Counterfactual Worlds*

Benjamin Brast-McKie

Abstract

This paper extends Kit Fine's [1–5] truthmaker framework to provide a novel semantics for tensed counterfactual conditionals. Instead of taking possible worlds to be primitive elements in a model, possible worlds will be defined in terms of *states*, *parthood*, and *tasks* where the latter encode the possible transitions between states. Rather than invoking primitive similarity or imposition relations, possible worlds will be compared at a time independent of that time's past and future where the comparison will be carried out in mereological and modal terms. After reviewing the motivations for this approach, I will provide the hyperintensional semantics for counterfactuals that is implemented in the `model-checker` software for this project. I will then extend the language to include tense operators in order to analyze forwards, backwards, and backtracking counterfactuals.

Keywords: Counterfactual Conditionals, Truthmaker Semantics, Hyperintensionality

1 Introduction

Drawing on Kripke's [6] semantics for the metaphysical modal operators, intensional semantics theories of counterfactual conditionals model *possible worlds* as structureless points. Although taking possible worlds to be primitive is a harmless idealization when interpreting modal claims, the same cannot be said for counterfactual claims of the form ⌜If it were the case that A, then it would be the case that B⌝, or in symbols, ⌜$A \boxright B$⌝. Rather, I will define possible worlds as histories of moments where each moment is rich in mereological structure. In order to motivate this approach, this section reviews Fine's [7] early criticisms of the *similarity theory* that Lewis [8] developed as well the *imposition theory* that Fine [1, 2] went on to provide. Despite their differences, these views are not equipped to interpret tensed counterfactuals and may be criticized for including primitives in the metalanguage that are at least as hard to understand as the object language that they aim to interpret.

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Instead of positing primitive similarity or imposition relations, §2 defines Fine's imposition relation in purely modal and mereological terms. In order to also interpret tense operators, I will introduce a primitive *task relation* which encodes the possible transitions between *states*. Given this additional resource, §3 defines the *possible states* that Fine takes to be primitive as well as the temporally extended *world histories* needed to interpret tensed counterfactuals. By drawing on these resources, §4 will provide a unified semantics for tensed counterfactuals, employing this framework to analyze forwards, backwards, and backtracking counterfactuals in §5. The paper will conclude by considering objections and extensions in §6 and presenting minor results in §7. The following subsections will motivate these developments by presenting the shortcomings that similarity and imposition theories of counterfactual face.

1.1 Totality

In addition to a primitive set of *possible worlds* W and an *accessibility relation* R over W, Stalnaker [9] and Lewis [8] equip models with a *selection function* or *comparative similarity relation*, respectively. Abstracting from their differences, similarity theories take $A \boxright B$ to be true in a world w just in case B is true in the most similar accessible world(s) to w in which A is true.[1] Independent of how similarity is to be understood, modeling possible worlds as structureless points imposes the following limitation:

> *Totality:* Similarity selects/relates possible worlds considered in their entirety.

Although the worlds in W are simple points, this does not stop similarity theorists from assuming otherwise while working over an intended model which includes the possible worlds themselves along with all of their spatiotemporal structure and other relevant features. However, the various features that one might take possible worlds to have in an intended model are not included in the definition of a model. For the purposes of logic, the models of a given object language only have the structure that they are given, and standard intensional models do not include any internal structure in the possible worlds that are used to interpret the language. Focusing on Lewis' account, W is any nonempty set where the similarity relation stipulates an ordering over the worlds in W by treating each world as a structureless point.

Insofar as worlds are taken to model complete histories of everything, *Totality* leads similarity theories to predict the wrong results. For instance, consider the present tense analogue of the counterexample Fine [7, p. 425] raised against Lewis' account:

> `(N)` If Nixon were to press the button there would be a nuclear holocaust.

Imagine that Nixon is hovering over the detonator where one of his advisers asserts `N` to another. As Fine points out, Lewis' theory predicts that `N` is false since for any world in which Nixon pressed the button and a nuclear holocaust followed, there will be a much more similar world which includes, "a change that prevents the holocaust

[1] Whereas Stalnaker [9] assumes that there is always a closest A-world, Lewis [8] permits there to be equally close A-worlds as well as unending sequences of ever closer A-worlds. For simplicity, I will often leave the world of evaluation as well as the restriction to accessible worlds implicit in what follows, referring simply to the most similar world(s) in which the antecedent is true.

but that does not require such a great divergence from reality." For instance, perhaps the button is disconnected. Even so, it is natural to take `N` to be true since the results of Nixon pressing the button are irrelevant to the moment that we are considering in which Nixon did not press the button but easily could have. Rather than measuring the similarity of the actual world in its entirety against all of what would result if Nixon were to press the button, it is natural to consider the result of making minimal changes to the moment in which Nixon did not press the button to one in which he did. Given any such minimally changed moment, `N` makes a claim about the disaster to follow even if those futures end up diverging widely from the actual course of events. So long as the moments that result from changing the actual moment in which Nixon did not press the button to one in which he did do not include any other changes such as a disconnected button, etc., we may expect `N` to be true.

However natural it may be to restrict consideration to the moment in which the antecedent is to be evaluated while ignoring later moments, intensional models do not include the resources to do so.[2] If possible worlds are modeled by structureless points, a similarity relation can only order possible worlds considered in their entirety as stated by *Totality*. So long as possible worlds are taken to model temporally extended histories, *Totality* makes it easy to imagine possible worlds in which Nixon pushed the button that are much more similar to the actual world than to possible worlds in which he pushed the button and a nuclear holocaust took place.

Lewis [10] responded by dispensing with our pretheoretic similarity judgments, opting instead for weighted rules that constrain how the similarity of worlds is to be measured. Instead of precise rules which are easy to apply, Lewis' weighting system consists of open-ended suggestions for how to adjust our untutored intuitions about similarity while working over an intended model that includes the possible worlds themselves. In particular, Lewis sought to maximize the spatiotemporal regions of the counterfactual worlds which accord with actuality while reducing the extent to which those counterfactual worlds violate the actual laws of nature. Whether one adopts Lewis' rules or not, it is natural to worry that the metalanguage that Lewis used to articulate a semantics for counterfactuals is considerably harder to understand than the object language that he sought to interpret.[3] Rather, if a semantic theory is to assign meaningful truth-conditions to sentences, I will assume that we ought to develop that theory in a metalanguage that is understood at least as well as the object language under study, providing an adequate account of the theoretical terms it employs. By contrast, admitting vague or context sensitive terms into the metalanguage undermines the ambition to provide a predictively powerful semantics for the language.

Taking possible worlds to be structureless points despite modeling temporally extended histories raises an even more fundamental problem. Suppose that Nixon had pushed the button before it was activated and nothing happens. The button is then activated and Nixon's adviser asserts `N`. Although it is natural to take `N` to be true, Lewis' semantics predicts the opposite result. After all, the actual world is a world in

[2] Although intervals could be considered, I will restrict attention to a single moment of comparison for the majority of what follows, postponing consideration of possible extensions of the semantics to §6.

[3] A similar complaint may be raised for Stalnaker's [9] selection function which is assumed to specify the most similar antecedent-world while ignoring irrelevant details, as well as for theories that employ causal notions in the metalanguage such as [11] and [12].

which Nixon pushed the button, making the actual world closer than any other world. Since the actual world does not include a nuclear holocaust, `N` is false. What is missing from the semantics is the time at which the antecedent is to be evaluated.

In addition to predicting the wrong results for counterfactuals in the present tense, evaluating sentences at possible worlds on their own makes the semantics incapable of interpreting tense operators. For instance, consider Fine's [7] original example in which a tense operator takes wide scope over the counterfactual claim `N` from before:

(`N′`) If Nixon had pressed the button there would have been a nuclear holocaust.

Although `N′` is false if evaluated before the button is activated, `N′` is true if evaluated after the button is activated and the temporal operators shift to a not too distant time. Since counterfactuals often occur together with tense operators, it is natural to develop a common framework in which to model their interaction. In order to both interpret tense operators and provide a semantics for counterfactuals that compares worlds for similarity at specific times, the following subsection will present a similarity theory which rejects *Totality*. Instead of taking possible worlds to be primitive, I will define possible worlds to be histories of *moments*, evaluating sentences at world-time pairs. Despite answering the objections above, the resulting semantic theory faces the same problems that Fine [1, 2, 7] raised against Lewis' theory of counterfactuals. These considerations will motivate a departure not only from positing a primitive similarity relation, but from intensional semantic theories of counterfactuals altogether. Even so, it is by first defining worlds as histories of moments that we may later come to appreciate the mereological and modal structure that make up each moment.

1.2 Restriction

Given a primitive set of *possible moments* M, *times* T, and *accessibility relation* R over M, a similarity theorist may avoid the problems brought out above by taking *similarity* $\preceq$ to order moments instead of worlds. Whereas possible worlds model temporally extended histories of everything, possible moments are the time slices of those histories where each moment determines the truth-value of every sentence letter of the language. Rather than taking accessibility to encode the relative possibility of one complete possible history from another, a moment m is *accessible* to k (i.e., Rmk) just in case it is possible for m to transition to k. By taking the set of times T to be the integers for simplicity, a *possible world* may be defined as any function from times to moments where every moment is accessible from its predecessor.[4]

By defining possible worlds, a similarity theorist may evaluate counterfactuals at world-time pairs so that $A \mathrel{\Box\!\!\rightarrow} B$ is true in a world w at a time x just in case B is true in u at x whenever $u(x)$ is one of the most similar moments to $w(x)$ and A is true in u at x.[5] Reading '$\Diamond A$' as 'It has been the case that A', we may take $\Diamond A$ to be true in a world w and time x just in case A is true in w at y for some time $y < x$. Something similar may be said for '$\Diamond A$' which reads 'It is going to be the case that A'. Encoding

[4] I develop this framework in much greater detail elsewhere in order to provide a semantics for a bimodal language which includes both tense and modal operators. I develop the hyperintensional analogue in §3.

[5] Although I will assume a Lewis-style ordering relation $\preceq$ which relates moments instead of worlds, these details can be adapted to employ a Stalnaker-stle selection function for moments instead.

temporality in this way avoids the counterexamples brought out above. In particular, $\mathtt{N}$ is true in a world w and time x just in case there is a time $y > x$ in which a nuclear holocaust occurs in u at y whenever $u(x)$ is one of the most similar moments to $w(x)$ in which Nixon pushes the button. Moreover, $\mathtt{N}'$ is true in a world w and time x just in case there is a time $z < x$ where $\mathtt{N}$ is true in w at z.[6]

Whereas comparing possible worlds for similarity makes both $\mathtt{N}$ and $\mathtt{N}'$ false, the same cannot be said for comparing moments for similarity. With respect to an actual moment in which the button is activated and Nixon did not press the button but could have, the most similar moments in which Nixon did push the button do not include any broken wires or blown fuses, and so are sure to lead to a nuclear holocaust. Thus $\mathtt{N}$ is predicted to come out true, where something similar may be said for $\mathtt{N}'$. Even so, this semantics invalidates *Simplification of Disjunctive Antecedents* for familiar reasons. For simplicity, we may restrict consideration to tenseless counterfactuals, replacing world-time pairs with moments for the time being. Accordingly, $A \boxright B$ is true in a moment m just in case B is true in the most similar moment(s) to m in which A is true. As a result, the following principle is invalid:

SDA $\quad A \vee B \boxright C \;\vdash (A \boxright C) \wedge (B \boxright C).$

Assuming the consequent is true in the most similar moments in which the antecedent is true does not require the consequent to be true in the most similar moments in which a disjunct of the antecedent is true.[7] For instance, suppose that $A \vee B \boxright C$ is true in m. Since the most similar $A \vee B$-moments to m may all be $A \wedge C$-moments without also all being B-moments, nothing requires $B \boxright C$ to be true in m. Adding color, suppose that although Philip is often the life of the party, Peter avoids parties since he always gets in fights. Assuming neither Philip nor Peter are at the party, the most similar moments in which either of them is at the party are moments in which only Philip is at the party. It follows that if either Philip or Peter were at the party, it would have been better, though it is false to claim that if Peter were at the party, it would have been better. Thus **SDA** is predicted to be false.

The invalidity of **SDA** does not turn on anything peculiar to similarity or moments. For the sake of generality, let *points* be generic elements at which sentences are to be evaluated. We may then observe that any theory of counterfactuals that conforms to the following principle will invalidate **SDA** for the same reason given above:

> *Restriction:* The consequent is only required to be true in a restricted subset of the points at which the antecedent is true (e.g., the set of most similar moments).

I will assume that **SDA** is to be maintained, and so *Restriction* must be given up.[8] Taking this lesson to heart, *strict theories* of counterfactuals require the consequent to be true at every point at which the antecedent is true. Replacing points with moments for consistency with the above, a strict theory of counterfactuals may take $A \boxright B$

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[6] Such a theorist may regiment $\mathtt{N}$ as $B \boxright \Diamond H$ and $\mathtt{N}'$ as $\Diamond(B \boxright \Diamond H)$. See §5 for further discussion.

[7] Fine [1, 7] makes this observation for Lewis' account. See also [13] for related observations and discussion.

[8] This is not to claim that all apparent instances of **SDA** in natural language hold without exception. Rather, this paper concerns the development of a logic for counterfactuals stated in a formal language. See [14], [15], and [16] for relevant discussion of the invalidity of apparent instances of **SDA** in natural language.

to be true in a moment m just in case B is true in every moment accessible from m in which A is true.[9] It follows that **SDA** is valid for if C is true in every accessible moment in which $A \vee B$ is true, then C is also true in every accessible moment in which A (similarly B) is true. Even so, strict theories validate *Strengthening the Antecedent*:

STA $\quad A \boxright C \ \vdash A \wedge B \boxright C.$

If C is true in all accessible moments in which A is true, C is also true in all accessible moments in which $A \wedge B$ is true, and so a strict theory of counterfactuals validates **STA**. However, it has long been observed that there are natural counterexamples to **STA**. Just because the match would light if it were struck, it does not follow that the match would light if it were struck and wet.[10] In response, defenders of **STA** provide pragmatic explanations for why the counterexamples are only apparent by appealing to contextual shifts in R between the evaluation of the premise(s) and conclusion.[11] Nevertheless, strict theories of counterfactual claims admit that in any particular context, counterfactual conditionals have the same logic as the strict conditional, and so reject attempts to identify a distinct logic for counterfactual conditionals.

Rather than looking to pragmatics in order to provide an error theory for the apparent invalidity of **STA**, I will take the counterexamples to **STA** to show that an adequate logic for the counterfactual conditional cannot be identified with the logic for the strict conditional. Additionally, I will include **SDA** in the logic for counterfactuals in order to account for the apparent validity of simplifying disjunctive antecedents.[12] Although there would seem to be no space between invalidating **SDA** by accepting *Restriction* and validating **STA** by rejecting *Restriction*, this choice is not forced. Since **STA** follows from **SDA** in any intensional logic for a transparent language, the following section will motivate the hyperintensional alternative that Fine [1, 2] develops in order to validate **SDA** without also validating **STA**.

Despite these advantages, Fine's semantics is not equipped to interpret tensed counterfactuals and may be faulted for positing a primitive imposition relation with a counterfactual reading. In order to address this latter concern, the following section will conclude by defining Fine's imposition relation in terms of the purely modal and mereological primitives that Fine already includes. By deriving the constraints that Fine requires his primitive imposition relation to satisfy, the resulting semantics will be shown to validate a logic for counterfactuals that is at least as strong as the logic that Fine defends while positing strictly fewer primitives. In addition to simplifying Fine's semantics conceptually, defining imposition streamlines the implementation of the resulting semantics in the `model-checker` software that I developed for evaluating logical consequences in a language which includes counterfactual operators. Given these advantages, §3 will extend the framework to accommodate tense operators, providing a unified semantics and logic for counterfactual conditionals and tense.

