Distributed Systems & Control

Advanced Topics in Control 2022

Lecture 6: Laplacian & Incidence Matrix Dynamics

ETH zürich

AUTOMATIC CONTROL LABORATORY

Brief announcements

227-0690-12L Advanced Topics in Control (Spring 2022) FS2022

Dashboard / Meine Kurse / 227-0690-12L Advanced Topics in Control (Spring 2022) FS2022

General Information

Lecturers: Prof. Florian Dörfler, Dr. Mathias Hudoba de Badyn, Dr. Vahid Mamduhi

Assistants: Andrea Martinelli, Dr. Dominic Liao-McPherson, Alberto Padoan, Carlo Cenedese

Student assistants: Joudi Hajar, Aristomenis Sfetsos

	When	Where	Video link
Lectures:	Mondays, 16:00-18:00	HG D1.1 (also streamed online)	link
Tutorials:	Fridays, 10:00-12:00	HG D1.1 (also streamed online)	link

Grading: based on 3 homework assignments (50%) and a final project (50%). Here is the information on grading, homework, and the final project.

- **projects**: start forming groups & discuss topics
- **get in touch** with myself, Mathias, or Vahid anytime in the next **4-5 weeks**
- **contents** of a first meeting: pitch your idea & rough work plan

recap: the Laplacian matrix

(last lecture, exercise session, & chapter 6)

- **Laplacian matrix** of a weighted digraph G: $L = D_{\text{out}} - A$

$$L = \begin{bmatrix} \vdots & \ddots & \vdots & \iddots & \vdots \\ -a_{i1} & \cdots & \sum_{j=1, j\neq i}^{n} a_{ij} & \cdots & -a_{in} \\ \vdots & \iddots & \vdots & \ddots & \vdots \end{bmatrix}$$

- **error vector** $(Lx)_i = \sum_{j=1, j\neq i}^{n} a_{ij}(x_i - x_j) = \sum_{j \in \mathcal{N}^{\text{out}}(i)} a_{ij}(x_i - x_j)$

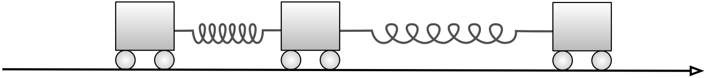
- **Laplacian potential** or **disagreement function**:

$$x^\top L x = \frac{1}{2} \sum_{i,j=1}^{n} a_{ij}(x_i - x_j)^2 = \sum_{\{i,j\} \in E} a_{ij}(x_i - x_j)^2$$

- **Laplacian flow** $\boxed{\dot{x} = -Lx}$ or $\dot{x}_i = -\sum_{j=1, j\neq i}^{n} a_{ij}(x_i - x_j)$

examples

The Laplacian in a mechanical network of springs



- each node i is subject to an **elastic force**

$$F_{\text{elastic},i} = \sum_{j \neq i} a_{ij}(x_j - x_i) = -(Lx)_i$$

- the **elastic energy** of a **single spring** is

$$E_{\text{elastic,ij}} = \frac{1}{2} a_{ij}(x_i - x_j)^2$$

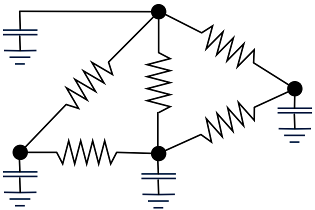
- the **total elastic energy** is

$$E_{\text{elastic}} = \frac{1}{2} \sum_{\{i,j\} \in E} a_{ij}(x_i - x_j)^2 = \frac{1}{2} x^\top L x$$

- the Newtonian **dynamics** are

$$M_i \ddot{x}_i = -D_i \dot{x}_i - \sum_{j \neq i} a_{ij}(x_j - x_i)$$

The Laplacian in an electrical network of resistors



- **Ohm's law** gives the current flowing from i to j as

$$c_{i \to j} = a_{ij}(v_i - v_j)$$

where $a_{ij} = 1/(\text{resistance})_{ij}$ is the conductance & v_i is the potential

- **Kirchhoff's current law** at node i:

$$c_{\text{inj},i} = \sum_{j \neq i} c_{i \to j} = \sum_{j \neq i} a_{ij}(v_i - v_j) \quad \text{or} \quad c_{\text{inj}} = L\,v$$

