Pauli Operator

$$\hat{S} = \frac{\hbar}{2}\sigma \qquad\qquad \mu_B = \mu_0 \hat{\sigma}$$

$$\begin{cases} |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} & |\uparrow\rangle \\ |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} & |\downarrow\rangle \end{cases}$$

$$\sigma_0 \triangleq \mathbb{1}_{2\times 2} = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$$

$$\sigma_1 = |0\rangle\langle 1| + |1\rangle\langle 0| = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$$

$$\sigma_2 = -i\left(|0\rangle\langle 1| - |1\rangle\langle 0|\right) = \begin{pmatrix} & -i \\ i & \end{pmatrix}$$

$$\sigma_3 = |0\rangle\langle 0| - |1\rangle\langle 1| = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

$$\begin{cases} \sigma_+ = |0\rangle\langle 1| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ \sigma_- = |1\rangle\langle 0| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{cases}$$

Other:

$$\sigma_\pm = \sigma_1 \pm i\sigma_y$$

$$\begin{cases} \sigma_1 = \sigma_+ + \sigma_- \\ \sigma_2 = -i(\sigma_+ - \sigma_-) \end{cases}$$

Properties of Pauli Operator

① $\hat{\sigma}_i \hat{\sigma}_j = \delta_{ij} \mathbb{1} + i \epsilon_{ijk} \hat{\sigma}_k$

② $[\hat{\sigma}_i, \hat{\sigma}_j] = 2i\epsilon_{ijk}\hat{\sigma}_k \qquad [\hat{S}_i, \hat{S}_j] = i\hbar \epsilon_{ijk} \hat{S}_k$

③ $\{\hat{\sigma}_i, \hat{\sigma}_j\} = 2 \mathbb{1} \delta_{ij}$

④ "Trinity" $\quad \hat{\sigma}_i^{+} = \hat{\sigma}_i = \hat{\sigma}_i^{-1} \qquad \hat{\sigma}_i^2 = \mathbb{1}$

⑤ Trace Less $\quad \boxed{Tr\, \hat{\sigma}_i \equiv 0}$

⑥ $\det \hat{\sigma}_i = -1$

⑦ Any a Hermitian Operator can be expressed

as: $H = aI + bX + cY + dZ$

$$H = a I_{2\times2} + |\vec{n}| \cdot \frac{\vec{n}\cdot\vec{\sigma}}{|\vec{n}|}$$

$$\exp\left(-\frac{iHt}{\hbar}\right) = \exp\left[-\frac{it}{\hbar}\left(a I_{2\times2} + |\vec{n}| \frac{\vec{n}\cdot\vec{\sigma}}{|\vec{n}|}\right)\right]$$

BCH: $e^{A+B} = e^{-\frac{1}{2}[A,B]} e^A e^B$

$$= \exp\left(-\frac{it}{\hbar} a I_{2\times2}\right) \exp\left[-\frac{it}{\hbar} |\vec{n}| \frac{\vec{n}\cdot\vec{\sigma}}{|\vec{n}|}\right]$$

And $I_{2\times2}$ Comute with each

$$= \exp\left[-\frac{it}{\hbar}a\right] I_{2\times2} \cdot \left(\cos\frac{t|\vec{n}|}{\hbar} I_{2\times2} - \frac{\vec{n}\cdot\vec{\sigma}}{|\vec{n}|} \sin\frac{t|\vec{n}|}{\hbar}\right)$$

Linear Expension

$$\hat{\vec{\sigma}} = \hat{\sigma}_i \vec{e}_i \qquad \vec{A} = A_i \vec{e}_i \qquad \vec{B} = B_i \vec{e}_i$$

① $(\hat{\vec{\sigma}} \cdot \vec{A})(\vec{\sigma} \cdot \vec{B}) = \vec{A} \cdot \vec{B} + i\hat{\vec{\sigma}} \cdot (\vec{A} \times \vec{B})$

[Proof]

$$LHS = (\hat{\sigma}_i A_i)(\hat{\sigma}_j B_j)$$

$$= (\hat{\sigma}_i \hat{\sigma}_j)(A_i B_j)$$

$$= (\delta_{ij} + i\epsilon_{ijk}\hat{\sigma}_k)(A_i B_j)$$

$$= \delta_{ij} A_i B_j + i\epsilon_{ijk} A_i B_j \hat{\sigma}_k$$

$$= \vec{A} \cdot \vec{B} + i\hat{\sigma}_k (\vec{A} \times \vec{B})$$

② $(\vec{\sigma} \cdot \vec{A})^2 = \vec{A}^2 \mathbb{1}$

$(\vec{\sigma} \cdot \vec{n})^2 = \mathbb{1} \quad |n| = 1 \qquad \sigma_n \stackrel{def}{=} \vec{\sigma} \cdot \vec{n}$

$\vec{n} = (\sin\theta \sin\phi, \cos\theta \sin\phi, \cos\phi)$

$\Rightarrow (\vec{\sigma} \cdot \vec{n})^{2n} = \mathbb{1} \qquad (\vec{\sigma} \cdot \vec{n})^{2n+1} = \vec{\sigma} \cdot \vec{n}$

