Advanced Topics in Control: Distributed Systems and Control

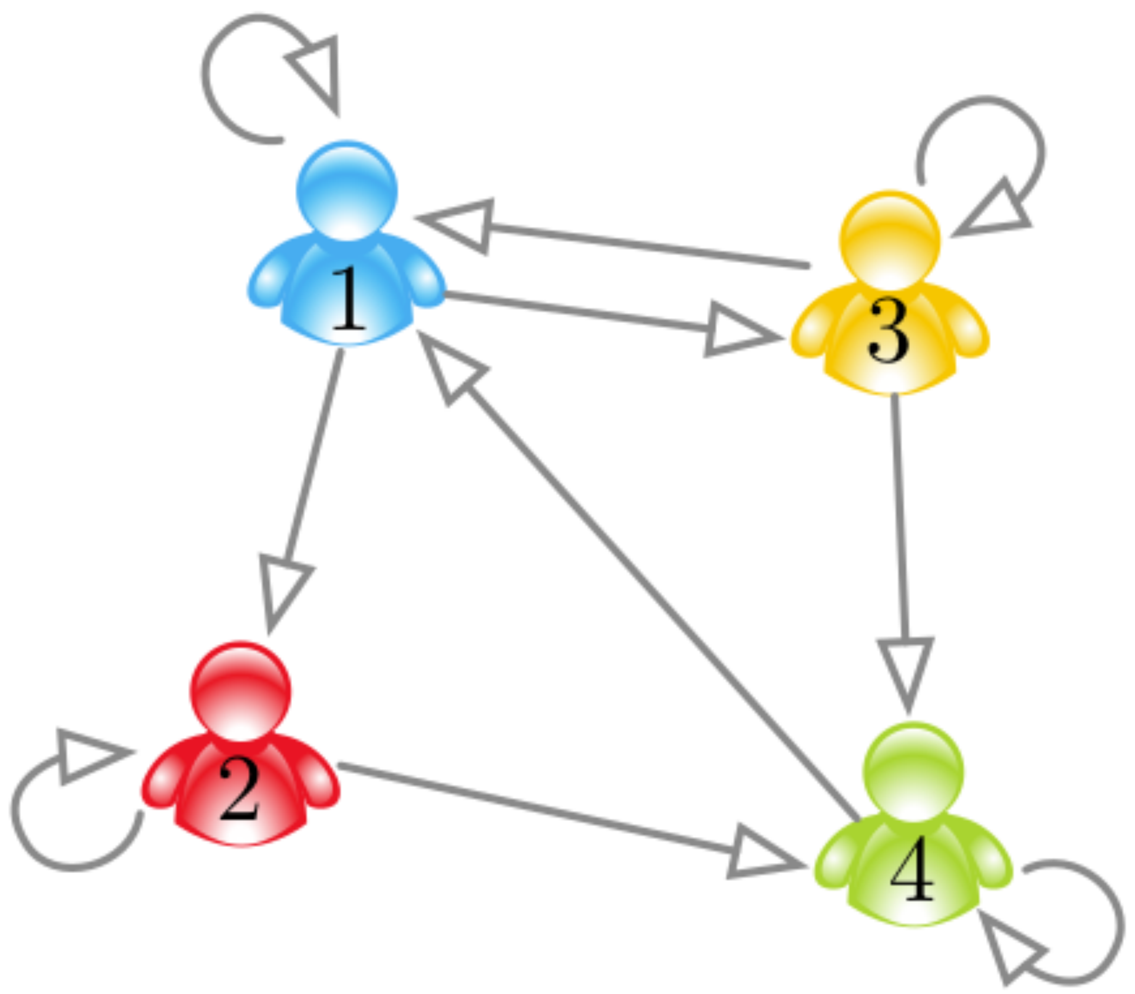
Lecture 3: Algebraic Graph Theory

Dr. Mohammad H. (Vahid) Mamduhi, March 7th, 2022

ETH zürich

AUTOMATIC CONTROL LABORATORY

Quick review of Lecture 2



Distributed averaging algorithm:

$$x^+ = Ax$$

Properties:

- sparsity pattern induced by graph
- nonnegative coefficients: $a_{ij} \geq 0$
- row-stochasticity: $\sum_j a_{ij} = 1$

Observations regarding the solution $x(k) = A^k x_0$:

$$A = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix} \quad A^2 = \begin{bmatrix} \frac{2}{9} & \frac{5}{18} & \frac{2}{9} & \frac{5}{18} \\ \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{2} \\ \frac{7}{18} & \frac{1}{9} & \frac{2}{9} & \frac{5}{18} \\ \frac{5}{12} & \frac{1}{6} & \frac{1}{6} & \frac{1}{4} \end{bmatrix} \quad \dots \quad A^\infty = \frac{6}{19} \begin{bmatrix} 1 & \frac{2}{3} & \frac{1}{2} & 1 \\ 1 & \frac{2}{3} & \frac{1}{2} & 1 \\ 1 & \frac{2}{3} & \frac{1}{2} & 1 \\ 1 & \frac{2}{3} & \frac{1}{2} & 1 \end{bmatrix}$$

$\Rightarrow$ agreement/sync/rendezvous: $\lim_{k \to \infty} x(k) = \left(\frac{6}{19} \begin{bmatrix} 1 & \frac{2}{3} & \frac{1}{2} & 1 \end{bmatrix} \cdot x_0 \right) \cdot \mathbb{1}_4$

Quick review of Lecture 2

Gershgorin disks theorem

Theorem 2.8 (Geršgorin Disks Theorem). *For any square matrix $A \in \mathbb{R}^{n\times n}$,*

$$\operatorname{spec}(A) \subset \bigcup_{i\in\{1,\dots,n\}} \underbrace{\left\{z \in \mathbb{C} \mid |z - a_{ii}| \le \sum\nolimits_{j=1,j\neq i}^{n} |a_{ij}|\right\}}_{\text{disk in the complex plane centered at } a_{ii} \text{ with radius } \sum_{j=1,j\neq i}^{n} |a_{ij}|}$$

$$x(t+1) = Ax(t) \Rightarrow \lim_{t\to\infty} x(t) = \lim_{t\to\infty} A^t x(0) \text{ exists if } A \text{ is convergent} \Leftarrow$$

or semi-convergent

$\Uparrow$

non-negative

dominant eigenvalue λ is simple

$\lambda > |\mu|$ for any other eigenvalue μ

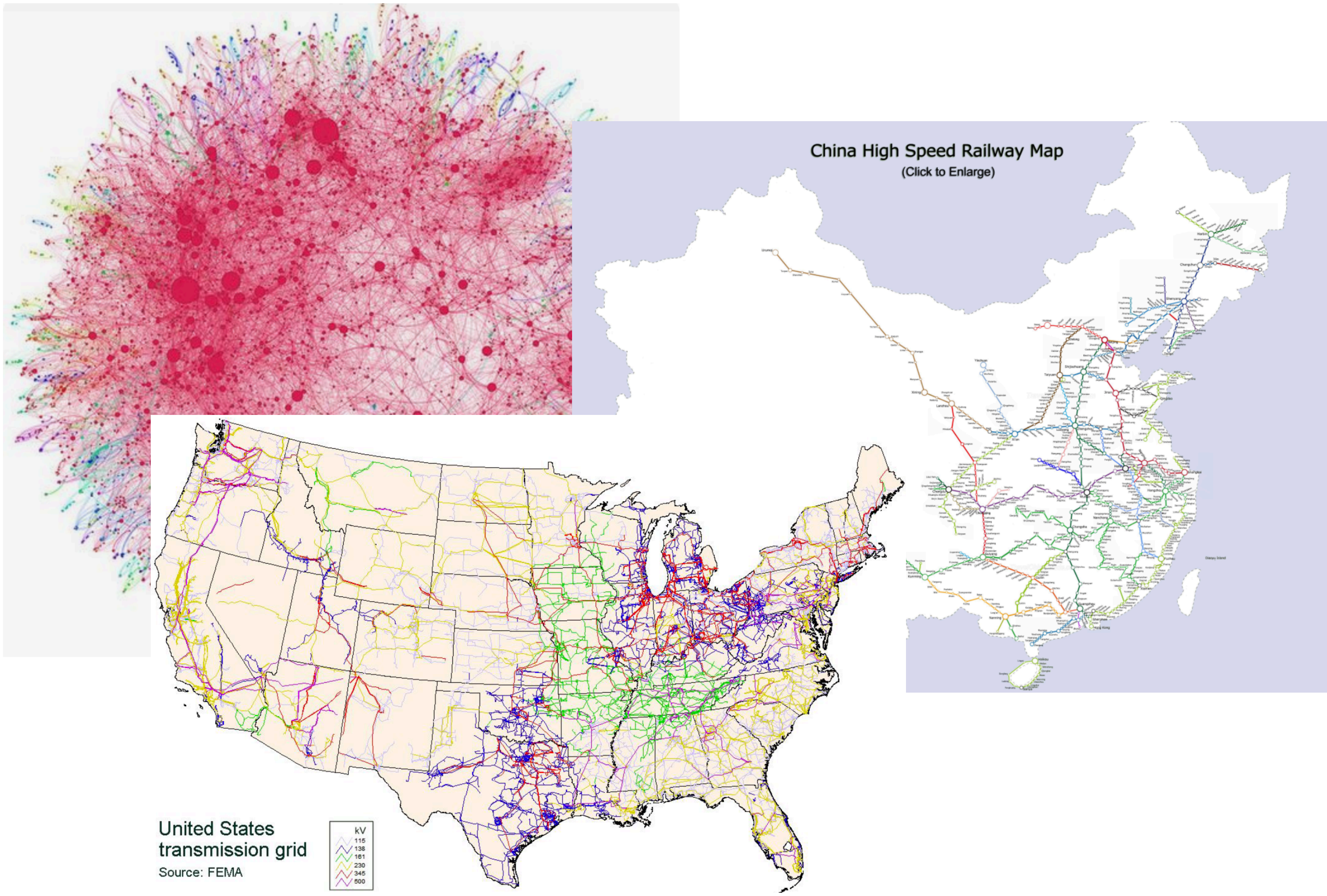
Theorem 2.13 (Powers of non-negative matrices with a simple and strictly dominant eigenvalue). *Let A be a non-negative matrix. Assume the dominant eigenvalue λ is simple and strictly larger, in magnitude, than all other eigenvalues. Then A/λ is semi-convergent and*

$$\lim_{k\to\infty} \frac{A^k}{\lambda^k} = vw^{\mathsf{T}},$$

where v and w are the right and left dominant eigenvectors of A normalized so that $v \geq 0$, $w \geq 0$ and $v^{\mathsf{T}}w = 1$.

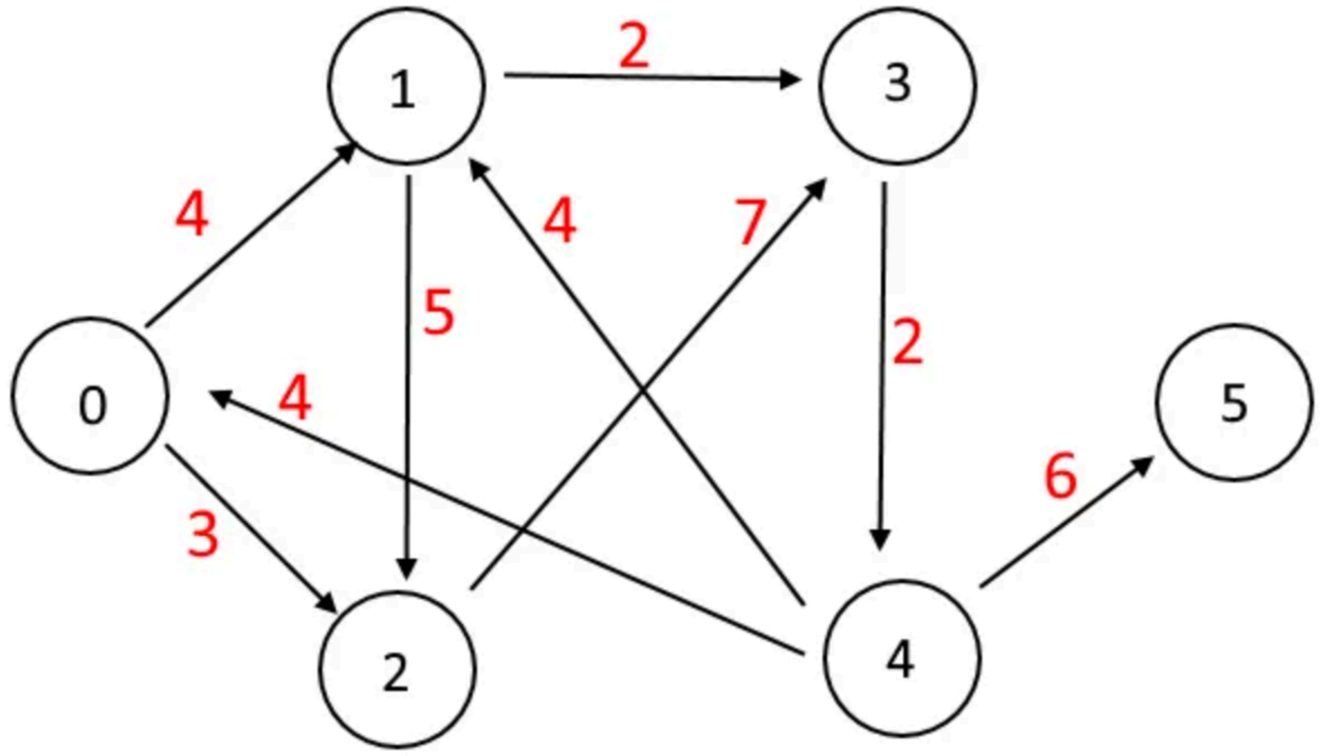
$\Uparrow$

Perron-Frobenius theorem: when A is primitive



Inter-connected systems: power grids
transportation networks
social networks, etc

real world (empirical observation)



graph + dynamics

"physics"
(modeling)