- **total dissipated power** is

$$E_{\text{dissipated}} = \sum_{\{i,j\} \in E} a_{ij}(v_i - v_j)^2 = v^\top L v$$

- **Faraday's law** at capacitor i is $C_i \dot{v}_i = -c_{\text{inj},i}$ & **network dynamics** are

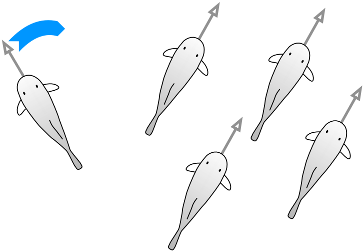
$$C_i \dot{v}_i = -\sum_{j \neq i} c_{i \to j} = -\sum_{j \neq i} a_{ij}(v_i - v_j)$$

Flocking behavior for a group of animals

$$\dot{\theta}_i = \begin{cases} (\theta_j - \theta_i), & \text{if one neighbor} \\ \frac{1}{2}(\theta_{j_1} - \theta_i) + \frac{1}{2}(\theta_{j_2} - \theta_i), & \text{if two neighbors} \\ \frac{1}{m}(\theta_{j_1} - \theta_i) + \cdots + \frac{1}{m}(\theta_{j_m} - \theta_i), & \text{if } m \text{ neighbors} \end{cases}$$

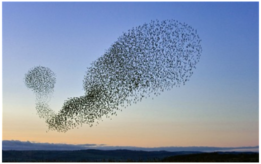
or

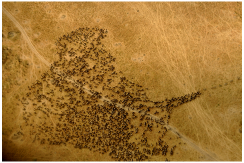
$$\dot{\theta}_i = \text{average}\left(\{\theta_j, j \in \mathcal{N}^{\text{out}}(i)\}\right) - \theta_i$$



or

$$\dot{\theta} = -L\theta$$





algebraic & spectral graph theory

Summary of key results

- **zero row sums:** $L\mathbb{1} = \mathbb{0}$
- **left eigenvectors:** $\mathbb{1}^\top L = \mathbb{0}^\top$ if and only if G is weight-balanced

$\Rightarrow$ general: $w^\top L = \mathbb{0}^\top$ for $w \geq 0$ & $w_i > 0$ iff i is globally reachable

- **spectrum:** eigenvalues other than 0 have strictly positive real part
- **rank condition:** $\operatorname{rank}(L) = n - \#(\text{sinks in condensation digraph})$

$\Rightarrow$ $\operatorname{rank}(L) = n - 1$ if and only if G has a globally reachable node

Symmetric case $L = L^\top$:

- **zero row/column sums:** $L\mathbb{1} = L^\top\mathbb{1} = \mathbb{0}$
- **spectrum:** $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq 2 \cdot \max_i d_{\text{out}}(i)$
- **algebraic connectivity** $\lambda_2 > 0$ if and only if G is connected

Organization of today's lecture

Continuous-Time Averaging

(ctd from last week)

(Chapter 7)

Diffusively Coupled Linear Systems

(selected topics)

(Chapter 8)

Organization of today's lecture cont'd

Incidence Matrix & Applications

(selected topics)

(Chapter 9)

Extra Notes on Circuits

(on moodle)

+ review paper on

"Electrical Networks and Algebraic Graph Theory . . . "

Continuous-Time Averaging Algorithms or Laplacian Flows

$$\dot{x} = -Lx$$

(Chapter 7)

Symmetric case from last lecture

- **dynamics:** $\dot{x} = -Lx$ with $L = L^\top$ & connected graph
- **spectrum:** $L = V \begin{bmatrix} 0 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} V^\top$ where $0 < \lambda_2 \leq \cdots \leq \lambda_n$

  are non-zero Laplacian eigenvalues & $V = \begin{bmatrix} \frac{1}{\sqrt{n}}\mathbb{1} & v_2 & \dots & v_n \end{bmatrix}$
  is a matrix collecting the associated orthonormal eigenvectors
- **modal decomposition** of the solution:

$$x(t) = \sum_{i=1}^{n} \left(v_i^\top x_0\right) \cdot v_i e^{-\lambda_i t} \quad \xrightarrow{t \to \infty} \quad \frac{1}{n}\left(\mathbb{1}^\top x_0\right)\mathbb{1}$$

