Type II: If the equation involves only p and q then

$$f_x = 0, \quad f_y = 0, \quad f_z = 0$$

$$\therefore \quad \frac{dp}{0} = \frac{dq}{0} = c$$

$\Rightarrow dp = 0$

$$\boxed{p = a}$$

$dq = 0$

$$\boxed{q = b}$$

Type 2 CLAIRAUT'S FORM

$$z = px + qy + f(p,q) \quad \text{--- (1)}$$

Procedure

1. Put $p = a$, $q = b$ in (1)

$$z = ax + by + f(a,b) \quad \text{--- (2)}$$ is called

Complete Solution.

Differentiating (2) partially with respect to 'a' and 'b'.

$$0 = x + \frac{\partial f}{\partial a} \quad \text{--- (3)}$$

$$0 = y + \frac{\partial f}{\partial b} \quad \text{--- (4)}$$

Solve (3) & (4) for 'a' and 'b' to get the Singular Integral

Ex

1. Find the singular integral of the pde

$$z = px + qy + p^2 - q^2 \quad \text{— (1)}$$

Soln :-

put $p = a$ and $q = b$ in (1)

$$z = ax + by + a^2 - b^2 \quad \text{— (2)}$$ is a

Complete Integral

Differentiating 2 partially with respect to 'a' and 'b'.

$$0 = x + 2a$$

$$\Rightarrow 2a = -x$$

$$a = -x/2 \quad — \quad 3$$

Now,

$$0 = y - 2b$$

$$\Rightarrow b = y/2 \quad — \quad 4$$

(2) $\Rightarrow$

$$z = \left[\frac{-x}{2}\right] x + \left(\frac{y}{2}\right) y + \left(-x/2\right)^2 - \left(y/2\right)^2$$

$$= \frac{-x^2}{2} + \frac{y^2}{2} + \frac{x^2}{4} - \frac{y^2}{4}$$

$$= -\frac{x^2}{4} + \frac{y^2}{4}$$

$\Rightarrow \underline{\underline{4z = y^2 - x^2}}$ is - S.I

2. Solve : $z = px + qy + \sqrt{1+p^2+q^2}$ — ①

Soln

Put $p = a$, $q = b$ in ①

We get,

$z = ax + by + \sqrt{1+a^2+b^2}$ — ② is a

complete Integral of ①.

Now, Differentiate ② with respect to 'a' and 'b'.

$0 = x + \frac{1}{2}(1+a^2+b^2)^{-\frac{1}{2}}[2a]$

$$x = \frac{-a}{\sqrt{1+a^2+b^2}} \quad — \quad 3$$

$$0 = y + \frac{r}{2}(1+a^2+b^2)^{-1/2}(2b)$$

$$\Rightarrow \quad y = \frac{-b}{\sqrt{1+a^2+b^2}} \quad — \quad 4$$

$3^2 + 4^2$

$$x^2 + y^2 = \frac{a^2}{(1+a^2+b^2)} + \frac{b^2}{(1+a^2+b^2)}$$

$$x^2 + y^2 = \frac{(a^2+b^2)}{(1+a^2+b^2)}$$

$$1 - (x^2+y^2) = 1 - \frac{(a^2+b^2)}{(1+a^2+b^2)}$$

$$1 - x^2 - y^2 = \frac{(1+a^2+b^2) - (a^2+b^2)}{(1+a^2+b^2)}$$

$$1 - x^2 - y^2 = \frac{1}{(1+a^2+b^2)}$$

and

$$\Rightarrow (1+a^2+b^2) = \frac{1}{1-x^2-y^2}$$

$$\sqrt{1+a^2+b^2} = \frac{1}{\sqrt{1-x^2-y^2}}$$

(3) $\Rightarrow$

$$x\sqrt{1+a^2+b^2} = -a$$

$$\frac{x}{\sqrt{1-x^2-y^2}} = -a$$

$$\Leftrightarrow \quad a = \frac{-x}{\sqrt{1-x^2-y^2}}$$

Similarly

$$b = \frac{-y}{\sqrt{1-x^2-y^2}}$$

now, $c \cdot r \Rightarrow$

$$z = \frac{-x}{\sqrt{1-x^2-y^2}} \cdot x - \frac{y \cdot y}{\sqrt{1-x^2-y^2}} + \frac{1}{\sqrt{1-x^2-y^2}}$$