[9] In order to accommodate tense, a strict theorist may take $A \boxright B$ to be true in a world w at time x just in case B is true in w' at x whenever $w'(x)$ is accessible from $w(x)$ and A is true in w' at x.

[10] See Goodman's [17] discussion of this classic case.

[11] For instance, both [18] and [19, 20] provide strict conditional views in intensional frameworks.

[12] See Fine [1] for relevant discussion.

2 Imposition Semantics

Intensional semantic theories model propositions as subsets of a set of points. Letting '$\Box$' express *metaphysical necessity*, '$\equiv$' express *propositional identity*, and $\ulcorner C_{[B/A]} \urcorner$ be the result of uniformly substituting B for A in C, intensional theories validate **INT** where **LL** is valid in any transparent language which excludes opaque operators:

INT $\quad \Box(A \leftrightarrow B) \vdash A \equiv B.$[13] $\qquad$ **LL** $\quad A \equiv B,\ C \vdash C_{[B/A]}.$

Since $A \leftrightarrow A \vee (A \wedge B)$ is a classical tautology, $\Box[A \leftrightarrow A \vee (A \wedge B)]$ is a theorem of any normal modal logic. Accordingly, we may derive $A \vee (A \wedge B) \mathrel{\Box\!\!\rightarrow} C$ from the premise $A \mathrel{\Box\!\!\rightarrow} C$ by **INT** and **LL**, and so $A \wedge B \mathrel{\Box\!\!\rightarrow} C$ follows by **SDA**.[14] Thus **STA** is derivable from **SDA**. In order to avoid this inference, **INT** must be given up since **LL** is unassailable in a language that does not include opaque operators. In order to validate **SDA** without also validating **STA**, Fine [1, 2] provides a hyperintensional semantics which distinguishes necessarily equivalent propositions. After presenting the details of this account in the following subsection, §2.2 will define the imposition relation that Fine takes to be primitive in order to avoid positing counterfactual primitives. Later, §3 will present a revised theory of hyperintensional propositions which I will draw on in order to interpret the sentences of a counterfactual language.

2.1 Outcomes

Instead of maintaining *Restriction*, Fine [1] introduces a primitive *imposition relation* where $t \rightarrow_w u$ indicates that, "u is a possible outcome of imposing the change t on the world [state] w" (p. 237).[15] Whereas w and u are *world states* where each, "either contains or is incompatible with any other state" (p. 236), t is a *possible state* which Fine glosses as a "fragment" of a world state.[16] Accordingly, Fine takes *parthood* $\sqsubseteq$ to partially order the space of all states, referring to the least upper bound of a set of states as a *fusion state*.[17] In general, states only settle the truth-values of those sentences that they *exactly verify* or *exactly falsify*, "tell[ing] us what it is *in* the world that makes the statement true if it is true or what it is *in* the world that makes it false if it is false" (p. 235). Building on Fine's [3, 4] more recent work, I will have more to say about states and world states in the following subsection. For the time being, it will be enough to consider the broad role that states play in Fine's semantics.

Given the addition of states and the imposition relation, Fine takes $A \mathrel{\Box\!\!\rightarrow} C$ to be true in a world state w just in case C is true in any world state u which is the outcome of imposing any exact verifier state t for A on w. Fine is clear that his *imposition theory* for counterfactuals is based on the following ideas:

[13] *Intensionalists* typically assume $A \equiv B := \Box(A \leftrightarrow B)$ instead of taking '$\equiv$' to be primitive, or else do not introduce '$\equiv$' at all. See [21] for a clear statement of intensionalism.

[14] See Fine [1, 2, 22], [13], and Ellis et al. [23] for closely related arguments and [24] for discussion of **LL**.

[15] Whereas Fine [1] refers to world states simply as *worlds*, I will take worlds to be temporally extended histories for consistency throughout what follows.

[16] As brought out below, impossible states are not parts of world states, but states nonetheless.

[17] Whereas Fine [1] assumes that a set of states has a least upper bound if it has an upper bound, the next section will follow Fine [4] in taking the set of all states to form a complete lattice.

Universal Realizability of the Antecedent: There is no restriction on the exact verifiers for the antecedent which are to be imposed on the evaluation world state.

Universal Verifiability of the Consequent: There is no restriction on the outcomes of imposing an exact verifier for the antecedent on the evaluation world state.

Fine's semantics rejects *Restriction* without identifying the exact verifier states for the antecedent with the outcome world states that they produce when imposed on the evaluation world state. An imposition theory of this kind is to be contrasted with strict theories of counterfactuals which require the consequent to be true at all accessible antecedent points themselves. Even so, the principles above preserve the unrestricted spirit of strict theories of counterfactuals while providing a means by which to validate **SDA** without validating **STA**. Since the exact verifiers for $A \vee B$ include the exact verifiers for A, it follows that if C is true in every outcome that results from imposing an exact verifier for $A \vee B$ on a world state w, then C is also true in every outcome that results from imposing an exact verifier for A (similarly B) on w, and so **SDA** is valid. By contrast, the outcomes that result from imposing an exact verifier for A on w may differ considerably from the outcomes that result from imposing an exact verifier for $A \wedge B$ on w, and so even if C is true in all of the former outcomes, C may fail to be true in all of the latter outcomes, thereby invalidating **STA**.

Although Fine's imposition semantics validates **SDA** without validating **STA**, a number of important questions remain to be answered. Postponing discussion of the precise relationship between states and world states to the following subsection, we may consider an objection that Fine [1, p. 241] addresses in defense of his account:

> It has been argued, in the second place, that the present semantics is relatively problematic in its conceptual commitments. For the transition relation must itself be understood in terms of counterfactuals. Thus to say that u is a possible outcome of t in w is just to say that, in w, u might obtain if t were to obtain (and also that u is maximal in this respect).

In response, Fine observes that Lewis is guilty of the same fault since counterfactual assumptions will be difficult to avoid in providing a suitable account of similarity. Instead of requiring a semantics to, "provide an analysis of the locutions with which it deals," Fine settles for the position that his semantics provides, "a perspicuous account of how the truth-conditions of the sentences containing the locutions are to be determined." Although analysis is by no means the only standard for productive semantic theorizing, it is nevertheless important to establish a solid understanding of the primitive terms used to articulate a semantics in order to provide an unambiguous specification of the truth-conditions for the locutions in question.

In order to clarify the interpretation of the imposition relation as well as validate a number of important counterfactual inferences, Fine [1] provides four constraints on the imposition relation where $t.v$ is the fusion of the states t and v:

Inclusion: If $t \rightarrowtriangle_w u$, then $t \sqsubseteq u$.

Actuality: If $t \sqsubseteq w$, then $t \rightarrowtriangle_w u$ for some $u \sqsubseteq w$.

Incorporation: If $t \rightarrowtriangle_w u$ and $v \sqsubseteq u$, then $t.v \rightarrowtriangle_w u$.

Completeness: If $t \rightarrowtriangle_w u$, then u is a world-state.

Although Fine [1] offers a plausible defense for the principles above, it is unnecessary to take imposition to be primitive. Rather, the following subsection defines imposition in purely modal and mereological terms. By deriving the constraints that Fine assumes above I will avoid positing a counterfactual primitive in the metalanguage at no cost to the strength of the resulting logic. Moreover, simplifying the semantics reduces the computational complexity of its implementation in the `model-checker` software, expanding the range of inferences that the software can effectively evaluate.

2.2 Defining Imposition

I will follow Fine [3–5] in taking a *state space* $\mathcal{S} = \langle S, \sqsubseteq \rangle$ to be any complete lattice where S is the set of *states* and $\sqsubseteq$ is the *parthood relation*. For instance, the state of Sanna sitting s includes a state of Sanna's legs being bent t as a proper part, where t is a *proper part* of s— i.e., $t \sqsubset s$— just in case $t \sqsubseteq s$ and $s \not\sqsubseteq t$. Since the least upper bound of any set of states $X \subseteq S$ is unique, we may refer to the least upper bound of the set of states X as the *fusion* $\bigsqcup X$ of X. In particular, $\bigsqcup \varnothing \coloneqq \square$ is the *null state* and $\bigsqcup S \coloneqq \blacksquare$ is the *full state*. When $X = \{s, t, \ldots\}$ is finite, it will be convenient to represent the fusion of the states in X as $s.t.\cdots \coloneqq \bigsqcup\{s, t, \ldots\}$. For instance, the fusion of a state s of Sanna sitting and a state k of Kevin cooking is the state $s.k$ of Sanna sitting and Kevin cooking. In general, a fusion obtains just in case all of its parts obtain, where this interpretation will play a critical role below.

In order to encode modal structure, Fine goes on to take a *modalized state space* to be any ordered triple $\mathcal{S}^{\Diamond} = \langle S, P, \sqsubseteq \rangle$ where $\langle S, \sqsubseteq \rangle$ is a state space and $P \subseteq S$ is a primitive subset of *possible states* which satisfies the following constraints:

Nonempty: $P \neq \varnothing$.

Possibility: If $s \in P$ and $t \sqsubseteq s$, then $t \in P$.

Whereas every state in P can obtain, the states in S/P are *impossible* and so cannot obtain.[18] Given that every part of a possible state is possible, the null state $\square$ is possible. Additionally, Fine takes the states s and t to be *compatible* just in case their fusion is possible, or in symbols, $s \circ t \coloneqq s.t \in P$. For instance, although any state of Sanna sitting is compatible with any state of Kevin cooking, the same cannot be said for the states of Sanna standing which are incompatible with the states of her sitting. I will follow Fine in defining the *world states* as follows:

World States: $W \coloneqq \{w \in P \mid \forall s \circ w (s \sqsubseteq w)\}$.

World states are maximal possible states which include all compatible states as parts. Fine restricts attention to *world spaces* which are modalized state spaces that satisfy:

World Space: If $s \in P$, then $s \sqsubseteq w$ for some $w \in W$.

[18] Without admitting impossible states, the exact verifiers and falsifiers for a sentence could not be closed under fusion, and so $A \equiv A \wedge A$ and $A \equiv A \vee A$ would admit counterexamples. See [24] for discussion.

This principle amounts to identifying the possible states with the parts of world states in accordance with the gloss above, thereby ruling out infinite sequences of ever bigger possible states which do not belong to a maximal possible world state.

Given any world space, it is straightforward to define Fine's imposition relation in terms of the primitive states S, possible states P, and parthood relation $\sqsubseteq$ that it includes. Letting '$s \sqsubseteq_t w$' read 's is a *t-compatible part* of w', I will define the following:

Compatible Part: $s \sqsubseteq_t w \coloneqq s \sqsubseteq w \wedge s \circ t$.

Maximal Compatible Parts: $w_t \coloneqq \{s \sqsubseteq_t w \mid \forall r \sqsubseteq_t w(s \sqsubseteq r \rightarrow r \sqsubseteq s)\}$.

Imposition: $t \rightarrow_w u \coloneqq u \in W \wedge \exists s \in w_t(s.t \sqsubseteq u)$.

Whereas a t-compatible part of w is a part of w that is compatible with t, a maximal t-compatible part of w is any t-compatible part of w that is not a proper part of any t-compatible part of w. Accordingly, an outcome u of imposing t on w is any world state which includes as parts both t as well as a maximal t-compatible part of w. Intuitively, the outcome world states are the results of minimally changing a given world state to include the imposed state, where outcomes need not be unique.

Whereas Fine admits that his primitive imposition relation has a counterfactual reading, this complaint cannot be raised against the primitive states S, possible states P, and parthood relation $\sqsubseteq$ employed in the definitions above. Rather, the primitives included in a world space have purely modal and mereological readings. Given these resources, the definitions above provide an analysis of the imposition relation, reducing the number of primitives included in the semantics. Since **P1 - P4** in §7 derive the constraints on imposition that Fine assumes, the resulting semantics validate a logic for counterfactuals that is at least as strong as the logic that Fine defends.

In addition to excluding counterfactual primitives from the metalanguage, defining the imposition relation improves the efficiency of the semantics' implementation in the `model-checker` software that I developed for evaluating counterfactual inferences.[19] In order to provide evidence that an inference is valid, the `model-checker` may be used to show that the inference does not have countermodels below a certain finite level of complexity, where the strength of that evidence is proportional to the number of models surveyed. Although the `model-checker` draws on Microsoft's state of the art SMT solver Z3, every computational system has its limits. By simplifying the semantics implemented in Z3, the `model-checker` is able to evaluate inferences of greater complexity as well as surveying countermodels with a greater number of atomic elements. As a result, simplifying the semantics strengthens the evidence that the `model-checker` can provide for a wider range of inferences.

Despite overcoming the problems that intensional similarity theories face without positing a counterfactual primitive in the metalanguage, the resulting semantics does not accommodate tense operators which often combine with counterfactuals. In order to provide a unified semantic framework for studying tensed counterfactuals, I will replace Fine's primitive set of possible states with a set of *times* and a two-place *task relation* in the following section. In addition to providing temporal structure, these

[19]I am grateful to Miguel Buitrago who helped me develop the software during his UROP at MIT. The `model-checker` software is free and open source: https://pypi.org/project/model-checker/

resources provide the modal structure needed to define the possible states P that Fine takes to be primitive, deriving Nonempty, Possibility, and World Spaces.

3 The Construction of Possible Worlds

Although states may be variously interpreted for different applications, it will help to fix ideas by taking states to be *some things being a specific way*. Despite taking states to be primitive, we may draw conceptual connections between states and possible worlds by identifying the *possible states* with restrictions of possible worlds. Whereas possible worlds are complete histories of everything, states are *static* and typically *partial*, concerning some limited way for certain things to be at an instant. For instance, given a possible world in which Sanna is sitting at time x, restricting to the part of that world which occurs at time x and makes it true that Sanna is sitting is a state of Sanna sitting. In general, states are *wholly relevant* to the sentences that they exactly verify or falsify, and so most states will neither exactly verify nor falsify a given sentence. Accordingly, state are much more discriminating than moments or worlds.

States are also many. Just as the world-time pairs in which Sanna is sitting may be taken to cover all of the different ways for it to be true that Sanna is sitting, the states of Sanna sitting are the parts of the world time-slices which make it true that Sanna is sitting in all of those different ways. Moreover, states are *specific*: just as there is exactly one way for a possible world to be actual, there is exactly one way for a state to *obtain*. Put otherwise, neither possible worlds nor states are multiply realizable but rather model specific realizations. It is nevertheless important not to conflate the specificity of states with their possibility. Given a state s of Sanna sitting and a state t of her standing, the fusion state $s.t$ is just as specific as s and t considered on their own, amounting to a precise way for Sanna's body to be arranged despite it being impossible for $s.t$ to obtain. For instance, $s.t$ includes Sanna's legs being both bent as they are in her sitting state s and straight as they are in her standing state t. Insofar as t makes it false that Sanna is sitting, t makes it true that Sanna is not sitting, and so $s.t$ makes the conjunction true that Sanna is sitting and not sitting. Just as s obtaining is sufficient for it to be true that Sanna is sitting independent of whether s can obtain, $s.t$ obtaining is sufficient for it to be true that Sanna is sitting and not sitting despite the fact that $s.t$ cannot obtain. Instead of taking an impossible proposition to have no exact verifiers, the present framework models impossibility by way of the impossible states themselves. It is for this reason that the proposition that Sanna is sitting and standing differs from the proposition that Kevin is cooking and sleeping, or any other impossible proposition. In general, each impossible proposition will have its own range of impossible exact verifier states, where each exact verifier state specifies an impossible way for that proposition to be true.

