QHO Quantum Harmonic Oscillator

1. Basic
2. Number Operator
3. Fock State
4. Coherent State

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}) \qquad 1D$$

$$V(\vec{x}) = \frac{1}{2} m\omega^2 \hat{x}^2 \quad (\text{for harmonic})$$

$$\begin{cases} \hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\,\hat{x} + i\frac{1}{\sqrt{2m\omega\hbar}}\,\hat{p} \\ \hat{a}^+ = \sqrt{\frac{m\omega}{2\hbar}}\,\hat{x} - i\frac{1}{\sqrt{2m\omega\hbar}}\,\hat{p} \end{cases}$$

$$[a^+, a] = 1$$

$$\begin{cases} \hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\,(\hat{a} + \hat{a}^+) \\ \hat{p} = i\sqrt{\frac{m\hbar\omega}{2}}\,(\hat{a} - \hat{a}^+) \end{cases}$$

$$\hat{N} \overset{\text{Def}}{=\!=} a^+ a \qquad \text{Number Operater}$$

$$[\hat{N}, \hat{a}] = \cancel{a^+[a, a]} + [a^+, a]a = -\hat{a}$$

$$[\hat{N}, \hat{a}^+] = \widehat{a^+}[\hat{a}, \hat{a}^+] = \hat{a}^+$$

Let $\hat{N}|\lambda\rangle = \lambda|\lambda\rangle$

$$\hat{N}(\hat{a}|\lambda\rangle) = (\lambda-1)(\hat{a}|\lambda\rangle)$$

$$\Rightarrow \hat{a}|\lambda\rangle = C_-|\lambda-1\rangle$$

annihiliation operator

Similarily

$$\hat{a}^+|\lambda\rangle = C_+|\lambda+1\rangle$$

Creation operator

While

$$\langle\lambda|N|\lambda\rangle = \lambda$$

$$\|$$

$$\langle\lambda|a^+a|\lambda\rangle = \|a|\lambda\rangle\|^2 \geqslant 0$$

$$\Rightarrow \|C_-\|^2 = \lambda \quad \Rightarrow \quad C_- = \sqrt{\lambda}$$

Similarily

$$C_+ = \sqrt{\lambda+1}$$

Hint that λ is an non-negative $\mathbb{Z}$

$$\underline{\underline{\hat{a}|0\rangle}} \equiv 0$$

$\downarrow$

$|0\rangle$: ground state

$|\lambda\rangle \rightarrow |n\rangle$

Zero Point Energy

$$\Rightarrow \hat{H} = \hbar\omega(a^+a + \tfrac{1}{2}) \quad \text{when } |\varphi\rangle = |0\rangle$$

$$\Rightarrow \boxed{E_0 = \frac{1}{2}\hbar\omega}$$

Folk State $|n\rangle$

$|n\rangle$: ① Eigenstate of $\hat{n}$
② Folk State
③ Accupation Numbers States.

$$\langle \hat{x} \rangle = \left(\frac{\hbar}{2m\omega}\right)^{1/2} \langle n | a^{+} + a | n \rangle$$

while $\begin{cases} \langle n | a^{+} | n \rangle = \langle n | n+1 \rangle \equiv 0 \\ \langle n | a | n \rangle = \langle n | n-1 \rangle \equiv 0 \end{cases}$

$\Rightarrow \langle \hat{x} \rangle = 0$

Similarily

$\langle \hat{p} \rangle = 0$

$$\boxed{|n\rangle = \frac{\hat{a}^{+}}{\sqrt{n}} |n-1\rangle = \frac{(\hat{a}^{+})^{n}}{\sqrt{n!}} |0\rangle}$$

Futhermore

$\langle n | a^{+2} | n \rangle$

$= \langle n | a^{2} | n \rangle \equiv 0$

$\langle n | a^{+} a | n \rangle = n$

$\langle n | a a^{+} | n \rangle = n+1$

$$\Delta x^2 = \langle x^2 - \langle x \rangle^2 \rangle = \langle x^2 \rangle - \langle x \rangle^2 = \langle x^2 \rangle$$

$$= \frac{\hbar}{2m\omega} \langle n | (a^{+} + a)^2 | n \rangle = \frac{\hbar}{2m\omega} \langle n | a^2 + a^{+2} + a^{+}a + a a^{+} | n \rangle = \boxed{\frac{\hbar}{2m\omega}(2n+1)}$$

$$\Delta p^2 = \frac{m\hbar\omega}{2}(2n+1) \quad \Rightarrow \boxed{\Delta x \cdot \Delta p = \frac{\hbar}{2}(2n+1) \geq \frac{\hbar}{2}} \quad \text{MUS}$$

