Split Monic Morphisms

A morphism $\alpha : x \to y$ is *split monic* iff there exists a morphism $\beta : y \to x$ such that $\beta\alpha = 1_x$:

$$
\begin{tikzcd}
x \arrow[loop left, "1_x"] \arrow[r, shift right, "\alpha"'] & y \arrow[l, dashed, shift right, "\exists\beta"']
\end{tikzcd}
$$

i.e. it has a left inverse.

Remark. This definition is dual to that of split epic morphisms.

Theorem 1. *Split monic* $\to$ *monic*

Proof. Let $\alpha : x \to y$ be a *split monic* morphism in $\mathcal{C}$, so there exists $\beta : y \to x$ such that $\beta\alpha = 1_x$. For some $z \in \mathcal{C}$, let $\beta_1, \beta_2 : z \to x$ such that $\alpha\beta_1 = \alpha\beta_2$. Then

$$\beta_1 = 1_x\beta_1 = \beta\alpha\beta_1 = \beta\alpha\beta_2 = 1_x\beta_2 = \beta_2,$$

meaning α is monic. $\square$

Theorem 2. *In a concrete category, every split monic morphism is injective.*

Proof. Let $\alpha : X \to Y$ be a *split monic* morphism in $\mathcal{C}$, so there exists $\beta : Y \to X$ such that $\beta\alpha = 1_X$. Let $x_1, x_2 \in X$ such that $\alpha(x_1) = \alpha(x_2)$. Then

$$
\begin{aligned}
\alpha(x_1) &= \alpha(x_2) \\
\beta\alpha(x_1) &= \beta\alpha(x_2) \\
x_1 &= x_2. \quad \square
\end{aligned}
$$