Rather than relying on a primitive set of possible states P to distinguish the possible and impossible states, the following section will introduce the *task relation* $\rightarrow$ to model the possible state transitions, defining the set of possible states in these terms. By also including a set of *times* T, I will demonstrate how to construct possible worlds. By contrast, merely specifying a primitive set of possible states P does not encode any temporal structure on its own. Insofar as states are static, they are entirely devoid of temporal structure. Even including a set of times T by which to index states into a time series, there is still no telling which time series are possible worlds and which are not. It is this theoretical role which the task relation is intended to fill.

3.1 Task Space

Given any two states s and t, we may ask whether it is possible for s to transition to t. For instance, although it is possible for the state of Sanna sitting to transition to a state of Sanna standing, it is not possible for the state of Sanna sitting to transition to a state of Kevin cooking. In order to capture these differences, I will take any ordered pair of states $\langle s, t\rangle$ to represent a *state transition* from s obtaining to t obtaining. Since not all state transitions are possible, I will take there to be a *task* $s \to t$ just in case the transition $\langle s, t\rangle$ from s obtaining to t obtaining is possible.

For any state s, the *trivial transition* $\langle s, s\rangle$ leaves s unchanged. So long as it is possible for s to obtain, then making no change to s amounts to a possible transition from s obtaining to s obtaining, and so there is a *trivial task* $s \to s$. Letting the states s and t be *connected* $s \sim t$ just in case either $s \to t$ or $t \to s$, it follows that every possible state is connected to some state, in particular to itself. Conversely, if it is impossible for s to obtain, then there is no state t where either the transition from s obtaining to t obtaining nor the transition from t obtaining to s obtaining are possible, and so there is no state t where $s \sim t$. It follows that it is possible for s to obtain just in case s is connected to some state. I will define the set of *possible states* $P \coloneqq \{s \in S \mid \exists t(s \sim t)\}$ to include all states connected to some state.

Continuing with the example above, we may imagine that Sanna stands up, thereby transitioning from sitting s to standing t. Insofar as this transition is possible— i.e., $s \to t$— it follows that that it is possible for Sanna to sit and also possible for her to stand, i.e., both s and t are possible. More generally, $s, t \in P$ whenever $s \to t$. Although it follows that $s \in P$ whenever $s \to s$, the converse need not hold:

Restricted Reflexivity: If $s \in P$, then $s \to s$.

The principle above is not required for what follows, and so will not be imposed.[20] Nevertheless, we may observe that insofar as there are impossible states, it follows that the task relation is not reflexive since impossible states are not connected to any state, and so are inaccessible even to themselves. For instance, consider the fusion state $s.t$ of Sanna sitting and standing: since $s.t$ cannot obtain, no transition to $s.t$ obtaining is possible, nor is any transition from $s.t$ obtaining possible.

In order to constrain the interpretation of $\to$ in support of the intended reading of P as the set of possible states, recall that a state fusion is said to obtain just in case all of its parts obtain. Since a possible state can obtain, we ought to expect that each of its parts can obtain as a result. It is for this reason that Fine requires modalized state spaces to satisfy Possibility. Given that P is defined rather than primitive, I will derive Possibility by constraining the interaction between the task and parthood relations. In particular, tasks between fusions must be decomposable into subtasks

[20] A state is *transient* just in case it is possible but does not have a trivial task. Although such states will not be needed below, neither will they be ruled out by imposing Restricted Reflexivity.

between their respective parts, where $s \to t$ is a *subtask* of $s' \to t'$ just in case $s \sqsubseteq s'$ and $t \sqsubseteq t'$. More specifically, I will assume the following interaction constraints:

Parthood: (L) If $d \sqsubseteq s$ and $s \to t$, then $d \to r$ for some $r \sqsubseteq t$.[21]
(R) If $r \sqsubseteq t$ and $s \to t$, then $d \to r$ for some $d \sqsubseteq s$.

These constraints ensure that every part is accounted for in any task between fusions. For example, given the red brake lights ahead (state r), we may take there to be a task from Nicky's driving state $d.r$ to a state $b.r'$ in which she brakes (state b) where r' is a similar arrangement of red brake lights. Since $d \sqsubseteq d.r$, it follows by Parthood that there is some suitable $t \sqsubseteq b.r'$ where $d \to t$. Moreover, since $b \sqsubseteq b.r'$, there is some $s \sqsubseteq d.r$ where $s \to b$. Although Parthood only specifies the existence of such states, we may expect that $s = d$ and $t = b$ so that in this case both $d \to b$ and $r \to r'$. Given Parthood, **P5** in §7 shows that it is easy to derive Possibility.

Given that the null state $\square$ is a part of every state, Parthood is trivialized if $s \to \square$ and $\square \to s$ for any state s. In order to avoid instances of this scenario, it suffices to require the null state $\square$ to be necessary. A state s is *contingent* just in case s is connected to a distinct state, i.e., $s \sim t$ for some $t \neq s$. Intuitively, the contingent states are those states which can change on account of possibly transitioning to or from a distinct state. It follows immediately that all contingent states are possible. By contrast, there are no tasks between impossible states. Rather, impossible states are non-contingent insofar as they cannot transition to or from any state at all, much less to or from distinct states. Despite being possible, the necessary states are also non-contingent since they must obtain. Thus s is *necessary* just in case s is only connected to itself where $N \coloneqq \{s \in S \mid \forall t \in S(s = t \Leftrightarrow s \sim t)\}$ is the set of all necessary states. We may avoid trivializing instances of Parthood by assuming:

Nullity: $\square = t$ just in case $\square \sim t$.

It follows that the null state $\square$ is necessary where this fits with its definition as the fusion of the empty set. Since a fusion obtains just in case all of its parts obtain, nothing is required for $\square$ to obtain, and so $\square$ obtains trivially and thus of necessity. In addition to avoiding trivializing instances of Parthood, it follows that every necessary state is possible, and so **P6** in §7 derives Nonempty from Nullity.[22]

It remains to show that every possible state is a part of a world state in accordance with Fine's World Space constraint. Given any state s and possible state t, we may consider the parts of s that are compatible with t. In order to prevent there from being bigger and bigger parts $r_1 \sqsubset r_2 \sqsubset \ldots$ of s without end which are all compatible with t, I will draw on the definitions given above in §2.2 to provide the following constraint:

Maximality: If $s \in S$ and $t \in P$, there is some maximal t-compatible part $r \in s_t$.

[21] These principles are interderivable if $\to$ is symmetric. Although it is natural to take the task relation to be symmetric and transitive for certain applications, these constraints will not be required below.

[22] It is also natural to assume Compatibility: if $s \in N$ and $t \in P$, then $s \circ t$. Additionally, one might impose Necessity: if $s \in N$ and $t \sqsubseteq s$, then $t \in N$. Neither constraint will be required for what follows.

Consider the total state s of the strategy room at the moment Nixon nearly pressed the button. We may imagine s to include parts that fix all the features of the objects which make up the room and its occupants. Although any state of Nixon pressing the button is incompatible with s, this is not true for the parts of s. For instance, the state a of Nixon's advisers sitting just as they are is a part of s and perfectly compatible with a state of Nixon pressing the button. Given a state b of Nixon pressing the button, MAXIMALITY requires there to be a maximal part of s which is compatible with b, ruling out unending chains of bigger and bigger b-compatible parts of s.[23]

Whereas Fine restricts attention to world spaces as a basic assumption, **P7** in §7 derives WORLD SPACE from MAXIMALITY. Nevertheless, WORLD SPACE ensures that every proposition has exactly one truth-value at every world state in accordance with Fine's [3] original observations. Since I will identify possible worlds at a time with world states, it follows that the tautologies of classical logic are true in every model at every world at every time. This result provides a powerful reason to maintain WORLD SPACE and given that WORLD SPACE may be derived from MAXIMALITY, a corresponding reason to adopt MAXIMALITY. Moreover, MAXIMALITY is in keeping with an intuitive understanding of states, ruling out exotic cases in which there is no biggest part of a state s that is compatible with a possible state t.[24]

Given the above, I will define a *task space* to be any triple $\mathcal{T} = \langle S, \sqsubseteq, \rightarrow \rangle$ where $\langle S, \sqsubseteq \rangle$ is a state space and $\mathcal{T}$ satisfies PARTHOOD, NULLITY, and MAXIMALITY. Given **P5** - **P7**, it follows that every task space determines a world space. Whereas the world spaces that Fine defines are rich with mereological and modal structure, task spaces provide the additional resources needed to encode the temporal structure that I will take possible worlds to include. By drawing on these elements, the following subsection will present the construction of possible worlds which will play a critical role in the semantics that I will present in the following section.

3.2 Possible Worlds

Assuming a discrete theory of time for simplicity, I will model *times* by the set of integers $\mathbb{Z}$ where every time x differs from its successor time $x+1$ by a unit duration specified for the application at hand.[25] Letting $\mathcal{T}_{\mathbb{Z}} = \langle S, \sqsubseteq, \rightarrow, \mathbb{Z} \rangle$ be a *discrete task space*, a *discrete evolution* $\tau : \mathbb{Z} \rightarrow S$ is any function from times to states. A discrete evolution τ is *possible* just in case $\tau(x) \rightarrow \tau(x+1)$ for all times $x \in \mathbb{Z}$. It follow immediately that every state in a possible discrete evolution is possible.

A *world history* parameterized by $\mathbb{Z}$ is any possible evolution $\alpha : \mathbb{Z} \rightarrow W$ so that $\alpha(x) \in W$ for every time $x \in \mathbb{Z}$. We may then take $H_{\mathbb{Z}}$ to be the set of all world histories parameterized by $\mathbb{Z}$. Given a history α, we may refer to the range of values that α occupies at different times as the *moments* of that history. Although officially the moments of a history α are its members which take the form $\langle x, \alpha(x) \rangle$, it is convenient to write them as $\alpha(x)$. For instance, suppose that α is a history which occupies the

[23]Recall that states are static. Since pushing the button is not static, one might deny that any state makes it true that Nixon pushed the button. I will return to this objection in the conclusion. For now, think of the state of Nixon pushing the button as the point of no return in which the circuit is closed.

[24]For certain applications, also impose DETERMINISM: if $s \rightarrow t$ and $s \rightarrow r$, then $t = r$. Alternatively, one might reject DETERMINISM while maintaining WORLD DETERMINISM: if $s \rightarrow t$ and $s \rightarrow r$ where $s, t, r \in W$, then $t = r$. Neither of these constraints are required or particularly natural for the present application.

[25]I will consider an objection to the arbitrariness of the unit in §6 along with a continuous analogue.

same world state $\alpha(x) = \alpha(y)$ at different times $x \neq y$. Even though the world state is the same at both times, the moments differ since they occur at different times in α's history. Letting $\alpha \approx \beta$ just in case there is some $z \in \mathbb{Z}$ where $\alpha(x) = \beta(x + z)$ for all $x \in \mathbb{Z}$, we may identify the *possible worlds* parameterized by $\mathbb{Z}$ with the set of equivalence classes $W_{\mathbb{Z}} = \{[\alpha]_{\mathbb{Z}} \mid \alpha \in H_{\mathbb{Z}}\}$ where $[\alpha]_{\mathbb{Z}} = \{\beta \in H_{\mathbb{Z}} \mid \alpha \approx \beta\}$. Whereas possible worlds represent genuinely distinct sequences of world states, world histories may be compared to coordinate planes which stipulate an origin while acknowledging that the choice of origin does not represent anything significant about the space it represents. Although the set of possible worlds may claim to hold a metaphysical standing that the set of histories cannot, it is the histories that will play an important role in the semantics presented below rather than possible worlds. Nevertheless, I will refer to histories as worlds, though officially they are to be distinguished.

In order to evaluate counterfactual conditionals in a language that includes tense operators, it will be important to consider the range of counterfactual worlds to result from making minimal changes to the world of evaluation at the time of evaluation. Roughly, the semantics for counterfactual conditionals presented in §4 will quantify over the worlds that result from imposing an exact verifier state for the antecedent on the world of evaluation at the time of evaluation, checking to see if the consequent is true. Before presetting these details, the following section will conclude by motivating the hyperintensional theory of proposition that I will use to interpret the language.

3.3 Propositions

Intuitively, a counterfactual conditional is true at a moment of evaluation just in case the consequent is true in any counterfactual moment to result from minimally changing the evaluation moment so that the antecedent is true. However, if the antecedent is interpreted by a set of moments, it is unclear what the result of making a minimal change would be. After all, a counterfactual moment in which the antecedent is true may differ from the evaluation moment in unrelated ways. The reason the exact verifier states for the antecedent are of greater use in this application than moments is that they amount to specific ways for the antecedent to be true without including anything irrelevant. Put otherwise, the exact verifier states for the antecedent belong to the *subject-matter* of the antecedent. By contrast, the moments in which the antecedent is true contain elements reaching far beyond the subject-matter of the antecedent. This motivates a departure from intensional theories of propositions to a hyperintensional theory of propositions which is sensitive to differences in subject-matter in addition to modal profile. By taking moments to be world states which have mereological structure, we may make minimal changes to the evaluation moment without including irrelevant details, checking to see if the consequent is true as a result.

In [24], I draw on Fine's [3, 4] recent work in order to identify and defend a bilateral theory of propositions which distinguishes propositions that differ in either subject-matter or modal profile without positing any unnecessary distinctions between propositions. Given that the present application calls for a theory of propositions which is sensitive to both modal profile and subject-matter, I will employ that bilateral theory here without reproducing its defense.[26] The theory is said to be *bilateral* on

[26]See Fine [25] for a comparison of a related approach to Yablo's [26] theory of subject-matter.

account of taking propositions to consist of both a positive and negative content. For instance, 'John is sitting down' expresses a proposition that consists of a set of exact verifier states which make that sentence true along with a set of exact falsifier states which make that sentence false. However, not all ordered pairs of sets of states are propositions. In order to define the space of propositions, let a set of states X be *closed* just in case the fusion of any nonempty subset $Y \subseteq X$ belongs to X. Given any sets of states V and F, the ordered pair $\langle V, F\rangle$ is *exclusive* just in case the states in V are incompatible with the states in F, and *exhaustive* just in case every possible state is compatible with some state in either V or F. An ordered pair $\langle V, F\rangle$ is a *bilateral proposition* just in case $\langle V, F\rangle$ is exclusive and exhaustive where V and F are both closed under fusion. Given any proposition $\langle V, F\rangle$, every world state includes a part which belongs to V or F but not both, thereby validating classical logic.[27] Letting $\mathbb{P}$ be the set of bilateral propositions, we may now turn to interpret a propositional language with extensional, modal, counterfactual conditional, and tense operators.

4 Counterfactual Logic

Reading '$\boxed{\scriptstyle P}A$' as 'It has always been the case that A' and '$\boxed{\scriptstyle F}A$' as 'It is always going to be the case that A', I will take $\mathcal{L} = \langle \mathbb{L}, \top, \bot, \neg, \boxed{\scriptstyle P}, \boxed{\scriptstyle F}, \wedge, \vee, \Box\!\!\rightarrow\rangle$ to be a propositional language where $\mathbb{L} \coloneqq \{p_i \mid i \in \mathbb{N}\}$ is the set of *sentence letters*, and $\top$ and $\bot$ are the *top* and *bottom* constants. We may define the *extensional sentences* of $\mathcal{L}$ as follows:

$$\varphi \ ::= \ p_i \mid \top \mid \bot \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi.$$

Letting $\texttt{ext}(\mathcal{L})$ be the set of extensional sentences of $\mathcal{L}$, we may define the *well-formed sentences* of $\mathcal{L}$ by restricting the antecedent of a counterfactual to $\varphi \in \texttt{ext}(\mathcal{L})$:

$$A \ ::= \ p_i \mid \top \mid \bot \mid \neg A \mid A \wedge A \mid A \vee A \mid \boxed{\scriptstyle P}A \mid \boxed{\scriptstyle F}A \mid \varphi \Box\!\!\rightarrow A.$$

Although the consequent of a counterfactual may be any well-formed sentence of $\mathcal{L}$, only the extensional operators may occur in the antecedent. We may then define a *discrete model* $\mathcal{M} = \langle S, \sqsubseteq, \rightarrow, \mathbb{Z}, |\cdot|\rangle$ of $\mathcal{L}$ to be any ordered tuple where $\langle S, \sqsubseteq, \rightarrow, \mathbb{Z}\rangle$ is a discrete task space, $|p_i| \in \mathbb{P}$ for all $p_i \in \mathbb{L}$, $|\top| = \langle S, \{\blacksquare\}\rangle$, and $|\bot| = \langle \varnothing, \{\square\}\rangle$.[28] After presenting a compositional semantics in the following subsection, I will provide a theory of logical consequence in §4.2, discussing countermodels in §4.3, applying these resources to analyze forwards, backwards, and backtracking counterfactuals in §5.

