Newtonian Mechanics   Physics - Space

$$\vec{F} = m\frac{d^2x^2}{dt^2} \qquad \frac{d\vec{p}}{dt} = \vec{F}$$

Configeration Space (位形空间) $q_a(t)$ $\xrightarrow{\text{Legendra}}$ Transformation Phase Space $q_k(t)$ $p_k(t)$

$p_a = \frac{\partial L}{\partial \dot{q}_a}$ $H(q,p,t) = \sum p_k \dot{q}_k - L$

Lagragian Mechanics $\xrightarrow{\delta S \equiv 0}$ Euler-Lagragian Mechanics

$$\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} = \frac{\partial L}{\partial q_k}}$$

Hamiltonian Canonical Eq.

$$\begin{cases} \dot{q}_a = \frac{\partial H}{\partial p_a} \\ \dot{p}_a = -\frac{\partial H}{\partial q_a} \end{cases}$$

(Dirac → Feynman)

Path Intergreted &

Lagragian based QM, QFT

$\forall f(p,q,t)$,

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + [f, H]_{PB}$$

$$[A,B]_{PB} = \sum_a \begin{vmatrix} \frac{\partial A}{\partial p_a} & \frac{\partial A}{\partial q_a} \\ \frac{\partial B}{\partial p_a} & \frac{\partial B}{\partial q_a} \end{vmatrix}$$

Hamiltonian QM

(Non-relativitisic)

For holonomic Constrait (定常约束：约束与速度无关)

Lagrangian Mechanics

Configuration $q(t)$

$$\frac{\partial L}{\partial q_k} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k}$$

$\downarrow$ Legendre Transformation

Hamiltonian Mechanics

Phase Space (Canonical)

$q(t)$ Canonical Position

$p(t)$ Canonical Momentum

$$\begin{cases} \dot{q} = \frac{\partial H}{\partial p} \\ \dot{p} = -\frac{\partial H}{\partial \dot{q}} \end{cases}$$

For Non-holonomic Constraints (like CED)

- d'Alenbert Priciple

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q_i} = Q_i$$

T: Kinetic Energy Q_i : Generalize Force

Genelized Coordinate

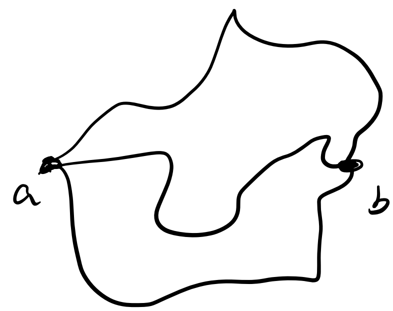
$q_k(t) \qquad k = 1, 2, \cdots D$

$L[q] = L(q(t), \dot{q}(t); t)$ Lagrangain

$S = \int_{t_1}^{t_2} L(q(t), \dot{q}(t); t)$ Action

Priciple of Least Action :

$$\boxed{\delta S = 0}$$



Euler-Lagrange Eq.

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \frac{\partial L}{\partial q_k}} \qquad k = 1, 2, \cdots, D$$

[Proof]

$$\delta S = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q_k} \delta q_k + \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k \right) dt$$

$$= \int_{t_1}^{t_2} \left\{ \frac{\partial L}{\partial q_k} \delta q_k + \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_k} \delta q_k\right] - \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)\right] \delta q_k \right\} dt$$

$$= \cancel{\frac{\partial L}{\partial \dot{q}_k} \delta q_k \Big|_{t_1}^{t_2}} + \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)\right] \delta q_k \, dt$$

$$\equiv 0$$

$$\Longleftrightarrow \quad \frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) \equiv 0$$

Hamiltonian

$$\boxed{H = \sum p\dot{q} - L} \qquad \boxed{p_i = \frac{\partial L}{\partial \dot{q}}}$$

$$dH = \frac{\partial H}{\partial t}dt + \sum_i \left(\frac{\partial H}{\partial q_i}dq_i + \frac{\partial H}{\partial p_i}dp_i\right)$$

