Metric Balls and Infinitely Degenerate Elliptic PDEs

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I am grateful to the people who once accompanied me in this journey. They remind me that I could be trusted and loved.

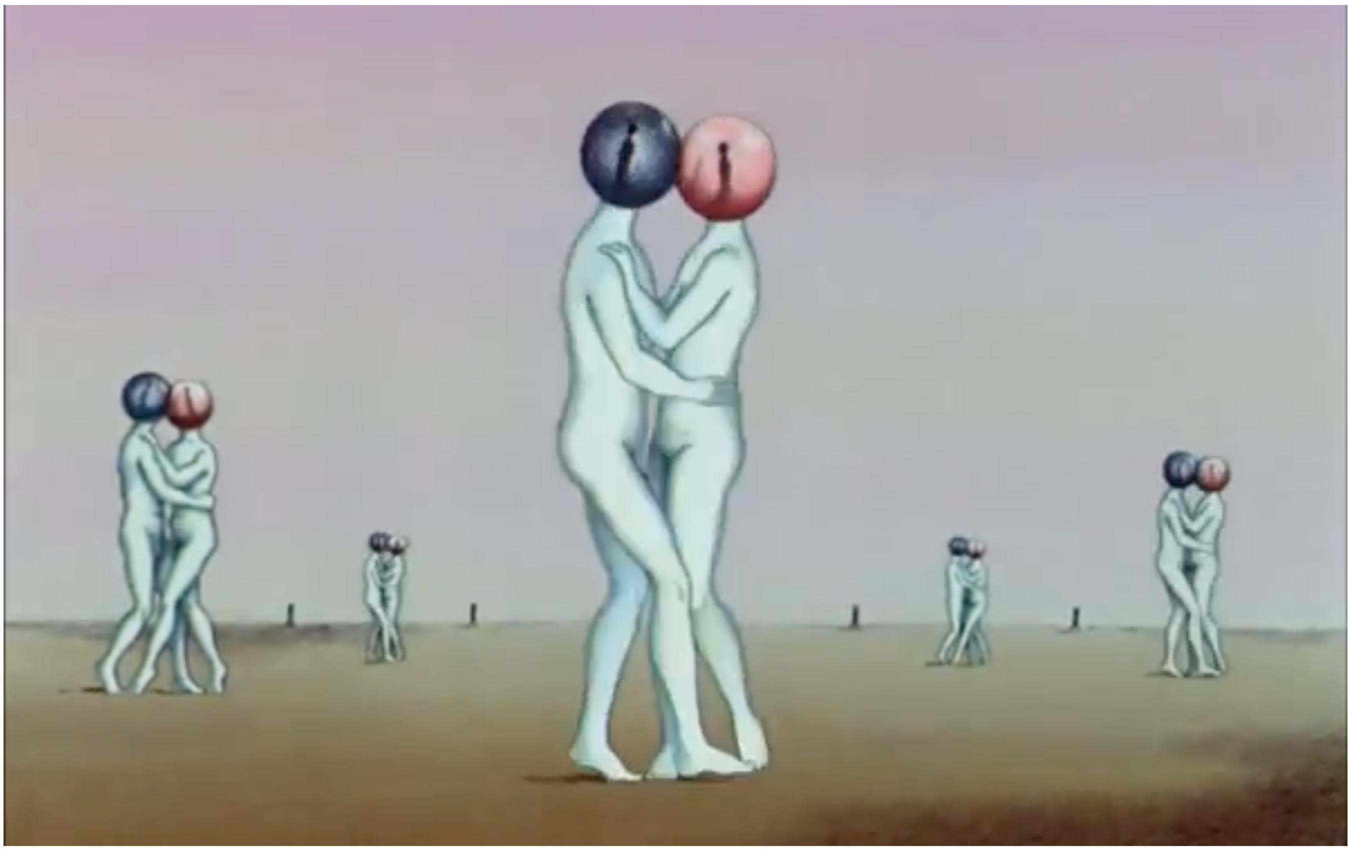

Figure 1: Dancing with headless giants (Fantastic Planet, Laloux).

List of Abbreviations

$\mathbb{N}$	Natural numbers.
$\mathbb{R}$	Real numbers.
$\mathbb{R}^n$	Euclidean space of dimension n.
C^k	k times continuously differentiable.
C^∞	smooth.
C_0^k	compactly supported and in C^k.
∇	gradient.
div	divergence.
diam	diameter.
PDE	partial differential equation.

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Abstract

This thesis is done under the guidance of my advisor Dr. Lyudmila Korobenko. The thesis is concerned with the computation of some particular objects and conditions that are important in the regularity theory of partial differential equations (PDEs). The thesis has two chapters. The first chapter is a detailed exposition of some basic results regarding the Sobolev spaces and the Carathéodory metric. Unless otherwise cited, the organization of theorems and proofs presented here closely follows the materials presented in Hajłasz & Koskela (2000). The second chapter presents the objects I've been working with and some computations. While the preliminary descriptions of the computation setting follow Korobenko et al. (2021), various original computations are presented as an extension to the previous results. These include a formula for the boundary of the metric ball, and an exact approximation of its area, both hold when $r \to 0$ asymptotically.

Introduction

The introduction will consist of three sections. The first section is for friends and family and non-math people who are interested in learning what I am doing in this thesis. I will motivate the notion of metrics, and talk about metric balls in general. The second section gives a sketch of what Carathéodory space is and how it is related to the distributional derivatives and regularity theories. I will also give an overview of the results of my computation presented in the second chapter.

0.1 Rants for total beginners and non-math readers

I assume my readers went to school at some point in their life and know what numbers are.

The most well-known ones are the integers $\mathbb{Z}$. Rational numbers $\mathbb{Q}$ is an extension of integers where fractions with non-zero denominators are allowed. Real numbers $\mathbb{R}$ is an extension to the rational numbers where you would find funky things like $\sqrt{2}$ or π, but not others like $\sqrt{-1}$ or ∞.

One can define n-dimensional (!) Euclidean spaces $\mathbb{R}^n$ for $n \in \mathbb{Z}$ be any positive integer. This is done by the construction of a Cartesian coordinate system, namely, for any point in Euclidean $n-$space, it has a coordinate $(a_1, ..., a_n)$ where $a_1, ..., a_n \in \mathbb{R}$.

The Euclidean space is normally endorsed with the Euclidean distance, namely, the distance between $a = (a_1, ..., a_n)$ and $b = (b_1, ..., b_n)$ is computed as

$$|a - b|^2 = \sum_{i=1}^{n} (a_i - b_i)^2.$$

In particular, if one is only concerned with the two dimensional $\mathbb{R}^2$, then

$$|(a_1, a_2), (b_1, b_2)| = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}.$$

In particular, the shortest path between two points on the plane is a straight line. This is the conventionally assumed *Euclidean metric.*

Back when the Cartesian coordinate and the now-called Euclidean metric came out, it was a big hit. It provides tools for quantitatively describing the positions of points and lines in the plane (or the space, etc.), and it is so powerful such that most classical Euclidean geometric problems can nowadays be simulated by algorithms using the Cartesian coordinates and techniques from calculus. However, back in the days when the computing powers are not that great, sometimes people studying geometry problems still scratch their heads a bit trying to come up with actual proofs and deductions of certain geometric propositions or theorems.

In western civilization, geometry has been studied alongside religion and philosophy since ancient times. One of the most celebrated works is the *Elements of Geometry* by Euclid from ancient Greece. The book starts by listing five axioms of geometry that are presumed to be satisfied and proceeds to deduce every other classically known geometric fact from these axioms. For a long time, mathematicians were intrigued to know whether the fifth axiom - the parallel postulate - is a consequence of the first four axioms. In the 16th and the 17th century, while the Cartesian coordinates and methods of analytic geometry improved the computing powers within the framework of geometries satisfying the five axioms, the problem remained unsolved as to whether the five axioms themselves contain redundancy.

Interested readers may do their own research on this topic, as I am giving spoilers now. As it turns out, there exist mathematical constructions that satisfy the first four axioms but not the fifth one. Nowadays, mathematicians call the geometries that satisfy all five axioms the *Euclidean geometry*, and the ones that don't satisfy them all the *non-Euclidean geometry*.

Famous examples of well-studied non-Euclidean geometries include hyperbolic geometry and spherical geometry. In two dimensions, Euclidean geometry can be thought of as the geometry defined on a rectangle with infinite height and infinite width (so it is a not-curvy plane). Spherical geometry can be modeled on the surface of a sphere, and hyperbolic geometry can be modeled on the surface of a hyperboloid. Both the hyperbolic geometry and the spherical geometry do not satisfy the fifth axiom. Given a point and a line not passing through it, in hyperbolic geometry, there are infinitely many lines passing through the point that does not intersect the given line; and in spherical geometry, every line passing through the point will intersect the given line.[1]

The intuition of geometry inherited from Euclidean geometry is challenged when interests in non-Euclidean geometry grow. In the previous two examples, the hyperboloid and the sphere both count as "physical intuitions" to their respective geometry in two dimensions, that is, both the sphere and the hyperboloid can still be viewed as visible objects embedded in the Euclidean space. However, it is plain to see that such intuitions cannot be easily visualized when the dimension n is greater than 2. Moreover, take hyperbolic geometry, there are more models for hyperbolic geometry than modeling it on a hyperboloid (for example, the Klein disk model, or the Poincaré disk model). As long as there is a valid notion of equivalency between different models, one can perform computation on other models without direct referrals to the hyperboloid.

It is also true that when mathematicians started building models that violate the

[1] Lines, or geodesics, on the sphere are the "great circles", i.e., the intersection of a plane that passes through the sphere center with the sphere. On the two sphere (in three dimension space) they are equators when rotating the sphere to different orientations.

fifth axiom, it is apparent that it is difficult to hold them back from making models that do not obey the first four axioms as well. Then, a new definition of what counts as geometry is needed.

The computations I have done in chapter 2 of this thesis study metric geometry, that is, I consider geometry that has a valid notion of distance between objects. I will also only consider objects that are points in the Euclidean plane $\mathbb{R}^2$. By a valid distance, I mean that the distance satisfies the following four axioms: let A, B, C be any three points in $\mathbb{R}^2$, then

1. The distance from A to B is zero if and only if A and B are the same point,

2. The distance between two distinct points is positive (in particular, less than infinity),

3. (Triangle inequality) the distance from A to B is the same as the distance from B to A, and the distance from A to B is less than or equal to the distance from A to B via any third point C.

It is straightforward to check that the Euclidean metric satisfies the axioms of the metrics above. But it is not the only metric defined on $\mathbb{R}^2$ that will satisfy these axioms. For example: the metric I study also satisfies them - since it is assumed a metric - but it is not the Euclidean metric.

Here's a mini-algorithm that provides some tools that will help me explain of what my metric is. I will use the Euclidean metric in this paragraph. For $A, B \in \mathbb{R}^2$, although there is a direct way of computing the distance between A, B, I will use the following method

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1. Find the expression of the shortest curve between point A, B.

In the Euclidean metric the shortest curve is a straight line. Suppose that the line connecting A, B is denoted l. Suppose l has formula $y = mx + b$ for every x, y on the line, and where $m, b \in \mathbb{R}$. Then some consequences are

2. The slope of l is m. So if the line l travels x horizontally, l travels mx vertically.

3. Suppose $\mathrm{d}x$ is a very small distance covered by line l horizontally, and assume in the meantime $\mathrm{d}y$ is traveled vertically. Then the Euclidean metric asserts that the distance traveled is

$$\mathrm{d}t^2 = \mathrm{d}x^2 + \mathrm{d}y^2 = \mathrm{d}x^2 + (m\,\mathrm{d}x)^2 = (1 + m^2)\,\mathrm{d}x^2$$

Now, measure the difference between the $x-$coordinates of A, B, denote it Δx. Then because the slope of l is constant, one has computed that the distance between A, B is $\Delta x\sqrt{1 + m^2}$.

In my metric, it is given that along the *geodesic*, i.e., the curve that minimizes the distance between any two points on the curve, it holds in infinitesimal scale that

$$\mathrm{d}t^2 = \mathrm{d}x^2 + \frac{1}{f(x)^2}\,\mathrm{d}y^2.$$

Notice that this alone does not give me sufficient information for computing the distance between any two points. In the example above in the Euclidean metric, I would need to know the formula of the line connecting the points, and, in particular, the slope of the line. So the first step in my scheme of computation is to compute (locally) the slope of the geodesic. That is, I would need to know along a geodesic, denoted λ, if the geodesic moves an infinitesimal $\mathrm{d}x$, how much $\mathrm{d}y$ the geodesic will also move. Knowing this, I can plug them back into the formula above to obtain $\mathrm{d}t$. Obtaining the formula for $\mathrm{d}t$ is shown in section 2.1.1, and obtaining the slope $\frac{\mathrm{d}y}{\mathrm{d}x}$ is shown in section 2.1.2. Section 2.1.2 as well gives some descriptions of what my geodesics look like (in particular, see figure 2.2).

The reader should note that one significant drawback of the method I used, i.e., the Euler-Lagrange equations, is that while it gives a general formula that all geodesics must satisfy, it does not give specific formulas for the geodesic connecting any two points I designated. In general, no analytical or numerical method can help me compute this distance. Luckily, my goal is not to obtain a precise distance between points, but rather to compute the area of metric balls centered at the origin. Here is the motivation for the metric ball.

In the Euclidean metric, the notion of a circle of radius r around the origin denotes the collection of points that are distance r away from the origin. The circle bounds a disk, and this disk is the *metric ball* in the Euclidean metric. A metric ball centered at the origin with radius r, by definition, is the collection of points at distance less than r away from the origin. The metric ball may look nothing like a disk, however. In my case, it vaguely reminds me of bat wings (refer to figure 2.1).

The entire chapter 2 is dedicated to computing the area of the metric ball centered at the origin when the radius is very small. But again, for any non-zero positive r, it is very hard to obtain an exact computation of the actual area of the metric ball. My result will only hold asymptotically when r goes to 0. Think of it this way, suppose I keep shrinking the radius, but for each radius always scale the width of the metric ball back to 1. Then as r becomes smaller, my model will fit better. The model will fit the best when r is infinitesimal. For the simulated fitting process, see figures 2.7, 2.8, and 2.9. For the actual model, see figure 2.6 and read the paragraph of descriptions above it.

0.2 A general introduction of background

To study solutions to certain differential equations it turns out to be useful to define a different notion of distance. The new distance is related to the equation at hand, and we would like to study the properties of balls constructed using this new distance, to learn more about solutions to our differential equation. More precisely, given a differential equation, there is often a family of vector fields associated with it, and the distance studied in this thesis is defined as follows

Definition 1.2.1. *For a family of vector fields $\boldsymbol{X} = (X_1, ..., X_m)$, say that an absolutely continuous curve $\gamma : [a, b] \to \Omega$ is admissible if there exists measurable functions $c_1(t), ..., c_k(t)$, $t \in [a, b]$ such that both conditions below are satisfied: for all t,*

$$\sum_{j=1}^{k} c_j(t)^2 \leq 1 \qquad \text{and} \qquad \dot{\gamma}(t) = \sum_{j=1}^{k} c_j(t) X_j(t).$$

Define the Carathéodory distance $\rho(x, y)$ between $x, y \in \Omega$ as the infimum of $T > 0$ such that there exists an admissible curve γ with $\gamma(0) = x$ and $\gamma(T) = y$. If there's no admissible curve between the two points, denote $\rho(x, y) = \infty$.

If $\rho(a, b) < \infty$ for all $a, b < \infty$, then the Carathéodory distance is called the Carathéodory metric. Denote a space equipped with Carathéodory metric Carathéodory space (Ω, ρ).

Although Carathéodory metric is not necessarily defined on the vector fields as long as the notion of the admissible curve is not based on vector fields (for example, see Proposition 1.2.1), this thesis follows the structure of the contents presented in Chapters 10, 11 of Hajłasz & Koskela (2000).

The upshots of this definition lead to some important results in regularity theories for the distributional derivatives as discussed below. The first chapter of the thesis is dedicated to presenting these regularity theorems related to the Carathéodory metric.