4.1 A Unified Semantics

In [24] I argue that Fine's [1, 3] exact inclusive semantics for the extensional operators preserves differences in both subject-matter and modal profile without drawing any unnecessary distinctions. Given the present aim to preserve differences in the subject-matter and modal profile of a proposition, I will reproduce that semantics here:

[27] See **P10** below and Fine [3, p. 630] for the original observation and definitions.

[28] Although $\bot$ is not needed below, it is worth observing that $|\bot| \neq |\neg\top|$. Rather, $\neg\bot$ and $\neg\top$ are the other two top and bottom elements making for a total of four *extremal elements*. See [27] for further details.

Product: $X \otimes Y := \{s.t \mid s \in X,\ t \in Y\}$.

Sum: $X \oplus Y := X \cup Y \cup (X \otimes Y)$.

Conjunction: $\langle V, F\rangle \wedge \langle V', F'\rangle := \langle V \otimes V',\ F \oplus F'\rangle$

Disjunction: $\langle V, F\rangle \vee \langle V', F'\rangle := \langle V \oplus V',\ F \otimes F'\rangle$

Negation: $\neg\langle V, F\rangle := \langle F, V\rangle$.

Whereas the product $X \otimes Y$ is the set of pairwise fusions of the states in X with the states in Y, the sum $X \oplus Y$ is the union of X, Y, and their product $X \otimes Y$. Thus conjunction takes the product of the verifiers for the conjuncts followed by the sum of the falsifiers for the conjuncts, and disjunction does the reverse. By contrast, negation exchanges the verifiers and falsifiers of the negated proposition.[29]

In §7, **P9** shows that $\mathbb{P}$ is closed under the propositional operators defined above. Thus we may extend the interpretation provided by a model $\mathcal{M}$ to all extensional sentences with the following *exact inclusive semantic* clauses where $|\varphi| \in \mathbb{P}$ by **P10**:

[$\neg$] $|\neg\varphi| = \neg|\varphi|$.

[$\wedge$] $|\varphi \wedge \psi| = |\varphi| \wedge |\psi|$.

[$\vee$] $|\varphi \vee \psi| = |\varphi| \vee |\psi|$.

Since there is no confusing 'φ' with '$|\varphi|$', the abuse of notion for '$\neg$', '$\wedge$', and '$\vee$' may be forgiven. Having fixed the interpretation of the extensional sentences, we may interpret all well-formed sentences of $\mathcal{L}$ where $\alpha, \beta \in H_{\mathbb{Z}}$ are worlds, $x, y \in T$ are times, and $q \in \mathbb{L} \cup \{\top, \bot\}$ is either a sentence letter, top element, or bottom element:

$\mathcal{M}, \alpha, x \vDash q$ *iff* there is some $s \sqsubseteq \alpha(x)$ where $s \in |q|^+$.

$\mathcal{M}, \alpha, x \vDash \neg A$ *iff* $\mathcal{M}, \alpha, x \nvDash A$.

$\mathcal{M}, \alpha, x \vDash A \wedge B$ *iff* $\mathcal{M}, \alpha, x \vDash A$ and $\mathcal{M}, \alpha, x \vDash B$.

$\mathcal{M}, \alpha, x \vDash A \vee B$ *iff* $\mathcal{M}, \alpha, x \vDash A$ or $\mathcal{M}, \alpha, x \vDash B$.

$\mathcal{M}, \alpha, x \vDash \boxed{\scriptstyle P}A$ *iff* $\mathcal{M}, \alpha, y \vDash A$ for all $y < x$.

$\mathcal{M}, \alpha, x \vDash \boxed{\scriptstyle F}A$ *iff* $\mathcal{M}, \alpha, y \vDash A$ for all $y > x$.

$\mathcal{M}, \alpha, x \vDash \varphi \;\Box\!\!\rightarrow C$ *iff* $\mathcal{M}, \beta, x \vDash C$ whenever $t \in |\varphi|^+$ and $t \rightarrow_{\alpha(x)} \beta(x)$.

Whereas the exact semantics took a bilateral form in order to specify a propositional operation for negation that respects differences in subject-matter, evaluating sentences for truth does not require the same sensitivity to subject-matter. Suppressing reference to the model, **P11** in §7 shows that $\varphi \in \texttt{ext}(\mathcal{L})$ is true at a world-time pair just in case there is a part of that world at that time which exactly verifies φ. Given the definition of a proposition in §3.3, it follows from World Space that every extensional sentence has exactly one truth-value at every world-time pair, where the remaining semantic clauses extend this property to all well-formed sentences of the language.[30]

[29]The hyperintensional space of propositions $\mathbb{P}$ forms a *non-interlaced bilattice* rather than a Boolean lattice as in intensional theories. See [28] and [29] for definitions and [24] for further discussion.

[30]See Fine [3, pp. 665-7] for related results.

In addition to maintaining standard semantic clauses for the extensional and tense operators, the semantics for the counterfactual conditional has been presented in terms of imposition in order to facilitate comparison with Fine's account. However, given the definition of imposition provided in §2.2 above, we may restate the semantics for the counterfactual conditional in more basic terms:

$\mathcal{M}, \alpha, x \vDash \varphi \boxright C$ *iff* $\mathcal{M}, \beta, x \vDash C$ whenever $\beta \in H_{\mathbb{Z}}$, $t \in |\varphi|^+$, and there is a maximal t-compatible part $s \in \alpha(x)_t$ where $s.t \sqsubseteq \beta(x)$.

The clause above articulates the semantics for counterfactuals in purely modal and mereological terms. Intuitively, a counterfactual is true in a world α at a time x just in case the consequent is true in any world β at x where $\beta(x)$ is the result of minimally changing $\alpha(x)$ to make the antecedent true. This approach is to be contrasted with similarity theories which compare temporally extended worlds for similarity in their entirety, as well as with imposition theories which posit a counterfactual primitive in the metalanguage while restricting consideration to the present tense. Despite its definition in other terms, it will often be convenient to continue to speak of the outcome worlds to result from imposing a state on a world at a given time.

Having extended the interpretations provided by the models of $\mathcal{L}$ to all well-formed sentences, we may present a theory of logical consequence for any $\Gamma \cup \{C\} \subseteq \texttt{wfs}(\mathcal{L})$ in order to study the interactions between the operators included in the language:

Logical Consequence: $\Gamma \vDash C$ *iff* for any model $\mathcal{M}$ of $\mathcal{L}$, world α, and time x, if $\mathcal{M}, \alpha, x \vDash A$ for all $A \in \Gamma$, then $\mathcal{M}, \alpha, x \vDash C$.

As usual, a rule schema is *valid* just in case for any instance, its conclusion is a logical consequence of its premises, and a metarule is *valid* just in case it preserves validity. Without attempting to provide a complete logic for the semantics above, the following subsection will present three extensions of classical propositional logic in order to study the interactions between the operators included in the language.

4.2 Logics

Letting $\mathcal{L}^{\text{PL}} = \langle \mathbb{L}, \neg, \wedge, \vee \rangle$ be an extensional language without top and bottom, I will take the deduction relation of classical propositional logic $\vdash_{\text{PL}}$ to be defined over a restriction to the well-formed sentences of $\mathcal{L}^{\text{PL}}$. By taking $\mathcal{L}^{\text{CL}} = \langle \mathbb{L}, \neg, \wedge, \vee, \boxright \rangle$, I will assume that $\varphi, \psi, \chi \ldots \in \texttt{ext}(\mathcal{L}^{\text{CL}})$ and $A, B, C, \ldots \in \texttt{wfs}(\mathcal{L}^{\text{CL}})$ as above, where $\varphi \boxright \Gamma \coloneqq \{\varphi \boxright A \mid A \in \Gamma\}$. I will then define $\vdash_{\text{CL}}$ to be the smallest extension of $\vdash_{\text{PL}}$ to be closed under the standard structural rules and all instances of the following:

R1 *If* $\Gamma \vdash C$, *then* $\varphi \boxright \Gamma \vdash \varphi \boxright C$

C1 $\varphi \boxright \varphi$

C2 $\varphi,\ \varphi \boxright A \vdash A$

C3 $\varphi \boxright \psi,\ \varphi \wedge \psi \boxright A \vdash \varphi \boxright A$

C4 $\varphi \vee \psi \boxright A \vdash \varphi \wedge \psi \boxright A$

C5 $\varphi \vee \psi \boxright A \vdash \varphi \boxright A$

C6 $\varphi \vee \psi \boxright A \vdash \psi \boxright A$

C7 $\varphi \boxright A,\ \psi \boxright A,\ \varphi \wedge \psi \boxright A \vdash \varphi \vee \psi \boxright A$

Whereas **R1** requires the consequences of a counterfactual hypothesis to be closed under deduction, **C1** is the *Identity* schema, **C2** is *Counterfactual Modus Ponens*, and **C3** is *Weakened Transitivity*.[31] Additionally, **C4** - **C6** eliminate a disjunction from the antecedent of a counterfactual, and **C7** introduces a disjunction to the antecedent of a counterfactual.[32] It is easy to see that **SDA** follows from **C5** and **C6**. Referring to the resulting *Counterfactual Logic* as **CL**, §7 provides elements of the proof that **CL** is sound. Additionally, we may derive the following consequences from **R1**:

D1 $\varphi \boxright A,\ \varphi \boxright B \vdash \varphi \boxright A \wedge B$

D2 *If* $A \vdash B$*, then* $\varphi \boxright A \vdash \varphi \boxright B$

These rules correspond to Fine's [1, 2] *Finite Conjunction* and *Classical Weakening* rules respectively.[33] Whereas the following section will present a number of inferences that are excluded from **CL** on account of admitting countermodels, the remainder of the present section extends **CL** to include modal and tense operators.

Restricting consideration to the expressive resources included in $\mathcal{L}^{\text{CL}}$, one might attempt to define metaphysical necessity in terms of the counterfactual conditional operator as $\Box\varphi \coloneqq \neg\varphi \boxright A \wedge \neg A$. However, given the limitation that $\varphi \in \texttt{ext}(\mathcal{L}^{\text{CL}})$, it follows that iterated modalities are not well-formed.[34] Rather, I will include the top and bottom elements in $\mathcal{L}^{\text{CML}} = \langle \mathbb{L}, \top, \bot, \neg, \wedge, \vee, \boxright \rangle$ in order to define *metaphysical necessity* as $\Box A \coloneqq \top \boxright A$. By also defining the *might counterfactual conditional* as $\varphi \diamondright A \coloneqq \neg(\varphi \boxright \neg A)$, I will take $\Diamond A \coloneqq \top \diamondright A$ to define *metaphysical possibility*, letting $\vdash_{\text{CML}}$ minimally extend $\vdash_{\text{CL}}$ to include all instances of the following:

M1 $\top$

M2 $\neg\bot$

M3 $A \to \Box\Diamond A$

M4 $\Box A \to \Box\Box A$

M5 $\Box(\varphi \to A) \vdash \varphi \boxright A$

Having defined the modal operators in $\mathcal{L}^{\text{CML}}$ instead of $\mathcal{L}^{\text{CL}}$, all instances of the axioms given above are well-formed. It is important to observe that **M5** together with **C2** entail that the counterfactual conditional is of intermediate strength between the strict conditional and material conditional. We may then derive the following:

D3 *If* $\vdash A$*, then* $\vdash \Box A$*.*

D4 $\Box(A \to B) \to (\Box A \to \Box B)$

D5 $\Box A \to A$

D6 $\neg A,\ \varphi \boxright A \vdash \neg\varphi$

D7 $\Box A \leftrightarrow \neg\Diamond\neg A$

D8 $\Box\varphi \leftrightarrow \neg\varphi \boxright \bot$

D9 $\varphi \boxright \top$

D10 $\bot \boxright A$

[31] See Lewis [8, p. 35] and Fine [1] for discussion of **C3**.

[32] Fine's [1, 2] provides **C1** - **C7** but does not include **R1**.

[33] Were infinite conjunction included in the language, we could also derive the infinite conjunction rule that Fine [2] assumes. I will restrict consideration to sentences with finite length for simplicity.

[34] Although it is possible to extend the semantics in order to permit the antecedent to be any well-formed sentence, interpreting counterfactuals with counterfactual antecedents remains unintuitive.

Given **D3** - **D5** together with **M3** - **M4**, *Counterfactual Modal Logic* **CML** entails an S5 logic. Whereas S5 is only valid given appropriate constraints on Kripke frames in an intensional semantics, the validity of **M3** - **M4** as well as **M1**, **R1**, and **C2** from which **D3** - **D5** follow does not depend on constraining the range of task spaces. Although a primitive accessibility relation between states could be added in order to weaken the logic, I will assume that metaphysical modality has an S5 logic, undermining the motivation to complicate the semantics in this way. Additionally, **D6** is *Counterfactual Modus Tollens*, **D7** derives the standard duality between the metaphysical modals, **D8** entails the equivalence that one might take to define metaphysical necessity in $\mathcal{L}^{\text{CL}}$, and **D9** - **D10** derive the *Triviality* axioms that Fine [1] assumes.

Having considered $\mathcal{L}^{\text{PL}}$, $\mathcal{L}^{\text{CL}}$, and $\mathcal{L}^{\text{CML}}$, it remains to include temporal operators, presenting a range of further axiom schemata within the unrestricted language $\mathcal{L}$. In order to present these additions, I will take $\boxed{\scriptstyle\mathrm{F}}\Gamma \coloneqq \{\boxed{\scriptstyle\mathrm{F}}A \mid A \in \Gamma\}$ and $\ulcorner A_{\langle\mathrm{P}|\mathrm{F}\rangle}\urcorner$ to be the result of exchanging '$\boxed{\scriptstyle\mathrm{P}}$' and '$\boxed{\scriptstyle\mathrm{F}}$' in A where $\Gamma_{\langle\mathrm{P}|\mathrm{F}\rangle} \coloneqq \{A_{\langle\mathrm{P}|\mathrm{F}\rangle} \mid A \in \Gamma\}$. I will also assume the standard metalinguistic abbreviations $\Diamond_{\mathrm{P}}A \coloneqq \neg\boxed{\scriptstyle\mathrm{P}}\neg A$ which may be read 'It was the case that A' and $\Diamond_{\mathrm{F}}A \coloneqq \neg\boxed{\scriptstyle\mathrm{F}}\neg A$ for 'It is going to be the case that A'. We may then take $\vdash_{\text{CTL}}$ to minimally extend $\vdash_{\text{CML}}$ to include all instances of the following:

R2 *If* $\Gamma \vdash C$, *then* $\boxed{\scriptstyle\mathrm{F}}\Gamma \vdash \boxed{\scriptstyle\mathrm{F}}C$	**R3** *If* $\Gamma \vdash C$, *then* $\Gamma_{\langle\mathrm{P}|\mathrm{F}\rangle} \vdash C_{\langle\mathrm{P}|\mathrm{F}\rangle}$
T1 $A \to \boxed{\scriptstyle\mathrm{F}}\Diamond_{\mathrm{P}}A$	**T2** $\boxed{\scriptstyle\mathrm{F}}A \to \boxed{\scriptstyle\mathrm{F}}\boxed{\scriptstyle\mathrm{F}}A$
T3 $\boxed{\scriptstyle\mathrm{F}}A \to \Diamond_{\mathrm{F}}A$	**T4** $\Diamond_{\mathrm{F}}\Diamond_{\mathrm{P}}A \to (\Diamond_{\mathrm{P}}A \vee A \vee \Diamond_{\mathrm{F}}A)$
T5 $(\boxed{\scriptstyle\mathrm{P}}A \wedge A \wedge \Diamond_{\mathrm{F}}\top) \to \Diamond_{\mathrm{F}}\boxed{\scriptstyle\mathrm{P}}A$	**T6** $\Box\boxed{\scriptstyle\mathrm{F}}A \leftrightarrow \Box A$.
T7 $\Diamond_{\mathrm{F}}\Box A \to \Box A$.	**T8** $\Diamond_{\mathrm{F}}\Diamond A \to \Diamond A$.

