Isomorphisms

A morphism $\alpha : x \to y$ is an *isomorphism* iff there exists a morphism $\alpha^{-1} : y \to x$ such that $\alpha^{-1}\alpha = 1_x$ and $\alpha\alpha^{-1} = 1_y$.

$$\begin{tikzcd} x \arrow[loop left, "1_x"] \arrow[r, shift right, "\alpha"'] & y \arrow[l, dashed, shift right, "\exists\alpha^{-1}"'] \arrow[loop right, "1_y"] \end{tikzcd}$$

i.e. it has a complete inverse.

A morphism $\alpha : x \to y$ is a *isomorphism* iff it is split monic and split epic.

Definition 1 (Isomorphic)**.** *Two objects x and y are* ***isomorphic*** *iff they have an isomorphism between them. This is notated by $x \cong y$.*

Theorem 2. *A morphism is isomorphic iff it is split monic and split epic.*

Proof. ($\implies$) α^{-1} directly gives β for the definitions of split monic and split epic morphisms.

($\impliedby$) Let $\alpha : x \to y$ be split monic and split epic, so there exists $\beta, \gamma : y \to x$ such that $\beta\alpha = 1_x$ and $\alpha\gamma = 1_y$. Then

$$\beta = \beta 1_y = \beta\alpha\gamma = 1_x\gamma = \gamma,$$

meaning α is an *isomorphism* with $\alpha^{-1} = \beta = \gamma$. □

Theorem 3. α^{-1} *is unique*

Proof. Let $\alpha : x \to y$ be an *isomorphism* with two inverses $\alpha_1^{-1}, \alpha_2^{-1} : y \to x$. Then

$$\alpha_1^{-1} = 1_x\alpha_1^{-1} = \alpha_2^{-1}\alpha\alpha_1^{-1} = \alpha_2^{-1}1_y = \alpha_2^{-1}$$