Traitement de signal
Analyse spectrale : Fourier, temps-fréquence, temps-échelle

Laurent Duval

IFP Energies nouvelles, Rueil-Malmaison, France
laurent.duval@ifpen.fr

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Motivation

- Standard framework
  - diversity and quantity of available data
    - wide audience audio, image and video (main drive); exponentially increasing digital measures from different sensors (industrial, mobile devices)
    - seismic data (petabytes)
  - efficiency of standard (multi-dimensional) signal processing techniques
    - data acquisition and processing generally performed via standard approaches (Nyquist-Shannon sampling, Fourier analysis, parametric modeling, etc.), astonishingly efficient for most applications *if properly used*

Motivation

- Present challenges
  - "Infobesity" : managing huge data volumes (information overload amplified by explosion in precision and sampling of digital data "made" cheaper by sensors and storage)
  - Complexity: achieve higher level tasks; more decision (less user dependent), more imitation; more and more complex, integrated environments, higher constraints
- Approaches
  - compression or sparsification aimed at high-fidelity rendering
  - attribute extraction aimed detection, diagnosis, classification, inference

Motivation

Figure 1: Example of a simple image ...

Motivation

Figure 2: Example of a simple image, yet...

Motivation

Figure 3: Example of an simple image, yet complex

Motivation

- Doxa, common belief: more data $\rightsquigarrow$ more understanding
- Pitfall: less time to analyze
- Recent finding: from the observer point of view, representative or relevant information form a very tiny subset of the set of n-sample, b-bit coded, m-dimensional data
- Thus, data could be faithfully represented (and processed) by a sparse combination of a few relevant coefficients. But which one? And how?
- May (even) depend on data themselves

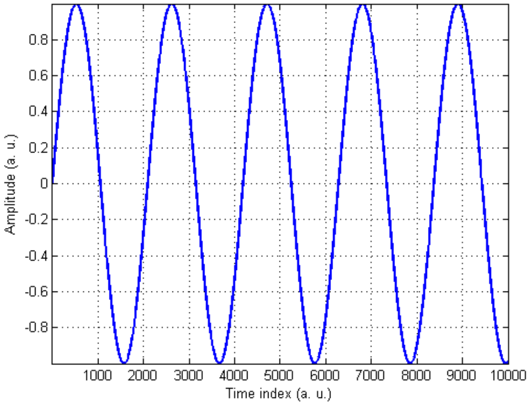
Motivation

- Rediscovery: Ockham's principle of parsimony, a.k.a Occam's razor, law of parsimony, law of economy or law of succinctness
- William of Ockham (c. 1285-1349) : *Numquam ponenda est pluralitas sine necessitate* (Plurality must never be posited without necessity)
- Modern short version: the simplest explanation/theory is most likely the correct one
- Refutable principle, indirect, highly heuristic
- As every principle, should be use with parsimony

Motivation

- Program: provide a panel of methods and tools aimed at more efficient, sparser data representations
  - Examples: time-scale and time-frequency (pyramids, wavelets, spectrograms, filter banks, lapped transforms), atomic representation (bases, trames and dictionaries), higher dimension generalizations (geometric multiscale transforms)
  - Not addressed: Fourier/Radon techniques extensions, modeling (AR, ARMA, ARMAX, Burg, Capon, Prony), Principal Component Analysis (PCA), Independent Component Analysis (or Latent Variable Analysis), clustering, K-means, neural networks
- Mathematical background (sic!)
  - linear algebra matrix analysis
  - selection, shrinkage & thresholding, non-linear filtering, robust statistics
  - operator theory, approximation theory (L_p norms and spaces, Sobolev, Besov)

Examples of “stationary” signals and images



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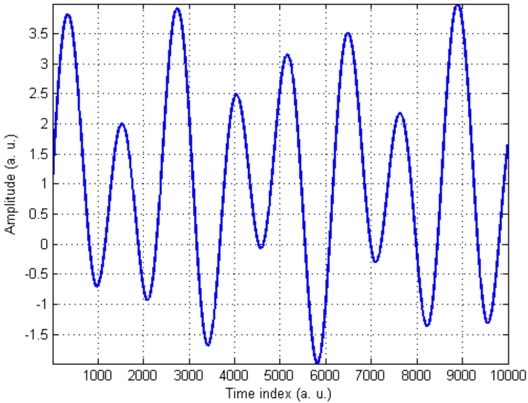

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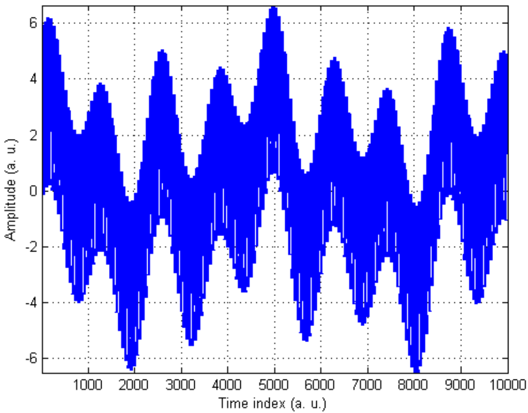

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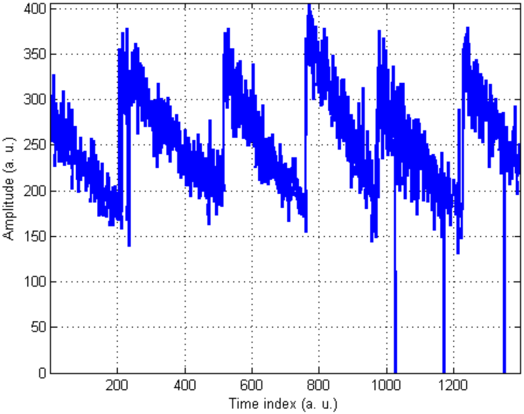