  $\rightarrow$ consensus to average(x_0)
- perfectly **analogous for a diagonalizable matrix**, where $V^\top$ is replaced by a matrix containing left eigenvectors as rows

Solutions & convergence of continuous-time LTI systems

Solutions

The **solution** of $\dot{x} = Ax$ is $x(t) = e^{At}x_0$, where $e^{A} = I + A + \frac{1}{2!}A^2 + \dots$

Diagonalizable case with eigenstructure: $Av_i = \lambda_i v_i$ & $w_i^\top A = \lambda_i w_i^\top$

$\longrightarrow x(t) = \sum_i e^{\lambda_i t} v_i \cdot \left(w_i^\top x_0\right) = \sum_i \text{mode}\, i \cdot \text{projection of } x_0$

Glimpse into **non-diagonalizable** case: $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \rightarrow e^{At} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$

Convergence

A is **convergent**: $\lim_{t\to\infty} e^{At} = 0 \Leftrightarrow A$ is Hurwitz $\Re(\lambda) < 0\ \forall \lambda \in \text{spec(A)}$

A is **semi-convergent**: $\lim_{t\to\infty} e^{At}$ exists $\Leftrightarrow$ if 0 an eigenvalue, then it is semi-simple, and all other eigenvalues have strictly negative real part

discussion of
$\dot{x} = -Lx$ on the board

Consensus with a globally reachable node

If a Laplacian matrix L has an associated digraph G with a globally reachable node, then the following statements hold:

1. The eigenvalue 0 is simple and all others have negative real part.
2. $\lim_{t\to\infty} e^{-Lt} = \mathbb{1} w^\top$, where w is the left eigenvector of L with eigenvalue 0 satisfying $w_1 + \cdots + w_n = 1$.

   Moreover, $w_i \geq 0$ for all nodes i, and $w_i > 0$ iff node i is globally reachable. Accordingly, $w_i = 0$ iff node i is not globally reachable.
3. The solution to $\frac{d}{dt}x(t) = -Lx(t)$ satisfies $\lim_{t\to\infty} x(t) = \mathbb{1} \cdot \left(w^\top x_0\right)$.
4. If additionally G is weight-balanced, then G is strongly connected, $\mathbb{1}^\top L = \mathbb{0}^\top$ and $w = \frac{1}{n}\mathbb{1}$ (because $\frac{1}{n}\mathbb{1}^\top\mathbb{1} = 1$) so that

$$\lim_{t\to\infty} x(t) = \mathbb{1}^\top x_0/n \cdot \mathbb{1} = \text{average}(x_0) \cdot \mathbb{1} \, .$$

Comparison with discrete-time averaging algorithms

Main result for discrete-time averaging algorithms

Let A be a row-stochastic matrix. Assume that the digraph associated to A contains a globally reachable node and the subgraph of globally reachable nodes is aperiodic. Then the solution to $x^+ = Ax$ satisfies

$$\lim_{k\to\infty} x(k) = \mathbb{1} \cdot \left(w^\top x_0 \right) ,$$

where $w_i > 0$ if and only if node i is globally reachable.

Main result for continuous-time averaging algorithms

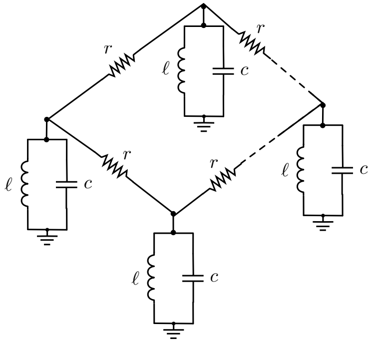
Let L be a Laplacian matrix. Assume that the the digraph associated to A contains a globally reachable node. Then the solution to $\dot{x} = -Lx$ satisfies

$$\lim_{t\to\infty} x(t) = \mathbb{1} \cdot \left(w^\top x_0 \right) ,$$

where $w_i > 0$ if and only if node i is globally reachable.

second-order Laplacian flows

Example: resistively coupled identical LC tank circuits



- **individual dynamics** of isolated LC tank circuit (derive at home)

