DDA2001: Introduction to Data Science
Midterm Review
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Part 1: Probability (Include Lecture 1~9)

1. Mutually exclusive (Disjoint) $\neq$ Independent
   Disjoint: $P(A \cup B) = P(A) + P(B)$, otherwise,
   $$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
   Independent: $P(A \cap B) = P(A)P(B)$
2. Zero-probability event ($P(E) = 0$) $\neq$ Impossible event ($E = \emptyset$)
   <u>Impossible Event $\Rightarrow$ Zero-Probability Event</u>, but the inverse is not true.
   Useful example: $P(A \cap B) = 0$,
   - {A: randomly pick a point from [0.1, 0.2]}
   - {A′: randomly pick a point from [0.1, 0.2)}
   - {B: randomly pick a point from [0.2, 0.3]}
3. De Morgan's law: $(A \cup B)' = A' \cap B', (A \cap B)' = A' \cup B'$
4. Functions:
   (1) PMF: For discrete R.V., $f(x_i) = P(X = x_i)$, $\sum_{i=1}^{n} f(x_i) = 1$.
   (2) PDF: For continuous R.V., $P(a \le x \le b) = \int_a^b f(x)dx$, $\int_{-\infty}^{\infty} f(x)dx = 1$.
   $$\begin{cases} f(x) > 0, & if\ x \in S \\ f(x) = 0, & if\ x \notin S \end{cases}$$
   (3) CDF: $F(x) = P(X \le x) = \sum_{\tilde{x} < x} f(\tilde{x})$, $F(x) = P(X \le x) = \int_{-\infty}^{x} f(u)du$. We can consider CDF are areas, and use them to do some calculations. (Draw pictures!)
   Use CDF to calculate PMF: $P(X = x) = F(x) - F(x^-)$, $F(x^-) = \lim_{y \to x} F(y)$.
5. Discrete and Continuous:
   - A sample space is discrete if it consists of a finite or countable infinite set of outcomes.
   - A sample space is continuous if it contains an interval or a union of multiple intervals of real numbers.
6. Mean: $E[X] = \sum_x x\, f(x)$, $E[X] = \int_{-\infty}^{\infty} x f(x) dx$
   Variance: $Var[X] = \sum_x (x - E[X])^2 f(x) = E[(X - E[X])^2] = E[X^2] - E[X]^2$
7. Linearity (Hat Check Problem): $E[\sum_i C_i X_i] = \sum_i C_i E[X_i]$
   Expectation of a function of X: $E[g(X)] = \sum_x g(x) P(X = x) = \sum_x g(x) f(x)$
8. Distributions:
   <u>Discrete R.V. distributions:</u>
   (1) Bernoulli Distribution: If $X \sim Bernoulli(p)$,
   $f(x) = P(X = k) = p^k (1 - p)^{1-k}$, $k = 0/1$, p is the possibility when $k = 1$
   $E(X) = p$, $VarX = p(1 - p)$.

(2) Binomial Distribution: If $X \sim Binomial(N, p)$, X denotes the **number** of success/failures during the first N experiments,

$$f(x) = P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$E(X) = Np,\ VarX = Np(1-p)$.

(3) Geometric Distribution: If $X \sim Geometric(p)$, the k^th^ sample is the first success,

$$f(x) = P(X = k) = (1-p)^{k-1} \cdot p$$

$E(X) = \frac{1}{p},\ VarX = \frac{1-p}{p^2}$. (Remember how to calculate them!)

(4) Discrete Uniform Distribution: If $X \sim U(n)$,

$$f(x) = \frac{1}{n}$$

$E(X) = \frac{n+1}{2},\ VarX = \frac{n^2-1}{12}$.

Continuous R.V. distributions:

(5) Uniform Distribution: If $X \sim Uniform(a, b)$,

$$f(x) = \begin{cases} \frac{1}{b-a}, & a \le x \le b \\ 0, & otherwise \end{cases}$$

$E(X) = \frac{a+b}{2},\ VarX = \frac{(b-a)^2}{12}$.

(6) Normal Distribution: If $X \sim Normal(\mu, \sigma^2)$,

$$f(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$E(X) = \mu,\ VarX = \sigma^2$.

Empirical Rule (68-95-99.7 Rule): Within one std (standard deviation of the mean), 68%, two $\rightarrow$ 95%, three $\rightarrow$ 99.7%.

Standard Normal Distribution: $X \sim Normal(0,1)$,

$$f(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

If an integral contains both e and x^2, we can use the standard normal distribution to calculate the integral.

(7) Exponential Distribution: If $X \sim Exponential(\beta)$,

$$f(x) = \begin{cases} \frac{1}{\beta} e^{-\frac{x}{\beta}}, & 0 < x < \infty \\ 0, & otherwise \end{cases}$$

$E(X) = \beta,\ VarX = \beta^2$.

