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Module - 3 Laplace Transform

Defn :-

Let $f(t)$ be a function of t defined for all possible positive values of t. Then the Laplace transforms of $f(t)$ is denoted by $L[f(t)]$ is defined by

$$\bar{F}(s) = L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt,$$

s is a parameter which may be a real or complex

$L[f(t)]$ being clearly a function of s is briefly written as $\bar{F}(s)$.

That is $L[f(t)] = \bar{F}(s)$ which can also be written as $f(t) = L^{-1}[\bar{F}(s)]$, is called the inverse Laplace transform of $\bar{F}(s)$.

Transforms of elementary functions

Laplace transform – Power Function

	$f(t)$	$\bar{F}(s) = L\{f(t)\}$
Integer powers	1	$1/s$
	t	$1/s^2$
	t^2	$2!/s^3$
	$\vdots$	$\vdots$
	t^n	$n!/s^{n+1}$
	$t^p, p > 0$	$\Gamma(p+1)/s^{p+1}$, where $\Gamma(p+1) = \int_0^\infty x^p e^{-x} dx$
Positive real powers	$\sqrt{t}$	$\Gamma\left(\frac{3}{2}\right)\frac{1}{s^{3/2}} = \frac{1}{2}\Gamma\left(\frac{1}{2}\right)\frac{1}{s^{3/2}} = \frac{\sqrt{\pi}}{2\, s^{3/2}}$

$$\Gamma(n+1) = n!$$

Laplace transform – Exponential and Trigonometric Functions

	$f(t)$	$\bar{F}(s) = L\{f(t)\}$
Exponential	e^{at}	$\frac{1}{s-a}$
	e^{-at}	$\frac{1}{s+a}$
Trigonometric	$\sin at$	$\frac{a}{s^2+a^2}$
	$\cos at$	$\frac{s}{s^2+a^2}$

Sinh at $\qquad \frac{a}{s^2-a^2}$

cosh at $\qquad \frac{s}{s^2-a^2}$

Prove that $L[1] = \frac{1}{s}, \quad s > 0$

<u>Proof</u>:-

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^\infty e^{-st} (1)\, dt \qquad \{\because f(t) = 1$$

$$= \left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{-1}{s}\left[0 - 1\right] = 1/s.$$

2. $L[t^n] = \dfrac{n!}{s^{n+1}}$, $n > -1$ and $s > 0$.

Proof

$$L[t^n] = \int_0^\infty e^{-st}\, t^n \, dt \quad -(1)$$

Let $p = st \Rightarrow t = p/s$

$$\frac{dt}{dp} = \frac{1}{s}$$

$$dt = \frac{dp}{s}$$

$$① \Rightarrow \int_0^\infty e^{-p} \left(\frac{p}{s}\right)^n \frac{dp}{s}$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-p} p^n \, dp$$

$$= \frac{1}{s^{n+1}} \Gamma(n+1) = \frac{n!}{s^{n+1}} \quad //$$

In particular

$$L[t^{-1/2}] = \frac{\Gamma 1-\frac{1}{2}}{s^{1-\frac{1}{2}}} = \frac{\Gamma 1/2}{s^{1/2}} = \frac{\sqrt{\pi}}{\sqrt{s}} \quad \left\{ \because \Gamma \tfrac{1}{2} = \sqrt{\pi} \right.$$

$$= \sqrt{\frac{\pi}{s}} \quad //.$$

3 $\displaystyle L[e^{at}] = \frac{1}{s-a}$ if $s > a$

Soln:

$$L[e^{at}] = \int_0^\infty e^{-st} \cdot e^{at} \, dt$$

$$= \int_0^\infty e^{-t[s-a]} \, dt$$

$$= \left. \frac{e^{-t(s-a)}}{-(s-a)} \right]_0^{\infty}$$

$$= \frac{-1}{(s-a)} \left\{ e^{-\infty} - e^{0} \right]$$

$$= \frac{-1}{s-a} \left\{ 0 - 1 \right\} = \frac{1}{s-a} \quad //$$

4) $L[\sin at] = \int_0^\infty e^{-st} \sin at \, dt$

$\int e^{ax} \sin bx \, dx$

$= \dfrac{e^{ax}}{a^2+b^2}\left[a \sin bx - b \cos bx\right]$

$$= \left[\frac{e^{-st}}{s^2+a^2}\left[-s \sin at - a \cos at\right]\right]_0^\infty$$

$$= \frac{1}{s^2+a^2}\left[0 - (1)\left[-a\right]\right]$$

$$L[\sin at] = \frac{a}{s^2+a^2}$$

5) $L[\sinh at] = \int_0^\infty e^{-st} \sinh at \, dt$

$\because \quad \sinh at = \dfrac{e^{at} - e^{-at}}{2}$

$\therefore \quad \Rightarrow \int_0^\infty e^{-st} \left[\dfrac{e^{at} - e^{-at}}{2} \right] dt$

$= \dfrac{1}{2} \left\{ \int_0^\infty e^{-st} e^{at} \, dt - \int_0^\infty e^{-st} e^{-at} \, dt \right\}$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right]$$

$$= \frac{1}{2}\left[\frac{\cancel{s}+a-\cancel{s}+a}{s^2-a^2}\right] = \frac{a}{s^2-a^2}$$

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Properties of Laplace Transforms

1. Linearity Property:

If a, b, c be any constants and f, g, h any functions of t, then

$$L[a f(t) + b g(t) - c h(t)] = a L(f(t)) + b L[g(t)] - c L[h(t)]$$

2. First Shifting Property :

If $L[f(t)] = \overline{F}(s)$ then

$$L[e^{at} f(t)] = \overline{F}(s-a)$$

Results

1. $L[1 \cdot e^{at}] = \dfrac{1}{s-a}$ $\quad \left\{ \because L(1) = \dfrac{1}{s} \right.$

2. $L[e^{at} t^n] = \dfrac{n!}{(s-a)^{n+1}}$ $\quad \because L[t^n] = \dfrac{n!}{s^{n+1}}$

3. $L[e^{at}\sin bt] = \dfrac{b}{(s-a)^2+b^2}$ $\quad \because L[\sin bt] = \dfrac{b}{s^2+b^2}$

4. $L[e^{at}\cosh bt] = \dfrac{s-a}{(s-a)^2-b^2}$

Ex

1. Find the Laplace transforms of

$\sin 2t \sin 3t$.

Soln:-

$$L[\sin 2t \sin 3t] = L\left[\frac{1}{2}\{\cos t - \cos 5t\}\right]$$

$$= \frac{1}{2}\left\{L[\cos t] - L[\cos 5t]\right\}$$

$$= \frac{1}{2}\left\{\frac{s}{s^2+1} - \frac{s}{(s^2+25)}\right\}$$

$$= \frac{s}{2}\left\{\frac{s^2+25-s^2-1}{(s^2+1)(s^2+25)}\right\}$$

$$= \frac{12s}{(s^2+1)(s^2+25)} \quad //$$

2. Find $L[\cos^2 2t]$

Sol^n :- $\cos^2 2t = \dfrac{1+\cos 4t}{2}$

$$L[\cos^2 2t] = \frac{1}{2} L(1 + \cos 4t)$$

$$= \frac{1}{2} \left\{ L(1) + L(\cos 4t) \right\}$$

$$= \frac{1}{2} \left\{ \frac{1}{s} + \frac{s}{s^2+16} \right\} //$$