①

②

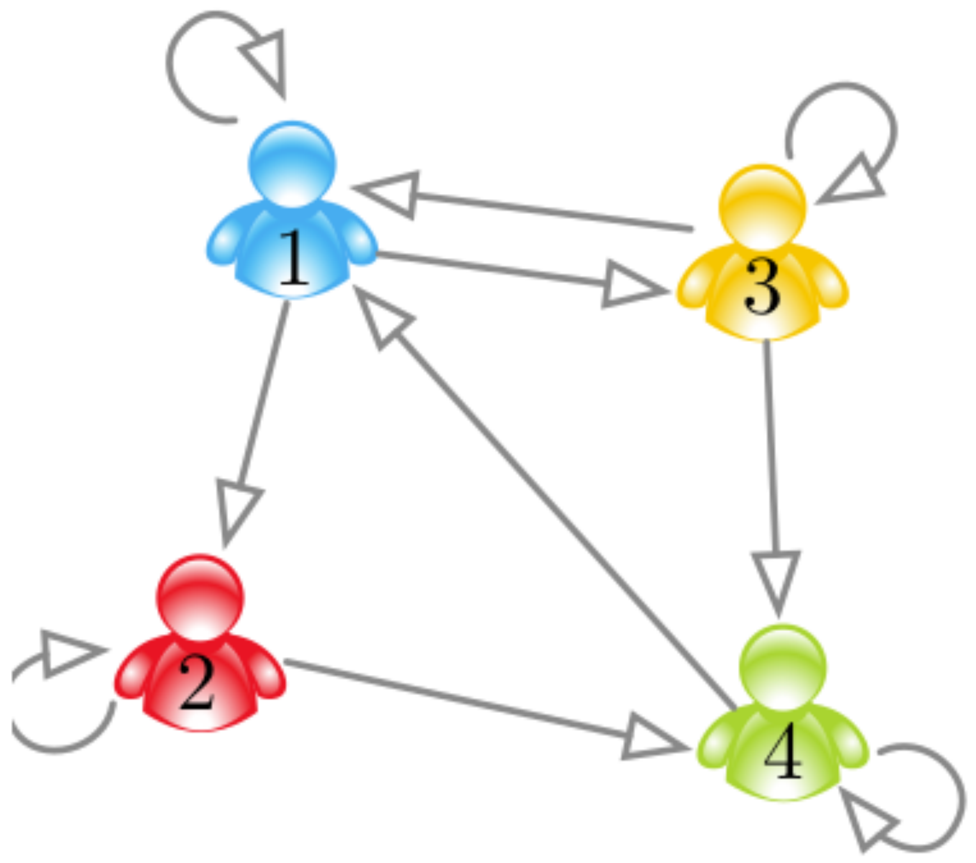
algebraic graph theory

① properties of graphs → properties of matrices

② graph structure → system's dynamic behaviour

mathematics (analysis)

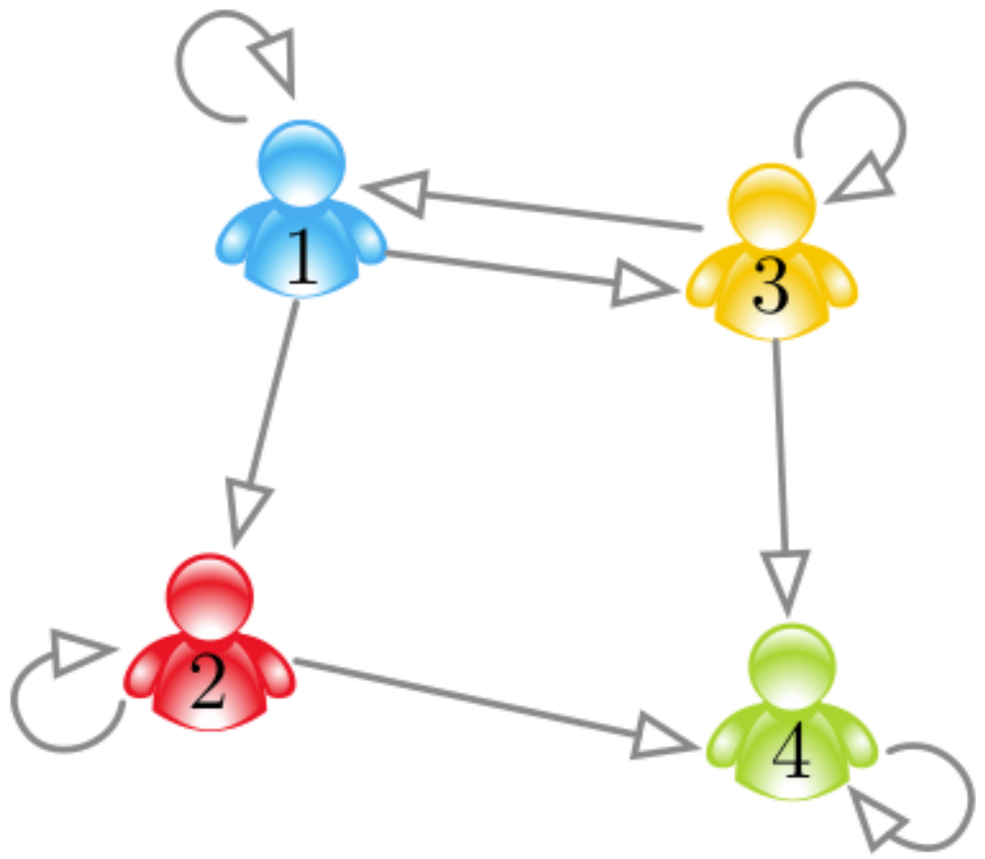
This week and next week: deduce convergence from graph



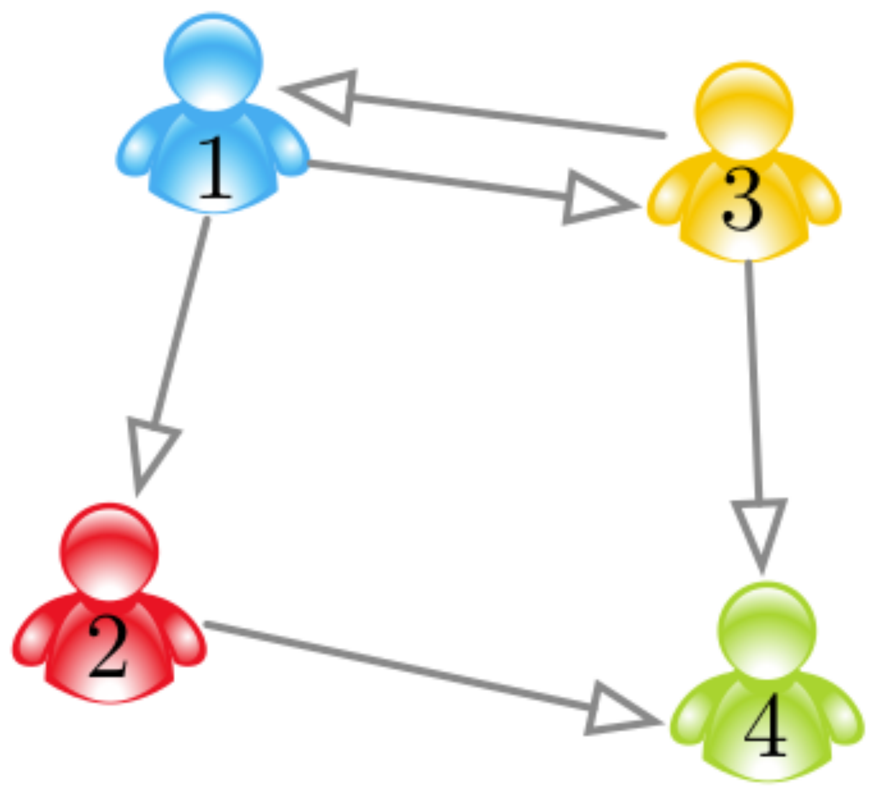
$$A = \begin{bmatrix} \star & \star & \star & \\ & \star & & \star \\ \star & & \star & \star \\ \star & & & \star \end{bmatrix}$$

nonnegative and row-stochastic

$\Rightarrow$ converges to $\mathbb{1} \cdot [\star \ \star \ \star \ \star]$



converges to $\mathbb{1} \cdot [0 \ 0 \ 0 \ 1]$



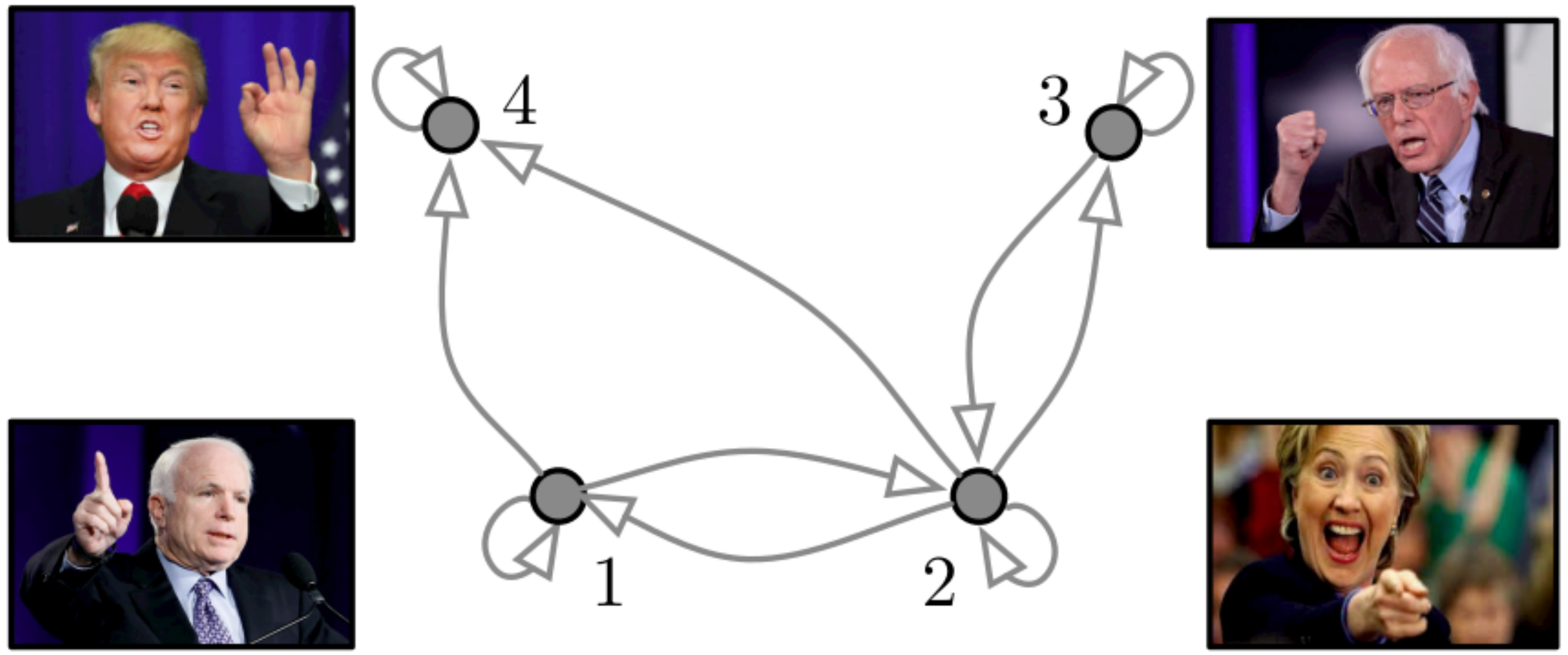
diverges & oscillates ...

