14/3/22

Variable Seperable Method

Assume that the solution of the given pde is of the form

$$u(x,y) = X(x) \cdot Y(y) \quad -①$$

, neither of the factors $X(x)$ and $Y(y)$ being identically zero.

Solve these $X(x)$ and $Y(y)$ to find the Solution of the given pde.

Ex

1. Solve $3\frac{\partial u}{\partial x} - 4\frac{\partial u}{\partial t} = 0$. — ①, with $u(0,t) = 5e^{-t}$.

Soln :-

$$u(x,t) = X(x) \cdot T(t) \quad — ②$$

$$\frac{\partial u}{\partial x} = X'(x) \cdot T(t)$$

$$\frac{\partial u}{\partial t} = X(x) \cdot T'(t)$$

$$\left\{ \begin{array}{l} \text{Let } X(x) = X \\ \therefore T(t) = T \end{array} \right.$$

Substitute these $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial t}$ in ①

$$3X'T - 4XT' = 0$$

$$3X'T = 4XT'$$

$$\frac{1}{4}\frac{X'}{X} = \frac{1}{3}\frac{T'}{T} = K \text{ [constant]}$$

$$\Rightarrow \frac{X'}{4X} = K \quad ; \quad \frac{T'}{3T} = K$$

$$\Rightarrow X' - 4KX = 0 \quad \text{——} \quad (3)$$

$$T' - 3KT = 0 \quad \text{—} \quad (4)$$

(3) and (4) are 1st order ODE.

Now $X' - 4KX = 0$

<u>A.E</u> $m - 4k = 0$

$m = 4K$

$\therefore X(x) = A e^{4Kx}$

(4) $\Rightarrow T' - 3KT = 0$

A.E

$m - 3K = 0$

$m = 3K$

$\therefore T(t) = B e^{3Kt}$

Now, (2) $\Rightarrow u(x,t) = A e^{4Kx} \cdot B e^{3Kt}$

$u(x,t) = C e^{K[4x+3t]}$ — (3)

2) Solve: $2\frac{\partial u}{\partial x} + t\frac{\partial u}{\partial t} = 0$, $u(x,1) = \frac{1}{3}e^{-x}$ — (1)

<u>Soln</u>:

Let $u(x,t) = X(x) \cdot T(t)$ — (2)

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= X' T \\ \frac{\partial u}{\partial t} &= X T' \end{aligned}\right\} \quad \text{— (3)}$$

2 in 1

$$2X'T + tXT' = 0$$

$$X'T = -\frac{t}{2}XT'$$

$$\frac{X'}{X} = -\frac{tT'}{2T} = K$$

$$\therefore \frac{X'}{X} = K \Rightarrow \underline{\underline{X' - KX = 0}} \quad - \quad 3$$

$$\frac{-tT'}{2T} = k$$

$$-tT' - 2kT = 0$$

$$tT' + 2kT = 0 \qquad (4)$$

$$(3) \Rightarrow \quad X' - kX = 0$$

A.E $\quad m - k = 0$

$$m = k$$

$$\therefore X(x) = Ae^{Kx} \quad \text{—} \quad (5)$$

$$(4) \Rightarrow$$

$$tT' + 2KT = 0 \qquad \left\{ \begin{array}{l} x\dfrac{dy}{dx} + 2Kxy = 0 \\[2ex] x\dfrac{d}{dx} = \alpha \end{array} \right.$$

$$\alpha + 2K = 0$$

$$\alpha = -2K$$

$$\left\{ \begin{array}{l} \alpha = \dfrac{d}{ds} \\[2ex] t = e^{s} \\ s = \log t \end{array} \right.$$

$$cf = Be^{-2Ks}$$

$$= Be^{-2K\log t}$$

$$= Be^{\log t^{-2K}}$$

$T(t) = B\left(t^{2\kappa}\right)^{-1}$ (or) $\dfrac{B}{t^{2k}}$

$\therefore$ ① $\Rightarrow$

$u(x,t) = Ae^{kx} \cdot \dfrac{B}{t^{2k}}$ is the ——— ②

soln of pde (1).

put $t=1$ in (b)

$$\frac{1}{3}e^{-x} = c\,e^{kx}$$

comparing,

$$c = \frac{1}{3}, \quad k = -1.$$

$$\therefore\; u(x,t) = \frac{1}{3}e^{-x}\cdot\frac{1}{t^{-2}} = \frac{1}{3}t^{2}e^{-x}$$

Given : $u(0,t) = 5e^{-t}$

(5) $\Rightarrow$

$$5e^{-t} = c\, e^{3kt}$$

comparing both sides

$c = 5$, $3k = -1$

$$\boxed{k = -\frac{1}{3}}$$

$\therefore$ (5) $\Rightarrow$ $u(x,t) = 5e^{-\frac{1}{3}[4x+3t]}$ //

3 Solve :

$$\frac{\partial u}{\partial x} - \frac{\partial u}{\partial t} = 3u. \quad \text{--- (1)}$$

Soln

$$u(x,t) = X(x) \cdot T(t) \quad \text{--- (2)}$$

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= X' T \\ \frac{\partial u}{\partial t} &= X T' \end{aligned}\right\} \quad \text{--- (3)}$$

Put 2 in 1

$$X'T = XT' + 3XT$$

$$X'T - XT' - 3XT = 0$$

$\div\ 3XT$

$$\frac{X'\cancel{T}}{3X\cancel{T}} - \frac{\cancel{X}T'}{3\cancel{X}T} - 1 = 0$$

$$\frac{X'}{3X} = 1 + \frac{T'}{3T} = K$$

$$\frac{X'}{3X} = K$$

$\therefore\ X' - 3KX = 0$ — 4 ' is an ODE.

Now, $1 + \frac{T'}{3T} = K$

$$\frac{3T + T'}{3T} = K \implies T' - 3TK + 3T = 0$$

$$\implies T' + 3T[1 - K] = 0 \quad \text{— } 5$$

Solve (4)

$$X' - 3KX = 0$$

<u>A.E</u>

$$m - 3K = 0$$

$$m = 3K$$

$$\therefore X(x) = A e^{3Kx}$$

Solve (3)

$$T' + 3T(1-K) = 0$$

A.E

$$m + 3(1-k) = 0$$

$$m = -3(1-k)$$

$$T(t) = Be^{-3(1-k)t}$$

$$\therefore \quad u(x,t) = Ae^{3kx} \cdot Be^{-3(1-k)t}$$

$$= Ce^{3[kx-(1-k)t]}, \quad \text{where } C = AB$$

4) Solve: $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial t} = 2u$. **HW**

Sol^n:

$$u(x,t) = c\, e^{kx + 2(k-1)t}$$

$$X'T + XT' = 2XT$$

$$\frac{X'}{X} + \frac{T'}{T} = 2$$

$$\frac{X'}{X} = 2 - \frac{T'}{T} = k$$

$$2 - \frac{T'}{T} = k$$

$$\frac{T'}{T} = 2 - k$$

$$= -2(k-1)$$