用一个例子描述了LMB filter for multi-targets的framework如何工作
(但没有描述如何对于每一个labeled extended target ℓ with component i of GLMB
RFS & partition j of detection points，如何建模，求prediction，update)
这部分对应之probability density可通过，比如GGIW，来modeling $p^{(l,i,\theta_i^j)}(x)$

Labeled Multi-Bernoulli Filter – Key Concept

(Internal Notes)

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1 Motivation

Traditional multi-target tracking algorithms are based on the concatenation of individual target states, i.e., the target states $\mathbf{x}_1, \ldots, \mathbf{x}_n$ for n targets are concatenated to form the joint vector $[\mathbf{x}_1^T, \ldots, \mathbf{x}_n^T]^T$. However, this representation suffers from severe problems, e.g.,

- there is no natural order of the target states, e.g., $[\mathbf{x}_n^T, \ldots, \mathbf{x}_1^T]^T$ would also be a reasonable concatenation, and
- the number of targets n is unknown and can vary over time.

For this reason, modern multi-target trackers model the multi-target state as a set of single target states, i.e.,

$$X = \{(\mathbf{x}_1, l_1), \ldots, (\mathbf{x}_n, l_n)\} \ ,$$

where $\mathbf{x}_i$ is the target state and l_i its label (for $i = 1, \ldots, n$).

2 Random Sets

The objective is to estimate the multi-target set X, where the (uncertain) information about X is encoded with a so-called random finite set (RFS), i.e., a set with a random number of random vectors.

A convenient RFS is the Labeled Multi-Bernoulli (LMB) RFS, which can be written as

$$X = \{(r^{(l)}, p^{(l)})\}_{l \in L} \ ,$$

where $r^{(l)}$ is the existence probability and $p^{(l)}$ is the probability density of the target with label l. Intuitively, the meaning is that with probability $r^{(l)}$ a target with label l exists and its state density is given by $p^{(l)}(\mathbf{x})$.

As an example, consider the scenario in Figure 1, where two vehicles and three detections are depicted. A multi-target RFS for the labels $L = \{1, 2\}$ could be given by the LMB RFS

$$\{(r^{(1)}, p^{(1)}), (r^{(2)}, p^{(2)})\} \tag{1.1}$$

as visualized in Figure 1. For example, if $r^{(1)}$ is close to zero, there is a low probability that target 1 exists. Furthermore

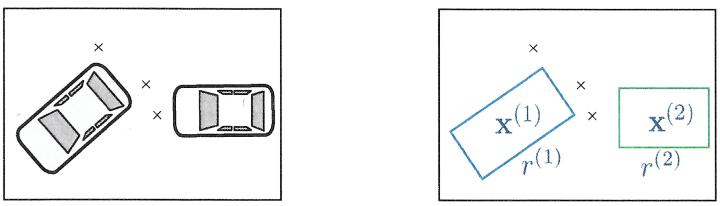
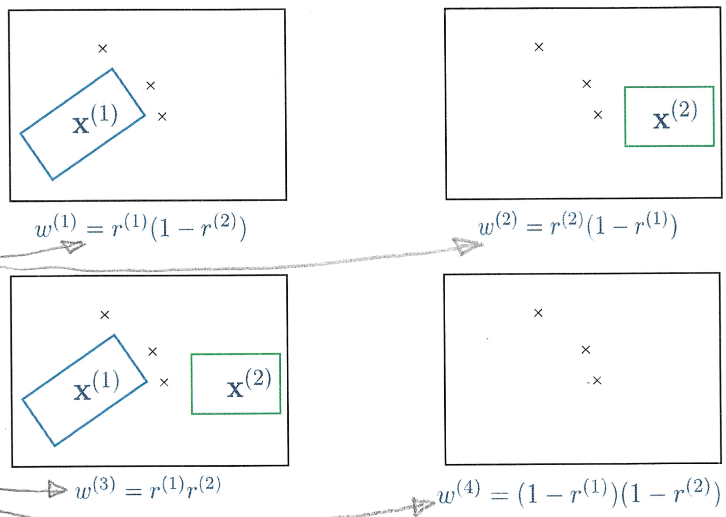


Figure 1: Two vehicles and three detections (left). A multi-target estimate given by an LMB RFS (right). $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ illustrate the means of the corresponding state distributions $p^{(1)}(\mathbf{x})$ and $p^{(2)}(\mathbf{x})$, respectively. Furthermore, $r^{(1)}$ and $r^{(2)}$ are the existence probabilities.