Deduce convergence from graph: An example

1. iterative algorithm: $x^+ = Ax$
2. non-negativity: $A \geq 0$
3. row-stochasticity: $A\mathbb{1} = \mathbb{1}$
4. diagonal self-weights: $a_{ii} > 0$

$$A = \begin{pmatrix} \star & \star & \star & \star \\ \star & \star & \star & \star \\ \star & \star & \star & \\ & & & \star \end{pmatrix}$$

($\star > 0$ unspecified)



Q1: Does the iteration converge? $\Rightarrow$ yes ✓

Q2: What does it converge to? $\Rightarrow \lim_{k\to\infty} x(k) = \mathbb{1} \cdot \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \cdot x_0$

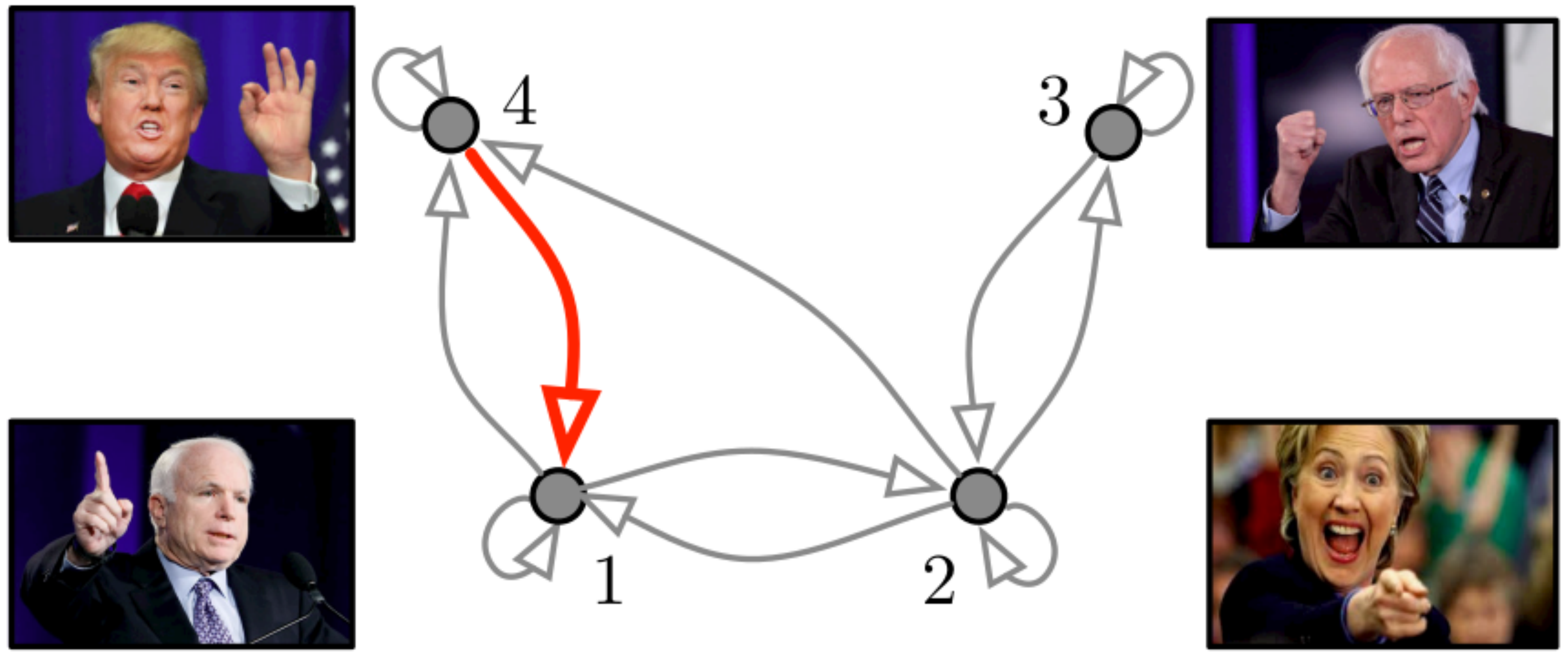
Deduce convergence from graph: An example

What if node 4 was also listening to somebody?

1. iterative algorithm: $x^{+} = Ax$
2. non-negativity: $A \geq 0$
3. row-stochasticity: $A\mathbb{1} = \mathbb{1}$
4. diagonal self-weights: $a_{ii} > 0$

$$A = \begin{pmatrix} \star & \star & \star & \star \\ \star & \star & \star & \star \\ \star & \star & \star & \\ {\color{red}\star} & & & \star \end{pmatrix}$$

$(\star > 0$ unspecified$)$

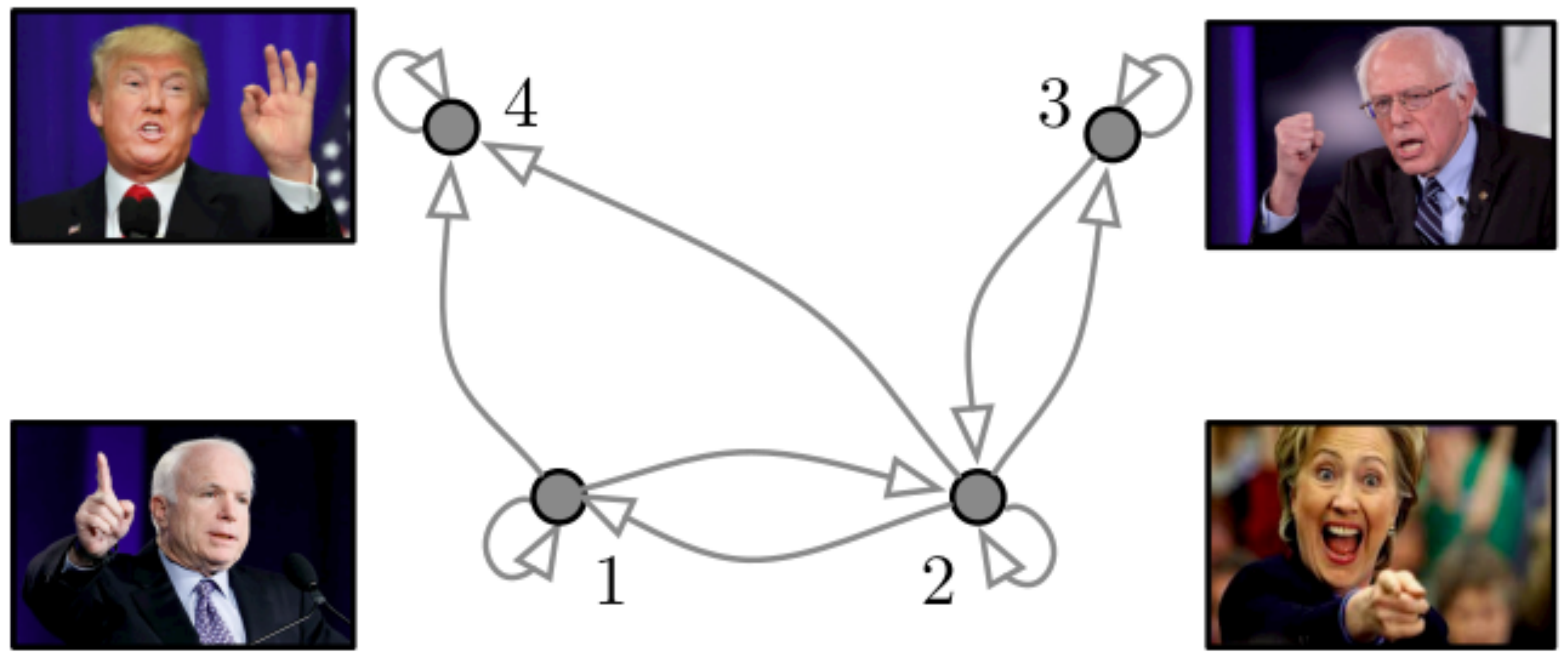


Q1: Does the iteration converge? $\Rightarrow$ yes ✓

Q2: What does it converge to? $\Rightarrow \lim_{k\to\infty} x(k) = \mathbb{1} \cdot {\color{red}\begin{bmatrix} \star & \star & \star & \star \end{bmatrix}} \cdot x_0$

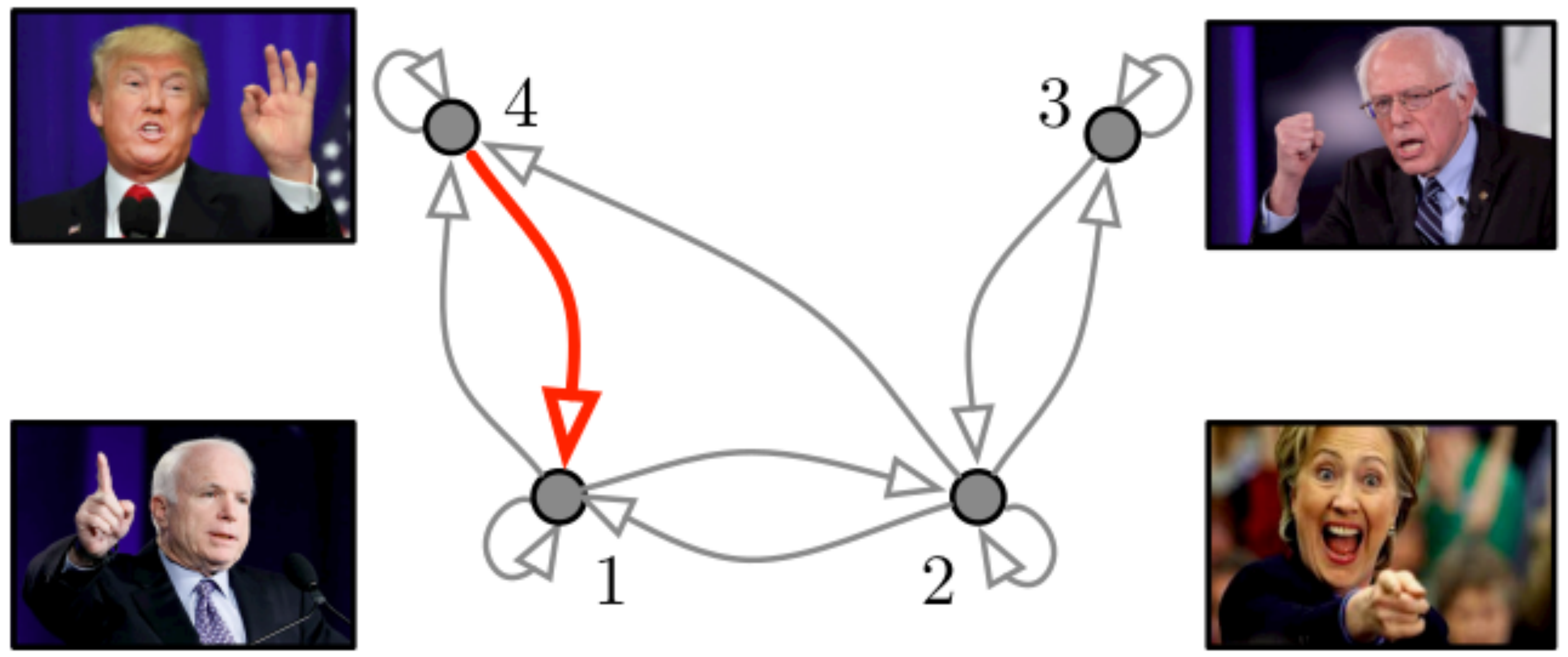
Deduce convergence from graph: An example

What is the distinguishing feature? Remember this example!



$$A = \begin{pmatrix} \star & \star & \star & \star \\ \star & \star & \star & \star \\ \star & \star & \star & \\ & & & \star \end{pmatrix}$$

$$\lim_{k\to\infty} x(k) = \mathbb{1} \cdot \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \cdot x_0$$



$$A = \begin{pmatrix} \star & \star & \star & \star \\ \star & \star & \star & \star \\ \star & \star & \star & \\ \star & & & \star \end{pmatrix}$$

$$\lim_{k\to\infty} x(k) = \mathbb{1} \cdot \begin{bmatrix} \star & \star & \star & \star \end{bmatrix} \cdot x_0$$

Graph theory: basic concepts

1. Basic elements: nodes (labels as 1,2,…,n)

links (undirected/directed, unweighted/weighted)

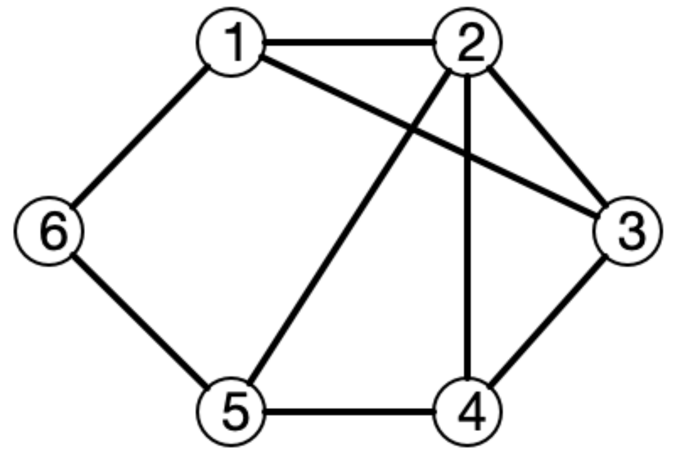
Unweighted $G = (V, E)$: node set $V = \{1, \dots, n\}$

e.g. coauthor net. $\leftarrow$ link set $E \subseteq \{\{i, j\} \mid i, j \in V\}$ (undirected graphs)