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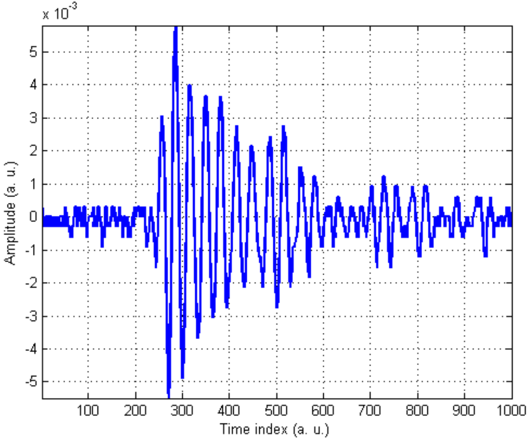

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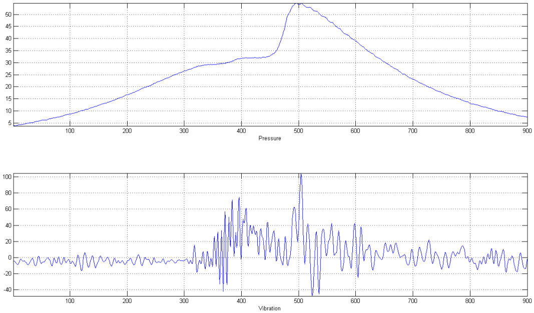

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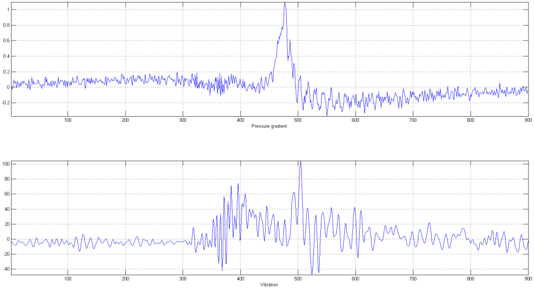

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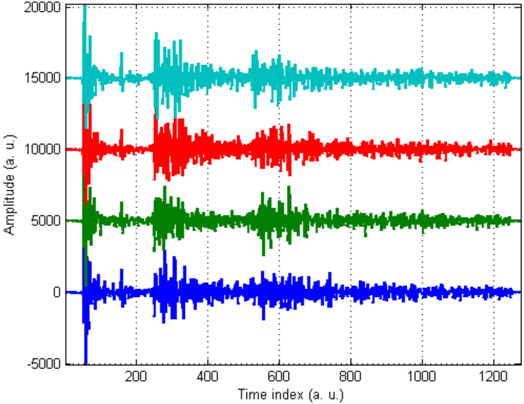

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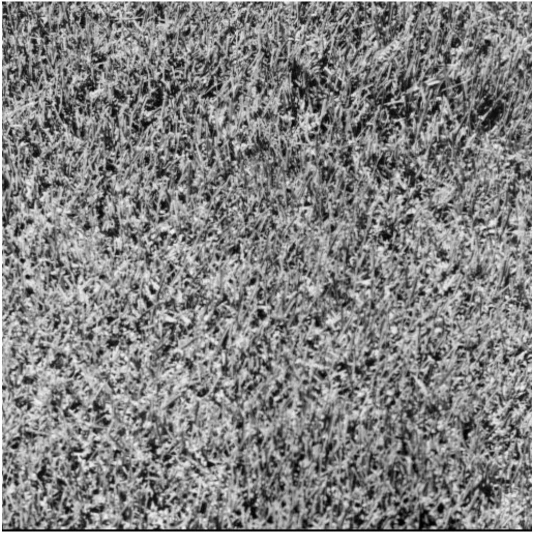

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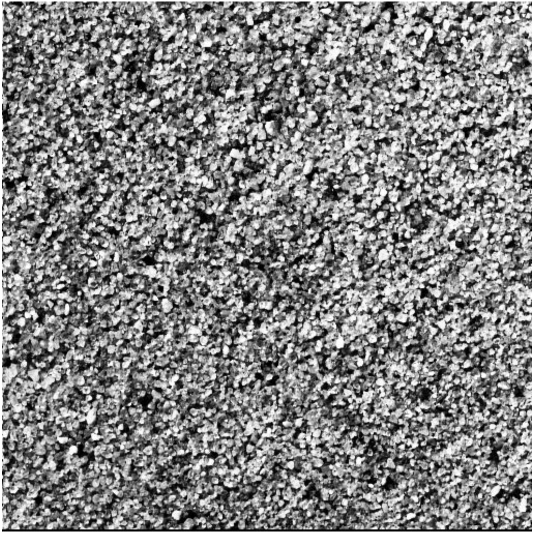