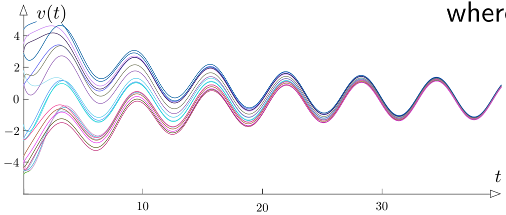
$$\ddot{v} = -\frac{1}{\ell c} v$$

$\rightarrow$ oscillator with resonance freq. $\sqrt{\frac{1}{\ell c}}$

- **coupled dynamics** of connected tanks

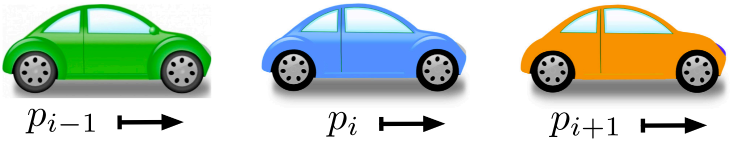
$$\ddot{v} = -\frac{1}{\ell c} v - L\dot{v}$$

where L is Laplacian with weights $1/r$



discussion on board

Another example: the platoon problem



model of car $\ddot{p}_i = u_i$ with front/rear **cameras** for nearest-neighbor sensing

objective: design a distributed controller so that $\|p_i - p_{i+1}\| = d$

control based on artificial springs/dampers + velocity damping + GPS:

$$u_i = -\gamma_d(\dot{p}_i - \dot{p}_{i+1}) - \gamma_p(p_i - p_{i+1} - d) - k_d\dot{p}_i - k_p(p_i - p_{i,\mathrm{ref}}) - \dots$$

Linearization is *second-order Laplacian flow*:

$$\ddot{x} + \underbrace{k_d\dot{x}}_{\text{absolute velocity}} + \underbrace{\gamma_d L\dot{x}}_{\text{relative velocity}} + \underbrace{k_p x}_{\text{absolute position}} + \underbrace{\gamma_p L x}_{\text{relative position}} = \mathbb{0}$$

more results on diffusively coupled systems in Chapter 8

the incidence matrix

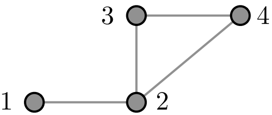
Setup: undirected and unweighted graph G with n nodes and m edges

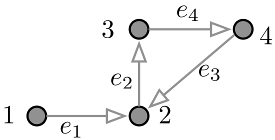
Numbering & orientation: number the edges with $e \in \{1, \ldots, m\}$ (in arbitrary order) and assign an arbitrary direction to each edge.

Definition: the (oriented) **incidence matrix** $B \in \mathbb{R}^{n \times m}$ of the graph G is

$$B_{ie} = \begin{cases} +1, & \text{if node } i \text{ is the source node of edge } e, \\ -1, & \text{if node } i \text{ is the sink node of edge } e, \\ 0, & \text{otherwise.} \end{cases}$$

$$= \text{which nodes are incident to which edges plus orientation}$$





$$B = \begin{bmatrix} +1 & 0 & 0 & 0 \\ -1 & +1 & -1 & 0 \\ 0 & -1 & 0 & +1 \\ 0 & 0 & +1 & -1 \end{bmatrix}$$

discussion on board

Summary of facts so far

- **error vector:** $e = B^\top x = \begin{bmatrix} \vdots \\ x_i - x_j \\ \vdots \end{bmatrix}$ for each edge $\{i, j\}$ $\Rightarrow$ $B^\top \mathbb{1} = \mathbb{0}$

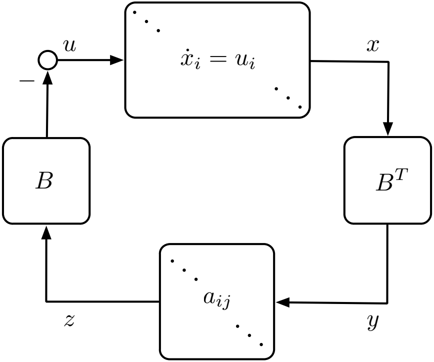
$\Rightarrow$ **Laplacian potential:**

$$\sum\nolimits_{\{i,j\}\in E} a_{ij}(x_i - x_j)^2 = \frac{1}{2}\|B^\top x\|^2_{\mathrm{diag}(\{a_{ij}\}_{\{i,j\}\in E})} = \frac{1}{2}\|e\|^2_{\mathrm{diag}(\{a_{ij}\}_{\{i,j\}\in E})}$$