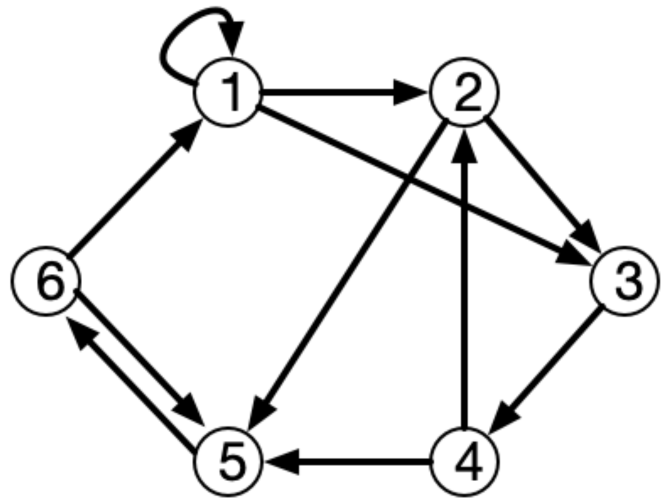
e.g. citation net. $\leftarrow$ link set $E \subseteq V \times V = \{(i, j) \mid i, j \in V\}$ (directed)

Weighted $G = (V, E, \{a_e\}_{e \in E})$: a_e is the weight of link $e \in E$

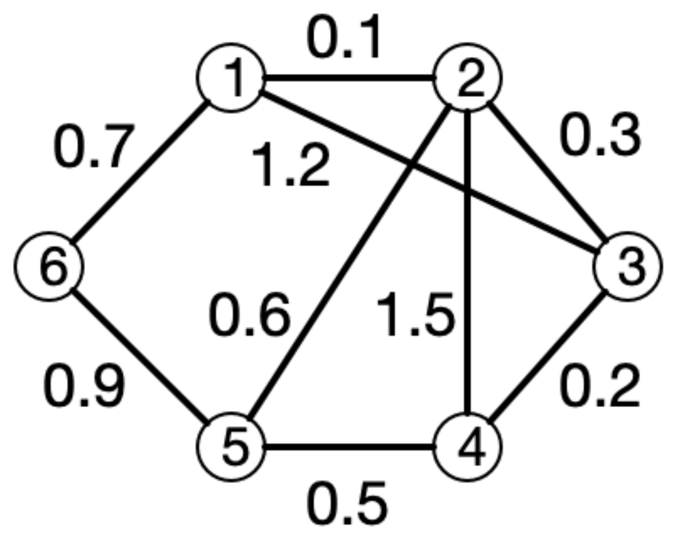
usually assumed to be **non-negative**



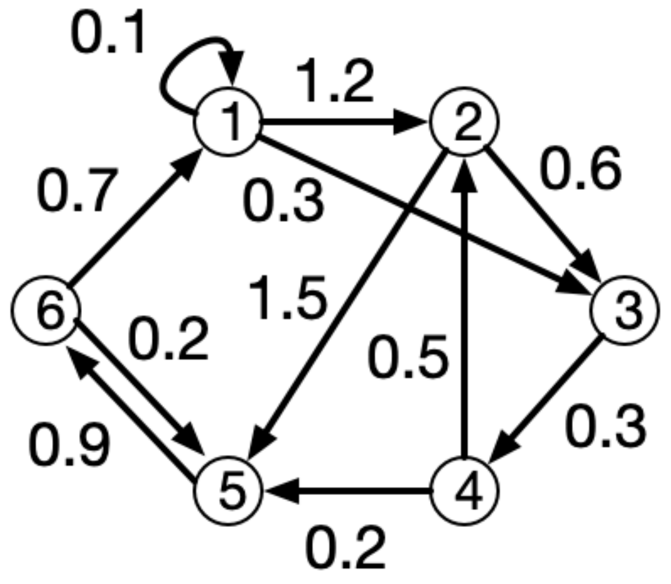
undirected unweighted



directed unweighted



undirected weighted

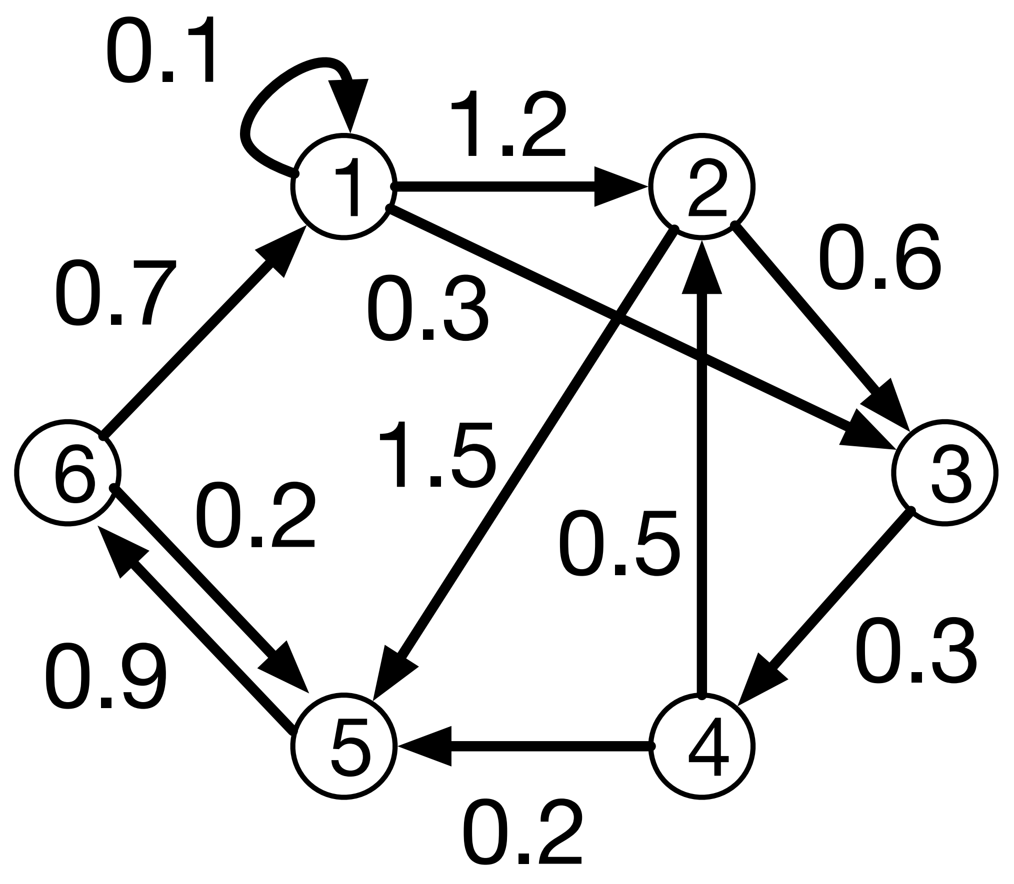


directed weighted

* Conventionally, self-loops are not allowed in undirected graphs.

Graph theory: basic concepts

$G = (V, E, \{a_e\}_{e \in E})$



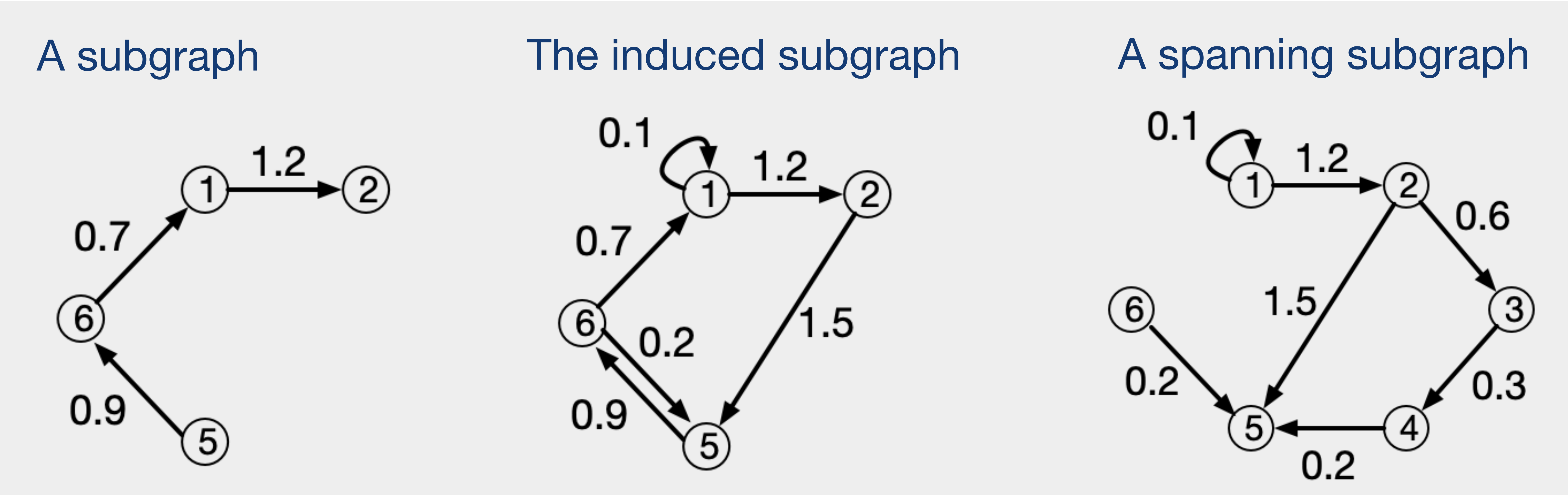
2. Subgraph $G' = (V', E')$: $V' \subseteq V$, $E' \subseteq E \cap \{\{i,j\} \mid i,j \in V'\}$ (undirected)

$E' \subseteq E \cap V' \times V'$ (directed)

Induced subgraph $G' = (V', E')$: $V' \subseteq V$, $E' = E \cap \{\{i,j\} \mid i,j \in V'\}$ (undirected)

$E' = E \cap (V' \times V')$ (directed)

Spanning subgraph $G' = (V, E')$: $E' \subseteq E$



Graph theory: basic concepts

1. Basic elements: nodes (labels as 1,2,…,n)

links (undirected/directed, unweighted/weighted)

Unweighted $G = (V, E)$: node set $V = \{1, \ldots, n\}$

link set $E \subseteq \{\{i, j\} \mid i, j \in V\}$ (undirected graphs)

link set $E \subseteq V \times V = \{(i, j) \mid i, j \in V\}$ (directed)