从 LMB convert 到 GLMB 的方案中，用 LMB 的每个 component 的 existence probability r 来计算 GLMB 的每个 component 的 weight 的例子



** Figure 2: δ-GLMB visualization of the example LMB (1.1) from Figure 1. For each subset L_i of $L = \{1, 2\}$, a weight $w^{(i)}$ specifies the probability of the subset L_i.

A labeled RFS can also be characterized by the probabilities for all possible subsets of labels (plus the individual state densities). For example, the LMB (1.1) can be characterized as follows (see Figure 2 for a visualization):

Component i	Set of Label L_i	Probability w^i	State densities
1	$\{1\}$	$r^{(1)}(1 - r^{(2)})$	$p^{(1)}$
2	$\{2\}$	$(1 - r^{(1)})r^{(2)}$	$p^{(2)}$
3	$\{1, 2\}$	$r^{(1)}r^{(2)}$	$p^{(1)}, p^{(2)}$
4	$\{\}$	$(1 - r^{(1)})(1 - r^{(2)})$	-

** Table 1.1: The δ-GLMB representation of the example LMB (1.1).

这个表格表示了所有可能的场景对应的hypothes
以及其相关概率
每一个component
(hypothesis)对应
场景见figure 2

The above representation is also called δ-Generalized Multi-Bernoulli (δ-GLMB) RFS.
In the same way as for random vectors, the probability distribution of an RFS can be specified by a density. For a RFS, the density is a function of sets. The formulas for the corresponding RFS density $\pi(X)$ of the above example LMB (1.1) is given in Section 7.

3 Bayes Rule for Random Sets

In order to incorporate a new set of detections Y, Bayes rule for random sets

$$f(X|Y) = \frac{f(Y|X) \cdot f(X)}{\int f(Y|X) f(X) dX}$$

can be used, where

- $f(X)$ is the prior RFS density,
- $f(Y|X)$ is the multi-target likelihood, and
- $\int f(Y|X) f(X)\, \delta X$ is the predicted measurement set density, which involves a *set* integral (see [1] for a definition).

P_j : partitions of detection points

θ_i^j : hypothesis of partition j associated to component i

$w^{(i,j)}$: weight for hypothesis j associated to i

component i : the possible "scenario" (component) of GLMB RFS

prediction of component 1

update of component 1

prediction of component 2

update of component 2

prediction of component 3

update of component 3

prediction of component 4

这个例子中，2个targets与3个detections一共可以存在4种可能的components，每一种component中的detections将会被partition。之后再进行update step

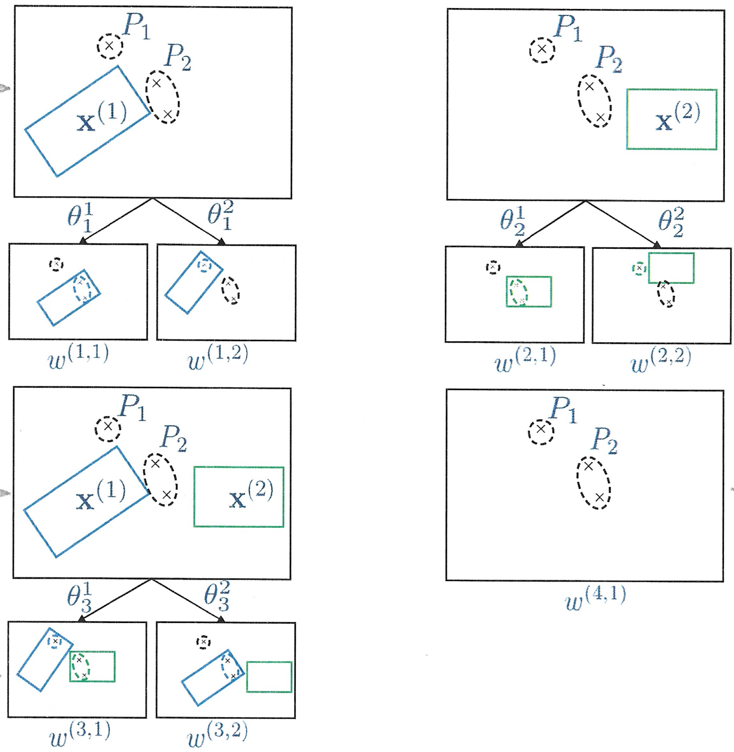


Figure 3: Updating the δ-GLMB using 1 feasible partitioning (that consists of the partitions P_1 and P_2) and the 2 best associations (θ_i^1 and θ_i^2 for each component i).

For extended targets [2], the multi-target likelihood function can incorporate, e.g., detection probabilities, Poisson distributed clutter and the possibility of multiple measurements per object.

