Inverted pendulum model

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1 Physics model of inverted pendulum

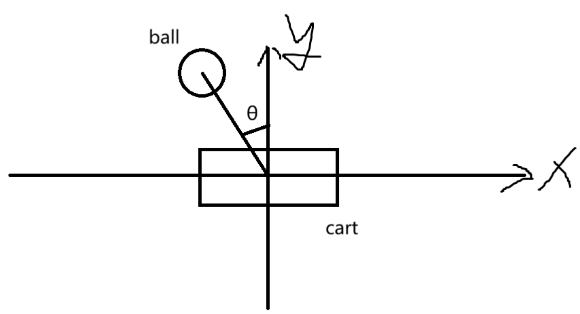

Figure 1: simplified figure

We assume that the mass of the cart is m_1,the mass of the ball is m_2,the length of pendulum id l,the angle between pendulum and y axis is θ.

2 Lagrange's Equation

The main purpose of Lagrange's Equation is to derive the equation of motion of a mechanical system by using generalized coordinates and least action principle, without directly applying Newton's Law.
T means kinetic energy, V means potential energy.
$Q_i$means force,$q_i$means different kind of generalized coordinate.

$$L = T - V$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i$$

3 Dynamic model of inverted pendulum

There are only 2 movable points.

cart position:(x_1, y_1)
ball position:(x_2, y_2)

According to the graph,we can easily calculate the ball position:

$$\begin{cases} x_2 = x_1 + l\sin\theta \\ y_2 = y_1 + l\cos\theta = l\cos\theta \end{cases}$$

so we will use x_1 and y_1 to represent ball position in the following deduction.

$$\begin{aligned} T &= \frac{1}{2}m_1\left(\dot{x}_1^2 + \dot{y}_1^2\right) + \frac{1}{2}m_2\left(\dot{x}_2^2 + \dot{y}_2^2\right) \\ &= \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\left(\dot{x}_1^2 + l^2\dot{\theta}^2 - 2l\dot{x}_1\dot{\theta}\cos\theta\right) \end{aligned} \tag{1}$$

$$V = m_2 g l\cos\theta \tag{2}$$

$$L = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\left(\dot{x}_1^2 + l^2\dot{\theta}^2 - 2l\dot{x}_1\dot{\theta}\cos\theta\right) - m_2 g l\cos\theta \tag{3}$$

As mentioned above, to calculate force in coordinate x, let $q_i = x_1$.

$$\frac{\partial L}{\partial x_1} = 0 \tag{4}$$

$$\frac{\partial L}{\partial \dot{x}_1} = (m_1 + m_2)\,\dot{x}_1 - m_2 l\dot{\theta}\cos\theta \tag{5}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = (m_1 + m_2)\,\ddot{x}_1 - m_2 l\left(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta\right) \tag{6}$$

$$\begin{aligned} F_x &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} \\ &= (m_1 + m_2)\,\ddot{x}_1 - m_2 l\left(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta\right) \end{aligned} \tag{7}$$

To portray this system in angle dimension, let $q_i = \theta$.

$$\frac{\partial L}{\partial \theta} = m_2 l \dot{x}_1 \dot{\theta} \sin\theta + m_2 g l \sin\theta \tag{8}$$

$$\frac{\partial L}{\partial \dot{\theta}} = m_2 l^2 \dot{\theta} - m_2 l \dot{x}_1 \cos\theta \tag{9}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = m_2 l^2 \ddot{\theta} - ml\left(\ddot{x}_1 \cos\theta - \dot{\theta}\dot{x}_1 \sin\theta\right) \tag{10}$$

$$F_\theta = m_2 l^2 \ddot{\theta} - ml\left(\ddot{x}_1 \cos\theta - \dot{\theta}\dot{x}_1 \sin\theta\right) - m_2 l^2 \dot{\theta} + m_2 l \dot{x}_1 \cos\theta \tag{11}$$

It's known that $F_\theta = 0$(the force F_x only acts on cart, neither pendulum nor ball),so from formula(7)and(11),we can get equation set.

$$\begin{cases} \left(m_1 + m_2\right)\ddot{x}_1 - m_2 l\left(\ddot{\theta}\cos\theta - \dot{\theta}^2 \sin\theta\right) - F_x = 0 \\ m_2 l^2 \ddot{\theta} - ml\left(\ddot{x}_1 \cos\theta - \dot{\theta}\dot{x}_1 \sin\theta\right) - m_2 l^2 \dot{\theta} + m_2 l \dot{x}_1 \cos\theta = 0 \end{cases} \tag{12}$$

After simplification results are shown as follows,we will finally use these two equations to implement the part of status updater in simulation program.

$$\begin{cases} \ddot{x}_1 = \dfrac{F_x + m_2 g \sin\theta\cos\theta - m_2 l \dot{\theta}^2 \sin\theta}{m_1 + m_2 \sin^2\theta} \\ \ddot{\theta} = \dfrac{\left(m_1 + m_2\right) g \sin\theta + F_x \cos\theta - m_2 l \dot{\theta}^2 \sin\theta\cos\theta}{l\left(m_1 + m_2 \sin^2\theta\right)} \end{cases} \tag{13}$$