Weighted $G = (V, E, \{a_e\}_{e \in E})$: a_e is the weight of link $e \in E$

2. Subgraph $G' = (V', E')$: $V' \subseteq V$, $E' \subseteq E \cap \{\{i, j\} \mid i, j \in V'\}$ (undirected)

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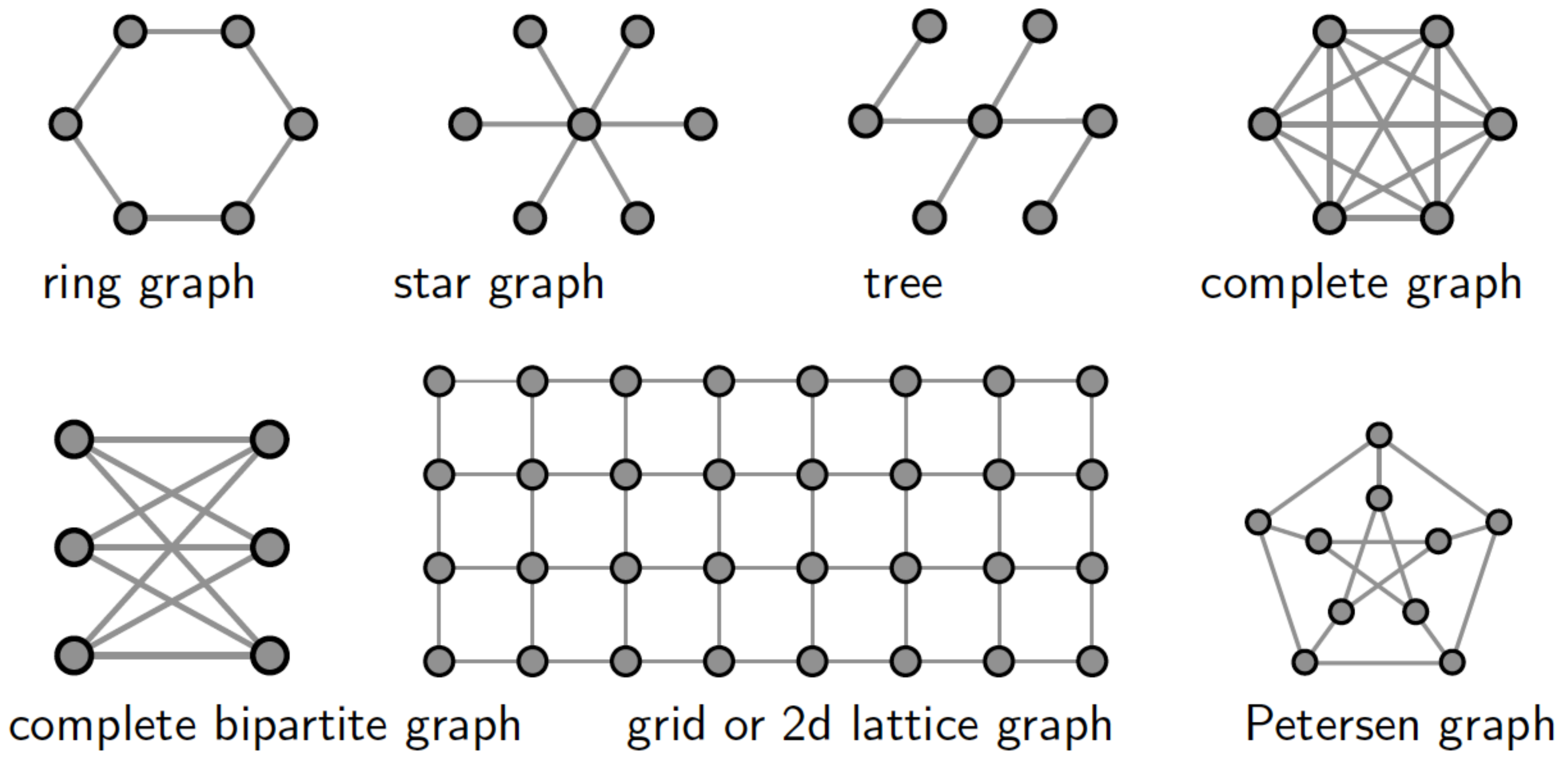
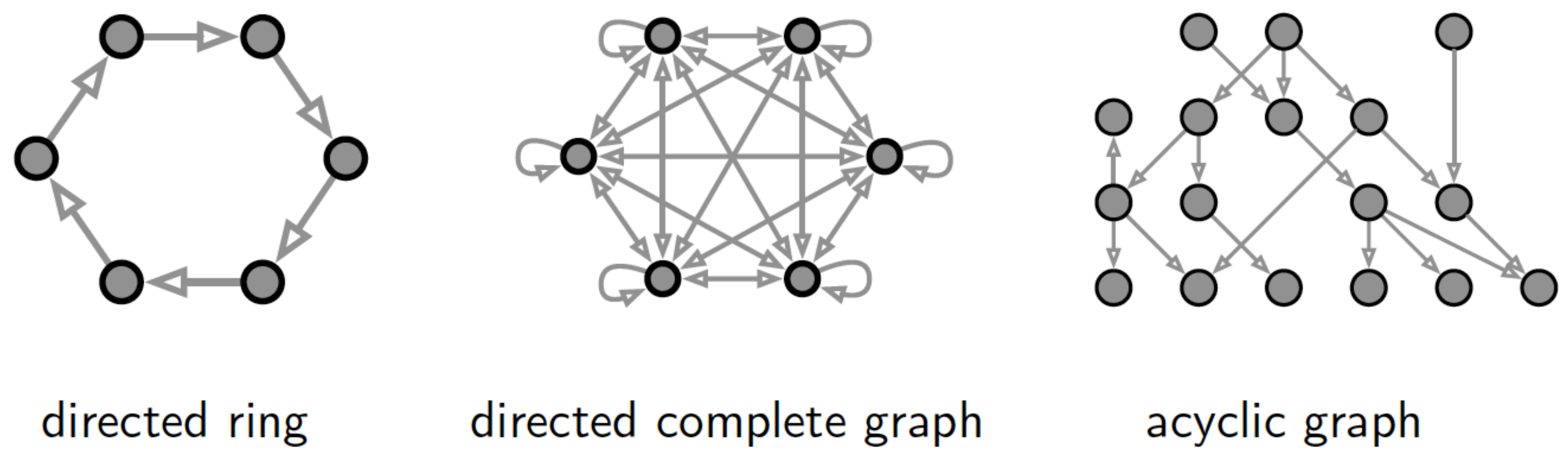
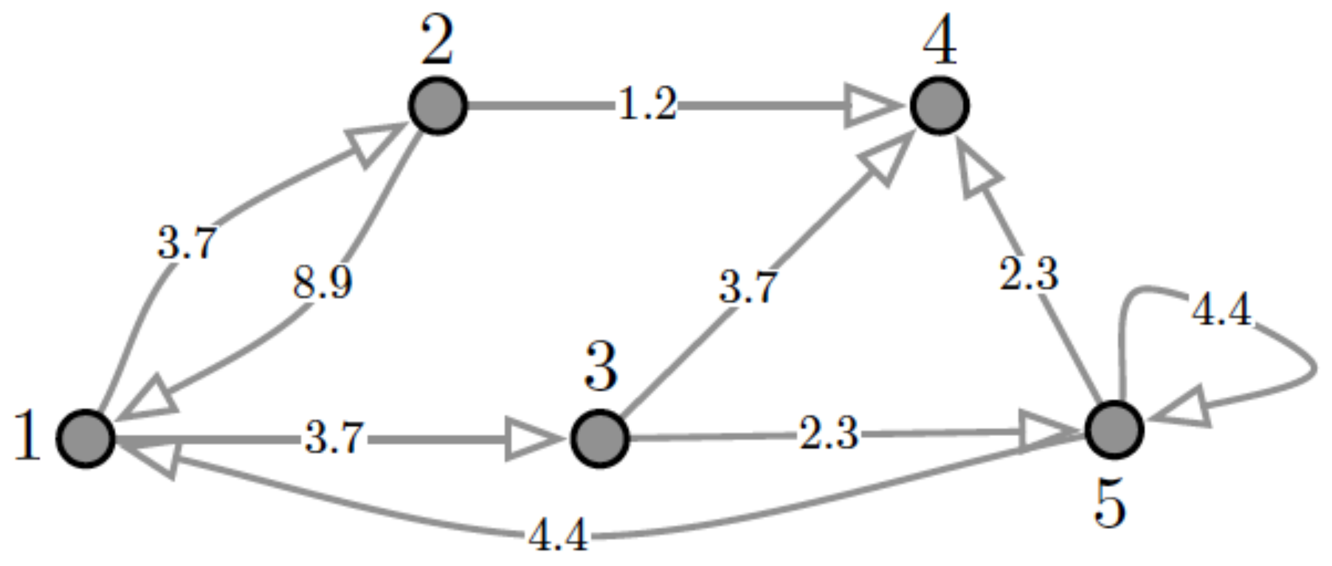






weighted directed graph

Graph theory: basic concepts

3. Directed graphs (weighted or unweighted)

— in-neighbor: node j is an in-neighbor of node i if $(j, i) \in E$

— out-neighbor: node j is an out-neighbor of node i if $(i, j) \in E$

— in-degree: number of in-neighbors (unweighted)

sum of the weights of links from in-neighbors (weighted)

— out-degree: number of out-neighbors (unweighted)

sum of the weights of links to out-neighbors (weighted)

— directed path: an ordered sequence of nodes $i_1, i_2, \ldots, i_m$ such that

$$(i_k, i_{k+1}) \in E, \text{ for any } k = 1,2,\ldots,m-1$$

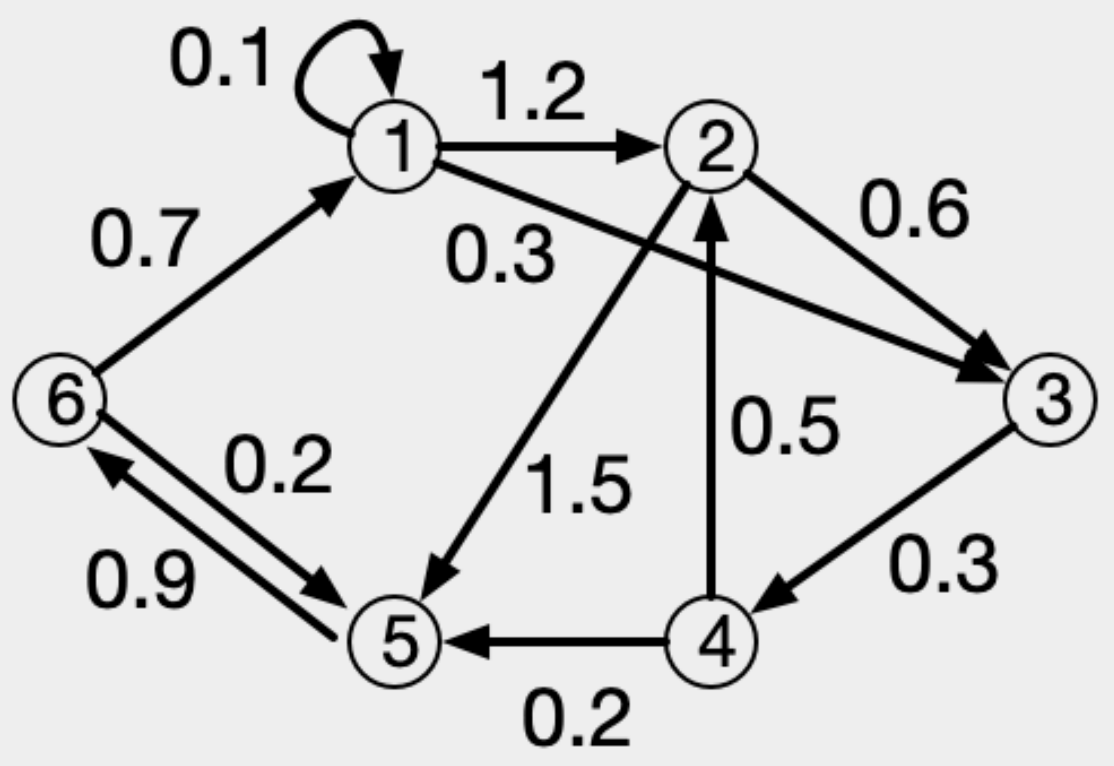
simple directed path: no repeating node, except possibly for $i_1 = i_m$

— cycle: simple directed path with $i_1 = i_m$, with length equal to m

— acyclic graph: a directed graph with no cycle

* A self loop at node i makes i both its own in-neighbor and own out-neighbour.

Please carefully read Ch. 3 of Prof. Bullo's lecture notes regarding undirected graphs & concepts not covered in this lecture.



in-neighbors of node 2: {1,4}
in-degree of 2 = 1.2+0.5 = 1.7

out-neighbors of node 2: {3,5}
out-degree of 2 = 0.6+1.5 = 2.1

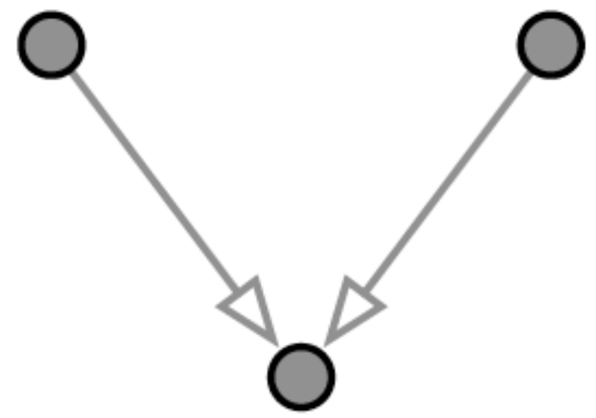
a directed path: (1,2,3,4,2,5)
a simple directed path: (1,3,4,2,5)
a cycle: (1,3,4,2,5,6,1)

Graph theory: basic concepts

3. Directed graphs (weighted or unweighted)

— source: node with no in-neighbor

sink: node with no out-neighbor



(a) An acyclic digraph with one sink and two sources

— directed tree: an **acyclic** graph satisfying: ∃ a node (the root), such that it can reach any other node by one and only one directed path.

