx(t) + BLx(\tau_i) \\
&= (A + BL)x(t) + BL(x(\tau_i) - x(t)) \\
&= (A + BL)x(t) + BLe(t)
\end{aligned}$$

Lyapunov Stability for Linear Systems

Consider a continuous LTI system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad u(t) = Lx(t), \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m$$

- $L \in \mathbb{R}^{m \times n}$ is a stabilising gain making the closed-loop matrix $A + BL$ stable!

Definition (Positive definite function): A continuous function $V : \mathbb{R}^n \to \mathbb{R}$ is called positive definite if it satisfies

1- $V(0) = 0$

2- $V(x) > 0, \forall x \neq 0, x \in \mathbb{R}^n$

Lyapunov Stability Theorem:

For the given linear closed-loop system $\dot{x}(t) = (A + BL)x(t)$, if exists a positive definite function $V: \mathbb{R}^n \to \mathbb{R}$ s.t.

$$\dot{V} = \frac{dV}{dx}\frac{dx}{dt} = \frac{dV}{dx}((A + BL)x(t)) = -W(x) \leq 0$$

1. if $W(x) \geq 0$, i.e., positive semi-definite, the equilibrium point is Lyapunov stable.
2. If $W(x) > 0$, i.e., positive definite, the equilibrium is asymptotically stable.

Event-triggered Control

Lemma: For a linear time-invariant system $\dot{x}(t) = Ax(t) + Bu(t)$, $u(t) = Lx(t)$ where L is a stabilising gain and $A + BK$ is a stable matrix, a Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ exists, such that for some $\{\underline{a}, \overline{a}, a, g\} \in \mathbb{R}_{>0}$

$$\underline{a}|x|^2 \leq V(x) \leq \overline{a}|x|^2$$

$$\frac{dV}{dx}(Ax + BLx + BLe) \leq -a|x|^2 + g|e||x|$$

Define the constant positive parameter σ such that $-a + g\sigma < -b$, and $b > 0$. If it is guaranteed that:

$$|e| \leq \sigma|x|$$

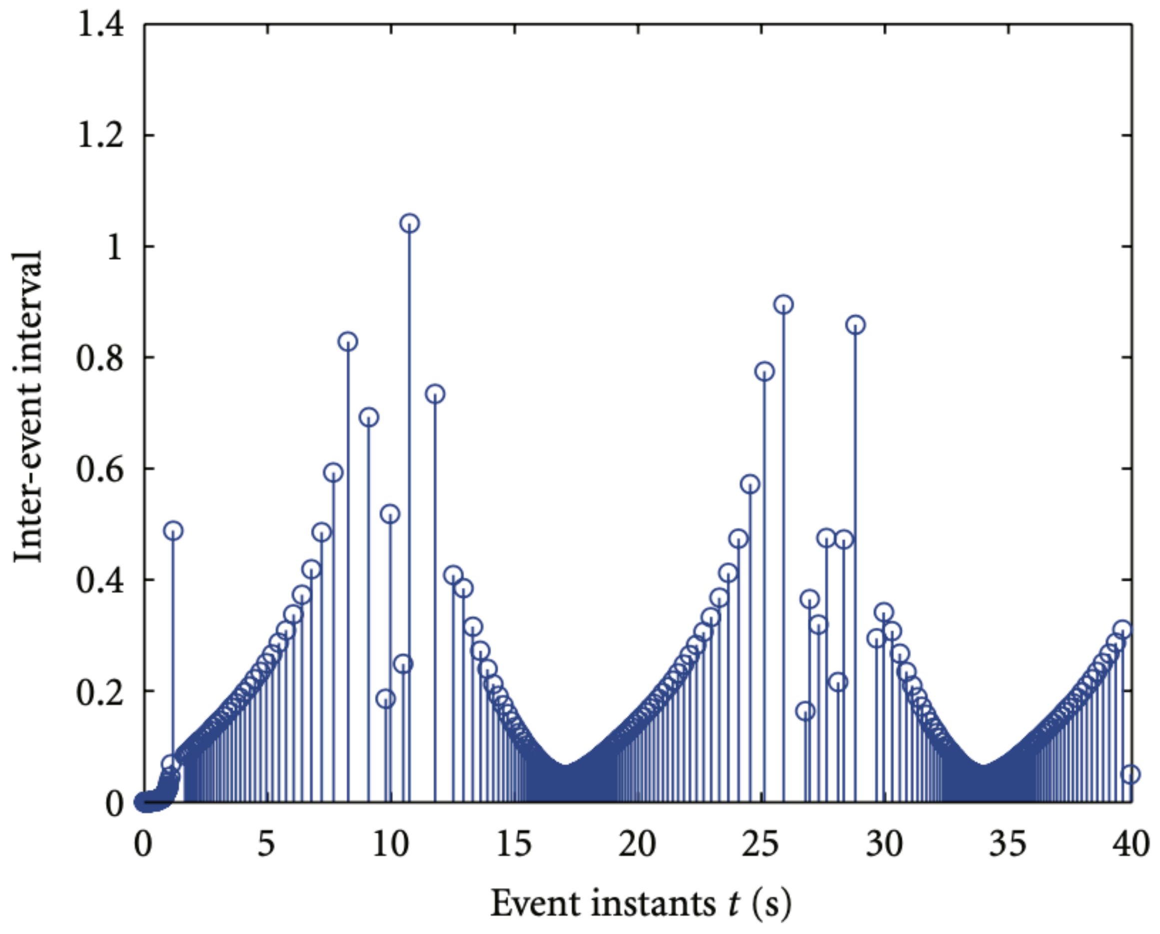
We then have:

$$\frac{dV}{dt} \leq -a|x|^2 + g|e||x| \leq -a|x|^2 + g\sigma|x|^2 \leq (-a + g\sigma)|x|^2 < -b|x|^2$$

☑ We can execute the control input when the error value $|e|$ reaches to $\sigma|x|$ which guarantees asymptotic stability of the closed-loop control system: $\rightarrow$ **we call $|e| = \sigma|x|$ a triggering law!**

Inter-event Times and Zeno Behavior

Inter-event time: the temporal duration $\tau_{i+1} - \tau_i$ between two consecutive triggering times is called inter-event time.