- **Laplacian factorization:** $\boxed{L = B \cdot \mathrm{diag}\left(\{a_{ij}\}_{\{i,j\}\in E}\right) \cdot B^\top}$

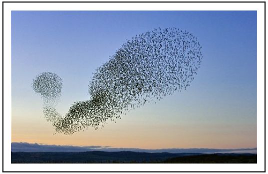
$\Rightarrow$ **rank:** $\mathrm{rank}(B) = n - \#\{\text{connected components}\}$

Laplacian flow in relative sensing networks



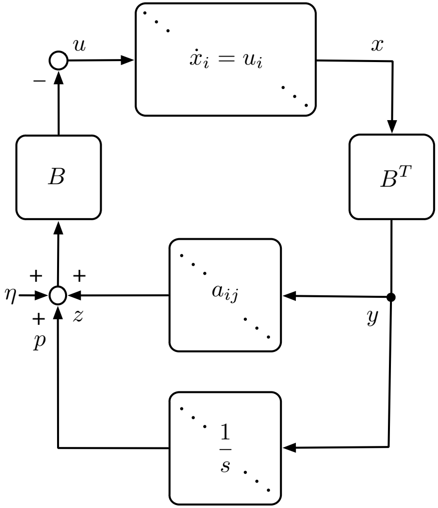
decomposition of Laplacian flow:

$$\begin{aligned}\dot{x} &= -Lx \\ &= -B\,\mathrm{diag}(\{a_e\}_{e\in E})B^\top x\end{aligned}$$



open-loop plant:	$\dot{x}_i = u_i\,,$	$i \in \{1,\dots,n\}\,,$	or	$\dot{x} = u\,,$
measurements:	$y_{ij} = x_i - x_j\,,$	$\{i,j\} \in E\,,$	or	$y = B^\top x\,,$
control gains:	$z_{ij} = a_{ij} y_{ij}\,,$	$\{i,j\} \in E\,,$	or	$z = \mathrm{diag}(a_e) y\,,$
control inputs:	$u_i = -\sum_{\{i,j\}\in E} z_{ij}\,,$	$i \in \{1,\dots,n\}\,,$	or	$u = Bz\,.$

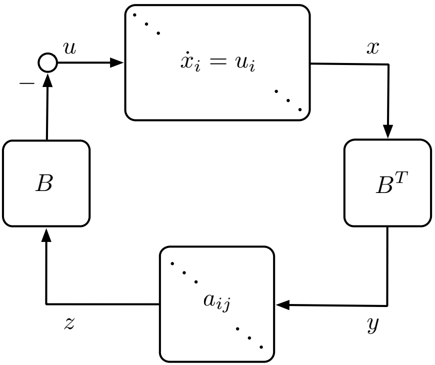
A very versatile framework



- Laplacian flow: $\dot{x} = -Lx$
  (see last lecture)
- PI consensus with constant disturbances
  (see Exercise 9.4)
- circuits
  (later today)
- distributed optimization
  (in a few weeks)
- nonlinear consensus, formation control, collision-avoidance
  (in a few weeks)

$\vdots$

Summary: properties of the incidence matrix



- **Laplacian factorization:**

$$L = B \cdot \mathrm{diag}\left(\{a_{ij}\}_{\{i,j\}\in E}\right) \cdot B^\top$$

- nearest-neighbor **error vector**: $y = B^\top x$ with $B^\top \mathbb{1} = \mathbb{0}$
- **rank** for a connected graph: $\mathrm{rank}(B) = n - 1$

Not derived (but often important):

- kernel(B) = **cycle space** spanned by signed-closed-path vectors
- image$(B^\top)$ = **cutset space** spanned by cutset orientation vectors

circuits

See extra notes & review paper for reference

INVITED PAPER

Electrical Networks and Algebraic Graph Theory: Models, Properties, and Applications

This article provides an overview of the connections of algebraic graph theory and the design and analysis of electric circuits, from integrated circuits to large distribution grids.