$p_i dq_i$ // $\quad p_i d\dot{q}_i$ //

$$dH = \sum_i \left((dp_i)\dot{q}_i + (dq_i)p_i\right) - \sum_i \left(\frac{\partial L}{\partial p_i}dp_i + \frac{\partial L}{\partial \dot{q}}d\dot{q}\right) - \frac{\partial L}{\partial t}dt$$

$$= \sum_i \left(\dot{q}_i dp_i - \dot{p}_i dq_i\right) - \frac{\partial L}{\partial t}dt$$

$$\Rightarrow \begin{cases} \dfrac{\partial H}{\partial t} = -\dfrac{\partial L}{\partial t} \\ \left.\begin{array}{l} \dfrac{\partial H}{\partial q_i} = -\dot{p}_i \\ \dfrac{\partial H}{\partial p_i} = \dot{q}_i \end{array}\right\} \end{cases}$$

$\forall f(p,q,t)$

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \sum\left(\frac{\partial f}{\partial q}\dot{q} + \frac{\partial f}{\partial p}\dot{p}\right)$$

$$= \frac{\partial f}{\partial t} + \sum\left(\frac{\partial f}{\partial q}\cdot\frac{\partial H}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial H}{\partial q}\right)$$

$$= \frac{\partial f}{\partial t} + [f, H]_{PB}$$

CCC

Classical Canonical Commutator

While for any observable:

$$\frac{d\hat{A}}{dt} \doteq \frac{\partial \hat{A}}{\partial t} + \frac{1}{i\hbar}[A, H] \quad \text{QCC}$$

Fundamental CCC

$$\begin{cases} [q_i, p_j]_{PB} = \delta_{ij} \\ [q_i, q_j]_{PB} = [p_i, p_j]_{PB} \equiv 0 \end{cases}$$

Angular Momentum:

$$\vec{L} = \vec{x} \times \vec{p}$$

$$L_i = (\vec{x} \times \vec{p})_i = \varepsilon_{ijk}\, x_j p_k$$

$$\begin{cases} [L_i, L_j]_{PB} = \varepsilon_{ijk} L_k \\ \Downarrow \\ [L_i, L^2]_{PB} \equiv 0 \end{cases}$$

Leibniz:

$$[A, BC]_{PB} = [A, B]_{PB} C + B [A, C]_{PB}$$

$$[AB, C]_{PB} = [A, C]_{PB} B + A [B, C]_{PB}$$

Jacobi [ ] mean PB

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] \equiv 0$$

Defferentiate

$$[A, B^n]_{PB} = n [A, B]_{PB} B^{n} \quad \text{if } [B, [A, B]] = 0$$

d'Alenbert Priciple (Non-holonomic) 非定常约束

If $U = f(q, \dot{q}, t)$ → 约束与广义速度有关

$L = T - U \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q}$

$$Q_i = \begin{cases} \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q_i} \\ \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}}\right) - \frac{\partial U}{\partial q} \end{cases}$$

(广义力)

$$\vec{F} = q\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right)$$

Legendra transformation

$$f(x) \xrightarrow{p = f'(x)} g(p) = f(x) - x f'(x)$$

If $\left|\frac{\partial^2 L}{\partial q_i \partial q_j}\right| \neq 0 \Rightarrow$

$L = T - V$

$H = T + V$

Canonical → "Right"

→ coordinate $q(t)$, $p(t) = \frac{\partial L}{\partial \dot{q}}$

→ Equation

→ Commutation $[A,B]_{PB}$

→ Transformation 正则变换

$$H(q,p;t) \xrightarrow{CT} K(Q,P;t)$$

$$\left([A,B]_{PB}\right)_{q,p} = \left([A,B]_{PB}\right)_{Q,P}$$

Canonical transformation

$$\delta \int L(q,\dot{q};t)\,dt = \delta \int \sum_a p_a \dot{q}_a - H(p,q;t)\,dt$$

$$= \delta \int \sum_a P_a \dot{Q}_a - K(P,Q;t)\,dt + dG$$

$$\Rightarrow dG = \left(\sum_a p_a dq_a - H(p,q;t)dt\right) - \left(\sum_a P_a dQ_a - K(P_a,Q_a;t)dt\right)$$