Wave function of Folk State

$$\varphi(x) = \langle x | n \rangle$$

$$\begin{cases} |\hat{n}\rangle = \dfrac{(\hat{a}^+)^n}{\sqrt{n!}} |0\rangle \\ \hat{a}|0\rangle \equiv 0 \end{cases}$$

$$\langle x | \hat{a} | 0 \rangle = \langle x | \left(\sqrt{\frac{m\omega}{2\hbar}} \hat{x} + i \frac{1}{\sqrt{2m\hbar\omega}} \hat{p} \right) |0\rangle$$

x

$-i\hbar\frac{\partial}{\partial x}$

$$= \left(\sqrt{\frac{m\omega}{2\hbar}} x + \sqrt{\frac{\hbar}{2m\omega}} \frac{\partial}{\partial x} \right) \varphi_0(x) = 0$$

$$\sqrt{\frac{m\omega}{2\hbar}} x\, dx = -\sqrt{\frac{\hbar}{2m\omega}} \frac{d\varphi_0(x)}{\varphi(x)}$$

$$\Rightarrow \frac{d\varphi_0(x)}{\varphi(x)} = -\frac{m\omega}{\hbar} x\, dx$$

$$\Rightarrow \ln \varphi(x) = -\frac{m\omega}{2\hbar} x^2$$

$$\sigma x = \left(\frac{\hbar}{2m\omega}\right)^{1/2}$$

$$\Rightarrow \varphi_0(x) = A e^{-\frac{m\omega}{2\hbar} x^2} = A e^{-\frac{1}{2}\left(\frac{x}{\sigma x}\right)^2}$$

Gaussian function

$$\varphi_n(x) = \langle x | \frac{(a^+)^n}{\sqrt{n!}} |0\rangle$$

$$= \frac{1}{\sqrt{n!}} \langle x | \left(\sqrt{\frac{m\omega}{2\hbar}} \hat{x} - \sqrt{\frac{1}{2\hbar m\omega}} \hat{p} \right)^n |0\rangle$$

$$= \frac{1}{\sqrt{2^n n!}} \langle x | \left(\sqrt{\frac{m\omega}{2\hbar}} \hat{x} - \sqrt{\frac{\hbar}{2m\omega}} \frac{d}{dx} \right)^n |0\rangle$$

$$= \frac{1}{\sqrt{2^n n!}} \left(\alpha x + \frac{d}{d(\alpha x)} \right)^n \varphi_0(x)$$

$$= \frac{1}{\sqrt{2^n n!}} H_n(\alpha) e^{-\frac{\alpha^2 x^2}{2}} \qquad \alpha = \sqrt{\frac{m\omega}{\hbar}}$$

Gaussian Still

Coherent State

① $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$

② $\langle\hat{N}\rangle = \langle\alpha|\hat{a}^{+}a|\alpha\rangle = |\alpha|^2$

③ $P_n(\alpha) = \frac{e^{-\lambda}\lambda^n}{n!}$, $\lambda = \langle\hat{N}\rangle$

Conside

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$$

where $C_n(\alpha) = \langle n|\alpha\rangle$

$$|\alpha\rangle = \sum_n C_n(\alpha)|n\rangle$$

$$\hat{a}|\alpha\rangle = \sum_n C_n(\alpha)\sqrt{n}\,|n-1\rangle$$

$$\alpha|\alpha\rangle = \sum \alpha C_n(\alpha)|n\rangle$$

$$\Rightarrow C_{n+1}(\alpha) = \frac{\alpha}{\sqrt{n+1}} C_n(\alpha)$$

$$\Rightarrow C_n(\alpha) = \frac{\alpha^n}{\sqrt{n!}} C_0(\alpha)$$

while $\langle\alpha|\alpha\rangle = 1 = \sum_{n,m} C_n^*(\alpha) C_m(\alpha)\langle n|m\rangle$

