9/3/22 Linear Equation of first order

A linear pde of the 1st order known as Lagrange's linear equation of the form

$$Pp + Qq = R \quad \text{—— (1)}$$

Where P, Q and R are of functions of x, y, z. This equation is called as quasi-linear linear equation. When P, Q R are independent of z, it is known as linear equation.

To solve (1) we have to form the subsidiary equation

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}.$$ Now

1. Method of Grouping :-

Solve these simultaneous equations by equating $\frac{dx}{P} = \frac{dy}{Q}$ and eliminating the z factor to form an exact differential eqn.

on integrating,

$$u(x, y, z) = a \quad - \quad ②$$

Similarly, equate $\frac{dy}{Q} = \frac{dz}{R}$ and Remove all the x-factor to form an exact differential equation. on integrating we get

$$V(x, y, z) = b \quad - \quad ③$$

This is called Grouping method and finally write the general solution as $\phi(u, v) = 0$ (or $u = f(v)$).

Ex Solve: $px + qy = z$

Soln: Here $P = x$, $Q = y$, $R = z$

A.E

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z} \quad \text{--- (1)}$$

$$\frac{dx}{x} = \frac{dy}{y}$$

Integrate, $\int \frac{dx}{x} = \int \frac{dy}{y}$

$\log x = \log y + \log a$

$$\boxed{x/y = a}$$

$\{$ form : $u(x, y, z) = a$

Now $\frac{dy}{y} = \frac{dz}{z}$

$\log y = \log z + \log b$

$\Rightarrow \boxed{y/z = b}$

$\{$ form : $v(x, y, z) = b$

$\therefore$ The General is

$$\phi\left(x/y, \frac{y}{z}\right) = 0$$

Con

$$\phi\left(x/y, x/z\right) = 0$$

2) $p\sqrt{x} + q\sqrt{y} = \sqrt{z}$

Sol :-

$P = \sqrt{x},\ Q = \sqrt{y},\ R = \sqrt{z}$

A.E

$$\frac{dx}{\sqrt{x}} = \frac{dy}{\sqrt{y}} = \frac{dz}{\sqrt{z}}$$

Take

$$\frac{dx}{\sqrt{x}} = \frac{dy}{\sqrt{y}}$$

on integrating,

$$2\sqrt{x} = 2\sqrt{y} + c$$

$$\sqrt{x} - \sqrt{y} = c_1 \quad , \text{ where } c_1 = \frac{c}{2}$$

now

$$\frac{dx}{\sqrt{x}} = \frac{dz}{\sqrt{z}}$$

Integrating

$$2\sqrt{x} = 2\sqrt{z} + d \Rightarrow \sqrt{x} - \sqrt{z} = d_1$$

$\therefore$ The General Solution is

$$\phi(\sqrt{x}-\sqrt{y}\,,\ \sqrt{x}-\sqrt{z})=0$$

3) $y^2zp+x^2zq=y^2x$,

Soln: $P=y^2z$, $Q=x^2z$, $R=y^2x$

A.E

$$\frac{dx}{y^2z}=\frac{dy}{x^2z}=\frac{dz}{y^2x}$$

$$\frac{dx}{y^2 \cancel{z}} = \frac{dy}{x^2 \cancel{z}}$$

$x^2\,dx = y^2\,dy$, an exact D.E.

on integrating,

$$\int x^2\,dx = \int y^2\,dy$$

$$\frac{x^3}{3} = \frac{y^3}{3} + a$$

(or) $x^3 - y^3 = a_1$, where $\underline{\underline{a_1 = 3a}}$

Now

$$\frac{dx}{\cancel{x}z} = \frac{dz}{\cancel{x}x}$$

$$x\,dx = z\,dz$$

Integrating,

$$\frac{x^2}{2} = \frac{z^2}{2} + b \quad \text{Con}$$

$x^2 - z^2 = b_1$, where $\underline{\underline{b_1 = 2b}}$

$\therefore$ The G.S is $\phi(x^3-y^3,\ x^2-z^2)=0$

4. Solve : $\underbrace{\frac{y^2z}{x}p + xzq = y^2}$

Soln :- Here $P=\frac{y^2z}{x}$, $Q=xz$, $R=y^2$

Rewriting the equation,

$$y^2zp + x^2zq = y^2x$$

Now, $P=y^2z$, $Q=x^2z$, $R=y^2x$

AE

$$\frac{dx}{y^2 z} = \frac{dy}{x^2 z} = \frac{dz}{y^2 x}$$

$$\frac{dx}{\cancel{y^2} z} = \frac{dz}{\cancel{y^2} x}$$

$$x\,dx = z\,dz$$

$$x^2 - z^2 = c$$

$$\frac{dx}{y^2 \cancel{z}} = \frac{dy}{x^2 \cancel{z}}$$

$$x^2\,dx = y^2\,dy$$

$$x^3 - y^3 = d$$

$\therefore$ The G.S is $\phi(x^2 - z^2,\ x^3 - y^3) = 0.$

5. Solve $p - q = \log(x+y)$

Soln:- $P = 1, \quad Q = -1, \quad R = \log(x+y)$

A.E

$$\frac{dx}{1} = \frac{dy}{-1} = \frac{dz}{\log(x+y)}$$

Now,

$dx = -dy$

Integrating,

$x = -y + c \Rightarrow \underline{\underline{x + y = c}}$

$$-\frac{dy}{1} = \frac{dz}{\log(x+y)}$$

But, we know that $x+y = c$ from the previous solution

$$-dy = \frac{dz}{\log c}$$

$$-\log c \, dy = dz$$

Integrating,

$$-y\log c = z + d$$

$$\underline{\underline{y\log c + z = d_1}} \quad \Rightarrow y\log(x+y) + z = d_1$$

$$\therefore \quad \phi\left(x+y,\ y\log(x+y)+z\right) = 0$$

1. $xp - yq = y^2 - x^2$

Sol:-

$P = x, \quad Q = -y, \quad R = y^2 - x^2$

AE

$$\frac{dx}{x} = \frac{dy}{-y} = \frac{dz}{y^2 - x^2}$$

Now

$$\frac{dx}{x} = \frac{dy}{-y}$$

$$\log x = -\log y + \log c$$

$$\log x + \log y = \log c$$

$$\Rightarrow \boxed{xy = c}$$

Now

$$\frac{dy - dx}{-y - x} = \frac{dz}{y^2 - x^2}$$

$$\frac{dy - dx}{-\cancel{(y+x)}} = \frac{dz}{\cancel{(y+x)}(y-x)}$$

$$-d[y-x] = \frac{dz}{(y-x)}$$

$$d[y-x] = dy - dx$$

$$-(y-x)\,d(y-x) = dz$$

Integrate

$$-\frac{(y-x)^2}{2} = z + b$$

or $(y-x)^2 + z = b_1$

$\therefore \quad \Phi\left(x+h(J-x)^2+z\right)=0 \quad //$