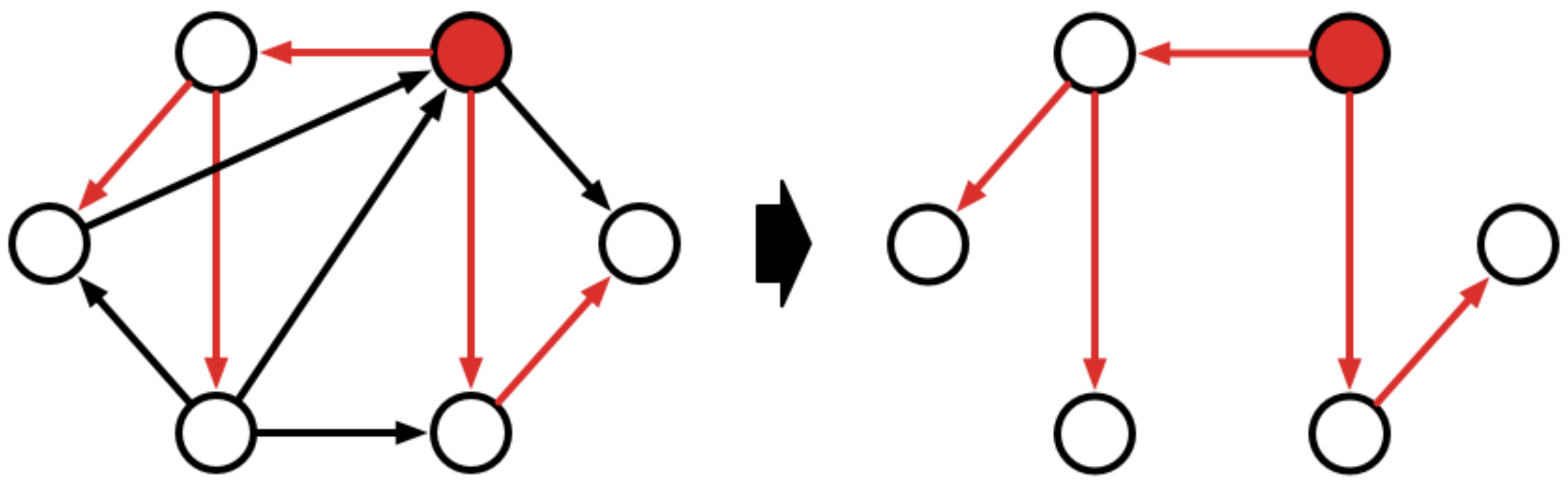
e.g. academic genealogy: $i \to j \Leftrightarrow$ individual i is the PhD advisor of j

— directed spanning tree of a directed graph: spanning subgraph & directed tree

Q1: Every acyclic digraph has at least one source and at least one sink. Why?

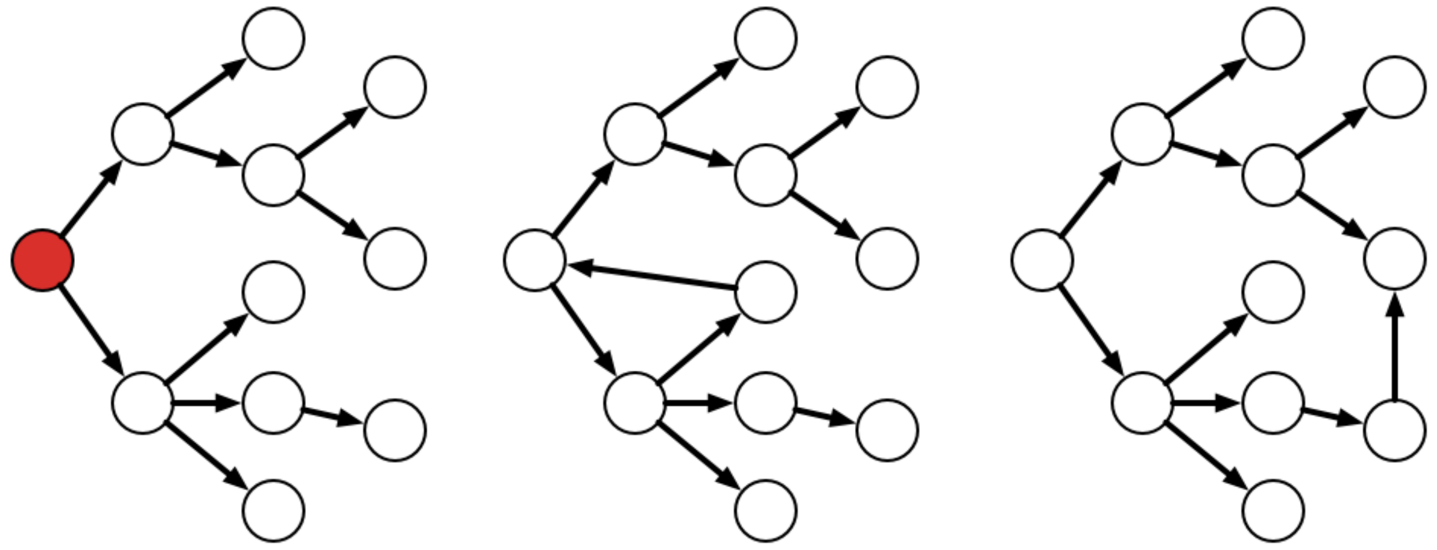
Q2: The root is the only source in a directed tree. Why?

Q3: Does every digraph have a directed spanning tree?



$G = (V, E)$ $\qquad$ $G' = (V, \{\text{red links}\})$

G' is a directed spanning tree of G (but not the only one).



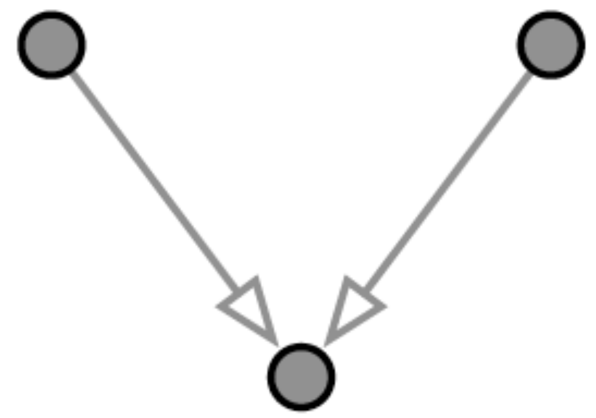
A directed tree $\qquad$ Not a directed tree $\qquad$ Not a directed tree

Graph theory: basic concepts

3. Directed graphs (weighted or unweighted)

— source: node with no in-neighbor

sink: node with no out-neighbor



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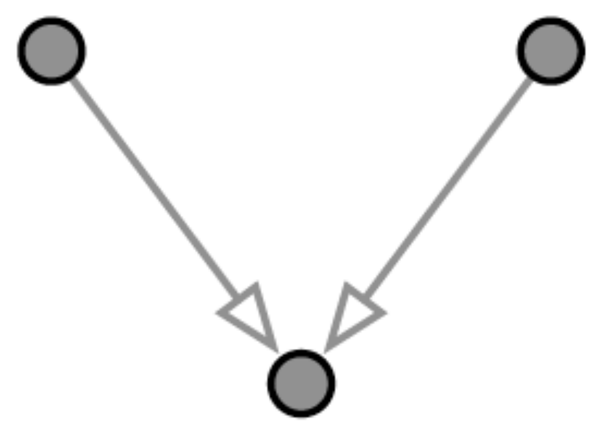
Graph theory: basic concepts

3. Directed graphs (weighted or unweighted)

— source: node with no in-neighbor

sink: node with no out-neighbor



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Graph theory: basic concepts

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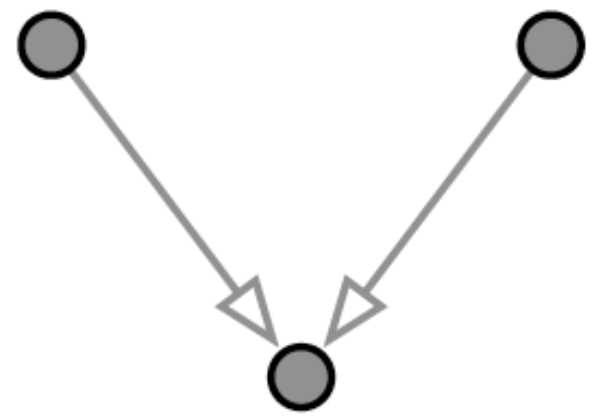
— source: node with no in-neighbor

sink: node with no out-neighbor



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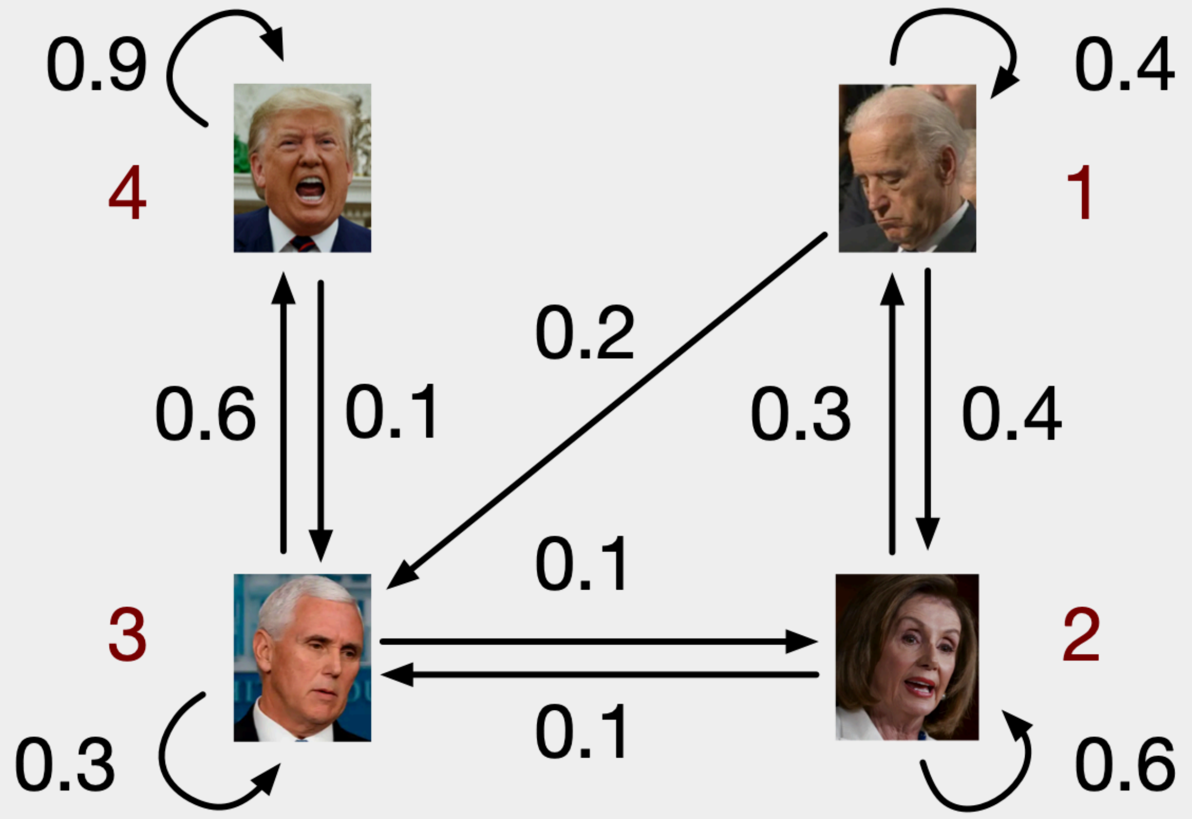
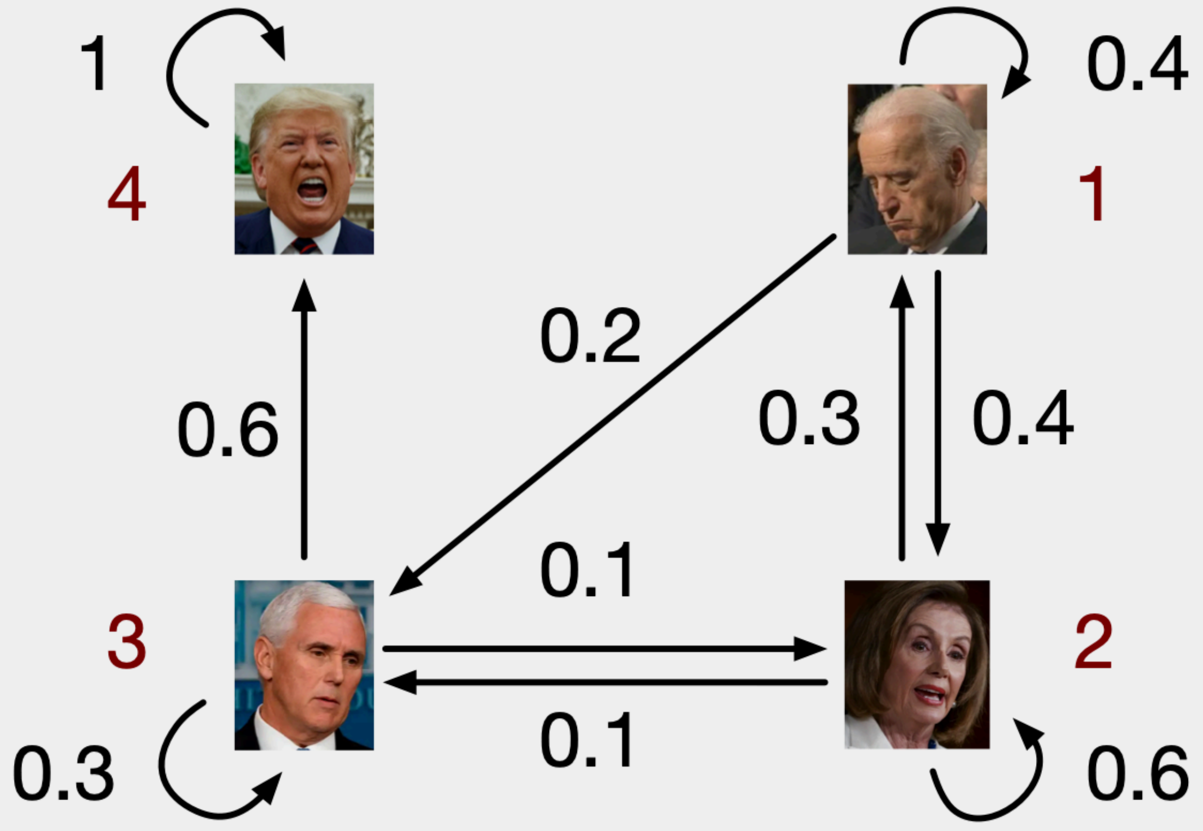
Graph theory: basic concepts