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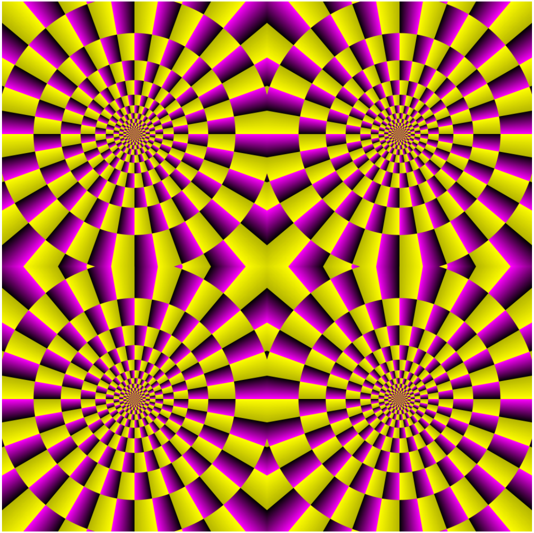


Reminders on analog signals

- Integrable signals

$$s \in L_1(\mathbb{R}) \longleftrightarrow \int_{-\infty}^{\infty} |s(t)|dt < \infty$$

- Finite energy signals

$$s \in L_2(\mathbb{R}) \longleftrightarrow ||s||^2 = \int_{-\infty}^{\infty} |s(t)|^2 dt < \infty$$

- Scalar product

$$\langle s_1, s_2 \rangle = \int_{-\infty}^{\infty} s_1(t) s_2^*(t) dt$$

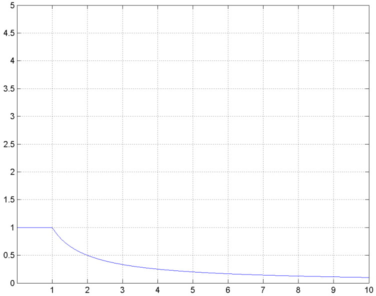
- Remark

$$L_1(\mathbb{R}) \not\subset L_2(\mathbb{R}) \quad \text{and} \quad L_2(\mathbb{R}) \not\subset L_1(\mathbb{R})$$

Reminders - Counter-example 1

- Signal in $L_2(\mathbb{R})$ but not in $L_1(\mathbb{R})$

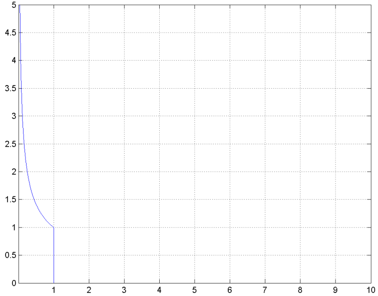
$$x = \begin{cases} 1 & \text{if } |t| < 1 \\ \frac{1}{|t|} & \text{otherwise.} \end{cases}$$



Reminders - Counter-example 1

- Signal $L_1(\mathbb{R})$ but not in $L_2(\mathbb{R})$

$$x = \begin{cases} \frac{1}{\sqrt{t}} & \text{if } |t| < 1 \\ 0 & \text{otherwise.} \end{cases}$$



History

A few names in the history:

- Joseph Fourier: 1807 (heat equation) [1768-1830]
- Gauß: 1805 (interpolation of orbits of celestial bodies)
- Runge: 1903
- Danielson-Lanczos: 1942
- Good: 1958
- Cooley-Tuckey: 1965 (FFT)
- Frigo-Johnson: 1998 (Fastest Fourier Transform in the West)

Source: Gauß and the history of the Fourier transform. Heideman, Johnson, Burrus, IEEE ASSP Magazine, 1984

History

Lettre de C. Jacobi à A. Legendre, 1830

> *M. Fourier avait l'opinion que le but principal des mathématiques était l'utilité publique et l'explication des phénomènes naturels; mais un philosophe comme lui aurait dû savoir que le but unique de la science, c'est l'honneur de l'esprit humain, et que sous ce titre, une question de nombres vaut autant qu'une question du système du monde.*

Continuous Fourier transform

- Definition

$$s(t) \xrightarrow{\text{FT}} S(f) = \int_{-\infty}^{\infty} s(t) e^{-\imath 2\pi f t} dt$$

function of the (dual) variable frequency $f \in \mathbb{R}$

- Existence 1: if $s \in L_1(\mathbb{R})$, then
    - S(f) is continuous and bounded
    - $\lim_{|f| \to \infty} S(f) = 0$
- Existence 2:

$$s \in L_2(\mathbb{R}), \text{ iff } S \in L_2(\mathbb{R})$$

- Inversion:

$$s(t) = \int_{-\infty}^{\infty} S(f) e^{\imath 2\pi f t} df$$

for almost every t

Fourier transform - properties 1

- Linearity

$$s_1(t) \xrightarrow{\text{FT}} S_1(f), \quad s_2(t) \xrightarrow{\text{FT}} S_2(f)$$

then

$$\forall(\lambda,\mu) \in \mathbb{C}^2 \quad \lambda s_1(t) + \mu s_2(t) \xrightarrow{\text{FT}} \lambda S_1(f) + \mu S_2(f)$$

- Delay/translation

$$\forall b \in \mathbb{R}, \quad s(t-b) \xrightarrow{\text{FT}} e^{-\imath 2\pi b f} S(f)$$

delay/translation invariance in amplitude spectrum

- Modulation:

$$\forall \nu \in \mathbb{R}, \quad e^{\imath 2\pi \nu t} s(t) \xrightarrow{\text{FT}} S(f-\nu)$$

