Einstein Convention $\vec{x} = \underset{=}{x_i} \underset{=}{\vec{e}_i}$ Dummy Index

$$\vec{e_i} \cdot \vec{e_j} = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Symmetric Tensor

$$\vec{e_i} \times \vec{e_j} = \underline{\underline{\epsilon_{ij}}}{}_k \vec{e_k}$$

Levi-Civita Tensor

$$\epsilon_{ijk} = \begin{cases} +1 & : \text{even permutation} \\ 0 & : \text{no order} \\ -1 & : \text{odd permutation} \end{cases}$$

① Symmetric $\delta_{ij} = \delta_{ji}$

② Anti-Symmetric $\epsilon_{ijk} = -\epsilon_{jik}$

③ If $\underline{T_{ij} = T_{ji}}$, then $\underline{\epsilon_{ijk} T_{ij}} \equiv 0$

Anti-Symmetric × Symmetric

$\forall \vec{A}, \vec{B}$, $\vec{A} = A_i \vec{e}_i \qquad \vec{B} = B_i \vec{e}_i$

$\vec{A} \cdot \vec{B} = A_i B_i \qquad \vec{A} \times \vec{B} = \varepsilon_{ijk} A_i B_j \vec{e}_k$

$\vec{\nabla} \phi = \partial_i \phi \qquad \vec{\nabla} \cdot \vec{A} = \partial_i A_i \qquad \vec{\nabla} \times \vec{A} = \epsilon_{ijk} \partial_i A_j e_k$

$$\vec{\nabla} \times \vec{\nabla} \phi = \epsilon_{ijk} \underbrace{\partial_j \partial_k}_{\text{Symmetric}} \phi$$

$\vec{\nabla} \times \vec{\nabla} \phi = 0$ Curl-free

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = \partial_i (\vec{\nabla} \times \vec{A})_i$$
$$= \partial_i \epsilon_{ijk} \partial_j A_k$$
$$= \epsilon_{ijk} \underbrace{\partial_i \partial_j}_{\text{Symmetric}} A_k$$

$\vec{\nabla} \cdot \vec{\nabla} \times A = 0$ Divergence-Free

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B})$$

$\quad \|$

$A_i (\vec{B} \times \vec{C})_i \qquad \| \qquad \|$

$\quad \|$

$A_i \epsilon_{ijk} B_j C_k$

$\quad \|$

$\epsilon_{ijk} A_i B_j C_k = \cdots =$

$\blacksquare$

$$\epsilon_{ijk} = \vec{e}_i \cdot (\vec{e}_j \times \vec{e}_k) = \begin{vmatrix} \delta_{1i} & \delta_{2i} & \delta_{3i} \\ \delta_{1j} & \delta_{2j} & \delta_{3j} \\ \delta_{1k} & \delta_{2k} & \delta_{3k} \end{vmatrix}$$

$$\epsilon_{ijk}\epsilon_{lmn} = \begin{vmatrix} \delta_{il} & \delta_{jl} & \delta_{kl} \\ \delta_{im} & \delta_{jm} & \delta_{km} \\ \delta_{in} & \delta_{jn} & \delta_{kn} \end{vmatrix}$$

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$ (frequently used !)

$$\vec{\nabla}(uv) = u\vec{\nabla} v + v\vec{\nabla} u$$

$$\vec{\nabla}(\vec{A}\cdot\vec{B}) = \vec{A}\times(\vec{\nabla}\times\vec{B}) + (\vec{A}\cdot\vec{\nabla})\vec{B} + \vec{B}\times(\vec{\nabla}\times\vec{A}) + (\vec{B}\cdot\vec{\nabla})\vec{A}$$

$$\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\vec{\nabla}\times\vec{A}) - \vec{A}\cdot(\vec{\nabla}\times\vec{B})$$

$$\vec{\nabla}\times(\vec{A}\times\vec{B}) = (\vec{B}\cdot\vec{\nabla})\vec{A} + (\vec{\nabla}\cdot\vec{B})\vec{A} - (\vec{A}\cdot\vec{\nabla})\vec{B} - (\vec{\nabla}\cdot\vec{A})\vec{B}$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \vec{\nabla}^2\vec{A}$$

$$\begin{cases} \vec{\nabla}\times\vec{\nabla}\phi \equiv 0 \\ \vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) \equiv 0 \end{cases}$$

$$\vec{\nabla}\left(\frac{1}{r}\right) = -\frac{\vec{x}}{r^3} \qquad r = \|\vec{x}\|$$

$$\vec{\nabla}^2\left(\frac{1}{r}\right) = 4\pi\,\delta^{(3)}(\vec{x})$$

$$\boxed{\begin{array}{l} \forall \vec{F}, \quad \vec{F} = \vec{F}_{\parallel} + \vec{F}_{\perp} \\ \nabla\times\vec{F}_{\parallel} \equiv 0 \qquad \vec{\nabla}\cdot\vec{F}_{\perp} = 0 \end{array}}$$