By Florian Dörfler, *Member IEEE*, John W. Simpson-Porco, *Member IEEE*, and Francesco Bullo, *Fellow IEEE*

KCL & KVL on board

Kirchhoff & Ohm's laws in a resistive circuit

- **setup:** circuit with nodes $\mathcal{V} \in \{1, \dots, n\}$ & branches $\{i, j\} \in \mathcal{E}$
- **nodal variables:** external injections I_i & potentials V_i
- directed **branch variables:** branch currents f_{ij} & voltage drops u_{ij}

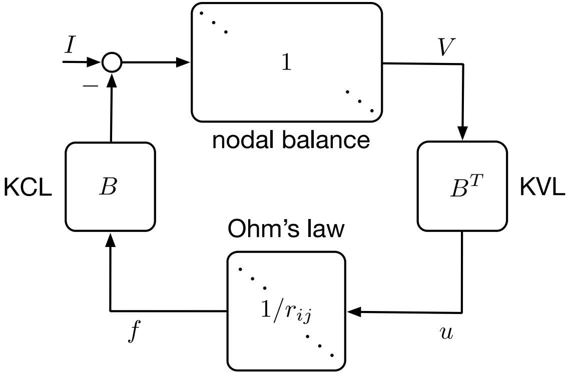
(i) **Kirchhoff's current law** (KCL): $\sum_{j \in N_{out}(i)} f_{ij} - \sum_{j \in N_{in}(i)} f_{ji} = I_i$ or $\boxed{I = Bf}$

(ii) **Kirchhoff's voltage law** (KVL): $\sum_{(i,j) \in \text{cycle}} \pm u_{ij} = 0$ or $\boxed{u = B^\top V}$

(iii) **Ohm's law** for resistors: $u_{ij} = r_{ij} f_{ij}$ or $\boxed{f = \mathrm{diag}(1/r_{ij}) u}$

$\Rightarrow$ **nodal balance equations:** $\boxed{I = B\,\mathrm{diag}(1/r_{ij}) B^\top V = \mathcal{L}_R V}$

with network conductance (Laplacian) matrix $\mathcal{L}_R = B\,\mathrm{diag}(1/r_{ij}) B^\top$



- **Kirchhoff's voltage law** (KVL)

  $\boxed{u = B^\top V}$

- **Ohm's law** for resistors:

  $\boxed{f = \mathrm{diag}(1/r_{ij}) u}$

- **Kirchhoff's current law** (KCL)

  $\boxed{I = Bf}$

$\Rightarrow$ **nodal balance equations:**

$$\boxed{I = B\,\mathrm{diag}(1/r_{ij}) B^\top V = \mathcal{L}_R V}$$

dynamic circuits

(optional: time permitting)

Fundamental circuit laws

- **setup:** circuit with nodes $\mathcal{V} \in \{1, \ldots, n\}$ & branches $\{i, j\} \in \mathcal{E}$
- **nodal variables:** external injections I_i & potentials V_i
- **branch variables:** branch currents f_{ij} & voltage drops u_{ij}

(i) **Kirchhoff's current law** (KCL):

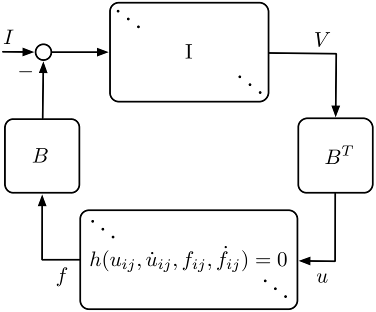
$$\boxed{I = Bf}$$

(ii) **Kirchhoff's voltage law** (KVL):

$$\boxed{u = B^\top V}$$

(iii) **constitutive relations** for branches:

$$\boxed{h(u_{ij}, \dot{u}_{ij}, f_{ij}, \dot{f}_{ij}) = 0}$$



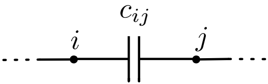
(iii) linear branch **constitutive relations**

resistor (Ohm's law):

$$\boxed{u_{ij} = r_{ij} f_{ij}}$$

inductor (Faraday):

$$\boxed{\ell_{ij} \dot{f}_{ij} = u_{ij}}$$

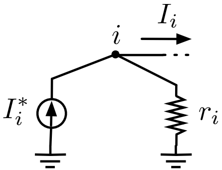


capacitor (charge):