Fourier transform - properties 2

- Scale change

$$\forall \alpha \in \mathbb{R}^*, \quad s(\alpha t) \xrightarrow{\mathrm{FT}} \frac{1}{|\alpha|} S(\frac{f}{\alpha})$$

  turns dilatation onto contraction
- Time inverse: a special case

$$s(-t) \xrightarrow{\mathrm{FT}} S(-f)$$

- Conjugation

$$s^*(t) \xrightarrow{\mathrm{FT}} S^*(-f)$$

- Hermitian symmetry: if s is real, then:
  - $\mathrm{Re}(S(f))$ even, $\mathrm{Im}(S(f))$ odd
  - $|S(f)|$ even, $\arg S(f)$ odd $\pmod{2\pi}$

Fourier transform - properties 3

- Convolution

$$(s_1 * s_2)(t) = \int_{-\infty}^{\infty} s_1(u) s_2(t-u) du = (s_2 * s_1)(t)$$

- Conditions
  - $s_1 \in L_1(\mathbb{R}), s_2 \in L_1(\mathbb{R}) \Rightarrow s_1 * s_2 \in L_1(\mathbb{R})$
  - $s_1 \in L_2(\mathbb{R}), s_2 \in L_2(\mathbb{R}) \Rightarrow s_1 * s_2 \in L_2(\mathbb{R})$

$$(s_1 * s_2)(t) \xrightarrow{\text{FT}} S_1(f) S_2(f)$$

- Parseval-Plancherel equalities: if $s_1, s_2 \in L_1(\mathbb{R})$:
  - $\langle s_1, s_2 \rangle = \langle S_1, S_2 \rangle$
  - $\|s\|^2 = \|S\|^2$

Fourier transform - examples

- $s(t) = e^{-at}u(t), \Re(a) > 0$:
  $S(f) = \frac{1}{a+2\imath\pi f}$
- $s(t) = e^{-at}\cos(2\pi f_0 t)u(t)$:
  $S(f) = \frac{1}{2}\left[\frac{1}{a+2\pi\imath(f-f_0)} + \frac{1}{a+2\pi\imath(f+f_0)}\right]$
- $s(t) = e^{-a|t|}$:
  $S(f) = \frac{2|a|}{a^2+(2\pi f)^2}$

Digital signals - sampling

- For a signal $s(t)$, regularly sampled:

$$s[k] = s(kT)$$

with T: sampling period; $f_s = 1/T$: sampling frequency

- Sampling theorem: if $S(f) = 0$ for $|f| \geq B$ and $f_s \geq 2B$, the the signal is sampled without information loss (theoretically), with Shannon-Nyquist formula:

$$s(t) = \sum_{k=-\infty}^{\infty} s[k]\text{sinc}\left(\frac{\pi(t-kT)}{T}\right), \text{sinc}(t) = \begin{cases} \frac{\sin(t)}{t} & \text{if } t \neq 0 \\ 1 & \text{otherwise.} \end{cases}$$

- cardinal theorem of interpolation theory: (Cauchy)-Whittaker, 1915; Nyquist, 1928; Kotelnikov, 1933; Whittaker, 1935; Raabe, 1938; Gabor, 1946; Shannon, 1948; Someya, 1949
- Note: Balian-Low theorem for time-frequency analysis

Digital signals - sampling

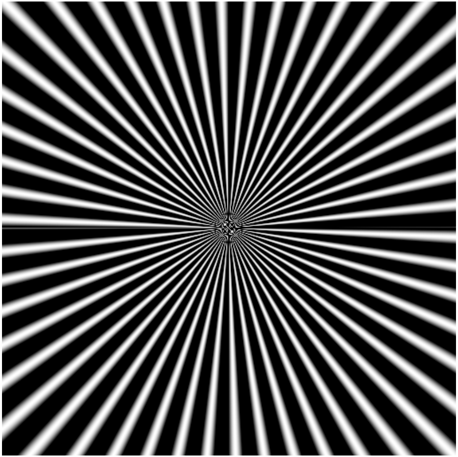


Figure 4: Example of aliasing

Digital signals - sampling

- Comments
  - sufficient but non-necessary condition
  - two-part theorem: (1) sampling, (2) reconstruction
  - caution: jitter, amplitude quantization, noise
  - slow convergence of the sinc, instability
  - signals cannot be time-limited and frequency-bounded
  - extensions to band-limited signals exist (iterative Papoulis-Gerchberg); optimally $2(f_u - f_l)$. Aliases in $\pm$ freq. content do not overlap for integer k:

$$\frac{2f_u}{k} \leq f_s \leq \frac{2f_l}{k-1}$$

  - alternatives: non-regularly sampled data (Lomb-Scargle periodogram); sparse/finite-rate-of-innovation signals (no more than n events per unit of time): compressive sensing, sparse sampling
- Beware of weak analogies between continuous/discrete

Digital signals

- Absolutely convergent sequences

$$(s[k])_{k\in\mathbb{Z}} \in l^1(\mathbb{R}) \Leftrightarrow \sum_{k=-\infty}^{\infty} |s[k]| < \infty$$

- Square-summable sequences

$$(s[k])_{k\in\mathbb{Z}} \in l^2(\mathbb{R}) \Leftrightarrow \sum_{k=-\infty}^{\infty} |s[k]|^2 < \infty$$

- Remark

$$l^1(\mathbb{R}) \subset l^2(\mathbb{R})$$

Discrete-Time Fourier transform (DTFT)

- Definition 1: z-transform

$$(s[k])_{k\in\mathbb{Z}} \xrightarrow{\text{ZT}} S(z) = \sum_{k=-\infty}^{\infty} s[k]z^{-k}$$

- Definition 2: Discrete-Time Fourier transform

$$(s[k])_{k\in\mathbb{Z}} \xrightarrow{\text{DTFT}} S(f) = \sum_{k=-\infty}^{\infty} s[k]z^{-k}, \quad z = e^{\imath 2\pi f}$$

