Distributed Systems & Control

Advanced Topics in Control 2022

Lecture 5: The Laplacian Matrix

ETH zürich

AUTOMATIC CONTROL LABORATORY

Brief announcements

227-0690-12L Advanced Topics in Control (Spring 2022) FS2022

Dashboard / Meine Kurse / 227-0690-12L Advanced Topics in Control (Spring 2022) FS2022

General Information

Lecturers: Prof. Florian Dörfler, Dr. Mathias Hudoba de Badyn, Dr. Vahid Mamduhi

Assistants: Andrea Martinelli, Dr. Dominic Liao-McPherson, Alberto Padoan, Carlo Cenedese

Student assistants: Joudi Hajar, Aristomenis Sfetsos

	When	Where	Video link
Lectures:	Mondays, 16:00-18:00	HG D1.1 (also streamed online)	link
Tutorials:	Fridays, 10:00-12:00	HG D1.1 (also streamed online)	link

Grading: based on 3 homework assignments (50%) and a final project (50%). Here is the information on grading, homework, and the final project.

- **first homework** is due March 25
- **project**: start to get ready, form groups, think of topics, . . .

recap: discrete averaging

(last lecture, exercise session, & Chapters 4–5)

Our main result for discrete-time averaging algorithms

Main result for discrete-time averaging algorithms

Let A be row-stochastic. The following statements are equivalent:

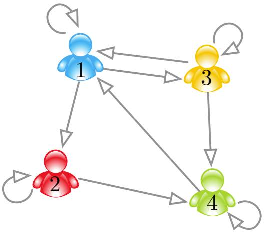
1. the eigenvalue 1 is simple, and all other eigenvalues have magnitude strictly smaller than 1.
2. $\lim_{k\to\infty} A^k = \mathbb{1} w^T$ for some $w \geq 0$ satisfying $\mathbb{1}^T w = 1$.
3. the digraph associated to A contains a globally reachable node, and the subgraph of globally reachable nodes is aperiodic.

In either of the three equivalent cases, the solution to $x^+ = A\,x$ satisfies

$$\lim_{k\to\infty} x(k) = (w^T x_0)\,\mathbb{1}\,,$$

where $w_i > 0$ if and only if node i is globally reachable.

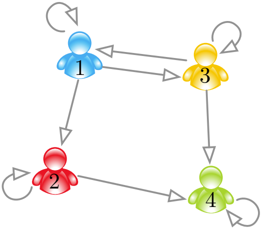
Now we have a full picture of discrete averaging



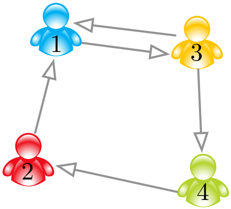
converges to $\mathbb{1} \cdot [\star \;\; \star \;\; \star \;\; \star]$

$$A = \begin{bmatrix} \star & \star & \star & \\ & \star & & \star \\ \star & & \star & \star \\ \star & & & \star \end{bmatrix}$$

nonnegative and row-stochastic



converges to $\mathbb{1} \cdot [0 \;\; 0 \;\; 0 \;\; 1]$



diverges & oscillates

Organization of today's lecture

The Laplacian Matrix

(Chapter 6)

Continuous-Time Averaging

(see how far we get)

(Chapter 7)

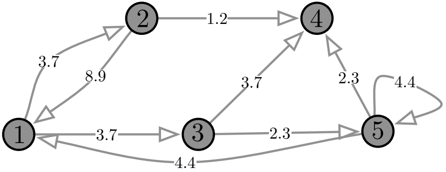
The Laplacian Matrix

(Chapter 6)

The Laplacian matrix

Laplacian matrix of a weighted digraph G

$$L = \begin{bmatrix} \vdots & \ddots & \vdots & \iddots & \vdots \\ -a_{i1} & \cdots & \sum_{j=1, j\neq i}^{n} a_{ij} & \cdots & -a_{in} \\ \vdots & \iddots & \vdots & \ddots & \vdots \end{bmatrix} \quad \text{or} \quad \boxed{L = D_{\text{out}} - A}$$



$$L = \begin{bmatrix} 7.4 & -3.7 & -3.7 & 0 & 0 \\ -8.9 & 10.1 & 0 & -1.2 & 0 \\ 0 & 0 & 6.0 & -3.7 & -2.3 \\ 0 & 0 & 0 & 0 & 0 \\ -4.4 & 0 & 0 & -2.3 & 6.7 \end{bmatrix}$$

properties of Laplacian matrices

discussion on board

- **Laplacian matrix** of a weighted digraph G: $L = D_{\text{out}} - A$

$$L = \begin{bmatrix} \vdots & \ddots & \vdots & \iddots & \vdots \\ -a_{i1} & \cdots & \sum_{j=1, j \neq i}^{n} a_{ij} & \cdots & -a_{in} \\ \vdots & \iddots & \vdots & \ddots & \vdots \end{bmatrix}$$

