21/3/22

Ex 1.

Find the pde of all spheres whose centres lie on the z-axis.

Soln:

The equation of a sphere with centre lying on z axis is

$$x^2 + y^2 + (z-c)^2 = r^2 \quad \text{--- } (1)$$

Differentiating (1) partially with respect to x and y,

$$2x + 2(z-c)\cdot\frac{\partial z}{\partial x} = 0$$

$\div 2$

$$x + (z-c)p = 0$$

$$(z-c) = -\frac{x}{p} \quad \text{--- (2)}$$

Now,

$$2y + 2(z-c)\,\partial z/\partial y = 0$$

$\div 2$ $\quad y + (z-c)q = 0$

$$(z-c) = \frac{-y}{q} \quad - (5)$$

$\therefore$ L.H.S. of (4) & (5) are

same

$$\Rightarrow \frac{-x}{p} = \frac{-y}{q}$$

$\Rightarrow xq - yp = 0$ is a reqd. p

2) Form a pde of the family of spheres of constant Radius k and centres lying on the line $y=x$ in the xy plane

Soln :-

$$(x-a)^2+(y-a)^2+z^2=k^2 \quad -(1)$$

$$2(x-a)+2z\frac{\partial z}{\partial x}=0$$

$$(x-a)=-zp$$

Now, $2(y-a) + 2z\frac{\partial z}{\partial y} = 0$

$\Rightarrow (y-a) = -zq$

put $(x-a)$ and $(y-b)$ in ①

$\therefore$ ① $\Rightarrow$

$z^2p^2 + z^2q^2 + z^2 = k^2$

$z^2(p^2+q^2+1) = k^2$ is the reqd

pde.

Formation of pde - by eliminating the arbitrary functions.

Consider the relation:

$\phi(u, v) = 0$, where u, v

$- (1)$

are functions of independent variables x and y and dependent variable z.

Diff (1) partially with respect to x and y, we get.

$$\frac{\partial \phi}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial v}\cdot\frac{\partial v}{\partial x} = 0 \qquad — (2)$$

$$\frac{\partial \phi}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial \phi}{\partial v}\cdot\frac{\partial v}{\partial y} = 0 \qquad — (2)$$

Then eliminating $\frac{\partial \phi}{\partial u}$ and $\frac{\partial \phi}{\partial v}$ from these relations gives the reqd. pde in the form of

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = 0 .$$

Ex

1. Eliminate the arbitrary function 'ϕ' from the relation $\phi(x^2+y^2+z^2, xyz) = 0$

Soln

Given : $\phi(x^2+y^2+z^2, xyz) = 0$

Here $u = x^2+y^2+z^2$, $v = xyz$

$\frac{\partial u}{\partial x} = 2x + 2zp$; $\frac{\partial v}{\partial x} = y[xp + z]$

$\frac{\partial u}{\partial y} = 2y + 2zq$ $\frac{\partial v}{\partial y} = x[yq + z]$

$$\therefore \quad \begin{vmatrix} 2x+2pz & y(xp+z) \\ 2y+2qz & x(yq+z) \end{vmatrix} = 0$$

$$\Rightarrow \quad 2(x+pz)\cdot x(yq+z) - 2(y+qz)\cdot y(xp+z) = 0$$

$$x(x+pz)(yq+z) - y(y+qz)(xp+z) = 0$$

$$(x^2+pxz)(yq+z) - [y^2+qyz](xp+z) = 0$$

$(x^2yq + x^2z + \cancel{xyz}pq + px z^2) - (xpy^2 + zy^2 + \cancel{xyz}pq + qyz^2) = 0$

$$px[z^2 - y^2] + qy[x^2 - z^2] + (x^2 - y^2)z = 0$$

is a reqd. pde [linear]

Ex 2. $f(x^2+y^2,\ z-xy) = 0$

Soln: $u = x^2+y^2$, $v = z - xy$

$\frac{\partial u}{\partial x} = 2x$, $\frac{\partial v}{\partial x} = \frac{\partial z}{\partial x} - y = p - y$

$\partial u/\partial y = 2y$ $\frac{\partial v}{\partial y} = \frac{\partial z}{\partial y} - x = q - x$

$$\begin{vmatrix} 2x & p-y \\ 2y & q-x \end{vmatrix} = 0$$

$$2x(q-x) - 2y(p-y) = 0$$

$$2qx - 2x^2 - 2py + 2y^2 = 0$$

$\div 2$ $$qx - x^2 - py + y^2 = 0$$

or $py - qx = y^2 - x^2$ is the reqd pde.