SU(2) group

③ $\exp(i\theta\, \vec{n} \cdot \vec{\sigma}) = \mathbb{1} \cos\theta + i(\vec{n} \cdot \vec{\sigma}) \sin\theta$

④ Eig-state

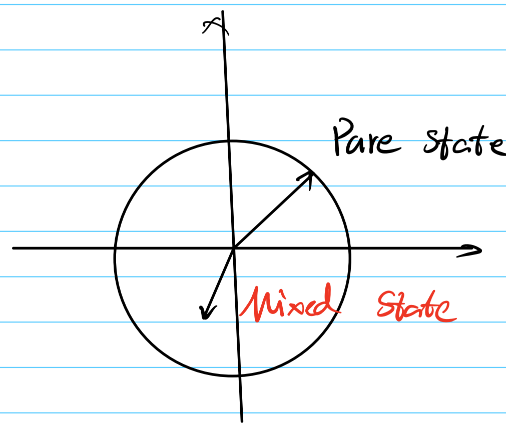
$$\cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2} e^{i\phi} |1\rangle$$

Arbitary | Matrix on 2x2 space

① Unitary
② Hermitian
③' Anti-Unitary

$$\hat{U} = a\mathbb{1} + i\vec{b}\cdot\vec{\sigma}$$

$$\hat{H} = a\mathbb{1} + \vec{b}\cdot\vec{\sigma}$$



For Two State System

$$\hat{\rho} = \frac{1}{2}(\mathbb{1} + \vec{b}\cdot\vec{\sigma})$$

Bloch Sphere

Density Matrix

$$\boxed{\hat{\rho} = \sum_i p_i |\varphi_i\rangle\langle\varphi_i|} \quad \text{(DV)}$$

$$\hat{\rho} = \int d\lambda\, f(\lambda)\, |\varphi(\lambda)\rangle\langle\varphi(\lambda)| \quad \text{(CV)}$$

Properties

① $\mathrm{Tr}\,\hat{\rho} = \mathrm{Tr}\left(\sum_i p_i |\varphi_i\rangle\langle\varphi_i|\right) = \sum_i p_i \mathrm{Tr}(|\varphi_i\rangle\langle\varphi_i|)$
$= 1$

② $\mathrm{Tr}\,\hat{\rho}^2 \leq 1$

"=" only when $\hat{\rho}$ denote a state of purity

If $Tr\,\hat{\rho}^2 = 0 \Rightarrow$ Maximum Mixed State

③ $\hat{\rho}^{\dagger} = \hat{\rho}$

④ Entropy

$$S = -k_B\, Tr(\hat{\rho} \log \hat{\rho})$$

⑤ Eigan State

$$\hat{\rho}\,|\varphi_n\rangle = p_n\,|\varphi_n\rangle$$

$$\Rightarrow S = -k_B\, Tr(\hat{\rho} \log \hat{\rho})$$

$$= -k_B \sum_n \langle \varphi_n | \hat{\rho} \log \hat{\rho} | \varphi_n \rangle$$

$$= \boxed{-k_B \sum_n p_n \log p_n}$$

⑥
$$\begin{cases} \hat{\rho} = \frac{1}{2}\left(\mathbb{1} + \frac{2}{\hbar}\langle \hat{S} \rangle \cdot \vec{\sigma}\right) \\ \vec{\rho} = \frac{1}{2}\left(\mathbb{1} + \vec{b} \cdot \vec{\sigma}\right) \end{cases}$$

$$|\varphi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix}$$

$$|\varphi\rangle\langle\varphi| = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix} \left(\cos\frac{\theta}{2} \,,\; e^{-i\phi}\sin\frac{\theta}{2} \right)$$

$$= \begin{pmatrix} \cos^2\frac{\theta}{2} & e^{-i\phi}\sin\frac{\theta}{2}\cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2}\cos\frac{\theta}{2} & \sin^2\frac{\theta}{2} \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1+\cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & 1-\cos\theta \end{pmatrix}$$

$$= \frac{1}{2}\left(\mathbb{1} + \vec{n}\cdot\vec{\sigma} \right) \triangleq \hat{\rho}$$

$$\vec{n} = (\sin\theta\cos\phi \,,\; \sin\theta\sin\phi \,,\; \cos\theta)$$

$$\langle\hat{\vec{\sigma}}\rangle = \langle\varphi|\hat{\vec{\sigma}}|\varphi\rangle \qquad \hat{\vec{\sigma}} = \hat{\sigma}_i\,\vec{e}_i \qquad i = 1,2,3$$

$$= \langle\varphi|\hat{\sigma}_i|\varphi\rangle\,\vec{e}_i$$

$$= n_i\,\vec{e}_i = \vec{n}$$

$$\langle\hat{\vec{S}}\rangle = \frac{\hbar}{2}\langle\hat{\vec{\sigma}}\rangle = \frac{\hbar}{2}\vec{n}$$

So while $\hat{\rho} = \frac{1}{2}\left(\mathbb{1} + \vec{n}\cdot\vec{\sigma} \right)$

$$= \frac{1}{2}\left(\mathbb{1} + \frac{2}{\hbar}\langle\hat{\vec{S}}\rangle\cdot\vec{\sigma} \right)$$