Whereas **R2** entails analogues of **D3** - **D4** for the operator $\boxed{\scriptstyle\mathrm{F}}$, **R3** exchanges '$\boxed{\scriptstyle\mathrm{P}}$' and '$\boxed{\scriptstyle\mathrm{F}}$' throughout any inference, making the past and future have the same structure at every time in each world. Focusing on the future, **T1** captures the idea that every present moment is past in any future moment and **T2** - **T5** require that the temporal ordering be transitive, endless, linear, and discrete. The remaining axiom schemata describe the interactions between the metaphysical modal and tense operators. I will characterize this relationship by way of the derived schema **D11** $\Box A \to \triangledown A$ in §7 where $\triangledown A \coloneqq \boxed{\scriptstyle\mathrm{P}}A \wedge A \wedge \boxed{\scriptstyle\mathrm{F}}A$ reads 'It is always the case that A'. Much more generally, we may assert the slogan: *modal truths are necessarily always the case.*

I will refer to the system including all axiom schemata and metarules defended above as *Counterfactual Tense Logic* **CTL** where I will derive **D1** - **D11** in §7. In addition to being sound over the discrete models of $\mathcal{L}$, **CTL** is strong enough to justify a counterfactual conditional reading of '$\boxright$', a metaphysical modal reading of '$\Box$', as well as the temporal readings of '$\boxed{\scriptstyle\mathrm{P}}$' and '$\boxed{\scriptstyle\mathrm{F}}$' presented above. The following subsection will complement these findings by considering a number of invalid inferences that are excluded from **CTL** (and so also from **CL**) on account of admitting countermodels.

4.3 Countermodels

In order to get a better sense of the semantics for counterfactual conditionals, it will help to review a number of invalid schemata. Consider the following:

#1 $\varphi, A \vdash \varphi \boxright A$.

#2 $\varphi \boxright \psi, \psi \boxright A \vdash \varphi \boxright A$.

#3 $(\varphi \boxright A) \vee (\varphi \boxright \neg A)$.

#4 $\varphi \boxright A \vee B \vdash (\varphi \boxright A) \vee (\varphi \boxright B)$.

#5 $\varphi \boxright A, \psi \boxright A \vdash \varphi \wedge \psi \boxright A$.

#6 $\varphi \boxright \psi, \neg\varphi, \neg\psi \vdash \neg\psi \boxright \neg\varphi$.

#7 $\varphi \wedge \psi \boxright A \vdash \varphi \boxright (\psi \boxright A)$.

#8 $\varphi \boxright (\psi \boxright A) \vdash \varphi \wedge \psi \boxright A$.

#9 $\varphi \boxright A \vdash \varphi \wedge \psi \boxright A$.

#10 *If* $\Gamma, \varphi \vdash C$, *then* $\Gamma \vdash \varphi \boxright C$.

#11 $\varphi \boxright \psi \vdash \neg\psi \boxright \neg\varphi$.

For brevity, I will focus on **#1**, **#6**, and **#9**, discussing a countermodel for each in order to shed light on the nature of counterfactual reasoning.

#1 $\varphi, A \vdash \varphi \boxright A$

The ball is red and Mary likes it. Even so, it would be wrong to claim that if the ball were red Mary would like it since there are certain shades of red Mary does not like. In accordance with these assumptions, we may draw on the task semantics in order to provide a countermodel for the instance $p_1, p_2 \vdash p_1 \boxright p_2$:

$$|p_1|^+ = \{a, c, a.c\} \qquad \alpha(x) = a.b \qquad W = \{a.b, c.d\}$$
$$|p_2| = \langle\{b\}, \{d\}\rangle \qquad \beta(x) = c.d$$

The world state $a.b$ includes an exact verifier for both p_1 and p_2, making both sentences true in α at x. However, c is also an exact verifier for p_1 and is incompatible with any exact verifier for p_2. Thus the maximal c-part of $\alpha(x)$ is $\square$, and so β is an outcome of imposing c on α at x which does not include a part that exactly verifies p_2.

Adding color, we may take a to be a state of the ball being a shade of red that Mary likes, b to be the state of Mary liking the ball, c to be a state of the ball being a shade of red that Mary does not like, and d to be the state of Mary disliking the ball. It follows by the semantics for counterfactuals that $p_1 \boxright p_2$ is not true in α at x. The rest of the details needed to complete the model do not matter and so will be omitted.

#6 $\varphi \boxright \psi, \neg\varphi, \neg\psi \vdash \neg\psi \boxright \neg\varphi$

If Boris had gone to the party, Olga would have gone too.[35] Neither Olga nor Boris went in the end. Even so, it would be wrong to claim that if Olga were to not go to the party then Boris would not go to the party. For even though Olga likes to go to the parties Boris attends, Boris prefers to socialise without Olga. Drawing on the task semantics, we may present a countermodel to the instance $p_1 \boxright p_2, \neg p_1, \neg p_2 \vdash \neg p_2 \boxright \neg p_1$:

$$|p_1| = \langle\{a\}, \{c\}\rangle \qquad \alpha(x) = c.d.f \qquad \gamma(x) = a.e.g$$
$$|p_2| = \langle\{b\}, \{d, e, d.e\}\rangle \qquad \beta(x) = a.b.f \qquad W = \{a.b.f, c.d.f, e.a.g\}$$

[35] This example has been adapted from Lewis' [8, p. 35] case against counterfactual contraposition.

Since $\alpha(x)$ contains the exact falsifiers c for p_1 and d for p_2, both $\neg p_1$ and $\neg p_2$ are true in α at x. Moreover, the only outcome of imposing an exact verifier for p_1 on $\alpha(x)$ is $\beta(x)$ which includes an exact verifier for p_2, and so $p_1 \boxright p_2$ is true in α at x. However, if the exact verifier e for $\neg p_2$ is imposed on $\alpha(x)$, the outcome $\gamma(x)$ does not include an exact verifier for $\neg p_1$, and so $\neg p_2 \boxright \neg p_1$ is false in α at x.

Adding substance, we may take d and e to be states of Olga staying home where d is only compatible with the state f of Boris not knowing where Olga is and e is only compatible with the state g of Boris believing her to be home. Since the state of Boris being at the party a is compatible with him not knowing where Olga is f and the only world state to include both a and f as parts also includes the state of Olga going to the party b, imposing a on $\alpha(x)$ results in the world state $\beta(x)$ in which Olga goes to the party. Thus it is true at α at x that if Boris had gone to the party, then Olga would have gone. However, the state of Olga staying at home e is only compatible with Boris believing her to be home g, where the only world state $\gamma(x)$ to include both e and g as parts also includes Boris going to the party a. Thus it is false to claim that if Olga were to not go to the party, then Boris would not go either. All that is required is for Olga to not go to the party in a way that is compatible with Boris believing her to be home. For instance, perhaps Boris can see that the lights are on in her house.

The countermodel to **#6** also invalidates **#1** and *Counterfactual Contraposition* **#11** $\varphi \boxright \psi \vdash \neg\psi \boxright \neg\varphi$. Nevertheless, I show in §7 that **C2** is valid where this entails *Counterfactual Modus Tollens* **D6**. It follows that **#10** does not preserve validity for otherwise we may derive **#11** from **D6**. However, **#10** is sometimes defended, at least for certain restricted subject-matters.[36]

#9 (STA) $\varphi \boxright A \vdash \varphi \wedge \psi \boxright A$

Judy and Joey are at the party. If Cam were to join, he would argue with Judy, making her angry. Even so, if both Cam and Casey were to go to the party, Casey would keep Cam occupied, making Joey jealous. Consider the following countermodel:

$$|p_1| = \langle\{a\},\{b\}\rangle \qquad |p_2| = \langle\{c\},\{d\}\rangle \qquad |p_3| = \langle\{e\},\{f\}\rangle$$
$$\alpha(x) = b.d.f.g \qquad \beta(x) = a.d.e.g \qquad \gamma(x) = a.c.f.h$$
$$W = \{b.d.f.g, a.d.e.g, a.c.f.h\}$$

Letting $p_1 \boxright p_3 \vdash p_1 \wedge p_2 \boxright p_3$ be the instance of **#9**, we may take a to be a state of Cam being at the party, b to be a state of Cam being at home, c to be a state of Casey being at the party, d to be a state of Casey being at home, e to be a state of Judy being angry, f to be a state of Judy being happy, g to be a state of Joey being content, and h to be a state of Joey being jealous. Since $\beta(x)$ is the only world state to result from imposing an exact verifier for p_1 on $\alpha(x)$ and includes an exact verifier for p_3, the claim $p_1 \boxright p_3$ is true in α at x. Nevertheless, the world state $\gamma(x)$ results from imposing an exact verifier for $p_1 \wedge p_2$ on $\alpha(x)$ and does not include an exact verifier for p_3, and so $p_1 \wedge p_c \boxright p_3$ is false in α at x.[37] Thus **STA** is invalid.

[36] See [30] for such a view and [31] for further discussion.

[37] This model can be extended to make $p_1 \wedge p_2 \wedge p_4 \boxright p_3$ true while still making $p_1 \boxright p_3$ true and $p_1 \wedge p_2 \boxright p_3$ false. To do so, let $|p_4| = \langle\{i\},\{j\}\rangle$ where i is the state of Daniel attending the party, j is the

It is worth pausing to consider the role that states as opposed to moments play in invalidating **STA**. Although all $A \wedge B$-moments are A-moments, imposing an exact verifier state for $A \wedge B$ on an evaluation moment $\alpha(x)$ need not be an outcome of imposing an exact verifier states for A on $\alpha(x)$. The sensitivity of the outcomes induced by imposing a state on a moment of evaluation is made possible not only by the mereological structure that each moment includes, but also by the requirement that the exact verifier states for the antecedent belong to the subject-matter of the antecedent, where this requirement constitutes the *exactness* of the state semantics. It is for this reason that it is appropriate to assume a hyperintensional theory of propositions which is sensitive to differences in subject-matter in addition to modal profile.

Although the `model-checker` software may be used to find simpler countermodels than those presented above, it is often helpful to add states to a countermodel in support of an intuitive interpretation of the countermodel in question. Nevertheless, using the `model-checker` to find minimal countermodels can vastly accelerate the process of finding interpreted countermodels to counterfactual conditional reasoning.

5 Counterfactuals and Tense

Although many of the examples above were stated in simple present tense, this was by contrivance rather than a reflection of standard practice. By contrast, past tense counterfactuals are commonplace where Fine's [22] original example is one such case:

($\mathtt{N}'$) If Nixon had pressed the button there would have been a nuclear holocaust.

Suppose that the button is connected for a ten minute period after a security meeting. A year later, one of Nixon's advisers asserts $\mathtt{N}'$ to another. Whereas it is natural to take this assertion to be true, regimenting $\mathtt{N}'$ as $B \mathrel{\Box\!\!\rightarrow} H$ where B reads 'Nixon pushes the button' and H reads 'There is a nuclear holocaust' predicts that the assertion is false. Not only is the button disconnected at the time of the assertion, the nuclear holocaust would not occur immediately even if the button were connected and pushed. Rather, consider the following alternative:

(N) $\Diamond_P(B \mathrel{\Box\!\!\rightarrow} \Diamond_F H)$.

If Nixon were to push the button during the activation period there would be a future time in which a nuclear holocaust occurs. Since there is a time that is both during the activation period and before the adviser's assertion, N is predicted to be true, and so may be taken to provide a better regimentation of $\mathtt{N}'$ than $B \mathrel{\Box\!\!\rightarrow} H$.

Although there would have been a nuclear holocaust if Nixon had pressed the button after finishing the security meeting when the button was activated, there would not have been a nuclear holocaust had Nixon pressed the button before it was activated or anytime after it was deactivated. As a result, $\Diamond_P(B \mathrel{\Box\!\!\rightarrow} \neg\Diamond_F H)$ is also true if it were asserted a year after the incident. Instead of taking $\mathtt{N}'$ to assert the existence of any past time where $B \mathrel{\Box\!\!\rightarrow} \Diamond_F H$ is true, we may take $\mathtt{N}'$ to assert that $B \mathrel{\Box\!\!\rightarrow} \Diamond_F H$ is

state of Daniel being at home, and $\alpha(x) = b.d.f.g.j$, $\beta(x) = a.d.e.g.j$, $\gamma(x) = a.c.f.h.j$, and $\delta(x) = a.c.e.h.i$ are world states. Repeating this strategy generates a Sobel sequence.

true and $B \boxright \neg\Diamond H$ is false at a specific time. In order to capture this temporally specific reading, the following subsection will further extend the language to include two additional tense operators for speaking about specific times. Given these resources, §5.2 will analyze forwards, backwards, and backtracking counterfactuals.

5.1 Temporal Operators

Letting $\mathcal{L}^*$ extend $\mathcal{L}$ to include the unary *store operator* $\uparrow_i$ and unary *recall operator* $\downarrow^i$ for all $i \in \mathbb{N}$, we may define $\texttt{ext}(\mathcal{L}^*)$ and $\texttt{wfs}(\mathcal{L}^*)$ as before. By including a vector $\vec{v} = \langle v_1, v_2, \ldots \rangle$ of stored times in the point of evaluation, we may present the following:

($\uparrow$) $\mathcal{M}, \alpha, x, \vec{v} \vDash \uparrow^i A$ *iff* $\mathcal{M}, \alpha, x, \vec{v}_{[x/v_i]} \vDash A$.

($\downarrow$) $\mathcal{M}, \alpha, x, \vec{v} \vDash \downarrow^i A$ *iff* $\mathcal{M}, \alpha, v_i, \vec{v} \vDash A$.

Whereas $\uparrow^i A$ stores the current time of evaluation in the i^{th} value of $\vec{v}$, the sentence $\downarrow^i A$ shifts the time of evaluation to the i^{th} value stored in $\vec{v}$. By adding $\vec{v}$ as a parameter to the point of evaluation, the semantic clauses given for $\mathcal{L}$ may otherwise be maintained. Supposing Nixon's adviser asserts $\texttt{N}'$ a year after Nixon almost pushed the button, we may draw these expressive resources to regiment $\texttt{N}'$ as follows:

($\textsc{n}'$) $\downarrow^1(B \boxright \Diamond H)$.