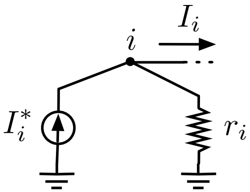
$$\boxed{c_{ij} \dot{u}_{ij} = f_{ij}}$$

(iv) **ground:** node with 0 potential $V_0 = 0$
(we drop the index "0" in r_{i0}, l_{i0}, c_{i0})

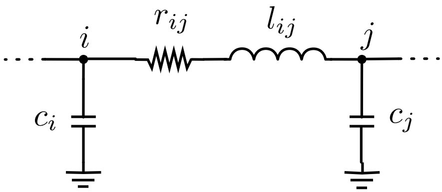
(v) **current sources/loads** with
constant current injection $I_i^* \in \mathbb{R}$



A prototypical circuit



RI-load model



Π-branch model

ground equation: $I_i = I_i^* - c_i \dot{V}_i - g_i V_i$

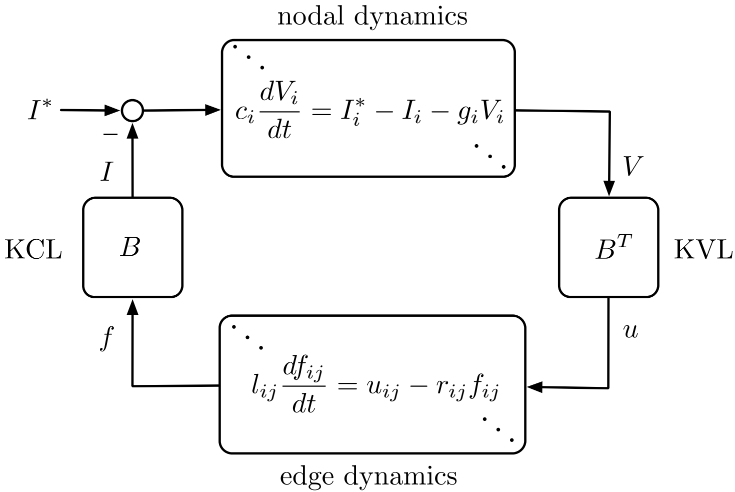
$(g_i = 1/r_i)$

branch equation: $\ell_{ij} \dot{f}_{ij} = u_{ij} - R_{ij} f_{ij}$

KCL: $I = Bf$

KVL: $u = B^\top V$

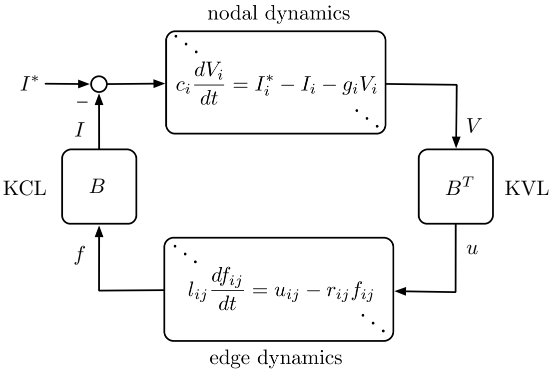
A prototypical circuit con'td



KCL:	$I = Bf$	ground equation:	$I = I^* - C\dot{V} - GV$
KVL:	$u = B^\top V$	branch equation:	$L\dot{f} = u - Rf$

circuit analysis on board

Energy analysis of circuits I

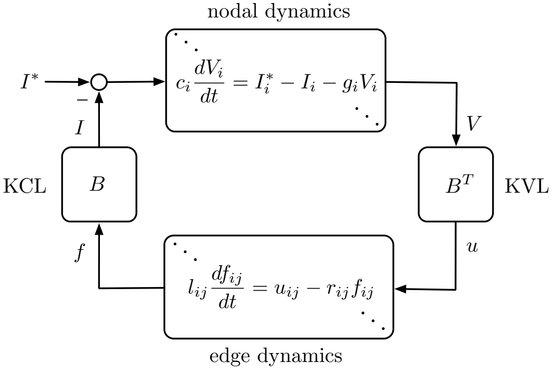


Lemma: Circulations in non-dissipative circuit

Consider the case $I^* = \mathbb{O}$, $G = \mathbb{O}$, $R = \mathbb{O}$. Then

1. the solution is a superposition of n undamped harmonic signals; and
2. if $C = I_n$, the frequencies of the harmonics are $\sqrt{\lambda_i}$, where λ_i are the eigenvalues of the Laplacian $\mathcal{L}_L = BL^{-1}B^\top$.