4. Connectivity properties of directed graphs

- G is strongly connected: $\exists$ at least one directed path from any node to any other node
- G has a globally reachable node: $\exists$ a node that can be reached from any other node via at least one directed path
- G has a directed spanning tree $\Leftrightarrow$ $\exists$ a node that can reach any other node via at least one directed path



Strongly connected



Not strongly connected

one globally reachable node: 4

multiple directed spanning trees:

$G' = (V, \{(1,2), (1,3), (3,4)\})$

$G'' = (V, \{(2,1), (2,3), (3,4)\})$

$\cdots$

Graph theory: basic concepts

4. Connectivity properties of directed graphs

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- G has a globally reachable node: $\exists$ a node that can be reached from any other node via at least one directed path
- G has a directed spanning tree $\Leftrightarrow$ $\exists$ a node that can reach any other node via at least one directed path

A strongly connected graph always has at least one directed spanning tree.

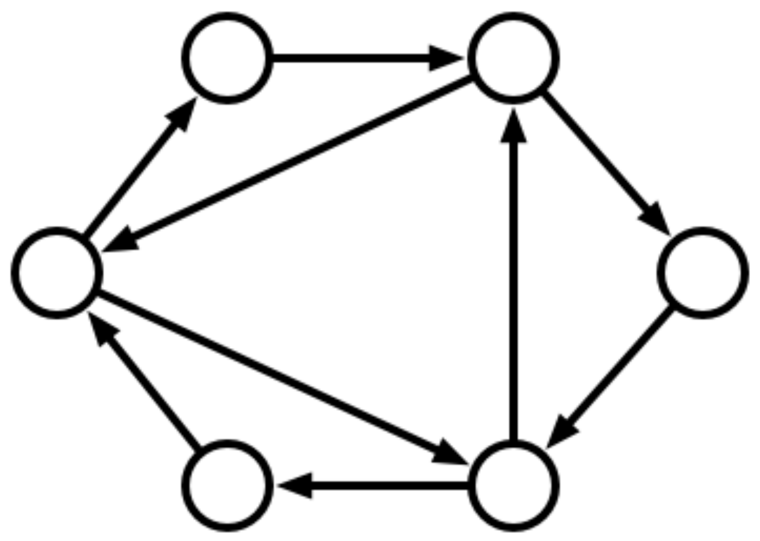
Graph theory: basic concepts

5. Periodicity of strongly connected directed graphs

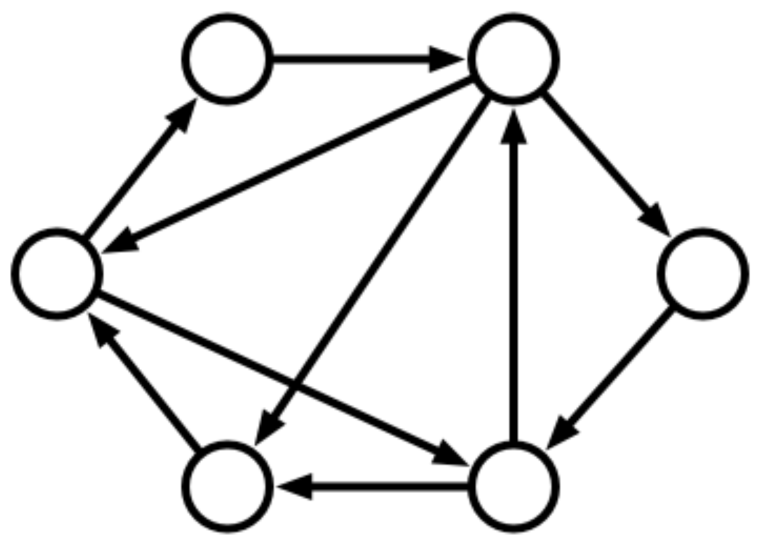
— period: the largest common divisor of the lengths of all the cycles

— G is periodic if its period > 1

— G is aperiodic if its period =1



period = 3



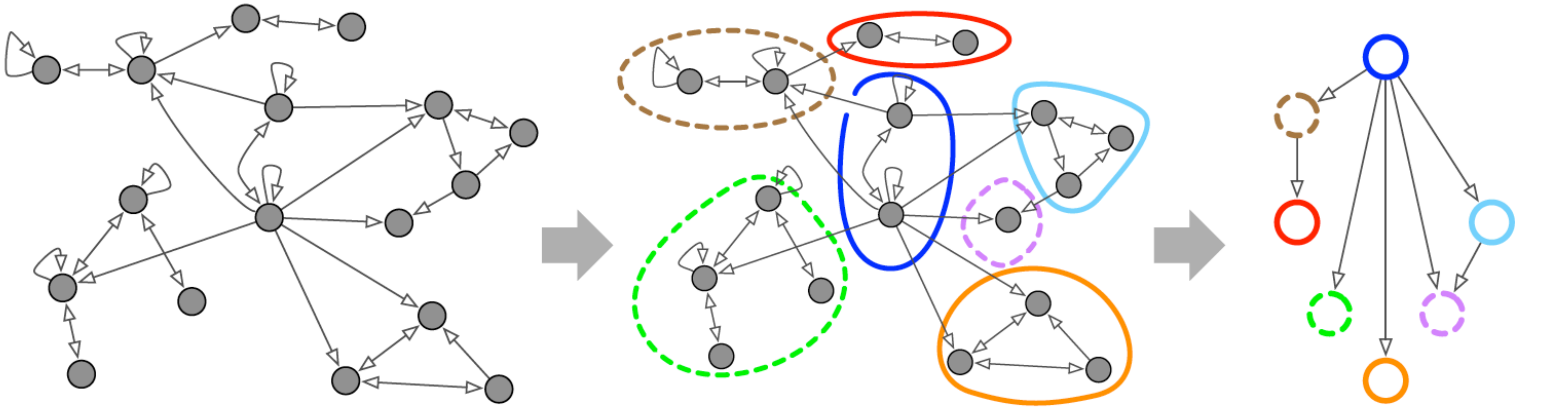
aperiodic

6. Condensation of directed graphs

— strongly connected component: induced subgraph H & strongly connected & any other subgraph strictly containing H is not strongly connected

— condensation digraph: $C(G)$ is a directed graph

- Each node H is a strongly connected component in G
- $\exists$ a link from H_i to H_j if in G there exists a link from one node in SCC H_i to another node in SCC H_j



(a) An example digraph G

(b) The strongly connected components of the digraph G

(c) The condensation digraph $C(G)$

Figure 3.7: An example digraph, its strongly connected components and its condensation digraph.

Properties of the condensation digraph:

1) $C(G)$ is acyclic;
2) G has a globally reachable node $\Leftrightarrow$ $C(G)$ has a globally reachable node $\Leftrightarrow$ $C(G)$ has a unique sink;
3) G has a directed spanning tree $\Leftrightarrow$ $C(G)$ has a directed spanning tree $\Leftrightarrow$ $C(G)$ has a unique source;

Algebraic graph theory

- graph properties $\leftrightarrow$ matrix properties
- graph properties $\leftrightarrow$ (matrix properties) $\leftrightarrow$ dynamic behaviour of $x(t+1) = Ax(t)$ (Next lecture)

1. adjacency matrix: matrix representation of directed weighted graphs

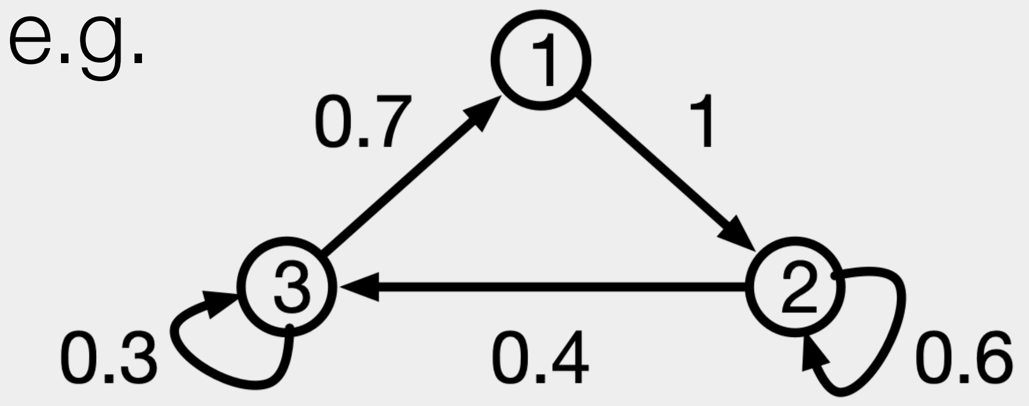
Graph $G = (V, E, \{a_e\}_{e \in E})$ $\leftrightarrow$ Adjacency matrix $A = (a_{ij})_{n \times n} \in \mathbb{R}^{n \times n}_{\geq 0}$, where

$$a_{ij} = \begin{cases} \text{the weight of the link from i to j}, & \text{if } (i,j) \in E \\ 0, & \text{if } (i,j) \notin E \end{cases}$$

$\rightarrow$ Binary adjacency matrix $[A] = ([a]_{ij})_{n \times n} \in \{0,1\}^{n \times n}$, where

$$[a]_{ij} = \begin{cases} 1, & \text{if } (i,j) \in E \\ 0, & \text{if } (i,j) \notin E \end{cases}$$

e.g.



$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0.6 & 0.4 \\ 0.7 & 0 & 0.3 \end{bmatrix} \quad [A] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

*[A] is just for simplicity of notations in this lecture, not a conventional symbol for binary adjacency matrices.

Algebraic graph theory

2. properties of graphs $\leftrightarrow$ properties of adjacency matrices

(1) in-neighbor set of node i, out-neighbour set of node i	(1) $\{j \in V \mid a_{ji} > 0\}, \quad \{j \in V \mid a_{ij} > 0\}$
(2) in-degree of node i, out-degree of node i	(2) $\sum_{j=1}^{n} a_{ji} = (\mathbb{1}_n^\top A)_i, \quad \sum_{j=1}^{n} a_{ij} = (A\mathbb{1}_n)_i$
(3) strongly connected	(3) $I + A + A^2 + \ldots + A^{n-1} > 0$
(4) The out-degree of every node is 1	(4) A is row-stochastic

Key Results:

Corollary 4.5 (Connectivity properties of the digraph and positive powers of the adjacency matrix: cont'd). *Let G be a weighted digraph with n nodes, weighted adjacency matrix A and a self-loop at each node. The following statements are equivalent:*

(i) *G is strongly connected; and*

(ii) *A^{n-1} is positive, so that A is primitive.*

Algebraic graph theory

2. properties of graphs ↔ properties of adjacency matrices

Lemma 4.2 (Directed paths and powers of the adjacency matrix). *Let G be a weighted digraph with n nodes, with adjacency matrix A and binary adjacency matrix $A_{0,1} \in \{0,1\}^{n\times n}$. For all $i, j \in \{1,\dots,n\}$ and $k \in \mathbb{N}$*

(i) the (i,j) entry of $A_{0,1}^k$ equals the number of paths of length k from node i to node j; and

(ii) the (i,j) entry of A^k is positive if and only if there exists a path of length k from node i to node j.

(Paths here are directed paths that possibly include self-loops.)

Lemma 4.4 (Global reachability and powers of the adjacency matrix). *Let G be a weighted digraph with $n \geq 2$ nodes and weighted adjacency matrix A. For any $j \in \{1,\dots,n\}$, the following statements are equivalent:*

(i) the jth node of G is globally reachable, and

(ii) the jth column of $\sum_{k=0}^{n-1} A^k$ is positive.