Let v_1 store the time at which Nixon almost pressed the button. Although this time may be forever imprinted on the minds of Nixon's advisers, the time at which the nuclear holocaust would have taken place may be unknown. It might have taken days for the disaster to unfold, or perhaps only hours. Instead of assuming there is a specific time, we may take $\textsc{n}'$ to assert that there is a future time at which the nuclear holocaust would have taken place. Given the semantics above, $\textsc{n}'$ is true at a world α, time x, and vector $\vec{v}$ just in case for every world β which results from imposing an exact verifier for B on α at v_1, there is some future time $y > v_1$ at which H is true in β.

Letting α be the actual world and x be the time of the adviser's assertion a year after the incident, $\textsc{n}'$ is true whenever v_1 is a time where the button is activated. Even so, we may observe that $\downarrow^1(B \boxright \Diamond\neg H)$ is also true assuming there is a time shortly after v_1 which occurs before the nuclear holocaust has taken place. Although this may be admitted, in certain circumstances there may be a specific future time at which it is appropriate to evaluate the consequent. For instance, perhaps Nixon's adviser had intended to specify that a nuclear holocaust is underway at the time of his assertion in all of the counterfactual worlds in which Nixon pushed the button a year prior:

($\textsc{n}''$) $\uparrow^2 \downarrow^1(B \boxright \downarrow^2 H)$.

By first storing the time of assertion and then reverting back to the time at which the button was engaged, $\textsc{n}''$ asserts that if Nixon were to have pressed the button at that time of engagement, then a nuclear holocaust would have occur at the stored time of assertion. In certain circumstances, it may be enough to assert the weaker claim that $\uparrow^2 \Diamond(B \boxright \downarrow^2 H)$ which merely requires there to be some past time where if Nixon were to push the button at that time we would now be in a nuclear holocaust. Which regimentation is right may depend on what the speaker intends.

Although typical, first going back in time and then going forward is not the only way to interpret tensed counterfactual claims, and sometimes not the most natural. The following subsection will consider both backwards and backtracking counterfactuals, providing their analysis with the resources of the present framework.

5.2 Backtracking

Following tradition, I will focus on an example presented by Jackson [32] in which Smith is standing on the edge of a building threatening to jump. Standing behind him in safety, Beth and Bill watch in fear for their friend. Moments later, Smith steps back from the edge. We may then consider the following claims:

(J) If Smith had jumped, he would have died.

(U) If Smith had jumped, a net would have been installed beneath him.

(L) If Smith had jumped, he would have lived.

Kicking things off, suppose that Bill asserts J. Beth responds by insisting on U. After all, they both know that Smith wants to live so the only way Smith would jump is if jumping would not kill him. For instance, perhaps there would have been a net there to catch him. Having made her case for U, Beth goes on to assert L. Unconvinced, Bill points out that nobody would install a net on the building, reasserting J. This prompts Beth to enumerate further reasons why Smith would never jump without a net, insisting on U and L once more. This may be imagined to continue round after round. The challenge is to identify what they are disagreeing about and to provide a theory of counterfactuals that adequately models their disagreement.

Whereas J is a forward counterfactual and can be regimented in a manner similar to N′ above, U and L express *backwards* and *backtracking* counterfactuals, respectively. Rather than changing the world of evaluation to make the antecedent true at one time and evaluating the consequent at a later time, backwards counterfactuals such as U evaluate the consequent at an earlier time than the time at which the antecedent is assumed to be true. Consider the following regimentations:

(D) $\downarrow^1(J \mathrel{\Box\!\!\rightarrow} \downarrow^2 D)$.

(U) $\downarrow^1(J \mathrel{\Box\!\!\rightarrow} \downarrow^3 U)$.

(L) $\downarrow^1(J \mathrel{\Box\!\!\rightarrow} \downarrow^2 L)$.

Holding the world α, time x, and stored times $\vec{v}$ fixed for the purposes of evaluating Bill and Beth's conversation, suppose that $v_3 < v_1 < v_2$ where Bill and Beth agree about this much. Nevertheless, Bill takes D to be true since any world to result from imposing a state in which Smith jumps in α at v_1 will be one in which Smith dies at v_2. Beth disagrees, claiming that L is true since any world to result from imposing a state in which Smith jumps in α at v_1 will be one in which Smith lives at v_2. Since the states of Smith being alive are incompatible with the states of Smith being dead, the worlds that Beth predicts will result from imposing a state of Smith jumping at v_1 make it false that Smith dies at v_2, and so Beth and Bill disagree.

In defense of her view, Beth asserts U which requires the worlds that result from imposing a state of Smith jumping in α at v_1 to include the placement of a net at v_3 prior to the time v_1 of Smith's jump. Bill disagrees. Beth's claim U helps to bring the source of the disagreement to light: whereas Beth assumes that making minimal changes to $\alpha(v_1)$ to include a state of Smith jumping will also include a net having been placed there at an earlier time v_3, Bill does not. What they disagree about are the outcomes of imposing a state of Smith jumping on the actual world at the time v_1 at which Smith almost jumped. Perhaps Beth's optimism reflects the emphasis she places on Smith's love for live, assuming that more would have to change about the actual world at v_1 for Smith to be willing to die than merely including a net there to catch him. Whereas jumping without a net is incompatible with Smith's love for life, jumping with a net is perfectly compatible. By contrast, Bill's pessimism may be attributed to his willingness to see Smith change his outlook on life, taking this to be a smaller change than including a net there to catch him.

Although the task semantics can help model counterfactual claims, the semantics does not provide information about the structure of an intended model, nor does it presume that there is a uniquely privileged intended model with a special metaphysical status. In the disagreement between Bill and Beth, each believes different worlds are the outcomes of imposing a state of Smith jumping in the actual world α at v_1 when he was poised to do so. Nevertheless, the task semantics helps to shed light on the nature of backtracking counterfactuals by providing an account of what this commits each speaker to in any given model. Even though backtracking counterfactuals like L are forward counterfactuals just as much as D, the proposed changes at the time of evaluation propagate backwards on account of requiring an altered past. Assuming that a net is present at v_1 requires that it was installed on the building at an earlier time v_3. By contrast, agreeing with Bill that Smith would have died only requires that Smith give up his love for life in the moment of jumping without requiring any substantial changes to the past. Given that the task semantics only selects outcome worlds by requiring them to occupy moments that differ minimally from the moment of evaluation, there is no requirement whatsoever that the worlds which result from imposing some state on the world of evaluation at a given time agree with the world of evaluation at earlier times. Indeed, small changes at the time of evaluation may require massive changes to the past leading up to the point of evaluation.

In order to emphasize this final point, I will conclude the present section with another backwards counterfactual along similar lines. To begin with, suppose that there are no alien civilizations anywhere in the universe. Consider the following case:

> `Icosahedron:` Deep inside an Egyptian tomb, Harry discovered a perfectly formed icosahedron that appeared to be made out of pure titanium. In fact, the icosahedron was only a crude iron alloy. Relieved, Harry couldn't help but say out loud, "If the icosahedron had been titanium, then advanced aliens put it there."

Although the icosahedron is not in fact made of titanium, supposing otherwise requires massive changes to the past. Even if aliens did not put it there, assuming humans had developed the metallurgy needed to refine pure titanium at the time of the pyramids still requires a radical shift from the actual past. Nevertheless, changing the moment of Harry's discovery to include a titanium icosahedron instead of the crude alloy that he found does not require much change at all. It is worth comparing such a case to Fine's Nixon example given above. Whereas $\mathtt{N}'$ makes a small change to the moment of supposition that results in a drastically altered future, `Icosahedron` makes a small change to the moment of supposition which requires a drastically altered past. Despite these differences, the two cases are handled by the task semantics in the same way: any world which differs minimally from the actual world at the time of evaluation is to be considered, no matter how far that world may diverge in its past or future.

Whereas Lewis' semantics compares worlds in their entirety for similarity, Fine's semantics considers the world states that result from imposing an exact verifier state for the antecedent of a counterfactual on the world state at which the counterfactual is evaluated. In this respect, Fine's semantics restricts consideration to the moment of evaluation. Despite strengthening the logic for counterfactuals, no consideration is paid to the effects that changing the present may have on the past or future.

6 Conclusion

Instead of taking possible worlds to be primitive points, the task semantics constructs worlds from states, parthood, tasks, and times. Worlds may then be compared at a time where that comparison is carried out in purely mereological and modal terms. By contrast, Lewis takes the similarity relation to be primitive and Fine posits a primitive imposition relation. Although Fine's semantics validates a stronger logic for counterfactuals than intensional semantic theories are capable of by validating **SDA** without also validating **STA**, taking the imposition relation to be primitive makes the semantics homophonic by including a counterfactual primitive in the metalanguage. Although a homophonic semantics may be tolerated when there is no better option, §2.2 shows that it is unnecessary to take imposition to be primitive by providing its definition in purely modal and mereological terms. In addition to simplifying the semantics conceptually, defining the imposition relation reduces the computational complexity of the semantics' implementation, strengthening the evidence that the `model-checker` can provide by surveying a broader range of models.[38]

In addition to taking imposition to be primitive rather than defined, Fine does not provide the temporal structure needed to interpret tensed counterfactuals. By contrast, I have taken the task relation to be primitive, defining the possible states in its terms. Intuitively, there is a task $s \to t$ just in case the transition from the state s obtaining to the state t obtaining is possible. Whereas the imposition relation has a counterfactual reading, the task relation has a purely modal reading. Given a set of times, I show how to construct possible worlds, drawing on these resources in order to provide a semantics for tensed counterfactuals. Although focusing attention on an intended model may help to keep track of the working parts of the theory, nothing about the semantics turns on this reading or its supposed metaphysics. Nevertheless, a number of difficulties remain, inspiring extensions of the present framework.

[38] I discuss the methodology for using the `model-checker` in formal semantics elsewhere.

6.1 Events

It is easy to consider exact verifiers states for sentences like 'The coin is heads up' and 'The icosahedron is pure titanium'. Such states are specific, perfectly static ways for certain things to be. For instance, we may consider the precise arrangement of titanium atoms which make up the icosahedron, or the exact orientation of the coin relative to the table. By contrast, consider the antecedent of `N` from before:

(`B`) Nixon is pressing the button.

It is natural to deny that there is any individual state of Nixon pressing the button. Rather, we may model Nixon's action by a sequence of states in which he moves his finger towards the button, presses it, and then slowly pulls his finger back. Letting a *discrete process* $\bar{e} \coloneqq \langle e_1, \ldots, e_n \rangle$ be any sequence of states where $\bar{e}$ is *possible* just in case $e_i \to e_{i+1}$ for all $1 \leqslant i < n$, we may take the *event* $\langle \overline{V}, \overline{F} \rangle$ of Nixon pressing the button to include the set of processes $\overline{V}$ which make the sentence `B` true together with the set of processes $\overline{F}$ which make `B` false. Given a counterfactual conditional whose antecedent is an event, one might hope to impose the exact verifier processes in $\overline{V}$ on a world over a duration rather than imposing any individual state at a time.

Before attempting to provide a theory of process imposition, it is important to consider the merits of doing so. For instance, if the result is a satisfying metaphysics which distinguishes events from static propositions but which does not otherwise affect the semantics for counterfactuals, no harm will come in taking at least some states to represent processes as a simplifying idealization. Nevertheless, it is natural to worry that conflating states with processes is incompatible with the intended interpretation guiding the construction of the semantics as well as the resulting principles. After all, it is not clear that any part of a possible process is possible, where the analogous principle for states played an important role in the semantics above. For instance, perhaps the possibility of some processes depend on the presence and coordination of their parts. Moreover, exact verifier processes may have different lengths and so will not have unique fusions on account of admitting fusions in which the processes being fused overlap in different ways. At the very least, considerably more would have to be said in order to provide a coherent process-theoretic semantics.

Instead of facing this challenge, one might take events to be a convenient proxy for related static propositions which might otherwise be hard to express. For instance, one might take any process of Nixon pressing the button $\bar{e}$ to include a critical state $\hat{e}$ in which the electrodes in the button connect, completing the circuit. More generally, a critical state in an exact verifier process is a *point of no return*. So long as every exact verifier process for an event has a unique critical state, that event may be said to be a proxy for a static proposition consisting only of the corresponding critical states. Nevertheless, it is natural to worry that critical states can only be identified by appealing to counterfactuals, and so faces the same criticisms brought out before. Moreover, it is far from obvious what it is to be a critical state in an exact falsifier process, nor is it clear that the critical states will be closed under fusion. Given my purposes here, I will leave the development of a theory of events and critical states for another time, idealizing processes as states in the present semantics.

6.2 Containment

Given a discrete task space $\mathcal{T} = \langle S, \sqsubseteq, \rightarrow, \mathbb{Z}\rangle$, the world histories in $H_{\mathbb{Z}}$ were defined to be functions $\alpha : \mathbb{Z} \rightarrow S$ where $\alpha(x) \in W$ and $\alpha(x) \rightarrow \alpha(x+1)$ for all $x \in \mathbb{Z}$. The space of world states W was then defined in terms of the possible states P which in turn was defined in terms of the task relation. However, we may provide a much more direct definition by taking world histories to be maximal possible evolutions. Letting a discrete evolution $\tau : \mathbb{Z} \rightarrow S$ be *possible* just in case $\tau(x) \rightarrow \tau(x+1)$ for all $x \in \mathbb{Z}$ as before, we may define *evolution parthood* by taking $\tau \sqsubseteq \sigma$ just in case $\tau(x) \sqsubseteq \sigma(x)$ for all $x \in \mathbb{Z}$. We may then take a possible discrete evolution $\tau : \mathbb{Z} \rightarrow S$ to be *maximal* just in case $\sigma \sqsubseteq \tau$ for any possible evolution $\sigma : \mathbb{Z} \rightarrow S$ where $\tau \sqsubseteq \sigma$.

By contrast with the definition of world histories given before, the definition of the maximal possible discrete evolutions is stated in entirely task theoretic terms. In addition to its conceptual simplicity, this definition does not rely on the possible states P defined before. Nevertheless, so far nothing requires the maximal possible discrete evolutions to be world histories. This motivates the following constraints:

Containment: (L) If $s \in P$, $d \sqsubseteq s$, and $d \rightarrow r$, then $s \rightarrow t.r$ for some $t \in S$.
(R) If $t \in P$, $r \sqsubseteq t$, and $d \rightarrow r$, then $s.d \rightarrow t$ for some $s \in S$.

Whereas the Parthood constraints ensure that every part is accounted for in any task between fusions, the constraints above require that for any task to or from the parts of a possible fusion, there will be a corresponding task to of from that fusion. Given this constraint, **P8** in §7.3 shows that the maximal possible discrete evolutions are coextensive with the world histories parameterized by $\mathbb{Z}$. In addition to motivating Containment, this result brings together two independent ways of thinking about world histories, shedding further light on the nature of possible worlds.

6.3 Continuous Time

Taking time to be discrete introduces a degree of arbitrariness since we must fix the unit in order to interpret a world history at all. Whereas some histories string together moments separated by mere seconds, others may take moments to be separated by minutes or even hours. Instead of making an arbitrary choice, we may take this decision to be practical in nature, choosing a temporal resolution which is as low as possible while preserving enough temporal features for the application at hand.

By contrast, taking time to be continuous requires amending the present theory. For instance, suppose that time has the structure of the real numbers $\mathbb{R}$. Given any state at a time, there is no next time which we may assign to an accessible state. As a result, the definition of an evolution cannot be maintained in its present form. Letting a *continuous evolution* by any function $\tau : \mathbb{R} \rightarrow S$ from the real numbers $\mathbb{R}$ to states S, we may take τ to be *possible* at x just in case there is some $y > 0$ where $\tau(u) \rightarrow \tau(x)$ for all $u \in (x-y, x)$ and $\tau(x) \rightarrow \tau(v)$ for all $v \in (x, x+y)$. A continuous evolution $\tau : \mathbb{R} \rightarrow S$ is *possible* just in case τ is possible at all $x \in \mathbb{R}$, and a *continuous world history* is any continuous possible evolution α where $\alpha : \mathbb{R} \rightarrow W$ (or, alternatively, maximal as in §6.2). We may then replace **T5** with the density axiom:

T5.1 $\boxed{\scriptstyle F}\boxed{\scriptstyle F}A \to \boxed{\scriptstyle F}A$.