Discrete-Time Fourier transform (DTFT)

- Normalized frequency

$$f = T f_{\text{phys}}$$

- Periodicity

$$f = 1 \ : S(f+1) = S(f)$$

- Existence

$$(s[k])_{k\in\mathbb{Z}} \in l^2(\mathbb{R})$$

- Special case: if $(s[k])_{k\in\mathbb{Z}} \in l^1(\mathbb{R})$, then $S(f)$ is continuous and bounded

DTFT - properties 1

- Linearity

$$\lambda s_1[k] + \mu s_2[k] \xrightarrow{\text{DTFT}} \lambda S_1(f) + \mu S_2(f)$$

- Integer delay/translation

$$s[k-n] \xrightarrow{\text{DTFT}} e^{-\imath 2\pi n f} S(f)$$

- Modulation

$$e^{\imath 2\pi \nu} s[k] \xrightarrow{\text{DTFT}} S(f-\nu)$$

- Time inversion

$$s[-k] \xrightarrow{\text{DTFT}} S(-f)$$

DTFT - properties 2

- Conjugaison

$$s^*[k] \xrightarrow{\text{DTFT}} S^*(-f)$$

- Parseval-Plancherel equalities

$$\sum_{k=-\infty}^{\infty} s_1[k] s_2^*[k] = \int_{-1/2}^{1/2} S_1(f) S_2^*(f) df$$

$$\sum_{k=-\infty}^{\infty} |s[k]|^2 = \int_{-1/2}^{1/2} |S(f)|^2 df$$

DTFT - properties 3

- Convolution

$$(s_1 * s_2)[k] = \sum_{l=-\infty}^{\infty} s_1[l] s_2[k-l] \xrightarrow{\text{DTFT}} S_1(f) S_2(f)$$

- Inversion

$$s[k] = \int_{-1/2}^{1/2} S(f) e^{\imath 2\pi f k} df$$

Fourier series

- If s is periodic (and continuous), $s(t+2\pi) = s(t)$ define:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{-\pi} s(t)$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{-\pi} s(t)\cos(kt)dt$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{-\pi} s(t)\sin(kt)dt$$

then the infinite Fourier series is:

$$s(t) \simeq \frac{a_0}{2} + \sum_{k=1}^{\infty}[a_k\cos(kt) + b_k\sin(kt)]$$

Discrete Fourier transform

- Definition

$$(s[k])_{0\leq k\leq K-1} \xrightarrow{\mathrm{DFT_K}} \hat{s}[p]_{0\leq p\leq K-1}$$

with

$$\hat{s}[p]_{0\leq p\leq K-1} = \sum_{k=0}^{K-1} s[k]e^{-\imath 2\pi \frac{kp}{K}}$$

- Link to the DTFT : if $s[k] = 0$ pour $k < 0$ et $k \geq K$, then

$$\hat{s}[p] = S\left(\frac{p}{K}\right)$$

i.e. K-sample sampling of DTFT $S(f)$ on $[0, 1]$

- Inversion

$$s[k] = \frac{1}{K}\sum_{p=0}^{K-1} \hat{s}[p]e^{\imath 2\pi \frac{kp}{K}}$$

All Fourier transforms unite: Pontryagin duality

Transform	Original domain	Transform domain
Fourier transform	$\mathbb{R}$	$\mathbb{R}$
Fourier series	T	$\mathbb{Z}$
Discrete-time Fourier transform (DTFT)	$\mathbb{Z}$	T
Discrete Fourier transform (DFT)	$\mathbb{Z}/n\mathbb{Z}$	$\mathbb{Z}/n\mathbb{Z}$

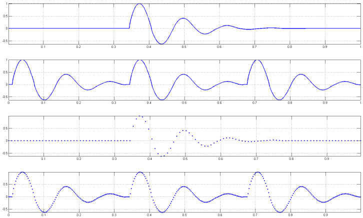


Figure 5: Signals in the different domains

Fast Fourier transform - FFT

- Fast algorithms exist (since Gauss)

$$\mathrm{FFT_K} \Rightarrow \text{complexity of } O(K \log_2(K)) \text{ operations}$$

- Cyclic or periodic convolution: Let $(s_1[k])_{0 \leq k \leq K-1}$ and $(s_2[k])_{0 \leq k \leq K-1}$

$$(s_1 \circledast s_2)[k] \xrightarrow{\mathrm{DFT_K}} \hat{s}_1[p] \cdot \hat{s}_2[p]$$

where $(s_1 \circledast s_2)[k]$ represents the K-point convolution of the periodized sequences:

$$(s_1 \circledast s_2)[k] = \sum_{l=0}^{k} s_1[l] s_2[k-l] + \sum_{l=k+1}^{K-1} s_1[l] s_2[K+k-l]$$

Reminders - Key message

- Nature of the data and the transform
  - Continuous and discrete natures ARE different
  - Generally stuff works
  - Intuition may be misleading (ex.: FFT on 8-sample signals, non proper windows)
  - Sometimes special care is needed: re-interpolation, pre-processing to avoid edge effects, instabilities, outliers

Windows

- Several uses
  - apodization, tapering (edges, jumps)
  - "stationarizing"
  - spectral estimation, filter design