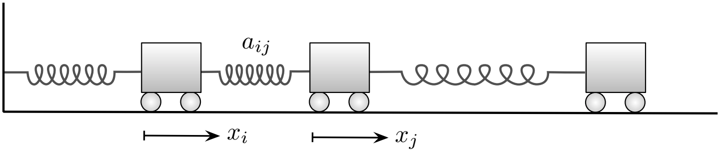
$\Rightarrow$ in components $\ell_{ij} = \begin{cases} -a_{ij}, & \text{if } i \neq j, \\ \sum_{j=1, j \neq i}^{n} a_{ij}, & \text{if } i = j. \end{cases}$

- **error vector** $(Lx)_i = \sum_{j=1, j \neq i}^{n} a_{ij}(x_i - x_j) = \sum_{j \in \mathcal{N}^{\text{out}}(i)} a_{ij}(x_i - x_j)$

- **Laplacian potential** or **disagreement function**:

$$x^T L x = \frac{1}{2} \sum_{i,j=1}^{n} a_{ij}(x_i - x_j)^2 = \sum_{\{i,j\} \in E} a_{ij}(x_i - x_j)^2$$

The Laplacian in a mechanical network of springs



- each node i is subject to an **elastic force**

$$F_{\text{elastic},i} = \sum\nolimits_{j \neq i} a_{ij}(x_j - x_i) = -(Lx)_i$$

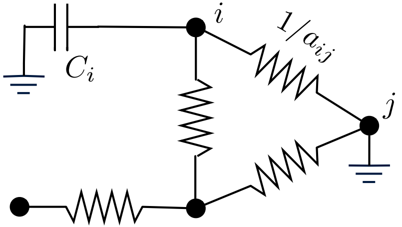
- the **total elastic energy** is

$$E_{\text{elastic}} = \frac{1}{2} \sum\nolimits_{\{i,j\} \in E} a_{ij}(x_i - x_j)^2 = \frac{1}{2} x^T L x$$

- the Newtonian **dynamics** are

$$M_i \ddot{x}_i = - \underbrace{D_i \dot{x}_i}_{\text{viscous friction}} - \underbrace{\sum\nolimits_{j \neq i} a_{ij}(x_j - x_i)}_{\text{spring force}}$$

The Laplacian in an electrical network of resistors



- **Ohm's law** gives the current flowing from i to j as

$$\mathrm{c}_{i\to j} = a_{ij}(\mathrm{v}_i - \mathrm{v}_j)$$

where $a_{ij} = 1/(\text{resistance})_{ij}$ is the conductance & v_i is the potential

- **Kirchhoff's current law** says that at each node i:

$$\mathrm{c}_{\mathrm{inj},i} = \sum_{j\neq i} \mathrm{c}_{i\to j} = \sum_{j\neq i} a_{ij}(\mathrm{v}_i - \mathrm{v}_j) \quad \text{or} \quad \mathrm{c}_{\mathrm{inj}} = L\,\mathrm{v}$$

- dissipation on resistor $\{i,j\}$ is $\mathrm{c}_{i\to j}(\mathrm{v}_i - \mathrm{v}_j)$ & **total dissipated power** is

$$E_{\mathrm{dissipated}} = \sum_{\{i,j\}\in E} a_{ij}\,(\mathrm{v}_i - \mathrm{v}_j)^2 = \mathrm{v}^T L \mathrm{v}.$$

- **Faraday's law** at capacitor i is $C_i\dot{\mathrm{v}}_i = -\mathrm{c}_{\mathrm{inj},i}$ & **network dynamics** are