Algebraic graph theory

2. properties of graphs ↔ properties of adjacency matrices

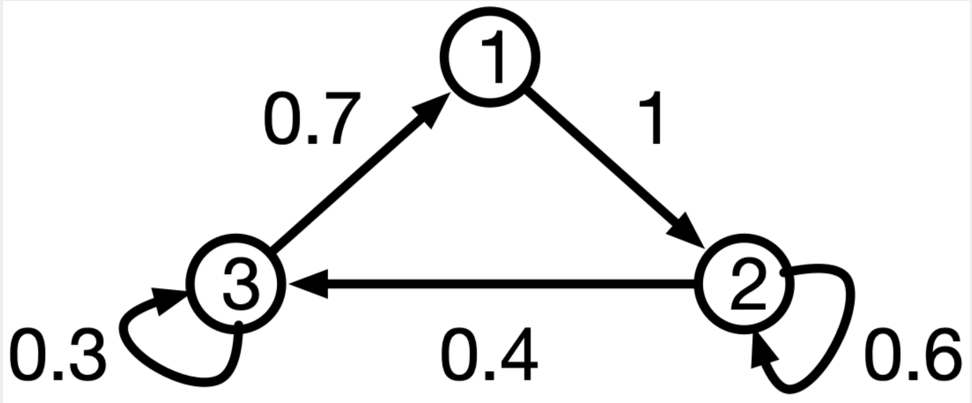
Theorem 4.7 (Strongly connected and aperiodic digraphs and primitive adjacency matrices). *Let G be a weighted digraph with $n \geq 2$ nodes and with weighted adjacency matrix A. The following two statements are equivalent:*

(i) G is strongly connected and aperiodic; and

(ii) A is primitive, that is, there exists $k \in \mathbb{N}$ such that A^k is positive.

Example:

An aperiodic directed graph



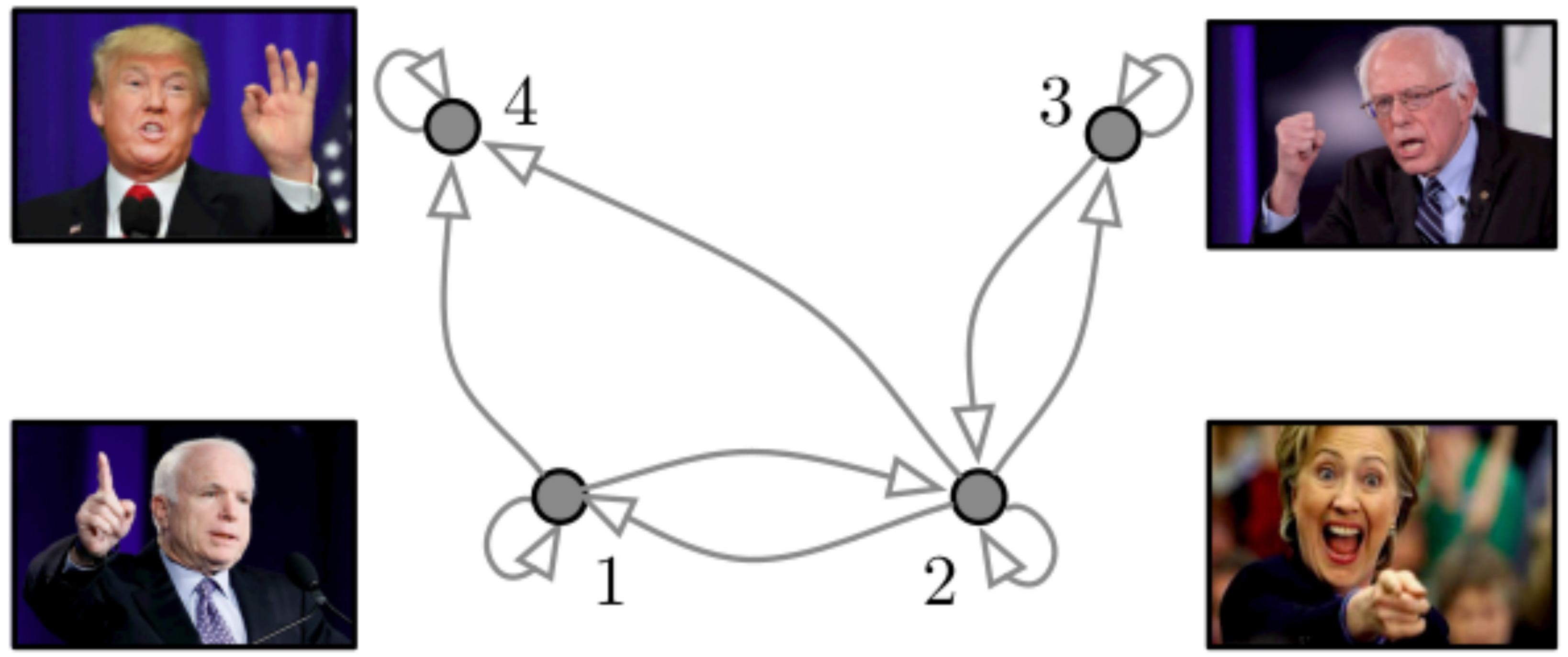
$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0.6 & 0.4 \\ 0.7 & 0 & 0.3 \end{bmatrix}$$

Graph is strongly connected and aperiodic → there should be a $k \in \mathbb{N}$ such that A^k is positive!

Compute $A^2 = \begin{bmatrix} 0 & 0.6 & 0.4 \\ 0.28 & 0.36 & 0.36 \\ 0.21 & 0.7 & 0.09 \end{bmatrix}$ → Not positive!

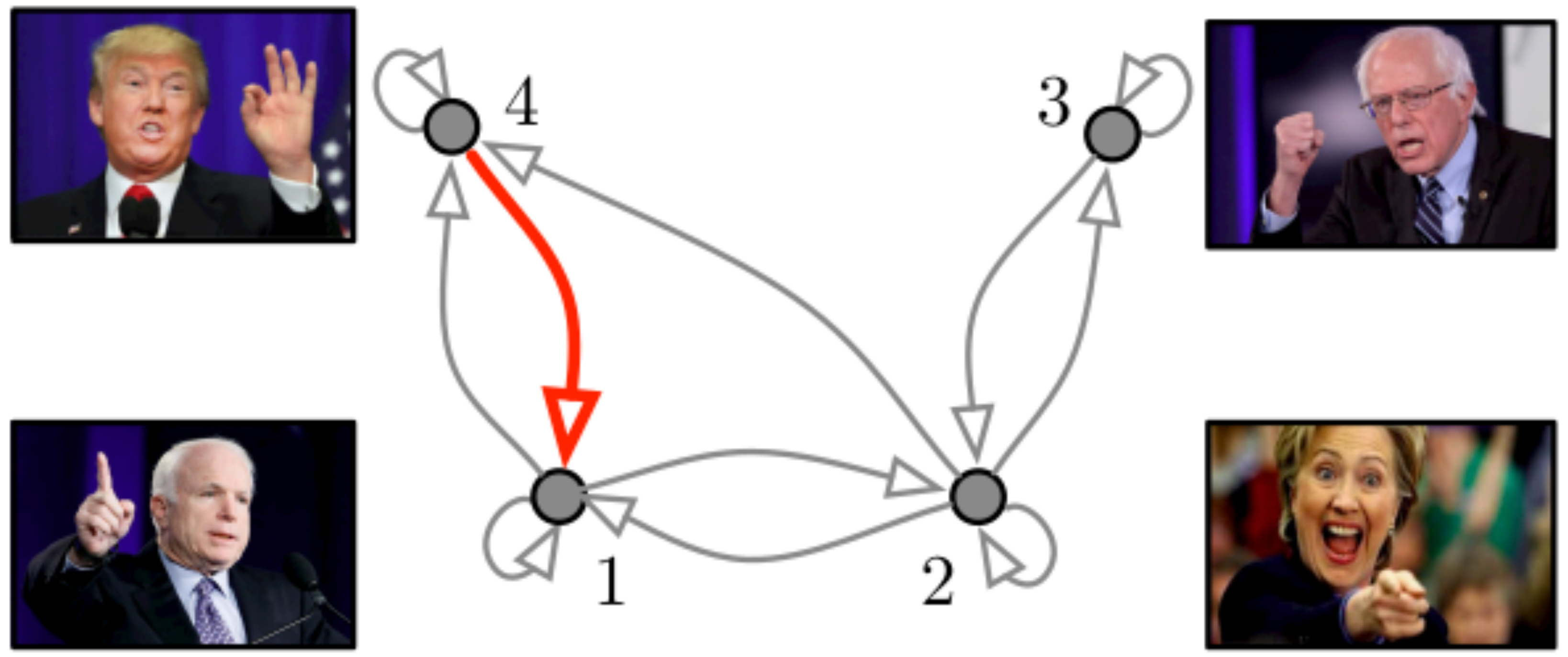
Compute $A^3 = \begin{bmatrix} 0.28 & 0.36 & 0.36 \\ 0.252 & 0.496 & 0.252 \\ 0.063 & 0.63 & 0.307 \end{bmatrix}$ → Positive!

Remember this example? Do you see the distinguishing feature now?



$$A = \begin{pmatrix} \star & \star & \star & \star \\ \star & \star & \star & \star \\ \star & \star & \star & \\ & & & \star \end{pmatrix}$$

$$\lim_{k\to\infty} x(k) = \mathbb{1} \cdot \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \cdot x_0$$



$$A = \begin{pmatrix} \star & \star & \star & \star \\ \star & \star & \star & \star \\ \star & \star & \star & \\ \star & & & \star \end{pmatrix}$$

$$\lim_{k\to\infty} x(k) = \mathbb{1} \cdot \begin{bmatrix} \star & \star & \star & \star \end{bmatrix} \cdot x_0$$

Summary

1. basic concepts in graph theory

   — different types of graphs

   — neighbour, degree, subgraphs

   — path, cycle, period, tree, etc

2. connectivity conditions

   — strongly connected

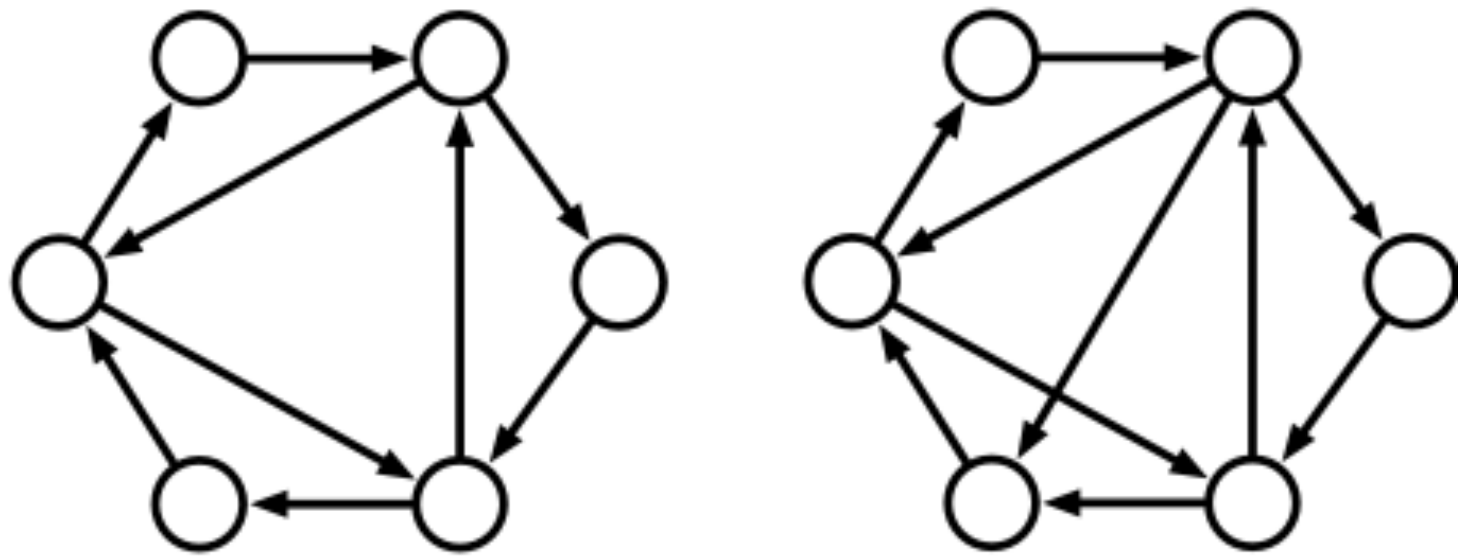
   — spanning tree, globally reachable node

3. algebraic graph theory I: describe graph properties

   — adjacency matrix

   — graph properties ↔ matrix properties, e.g., existence of paths, number of path, strongly connected, aperiodic

Next lecture: $x(t+1) = Ax(t)$, convergence? consensus?



Algebraic graph theory II

Infer the behaviour of $x(t+1) = Ax(t)$ directly from the graph structure.