Definition. *Let $\Omega \subset \mathbb{R}^n$ be connected. Consider $u \in Lip(\Omega)$, and a family of vector fields $X_1, ..., X_m : \mathbb{R}^n \to \mathbb{R}^n$. Say*

$$X_i u(x) = \langle X_i, \nabla u(x) \rangle, \qquad i = 1, ..., k$$

is the distributional derivative of u. I will assume $X_1, ..., X_m$ have locally Lipschitz continuous coefficients, and write $\boldsymbol{X}u = (X_1 u, ..., X_m u)$. Equip it with norm:

$$|\boldsymbol{X}u(x)| = \left(\sum_{l=1}^{k} |X_l u(x)|^2 \right)^{\frac{1}{2}}.$$

First, one can relate the Carathéodory metric to the notion of upper gradients, which is a generalized notion of the magnitude of gradients when the function is not necessarily differentiable.

Definition 1.1.11. *Consider $u : \Omega \to \mathbb{R}$ be a Borel function. Say the Borel function $g : \Omega \to \mathbb{R}$ is an upper gradient on Ω of u, if for every 1-Lipschitz curve $\gamma : [a, b] \to \Omega$, there holds*

$$|u(\gamma(b)) - u(\gamma(a))| \leq \int_a^b g(\gamma(t))\,\mathrm{d}t$$

Then using the notion of upper gradients one obtains the following regularity theorem, in parallel to situations when $|\nabla u|$ does exist.

Theorem 1.2.8. *Let $0 \leq g \in L^1_{loc}(\Omega)$ be an upper gradient on (Ω, ρ) of a function u which is continuous with respect to the Euclidean metric. Then the distributional derivatives $X_j u$, $j = 1, 2, ..., k$ are locally integrable and $|\boldsymbol{X}u| \leq g$ a.e..*

It turns out that the Carathéodory metric encodes information about the distributional derivative $|\mathbf{X}u|$ intrinsically. The first chapter of this thesis will culminate in the theorem below, which shows Carathéodory metric characterizes the norm of distributional derivatives.

Theorem 1.2.10. *Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two families of vector fields with locally Lipschitz coefficients in Ω and such that H1 holds for induced Carathéodory metrics $\rho_{\boldsymbol{X}}, \rho_{\boldsymbol{Y}}$. Then the following conditions are equivalent:*

1. *$\rho_{\boldsymbol{X}}$ is equivalent to $\rho_{\boldsymbol{Y}}$, i.e., there exists a constant $C \geq 1$ such that*

$$C^{-1}\rho_{\boldsymbol{X}} \leq \rho_{\boldsymbol{Y}} \leq C\rho_{\boldsymbol{X}}.$$

2. *There exists a constant $C \geq 1$ such that*

$$C^{-1}|\boldsymbol{X}u| \leq |\boldsymbol{X}u| \leq C|\boldsymbol{X}u|$$

for all $u \in C^\infty(\Omega)$.

The second part of the thesis concerns the regularity theory of the weak solutions to the following partial differential equation (PDE):

$$\operatorname{div}(A(x,y)\nabla u(x,y)) = 0. \tag{*}$$

One say the PDE is elliptic if $A(x,y)$ is positive definite for all (x,y), and degenerate elliptic if $A(x,y)$ is semi-positive definite for all (x,y). A special case is considered in section 1.3 following Hajłasz & Koskela (2000) when the matrix $A(x)$ arises from vector fields. In that case the following PDE is considered.

$$Lu(x) = -\sum_{i=1}^{k} X_i^* X_i(x)u(x) = 0. \tag{$\triangle$}$$

Definition 1.3.1. *Let μ be a measure of $\mathbb{R}^n$, and in this thesis I will assume it to be the Lebesgue measure. Say a function $u \in C^1(\Omega)$ is a weak solution to (*) if*

$$\int_\Omega \nabla u \cdot A\nabla\varphi \,\mathrm{d}\mu = 0$$

for all $\varphi \in C_0^\infty(\Omega)$. In particular, a weak solution to ($\triangle$) satisfies

$$\int_\Omega \sum_{i=1}^{k} X_i u \cdot X_i\varphi \,\mathrm{d}\mu = 0$$

for all $\varphi \in C_0^\infty(\Omega)$, and where $\boldsymbol{X}u$ is the distributional derivative.

The general strategy to obtain regularity is by applying the Moser iteration. For ($\triangle$), the *doubling condition* is one of the essential conditions to be met to initiate the Moser iteration.

Definition. *Consider $(\mathbb{R}^n, \rho, \mu)$ where ρ is the Carathéodory metric, and μ is the Lebesgue measure on $\mathbb{R}^n$. The doubling condition is satisfied if*

$$\mu(B_\rho(x, 2r)) \leq C\mu(B_\rho(x, r))$$

for all $x \in \mathbb{R}^n$ and radius $r > 0$, where $C > 0$ is some constant independent of x and r.

In general and not limited to the PDEs obtained through vector fields, the doubling condition is often assumed to hold. Using Moser iterations, existing literature has established regularity theories when the PDE is elliptic (see Moser (1961)) or degenerate elliptic and vanishing polynomial degrees (see Franchi & Lanconelli (1983)).

This thesis follows the treatment of infinitely degenerate PDEs in Korobenko et al. (2021), and is concerned with PDE of the form

$$\operatorname{div}(A(x,y)\nabla u(x,y)) = 0 \quad \text{where} \quad A(x,y) = \begin{pmatrix} 1 & 0 \\ 0 & f(x)^2 \end{pmatrix} \tag{0.2.1}$$

where $f \in C^2(\Omega)$ is even, $f(0) = 0$ and $f'(x) > 0$ for all $x \in \mathbb{R}_+$. The name"infinitely degenerate" refers to the setting that one further assumes f is smooth and without power series representations centered at zero. Typical examples of functions that satisfy these conditions includes $f(x) = e^{-\frac{1}{x^p}}$ or $f(x) = e^{-e^{\frac{1}{x^p}}}$. In these cases, the metric balls centered at the origin can never satisfy the doubling conditions for all $r > 0$. However, even if the doubling condition cannot be satisfied, obtaining an asymptote of the area of the metric ball at the origin when the radius is small is still a crucial component in the deducing of the regularity theories of infinitely degenerate PDEs.

In the study done by Korobenko et al. (2021), when computing the area of the metric ball, the following assumptions are made.

Assumption. *Fix $R > 0$ and let $F(x) = -\ln f(x)$ for $0 < x < R$, so that*

$$f(x) = e^{-F(|x|)}, \quad 0 < |x| < R.$$

Assume the following for some constants $C \geq 1$ and $0 < \varepsilon < 1$:

1. $\lim_{x\to 0^+} F(x) = +\infty$.
2. *$F'(x) < 0$ and $F''(x) > 0$ for all $x \in (0, R)$.*
3. $\frac{1}{C}|F'(r)| \leq |F'(x)| \leq C|F'(r)|, \quad \frac{1}{2}r < x < 2r < R.$
4. *$\frac{1}{-xF'(x)}$ is increasing in the interval $(0, R)$ and satisfy $\frac{1}{-xF'(x)}. \leq \frac{1}{\varepsilon}$ for $x \in (0, R)$.*
5. *$\frac{1}{Cx} \leq \frac{F''(x)}{-F'(x)} \leq \frac{C}{x}$ for $x \in (0, R)$.*

The computation obtained is the following (Conclusion 45 in Korobenko et al. (2021)).

Theorem. *There is constant $C' > 0$ such that*

$$\frac{1}{C'}\frac{f(r)}{F'(r)^2} \geq Area(B(0,r)) \geq C'\frac{f(r)}{F'(r)^2}.$$

In this thesis, I extended their results on two grounds. First, on one hand, the fifth assumption in Korobenko et al. (2021) limits the scope of functions that could be considered to functions that can be approximated by $e^{-\frac{1}{x}}$. On the other hand, the assumption I used can also approximate them as well as a wider range of other infinitely degenerate functions. Here is my assumption.

Assumption 2.2.1. *Let $F(x) = -\ln(f(x))$. Since f is strictly increasing and smooth at the origin, F strictly decreases from infinity. Then, I will assume that for $x > 0$,*

$$-F'(x) \text{ is decreasing}$$

and

$$\lim_{x\to 0} \frac{-F'(x)}{-F'(x \pm \frac{k}{-F'(x)})} = 1 \text{ for all constants } k > 0.$$

Instead of an approximation by uncertain constants, I also obtained an exact asymptotic of the area of the metric ball when r tends to zero. In the theorem below, B_r is the first quadrant of the metric ball centered at the origin with radius r. The symbol $a \to b$ denotes $\lim_{r\to 0} \frac{a}{b} = 1$.

Theorem 2.2.15.

$$Area(B_r) \to \frac{\pi}{2} \cdot \frac{f(r)}{F'(r)^2}.$$

I have also obtained an exact formula for the asymptotic boundary of B_r. In the corollary below r^* denotes the $x-$coordinate of the highest point on the boundary of B_r.

Corollary 2.2.14. *Define the variable s by*

$$s = -F'(r) \cdot ((x + \Delta x)(\omega) - r^*)$$

for $\omega \in (-\infty, \ln 2)$, and denote $f(s)$ be the label of the corresponding geodesic. Let $\varphi(s)$ be the height of which the geodesic $f(s)$ reaches distance r. Then

$$\varphi(s) \to \frac{1}{2}\operatorname{sech}(s) \cdot \frac{f(r)}{-F'(r)}.$$

In particular, it is noticeable that for any choice of functions f that satisfies the assumption, asymptotically their contour would have the same coefficient term dependent on ω, times a scaling term dependent on f and r.

Chapter 1

Background

The first section of this chapter covers the standard constructions of the Sobolev space in the context of functional analysis. The materials here are mainly referenced from Kinnunen (2017). The second and the third section cover the construction of the Carathéodory distance and its relations to PDEs, and they follow chapter 10, 11, 13 of Hajłasz & Koskela (2000).

1.1 Weak Derivatives, Sobolev Space

In this chapter, I will always consider Ω to be an open subset of $\mathbb{R}^n$, and $p \geq 1$. Denote $\Omega' \subset\subset \Omega$ if Ω' is open and $\overline{\Omega'} \subset \Omega$ is compact. A standard construction is the L^p space.

Definition 1.1.1. For $1 \leq p < \infty$, say a function $f : \Omega \to \mathbb{R}$ is in $L^p(\Omega)$ if $|f|_p < \infty$, with

$$|f|_p = \left(\int_\Omega |f|^p \right)^{\frac{1}{p}}.$$

There is no requirement on the continuity and differentiability of functions in $L^p(\Omega)$. In general, most functions in L^p are not differentiable, but many of them may still be nice enough to have weak derivatives. The intuition for weak derivative is as follows: suppose $u \in C^1(\Omega)$, i.e., u is continuously differentiable, then integration by parts gives the identity

$$\int_\Omega u\varphi_j \,\mathrm{d}x = -\int_\Omega u_j \varphi \,\mathrm{d}x, \quad j = 1, ..., n$$

for all *test functions* $\varphi \in C_0^\infty(\Omega)$, i.e., for all φ compactly supported and smooth over Ω.

More generally, I'll consider not only the first partial derivatives. Let $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n$ be a *multi-index*. Write $|\alpha| = \sum \alpha_i$. Define the differential operator D^α on function $u \in C^k(\Omega)$, $k \geq |\alpha|$, as following:

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} u.$$

Similar to the case of first partial derivatives, successive integrations by parts yields that for all test functions $\varphi \in C_0^\infty(\Omega)$,

$$\int u(D^\alpha \varphi)\,\mathrm{d}x = (-1)^{|\alpha|} \int (D^\alpha u)\varphi\,\mathrm{d}x$$

Now, loosing differentiability restrictions of u on Ω, I'll define the weak partial derivatives.

Definition 1.1.2. For $u \in L^1_{\mathrm{loc}}(\Omega)$, i.e., $u \in L^1(\Omega')$ for all $\Omega' \subset\subset \Omega$, call $v \in L^1_{\mathrm{loc}}(\Omega)$ the *αth weak partial derivative* of u if for all test functions $\varphi \in C_0^\infty(\Omega)$,

$$\int_\Omega u \cdot (D^\alpha \varphi_j)\,\mathrm{d}x = (-1)^{|\alpha|} \int_\Omega v\varphi\,\mathrm{d}x, \quad j = 1, ..., n.$$

With slight abuse of notation, in circumstances I will denote the weak gradient of u be

$$Du = (D_1 u, D_2 u, ..., D_n u)$$

where D_i is conventionally defined: $D_i = D^\alpha$ where $\alpha_i = 1$ and $\alpha_j = 0$ for $j \neq i$.

Remark 1.1.3. I will use the term *distributional derivative* indistinguishably with weak derivative, because in this thesis I am not concerned with distributional derivatives not represented by proper functions.

In general, when dealing with functions in L^p space I'll assume the following equivalence relationship: $f \equiv g$ if and only if $f = g$ almost everywhere. In this case, say g is a *representation* of f.

Proposition 1.1.4. Assume αth weak partial derivative of u exists, then it is unique up to equivalence relations.

Proof. Suppose $v, \tilde{v} \in L^1_{\mathrm{loc}}$ are both weak partial derivatives of u, this is to say

$$\int_\Omega u D^\alpha(\varphi)\,\mathrm{d}x = (-1)^{|\alpha|} \int_\Omega v\varphi\,\mathrm{d}x = (-1)^{|\alpha|} \int_\Omega \tilde{v}\varphi\,\mathrm{d}x$$

Then,

$$\int_\Omega (v - \tilde{v})\varphi\,\mathrm{d}x = 0 \tag{1.1}$$

holds for every test function $\varphi \in C_0^\infty(\Omega)$.

To show $v \equiv \tilde{v}$ on Ω, it is sufficient to show that $v \equiv \tilde{v}$ on every $\Omega' \subset\subset \Omega$. A standard result in analysis is that the space $C_0^\infty(\Omega')$ is dense in $L^p(\Omega')$ (ref. [Ziemer (1989)]), so there exists a sequence $\{\varphi_n\} \subset C_0^\infty(\Omega')$ such that $\{\varphi_n\} \to \mathrm{sgn}(v - \tilde{v})$ a.e.[1] I will assume $|\varphi_i| < 2$ for each term in the sequence. Then, because $|\varphi_i(v - \tilde{v})|_p \leq$

[1]sgn denote the usual sign function that reflects positivity or non-positivity.

$2(|v|_p + |\tilde{v}|_p) < \infty$ following $v, \tilde{v} \in L^1(\Omega')$, I will use dominated convergence theorem and (1.1) to show:

$$\begin{aligned} 0 = \lim_{i\to\infty} \int_{\Omega'} (v - \tilde{v})\varphi_i \,\mathrm{d}x &= \int_{\Omega'} (v - \tilde{v}) \lim_{i\to\infty} \varphi_i \,\mathrm{d}x \\ &= \int_{\Omega'} (v - \tilde{v})\mathrm{sgn}(v - \tilde{v}) \,\mathrm{d}x \\ &= \int_{\Omega'} |v - \tilde{v}| \,\mathrm{d}x \end{aligned}$$

Therefore $v \equiv \tilde{v}$ over every $\Omega' \subset\subset \Omega$, implying $v \equiv \tilde{v}$ over Ω.

□

The argument in the previous proof shows the following corollary.

Corollary 1.1.5. If $u \in L^1_{\mathrm{loc}}(\Omega)$ satisfies

$$\int_\Omega u\varphi \,\mathrm{d}x = 0$$

for all test functions $\varphi \in C_0^\infty(\Omega)$, then $f \equiv 0$ over Ω.

Classically there is the following construction in generalization to $L^p(\Omega)$ but with concerns to the weak derivatives.

Definition 1.1.6. Define the *Sobolev space* as

$$W^{k,p}(\Omega) := \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega), |\alpha| \le k\}.$$

This thesis will only be concerned with $W^{1,p}$, and endorse it with norm $|u|_{W^{1,p}} = |u|_p + \sum_j |D_j u|_p$.

A standard result shown in functional analysis is the following (ref. Brézis (2011)).

Theorem 1.1.7. *The Sobolev spaces are complete.*

Here are some terminologies to recall.

Definition 1.1.8. The *Borel $\sigma-$algebra* is the smallest $\sigma-$algebra containing all of the open sets. Say $f : \Omega \to \mathbb{R}$ is a *Borel function* if $f^{-1}(A)$ is Borel for all open sets in $\mathbb{R}$.

Definition 1.1.9. Let $I = [a, b] \subset \mathbb{R}$ be a closed interval. Say $f : I \to \mathbb{R}$ is *absolutely continuous* if f has a derivative f' almost everywhere, f' is Lebesgue integrable, and

$$f(x) = f(a) + \int_a^x f'(t) \,\mathrm{d}t$$

for all $x \in [a, b]$.

Definition 1.1.10. Say a function $f : I \to \mathbb{R}$ is $p-$*Lipschitz continuous* if

$$|f(x) - f(y)| \leq p|x - y|$$

for all $x, y \in \Omega$.