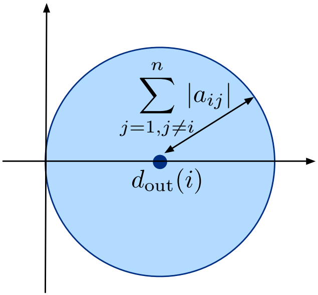
$$C_i\dot{\mathrm{v}}_i = -\sum_{j\neq i} \mathrm{c}_{i\to j} = -\sum_{j\neq i} a_{ij}(\mathrm{v}_i - \mathrm{v}_j)$$

spectrum of Laplacian matrices

discussion on board

Properties of the spectrum of Laplacian matrices

- **zero row sums:** $L\mathbb{1} = \mathbb{0}$
- **zero column sums:** $L^T\mathbb{1} = \mathbb{0}$ if and only if G is weight-balanced
- **Geršgorin:** eigenvalues other than 0 have strictly positive real part



Symmetric case:

- G is symmetric if and only if $L = L^T$
- spectrum: $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$
- Geršgorin: $\lambda_n \leq 2 \cdot \max_i d_{\mathrm{out}}(i)$
- zero eigenspace: $L\mathbb{1} = L^\top\mathbb{1} = \mathbb{0}$

$\rightarrow$ question: $\exists$ more zero eigenvalues ?

further discussion on board

Summary of discussion

- **discrete-time system** $\dot{x} = -Lx \quad \longrightarrow \quad x^+ = (I - \epsilon L)x \triangleq A_\epsilon x$

$\Rightarrow$ **eigenstructure**: (λ, v, w): eigenvalue & left/right eigenvector for A_ϵ

$\Leftrightarrow \quad (\frac{1-\lambda}{\epsilon}, v, w)$: eigenvalue & left/right eigenvector for L

$\Rightarrow$ **stochastic:** $A_\epsilon \mathbb{1} = \mathbb{0}$ & $[A_\epsilon]_{ij} \geq 0$ & $[A_\epsilon]_{ii} > 0$ for $\epsilon < \frac{1}{\max_i d_{\text{out}}(i)}$

$\Rightarrow$ **associated digraph** G_ϵ has same edges as G but additional self-loops

$\Rightarrow$ discrete-time averaging results for $A_\epsilon \quad \rightarrow$ **dramatic corollaries** for L

- **rank condition**: $\operatorname{rank}(L) = n - \#(\text{sinks in condensation digraph})$
- G has a **globally reachable node** if and only if $\operatorname{rank}(L) = n - 1$

  $\Rightarrow$ $\operatorname{spectrum}(L) = \{0, \lambda_2, \dots, \lambda_n\}$ with $\mathcal{R}(\lambda_i) > 0$ for $i \in \{2, \dots, n\}$

  $\Rightarrow$ $L\mathbb{1} = \mathbb{0}$ & $w^\top L = \mathbb{0}$ for $w \geq 0$ & $w_i > 0$ iff i is globally reachable
- **symmetric** G connected if and only if **algebraic connectivity** $\lambda_2 > 0$

Spectral graph theory

$=$ relation of graph and Laplacian spectrum

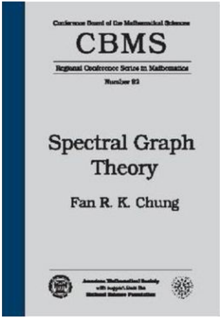
- $\lambda_n \leq 2 \cdot \max_i d_{\text{out}}(i)$
- $\lambda_2 > 0$ if and only if G is connected

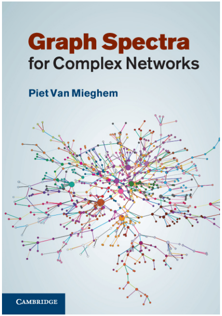
$\Rightarrow$ λ_2 is **algebraic connectivity**

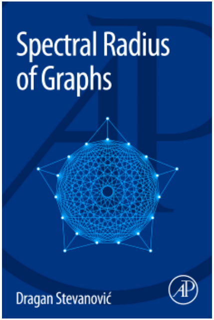
$\Rightarrow$ v_2 is **Fiedler eigenvector**

named after Miroslav Fiedler

$\Rightarrow$ some more insights on λ_2 and v_2 as the "**bottleneck**"







A crash course in spectral partitioning

- given: an undirected, connected, & weighted **graph**
- **partition:** $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2\,,\ \mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset\,,$ and $\mathcal{V}_1, \mathcal{V}_2 \neq \emptyset$
- **cut** is the size of a partition: $J = \sum_{i \in \mathcal{V}_1,\, j \in \mathcal{V}_2} a_{ij}$