Laplace-Runge-Lenz Vector (LRL)

① CED & CM

$$\vec{A} = \vec{p} \times \vec{L} - mk\frac{\vec{x}}{r}$$

② Quantum Machanics

$$\hat{\vec{A}} = \frac{1}{2}(\hat{\vec{p}} \times \hat{\vec{L}} - \hat{\vec{L}} \times \hat{\vec{p}}) - mk\frac{\vec{x}}{r}$$

Classical ElectroDynamics $\underline{\underline{CED}}$

① Scalar Potential

$$\Phi(\vec{x}, t) = \int_{V'} \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|} d^3\vec{x}'$$

② Vector Potential

$$\vec{A}(\vec{x}, t) = \frac{1}{c} \int \frac{\vec{j}(\vec{x}')}{|\vec{x} - \vec{x}'|} d^3\vec{x}'$$

③ $\vec{E} = -\vec{\nabla}\Phi - \frac{1}{c}\frac{\partial}{\partial t}\vec{A}$

$\vec{B} = \vec{\nabla} \times \vec{A}$

BCH Formula :

1. $e^{-a\hat{A}}\hat{B}e^{+a\hat{A}} = \hat{B} - a[A,B] + \frac{a^2}{2!}[A,[A,B]] + \cdots$

$$= \sum_{n=0}^{\infty} \frac{(-a)^n}{n!}[A^n, B]$$

where $[A^n,B] = [\underbrace{A,[A,[A,\cdots[A}_{n's\ A},B]]\cdots]$

2. $e^{A+B} = e^{-\frac{1}{2}[A,B]}e^{A}e^{B} = e^{\frac{1}{2}[A,B]}e^{B}e^{A}$

if $\begin{cases} [B,[A,B]] \equiv 0 \\ [A,[A,B]] \equiv 0 \end{cases}$

[Proof]

1. $f(a) = e^{-aA}\hat{B}e^{aA}$

$f(0) = B$

$f'(0) = e^{-aA}(-A)Be^{aA} + e^{-aA}BAe^{eA}\Big|_{a=0}$

$= -[A,B]$

If $f^{(k)}(a) = (-1)^k[A^k,B]$

then $f^{(k+1)}(0) = e^{-aA}(-A)[A^k,B]e^{aA} + e^{-a}(-1)^k[A^k,B]e^{aA}\Big|_{a=0}$

$\Rightarrow$ Proof 1

2. If $[A,[A,B]]=0$

$$\Rightarrow \underbrace{e^{-aA}Be^{aA}=B-a[A,B]}$$

let $g(a)=e^{aA}e^{aB}$

$$\Rightarrow \frac{dg(a)}{da}=e^{aA}Ae^{aB}+e^{aA}e^{aB}B$$

$$=e^{aA}e^{aB}\underbrace{e^{-aB}Ae^{aB}}+e^{aA}e^{aB}B$$

$$=e^{aA}e^{aB}(A-a[B,A]+B)$$

$$=e^{aA}e^{aB}(A+B+a[A,B])$$

$$=g(a)(A+B+a[A,B])$$

$$\Rightarrow \frac{dg(a)}{g(a)}=(A+B+a[A,B])da$$

$$\ln g(a)=[A+B+\frac{a}{2}[A,B]]a$$

$$\Rightarrow g(a)=e^{a[A+B+\frac{a}{2}[A,B]]}$$

$$\xrightarrow{a=1} e^Ae^B=e^{A+B+\frac{1}{2}[A,B]}$$

while by Comutator Conditon

$$e^{A+B}=e^Ae^Be^{-\frac{1}{2}[A,B]}$$

More Generally:

$$e^A e^B = e^C$$

$$\Longleftrightarrow C = A + B - \frac{1}{2}[A, B] + \cdots$$