Classical results regarding the continuity conditions shows that on compact intervals

Lipschitz continuous $\subset$ absolutely continuous $\subset$ differentiable almost everywhere.

Now, similar to the definition of weak partial derivatives, the goal is to define the upper gradient in an attempt to generalize the notion of the gradient.

Definition 1.1.11. Consider $u : \Omega \to \mathbb{R}$ be a Borel function. Say the Borel function $g : \Omega \to \mathbb{R}$ is an upper gradient on Ω of u, if for every 1-Lipschitz curve $\gamma : [a, b] \to \Omega$, there holds

$$|u(\gamma(b)) - u(\gamma(a))| \leq \int_a^b g(\gamma(t))\,\mathrm{d}t$$

Here is the immediate corollary following: Lipschitz continuous functions are absolutely continuous, therefore the derivatives exist a.e.

Corollary 1.1.12. If u is Lipschitz on Ω, then any measurable function satisfying $g \geq |\nabla u|$ everywhere is an upper gradient of u.

There is also a version of the opposite direction to the corollary above.

Proposition 1.1.13. If $g \in L^p(\Omega)$ is an upper gradient of $u \in L^p(\Omega)$, then $u \in W^{1,p}(\Omega)$ and $g \geq |\nabla u|$ almost everywhere.

Outline of the proof. Based on the existence of g, on each line γ parallel to one of the axis, one can construct a Lebesgue differentiable function g_γ so that

$$u(x) = u(a) + \int_a^x g_\gamma(t)\,\mathrm{d}t.$$

The existence of g_γ follows the existence of the upper bound g restricted to the line. This shows that u is absolutely continuous on any lines parallel to the coordinate axes. Now, use the ACL Property of the Sobolev space (stated below) to conclude point-wise gradient ∇u exists almost everywhere, and since $g \in L^p(\Omega)$, $|\nabla u| \in L^p(\Omega)$ and therefore $u \in W^{1,p}(\Omega)$. $\square$

Theorem 1.1.14 (Absolutely continuous on lines (ACL) property of Sobolev functions, cf Ziemer (1989))**.** *Suppose $u \in W^{1,p}(\Omega)$. Then there exists $\bar{u} \equiv u$ that is absolutely continuous on almost all line segments parallel to the coordinate directions in $\mathbb{R}^n$.*

Conversely, for $u \in L^p$, if there is $\bar{u} \equiv u$ that is absolutely continuous on almost all line segments parallel to the coordinate directions in $\mathbb{R}^n$, then the pointwise gradient ∇u exists almost everywhere. As usual, if $|\nabla u| \in L^p(\Omega)$, $u \in W^{1,p}(\Omega)$.

1.2 Carathéodory Spaces

Let $\Omega \subset \mathbb{R}^n$ be connected. Consider $u \in \mathrm{Lip}(\Omega)$, I will like to associate it with a family of vector fields $X_1, ..., X_m : \mathbb{R}^n \to \mathbb{R}^n$ in the following way:

$$X_i u(x) = \langle X_i, \nabla u(x)\rangle, \qquad i = 1, ..., k$$

I will assume $X_1, ..., X_m$ to have locally Lipschitz continuous coefficients.

For $\mathbf{X} = (X_1, ..., X_m)$. Write $\mathbf{X}u = (X_1 u, ..., X_m u)$. Then, the Euclidean norm is

$$|\mathbf{X}u(x)| = \left(\sum_{l=1}^{k} |X_l u(x)|^2\right)^{\frac{1}{2}}.$$

One can associate a family of vector fields with differential operators. Defining the formal adjoint X_l^* of vector field X_l in L^2 by the property

$$\int_\Omega (X_l \varphi)\psi \,\mathrm{d}x = \int_\Omega \varphi(X_l^* \psi)\,\mathrm{d}x$$

for all test functions $\varphi, \psi \in C_0^\infty(\Omega)$, then define the operator $Lu = -\sum_{i=1}^k X_l^* X_l u$. Given vector fields, one can always write [2]

$$Lu(x) = \mathrm{div}(A(x)\nabla u(x)).$$

Write $X_l = \sum_{i=1}^n a_{il} \frac{\partial}{\partial x_i}$, so $X_l^* = -\sum_{j=1}^n \frac{\partial}{\partial x_j} a_{jl}$, compute

$$-X_l^* X_l = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} a_{il} a_{jl} \frac{\partial}{\partial x_i}.$$

Thus one can see that

$$A(x) = \left(\sum_{l=1}^{m} a_{il}(x) a_{jl}(x)\right)_{ij}.$$

It is desirable for $|\mathbf{X}u|$ to be an upper gradient of u, however, this is mostly not the case if for the 1-Lipschitz curve γ considered for the upper gradient, $\dot\gamma$ is not spanned by the vector fields $X_1, ..., X_m$. For a simple demonstration, consider $\mathbf{X} = (X_1)$ is a single vector field

$$X_1 : \mathbb{R}^2 \to \mathbb{R}^2 : (x, y) \mapsto (1, 0), \quad \text{so } X_1 u = \frac{\partial}{\partial x} u$$

[2]For people who always confuse ∇, $\nabla\cdot$ and df_x:.

1. df_x is the Jacobian matrix $T_x(M_1) \to T_x(M_2)$, and essentially is a change of variables.

2. $\nabla f(x) = df_x(1, 1, ...1)$, $\nabla \cdot f = \langle \nabla f, (1, 1, ..., 1)\rangle$.

and $\gamma(t) = (0, t)$, then one could not expect the inequality required for upper gradient

$$|u(0, a) - u(0, b)| \le \int_\gamma \frac{\partial}{\partial x} u(\gamma(t)) = 0$$

to hold in general.

A slight modification to fix this issue would suffice for our purpose.

Definition 1.2.1. For a family of vector fields $\mathbf{X} = (X_1, ..., X_m)$, say that an absolutely continuous curve $\gamma : [a, b] \to \Omega$ is *admissible* if there exist measurable functions $c_1(t), ..., c_k(t)$, $t \in [a, b]$ such that both conditions below are satisfied: for all t,

$$\sum_{j=1}^{k} c_j(t)^2 \le 1 \quad \text{and} \quad \dot{\gamma}(t) = \sum_{j=1}^{k} c_j(t) X_j(t).$$

Define the *Carathéodory distance* $\rho(x, y)$ between $x, y \in \Omega$ as the infimum of $T > 0$ such that there exists an admissible curve γ with $\gamma(0) = x$ and $\gamma(T) = y$. If there's no admissible curve between the two points, denote $\rho(x, y) = \infty$.

If $\rho(a, b) < \infty$ for all $a, b < \infty$, then the Carathéodory distance is called the Carathéodory metric. Denote a space equipped with Carathéodory metric *Carathéodory space* (Ω, ρ).

The following proposition gives another characterization of the admissible curves. The first one is proved in Monti (2001).

Proposition 1.2.2. γ is admissible if and only if $\langle \dot{\gamma}, \xi \rangle^2 \le \xi^T A(x) \xi$ for all $\xi \in \mathbb{R}^n$.

Proposition 1.2.3. Every admissible curve $\gamma : [0, T] \to \Omega$ is Lipschitz (in Euclidean metric).

Proof. Cauchy-Schwarz inequality shows that along $\gamma[0, T]$,

$$|\dot{\gamma}(t)| = \left| \sum_{j=1}^{k} c_j(t) X_j(\gamma(t)) \right| \le |\mathbf{c}| \cdot \sup_\gamma |\mathbf{X} \circ \gamma| \le \sup_\gamma |\mathbf{X} \circ \gamma|$$

where $\mathbf{c} = (c_1, ..., c_k)$. Then for any segment $[a, b] \subset [0, T]$,

$$|\gamma(b) - \gamma(a)| \le \int_a^b |\dot{\gamma}(t)| \, \mathrm{d}t \le |b - a| \sup_\gamma |\mathbf{X} \circ \gamma|$$

shows γ is Lipschitz in the Euclidean metric. □

The theorem below characterizes the admissible curves.

Theorem 1.2.4. *A mapping $\gamma : [0, T] \to (\Omega, \rho)$ is admissible if and only if it is $1-$Lipschitz with respect to ρ, i.e., $\rho(\gamma(b), \gamma(a)) \le |b - a|$ for all a, b.*

The proof of the theorem is quite technical, so it will be omitted. But the following proposition leading up to it has topological implications.

Proposition 1.2.5. Let $G \subset\subset \Omega$. Then there exists constant $C > 0$ such that for all $x, y \in G$,

$$\rho(x, y) \geq C|x - y|.$$

I will first prove the following Lemma.

Lemma 1.2.6. Let $B(x, R) \subset\subset \Omega$, and denote $M = \sup_{p \in B(x,R)} |\mathbf{X}(p)|$. If $\gamma : [0, T] \to \Omega$ is an admissible curve with $\gamma(0) = x$, and γ includes a segment outside of $B(x, R)$, then the following inequality between parameters holds: $T \geq R/M$.

Proof. Take the smallest $t_0 \in (0, T]$ such that $|x - \gamma(t_0)| = R$. By Cauchy-Schwarz inequality, $|\dot{\gamma}(t)| \leq \sup_{\gamma(0,t_0)} |\mathbf{X} \circ \gamma| \leq M$ for all $t \in [0, t_0]$. Then,

$$R = |\gamma(0) - \gamma(t_0)| = \left| \int_0^{t_0} \dot{\gamma}(t)\, \mathrm{d}t \right| \leq MT.$$

□

Proof of Proposition 1.2.5. Assume $x, y \in G$. Since $G \subset\subset \Omega$ implies G is open, and $\bar{G}$ is compact in open set Ω, there is $\varepsilon > 0$ and a finite union such that $G(\varepsilon) = \bigcup_{p \in G} B(p, \varepsilon) \subset \Omega$. Now I will estimate $T = \rho(x, y)$. Assuming $\varepsilon < |x - y|$, construct

$$R = \min\{|x - y|, \varepsilon\} = \varepsilon \geq \frac{\varepsilon}{\operatorname{diam}(G)} |x - y|.$$

Then, any admissible curve from x to y must go outside the ball $B(x, R) \subset G(\varepsilon) \subset\subset \Omega$. Let $M = \sup_{p \in G(\varepsilon)} |\mathbf{X}(p)|$. By Lemma 1.2.6, $\rho(x, y) = T \geq R/M = C|x - y|$ where $C = \varepsilon/(M \operatorname{diam}(G))$. □

If $\rho(x, y) < \infty$ for every $x, y \in \mathbb{R}^n$, then ρ is a metric, and call it the *Carathéodory metric*. Proposition 1.2.5, then, implies the induced metric topology (Ω, ρ) is a finer topology than the Euclidean topology $(\Omega, |\cdot|)$, i.e., $\mathrm{id} : (\Omega, \rho) \to (\Omega, |\cdot|)$ is continuous. However, the two topologies need not be equivalent, for example consider vector fields ∂_x and $x_+ \partial_y$.

In order to avoid pathological situations, literatures often assume

$$\mathrm{id} : (\Omega, \rho) \to (\Omega, |\cdot|) \quad \text{is a homeomorphism.} \tag{H1}$$

However, in this thesis I will not implicitly assume H1.

The Carathéodory metric presents a natural setting for the distributional derivatives to be upper gradients.

Proposition 1.2.7. For $u \in C^\infty(\Omega)$, $|\nabla u|$ is an upper gradient of u on (Ω, ρ).

Proof. Let $\gamma : [a,b] \to (\Omega, \rho)$ be 1-Lipschitz hence admissible by Theorem 1.2.4. By Proposition 1.2.3 $u \circ \gamma$ is Lipschitz hence differentiable a.e. Compute

$$\begin{aligned}
|u(\gamma(b) - u(\gamma(a))| &= \left| \int_a^b \langle \nabla u(\gamma(t)), \dot{\gamma}(t) \rangle \, \mathrm{d}t \right| \\
&= \left| \int_a^b \langle \nabla u(\gamma(t)), \sum_j (c_j \cdot X_j)(\gamma(t)) \rangle \, \mathrm{d}t \right| \\
&= \left| \int_a^b \sum_j c_j(\gamma(t)) \langle \nabla u(\gamma(t)), \cdot X_j(\gamma(t)) \rangle \, \mathrm{d}t \right| \\
&\leq \int_a^b |\mathbf{X} u(\gamma(t))| \, \mathrm{d}t
\end{aligned}$$

with the last inequality following the Cauchy-Schwarz inequality. □

The following theorem generalizes Proposition 1.1.13.

Theorem 1.2.8. *Let $0 \leq g \in L^1_{loc}(\Omega)$ be an upper gradient on (Ω, ρ) of a function u which is continuous with respect to the Euclidean metric. Then the distributional derivatives $X_j u$, $j = 1, 2, ..., k$ are locally integrable and $|\boldsymbol{X} u| \leq g$ a.e..*

The proof in general is quite complicated. Here I will present the sketch of proof when assuming $u \in C^\infty(\Omega)$ and the vector fields have C^1 coefficients.

Sketch. Since u is smooth, it is sufficient to show that $|\mathbf{X}u| \leq g$ a.e. Because the set $\Omega' = \{x \in \Omega : |\mathbf{X}u(x)| > 0\}$ is open, and $g \geq |\mathbf{X}u| = 0$ holds trivially in $\Omega \backslash \Omega'$, it suffices to assume $\Omega = \Omega'$ so $|\mathbf{X}u| > 0$ everywhere.

Denote the vector fields $\mathbf{X} = (X_1, ..., X_m)$, and consider the curve $\gamma : [0, T] \to (\Omega, \rho)$ defined by

$$\dot{\gamma}(t) = \sum_j a_j(\gamma(t)) X_j(\gamma(t)) \quad , \quad a_j(x) = \frac{X_j u(x)}{|\mathbf{X}u(x)|}.$$

By construction γ is an admissible curve, then by Theorem 1.2.4 it is 1-Lipschitz with respect to the Carathéodory metric. Hence, suppose g is an upper gradient with respect to the Carathéodory metric, then by definition

$$|u(\gamma(t_2)) - u(\gamma(t_1))| \leq \int_{t_1}^{t_2} g(\gamma(t)) \, \mathrm{d}t$$

for all $0 < t_1 < t_2 < T$. On the other hand, compute that

$$\begin{aligned}|u(\gamma(t_2)) - u(\gamma(t_1))| &= \left|\int_{t_1}^{t_2} \langle \nabla u(\gamma(t)), \dot\gamma(t)\rangle \,\mathrm{d}t\right| \\ &= \left|\int_{t_1}^{t_2} \langle \nabla u(\gamma(t)), \sum_j a_j(\gamma(t)) X_j(\gamma(t))\rangle \,\mathrm{d}t\right| \\ &= \left|\int_{t_1}^{t_2} \sum_j \frac{X_j u(x)}{|\mathbf{X}u(x)|} \langle \nabla u(\gamma(t)), X_j(\gamma(t))\rangle \,\mathrm{d}t\right| \\ &= \int_{t_1}^{t_2} |\mathbf{X}u(x)| \,\mathrm{d}t.\end{aligned}$$

Notice that in general γ are integral curves (solutions) to the vector field $Y = \sum_j a_j X_j$.

Suppose that Y is parallel to one of the coordinate axis, then I have shown $|Yu| \leq g$ on almost every line parallel to that axis, hence $|Yu| \leq g$ almost everywhere in Ω. The general case can be reduced to this specific case by rectification theorems. The usual requirement for local rectification is assuming the vector field to be C^1, see Arnol'd (1992). The argument also works for the general Lipschitz case, see Hajłasz & Koskela (2000). □

The following two statements give more concrete descriptions of $|\mathbf{X}u|$. The first is a corollary proven in Franchi et al. (1999).

Corollary 1.2.9. Assume H1 holds, If u is $L-$Lipschitz with regard to the Carathéodory metric , then $X_j u$, $j = 1, ..., k$ are represented by bounded functions, and $|\mathbf{X}u| \leq L$ a.e.

The second statement asserts that, in a sense, the analysis of the Carathéodory metric implies regularity properties of the associated vector fields. This is related to the motivations of this thesis. In the next chapter, I will analyze the metric balls in the Carathéodory metric with the end goal of checking the admissible criteria for Moser iteration, which will assert regularity properties of solutions to degenerate elliptic equations.