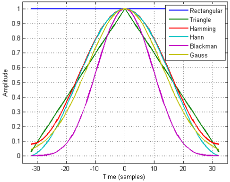


Figure 6: Windows in the original domain

- Many (parametric) designs:
  - Rect., Bartlett, Hann, Hamming, Kayser, Chebychev, Blackman-Harris,...
  - Gen. cosine: $a_0 - a_1 \cos(\frac{2\pi n}{M-1}) + a_2 \cos(\frac{4\pi n}{M-1}) - a_3 \cos(\frac{6\pi n}{M-1})$

Windows

- Several uses
  - apodization, tapering (edges, jumps)
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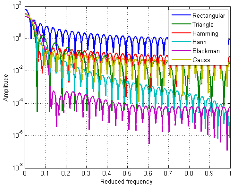


Figure 6: Windows in the frequency domain

- Many (parametric) designs:
  - Rect., Bartlett, Hann, Hamming, Kayser, Chebychev, Blackman-Harris,...
  - Gen. cosine: $a_0 - a_1 \cos(\frac{2\pi n}{M-1}) + a_2 \cos(\frac{4\pi n}{M-1}) - a_3 \cos(\frac{6\pi n}{M-1})$

Side dishes - Hilbert transform

- Linear operator (Cauchy kernel)
- Important tool in signal processing, mechanics, waves

$$Hf(t) = \frac{1}{\pi}\text{p.v.}\int_{-\infty}^{+\infty} \frac{f(s)}{t-s} ds$$

$$\mathcal{F}(Hf(t))(\omega) = (-\imath \text{sign}(\omega)).\mathcal{F}(f)(\omega)$$

- Analytic signal: $\hat{f}(t) = f(t) + \imath Hf(t)$
- Examples of Hilbert pairs:
  - $\cos(t) \to \sin(t) \quad (e^{\imath t})$ (cisoid)
  - $\frac{1}{1+t^2} \to \frac{t}{1+t^2}$
- Useful for envelope extraction, inst. freq. estimation, time-frequency processing

Side dishes - Hilbert transform and pairs

Reminders [Gabor-1946][Ville-1948]

$$\widehat{\mathcal{H}\{f\}}(\omega) = -\imath \mathrm{sign}(\omega)\widehat{f}(\omega)$$

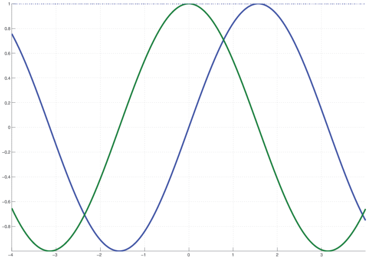


Figure 7: Hilbert pair 1

Side dishes - Hilbert transform and pairs

Reminders [Gabor-1946][Ville-1948]

$$\widehat{\mathcal{H}\{f\}}(\omega) = -\imath \mathrm{sign}(\omega)\widehat{f}(\omega)$$

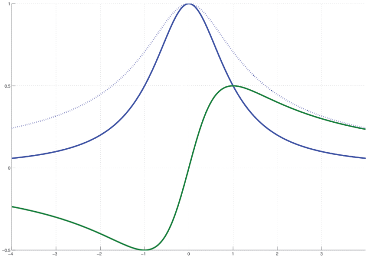


Figure 7: Hilbert pair 2

Side dishes - Hilbert transform and pairs

Reminders [Gabor-1946][Ville-1948]

$$\widehat{\mathcal{H}\{f\}}(\omega) = -\imath \operatorname{sign}(\omega)\widehat{f}(\omega)$$

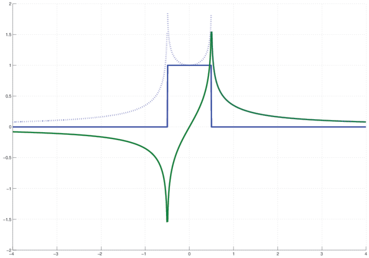


Figure 7: Hilbert pair 3

Side dishes - Hilbert transform and pairs

Reminders [Gabor-1946][Ville-1948]

$$\widehat{\mathcal{H}\{f\}}(\omega) = -\imath \mathrm{sign}(\omega)\widehat{f}(\omega)$$

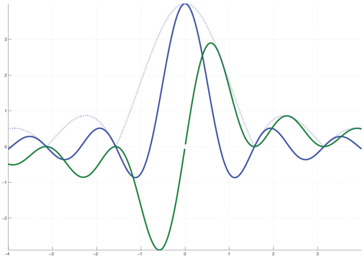


Figure 7: Hilbert pair 4

Reminders: set averages

- $s(n)$: discrete time random process (stationary stochastic process)
- expectation:
$$\mu_s(n) = E\{s(n)\}$$
- variance:
$$\sigma_s^2(n) = E\{|s(n) - \mu_s(n)|^2\}$$
- autocovariance:

$$c_s(k,l) = E\{(s(k) - \mu_s(k))(s(l) - \mu_s(l))^*\}$$

Reminders: power spectral density

For an autocorrelation ergodic process:

$$\lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} s(n+k)s(n) = r_{ss}(k)$$

- if $s(n)$ is known for every n, power spectrum estimation
- caveat 1: samples are not unlimited $[0, \dots, N-1]$, sometimes small
- caveat 2: corruption (noise, interfering signals)

Recast the problem: from the biased estimator of the ACF

$$\hat{r}_{ss}(k) = \sum_{n=0}^{N-1-k} s(n+k)s(n)$$

estimate power spectrum (periodogram)

$$\hat{P}_x(e^{\imath\omega}) = \sum_{k=-N+1}^{N-1} \hat{r}_{ss}(k)e^{\imath k\omega}$$