$$= \sum_n |C_n(\alpha)|^2 = \sum_n \frac{|\alpha|^{2n}}{n!}|C_0(\alpha)|^2$$

$$= |C_0(\alpha)|^2 \left(\sum_n \frac{|\alpha|^{2n}}{n!}\right)$$

$$\Rightarrow C_0(\alpha) = e^{-\frac{|\alpha|^2}{2}} = \langle n|\alpha\rangle$$

$$P_n(\alpha) = |\langle n|\alpha\rangle|^2$$

$$= \left(\frac{\alpha^n}{\sqrt{n!}} e^{-\frac{|\alpha|^2}{2}}\right)^2$$

$$= \boxed{\frac{\alpha^{2n}}{n!} e^{-|\alpha|^2}}$$

$$\downarrow \lambda = |\alpha|^2$$

$$= \boxed{\frac{\lambda^n}{n!} e^{-\lambda}}$$

Poissonian Distribution

$\lambda \gg 1$

$\simeq$ Gaussian

Displacement Operator

BCH : formula

$$\boxed{e^{\hat{A}+\hat{B}} = e^{-\frac{1}{2}[\hat{A},\hat{B}]} e^{\hat{A}} e^{\hat{B}}}$$

let $\hat{A} = \alpha \hat{a}^\dagger$

$B = \alpha^* \hat{a}$

$\Rightarrow$

$[A,B] = |\alpha|^2$

$\Rightarrow$

Equivelent :

$$\left\{ \begin{array}{l} a|\alpha\rangle = \alpha|\alpha\rangle \\ \Updownarrow \\ |\alpha\rangle = \hat{D}(\alpha)|0\rangle \end{array} \right.$$

[Proof] : " $\Rightarrow$ "

$\mathbb{1} = \hat{D}(\alpha)\hat{D}^\dagger(\alpha)$

$\hat{a}\mathbb{1}|\alpha\rangle = \alpha|\alpha\rangle$

$\hat{a}\hat{D}(\alpha)D^\dagger(\alpha)|\alpha\rangle = \alpha|\alpha\rangle$

$\hat{D}(\alpha)\hat{a}\hat{D}^\dagger(\alpha)|\alpha\rangle = 0$

$\Rightarrow \hat{a}\hat{D}^\dagger(\alpha)|\alpha\rangle = 0$

$\Rightarrow |\alpha\rangle = \hat{D}(\alpha)|0\rangle$

$|\alpha\rangle = \hat{D}(\alpha)|0\rangle$

$$\hat{D} = e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}}$$

$$= e^{-|\alpha|^2/2} e^{\alpha\hat{a}^\dagger} e^{-\alpha^*\hat{a}}$$

Properties of $D(\alpha)$

$\hat{D}^\dagger(\alpha) = \hat{D}^{-1}(\alpha) = \hat{D}(-\alpha)$ "Trinity"

$$\boxed{[\hat{a}, \hat{D}(\alpha)] = \alpha\hat{D}(\alpha)}$$

$\hat{a}\hat{D}(\alpha) = \alpha\hat{D}(\alpha) + \hat{D}(\alpha)\hat{a}$

" $\Leftarrow$ " $|\alpha\rangle = \hat{D}(\alpha)|0\rangle$

$\hat{a}|\alpha\rangle = \hat{a}\hat{D}(\alpha)|0\rangle$

$\Rightarrow |\alpha\rangle = \hat{D}(\alpha)|0\rangle$

How About Eigenstate of $\hat{a}^+$

Suppose $\hat{a}^+|\lambda\rangle = \lambda|\lambda\rangle$

$$|\lambda\rangle = \sum_n C_n(\lambda)|n\rangle$$

$$\hat{a}^+|\lambda\rangle = \sum_n C_n(\lambda)\sqrt{n+1}\,|n+1\rangle$$

$$\lambda|\lambda\rangle = \sum_n C_n(\lambda)\,\lambda\,|n\rangle$$

when $n=0$ $\quad C_0(\lambda)=0$

$n>0 \quad C_n(\lambda) = \frac{\sqrt{n}}{\lambda} C_{n-1}(\lambda) = \cdots = 0$

$\Rightarrow$ No such $|\lambda\rangle$!!!