Theorem 1.2.10. *Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two families of vector fields with locally Lipschitz coefficients in Ω and such that H1 holds for induced Carathéodory metrics $\rho_{\boldsymbol{X}}, \rho_{\boldsymbol{Y}}$. Then the following conditions are equivalent:*

1. *$\rho_{\boldsymbol{X}}$ is equivalent to $\rho_{\boldsymbol{Y}}$, i.e., there exists a constant $C \geq 1$ such that*

$$C^{-1}\rho_{\boldsymbol{X}} \leq \rho_{\boldsymbol{Y}} \leq C\rho_{\boldsymbol{X}}.$$

2. *There exists a constant $C \geq 1$ such that*

$$C^{-1}|\boldsymbol{X}u| \leq |\boldsymbol{X}u| \leq C|\boldsymbol{X}u|$$

for all $u \in C^\infty(\Omega)$.

I will postpone the proof until after I prove the following theorem and corollary concerning mollifiers. These are some necessary technicalities.

Theorem 1.2.11. *Let $\boldsymbol{X} = (X_1, ..., X_m)$ be a system of vector fields with locally Lipschitz coefficients in Ω, and let $1 \leq p < \infty$. If $u \in L^p(\Omega)$ and distributional derivatives $\boldsymbol{X}u \in L^p(\Omega)$, then there exists a sequence $u_k \in C^\infty(\Omega)$ such that $|u_k - u|_{L^p(\Omega)} + |\boldsymbol{X}u_k - \boldsymbol{X}u|_{L^p(\Omega)} \to 0$ as $k \to \infty$.*

Proof. Assume that u has compact support in Ω. The general case would follow an argument involving the partition of unity and Ω is separable, so the open cover has a countable sub-cover. Also, assume u is continuous. In general, one can approximate using continuous functions.

Define the mollifier $0 \leq \varphi \in C_0^\infty(\mathbb{R}^n)$ by

$$\varphi(x) = \begin{cases} ce^{\frac{1}{|x|^2-1}}, & |x| < 1 \\ 0, & |x| \geq 1 \end{cases}$$

where the constant c is chosen so that $\int_{\mathbb{R}^n} \varphi = 1$. Also define $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$. When understanding φ_ε as a distribution, a standard property of the mollifiers state that $\varphi_\varepsilon(x) \to \delta$ as $\varepsilon \to 0$, where one interpret δ as the Dirac delta function. A standard result in analysis states $|u - u * \varphi_\varepsilon|_{L^p} \to 0$.

Think of a single vector field $Y(x) = \sum_{j=1}^n c_j(x)\partial/\partial x_j$, where the coefficients $c_j(x)$ are locally Lipschitz. Since I am assuming $u \in L^p(\Omega)$, this implies the weaker assumption $u \in L^1(\Omega')$ where $\Omega' \subset\subset \Omega$. Since $\overline{\Omega'}$ is compact, there is a uniform choice of $\varepsilon > 0$ such that $B(x,\varepsilon) \subset \Omega$ for $x \in \Omega'$. Then, in the sense of distributions, compute

$$\begin{aligned} Y(u * \varphi_\varepsilon)(x) &= \langle Y(x), \int_{B(0,\varepsilon)} u(x-y)\varphi_\varepsilon(y)\,\mathrm{d}y\rangle \\ &= \sum_{j=1}^n \int_{B(0,\varepsilon)} c_j(x)\frac{\partial u}{\partial x_j}(x-y)\varphi_\varepsilon(y)\,\mathrm{d}y \\ &= \sum_{j=1}^n \int_{B(0,\varepsilon)} c_j(x-y)\frac{\partial u}{\partial x_j}(x-y)\varphi_\varepsilon(y)\,\mathrm{d}y \\ &+ \sum_{j=1}^n \int_{B(0,\varepsilon)} (c_j(x) - c_j(x-y))\frac{\partial u}{\partial x_j}(x-y)\varphi_\varepsilon(y)\,\mathrm{d}y \end{aligned}$$

Observe that the first term is just $(Yu) * \varphi_\varepsilon(x)$. Noting since $x \in \Omega$ is fixed,

$$\frac{\partial u}{\partial x_j}(x-y) = -\frac{\partial u}{\partial y_j}(x-y),$$

φ_ε vanishes on the boundary, and $u(x)$ is a constant. Use integration by parts to compute the second term

$$\sum_{j=1}^{n}\int_{B(0,\varepsilon)}(c_j(x)-c_j(x-y))\frac{\partial u}{\partial x_j}(x-y)\varphi_\varepsilon(y)\,\mathrm{d}y$$
$$=\sum_{j=1}^{n}\int_{B(0,\varepsilon)}(u(x-y)-u(x))\frac{\partial}{\partial y_j}\left((c_j(x)-c_j(x-y))\varphi_\varepsilon(y)\right)\mathrm{d}y := A_\varepsilon u(x).$$

Then $Y(u*\varphi_\varepsilon)(x)=(Yu)*\varphi_\varepsilon(x)+A_\varepsilon u(x)$.

Recall that the constants $c_j(x)$ of the vector field are by definition locally Lipschitz, ie., in $\bigcup_{x\in\Omega'}B(x,\varepsilon)$, there is $L>0$ such that

$$\frac{c_j(x-y)-c_j(x)}{y}\le L,$$

and φ is a bump function, so $\varphi_\varepsilon\le C_0\varepsilon^{-n}\chi_{B(0,\varepsilon)}$. Further φ' is also a bump function, so $\varphi_\varepsilon'(y)=(\varepsilon^{-n}\varphi(y\varepsilon))'\le K_0\varepsilon^{n-1}\varphi'(x/\varepsilon)\le K_1\varepsilon^{n-1}\chi_{B(0,\varepsilon)}$. Thus one can estimate

$$\left|\frac{\partial}{\partial y_j}(c_j(x-y)-c_j(x))\varphi_\varepsilon(y)\right|\le CL\varepsilon^{-n}\chi_{B(0,\varepsilon)}$$

for appropriate constant C. Hence,

$$|A_\varepsilon(x)|\le C_1L\varepsilon^{-n}\int_{B(x,\varepsilon)}|u(y)-u(x)|\,\mathrm{d}y\approx C_2L\fint_{B(x,\varepsilon)}|u(y)-u(x)|\,\mathrm{d}y$$

for appropriate constants C_1,C_2 independent of x.

Recall $u\in L^p(\Omega)$. If u is continuous in Ω', then by Lebesgue differentiation theorem $|A_\varepsilon(x)|\to 0$ as $\varepsilon\to 0$. In addition $Yu\in L^p(\Omega')$, hence

$$Y(u*\varphi_\varepsilon)\to Yu*\varphi_\varepsilon\to Yu \quad\text{ when }\quad \varepsilon\to 0.$$

in the sense of L^p convergence. This concludes the proof. □

The following corollary is a bookkeeping exercise left for the reader.

Corollary 1.2.12. Assume H1 holds. Let u be L-Lipschitz with respect to the Carathéodory metric in Ω. Then the standard mollifier approximation converges to u uniformly on compact sets of Ω, and

$$|\mathbf{X}(u*\varphi_\varepsilon)(x)|\le L+|A_\varepsilon u(x)|$$

where $|A_\varepsilon u(x)|\to 0$ as $\varepsilon\to 0$ uniformly on compact sets of Ω.

Now I will prove Theorem 1.2.10.

Proof of Theorem 1.2.10. $\rho_{\mathbf{X}}\approx\rho_{\mathbf{Y}}\Rightarrow|\mathbf{X}u|\approx|\mathbf{Y}u|$. Recall that the admissible curves in the Carathéodory metric are the Carathéodory 1-Lipschitz curves. Since $\rho_{\mathbf{X}}$ is equivalent to $\rho_{\mathbf{Y}}$, their admissible curves are the same up to re-parametrization by constant C. Suppose $u \in C^\infty(\Omega)$, then referring to the definition of upper gradients on Carathéodory metric space and following a change of variable between $t_{\mathbf{X}}, t_{\mathbf{Y}}$ all of the upper gradients are equivalent up to scaling by C. Use Theorem 1.2.8 to wrap up the argument.

$|\mathbf{X}u| \approx |\mathbf{Y}u| \Rightarrow \rho_{\mathbf{X}} \approx \rho_{\mathbf{Y}}$. Fix $x, y \in \Omega$, and let $u(z) = \rho_{\mathbf{X}}(x, z)$. Notice that u is by construction Carathéodory metric 1-Lipschitz. Let $\gamma : [0, T] \to (\Omega, \rho_Y)$ be an arbitrary admissible curve such that $\gamma(0) = x$, $\gamma(T) = y$.

Let $u_\varepsilon = u * \varphi_\varepsilon$ be the standard mollifier approximation, then by Proposition 1.2.7 , $|\mathbf{Y}u_\varepsilon|$ is an upper gradient of u_ε (existence follows φ_ε is a bump function). Invoke Corollary 1.2.12 for uniformity in the computation below:

$$\begin{aligned}\rho_{\mathbf{X}}(x,y) \xleftarrow{\varepsilon\to 0} |u_\varepsilon(\gamma(T)) - u_\varepsilon(\gamma(0))| &\le \int_0^T |\mathbf{Y}u_\varepsilon(\gamma(t))| \,\mathrm{d}t \\ &\le C\int_0^T |\mathbf{X}u_\varepsilon(\gamma(t))| \,\mathrm{d}t \le CT + C\int_0^T |A_\varepsilon(\gamma(t))| \,\mathrm{d}t \xrightarrow{\varepsilon\to 0} CT.\end{aligned}$$

Following the definition of Carathéodory distance, $\rho_{\mathbf{X}} \le C\rho_{\mathbf{Y}}$. The opposite inequality follows the same argument. $\square$

1.3 Applications to PDEs

At the beginning of the last section, I introduced partial differential equations based on vector fields, i.e.,

$$Lu(x) = -\sum_{i=1}^{k} X_i^* X_i(x) u(x) = 0. \tag{$\triangle$}$$

More generally, the degenerate elliptic PDEs have the form

$$\operatorname{div}(A(x)\nabla u(x)) = 0 \tag{*}$$

where $A(x) \in M_{n*n}(\mathbb{R})$ for $x \in \mathbb{R}^n$ has eigenvalues $\lambda_i(x) \ge 0$.

Definition 1.3.1. Let μ be a measure of $\mathbb{R}^n$, and in this thesis I will assume it to be the Lebesgue measure. Say a function $u \in C^1(\Omega)$ is a weak solution to (*) if

$$\int_\Omega \nabla u \cdot A\nabla\varphi \,\mathrm{d}\mu = 0$$

for all $\varphi \in C_0^\infty(\Omega)$. In particular, a weak solution to ($\triangle$) satisfies

$$\int_\Omega \sum_{i=1}^{k} X_i u \cdot X_i \varphi \,\mathrm{d}\mu = 0$$

for all $\varphi \in C_0^\infty(\Omega)$, and where $\mathbf{X}u$ is the distributional derivative.

The regularity results for (*) and (△) are usually obtained via Moser iteration. The topic is profound and I will not deal with it here. Below are the usual conditions required on the metric for the initiation of the Moser iteration.

Definition 1.3.2. Consider $(\mathbb{R}^n, \rho, \mu)$ where ρ is the Carathéodory metric. Say the measure μ is p-admissible if the measure satisfy the following four properties:

1. (Doubling Condition) $\mu(B_\rho(x, 2r)) \leq C_d \mu(B_\rho(x, r))$ where $C_d > 0$ is some constant.

2. (Uniqueness Condition) If $\Omega \subset \mathbb{R}^n$ is open and $\varphi_i \in C^\infty(\Omega)$ is a sequence such that $\int_\Omega |\varphi_i|^p \,\mathrm{d}\mu \to 0$ and $\int_\Omega |\mathbf{X}\varphi_i - v|^p \,\mathrm{d}\mu \to 0$, then $v \equiv 0$.

3. (Sobolev inequality) There exists constant $k > 1$ and $C_1 > 0$ such that for all metric balls $\tilde{B}(x, r) \subset \mathbb{R}^n$ and all $\varphi \in C_0^\infty(\tilde{B}(x, r))$,
$$\left(\fint_{\tilde{B}} |\varphi|^{kp} \,\mathrm{d}\mu\right)^{1/kp} \leq C_1 r \left(\fint_{\tilde{B}} |\mathbf{X}\varphi|^p \,\mathrm{d}\mu\right)^{1/p}.$$

4. (Poincaré inequality) If $\varphi \in C^\infty(\tilde{B}(x, r))$, then for constant $C_2 > 0$,
$$\int_{\tilde{B}} |\varphi - \varphi_{\tilde{B}}|^p \,\mathrm{d}\mu \leq C_2 r^p \int_{\tilde{B}} |\mathbf{X}\varphi|^p \,\mathrm{d}\mu$$
where $\varphi_{\tilde{B}}$ is the average value of φ over $\tilde{B}$

One can modify the definition above if only $\Omega \subset \mathbb{R}^n$ is considered, but the full generality will not be treated here.

Theorem 1.3.3. *Recall the assumption H1 from the last section. Assume H1, and* $\boldsymbol{X}$ *be a system of vector fields in* $\mathbb{R}^n$*. Then* μ *is* p*-admissible,* $1 < p < \infty$*, if and only if the doubling condition is satisfied, and there exists* $\sigma \geq 1$ *such that*
$$\fint_{\tilde{B}} |u - u_{\tilde{B}}| \,\mathrm{d}\mu \leq Cr \left(\fint_{\sigma\tilde{B}} |\boldsymbol{X}u|^p \,\mathrm{d}\mu\right)^{1/p}$$
for $C > 0$*, whenever* $\tilde{B} \subset \mathbb{R}^n$ *is a metric ball with radius* r*, and* $u \in C^\infty(\sigma\tilde{B})$.

The doubling property of metric balls will be of particular interest to this study. Most existing literature assumes the doubling condition is satisfied, however, this is not true in the general case this thesis is considering. Still, the idea of the thesis is to analyze the area of the metric balls geometrically and try to deduce sufficient conditions (though not doubling) for possible implementations of Moser iterations.

Chapter 2

Computations of Area of Balls

I will first introduce the set-up of the computations in section 1, and the materials there follow chapter 7 of Korobenko et al. (2021). Section 2 will first introduce some new assumptions which I will follow. Then, under these assumptions, computations of the area of the metric ball when the radius approaches zero will be presented. The arguments in section 2 are original. Section 3 presents some simulations of the model deduced in section 2 using Mathematica.

2.1 Descriptions of the Metric and the Geodesics

2.1.1 Description of the metric

Consider the degenerate elliptic partial differential equation (PDE)

$$\operatorname{div}(A(x)\nabla u(x,y)) = 0 \quad \text{where} \quad A(x) = \begin{pmatrix} 1 & 0 \\ 0 & f(x)^2 \end{pmatrix} \tag{*}$$

and assume that $f \in C^2(\Omega)$ is even, $f(0) = 0$ and $f'(x) > 0$ for all $x \in \mathbb{R}_+$.[1]

Consider the Carathéodory metric associated with the PDE above. Referring to Proposition 1.2.2, say that a curve γ is *admissible* if $\langle \dot{\gamma}, \xi \rangle^2 \leq \xi^T A(x) \xi$ for every $\xi \in \mathbb{R}^n$. Observe that when $x \neq 0$, A is invertible and symmetric, so there is the following proposition.

Proposition 2.1.1. The Cathéodory distance $\mathrm{d}t$ is given by

$$\mathrm{d}t^2 = \mathrm{d}x^2 + \frac{1}{f(x)^2}\,\mathrm{d}y^2.$$

Proof. I will show the claim that $(v \cdot \xi)^2 \leq \xi^T A \xi$ for all $\xi \in \mathbb{R}^n$ if and only if $v^T A^{-1} v \leq 1$.

[1] In general, say the PDEs are degenerate elliptic if the matrix A is positive semi-definite. This thesis is concerned with the infinitely degenerate PDEs, i.e., assuming f is smooth at the origin, i.e., f cannot be represented by power series centered at the origin.