$$z = \frac{-x^2 - y^2 + 1}{\sqrt{1 - x^2 - y^2}}$$

$$z = \sqrt{1 - x^2 - y^2}$$

$$z^2 = 1 - x^2 - y^2 \quad (\Rightarrow$$

$$\underline{x^2 + y^2 + z^2 = 1} \quad \text{is a} \quad S^2 .$$

3. Solve : $z = px + qy + \frac{p}{q} - p$

Soln :-

put $p = a$ and $q = b$,

$$z = ax + by + \frac{a}{b} - a \quad \text{—} \; (1) \text{ is}$$

the complete Integral

Now, Differentiating (1) with respect to 'a' and 'b'.

$0 = x + \frac{1}{b} - 1$

$\Rightarrow \frac{1}{b} = 1 - x$

$b = 1/_{1-x}$

Now,

$0 = y - a/b^2$

$yb^2 = a \quad \Rightarrow \quad a = y/(1-x)^2$

a) $\Rightarrow$

$$z = \frac{xy}{(1-x)^2} + \frac{y}{(1-x)} + \frac{y(1-x)^2}{1/(1-x)} - \frac{y}{(1-x)^2}$$

$$= \frac{xy}{(1-x)^2} + \frac{y}{(1-x)} + \frac{y}{(1-x)} - \frac{y}{(1-x)^2}$$

$$z = \frac{xy + 2y(1-x) - y}{(1-x)^2}$$

$$z = \frac{xy + 2y - 2xy - y}{(1-x)^2}$$

$$= \frac{y - xy}{(1-x)^2}$$

$$z = \frac{y[1-x]}{(1-x)^2}$$

$$\Rightarrow z(1-x) = y$$

is a S.I //

Type 3

(i) $F(x, p, q) = 0$, put $q = a$

(ii) $F(y, p, q) = 0$, put $p = a$

(iii) $F(z, p, q) = 0$, put $q = ap$

Ex

1. Solve : $p = 2qx$ — ①

$f(x, p, q) = 0$

So, $q = a$

Put $q = a$ in ①

$p = 2ax$ — ②

Now, the Exact Differential equation,

$dz = p\,dx + q\,dy$

$dz = 2ax\,dx + a\,dy$

$$z = 2a\left[\frac{x^2}{2}\right] + ay + c$$

$\Rightarrow z = ax^2 + ay + c$ — is a complete solution

2. $q = px + p^2$

Put $q = a$

$p^2 + px = a$ — (1) is a quadratic in p

$$p^2 + px - a = 0$$

$$p = \frac{-x \pm \sqrt{x^2 + 4a}}{2}$$

now,

$$dz = p\,dx + q\,dy$$

$$dz = \left[\frac{-x}{2} \pm \frac{\sqrt{x^2 + 4a}}{2}\right] dx + a\,dy$$

$$\int dz = -\frac{1}{2}\int x\,dx \pm \frac{1}{2}\int \sqrt{x^2+(2\sqrt{a})^2}\,dx + a\int dy + c$$

$$z = -\frac{1}{2}\left[\frac{x^2}{2}\right] \pm \frac{1}{2}\left\{\frac{4a}{2}\sinh^{-1}\left(\frac{x}{2\sqrt{a}}\right) + \frac{x}{2}\sqrt{x^2+4a}\right\}$$

$$+ ay + c$$

$$\because \int \sqrt{x^2+a^2}\,dx = \frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right) + \frac{x}{2}\sqrt{x^2+a^2} + c$$