3. $g\left(y/x,\ x^2+y^2+z^2\right)=0.$

Here $u = y/x$, $v = x^2+y^2+z^2$

$$\frac{\partial u}{\partial x} = \frac{-y}{x^2}, \quad \frac{\partial v}{\partial x} = 2x+2zp$$

$$\frac{\partial u}{\partial y} = \frac{1}{x} \qquad \frac{\partial v}{\partial y} = 2y+2zq$$

Now $\begin{vmatrix} \frac{-y}{x^2} & 2x+2zp \\ \frac{1}{x} & 2y+2zq \end{vmatrix} = 0$

$$\Rightarrow -\frac{2y}{x^2}(y+zq) - \frac{2}{x}(x+zp) = 0$$

$$-y(y+zq) - x(x+zp) = 0$$

$$y^2 + yzq + x^2 + xzp = 0$$

$$z[px+qy] + (x^2+y^2) = 0$$ is a

reqd. pde.

4) Eliminate 'f' from the equation

$$z = f(x^2+y^2) \quad \text{— } (1)$$

$$\frac{\partial z}{\partial x} = f'(x^2+y^2) \cdot 2x$$

$$\Rightarrow \quad \frac{p}{2x} = f'(x^2+y^2) \quad \text{— } (2)$$

Now, $$\frac{\partial z}{\partial y} = f'(x^2+y^2)(2y)$$

$$\frac{q}{2y} = f'(x^2+y^2) \quad \text{— } (3)$$

$\because$ R.H.S. are (2) & (3) are equal

$$\Rightarrow \frac{p}{2x} = \frac{q}{2y}$$

$$\Rightarrow py - qx = 0$$ is a reqd. pde.

5. Eliminate 'g' from $z = (x+y)\, g(x^2 - y^2)$ — (I)

$$\frac{z}{x+y} = g(x^2 - y^2) \quad \text{— (II)}$$

$$\frac{(x+y)\frac{\partial z}{\partial x} - z}{(x+y)^2} = g'(x^2-y^2)(2x)$$

$$d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}$$

$$\frac{(x+y)p - z}{2x(x+y)^2} = g'(x^2-y^2) \quad \text{--- } (3)$$

Now

$$\frac{(x+y)q - z}{(x+y)^2} = g'(x^2-y^2)(-2y)$$

$$\Rightarrow \frac{(x+y)q - z}{-2y(x+y)^2} = g'(x^2-y^2) \quad \text{--- } (4)$$

$$③ = ④$$

$$\frac{(x+y)p - z}{\cancel{2}x\cancel{(x+y)}^2} = \frac{(x+y)q - z}{-\cancel{2}y\cancel{(x+y)}^2}$$

$$-y((x+y)p - z] = x[(x+y)q - z]$$

$$-xyp - py^2 + zy = qx^2 + qyx - zx$$

$$py^2 + qx^2 + xyp + xyq - zx - zy = 0 \;//$$

6)
$$z = f\left(\frac{xy}{z}\right) \quad \text{--- (1)}$$

$$\frac{\partial z}{\partial x} = f'\left(\frac{xy}{z}\right)\left[y\left[\frac{z - xp}{z^2}\right]\right]$$

$$\Rightarrow p = f'(xy/z)\left[\frac{zy - xyp}{z^2}\right]$$

$$\Rightarrow \frac{pz^2}{zy - xyp} = f'(xy/z) \quad \text{--- (2)}$$

$$\frac{\partial z}{\partial y} = f'\left(\frac{xy}{z}\right)\left[x\left[\frac{z - yq}{z^2}\right]\right]$$

$$\frac{qz^2}{xz - xyq} = f'\left(\frac{xy}{z}\right) \quad \text{---} \quad 3$$

$$2 = 3$$

$$\frac{p\cancel{z^2}}{zy - xyp} = \frac{q\cancel{z^2}}{xz - xyq}$$

$$\Rightarrow \quad p(xz - xyq) = q(zy - xyp)$$

$$pxz - \cancel{pq}xy = qzy - \cancel{pqxy}$$

$$px\cancel{z} = q\cancel{z}y$$

$$\Rightarrow \quad px - qy = 0$$ is a pde //

—— x ——