- For time-triggered/periodic schemes, the inter-event times are known and constant
- For event-triggered scheme, the inter-event times are dynamic, not known in advance, sometimes even random!

Zeno behavior: If an infinite number of events are triggered within a finite time

- Implementation of such control input policy on a digital controller is not possible due to limited sampling rate
- The triggering law should be designed carefully to avoid Zeno behavior!

Corollary: For a LTI control system $\dot{x}(t) = Ax(t) + Bu(t)$, let $u(t) = Lx(t)$ be a control law ensuring the closed loop system is asymptotically stable. Let the control signals be executed according to the event-triggered law $|e| \geq \sigma |x|$. Then the inter-event times $\tau_{i+1} - \tau_i$, $i \in \mathbb{N}$ resulting from the event-triggered execution are lower-bounded.

Proof of Corollary

Event-triggered Control: Discrete Time Linear Systems

Consider a discrete LTI system

$$x(t+1) = Ax(t) + Bu(t), \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m, \quad t \in \mathbb{N}_0$$

- Pair (A, B) is controllable, i.e., the matrix $(B, AB, A^2B, \ldots, A^{n-1}B)$ has full rank.

Consider a triggering function $\xi : \mathbb{R}^n \to \mathbb{R}$, and define the event-triggered control input $u(t)$ as

$$u(t) = \begin{cases} Lx(t) & \text{if } \xi(x(t)) \geq 0 \\ u(t-1) & \text{if } \xi(x(t)) < 0 \end{cases}$$

L is the optimal feedback gain minimising the LQ cost function

$$J = \sum_{t=0}^{T} [x(t)^\top Q x(t) + u(t)^\top R u(t)]$$

- $L = -(R + B^\top PB)^{-1}B^\top PA$
- P solves the Algebraic Riccati equation: $P = Q + A^\top(P - PB(R + B^\top PB)^{-1}B^\top P)A$

Given $u(t)$, closed-loop system: $$x(t+1) = \begin{cases} (A+BL)x(t) & \text{if } \xi(x(t)) \geq 0 \\ Ax(t) + BLx(\bar{t}) & \text{if } \xi(x(t)) < 0 \end{cases}$$

- $\bar{t} \in \{0,1,\ldots,t-1\}$: time index of last triggered event!

Event-triggered Control

Define the Lyapunov function $V(x(t)) = x(t)^\top P x(t)$, $P > 0$, and compute $\Delta V = V(x(t+1)) - V(x(t))$:

- **Control input held constant:**

$$\Delta V_h = \left(Ax(t) + BLx(\bar{t})\right)^\top P \left(Ax(t) + BLx(\bar{t})\right) - V(x(t))$$

- **Control input updated:**

$$\Delta V_u = \left((A + BL)x(t)\right)^\top P \left((A + BL)x(t)\right) - V(x(t)) < 0 \quad \text{(closed-loop system is stable)}$$

Define the triggering function ξ as

$$\xi(x(t)) = \Delta V_h(x(t)) - \sigma \Delta V_u(x(t)), \quad \sigma \in (0,1]$$

- If $\xi(x(t)) \geq 0 \quad \rightarrow \quad \Delta V_u(x(t)) < 0$
- If $\xi(x(t)) < 0 \quad \rightarrow \quad \Delta V_h(x(t)) - \sigma \Delta V_u(x(t)) < 0 \quad \rightarrow \quad \Delta V_h(x(t)) < \sigma \Delta V_u(x(t)) < 0$

$\Rightarrow$ Triggering function ξ is stabilising as it ensures a monotonically decreasing Lyapunov function $V(x(t))$ along the trajectory of $x(t)$

$\Rightarrow$ σ is a tuning parameter and determines the triggering rate!

Event-triggered Control: A numerical Example

Consider the LTI control system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -2x_1 + 3x_2 + u$$

Controlled by the stabilising control inputs $u = x_1 - 4x_2$. Find a triggering value σ such that control input executions are governed by the law $|e| > \sigma|x|$ and the closed-system remains asymptotically stable.

Solution: Define Lyapunov function $V(x) = x^\top P x > 0$ where $P = \begin{pmatrix} 1 & 0.25 \\ 0.25 & 1 \end{pmatrix}$ and $x = [x_1, x_2]^\top$. We can derive

$$\frac{dV}{dt} = -x^\top Q x + e^\top L^\top B^\top P x + x^\top P B L e, \quad Q = \begin{pmatrix} 0.5 & 0.25 \\ 0.25 & 1.5 \end{pmatrix}$$

Event-triggered closed-loop system asymptotically stable: $\dfrac{dV}{dt} \leq -a|x|^2 + g|e||x|$

- Selecting $a = \lambda_{min}(Q) = 1 - \dfrac{\sqrt{5}}{4}$ and $g = |L^\top B^\top P + PBL| = 8$ ensure the Lyapunov inequality holds!
- Since $\sigma g < a$ should hold, we should select $\sigma < 0.0551$. Selecting $\sigma = 0.05$ results in triggering instances whenever $|e| > 0.05|x|$.

References

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