就是指的measurement model for multiple extended targets那些内容

A significant leap in multi-target tracking research has been made in 2013 when it was shown in [3] that the δ-GLMB is a conjugate prior for the multi-target likelihood, i.e., if the prior is a δ-GLMB density, then the posterior is a δ-GLMB density as well.

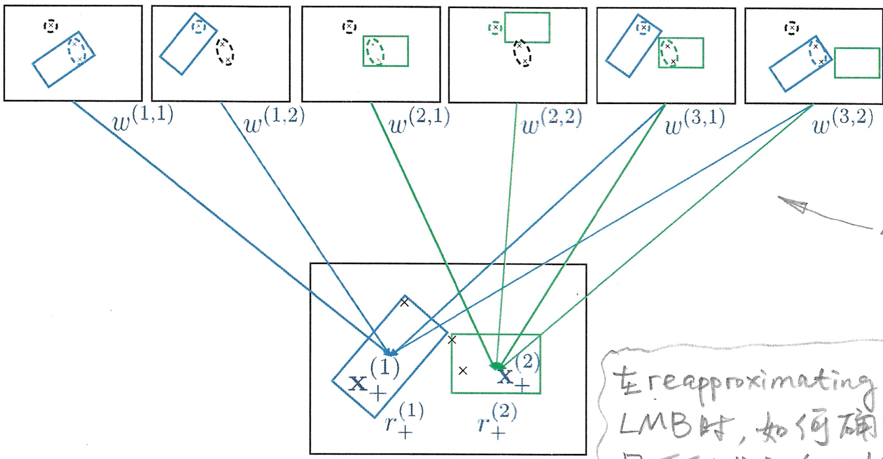
In order to illustrate this, we show how to apply Bayes rule for random sets to the δ-GLMB from Figure 1 given the three depicted detections. As there are spatially extended targets, multiple detections may stem from the same target. For this reason, the detections are partitioned in a first step, where all detections that come from the same target are in one partition. In our scenario, there are two partitions P_1 and P_2 as shown in Figure 3. In the second step, each component of the δ-GLMB is updated, i.e., each row in table 1. As it is unknown which partition belongs to which target (or clutter), it is necessary to (theoretically) consider all possibilities. For example, for the first component $i = 1$ of the δ-GLMB, the two best associations θ_1^1 and θ_1^2 are as follows:

- θ_1^1: P_2 is from target 1 and P_1 is clutter
- θ_1^2: P_1 is from target 1 and P_2 is clutter

among all possible association hypotheses (例如，some of association hypotheses have very "low probability"/"large distance" to be happened, will be abandoned, by using Murty's algorithm)

For each possible association, a new component is generated for the posterior RFS, where the state densities are updated with the detections accordingly.

This procedure has to be performed for all components of the prior δ-GLMB. As a final result, a δ-GLMB with 7 components is obtained, see Figure 3. Note that the prior δ-GLMB had 4 components. In general, the complexity for representing the posterior δ-GLMB distribution grows exponentially over time.



在reapproximating δ-GLMB back to LMB时，如何确定/计算 $r_+^{(l)}$ 和 $p_+^{(l)}$？见下列式子作例子

** Figure 4: Approximating the δ-GLMB as an LMB by using the individual components' weights.

4 Efficient RFS filter (LMB filter)

Further significant progress in multi-target tracking has been made in 2014 when an efficient implementation of the multi-target Bayes filter called the Labeled Multi-Bernoulli (LMB) filter [4] was proposed. The key idea is as follows:

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- Start with an LMB RFS (see Figure 1). LMB RFS at time k-1 → prediction → predicted LMB RFS at time k
- Convert it to a δ-GLMB RFS (see Figure 2) ← convert ← δ-GLMB RFS
- Perform a measurement update using Bayes rule for random sets to get the posterior δ-GLMB (see Figure 3). ← update ← updated (posterior) δ-GLMB RFS
- Reapproximate the (complex) posterior δ-GLMB with an LMB RFS again (see Figure 4). → reapproximate → updated (posterior) LMB RFS

By means of this procedure an *efficient* recursive multi-object tracking based on LMB densities is obtained.