9. General Property: For continuous R.V. distributions,

$$E(X) = \int_0^\infty P(X > x)dx$$

$$E(X^n) = \int_0^\infty n x^{n-1} P(X > x)dx$$

10. Approximate Problem: How to approximate $\int_a^b h(x)dx$?

Answer Template:

Step 1: If $X \sim Uniform(a,b)$, then the pdf of X will be … ($f(x)$)

Step 2: I can prove that $E[(b-a)h(X)] = \int_a^b (b-a)h(x)f(x)dx = \int_a^b h(x)dx$

Step 3: Thus, by Lindeberg-Lévy CLT, I can approximate the integral by

- Draw n samples of $X \sim Uniform(a,b)$= $x_1, x_2, x_3, \dots, x_n$
- Calculate $\frac{\sum_i (b-a)h(x_i)}{n}$

Example: How to approximate π?

Step 1: Draw a two-dimensional point from the square

- $X \sim Uniform(-1,1)$
- $Y \sim Uniform(-1,1)$

Then with the same chance (X, Y) is any point in the square

Step 2: Let $g(X,Y) = 1$ if $X^2 + Y^2 \leq 1$ and 0 otherwise. Then as the area represents probability, $E[g(X,Y)] = \frac{\pi}{4}$

Step 3: Thus, by Lindeberg-Lévy CLT, I can approximate π by

- Draw n samples of $X \sim Uniform(a,b)$= $x_1, x_2, x_3, \dots, x_n$
- Draw n samples of $Y \sim Uniform(a,b)$= $y_1, y_2, y_3, \dots, y_n$
- Calculate the proportion of $x_i^2 + y_i^2 \leq 1$

Step 4: Then π is approximated by the proportion calculated in step 3, multiplied by 4.

11. Conditional Probability (easy to ignore, read problems carefully!):

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Part 2: Statistics (Include Lecture10~11)

1. What is statistics? (Knowing) samples $\rightarrow$ (estimate) true model
2. Need: Samples, possible models, criterion for quantifying model performance
3. Notation: The point estimator used to estimate a parameter θ is usually denoted as $\hat{\theta}$.
4. $\hat{\theta}$ is a function of samples $(X_1, X_2, \dots, X_n)$ called statistic.
5. How to choose the statistic? In our course, use MLE (Maximum likelihood estimate)!

   Why? Maximize the probability.

   Given a model, the probability of generating such samples is called likelihood, notated as $L(\theta) = P(X_1, X_2, \dots, X_n|\theta)$. So, we want to find a parameter θ to maximize $L(\theta)$.

   $\rightarrow$ We define $l(\theta) = \log L(\theta)$, called log-likelihood. (In our course, log=ln).

   $\rightarrow$ To maximize $L(\theta)$ is to maximize $l(\theta)$, we can easily calculate $l'(\theta) = 0$ to find $\hat{\theta}$.
6. Likelihood Function:

   Given a model with an unknown parameter θ. Given samples: $X_1, X_2, \dots, X_n$.

   Continuous RV model (f is PDF):

   - Likelihood: $L(\theta) = \prod_i f(X_i|\theta)$
   - Log-likelihood: $l(\theta) = \sum_i \log f(X_i|\theta)$

   Discrete RV model (P is PMF):

   - Likelihood: $L(\theta) = \prod_i P(X_i|\theta)$
   - Log-likelihood: $l(\theta) = \sum_i \log P(X_i|\theta)$

7. Solve problem through MLE:
   Step 1: Judge the type of distribution, writing the PDF/PMF.
   Step 2: Use the formula in 6 to write the $L(\theta)$ and $l(\theta)$.
   Step 3: Find $l'(\theta)$, let it equals to 0, find $\hat{\theta}$. (Sometimes use partial derivatives)
   Examples are in the slides.
8. Linear Regression: Use a line to find the relationship between X and Y.
   We use the model $Y \sim N(\beta_0 + \beta_1 X, \sigma^2)$ with normal distribution (samples are centralized) to represent. So, we should then use MLE to find the best β_0, β_1 and σ^2.
   After least square regression and partial derivatives, we finally get the best parameters.
$$\begin{cases} \widehat{\beta_1} = \dfrac{\sum_i (X_i - \bar{X})\,(Y_i - \bar{Y})}{\sum_i (X_i - \bar{X})^2} \\ \widehat{\beta_0} = \bar{Y} - \widehat{\beta_1}\bar{X} \end{cases}$$
   Linear regression assumptions:
   (1) The relationship between X and Y is linear.
   (2) The variance of $Y - \beta_0 - \beta_1 X$ at every value of X is the same (the homogeneity of variances).
   (3) Different observations are independent of each other.
9. Residual Analysis: To check the two assumptions of linear regression, linear and constant variance.
$$e_i := Y_i - \widehat{\beta_0} - \widehat{\beta_1} X_i$$