Suppose $(v \cdot \xi)^2 \leq \xi^T A \xi$ for all $\xi \in \mathbb{R}^n$, then

$$(v^T A^{-1} v)^2 = (v \cdot A^{-1} v)^2 \leq (A^{-1} v)^T A (A^{-1} v) = v^T A^{-T} v = v^T A^{-1} v.$$

Now I will show the converse direction. Because A is positive semi-definite and symmetric, the matrix $\sqrt{A}$ exists and is symmetric. Suppose $v^T A^{-1} v \leq 1$, then following the Cauchy-Schwarz inequality,

$$(v \cdot \xi)^2 = \langle \sqrt{A} A^{-1} v, \sqrt{A}^T \xi \rangle \leq |\sqrt{A} A^{-1} v|^2 |\sqrt{A}^T \xi|^2 = (v^T A^{-1} v)(\xi^T A \xi) \leq (\xi^T A \xi).$$

Thus, writing $\gamma(t) = (x(t), y(t))$, one obtain

$$1 \geq (\frac{\partial x}{\partial t})^2 + \frac{1}{f^2}(\frac{\partial y}{\partial t})^2 \quad \iff \quad \mathrm{d}t^2 \geq \mathrm{d}x^2 + \frac{1}{f^2}\,\mathrm{d}y^2.$$

Recall that the Carathéodory distance $\rho(x, y)$ is $\inf\{T : \gamma[0, T] \to \mathbb{R}^n, \gamma \text{ connects } x, y\}$. Consider x, y to be infinitesimally close. For any admissible curve connecting x, y, one can always reparametrize the curve until the identity above is an equality. Then, indeed, abusing the notation to let t denote the Carathéodory distance, there is

$$\mathrm{d}t^2 = \mathrm{d}x^2 + \frac{1}{f^2}\,\mathrm{d}y^2.$$

□

Since the distance is defined by the Carathéodory metric, one can define Carathéodory metric balls. In this thesis, I will only consider balls centered at the origin. Due to the symmetry of the Carathéodory metric around the origin, the Carathéodory metric ball will be symmetrically consisting of its portion in the first quadrant and the reflections of that portion into other quadrants. Therefore I will only consider the metric ball in the first quadrant in the computations that follow.

Definition 2.1.2. Call $B_r := \{(x, y) \in \mathbb{R}^2 : \rho((0, 0), (x, y)) \leq r \text{ and } x, y > 0\}$ the *metric ball of radius r*. This is a shorthand for the first quadrant of the Carathéodory metric ball with radius r centered at the origin. See figure 2.1

Denote the area of B_r by $Area(B_r)$.

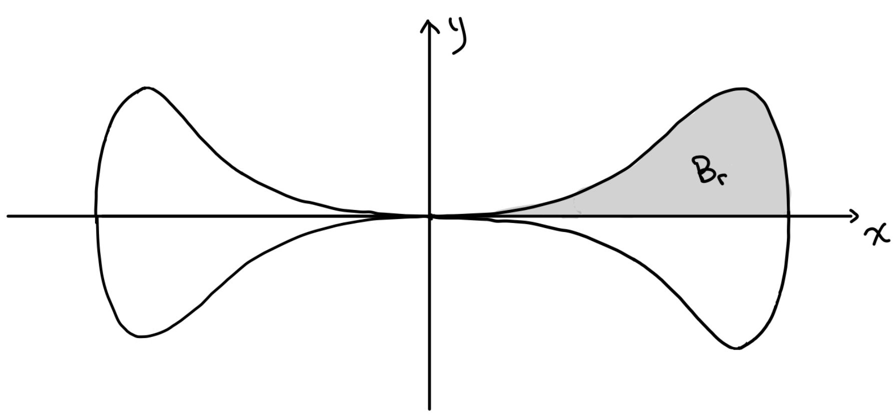


Figure 2.1: The metric ball of radius r and B_r.

2.1.2 Description of the geodesics

I am interested in calculating the asymptote of $Area(B_r)$ when $r \to 0$. One of the main tools that could be utilized is the description of the geodesics in the Carathéodory metric.

Geodesics are analogous to lines in the Euclidean metric. Lines minimize the distance between two points in the Euclidean metric, and geodesics in the Carathéodory metric minimize the distance between any two points in the Carathéodory metric. Formally, if γ are admissible curves from a to b, then γ_0 is called the geodesic if and only if for all γ,

$$\int_{\gamma_0} |\,\mathrm{d}t| \leq \int_{\gamma} |\,\mathrm{d}t|.$$

A general strategy to obtain the descriptions of geodesics is to apply the Euler-Lagrange equations. In our setting, by Euler Lagrange γ is a geodesic that minimizes

$$\int_{\gamma} |\,\mathrm{d}t| = \int_{\gamma} \sqrt{1 + \frac{1}{f(x)^2} \cdot \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}.$$

if and only if it is admissible and for every points (x, y) on γ,

$$\frac{\mathrm{d}}{\mathrm{d}x} \frac{\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)}{f(x)^2 \sqrt{1 + \frac{1}{f(x)^2}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}} = 0.$$

This makes it possible to define a constant λ invariant along each geodesic by simply defining

$$\lambda = \frac{f(x)^2 \sqrt{1 + \frac{1}{f(x)^2}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}}{\frac{\mathrm{d}y}{\mathrm{d}x}}. \tag{$\triangle$}$$

Rearrange the expression to obtain the slope of geodesic in terms of λ and coordinate x,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \pm \frac{f(x)^2}{\sqrt{\lambda^2 - f(x)^2}}. \tag{$\diamond$}$$

Definition 2.1.3. In ($\triangle$), call λ the *label* of the geodesic it denotes. Call the point $x > 0$ such that $f(x) = \lambda$ as the *turning point* of the geodesic λ, if the point exists. See figure 2.2.

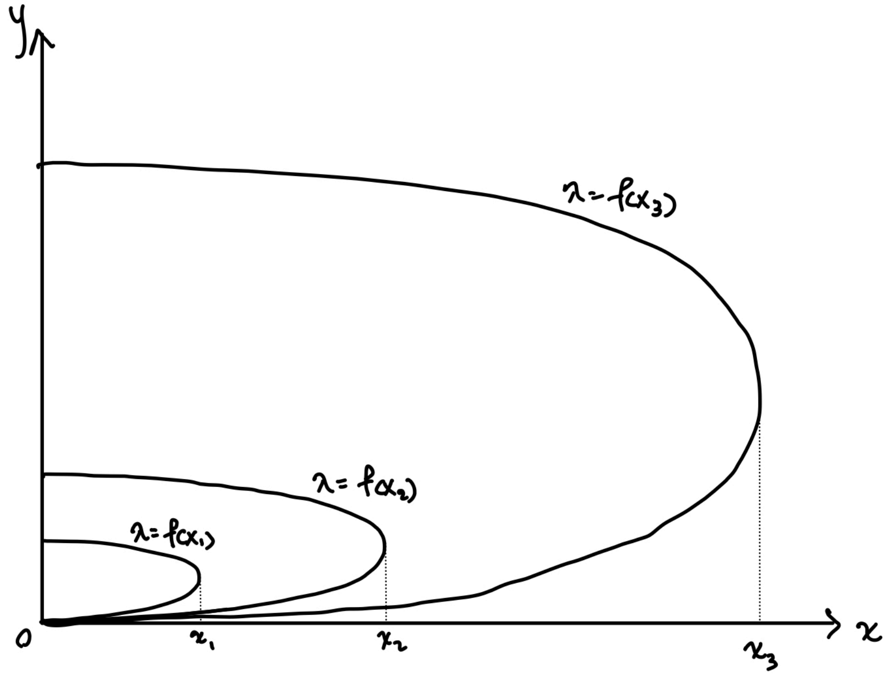


Figure 2.2: Geodesic tiling of $\mathbb{R}^2$.

Corollary 2.1.4. Some consequences are:

1. Labels $\lambda > 0$ are in bijection with the geodesics through the origin in the first quadrant.

2. Labels $\lambda > 0$ are in bijection with turning points $x > 0$.

Sketch of the proof. For (1), the only thing that should be checked is that the construction of the curves corresponding to $\lambda > 0$ is admissible. But since I am assuming the metric in Proposition 2.1.1, the only non-admissible curves possible are ones with non-zero slopes at the origin. The geodesics corresponding to all $\lambda \geq 0$ have zero slope at the origin. (2) follows the assumption that f is strictly increasing. $\square$

Proposition 2.1.5. Along geodesic λ,

$$\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{\lambda}{\sqrt{\lambda^2 - f(x)^2}}.$$

Proof. Following

$$\mathrm{d}t^2 = \mathrm{d}x^2 + \frac{1}{f(x)^2}\,\mathrm{d}y^2,$$

one computes

$$\frac{\mathrm{d}t}{\mathrm{d}x} = \sqrt{1 + \frac{1}{f(x)^2}\cdot\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} = \sqrt{1 + \frac{f(x)^4}{f(x)^2(\lambda^2 - f(x)^2)}} = \frac{\lambda}{\sqrt{\lambda^2 - f(x)^2}}.$$

$\square$

Definition 2.1.6. In the following sections, I will use the following functions extensively.

Define $X : \mathbb{R}_+ \to \mathbb{R}_+$ to be the identity function.

Define $Y : \mathbb{R}_+ \to \mathbb{R}_+$ to denote the $y-$coordinate of the turning point with parameter x, i.e.,

$$Y(x) = \int_0^x \frac{f(u)^2}{\sqrt{f(x)^2 - f(u)^2}}\,\mathrm{d}u = \int_0^x \frac{\mathrm{d}y}{\mathrm{d}u}(f(x))(u)\,\mathrm{d}u,$$

where $\frac{\mathrm{d}y}{\mathrm{d}u}(f(x))(u)$ denote that $\frac{\mathrm{d}y}{\mathrm{d}u}$ is defined along the geodesic $\lambda = f(x)$, and the argument is the variable u.

Define $R : \mathbb{R}_+ \to \mathbb{R}_+$ to denote the distance travelled along the geodesic $f(x)$ up to the turning point, i.e.,

$$R(x) = \int_0^x \frac{f(x)}{\sqrt{f(x)^2 - f(u)^2}}\,\mathrm{d}u = \int_0^x \frac{\mathrm{d}t}{\mathrm{d}u}(f(x))(u)\,\mathrm{d}u,$$

where again the parameter $f(x)$ denotes that $\frac{\mathrm{d}t}{\mathrm{d}u}$ is defined along the geodesic $f(x)$.

Proposition 2.1.7. Here are some qualitative descriptions of the behavior of the geodesics.

1. The geodesic $\lambda = \infty$ is horizontal.
2. Suppose the geodesic λ turns back at its turning point x characterized by $\lambda = f(x)$, then its path is symmetric about the line $y = Y(x)$.

Proof. For 2, the $y-$coordinate of the geodesic λ at x is given by[2]

$$y_\lambda(x) = \int_0^x \frac{f(u)^2}{\sqrt{\lambda^2 - f(u)^2}}\,\mathrm{d}u,$$

and after the geodesic turns back, the negative slope is assumed (when x increases). □

An important geometric identity first recorded in Lemma 48 of Korobenko et al. (2021) provides associations between different intrinsic quantities of the geodesics.

Proposition 2.1.8. Along any segments of geodesic λ, let $\mathrm{d}y$ denote the difference in height and $\mathrm{d}(t-x)$ represent the difference in distance r minus difference in coordinate x. Then

$$\mathrm{d}y \geq \lambda\,\mathrm{d}(t-x) \geq \frac{\mathrm{d}y}{2}.$$

Proof. Because $1 - \sqrt{1-x} = \frac{x}{1+\sqrt{1-x}}$, for $x < 1$

$$x \geq 1 - \sqrt{1-x} \geq \frac{x}{2}.$$

[2]This is true because I am only considering geodesics that go into the first quadrant.

Using the identity $1-\sqrt{1-x}\approx x$, it follows that locally

$$\frac{\mathrm{d}(t-x)}{\mathrm{d}x}=\frac{\mathrm{d}t}{\mathrm{d}x}-1=\frac{\lambda-\sqrt{\lambda^2-f(x)^2}}{\sqrt{\lambda^2-f(x)^2}}\approx\frac{1}{\lambda}\cdot\frac{f(x)^2}{\sqrt{\lambda^2-f(x)^2}}=\frac{1}{\lambda}\cdot\frac{\mathrm{d}y}{\mathrm{d}x}.$$

Then globally along any geodesic λ the same identity follows as well. □

2.2 Computations

I will assume the following assumption for the later computations.

Assumption 2.2.1. Let $F(x)=-\ln(f(x))$. Since f is strictly increasing and smooth at the origin, F strictly decreases from infinity. Then, I will assume that for $x>0$,

$$-F'(x)\text{ is decreasing}$$

and

$$\lim_{x\to 0}\frac{-F'(x)}{-F'(x\pm\frac{k}{-F'(x)})}=1\text{ for all constants }k>0.$$

Corollary 2.2.2. Fix $k>1$ to be a constant, then

$$kf(\alpha(x))=f(x)\quad\Rightarrow\quad\lim_{x\to 0}\frac{F'(\alpha(x))}{F'(x)}=1\tag{1}$$

where $\alpha(x)>0$ is a unique value dependent on x and k such that the first condition holds.

Proof. Compute that, for $\alpha(x)=x-\Delta x$,

$$\ln k=\ln\frac{f(x))}{f(\alpha(x))}=F(\alpha(x))-F(x)=\int_{\alpha(x)}^{x}-F'(u)\,\mathrm{d}u=\int_{x-\Delta x}^{x}-F'(u)\,\mathrm{d}u.$$

Since it is assumed that $-F'(x)$ is decreasing,

$$\ln k=\int_{x-\Delta x}^{x}-F'(u)\,\mathrm{d}u>-F'(x)\cdot\Delta x,$$

so $\alpha(x)=x-\Delta x>x-\frac{\ln k}{-F'(x)}$. Now, by assumption,

$$\lim_{x\to 0}\frac{F'(\alpha(x))}{F'(x)}=1.$$

□

By almost the exact same logic, one obtains another corollary.

Corollary 2.2.3. Suppose

$$\lim_{x\to 0}(x-\alpha(x))\cdot -F'(x)=\ln k$$

then

$$\lim_{x\to 0}\frac{f(x)}{f(\alpha(x))}=k.$$

Chapter 7 of Korobenko et al. (2021) studied functions that can be compared to $e^{-\frac{1}{x^p}}$ near the origin. The example below shows they also satisfy the assumption here.

Example 2.2.4. Let $f=e^{-\frac{1}{x^p}}$ for $p>0$, and let $k>0$ be a constant. Then $F(x)=\frac{1}{x^p}$ and $-F'(x)=\frac{p}{x^{p+1}}$. Thus $F'(x)$ is decreasing for $x>0$, and

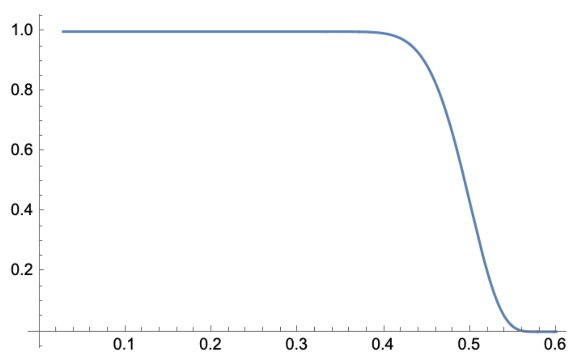
$$\lim_{x\to 0}\frac{-F'(x)}{-F'(x\pm\frac{k}{-F'(x)})}=\lim_{x\to 0}\frac{(x\pm k/p\cdot x^{p+1})^{p+1}}{x^{p+1}}=\lim_{x\to 0}(1\pm k/p\cdot x^p)^{p+1}\to 1.$$

In fact, the assumption should hold for any iterations of

$$F=e^{e^{\cdots e^{\frac{1}{x^p}}}},$$

where $p>0$. Below is a Mathematica plot of the convergence when $F=e^{e^{\frac{1}{x}}}$, and $k=1000$.

```
F[x_] := E^E^(1/x)
Plot[-F'[x] / -F'[x - 1000/-F'[x]], {x, 0, .6}]
```



A daring conjecture is that the assumption holds for all strictly increasing f smooth at the origin.[3]

In the contents that follows I will adopt the following notion. Say $a\to b$ when $r\to 0$ if and only if $\lim_{r\to 0}\frac{a}{b}=1$. The clause $r\to 0$ could be implicit if it is obvious.

[3] The assumption, however, is not true if one does not assume f does not have power series at the origin. One can easily check that the assumption does not hold for $f(x)=x$.

Proposition 2.2.5. For $x > 0$, and functions R, X as defined in Definition 2.1.6,

$$(R - X)(x) \cdot -F'(x) \leq \ln 2 \quad \text{and} \quad (R - X)(x) \to \frac{\ln 2}{-F'(x)} \quad \text{when} \quad x \to 0.$$

Also, $R - X$ is increasing, so R is increasing.

