11/3/22

Method of Multipliers

A.B

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Now choose (l, m, n) where l, m, n are functions of (x, y, z) or constant

$$\frac{dx}{P} = \frac{l\,dx + m\,dy + n\,dz}{Pl + Qm + Rn}$$

Here $Pl + Qm + nR = 0$ and

$l\,dx + m\,dy + n\,dz$ should be an exact differential equation.

$$l\,dx + m\,dy + n\,dz = (Pl + Qm + Rn)\left[\frac{dx}{P}\right]$$

$$l\,dx + m\,dy + n\,dz = 0$$

Integrating

$lx + my + nz = c$, is an Independent Soln.

Similarly, choose l', m', n'

$\Rightarrow$: $Pl' + Qm' + Rn' = 0$ and

$$l'dx + m'dy + n'dz = 0$$

on integrating,

$$l'x + m'y + n'z = d,$$ another independent

Solution:

$\therefore$ The General solution is,

$$\phi(lx + my + nz,\ l'x + m'y + n'z) = 0$$

Ex $\quad p + q = 2$

Here

$$P = 1, \quad Q = 1, \quad R = 2$$

A.E $\quad \dfrac{dx}{1} = \dfrac{dy}{1} = \dfrac{dz}{2}$

$$dx = dy$$

$$x = y + c$$

$$x - y = c$$

$$dy = \frac{dz}{2}$$

$$2\,dy = dz$$

$$2y = z + d$$

$$2y - z = d$$

$$\therefore \phi(x - y, 2y - z) = 0$$

Note

1. $\phi(x-y,\ 2y-z)=0$

$u=x-y,\qquad v=2y-z$

$u_x=1\qquad v_y=2-q$

$u_y=-1\qquad v_x=-p$

$$\begin{vmatrix} 1 & -p \\ -1 & 2-q \end{vmatrix}=0$$

$\Rightarrow\ 2-q-p=0\ \Rightarrow\ \boxed{p+q=2}$

2 $\phi(x-y,\ 2x-z) = 0$

$u = x-y \qquad v = 2x-z$

$u_x = 1 \qquad v_x = 2-p$

$u_y = -1 \qquad v_y = -q$

$$\begin{vmatrix} 1 & 2-p \\ -1 & -q \end{vmatrix} = 0$$

$$-q + 2 - p = 0 \Rightarrow \underline{\underline{p+q=2}}$$

Ex 2

Solve : $(x^2-y^2-z^2)p+2xyq = 2xz$

Sol :

$P = x^2-y^2-z^2$, $Q = 2xy$, $R = 2xz$

Now,

A.E,

$$\frac{dx}{x^2-y^2-z^2} = \frac{dy}{2xy} = \frac{dz}{2xz}$$

$$\frac{dy}{\cancel{2x}y} = \frac{dz}{\cancel{2x}z}$$

$$\Rightarrow \frac{dy}{y} = \frac{dz}{z}$$

Integrating,

$$\log y = \log z + \underline{\underline{\log a}}$$

$$\underline{\underline{y/z = a}}$$

use x, y, z as a multipliers

each fraction $= \dfrac{x\,dx + y\,dy + z\,dz}{x(x^2-y^2-z^2) + y(2xy) + z(2xz)}$

$x[x^2-y^2-z^2+2y^2+2z^2]$

$$\frac{dy}{2xy} = \frac{x\,dx + y\,dy + z\,dz}{\cancel{x}[x^2+y^2+z^2]}$$

$$\frac{dy}{y} = \frac{2x\,dx + 2y\,dy + 2z\,dz}{x^2+y^2+z^2} \Rightarrow \frac{dy}{y} = \frac{d(x^2+y^2+z^2)}{(x^2+y^2+z^2)}$$

on integrating

$$\log y = \log(x^2+y^2+z^2) + \log b$$

$$y/_{x^2+y^2+z^2} = b$$

$$\therefore \phi\left(y/z,\ y/_{x^2+y^2+z^2}\right) = 0.$$

(or)

$$\frac{y}{z} = f\left(y/_{x^2+y^2+z^2}\right)$$

3) Solve: $x^2(y-z)p + y^2(z-x)q = z^2(x-y)$

Sol^n

Here $P = x^2(y-z)$, $Q = y^2(z-x)$ and

$R = z^2(x-y)$

A.E

$$\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)}$$

using multiplier $\frac{1}{x^2}, \frac{1}{y^2}, \frac{1}{z^2}$

$$\frac{dx}{x^2(y-z)} = \frac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{\frac{1}{\cancel{x^2}}[\cancel{x^2}[y-z]] + \frac{1}{\cancel{y^2}}[\cancel{y^2}[z-x]] + \frac{1}{\cancel{z^2}}[\cancel{z^2}(x-y)]}$$

$$0 = \frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz$$

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = a$$

using $\left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right)$

$$\frac{dx}{x^2(y-z)} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{x(y-z) + y(z-x) + z(x-y)}$$

$$\Rightarrow \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

Integrating

$$\log x + \log y + \log z = \log b$$

$$\underline{\underline{xyz = b}}$$

$$\therefore \quad \phi\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z},\ xyz\right) = 0$$

3) Solve:

$$x(y-z)p + y(z-x)q = z(x-y)$$

Soln

A.E

$$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)}$$

The multipliers $\left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right)$ and $(1, 1, 1)$

$$\frac{dy}{y(z-x)} = \frac{dx+dy+dz}{xy-xz+yz-xy+zx-zy} \qquad (1,1,1)$$

$$\Rightarrow dx+dy+dz = 0$$

$$\underline{\underline{x+y+z = c}}$$

$$\left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right)$$

$$\frac{dy}{y(z-x)} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{y-z+z-x+x-y}$$

$$\Rightarrow \quad \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

$$\log x + \log y + \log z = \log b$$

$$(xyz) = b \quad //$$

$$\therefore \quad \phi(x+y+z,\ xyz) = 0.$$

5) Solve:

$$x(y^2-z^2)p + y(z^2-x^2)q = z(x^2-y^2)$$

(x, y, z) $\quad (1,1,1)$

$\left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right)$

Soln: $\phi(xyz,\ x^2+y^2+z^2) = 0$

6) Solve:

$$(y+z)p - (z+x)q = x-y$$

Multipliers:

$(1,1,1)$ $\quad (x, y, -z)$

$$\phi(x+y+z,\ x^2+y^2-z^2) = 0$$