$\Rightarrow$ if $x_i = +1$ for $i \in \mathcal{V}_1$ and $x_j = -1$ for $j \in \mathcal{V}_2$, then

$$J = \sum_{i \in \mathcal{V}_1,\, j \in \mathcal{V}_2} a_{ij} = \frac{1}{8} \sum_{i,j=1}^{n} a_{ij} (x_i - x_j)^2 = \frac{1}{4}\, x^T L x$$

- combinatorial **min-cut** problem: $\text{minimize}_{x \in \{-1,+1\}^n \setminus \{-\mathbb{1}_n, +\mathbb{1}_n\}}\ \frac{1}{4}\, x^T L x$
- **relaxed problem:** $\text{minimize}_{y \in \mathbb{R}^n,\, y \perp \mathbb{1}_n,\, \|y\|_2 = 1}\ \frac{1}{4}\, y^T L y$

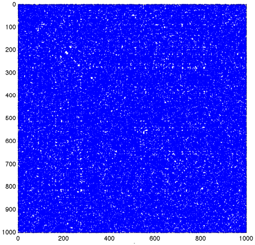
$\Rightarrow$ minimum is *algebraic connectivity* λ_2 and minimizer is *Fiedler vector* v_2

- **heuristic:** $x_i = \text{sign}(y_i) \ \Rightarrow$ "spectral partition"

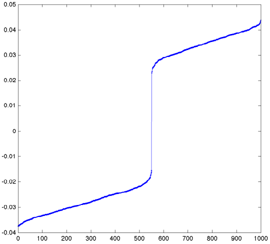
A quick example

```
% choose a graph size
n = 1000;

% randomly assign the nodes to two grous
x = randperm(n);
group_size = 450;
group1 = x(1:group_size);
group2 = x(group_size+1:end);

% assign probabilities of connecting nodes
p_group1 = 0.5;
p_group2 = 0.4;
p_between_groups = 0.1;

% construct adjacency matrix
A(group1, group1) = rand(group_size,group_size) < p_group1;
A(group2, group2) = rand(n-group_size,n-group_size) < p_group2;
A(group1, group2) = rand(group_size, n-group_size) < p_between_groups;
A = triu(A,1); A = A + A';

% can you see the groups?
subplot(1,3,1); spy(A);

% construct Laplacian and its spectrum
L = diag(sum(A))-A;
[V D] = eigs(L, 2, 'SA');

% plot the components of the algebraic connectivity sorted by magnitude
subplot(1,3,2); plot(sort(V(:,2)), '.-');

% partition the matrix accordingly and spot the communities
[ignore p] = sort(V(:,2));
subplot(1,3,3); spy(A(p,p));
```

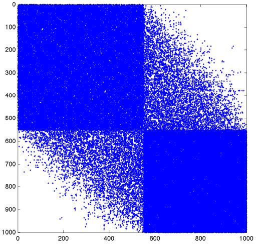
A quick example – cont'd



adjacency matrix



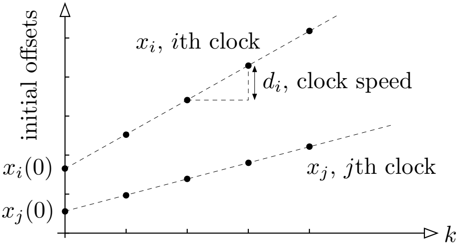
Fiedler vector v_2



re-arranged adj. matrix

interpretation: $\lambda_2 > 0$ & v_2 quantify the size & location of the *bottleneck*

Algorithmic application: clock sync (Appendix 6.5)



simple **clock model**:

$$x_i^+ = x_i + d_i + u_i$$

- x_i is counter (time) of clock i
- $x_i(0)$ is initial bias (unknown)
- d_i is clock skew/drift/speed
- u_i is control to be designed

objective: design a distributed control for u_i to synchronize the clocks

first try: **P-averaging:** $u = -Lx$ results in $x^+ = (I - L)x + d$

$=$ closed-loop consensus $+$ external drift (no sync unless $d \in \mathrm{span}(\mathbb{1})$)

second try: **PI-averaging:** $u = -Lx - w$ where $w^+ = w - \gamma Lx$ for $\gamma > 0$

integrator assures in steady state (when $w^+ = w$) that $\mathbb{0} = \gamma Lx \Rightarrow$ sync

discussion on board

Continuous-Time Averaging Algorithms or Laplacian Flows

$$\dot{x} = -Lx$$

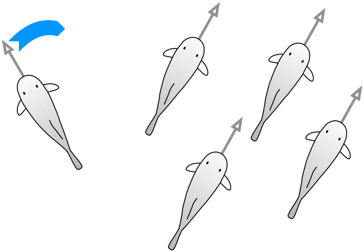
(Chapter 7)