All but the last step have been already illustrated above. The reapproximation with an LMB is illustrated in Figure 4. For example, in order to get the weight for target 1, the weights of all subset that contain target 1 are summed up:

$$r_+^{(1)} = w^{(1,1)} + w^{(1,2)} + w^{(3,1)} + w^{(3,2)}$$

The state distribution results from a weighted sum of all state distributions from the components:

$$p_+^{(1)}(\mathbf{x}) = \frac{1}{r_+^{(1)}} \cdot (w^{(1,1)} \cdot p_+^{(1,1,\theta_1^1)}(\mathbf{x}) + w^{(1,2)} \cdot p_+^{(1,1,\theta_1^2)}(\mathbf{x}) + w^{(3,1)} \cdot p_+^{(1,3,\theta_3^1)}(\mathbf{x}) + w^{(3,2)} \cdot p_+^{(1,3,\theta_3^2)}(\mathbf{x}))$$

with $p_+^{(l,i,\theta_i^j)}(\mathbf{x})$ being the updated density of the target with label l from component i using association θ_i^j. The weight and distribution for target 2 is obtained in the same way.

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假如模型服从GGIW，则可以根据prediction of GGIW for single hypothesis 来求出 $p_+^{(l,i,\theta_i^j)}(x)$。 l表示label，i表示可能的GLMB之component，j表示可能的detection points partition
θ_i^j 表示 hypothesis of "partition j of detection points" associated to component i of GLMB RFS

5 Conclusions

Random finite set theory provides a mathematically rigorous framework for deriving efficient multi-object trackers that provide labeled tracks and a systematic track management.

6 Bibliography

[1] R. P. S. Mahler, *Statistical Multisource-Multitarget Information Fusion*. Norwood, MA, USA: Artech House, Inc., 2007.

[2] M. Beard, S. Reuter, K. Granström, B. Vo, B. Vo, and A. Scheel, "Multiple Extended Target Tracking With Labeled Random Finite Sets," *IEEE Transactions on Signal Processing*, vol. 64, no. 7, pp. 1638–1653, April 2016.

[3] B.-T. Vo and B.-N. Vo, "Labeled Random Finite Sets and Multi-Object Conjugate Priors," *IEEE Transactions on Signal Processing*, vol. 61, no. 13, pp. 3460–3475, 2013.

[4] S. Reuter, B. Vo, B. Vo, and K. Dietmayer, "The Labeled Multi-Bernoulli Filter," *IEEE Transactions on Signal Processing*, vol. 62, no. 12, pp. 3246–3260, June 2014.

有关具体怎样求 GLMB $\rightarrow$ LMB $\rightarrow$ GLMB 的 r 和 w 的例子，见 [4] 的 section Ⅲ.C

7 Appendix

The RFS density for the LMB of two targets $\{r^{(i)}, p^{(i)}(\mathbf{x})\}$, $i \in \{1, 2\}$, is given by

$$\pi(\mathbf{X}) = \Delta(\mathbf{X}) \cdot (1 - r^{(1)})(1 - r^{(2)}) \cdot \prod_{l \in \mathcal{L}(\mathbf{X})} \frac{r^{(l)} p^{(l)}(\mathbf{x}_l)}{1 - r^{(l)}} 1_{\{l_1, l_2\}}(l) \ .$$

The corresponding δ-GLMB representation of the density is

$$\begin{aligned}\pi(\mathbf{X}) = \Delta(\mathbf{X}) \big(& \underbrace{r^{(1)}(1 - r^{(2)})}_{w^{(1)}} p^{(1)}(\mathbf{x}_1) \delta_{\{l_1\}}(\mathcal{L}(\mathbf{X})) \\ & + \underbrace{r^{(2)}(1 - r^{(1)})}_{w^{(2)}} p^{(2)}(\mathbf{x}_2) \delta_{\{l_2\}}(\mathcal{L}(\mathbf{X})) \\ & + \underbrace{r^{(1)} r^{(2)}}_{w^{(3)}} p^{(1)}(\mathbf{x}_1) p^{(2)}(\mathbf{x}_2) \delta_{\{l_1, l_2\}}(\mathcal{L}(\mathbf{X})) \\ & + \underbrace{(1 - r^{(2)})(1 - r^{(1)})}_{w^{(4)}} \delta_{\{\}}(\mathcal{L}(\mathbf{X})) \big) \ .\end{aligned}$$

Furthermore, we have the following definitions

$$\begin{aligned}\mathcal{L}((x, l)) &= l \\ \mathcal{L}(\mathbf{X}) &= \{\mathcal{L}(\mathbf{x}) : \mathbf{x} \in \mathbf{X}\} \\ \delta_{\mathbf{Y}}(\mathbf{X}) &= \begin{cases} 1, \text{ if } X = Y \\ 0, \text{ else} \end{cases} \\ 1_{\mathbf{Y}}(\mathbf{X}) &= \begin{cases} 1, \text{ if } X \subseteq Y \\ 0, \text{ else} \end{cases} \\ \Delta(\mathbf{X}) &= \delta_{|\mathbf{X}|}(|\mathcal{L}(\mathbf{X})|) \ .\end{aligned}$$