Time and frequency resolution

- Energy

$$E = \int |s(t)|^2 dt = \int |S(f)|^2 df$$

- Time or frequency location

$$\bar{t} = 1/E \int t|s(t)|^2 dt \qquad \overline{f} = 1/E \int f|S(f)|^2 df$$

- Energy dispersion

$$\Delta t = \sqrt{1/E \int (t - \bar{t})^2 |s(t)|^2 dt}$$

$$\Delta f = \sqrt{1/E \int (f - \overline{f})^2 |S(f)|^2 df}$$

Heisenberg-Gabor inequality

- Theorem (Weyl, 1931)
  If $s(t), ts(t), s'(t) \in \mathrm{L}^2$ then

$$\|s(t)\|^2 \leq 2\|ts(t)\|\|s'(t)\|$$

- Equality:
  Iff $s(t)$ is a modulated Gaussian/Gabor elementary function:

$$s'(t)/s(t) \propto t$$

$$s(t) = C\exp[-\alpha(t-t_m)^2 + \imath 2\pi\nu_m(t-t_m)]$$

- Proof
  Integration by part + Cauchy-Schwarz

Uncertainty principle (UP)

- Time-frequency UP
  For finite-energy every signal $s(t)$, with Δt and Δf finite:

$$\Delta t \Delta f \geq \frac{1}{4\pi}$$

  with equality for the modulated Gaussian only
- Principles

$$\|s'(t)\|^2 = |\imath 2\pi|^2 \|fS(f)\|^2$$

- Observations
  - Fourier (continuous) fundamental limit: arbitrary "location" cannot be attained both in time and frequency
  - have to choose between time and frequency locations
  - Gaussians are "the best"

Uncertainty principle (UP) for project management

Applies to other domains

Figure 8: Dilbert

Uncertainty principle - time

- One may write

$$s(t) = \int s(u)\delta(t-u)du$$

  - $\delta(t)$ is neutral w.r.t. convolution
  - interpreted as a decomposition of $s(t)$ onto a basis of shifted $\delta(t-u)$: $\Delta t = 0$ at
  - FT of basis functions: $e^{-\imath 2\pi f t}$: $\Delta f = \infty$

UP: as a limit of $0 \times \infty$

Uncertainty principle - frequency

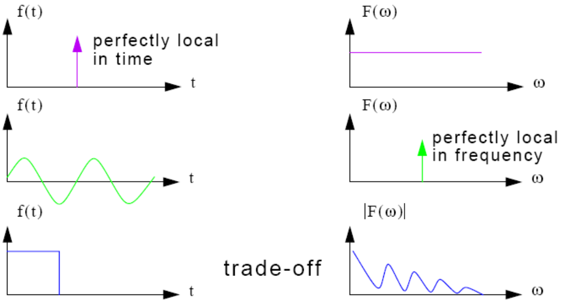
- One may write

$$s(t) = \int S(f) e^{\imath 2\pi f t} df$$

  - interpreted as a decomposition on pure waves $e^{\imath 2\pi f t}$: $\Delta t = \infty$
  - FT of basis functions: $\delta(f-t)$: $\Delta f = 0$

UP: as a limit of $\infty \times 0$

Uncertainty principle - illustration



Basis formalism interpretation

- Orthonormality

$$\langle e^{\imath 2\pi f t}, e^{\imath 2\pi g t}\rangle = \delta(f-g)$$

$$\langle \delta(f), \delta(g)\rangle = \delta(f-g)$$

- Scalar product

$$S(f) = \langle s(t), e^{\imath 2\pi f t}\rangle$$

$$s(t) = \langle s(t), \delta(f)\rangle$$

- Matching of a signal with a vector, a basis function (pure wave, Dirac)
    - Synthesis: continuous sum of orthogonal projection onto basis functions
    - Relative interest of the two bases? Other bases? (Walsh-Hadamard, DCT, eigenbase)
    - How to cope with mixed resolution?

Sliding window Fourier transform

- Principles
  Fourier analysis on time-space slices of the continuous $s(t)$ with a sliding window $h(t-\tau)$
- Short-term/short-time Fourier transform (STFT)

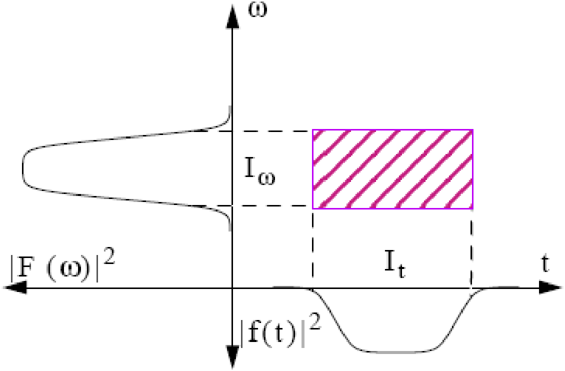
$$S_s(\tau, f; h) = \int s(t) h^*(t-\tau) e^{-\imath 2\pi f t} dt$$

- Wider domain of applications than FT
  - depend on h
  - FT as a peculiar instance (valid for other transforms: not a new tool, only a more versatile "leatherman"-like multi-tool)