Flocking behavior for a group of animals

$$\dot{\theta}_i = \begin{cases} (\theta_j - \theta_i), & \text{if one neighbor} \\ \frac{1}{2}(\theta_{j_1} - \theta_i) + \frac{1}{2}(\theta_{j_2} - \theta_i), & \text{if two neighbors} \\ \frac{1}{m}(\theta_{j_1} - \theta_i) + \cdots + \frac{1}{m}(\theta_{j_m} - \theta_i), & \text{if } m \text{ neighbors} \end{cases}$$

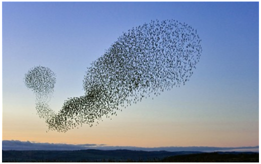
or

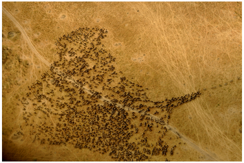
$$\dot{\theta}_i = \text{average}\left(\{\theta_j, j \in \mathcal{N}^{\text{out}}(i)\}\right) - \theta_i$$



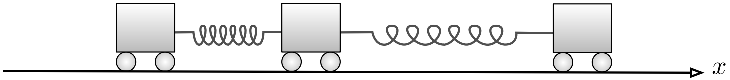
or

$$\dot{\theta} = -L\theta$$



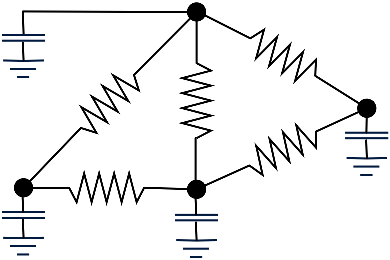


Laplacian flows in engineering



$\Rightarrow$ **Newtonian dynamics** are

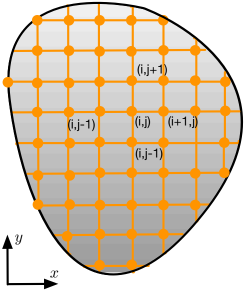
$$M_i \ddot{x}_i = -D_i \dot{x}_i - \sum_{j \neq i} a_{ij}(x_j - x_i)$$



- **Faraday's law** at capacitor i gives rise to

$$C_i \dot{\mathrm{v}}_i = -\sum_{j \neq i} c_{i \to j} = -\sum_{j \neq i} a_{ij}(\mathrm{v}_i - \mathrm{v}_j)$$

Laplacian flows and the *Laplacian operator*



continuous domain & finite-difference approximation via regular grid with coordinates (i, j)

finite-difference approximation (width h):

- spatiotemporal function $u(t, x, y) \approx u(t, x_i, y_j)$
- Laplace operator: $\Delta u(t, x, y) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$
  $\approx \frac{4u(t,x_i,y_j) - u(t,x_{i-1},y_j) - u(t,x_{i+1},y_j)}{h^2}$
  $\frac{-u(t,x_i,y_{j-1}) - u(t,x_i,y_{j+1})}{h^2}$

- **heat equation:** $\frac{\partial u}{\partial t} + \underbrace{\Delta u}_{\approx Lu} = 0 \;\Rightarrow\; \dot{u} = -Lu$

  where L is the Laplacian matrix of the grid

- **wave equation:** $\frac{\partial^2 u}{\partial t^2} + \Delta u = 0 \;\Rightarrow\; \ddot{u} = -Lu$

semi-convergence of linear systems in continuous-time on the board preliminary discussion of $\dot{x} = -Lx$ on the board

Reading assignment (lecture notes):

Chapter 6: The Laplacian Matrix

Chapter 7: Continuous-Time Averaging

(no need to read 7.3)

Next Exercise:

- led by Carlo Cenedese
- review of take-home messages
- examples & additional facts
- exercises & illustrations