Module - 2

Partial Differential Equations

$$u(x,t) = f(x) \cdot g(t)$$

$$\frac{\partial u}{\partial x} = f'(x)\, g(t)$$

$$\frac{\partial u}{\partial t} = f(x) \cdot g'(t)$$

$$\frac{\partial^2 u}{\partial x \partial t} = \frac{\partial^2 u}{\partial t \partial x} \;//$$

Defn

Let u be a function of independent variables $x_1, x_2, \ldots$. Then a relation of the form:

$$F\left[\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial^2 u}{\partial x_1^2}, \ldots, \frac{\partial^2 u}{\partial x_1 \partial x_2}, \ldots\right] = 0 \quad \text{— (1)}$$

which involves the partial derivatives of u is called partial differential equation and the order of the highest partial derivatives in the equation, its order.

A pde is said to be linear if the degree of each partial derivative is one otherwise it is a non-linear pde

Ex:

$$\left(\frac{\partial^2 u}{\partial x^2}\right)^1 + \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x \partial t} = f(x,t)$$

— linear

$$\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + \frac{\partial^2 u}{\partial t^2} + \left(\frac{\partial u}{\partial t}\right)^3 = e^{2x+3t}$$

— non-linear pde

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z \quad - \text{ linear pde}$$

Note

$$\left\{ \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial z}{\partial y}\right] \right.$$

Notation

If $z = f(x, y)$ then

$$\frac{\partial z}{\partial x} = p, \quad \frac{\partial z}{\partial y} = q, \quad \frac{\partial^2 z}{\partial x^2} = \frac{\partial p}{\partial x} = r$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial q}{\partial y} = t, \quad \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} = s$$

Formation of PDE - Eliminating arbitrary constants

Ex

1. $z = ax + by + a^2 + b^2$. Eliminate 'a' & 'b' — (1)

$$\frac{\partial z}{\partial x} = a$$

$$\Rightarrow \boxed{p = a}$$

$$\frac{\partial z}{\partial y} = b \Rightarrow \boxed{q = b}$$

$\therefore$ (1)

$z = px + qy + p^2 + q^2$ is a reqd pde

2) Eliminate a, b, c from

$z = a(x+y) + b(x-y) + abt + c$

$\frac{\partial z}{\partial x} = (a+b)$; $\frac{\partial z}{\partial y} = a - b$

$\boxed{q = a - b}$

$\Rightarrow p = a + b$

$$\frac{\partial z}{\partial t} = \underline{ab}$$

But, $(a+b)^2 - (a-b)^2 = 4ab$

$$\Rightarrow \quad p^2 - q^2 = 4\frac{\partial z}{\partial t} \quad //$$

3) $z = ax + a^2y^2 + b$

$$p = \frac{\partial z}{\partial x} = \underline{\underline{a}} \quad - (1) \quad ; \quad q = \frac{\partial z}{\partial y} = 2a^2y \quad - (2)$$

In ②

$$q = 2a^2 y$$

put $a = p$

$$\Rightarrow \underline{\underline{q = 2p^2 y}} \quad //$$

4) $z = (x^2+a)(y^2+b)$

$p = \frac{\partial z}{\partial x} = 2x(y^2+b) \Rightarrow \frac{p}{2x} = (y^2+b) \quad — ①$

$$q = \frac{\partial z}{\partial y} = (x^2 + a)\, 2y$$

$$\Rightarrow \frac{q}{2y} = (x^2 + a) \quad \text{--- } (2)$$

$$\therefore\ z = \frac{q}{2y} \cdot \frac{p}{2x}$$

$$\underline{\underline{4xyz = qp}}$$ is a required pde

5. $z = ax^2 + bxy + cy^2$

$$p = \frac{\partial z}{\partial x} = 2ax + by \qquad (1)$$

$$r = \frac{\partial z}{\partial y} = bx + 2cy \qquad (2)$$

$(1) \times x + (2) \times y$

$$px + ry = x(2ax + by) + y(bx + 2cy)$$

$$= 2ax^2 + bxy + bxy + 2cy^2$$

$$= 2[ax^2 + bxy + cy^2]$$

$\therefore\ \underline{\underline{px + qy = 2z}}$ is a pde

6. $z = a e^{bx} \sin by \quad \text{---} \ (1)$

$$p = \frac{\partial z}{\partial x} = ab e^{bx} \sin by$$

$$q = \frac{\partial z}{\partial y} = ab e^{bx} \cos by$$

$$\frac{\partial^2 z}{\partial x^2} = ab[b e^{bx} \sin by]$$

$$= b^2 [a e^{bx} \sin by)$$

$$= b^2 z \quad \text{---} \ (2)$$

$$\frac{\partial^2 z}{\partial y^2} = -ab^2 e^{bx} \sin by$$

$$= -b^2 [a e^{bx} \sin by]$$

$$= -b^2 z \quad \text{--- } (3)$$

(2) + (3)

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = b^2 z - b^2 z = 0$$

$\therefore$ $\underline{\underline{r + t = 0}}$ is a reqd pde

7. Find the partial differential equation of all planes which are at constant distance K from origin.

Sol

If $ax+by+cz+d=0$ is any plane which is at a constant distance K' unit from origin, we find

$$\frac{d}{\sqrt{a^2+b^2+c^2}} = K \quad \text{Con}$$

$$d = k\sqrt{a^2+b^2+c^2}$$

Hence

$$ax+by+cz+k\sqrt{a^2+b^2+c^2} = 0 \quad \text{--- (1)}$$

Diff (1) partially with respect to x and y

$$a + c\frac{\partial z}{\partial x} = 0 \quad \Rightarrow \quad a + cp = 0$$

$$\Rightarrow \underline{\underline{a = -cp}}$$

$$b + c\frac{\partial z}{\partial y} = 0 \quad \Rightarrow \quad b + cq = 0$$

$$\underline{\underline{b = -cq}}$$

$\therefore$ ①

$$\Rightarrow -cpx - cqy + cz + K\sqrt{(-cp)^2 + (-cq)^2 + c^2} = 0$$

$$-c\left[px + qy - z - K\cdot\sqrt{p^2 + q^2 + 1}\right] = 0$$

$$\Rightarrow px + qy - K\sqrt{p^2 + q^2 + 1} = z$$

is a reqd pde