Sliding window Fourier transform

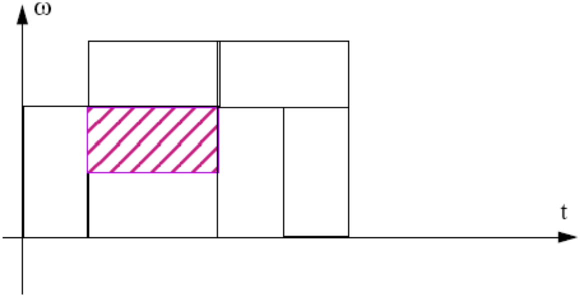
Figure 9: Leatherman wave black

Sliding window Fourier transform - illustration



Sliding window Fourier transform - time-freq. completude

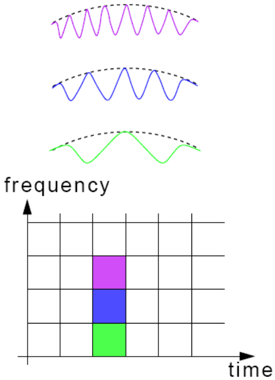
Notion of a "complete" descrition (*i.e.* somehow invertible)



Sliding window Fourier transform - windows

- Related to frequency analysis
  Depend on the window choice h (shape, length)
- Continuous time windows
  - rectangular: poor frequency resolution
  - gaussian: best time-frequency trade-off? (Gabor, 1946)
- Discrete time windows
  - τ discretized (jumps vs. redundancy)
  - different criteria: side lobes, *equiripple*, apodizing; Bartlett, Hamming, Hann, Blackmann-Harris, Blackmann-Nutall, Kaiser, Chebychev, Bessel, Generalized raised cosine, Lánczos, Flat-top,...

Sliding window Fourier transform - paving



Sliding window Fourier transform - reconstruction

- Simple analogy
  - synthesis: what two numbers add to result 3
  - $a + b = 3$: infinite number of solutions, *e.g.* $2945.75 + (-2942.75)$ but irrelevant
  - assume they are integers?
  - assume they are positive? $1 + 2 = 3$ or $2 + 1 = 3$
  - aim: increase interpretability, information compaction $(0 + 3 = 3)$, reduce overshoot

Sliding window Fourier transform - reconstruction

- Redundant transformation!
- Inversion

$$s(t) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} S_s(u, \xi; h) g(t-u) e^{\imath 2\pi t u} du d\xi,$$

provided that

$$\int_{-\infty}^{+\infty} g(t) h^*(t) dt = 1.$$

(perfect reconstruction, no information loss)

  - special case: admissible normalized window h

$$g(t) = h(t)$$

    but not the only solution (truncated $\sin$)
  - a bit more involved for discrete time, less if only approximate

Sliding window Fourier transform - spectrogram

- Definition

$$|S_s(\tau, f; h)|^2$$

- The spectrogram is a (bilinear) time-frequency distribution

$$E = \iint |S_s(\tau, f; h)|^2 d\tau df$$

for normalized admissible window h

- Parseval formula

$$\langle s_1, s_2 \rangle = \iint S_{s_1}(\tau, f; h) S_{s_2}(\tau, f; h) d\tau df$$

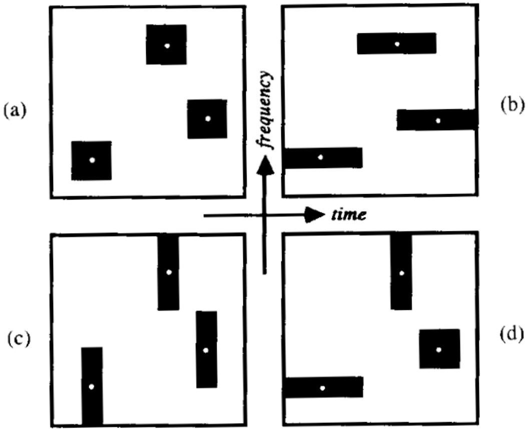
Sliding window Fourier transform - monoresolution

- Reason: basis functions

$$h(t-\tau)e^{\imath 2\pi f t}$$

  all possess similar resolution
- Examples:
  - $s(t) = \delta(t-t_0) \longrightarrow |S_s(\tau, f; h)|^2 = |h(t_0-\tau)|^2$
  - $s(t) = e^{\imath 2\pi f_0 t} \longrightarrow |S_s(\tau, f; h)|^2 = |H(f_0-f)|^2$
- Uses
  - for long range oscillatory signals, long windows are necessary
  - for short range transient, short windows needed
  - possibility to use several in parallel
  - incentive to use several ones simultaneously

Sliding window Fourier transform - illustration



Other time frequency distributions

- Quadratic or bilinear distributions
  - Wigner-Ville and avatars (smoothed, pseudo-, reweighted)
  - Cohen class (WV, Rihaczek, Born-Jordan, Choi-Williams)
  - property based: covariance, unitarity, inst. freq. & group delay, localization (for specific signals), support preservation, positivity, stability, interferences
  - Bertrand class, fractional Fourier transforms, linear canonical transformation (4 param.: FT, fFT, Laplace, Gauss-Weierstrass, Segal-Bargmann, Fresnel transforms)
  - generally not applied in more than 1-D

Conclusions and perpectives

- Time-scale/time-frequency
  - tools adapted to non-stationarity
    - may be adapted in scale/frequency
    - flexible choice of tools
    - flexible choice of sampling/redundancy
  - deterministic processing
    - 1D: trends, oscillations, singularities
    - nD: low-dimensional singularities, edges, textures
  - non-deterministic processing
    - noise, fractals, brownian processes
  - high hybridization potential
    - parametric modeling, statistics, heuristic, local learning
- Very active research