Proof. Recall that the unique geodesic turning around at x is labelled $f(x)$, so compute

$$\begin{aligned}
(R - X)(x) &= \int_0^x \frac{f(x) - \sqrt{f(x)^2 - f(u)^2}}{\sqrt{f(x)^2 - f(u)^2}} \, \mathrm{d}u \\
&= \int_0^x \frac{1 - \sqrt{1 - f(u)^2/f(x)^2}}{\sqrt{1 - f(u)^2/f(x)^2}} \, \mathrm{d}u \\
&= \int_0^x \frac{f(u)^2/f(x)^2}{(1 + \sqrt{1 - f(u)^2/f(x)^2})\sqrt{1 - f(u)^2/f(x)^2}} \, \mathrm{d}u \\
&= \int_0^x \frac{f(u)^2/f(x)^2}{(f(u)^2/f(x)^2)'} \cdot -\frac{\mathrm{d}}{\mathrm{d}u} 2\ln(1 + \sqrt{1 - f(u)^2/f(x)^2}) \, \mathrm{d}u \\
&= \int_0^x \frac{1}{-F'(u)} \cdot -\frac{\mathrm{d}}{\mathrm{d}u} \ln(1 + \sqrt{1 - f(u)^2/f(x)^2}) \, \mathrm{d}u.
\end{aligned}$$

An integration by parts gives

$$\begin{aligned}
(R - X)(x) = & \left.\frac{1}{-F'(u)} \cdot - \ln(1 + \sqrt{1 - f(u)^2/f(x)^2})\right|_0^x \\
& + \int_0^x \left(\frac{1}{-F'(u)}\right)' \cdot \ln(1 + \sqrt{1 - f(u)^2/f(x)^2}) \, \mathrm{d}u \\
= & \int_0^x \left(\frac{1}{-F'(u)}\right)' \cdot \ln(1 + \sqrt{1 - f(u)^2/f(x)^2}) \, \mathrm{d}u.
\end{aligned}$$

Observe that $R - X$ is increasing.

To show $\lim_{x\to 0}(R - X)(x) \cdot -F'(x) = \ln 2$, it is sufficient to show that for any $\varepsilon > 0$, when $x \to 0$ it holds

$$(R - X)(x) \leq \frac{\ln 2}{-F'(x)} \leq (1 + \varepsilon)(R - X)(x).$$

Since $\ln(1 + \sqrt{1 - f(u)^2/f(x)^2}) \leq \ln 2$, immediately

$$(R - X)(x) \leq \frac{\ln 2}{-F'(x)}.$$

To show the other inequality, use the contrapositive of Corollary 2.2.2. Fix $\varepsilon > 0$, and $\alpha(x)$ dependent on x and ε such that $F'(x) = (1 + \varepsilon)F'(\alpha(x))$. It is sufficient to show that

$$(R - X)(x) > (1 - \varepsilon)\frac{\ln 2}{-F'(x)},$$

or, equivalently, show that

$$\frac{\ln 2}{-F'(x)} - (R-X)(x) < C\varepsilon\frac{\ln 2}{-F'(x)},$$

where $C > 0$ is some constant.

Notice that

$$\frac{\ln 2}{-F'(x)} - (R-X)(x) = \int_0^x \left(\frac{1}{-F'(u)}\right)' \cdot (\ln 2 - \ln(1 + \sqrt{1 - f(u)^2/f(x)^2}))\,\mathrm{d}u$$

Split the integral into two parts: evaluated from 0 to $\alpha(x)$ and from $\alpha(x)$ to x. Compute that for the first part the difference is

$$\begin{aligned}
&\int_0^{\alpha(x)} \left(\frac{1}{-F'(u)}\right)' \cdot (\ln 2 - \ln(1 + \sqrt{1 - f(u)^2/f(x)^2})\,\mathrm{d}u. \\
&\le \int_0^{\alpha(x)} \left(\frac{1}{-F'(u)}\right)' \cdot (\ln 2 - \ln(1 + \sqrt{1 - f(\alpha)^2/f(x)^2})\,\mathrm{d}u \\
&= (\ln 2 - \ln(1 + \sqrt{1 - f(\alpha)^2/f(x)^2})F'(\alpha(x)).
\end{aligned}$$

But by construction of α, one can use Assumption 2.2.1 to deduce that $\lim_{x\to 0} \frac{f(\alpha(x))}{f(x)} \to 0$. Therefore the difference could be less than $\varepsilon \cdot \frac{1}{-F'(x)}$.

The difference in the second part is

$$\int_{\alpha(x)}^x \left(\frac{1}{-F'(u)}\right)' \cdot (\ln 2 - \ln(1 + \sqrt{1 - f(u)^2/f(x)^2})\,\mathrm{d}u) < \ln 2\left(\frac{1}{-F'(x)} - \frac{1}{-F'(\alpha(x))}\right).$$

Here one use the construction to finish the proof:

$$\frac{-F'(x)}{-F'(\alpha(x))} = 1 + \varepsilon \quad\Rightarrow\quad 1 - \frac{-F'(x)}{-F'(\alpha(x))} = \varepsilon \quad\Rightarrow\quad \frac{1}{-F'(x)} - \frac{1}{-F'(\alpha(x))} = \varepsilon\frac{1}{-F'(x)}.$$

□

Remark. In the following passages, I will use the notation $a \to b$ when $c \to 0$ to mean

$$\lim_{c\to 0}\frac{a}{b} = 1.$$

For example, I have just shown when $x \to 0$,

$$(R-X)(x) \to \frac{\ln 2}{-F'(x)}.$$

Also notice that in fact both sides of $\to$ go to zero. One trick that I will repeatedly use later is that by multiplying appropriate terms on both sides, for example, $-F'(r)$ in this case, both sides will converge to a non-zero value. So when $x \to 0$,

$$-F'(x) \cdot (R-X)(x) \to \ln 2.$$

Proposition 2.2.6. When $x \to 0$.

$$Y(x) \to \frac{f(x)}{-F'(x)}.$$

Proof. Compute that

$$\begin{aligned} Y(x) = \int_0^x \frac{f(u)^2}{\sqrt{f(x)^2 - f(u)^2}} \,\mathrm{d}u &= f(x) \int_0^x \frac{f(u)^2/f(x)^2}{\sqrt{1 - f(u)^2/f(x)^2}} \,\mathrm{d}u \\ &= f(x) \int_0^x \frac{1}{-2F'(u)} \cdot \frac{(f(u)^2/f(x)^2)'}{\sqrt{1 - f(u)^2/f(x)^2}} \,\mathrm{d}u \\ &= f(x) \int_0^x \left(\frac{1}{-F'(u)}\right)' \cdot \sqrt{1 - f(u)^2/f(x)^2} \,\mathrm{d}u. \end{aligned}$$

The error estimation follows the same logic as the proof of Proposition 2.2.5. □

Corollary 2.2.7. Consider $B(r)$, since $R-X$ is increasing - so R is strictly increasing - there is a unique geodesic $f(r^*)$ such that its intersection with the boundary of the metric ball is its turning point. It holds when $x \to 0$,

$$r - r^* \to \frac{\ln 2}{-F'(r)}$$

and

$$\frac{f(r^*)}{f(r)} \to \frac{1}{2}.$$

Proof. Fix $r > 0$. The trick is to notice that if $f(r^*)$ is the geodesic that crosses the boundary of B_r at the point with $x-$coordinate r^*, then by definition of the metric ball the distance the geodesic travelled up to the turning point is r, that is $R(r^*) = r$. Thus,

$$r - r^* = (R - X)(r^*).$$

Then by Proposition 2.2.5,

$$\lim_{r \to 0} (r - r^*) \cdot -F'(r^*) = \ln 2.$$

By Assumption 2.2.1, this implies

$$\lim_{r \to 0} \frac{-F'(r^*)}{-F'(r)} = 1$$

so indeed

$$\lim_{r \to 0} (r - r^*) \cdot -F'(r) = \ln 2.$$

Then, by Corollary 2.2.3,

$$\lim_{r \to 0} \frac{f(r^*)}{f(r)} = \frac{1}{2}.$$

□

Definition 2.2.8. Split the metric ball B_r vertically. Call the area left of r^* as *Area1* of the metric ball, and the area right of r^* as *Area2*. [4]

Here's a proposition listed as Proposition 47 (1) in Korobenko et al. (2021).

Proposition 2.2.9. Let $\varphi(x)$ denote the y-coordinate of the boundary of B_r at x. Then $\varphi(x)$ is well defined. Further, $\varphi(x)$ is decreasing when $x > r^*$.

Proof. Assume $x > 0$. As a direct corollary to $R(x)$ is increasing, concluded in Proposition 2.2.5, $\varphi(x)$ is well-defined in Area 2.

To check well-definedness in Area 1, for $\lambda_1 < \lambda_2$, check that $\frac{\mathrm{d}t}{\mathrm{d}x}(\lambda_1) > \frac{\mathrm{d}t}{\mathrm{d}x}(\lambda_2)$ at every x to the left of the turning point of λ_1. Then, if φ is not well defined, then there are two situations.

1. Geodesic λ_1 has turned back but λ_2 did not. This is impossible since $\frac{\mathrm{d}t}{\mathrm{d}x}(\lambda_1) > \frac{\mathrm{d}t}{\mathrm{d}x}(\lambda_2)$, the distance covered by λ_2 is strictly smaller.

2. Both geodesics have turned back. This is impossible. Since $\frac{\mathrm{d}t}{\mathrm{d}x}(\lambda_1) > \frac{\mathrm{d}t}{\mathrm{d}x}(\lambda_2)$, the distance not covered by λ_1 until the geodesic crosses the y-axis is strictly bigger than the distance not covered by λ_2. By symmetry of $\frac{\mathrm{d}t}{\mathrm{d}x}$ before and after turning back, this would imply that $2R_{\lambda_1} > 2R_{\lambda_2}$, a contradiction to Proposition 2.2.5.

At position $x \geq r^*$, let $\lambda(x)$ label the unique geodesic that reach distance r at x without turning back, i.e.,

$$r = \int_0^x \frac{\mathrm{d}t}{\mathrm{d}u}(\lambda(x))\,\mathrm{d}u = \int_0^x \frac{\lambda(x)}{\sqrt{\lambda(x)^2 - f(u)^2}}\,\mathrm{d}u.$$

Then by construction

$$\varphi(x) = \int_0^x \frac{\mathrm{d}y}{\mathrm{d}u}(\lambda(x))\,\mathrm{d}u = \int_0^x \frac{f(u)^2}{\sqrt{\lambda(x)^2 - f(u)^2}}\,\mathrm{d}u.$$

Differentiate both expressions above with respect to x to obtain

$$0 = \frac{\lambda(x)}{\sqrt{\lambda(x)^2 - f(x)^2}} - \left(\int_0^x \frac{f(u)^2}{(\lambda(x)^2 - f(u)^2)^{3/2}}\,\mathrm{d}u\right)\lambda'(x)$$

and

$$\varphi'(x) = \frac{f(x)^2}{\sqrt{\lambda(x)^2 - f(x)^2}} - \left(\int_0^x \frac{f(u)^2}{(\lambda(x)^2 - f(u)^2)^{3/2}}\,\mathrm{d}u\right)\lambda(x)\lambda'(x)$$

Combining equalities yields

$$\varphi'(x) = \frac{f(x)^2}{\sqrt{\lambda(x)^2 - f(x)^2}} - \lambda(x)\frac{\lambda(x)}{\sqrt{\lambda(x)^2 - f(x)^2}} = -\sqrt{\lambda(x)^2 - f(x)^2}.$$

□

[4] However, recall that B_r is only the first quadrant of the actual metric ball with radius r. By symmetry, the area of the actual metric ball is four times $Area(B_r)$.

Proposition 2.2.10. Let $x < r^*$. If the geodesic λ crosses the boundary of B_r at x, I will label $\lambda = f(x + \Delta x)$ for $x + \Delta x < r^*$ be the turning point of the geodesic. Then, for every $\varepsilon > 0$, the following inequality holds for r is sufficiently small:

$$-F'(x+\Delta x)(x+\Delta x)+\ln 2 \geq -F'(x+\Delta x)\cdot\frac{r+x}{2} \geq -F'(x+\Delta x)(x+\Delta x)+\frac{1}{2}\ln 2-\varepsilon.$$

See figure 2.3 for reference.[5]

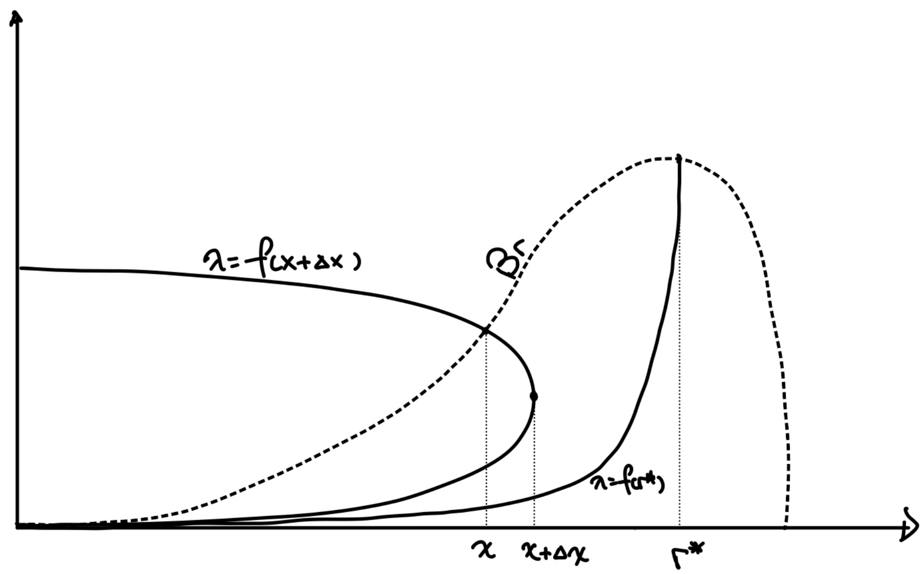


Figure 2.3: Demo 1.

Proof. Along the geodesic $f(x + \Delta x)$, define the function

$$(T_{f(x+\Delta x)} - X)(x) := \int_x^0 -\frac{\mathrm{d}t}{\mathrm{d}u}(f(x+\Delta x))\,\mathrm{d}u - x = \int_0^x \frac{\mathrm{d}t}{\mathrm{d}u}(f(x+\Delta x))\,\mathrm{d}u - x.$$

Then $(T_{f(x+\Delta x)} - X)(x)$ measures the magnitude of distance minus the $x-$coordinate traversed along the segment of the geodesic $f(x + \Delta x)$ from its intersection with the boundary of B_r to its intersection with the $y-$axis.[6] One can estimate that

$$0 \leq (T_{f(x+\Delta x)} - X)(x) \leq (R - X)(x + \Delta x).$$

[5] For small r, this proposition gives a fairly well estimate for the turning point of the unique geodesic that reaches distance r when intersecting the $y-$axis. Since $x >> \frac{1}{-F'(x)}$, the turning point would be roughly $r/2$.

[6] The main difference between T and R is that when specifying T, I have to give two arguments: the geodesic along which the measure is done, and the endpoint of the measure (recall that the measure starts from zero). However, when specifying R, I only give one argument, and that is the turning point of the geodesic. It is assumed that the measure is obtained from 0 up to the turning point.