Although time is modeled by the real numbers throughout the physical sciences, there is little evidence to suggest that time has the structure of the real numbers. For all we know, time could be discrete with a high resolution, and so well approximated by the real numbers.[39] It follows that even for the purposes of metaphysics, there is no reason to privilege a *continuous task space* $\mathcal{T} = \langle S, \sqsubseteq, \rightarrow, \mathbb{R}\rangle$ over the discrete task spaces considered before. Moreover, I will assume that there is nothing to be gained for the purposes of semantics by restricting consideration to continuous task spaces. Until compelling motivation can be provided for taking time to be continuous, I will maintain the definitions provided for discrete task spaces on account of their simplicity.

7 *Appendix*

The following subsection will derive a number of theorems within the various systems presented above. I will then derive the constraints that Fine [1] assumes the imposition relation and set of possible states satisfy in §7.2 and §7.3 respectively, presenting parts of the soundness proof for **CLT** in §7.4. In future work, I hope to establish the completeness for appropriate extensions of these systems.

7.1 Theorems

Classical tautologies and applications of the standard structural rules will be assumed throughout the following proofs.

D1 $\varphi \,\Box\!\!\rightarrow A,\ \varphi \,\Box\!\!\rightarrow B \vdash_{\text{CL}} \varphi \,\Box\!\!\rightarrow A \wedge B$.

Proof. Follows immediately from $A, B \vdash_{\text{CL}} A \wedge B$ by **R1**. □

D2 *If* $A \vdash_{\text{CL}} B$, *then* $\varphi \,\Box\!\!\rightarrow A \vdash_{\text{CL}} \varphi \,\Box\!\!\rightarrow B$.

Proof. Immediate from **R1**. □

D3 *If* $\vdash_{\text{CML}} A$, *then* $\vdash_{\text{CML}} \Box A$.

Proof. Immediate from **R1**. □

D4 $\vdash_{\text{CML}} \Box(A \to B) \to (\Box A \to \Box B)$

Proof. Immediate from **C2** by **R1**. □

D5 $\vdash_{\text{CML}} \Box A \to A$

Proof. Since $\top, \top \,\Box\!\!\rightarrow A \vdash_{\text{CML}} A$ by **C2**, $\top \,\Box\!\!\rightarrow A \vdash_{\text{CML}} A$ by **M1**, so $\vdash_{\text{CML}} \Box A \to A$. □

D6 $\varphi \,\Box\!\!\rightarrow A, \neg A \vdash_{\text{CL}} \neg\varphi$.

Proof. Immediate from **C2**. □

[39] See, however, [33] for discussion of discrete theories of time in recent physics.

D7 $\vdash_{\text{CML}} \Box\varphi \leftrightarrow \neg\varphi \boxright \bot$.

Proof. Since $\top \boxright \varphi, \neg\varphi \vdash_{\text{CML}} \neg\top$ by **D6** where $\neg\top \vdash_{\text{CML}} \bot$ by **M1**, it follows that $\top \boxright \varphi, \neg\varphi \vdash_{\text{CML}} \bot$, and so $\top \boxright \varphi \vdash_{\text{CML}} \neg\varphi \to \bot$. Equivalently, $\Box\varphi \vdash_{\text{CML}} \neg\varphi \to \bot$, and so $\Box\Box\varphi \vdash_{\text{CML}} \Box(\neg\varphi \to \bot)$ by **R1**. Thus $\Box\varphi \vdash_{\text{CML}} \Box(\neg\varphi \to \bot)$ given **M4**, and so $\Box\varphi \vdash_{\text{CML}} \neg\varphi \boxright \bot$ follows by **M5**.

Conversely, $\neg\varphi \boxright \bot, \neg\bot \vdash_{\text{CML}} \neg\neg\varphi$ by **D6**, and so $\neg\varphi \boxright \bot \vdash_{\text{CML}} \varphi$ given **M2** and $\neg\neg\varphi \vdash_{\text{CML}} \varphi$. Thus $\neg\varphi \boxright \bot, \top \vdash_{\text{CML}} \varphi$, and so $\neg\varphi \boxright \bot \vdash_{\text{CML}} \top \to \varphi$, which is equivalent to $\neg\varphi \boxright \bot \vdash_{\text{CML}} \Box\varphi$. Given the above, $\vdash_{\text{CML}} \Box\varphi \leftrightarrow \neg\varphi \boxright \bot$. $\square$

D8 $\vdash_{\text{CML}} \Box A \leftrightarrow \neg\Diamond\neg A$

Proof. Since $A \vdash_{\text{CML}} \neg\neg A$, we know that $\top \boxright A \vdash_{\text{CML}} \top \boxright \neg\neg A$ by **R1**, and so $\top \boxright A \vdash_{\text{CML}} \neg\neg(\top \boxright \neg\neg A)$. Equivalently, $\Box A \vdash_{\text{CML}} \neg\Diamond\neg A$. Running the same reasoning in reverse yields $\neg\Diamond\neg A \vdash_{\text{CML}} \Box A$, and so $\vdash_{\text{CML}} \Box A \leftrightarrow \neg\Diamond\neg A$. $\square$

D9 $\vdash_{\text{CML}} \varphi \boxright \top$

Proof. Immediate from **M1** by **R1**. $\square$

D10 $\vdash_{\text{CML}} \bot \boxright A$

Proof. Since $\neg A \vdash_{\text{CML}} \neg\bot$ given **M2**, we know $\bot \to A$, and so $\bot \boxright A$ by **M5**. $\square$

D11 $\vdash_{\text{CTL}} \Box A \to \triangledown A$

Proof. Since $\Box A,\ \Box\boxed{\scriptstyle \mathrm{F}}A \leftrightarrow \Box A \vdash_{\text{CTL}} \Box\boxed{\scriptstyle \mathrm{F}}A$, it follows from **T6** that $\Box A \vdash_{\text{CTL}} \Box\boxed{\scriptstyle \mathrm{F}}A$ where $\Box\boxed{\scriptstyle \mathrm{F}}A \vdash_{\text{CTL}} \boxed{\scriptstyle \mathrm{F}}A$ by **D5**, and so $\Box A \vdash_{\text{CTL}} \boxed{\scriptstyle \mathrm{F}}A$. Thus $\Box A \vdash_{\text{CTL}} \boxed{\scriptstyle \mathrm{P}}A$ by **R3**, where $\Box A \vdash_{\text{CTL}} A$ again by **D5**. Given that $\boxed{\scriptstyle \mathrm{P}}A,\ A,\ \boxed{\scriptstyle \mathrm{F}}A \vdash_{\text{CTL}} \triangledown A$, it follows that $\Box A \vdash_{\text{CTL}} \triangledown A$, and so we may conclude that $\vdash_{\text{CTL}} \Box A \to \triangledown A$. $\square$

7.2 Imposition

The proofs given below follow from the definition of imposition. For convenience, I have copied the relevant definitions here:

Compatible Part: $s \sqsubseteq_t w \coloneqq s \sqsubseteq w \wedge s \circ t$.

Maximal Compatible Parts: $w_t \coloneqq \{s \sqsubseteq_t w \mid \forall r \sqsubseteq_t w(s \sqsubseteq r \to r \sqsubseteq s)\}$.

Imposition: $t \rightarrowtriangle_w u \coloneqq u \in W \wedge \exists s \in w_t(s.t \sqsubseteq u)$.

World States: $W \coloneqq \{w \in P \mid \forall s \circ w(s \sqsubseteq w)\}$.

Instead of taking the imposition relation to be primitive and assuming the constraints that Fine [1] provides, the following results derive Fine's constraints from the definition of the imposition relation. Since these proofs do not concern the task relation, we may take the set of possible states P to be primitive for present purposes, working over the world spaces that Fine [3] introduces. Given that the constraints that Fine imposes on world spaces will also be derived in the following section, the results presented in this section will carry over to the task spaces that I define.

P1 (Inclusion) *If $t \rightarrowtriangle_w u$, then $t \sqsubseteq u$.*

Proof. Assuming $t \rightarrowtriangle_w u$, it follows that $u \in W$ where $s.t \sqsubseteq u$ for some $s \in w_t$. Since $t \sqsubseteq s.t$, it follows that $t \sqsubseteq u$ as desired. ☐

P2 (Actuality) *If $t \sqsubseteq w$ and $w \in W$, then $t \rightarrowtriangle_w u$ for some $u \sqsubseteq w$.*

Proof. Assume $t \sqsubseteq w$ for $w \in W$. Thus $w \in P$ where $w.t = w$, and so $w \circ t$. Since $w \sqsubseteq w$, we know that $w \sqsubseteq_t w$. Letting $r \sqsubseteq_t w$ where $w \sqsubseteq r$, it follows that $r \sqsubseteq w$, and so $w \in w_t$. Since $w.t \sqsubseteq w$, we know that $t \rightarrowtriangle_w w$, and so $t \rightarrowtriangle_w u$ for some $u \sqsubseteq w$. ☐

P3 (Incorporation) *If $t \rightarrowtriangle_w u$ and $v \sqsubseteq u$, then $t.v \rightarrowtriangle_w u$.*

Proof. Assuming $t \rightarrowtriangle_w u$ and $v \sqsubseteq u$, it follows that $u \in W$ where $s.t \sqsubseteq u$ for some $s \in w_t$. Thus $s.t.v \sqsubseteq u$ and $s \sqsubseteq_t w$ where (1): $r \sqsubseteq s$ whenever $r \sqsubseteq_t w$ and $s \sqsubseteq r$. It follows that $s \sqsubseteq w$. Since $u \in P$, we also know that $s \circ t.v$, and so $s \sqsubseteq_{t.v} w$.

Letting $q \sqsubseteq_{t.v} w$ where $s \sqsubseteq q$, it follows that $q \sqsubseteq w$ where $q \circ t.v$, and so $q.t.v \in P$. Thus $q.t \in P$, and so $q \circ t$. It follows that $q \sqsubseteq_t w$, and so $q \sqsubseteq s$ follows from (1). Generalizing on q, it follows that (2): $q \sqsubseteq s$ whenever $q \sqsubseteq_{t.v} w$ and $s \sqsubseteq q$.

Having already shown that $s \sqsubseteq_{t.v} w$, it follows from (2) that $s \in w_{t.v}$. Since $s.t.v \sqsubseteq u$ for $u \in W$, we may conclude that $t.v \rightarrowtriangle_w u$ as desired. ☐

P4 (Completeness) *If $t \rightarrowtriangle_w u$, then u is a world-state.*

Proof. Immediate from the definition of *Imposition*. ☐

7.3 World Space

The proofs given below follow from the constraints on the task relation along with the definitions copied here for convenience:

Parthood: If $d \sqsubseteq s$ and $s \rightarrow t$, then $d \rightarrow r$ for some $r \sqsubseteq t$.

Nullity: $\square = t$ just in case $\square \rightarrow t$ or $t \rightarrow \square$.

Maximality: If $s \in S$ and $t \in P$, there is some maximal t-compatible part $r \in s_t$.

Containment: If $s \rightarrow s$, $d \sqsubseteq s$, and $d \rightarrow r$, then $s \rightarrow t.r$ for some $t \in S$.

If $t \rightarrow t$, $r \sqsubseteq t$, and $d \rightarrow r$, then $s.d \rightarrow t$ for some $s \in S$.

Possibility: $P \coloneqq \{s \in S \mid \exists t(s \sim t)\}$.

Compatibility: $s \circ t$ *iff* $s.t \in P$.

Possible Evolutions: $E_{\mathbb{Z}}^{\Diamond} \coloneqq \{\tau : \mathbb{Z} \rightarrow S \mid \tau(x) \rightarrow \tau(x+1) \text{ for all } x \in \mathbb{Z}\}$.

Evolution Parthood: $\tau \sqsubseteq \sigma$ *iff* $\tau(x) \sqsubseteq \sigma(x)$ for all $x \in \mathbb{Z}$.

Maximal Possible Evolutions: $M_{\mathbb{Z}} \coloneqq \{\tau \in E_{\mathbb{Z}}^{\Diamond} \mid \sigma \sqsubseteq \tau \text{ for all } \sigma \in E_{\mathbb{Z}}^{\Diamond} \text{ where } \tau \sqsubseteq \sigma\}$.

Given an arbitrary task space $\mathcal{T} = \langle S, \sqsubseteq, \rightarrow\rangle$, we may establish the following results in order to show that every task space determines a corresponding world space.

P5 (Possibility) *If $s \in P$ and $t \sqsubseteq s$, then $t \in P$.*

Proof. Letting $s \in P$ and $t \sqsubseteq s$, there is some $r \in S$ where $s \sim r$, and so either $s \to r$ or $r \to s$. If $s \to r$, then $t \to d$ for some $d \sqsubseteq r$ by Parthood (l). Similarly, if $r \to s$, then $d \to t$ for some $d \sqsubseteq r$ by Parthood (r). Thus $t \sim d$ for some $d \in S$, and so $t \in P$. □

P6 (Nonempty) $P \neq \varnothing$.

Proof. Since $\square = \square$, it follows that $\square \to \square$ by Nullity, and so $\square \in P$. Thus $P \neq \varnothing$ □

P7 (World Space) *If $t \in P$, then $t \sqsubseteq w$ for some $w \in W$.*

Proof. Let $t \in P$. Since $\blacksquare \in S$, Maximality requires there to be some $r \in \blacksquare_t$, and so $r \sqsubseteq_t \blacksquare$ where (1): $q \sqsubseteq r$ whenever $q \sqsubseteq_t \blacksquare$ and $r \sqsubseteq q$. Thus $r \circ t$, and so $r.t \in P$.

Letting $s = r.t$, it follows that $s.t \in P$, and so $s \circ t$. Since $s \sqsubseteq \blacksquare$ where $r \sqsubseteq s$, we know that $s \sqsubseteq_t \blacksquare$, and so $s \sqsubseteq r$ by (1). Hence $s = r$, and so $r = r.t$.

Let $k \in S$ where $k \circ r$. Thus $k.r \in P$, and so $k.r.t \in P$ given the above. It follows that $k.r \circ t$ where $k.r \sqsubseteq \blacksquare$, and so $k.r \sqsubseteq_t \blacksquare$. Since $r \sqsubseteq k.r$, it follows that $k.r \sqsubseteq r$ by (1), and so $k \sqsubseteq r$. Generalizing on k, we may conclude that $r \in W$ where $t \sqsubseteq r$. □

P8 $M_{\mathbb{Z}} = H_{\mathbb{Z}}$.