Energy analysis of circuits II



Lemma: Fully dissipative circuit

Consider the case $I^* \in \mathbb{R}^n$ and G, R positive definite. Then the system admits a globally exponentially stable equilibrium point.

A useful matrix lemma (see Exercise 9.11)

for this lecture, the exercise, and the homework

Lemma : Consider a positive semidefinite matrix $P \in \mathbb{R}^{n\times n}$ and a matrix $A \in \mathbb{R}^{m\times n}$ with $n \geq m$ forming the composite matrix

$$\mathcal{A} = \begin{bmatrix} -P & -A^\top \\ A & \mathbb{0}_{m\times m} \end{bmatrix}.$$

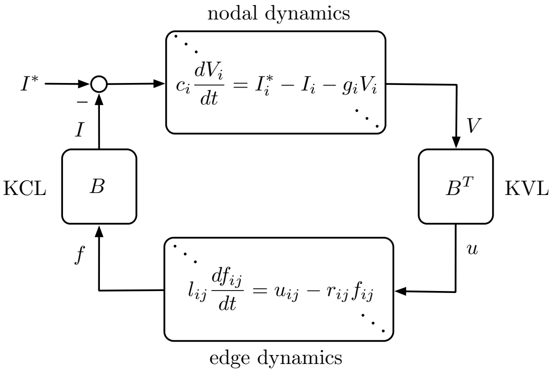
The matrix $\mathcal{A}$ has the following properties:

1) all eigenvalues are in the closed left half-plane: $\mathrm{spec}(\mathcal{A}) = \{\lambda \in \mathbb{C} \mid \mathcal{R}(\lambda) \leq 0\}$. Moreover, all eigenvalues on the imaginary axis have equal algebraic and geometric multiplicity;

2) if $\mathrm{kernel}(P) \cap \mathrm{image}(A^\top) \subseteq \{\mathbb{0}_n\}$, then $\mathcal{A}$ has no eigenvalues on the imaginary axis except for 0; and

3) if P is positive definite and A has full rank, then $\mathcal{A}$ has no eigenvalues on the imaginary axis.

Energy analysis of circuits III

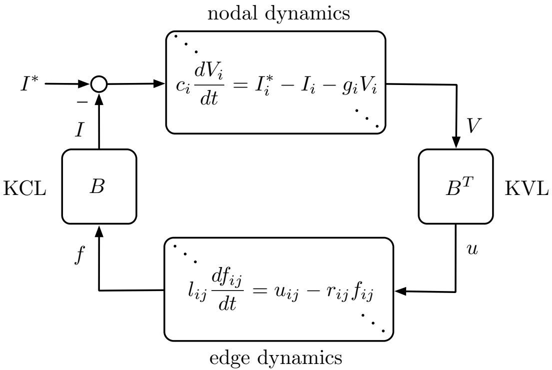


Lemma: Convergence to circulating currents

Consider the case $I^* = \mathbb{0}$, $R = \mathbb{0}$ and G is positive definite. Then

1. the equilibria are $(V^*, f^*) \in \big(\mathbb{0}, \mathrm{kernel}(B)\big)$, that is, the equilibria are defined up to a subspace corresponding to circulating currents; and
2. all trajectories converge to these equilibria.

Energy analysis of circuits IV



Lemma: Convergence to sync'd potentials

Consider the case $I^* = \mathbb{0}$, $G = \mathbb{0}$ and R is positive definite. Then

1. the equilibria are $(V^*, f^*) \in \big(\text{span}(\mathbb{1}), \mathbb{0}\big)$, that is, the potentials V^* reach consensus; and
2. all trajectories converge to these equilibria.

Reading assignment (lecture notes):

Chapter 7: Continuous-Time Averaging

Chapter 8: Diffusively Coupled Systems

Chapter 9: The Incidence Matrix

Next Exercise:

- led by Carlo Cenedese
- review of take-home messages
- examples & additional facts
- exercises & illustrations