Notice that by construction

$$r + x = 2R(x + \Delta x) - (T_{f(x+\Delta x)} - X)(x).$$

Thus, the following inequalities are obtained:

$$2R(x + \Delta x) \geq r + x \geq 2R(x + \Delta x) - (R - X)(x + \Delta x).$$

The inequalities above is equivalent to

$$2(R-X)(x+\Delta x) \geq r+x-2(x+\Delta x) \geq 2(R-X)(x+\Delta x)-(R-X)(x+\Delta x) = (R-X)(x+\Delta x).$$

When r is sufficiently small, and thus x is also sufficiently small, using Proposition 2.2.5 this implies

$$-F'(x+\Delta x)(x+\Delta x)+\ln 2 \geq -F'(x+\Delta x)\cdot\frac{r+x}{2} \geq -F'(x+\Delta x)(x+\Delta x)+\frac{1}{2}\ln 2-\varepsilon$$

for arbitrarily small $\varepsilon > 0$. So the statement in the proposition holds. $\square$

Corollary 2.2.11. For variable $\omega \geq 0$, define $x(\omega) = r^* - \frac{\omega}{-F'(r)}$. Then

$$\lim_{r\to 0} \frac{\Delta x(\omega)}{r^* - x(\omega)} \leq \frac{1}{2}.$$

Proof. Consider the inequality

$$-F'(x + \Delta x) \cdot \frac{r + x}{2} \geq -F'(x + \Delta x)(x + \Delta x) + \frac{1}{2}\ln 2$$

is equivalent to

$$-F'(x + \Delta x) \cdot (r - x) - \ln 2 \geq -2F'(x + \Delta x)\Delta x.$$

Notice that by Assumption 2.2.1 and Corollary 2.2.7,

$$\lim_{r\to 0} -F'(x + \Delta x) \cdot (r - x) = \lim_{r\to 0} \frac{-F'(x + \Delta x)}{-F'(r)} \cdot \lim_{r\to 0} -F'(r) \cdot (r - x) = \omega + \ln 2 > 0$$

so

$$\lim_{r\to 0} -F'(x + \Delta x) \cdot (r - x) - \ln 2 = \omega > 0.$$

Then, substitute this into the inequality above, cancel $-F'(x + \Delta x)$ on both sides and obtain

$$\lim_{r\to 0} \frac{\Delta x(\omega)}{r^* - x(\omega)} \leq \frac{1}{2}.$$

$\square$

Corollary 2.2.12.

$$\frac{F'(r)^2}{f(r)} Area1(B_r) \leq 8\ln 2.$$

Proof. By Assumption 2.2.1, for every $\varepsilon > 0$ there is $r > 0$ such that whenever $x < r$,

$$\frac{-F'(x - \frac{\ln 2}{-F'(x)})}{-F'(x)} > 1 - \varepsilon.$$

Repeating the idea in the proof of Corollary 2.2.2, the previous proposition allows the estimation [7]

$$F(x+\Delta x) - F\left(\frac{r+x}{2}\right) = \int_{x+\Delta x}^{\frac{r+x}{2}} -F'(u)\,\mathrm{d}u \geq \frac{\ln 2}{-2F'(x+\Delta x)} \cdot -F'\left(\frac{r+x}{2}\right) > (1-\varepsilon)\ln 2.$$

Thus

$$\frac{f(x+\Delta x)}{f\left(\frac{r+x}{2}\right)} = e^{-\left(F(x+\Delta x) - F\left(\frac{r+x}{2}\right)\right)} < c$$

where $c = 2^{-(1-\varepsilon)}$.

Using $Y(x)$ defined in Definition 2.1.6, an upper bound of $Area1(B_r)$ is

$$Area1(B_r) \leq \int_0^{r^*} 2Y(x+\Delta x)\,\mathrm{d}x$$

By Proposition 2.1.8 for the first inequality and the monotonicity of $R - X$ for the second, one can estimate

$$Y(x+\Delta x) \leq 2f(x+\Delta x)(R-X)(x+\Delta x) \leq 2cf\left(\frac{r+x}{2}\right)(R-X)\left(\frac{r+x}{2}\right).$$

So by a change of variable for the second equality, Proposition 2.2.5 for the third inequality, and recalling definition of $f(x) = e^{-F(x)}$ and the monotonicity of $\frac{1}{-F'(x)}$ for the fourth inequality

$$\begin{aligned}
Area1(B_r) &\leq 4c\int_0^{r^*} f\left(\frac{r+x}{2}\right)(R-X)\left(\frac{r+x}{2}\right)\mathrm{d}x \\
&= 8c\int_{\frac{r}{2}}^{\frac{r+r^*}{2}} f(u)(R-X)(u)\,\mathrm{d}u \\
&\leq 8c \cdot \ln 2 \int_0^r \frac{f(u)}{-F'(u)}\,\mathrm{d}u \\
&\leq c_1 \cdot \frac{1}{F'(r)^2}\int_0^r f'(u)\,\mathrm{d}u \\
&= c_1 \cdot \frac{f(r)}{F'(r)^2}
\end{aligned}$$

where $c_1 = \frac{8\ln 2}{2^{(1-\varepsilon)}} \leq 8\ln 2$ for all r.

□

[7] Note that by the previous proposition, $\frac{r+x}{2} > x + \Delta x$, so $F(x+\Delta x) > F(\frac{r+x}{2})$. Also recall $-F'(u)$ is positive and decreasing by Assumption 2.2.1.

Proposition 2.2.13. For variable $\omega \geq 0$ and for very small r, define $x(\omega) = r^* - \frac{\omega}{-F'(r)}$. Presume this denote the $x-$coordinate of the intersection point of a unique geodesic and the boundary of B_r, then denote the turning point of this geodesic be $(x + \Delta x)(\omega)$, i.e, the geodesic is labeled by $f((x + \Delta x)(\omega))$. Then,

$$-F'(r) \cdot \Delta x(\omega) \to \frac{1}{2}\omega - \frac{1}{2}\ln(2 - e^{-w}).$$

Proof. Assume $\omega \neq 0$. The geodesic $f((x + \Delta x)(\omega))$ has traveled distance $(R - X)((x + \Delta x)(\omega)) + (x + \Delta x)(\omega)$ up to the turning point, and distance r up to $x(\omega)$.[8] Then one can obtain a description of $\Delta x(\omega)$ be

$$r - (R - X)((x + \Delta x)(\omega)) - (x + \Delta x)(\omega) = \int_{x+\Delta x}^{x} -\frac{\mathrm{d}t}{\mathrm{d}u}(f((x + \Delta x)(\omega)))\,\mathrm{d}u.$$

Recall $x(\omega) = r^* - \frac{\omega}{-F'(r)}$, so by Assumption 2.2.1,[9] and recall $r - r^* = (R - X)(r)$ from Corollary 2.2.7. Multiply both side of the fraction by $-F'(r)$, the LHS (left hand side) can be approximated by

$$\begin{aligned}
&\lim_{r\to 0} \frac{-F'(r)(r - (R - X)((x + \Delta x)(\omega)) - (x + \Delta x)(\omega))}{-F'(r)(r^* - (x + \Delta x)(\omega))}\\
=&\lim_{r\to 0} \frac{-F'(r)(r^* - x(\omega)) - (-F'(r))\Delta x(\omega) + (-F'(r))((r - r^*) - (R - X)((x + \Delta x)(\omega))}{-F'(r)(r^* - x(\omega)) - (-F'(r))\Delta x(\omega)}\\
=&1 - \lim_{r\to 0} \frac{-F'(r)((R - X)(r) - (R - X)((x + \Delta x)(\omega)))}{\omega - (-F'(r))\Delta x(\omega)}.
\end{aligned}$$

The denominator is positive, since, by Corollary 2.2.11, $\lim_{r\to 0} \omega - (-F'(r))\Delta x(\omega) \geq \frac{1}{2}\omega > 0$. The numerator vanishes by Assumption 2.2.1 and Proposition 2.2.5. Therefore one obtain

$$\lim_{r\to 0} \frac{-F'(r)(r - (R - X)((x + \Delta x)(\omega)) - (x + \Delta x)(\omega))}{-F'(r)(r^* - (x + \Delta x)(\omega))} = 1.$$

Rearranging the previous two identities and times $-F'(r)$ on both the denominator and the numerator so they don't converge to zero, obtain

$$1 = \lim_{r\to 0} \frac{-F'(r)\int_x^{(x+\Delta x)(\omega)} \frac{1}{\sqrt{1 - f(u)^2/f(x)^2}}\,\mathrm{d}u}{-F'(r) \cdot (r^* - (x + \Delta x)(\omega))} = \lim_{r\to 0} \frac{-F'(r)\int_x^{(x+\Delta x)(\omega)} \frac{1}{\sqrt{1 - f(u)^2/f(x)^2}}\,\mathrm{d}u}{\omega - (-F'(r)) \cdot \Delta x(w)}.$$

Equivalently, by positivity, the following equality holds non-trivially:

$$\lim_{r\to 0} \omega - (-F'(r))\Delta x(w) = \lim_{r\to 0} -F'(r) \cdot \int_x^{(x+\Delta x)(\omega)} \frac{1}{\sqrt{1 - f(u)^2/f(x)^2}}\,\mathrm{d}u.$$

[8] That is, including entire the segment before the turning point, and the segment after the turning point up to $x(\omega)$

[9] It's important to note that when taking the limit $r \to 0$, w remains constant.

Now, again by Assumption 2.2.1, $\lim_{r\to 0}\frac{-F'(u)}{-F'(r)} = 1$ for all u such that $x(\omega) \le u \le (x+\Delta x)(\omega)$, so one can approximate the RHS via a change of variable

$$\begin{aligned}
&\lim_{r\to 0} -F'(r)\cdot \int_x^{(x+\Delta x)(\omega)} \frac{1}{\sqrt{1-f(u)^2/f(x)^2}}\,\mathrm{d}u \\
=&\lim_{r\to 0} -F'(r)\cdot \int_x^{(x+\Delta x)(\omega)} \frac{1}{-F'(u)}\cdot\frac{1}{\frac{f(u)}{f(x)}\sqrt{1-(\frac{f(u)}{f(x)})^2}}\,\mathrm{d}\frac{f(u)}{f(x)} \\
=&\lim_{r\to 0} \int_x^{(x+\Delta x)(\omega)} \frac{1}{\frac{f(u)}{f(x)}\sqrt{1-(\frac{f(u)}{f(x)})^2}}\,\mathrm{d}\frac{f(u)}{f(x)} \\
=&\lim_{r\to 0} \operatorname{arctanh}\sqrt{1-\left(\frac{f(x(\omega))}{f((x+\Delta x)(\omega))}\right)^2}.
\end{aligned}$$

Since both sides doesn't converge to zero, I can further apply tanh on both sides and rearrange to obtain

$$\lim_{r\to 0}\frac{f(x(\omega))}{f((x+\Delta(x))(\omega))} = \lim_{r\to 0}\sqrt{1-\tanh^2(\omega-(-F'(r))\Delta x(w))} = \lim_{r\to 0}\operatorname{sech}(\omega-(-F'(r))\Delta x(w)).$$

By Corollary 2.2.3 the following identity holds:

$$\lim_{r\to 0}\frac{f(x(\omega))}{f((x+\Delta x)(\omega))} = \lim_{r\to 0} e^{-(-F'(r))\Delta x(w)}.$$

Then, using positivity of $\lim_{r\to 0}\operatorname{sech}(\omega-(-F'(r))\Delta x(w))$, one can interchange a few more limits and ends up with the solution

$$\lim_{r\to 0} -F'(r)\cdot\Delta x(w) = \frac{1}{2}\ln\frac{e^{\omega}}{2-e^{-\omega}} = \frac{1}{2}\omega - \frac{1}{2}\ln(2-e^{-w}).$$

$\square$

Corollary 2.2.14. Here are some corollaries (see figure 2.4 for reference):

1. $$f((x+\Delta x)(\omega)) \to \frac{f(r)}{2\sqrt{2e^{\omega}-1}}.$$

2. With abuse of notation, let $Y(\omega) = Y((x+\Delta x)(\omega))$, where the latter is defined in Definition 2.1.6 and computed in Proposition 2.2.6, then as $r\to 0$,
$$Y(\omega) \to \frac{f(r)}{2\sqrt{2e^{\omega}-1}\cdot -F'(r)}$$

3. Let Δh denote the height that the geodesic $f((x+\Delta x)(\omega))$ continued to gain after turning back and until reaching the boundary of B_r at $x(\omega)$. Then
$$\Delta h(\omega) \to \frac{f(r)}{2\sqrt{2e^{\omega}-1}\cdot -F'(r)}\tanh(\frac{\omega}{2}+\ln(2-e^{-w})).$$

Proof. (1). By Corollary 2.2.3 and Proposition 2.2.13, compute

$$\lim_{r\to 0}\frac{f((x+\Delta x)(\omega))}{f(x(\omega))}=\lim_{r\to 0}e^{-F'(r)\cdot\Delta x(\omega)}=\sqrt{\frac{e^{\omega}}{2-e^{-\omega}}}=\frac{e^{\frac{\omega}{2}}}{\sqrt{2-e^{-\omega}}}.$$

By Corollary 2.2.3 and $x(\omega)=r^*-\frac{\omega}{-F'(r)}$, compute

$$\lim_{r\to 0}\frac{f(x(\omega))}{f(r^*)}=e^{-\omega}.$$

Now, by Corollary 2.2.7,

$$\lim_{r\to 0}\frac{f((x+\Delta x)(\omega))}{f(r)}=\lim_{r\to 0}\frac{f((x+\Delta x)(\omega))}{f(x(\omega))}\cdot\lim_{r\to 0}\frac{f(x(\omega))}{f(r^*)}\cdot\lim_{r\to 0}\frac{f(r^*)}{f(r)}=\frac{1}{2\sqrt{2e^{\omega}-1}}.$$

(2) follows Proposition 2.2.6 straightforwardly.

For (3), since Δh is the height that the geodesic $f(x)$ continued to gain after turning back and until reaching the boundary of B_r,

$$\Delta h(\omega)=f(x)\int_x^{x+\Delta x}\frac{\frac{f(u)^2}{f(x)^2}}{\sqrt{1-\frac{f(u)^2}{f(x)^2}}}\,\mathrm{d}u,$$

So compute

$$\begin{aligned}\lim_{r\to 0}\Delta h(\omega)\cdot\frac{-F'(x)}{f(x)}&=\lim_{r\to 0}\int_x^{x+\Delta x}\frac{\frac{f(u)}{f(x)}}{\sqrt{1-\frac{f(u)^2}{f(x)^2}}}\,\mathrm{d}\frac{f(u)}{f(x)}\\&=\lim_{r\to 0}\sqrt{1-\frac{f(x(\omega))^2}{f((x+\Delta x)(\omega))^2}}\\&=\lim_{r\to 0}\tanh(\omega-(-F'(r))\cdot\Delta x(\omega))\\&=\tanh(\frac{\omega}{2}+\frac{1}{2}\ln(2-e^{-w})).\end{aligned}$$

Use positivity and Assumption 2.2.1 to swap $-F'(x)$ for $-F'(r)$ on the LHS, and use positivity and Proposition 2.2.13 to obtain the last equality. $\square$

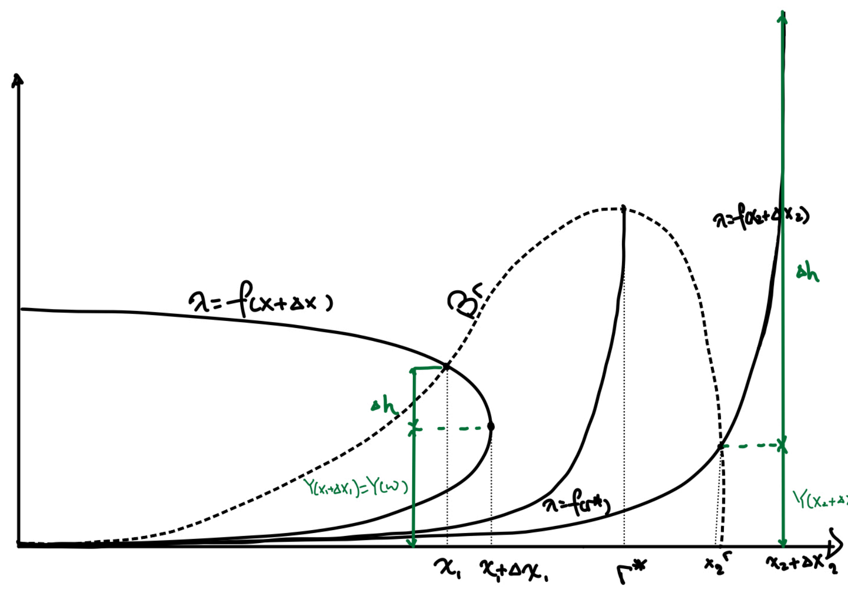


Figure 2.4: Demo 2.

