1. Picture: Heisenberg vs Schroedinger.

纠正 $\langle x | \varphi \rangle = \varphi(x)$

$\langle p | \varphi \rangle = \tilde{\varphi}(p)$ Fourier

H: State is constant, Operator changes

$$\frac{d}{dt}\hat{A} = \left(\frac{\partial A}{\partial t}\right)_{classical} + \frac{1}{i\hbar}[A, H]$$

$\boxed{\text{Similar to } \frac{d}{dt}A = \frac{\partial}{\partial t}A + [f, H]_{PB}}$

S: State is changeing, Operator remains

$$i\hbar \frac{d}{dt}|\varphi(t)\rangle = H|\varphi(t)\rangle$$

$$|\varphi(t)\rangle = e^{-\frac{iHt}{\hbar}}|\varphi(0)\rangle \quad \text{if } \frac{\partial H}{\partial t} \equiv 0$$

$$\begin{cases} \varphi(x) = \frac{1}{\sqrt{2\pi}} \int dp \, \langle x | p \rangle \, \tilde{\varphi}(p) \\ \tilde{\varphi}(p) = \frac{1}{\sqrt{2\pi}} \int dx \, \langle p | x \rangle \, \varphi(x) \end{cases}$$

2. Uncertainty Principle

GUP: $\Delta\hat{A}\cdot\Delta\hat{B} \geq \frac{1}{2}\left|\langle[\hat{A},\hat{B}]\rangle\right|$ $\Delta\hat{A} = \langle\hat{A}-\langle\hat{A}\rangle\rangle$

while $[x_i, p_i] = i\hbar$ then $\boxed{\Delta x_i \Delta p_i \geq \frac{\hbar}{2}}$

Proof: Let $|\psi\rangle = (A+i\lambda B)|\varphi\rangle$

$$\langle\psi|\psi\rangle = \langle\varphi|(A-i\lambda B)(A+i\lambda B)|\varphi\rangle \geq 0$$

$$\Rightarrow \quad \langle\varphi|A^2+\lambda^2B^2+i\lambda[A,B]|\varphi\rangle \geq 0$$

$$\Rightarrow \quad \langle A^2\rangle + i\lambda\langle[A,B]\rangle + \lambda^2\langle B^2\rangle \geq 0$$

$\because$ $i\langle[A,B]\rangle$ is real, because

$$\overline{i\langle[A,B]\rangle} = -i\langle[B,A]\rangle$$
$$= i\langle[A,B]\rangle$$

$$\therefore \quad \Delta = \langle i[A,B]\rangle^2 - 4\langle A^2\rangle\langle B^2\rangle \leq 0$$

$$\Rightarrow \langle A^2\rangle\langle B^2\rangle \geq \frac{1}{4}\langle i[A,B]\rangle^2 = \frac{1}{4}\langle|[A,B]|\rangle^2$$

Let $A' = A-\langle A\rangle$, $B' = B-\langle B\rangle$

$$\Rightarrow \langle A'\rangle = \langle A^2\rangle - \langle A\rangle^2 \equiv \Delta A^2 \quad \langle B'\rangle = \langle B^2\rangle - \langle B\rangle^2 \equiv \Delta B^2$$

and $[A',B'] = [A,B]$

$$\Rightarrow \quad \Delta A\,\Delta B \geq \frac{1}{2}\langle|[A,B]|\rangle^2$$

3. ✱ Ehrenfest's Theorem

$$i\hbar \frac{d}{dt}\langle \hat{A} \rangle = \left\langle \frac{\partial A}{\partial t} \right\rangle + \frac{1}{i\hbar}\langle [\hat{A}, \hat{H}] \rangle$$

1. If $[A, H] = 0$, $\frac{\partial A}{\partial t} = 0$

$\langle \hat{A} \rangle = \text{const}$

2. If def $\Delta T = \dfrac{\Delta \hat{A}}{\frac{d}{dt}\langle \hat{A} \rangle}$ $\Delta E = \Delta \hat{H}$

$\Rightarrow$ $\boxed{\Delta E \cdot \Delta T \geq \frac{\hbar}{2}}$

Proof: In classical

$$\frac{d}{dt} A = \frac{\partial}{\partial t} A + [A, H]_{PB}$$

In Quan $\Longrightarrow$ $\because$ $[A, H]_{PB} \rightarrow \frac{1}{i\hbar}[A, H]$

$$\therefore \frac{d}{dt}\hat{A} = \frac{\partial}{\partial t} A + \frac{1}{i\hbar}[A, H]$$

In H' picture, $|\psi\rangle$ is time-invariant

$$\therefore i\hbar \frac{d}{dt}\langle \hat{A} \rangle = \left\langle \frac{\partial}{\partial t} A \right\rangle + \frac{1}{i\hbar}\langle [A, H] \rangle$$

4. Represant Transformation is Unitary Transformation

$$|\varphi_n\rangle \xrightarrow{U} |\varphi_n\rangle$$

$$\hat{A} \xrightarrow{\hat{U}} \hat{A}'$$

where $\hat{A} = U^{\dagger} \hat{A}' U$

$$\hat{A}' \doteq U A U^{\dagger}$$

For Hermitian $\hat{H}$

① $\hat{H}' = \hat{H}$ ② $\boxed{[\hat{U}, \hat{H}] = 0}$

③ $\hat{U} = I + i\varepsilon\hat{F}$ $(\varepsilon \ll 1)$

where $\vec{F}$ is Hermitian

$[\hat{F}, \hat{H}] = 0$

$\Rightarrow \hat{F}$ is conserved.