↓

Generator function $G(\underline{q},p;\underline{Q},P)$ 4个中2个是独立的

$$= \frac{\partial G}{\partial t}dt + \sum_a \left(\frac{\partial G}{\partial q_a}dq_a + \frac{\partial G}{\partial Q_a}dQ_a\right)$$

dt-term: $K = H + \frac{\partial G}{\partial t}$

dq_a-term: $p_a = \frac{\partial G}{\partial q_a}$ dQ_a-term: $P_a = -\frac{\partial G}{\partial Q_a}$

$$\Rightarrow \quad \frac{\partial P_a}{\partial Q_b} = \frac{\partial^2 G}{\partial Q_b \partial q_a} \qquad \frac{\partial \mathbb{P}_a}{\partial q_b} = -\frac{\partial^2 G}{\partial q_b \partial Q_a}$$

$$\Rightarrow \boxed{\frac{\partial P_a}{\partial Q_b} + \frac{\partial \mathbb{P}_b}{\partial q_a} = 0} \quad \text{If } G_{,q_b Q_a} = G_{Q_a q_b}$$

$$\Downarrow$$

$$\boxed{\left([A,B]_{PB}\right)_{P,q} = \left([A,B]\right)_{\mathbb{P},Q}}$$

Let $K \equiv 0$, $Q = q_0$

Dynamics on Phase Space

$\rho(q,p;t)$ Density Distribution

$$\int \rho(q,p;t)\, d^D q\, d^D p = 1$$

① <u>Liouville's Theorem</u>: $\frac{d\rho}{dt} \equiv 0 \Rightarrow \frac{\partial \rho}{\partial t} = [H,\rho]_{PB}$

$\downarrow$ Liouvilles' Eq.

$$\boxed{\frac{\partial \hat{\rho}}{\partial t} = \frac{1}{i\hbar}[\hat{H},\hat{\rho}]}$$

Valid for closed system

and Both pure and mixed State

If $\hat{\rho} = |\varphi\rangle\langle\varphi|$ (Pure State)

$$i\hbar \frac{\partial}{\partial t} |\varphi\rangle\langle\varphi| = \hat{H}|\varphi\rangle\langle\varphi| - |\varphi\rangle\langle\varphi|\hat{H}$$

$$\left(i\hbar \frac{\partial}{\partial t}|\varphi\rangle\right)\langle\varphi| + |\varphi\rangle\, i\hbar\frac{\partial}{\partial t}\langle\varphi| = \hat{H}|\varphi\rangle\langle\varphi| - |\varphi\rangle\langle\varphi|\hat{H}$$

$$i\hbar\frac{\partial}{\partial t}|\varphi\rangle + |\varphi\rangle\left(i\hbar\frac{\partial}{\partial t}\langle\varphi|\right)|\varphi\rangle = \hat{H}|\varphi\rangle - |\varphi\rangle\langle\varphi|\hat{H}|\varphi\rangle$$

$$\langle\varphi|\left(\underbrace{i\hbar\frac{\partial}{\partial t}|\varphi\rangle - \hat{H}|\varphi\rangle}\right) = \left(\underbrace{-i\hbar\frac{\partial}{\partial t}\langle\varphi| - \langle\varphi|\hat{H}}\right)|\varphi\rangle = 0$$

$\hookrightarrow$ Schöreadinger !!

② Continue Eq. $\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{j} \equiv 0 \implies$ Liouville

In phase space, $\vec{j} = (\rho \dot{q}, \rho \dot{p})$

$$\nabla \cdot \vec{j} = \sum_a \frac{\partial}{\partial q_a}(\rho \dot{q}_a) + \frac{\partial}{\partial p_a}(\rho \dot{p}_a)$$

$$= \sum_a \left[\frac{\partial \rho}{\partial q_a} \dot{q}_a + \cancel{\rho \frac{\partial \dot{q}_a}{\partial q_a}} + \frac{\partial \rho}{\partial p_a} \dot{p}_a + \cancel{\rho \frac{\partial \dot{p}_a}{\partial p_a}} \right]$$

$$= \sum_a \frac{\partial \rho}{\partial q_a} \frac{\partial H}{\partial p_a} - \frac{\partial \rho}{\partial p_a} \frac{\partial H}{\partial q_a} = [\rho, H]_{PB}$$

$$\Rightarrow \frac{\partial \rho}{\partial t} + \nabla \cdot \vec{j} = \frac{\partial \rho}{\partial t} + [\rho, H]_{PB} \Rightarrow \text{Liouville}$$