Corollary 2.2.15. For $\omega \in (-\infty, \ln 2)$, define $x(\omega) = r^* + \frac{\omega}{-F'(r)}$. Then as $r \to 0$

$$-F'(r) \cdot \Delta x(\omega) \to -\frac{1}{2}\omega - \frac{1}{2}\ln(2 - e^{\omega}) \tag{1}$$

and, as $r \to 0$, the boundary of B_r is

$$\varphi(\omega) = (Y + \Delta h)(\omega) \to \frac{f(r)}{-F'(r) \cdot 2\sqrt{2e^{-\omega} - 1}}(1 - \tanh(\frac{\omega}{2} - \frac{1}{2}\ln(2 - e^{w}))). \tag{2}$$

Proof. Because I have already shown Proposition 2.2.13 and Corollary 2.2.14, it is sufficient to consider the case when $\omega > 0$. By analogy, there is identity

$$(R - X)((x + \Delta x)(\omega)) + (x + \Delta x)(\omega) - r = \int_{x+\Delta x}^{x} -\frac{\mathrm{d}t}{\mathrm{d}u}(f((x + \Delta x)(\omega)))\,\mathrm{d}u$$

where the LHS can be approximated by [10]

$$\lim_{r \to 0} \frac{(R - X)((x + \Delta x)(\omega)) + (x + \Delta x)(\omega) - r}{(x + \Delta x)(\omega) - r^*} = 1.$$

Then, as in Proposition 2.1.13, solve for

$$\lim_{r \to 0} \omega + (-F'(r))\Delta x(\omega) = \lim_{r \to 0} \operatorname{arctanh} \sqrt{1 - \left(\frac{f(x(\omega))}{f((x + \Delta x)(\omega))}\right)^2}$$

[10] Notice that I do not need an analogue for Corollary 2.2.11 for the positive ω when showing positivity by method similar to the one used in the proof of Proposition 2.2.13.

to obtain (1). Compute

$$\frac{f((x+\Delta x)(\omega))}{f(r)} = \frac{f((x+\Delta x)(\omega))}{f(x(\omega))} \cdot \frac{f(x(\omega))}{f(r^*)} \cdot \frac{f(r^*)}{f(r)} \to \frac{1}{2\sqrt{2e^{-\omega}-1}},$$

and notice that when $\omega > 0$, Δh is negative, and use that tanh is odd to obtain (2). □

See section 2.3 for what the ball looks like.

Corollary 2.2.16. Define the variable s by

$$s = -F'(r) \cdot ((x+\Delta x)(\omega) - r^*)$$

for $\omega \in (-\infty, \ln 2)$, and denote $f(s)$ be the label of the corresponding geodesic. With abuse of notation, also let $\varphi(s)$ be the height of which the geodesic $f(s)$ reach distance r. Then as $r \to 0$,

$$\varphi(s) \to \frac{1}{2}\operatorname{sech}(s) \cdot \frac{f(r)}{-F'(r)}.$$

Proof. First, compute

$$\begin{aligned}
\lim_{r\to 0} s &= \lim_{r\to 0} -F'(r) \cdot ((x+\Delta x)(\omega) - r^*) \\
&= -F'(r)\left(\frac{\omega}{-F'(r)} - \frac{\frac{1}{2}\omega + \frac{1}{2}\ln(2-e^{w})}{-F'(r)}\right) \\
&= \frac{\omega}{2} - \frac{1}{2}\ln(2-e^{\omega})
\end{aligned}$$

Then, for $\omega \in (0, \ln 2)$, the second equality follows the identity (they are the same function when expanded out)

$$\begin{aligned}
\lim_{r\to 0} \frac{-F'(r)}{f(r)}\varphi(s) &= \frac{1}{\sqrt{2e^{-\omega}-1}}\left(1+\tanh\left(\omega + \frac{1}{2}\ln\frac{e^{-2\omega}}{2e^{-\omega}-1}\right)\right) \\
&= \operatorname{sech}\left(\frac{\omega}{2} - \frac{1}{2}\ln(2-e^{w})\right) \\
&= \operatorname{sech}(\lim_{r\to 0} s) \\
&= \lim_{r\to 0} \operatorname{sech}(s).
\end{aligned}$$

□

Theorem 2.2.17. *As $r \to 0$,*

$$Area(B_r) \to \frac{\pi}{2} \cdot \frac{f(r)}{F'(r)^2}.$$

Proof. For each k, assign radius of the metric ball considered to be $r = r_k$. Assume that when $k \to \infty$, balls with smaller and smaller radius is measured, i.e., $r = r_k \to 0$. Let $a_k = r_k^* - \frac{k}{-F'(r_k)}$ and $b_k = r_k^* + \frac{\ln 2 - 1/(2k)}{-F'(r_k)}$. For each k in $\{k_i\} \to \infty$, associate a lower bound of $Area(B_{r_k})$ to be the area of the metric ball between a_k and b_k, that is,

$$LB_k = \int_{a_k}^{b_k} (Y(\omega) + \Delta h(\omega))\, \mathrm{d}x.$$

By a change of variable, note that $-F'(r_k)\, \mathrm{d}x = \mathrm{d}\omega$, obtain

$$-F'(r_k) \cdot LB_k = \int_{-k}^{k} (Y(\omega) + \Delta h(\omega))\, \mathrm{d}w.$$

Equivalently,

$$\frac{F'(r_k)^2}{f(r_k)} \cdot LB_k = \frac{-F'(r_k)}{f(r_k)} \left(\int_{-k}^{0} (Y(\omega) + \Delta h(\omega))\, \mathrm{d}w + \int_{0}^{k} (Y(\omega) + \Delta h(\omega))\, \mathrm{d}w \right).$$

Then LB_k measures the area of $Area1(B_{r_k})$ to the right of a_k plus the area of $Area2(B_{r_k})$ to the left of b_k, and omit the area to the left of a_k and to the right of b_k. Presume that I can apply dominated convergence theorem to the RHS to obtain, then, via Corollary 2.2.16,

$$\begin{aligned}
\lim_{k\to\infty} \frac{F'(r_k)^2}{f(r_k)} \cdot LB_k &= \frac{-F'(r_k)}{f(r_k)} \int_0^\infty \lim_{k\to\infty} (Y(\omega) + \Delta h(\omega))\, \mathrm{d}w \\
&= \int_{-\infty}^{\ln 2} \frac{1}{2\sqrt{2e^{-\omega} - 1}} (1 - \tanh(\frac{\omega}{2} - \frac{1}{2}\ln(2 - e^{w})))\, \mathrm{d}\omega \\
&= \frac{\pi}{2}.
\end{aligned}$$

I will now justify how the dominated convergence theorem could be applied. But the work is already done: an upper bound of $Area1(B_{r_k}) \cdot F'(r_k)^2/f(r_k)$ is shown in Corollary 2.2.12, and a simple rectangular upper bound (see figure 2.5) for $Area2(B_{r_k}) \cdot F'(r_k)^2/f(r_k)$ would follow Proposition 2.2.9 and Corollary 2.2.7.

It remains to show that this lower bound will accurately approximate the actual area of the ball when $k \to \infty$. I will show this separately for $Area1(B_{r_k})$ and $Area2(B_{r_k})$, so the error term concerned are $omitted\_Area1(B_{r_k}) \cdot F'(r_k)^2/f(r_k)$ and $omitted\_Area2(B_{r_k}) \cdot F'(r_k)^2/f(r_k)$. For $Area1(B_{r_k})$, for each k, only the area to the left of a_k is omitted. Then, using the same argument as in the proof of Corollary 2.2.12, one can estimate that the omitted area is less than $8\ln 2 \cdot f\left(\frac{r_k + a_k}{2}\right)/-F'(r_k)^2$. Compute using Proposition 2.2.10 and Corollary 2.2.3,

$$\lim_{k\to\infty} \frac{f\left(\frac{r_k+a_k}{2}\right)}{f(r_k)} = \lim_{k\to\infty} \frac{f\left(\frac{r_k - r_k*}{2} - \frac{k}{-2F'(r_k)}\right)}{f(r_k)} = \sqrt{\ln 2} \cdot e^{-k} = 0,$$

so

$$\lim_{k\to\infty} 8\ln 2 \cdot \frac{f\left(\frac{r_k+a_k}{2}\right)}{-F'(r_k)^2} \cdot \frac{-F'(r_k)^2}{f(r_k)} = 0.$$

For $Area2(B_{r_k})$, the undercounted area is to the right of b_k under the boundary of the metric ball. I will again use the rectangular upper bound (see figure 2.5) to bound the omitted area to the right of b_k. By construction, the width of the rectangle is $r_k - b_k$, and the height of the rectangle is less than $Y(r_k^*)$ because he boundary of B_{r_k}, is decreasing in $Area2$. Thus, by construction of b_k

$$
\begin{aligned}
\lim_{k\to\infty} omitted\_Area2(B_{r_k}) \cdot F'(r_k)^2/f(r_k) \leq & \lim_{k\to\infty} \left((-F'(r) \cdot (r_k - b_k)) \cdot (\frac{-F'(r)}{f(r)} \cdot Y(r_k^*)) \right) \\
= & \frac{1}{2} \lim_{k\to\infty} (-F'(r) \cdot (r_k - b_k)) \\
= & 0.
\end{aligned}
$$

I have shown the result holds when $k \to \infty$ so $r = r \to 0$. This is sufficient to show that the results hold when $r \to 0$. The proof is complete. $\square$

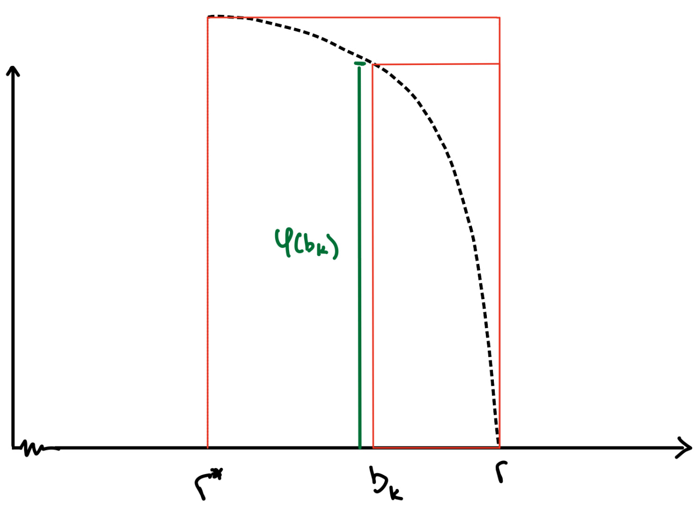


Figure 2.5: The bigger rectangle bounds Area 2, the smaller rectangle bounds the error term.

One should, however, recall that B_r is only the first quadrant of the actual metric ball with radius r. By symmetry, the area of the actual metric ball is four times $Area(B_r)$, i.e., when $r \to 0$, the area is approximately $2\pi \cdot \frac{f(r)}{F'(r)^2}$.

2.3 Gallery

Below is a picture of the "prototype ball" generated using the formula derived in Corollary 2.2.15. The variable of the horizontal axis is ω and the unit length of the vertical axis is $f(r)/F'(r)^2$. The boundary of the ball intersects the horizontal axis at $-\infty$ and $\ln 2$.

```
Plot[1/(2 Sqrt[2 E^(-w) - 1]) (1 - Tanh[w/2 - 1/2 Log[2 - e^w]]), {w, -10, Log[2]}]
```

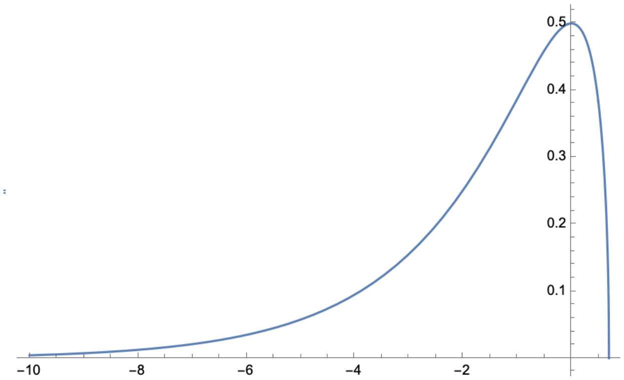


Figure 2.6: Prototype ball.

Using Mathematica, I also generated a series of pictures of how the prototype ball fits with actual balls with radius $r > 0$. I choose $f(x) = e^{-\frac{1}{x^2}}$, and $r^* = 1, .5, .25$ respectively.

Note: I adopted some changes to the original model when making these pictures.

1. In the last section I have defined $x = r^* + \frac{\omega}{-F'(r)}$. For a better fitting model I used instead $x = r^* + \frac{\omega}{-F'(r^*)}$. Because $-F'(r^*) > -F'(r)$, this gives more precision to the model. One can also recall that when $r \to 0$, Assumption 2.2.1 assumes they are asymptotic.

2. Recall that the prototype ball has left bounds at $-\infty$ and right bounds at $\ln 2$ while these conditions can never be achieved when fitting the model to actual balls with positive radius. So I rather adopted smaller upper and lower bounds for ω for each picture and truncated the model to fit the scale of the actual metric ball.

3. The picture is scaled so the unit length in the horizontal direction is r and the unit length in the vertical direction is $\frac{f(r)}{-F'(r^*)}$ (for the actual metric ball). *Blue* line is the actual metric ball and *orange* line is the model. The boundary of the actual metric ball should always reach and stop at 1 on the horizontal axis, but this is not always true because of the loss of precision in numerical computations.

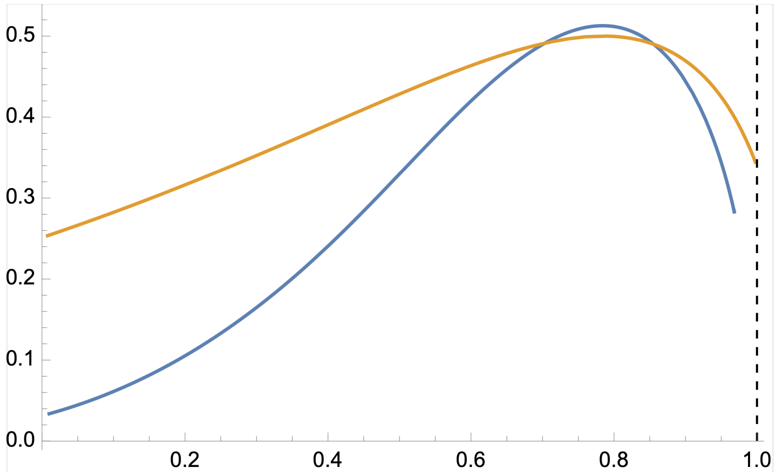


Figure 2.7: $r^* = 1$.

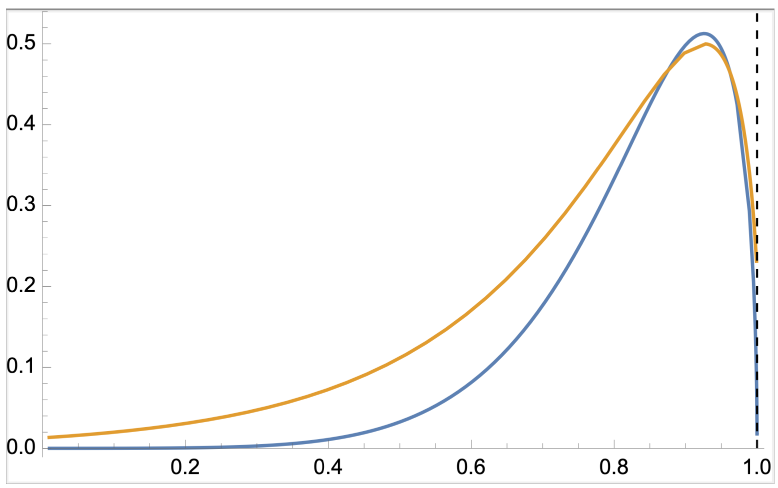


Figure 2.8: $r^* = .5$.

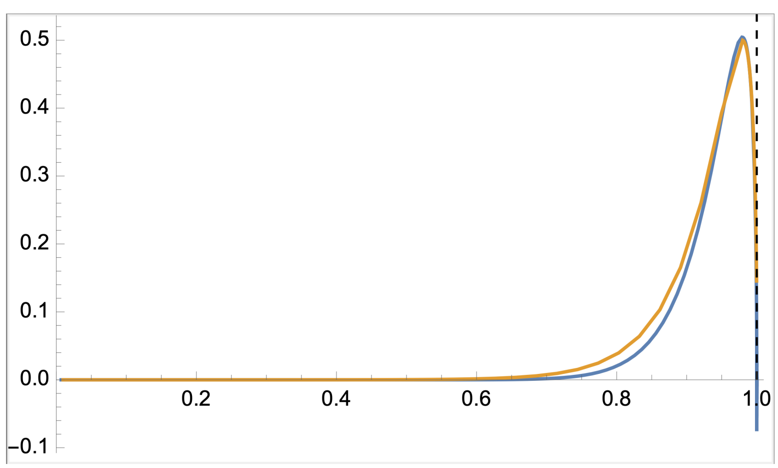


Figure 2.9: $r^* = .25$.

I used NIntegrate and NDSolve in Mathematica by Wolfram Research to obtain the boundary of the actual metric ball. The precisions of the algorithms will be lost if one further decrease r^* with the choice of $f(x) = e^{-\frac{1}{x^2}}$.

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