Categories Induced by Preorder

If you have a set P equipped with a preorder $\leq$, the category P *induced by the preorder* is defined such that

$$P(x,y) = \begin{cases} \{x \to y\} & x \leq y \\ \emptyset & x \not\leq y \end{cases}. \qquad \forall x, y \in P$$

Proof. Let $\alpha : x \to y$, $\beta : y \to z$. Then $x \leq y$ and $y \leq z$, so by the transitivity of preorder, $x \leq z$. Thus, $P(x,z) = \{x \to z\}$, so $\beta\alpha : x \to z$ can be defined, and it must inherently be unique.

Let $x \in P$. By the reflexivity of preorder, $x \leq x$, so $P(x,x) = \{x \to x\}$, meaning there is a unique morphism $1_x : x \to x$.