Proof. Let $\tau \in M_{\mathbb{Z}}$, $x \in \mathbb{Z}$, and $s_0 \in S$ where $s_0 \circ \tau(x)$. It follows that $s_0.\tau(x) \in P$ and so $s_0.\tau(x) \sim q$ for some $q \in S$. Assume for induction that $s_{-n}, s_n \in P$. Since $\tau \in M_{\mathbb{Z}}$, we know that $\tau(x-(n+1)) \to \tau(x-n)$ and $\tau(x+n) \to \tau(x+(n+1))$. Given that $\tau(x-n) \sqsubseteq s_{-n}.\tau(x-n)$ and $\tau(x+n) \sqsubseteq s_n.\tau(x+n)$, it follows by Containment that there are some $s_{-(n+1)}, s_{n+1} \in S$ where $s_{-(n+1)}.\tau(x-(n+1)) \to s_{-n}.\tau(x-n)$ and $s_n.\tau(x+n) \to s_{n+1}.\tau(x+(n+1))$. Letting $\alpha(y) = s_{x-y}.\tau(y)$ for all $y \in \mathbb{Z}$, it follows by induction that $\alpha(y) \to \alpha(y+1)$ for all $y \in \mathbb{Z}$, and so $\alpha \in E^{\Diamond}_{\mathbb{Z}}$. Since $\tau(y) \sqsubseteq \alpha(y)$ for all $y \in \mathbb{Z}$, we know that $\tau \sqsubseteq \alpha$, and so $\alpha \sqsubseteq \tau$ where $\alpha(x) \sqsubseteq \tau(x)$ in particular. Given that $\alpha(x) = s_0.\tau(x)$, it follows that $s_0 \sqsubseteq \tau(x)$. By generalizing on $s_0 \in S$, we may conclude that $\tau(x) \in W$, and so $\tau \in H_{\mathbb{Z}}$ follows from generalizing on $x \in \mathbb{Z}$.

Assuming instead that $\tau \in H_{\mathbb{Z}}$ and letting $x \in \mathbb{Z}$, we know that $\tau(x) \in W$. Given any $\sigma \in E^{\Diamond}_{\mathbb{Z}}$ where $\tau \sqsubseteq \sigma$, it follows that $\sigma(x) \to \sigma(x+1)$, and so $\sigma(x) \in P$. Since $\tau(x) \sqsubseteq \sigma(x)$, we know that $\sigma(x).\tau(x) = \sigma(x)$, and so $\sigma(x) \circ \tau(x)$. Given that $\tau(x) \in W$, we know that $\sigma(x) \sqsubseteq \tau(x)$, and so $\sigma \sqsubseteq \tau$ follows by generalizing on x. Thus $\tau \in M_{\mathbb{Z}}$. □

7.4 Soundness

Letting $\mathcal{M}$ be any model over a discrete task space $\mathcal{T}_{\mathbb{Z}} = \langle S, \sqsubseteq, \to, \mathbb{Z} \rangle$, I will assume the following semantic clauses where the others are standard, and so have been omitted:

$\mathcal{M}, \alpha, x \vDash q$ *iff* there is some $s \sqsubseteq \alpha(x)$ where $s \in |q|^+$.

$\mathcal{M}, \alpha, x \vDash \varphi \mathrel{\Box\!\!\rightarrow} C$ *iff* $\mathcal{M}, \beta, x \vDash C$ whenever $t \in |\varphi|^+$ and $t \rightarrowtriangle_{\alpha(x)} \beta(x)$.

For brevity, I will establish the validity of a collection of characteristic axiom schemata for **CTL** since the others are similar. In order to ease the proofs presented below, it will help to begin by proving a number of supporting lemmas and propositions.

L1 $\bigsqcup\{\bigsqcup E_i \mid i \in I\} = \bigsqcup\bigcup\{E_i \mid i \in I\}$ *where* $E_i \subseteq S$ *for all* $i \in I$.

Proof. Letting $e \in E_i$, it follows that $e \sqsubseteq \bigsqcup E_i$ where $\bigsqcup E_i \sqsubseteq \bigsqcup\{\bigsqcup E_i \mid i \in I\}$, and so $\bigsqcup\{\bigsqcup E_i \mid i \in I\}$ is an upper bound of $\bigcup\{E_i \mid i \in I\}$. By definition, we may conclude that $\bigsqcup\bigcup\{E_i \mid i \in I\} \sqsubseteq \bigsqcup\{\bigsqcup E_i \mid i \in I\}$. Since $E_i \subseteq \bigcup\{E_i \mid i \in I\}$ for any $i \in I$, it follows that $\bigsqcup E_i \sqsubseteq \bigsqcup\bigcup\{E_i \mid i \in I\}$ for all $i \in I$, and so $\bigsqcup\bigcup\{E_i \mid i \in I\}$ is an upper bound of the set $\{\bigsqcup E_i \mid i \in I\}$. Thus $\bigsqcup\{\bigsqcup E_i \mid i \in I\} \sqsubseteq \bigsqcup\bigcup\{E_i \mid i \in I\}$. □

L2 *If* $X, Y \subseteq S$ *are closed under nonempty fusion, then so are* $X \otimes Y$ *and* $X \oplus Y$.

Proof. Assume $X, Y \subseteq S$ are closed under nonempty fusion and $Z \subseteq X \otimes Y$ is nonempty. Let $Z = \{z_i \mid i \in I\}$ where $z_i = x_i.y_i$ for $x_i \in X$ and $y_i \in Y$ for each $i \in I$. Taking $Z_X = \{x_i \mid i \in I\}$ and $Z_Y = \{y_i \mid i \in I\}$, both $Z_X \subseteq X$ and $Z_Y \subseteq Y$ are nonempty, and so $\bigsqcup Z_X \in X$ and $\bigsqcup Z_Y \in Y$ by assumption. Thus $\bigsqcup\{\bigsqcup Z_X, \bigsqcup Z_Y\} \in X \otimes Y$ where:

$$\begin{aligned}
\bigsqcup\{\bigsqcup Z_X, \bigsqcup Z_Y\} &= \bigsqcup\{\bigsqcup\{x_i \mid i \in I\}, \bigsqcup\{y_i \mid i \in I\}\} \\
(*) \quad &= \bigsqcup\bigcup\{\{x_i, y_i\} \mid i \in I\} \\
&= \bigsqcup\{\bigsqcup\{x_i, y_i\} \mid i \in I\} \\
&= \bigsqcup\{z_i \mid i \in I\} \\
&= \bigsqcup Z.
\end{aligned}$$

The identities above are immediate with the exception of the starred line which follows by **L1**. Thus $\bigsqcup Z \in X \otimes Y$, and so $X \otimes Y$ is closed under nonempty fusion.

Assume instead that $Z \subseteq X \oplus Y$ is nonempty. By letting $Z^X = Z \cap X$, $Z^Y = Z \cap Y$, and $Z^{X\otimes Y} = Z \cap (X \otimes Y)$, it follows that (1): $Z = Z^X \cup Z^Y \cup Z^{X\otimes Y}$. If $X = \varnothing$, then $Z^X = Z^{X\otimes Y} = \varnothing$, and so $Z = Y$. By the same reasoning, $Z = X$ if $Y = \varnothing$. In either case, $\bigsqcup Z \in X \oplus Y$ since X and Y are both closed under nonempty fusion. Thus we may restrict attention to the case where both $X \neq \varnothing$ and $Y \neq \varnothing$.

Let $Z^{X\otimes Y} = \{z_j \mid j \in J\}$ where $z_j = x_j.y_j$ for $x_j \in X$ and $y_j \in Y$ for each $j \in J$. By setting $Z^{X\otimes Y}_X = \{x_j \mid j \in J\}$ and $Z^{X\otimes Y}_Y = \{y_j \mid j \in J\}$, we may observe as above that $\bigsqcup Z^{X\otimes Y}_X \in X$ and $\bigsqcup Z^{X\otimes Y}_Y \in Y$, and so $\bigsqcup\{\bigsqcup Z^{X\otimes Y}_X, \bigsqcup Z^{X\otimes Y}_Y\} \in X \otimes Y$. By the same reasoning above, $\bigsqcup\{\bigsqcup Z^{X\otimes Y}_X, \bigsqcup Z^{X\otimes Y}_Y\} = \bigsqcup Z^{X\otimes Y}$, and so $\bigsqcup Z^{X\otimes Y} \in X \otimes Y$.

Since $Z^X \subseteq X$ and $Z^Y \subseteq Y$ are both nonempty, it follows that $\bigsqcup Z^X \in X$ and $\bigsqcup Z^Y \in Y$ by assumption, and so $\bigsqcup\{\bigsqcup Z^X, \bigsqcup Z^Y\} \in X \otimes Y$. Having shown that $X \otimes Y$ is closed under nonempty fusion, $\bigsqcup\{\bigsqcup\{\bigsqcup Z^X, \bigsqcup Z^Y\}, \bigsqcup Z^{X\otimes Y}\} \in X \otimes Y$. Observe:

$$\begin{aligned}
\bigsqcup\{\bigsqcup\{\bigsqcup Z^X, \bigsqcup Z^Y\}, \bigsqcup Z^{X\otimes Y}\} &= \bigsqcup\bigcup\{\bigcup\{Z^X, Z^Y\}, Z^{X\otimes Y}\} \\
&= \bigsqcup\bigcup\{Z^X, Z^Y, Z^{X\otimes Y}\} \\
&= \bigsqcup Z.
\end{aligned}$$

Whereas the first identity follows from **L1**, the second identity follows from (1) above. Thus we may conclude that $\bigsqcup Z \in X \otimes Y$, and so $\bigsqcup Z \in X \oplus Y$ since $X \otimes Y \subseteq X \oplus Y$. □

P9 *If $J, K \in \mathbb{P}$, then $\neg J, J \wedge K, J \vee K \in \mathbb{P}$.*

Proof. Assuming $J, K \in \mathbb{P}$, both $J = \langle J^+, J^- \rangle$ and $K = \langle K^+, K^- \rangle$ are exclusive and exhaustive where $J^+, J-, K^+$, and $K-$ closed under nonempty fusion. By definition, $\neg J = \langle J^-, J^+ \rangle$, and so $\neg J$ satisfies the same properties. Thus $\neg J \in \mathbb{P}$.

In order to show that $J \wedge K \in \mathbb{P}$, we may recall that $J \wedge K = \langle J^+ \otimes K^+, J^- \oplus K^- \rangle$, assuming that $d \in J^+ \otimes K^+$ and $t \in J^- \oplus K^-$. It follows that $d = a.b$ where $a \in J^+$ and $b \in K^+$, and $t \in J^- \cup K^- \cup (J^- \otimes K^-)$. If $t \in J^-$, then $a.t \notin J$ by induction, and so $d.t \notin J$ by **P5** given that $a.t \sqsubseteq d.t$. By similar reasoning, $d.t \notin J$ if either $t \in K^-$ or $t \in J^- \otimes K^-$. Thus $J \wedge K$ satisfies exclusivity.

Assuming $s \in P$, it follows by **P7** that $s \sqsubseteq w$ for some $w \in W$, and so: (1) $w \in P$; and (2) $r \sqsubseteq w$ whenever $r \circ w$. By exhaustivity, there is some $j \in J^+ \cup J^-$ where $j \circ w$ and some $k \in K^+ \cup K^-$ where $k \circ w$, and so $j \sqsubseteq w$ and $k \sqsubseteq w$ by (2). Given that $s \sqsubseteq w$, we know that $s.j.k \sqsubseteq w$, and so $s.j, s.k, s.j.k \in P$ by **P5**. Thus $s \circ j$, $s \circ k$, and $s \circ j.k$. If $j \in J^-$ or $k \in K^-$, then there is some $m \in J^- \oplus K^-$ where $s \circ m$, and so some $m \in (J^+ \otimes K^+) \cup (J^- \oplus K^-)$ where $s \circ m$. If $j \notin J^-$ and $k \notin K^-$, it follows that $j \in J^+$ and $k \in K^+$, and so $j.k \in J^+ \otimes K^+$. Thus there is some $m \in (J^+ \otimes K^+) \cup (J^- \oplus K^-)$ where $s \circ m$. Hence $J \wedge K$ satisfies exhaustivity.

Given **L2**, it follows that $J \wedge K \in \mathbb{P}$. Since disjunction inverts the order of the product and sum of the exact verifiers and falsifiers of the disjuncts as compared with conjunction, analogous reasoning shows that $J \vee K \in \mathbb{P}$. □

P10 $|\varphi| \in \mathbb{P}$.

Proof. The proof goes by induction on the complexity of $\varphi \in \texttt{ext}(\mathcal{L})$ where the base case is given by the definition of a model.

Assume that $|\varphi| \in \mathbb{P}$ whenever $\texttt{comp}(\varphi) \leqslant n$. Letting $\texttt{comp}(\varphi) = n + 1$, there are three cases to consider: (1) $\varphi = \neg\psi$; (2) $\varphi = \psi \wedge \chi$; and (3) $\varphi = \psi \vee \chi$.

Case 1: Assume $\varphi = \neg\psi$. Since $|\psi| \in \mathbb{P}$ by induction, $\neg|\psi| \in \mathbb{P}$ by **P9**. Given that $\neg|\psi| = |\neg\psi|$ by the exact inclusive semantics, $|\varphi| \in \mathbb{P}$ by the case assumption.

Case 2: Assume $\varphi = \psi \wedge \chi$. Since $|\psi|, |\chi| \in \mathbb{P}$ by induction, $|\psi| \wedge |\chi| \in \mathbb{P}$ by **P9**. Given that $|\psi| \wedge |\chi| = |\psi \wedge \chi|$ by the semantics, $|\varphi| \in \mathbb{P}$ by the case assumption.

Case 3: Similar to *Case 2*. □

P11 *$\mathcal{M}, \alpha, x \vDash \varphi$ just in case $s \in |\varphi|^+$ for some $s \sqsubseteq \alpha(x)$.*

Proof. The proof goes by induction on the complexity of $\varphi \in \texttt{ext}(\mathcal{L})$ where the base case is given by the semantics. Thus we may assume for induction that the proposition holds whenever $\texttt{comp}(\varphi) \leqslant n$, letting $\texttt{comp}(\varphi) = n + 1$.

Case 1: Assume $\varphi = \neg\psi$. The equivalences below follow from the case assumption, induction hypothesis, and semantics for negation with the exception of $(*)$:

$$\begin{aligned}
\mathcal{M}, \alpha, x \vDash \varphi &\Leftrightarrow \mathcal{M}, \alpha, x \nvDash \psi \\
&\Leftrightarrow s \notin |\psi|^+ \text{ for any } s \sqsubseteq \alpha(x) \\
(*) \quad &\Leftrightarrow t \in |\psi|^- \text{ for some } s \sqsubseteq \alpha(x) \\
&\Leftrightarrow t \in |\varphi|^+ \text{ for some } s \sqsubseteq \alpha(x).
\end{aligned}$$

In support of ($*$), we may observe that $\alpha(x) \in W$, and so both: (1) $\alpha(x) \in P$; and (2) $r \sqsubseteq \alpha(x)$ whenever $r \circ \alpha(x)$. Additionally, $|\psi| \in \mathbb{P}$ follows by **P10**. By exhaustivity, it follows from (1) that there is some $t \in |\psi|^+ \cup |\psi|^-$ where $t \circ \alpha(x)$, and so $t \sqsubseteq \alpha(x)$ by (2). Given the $\Rightarrow$ assumption, we may conclude that $t \in |\psi|^-$ for some $t \sqsubseteq \alpha(x)$. For the $\Leftarrow$ direction, assume that there is some $t \in |\psi|^-$ where $s \sqsubseteq \alpha(x)$. Thus there cannot be any $s \in |\psi|^+$ where $s \sqsubseteq \alpha(x)$ since otherwise $s.t \sqsubseteq \alpha(x)$ where $s.t \notin P$ by exclusivity, and so $\alpha(x) \notin P$ by **P5**, thereby contradicting (1).

Case 2: Assume $\varphi = \psi \wedge \chi$. We may then observe the following:

$$\begin{aligned}
\mathcal{M}, \alpha, x \vDash \varphi \;&\Leftrightarrow\; \mathcal{M}, \alpha, x \vDash \psi \text{ and } \mathcal{M}, \alpha, x \vDash \chi \\
&\Leftrightarrow\; d \in |\psi|^+ \text{ and } t \in |\chi|^+ \text{ for some } d, t \sqsubseteq \alpha(x) \\
&\Leftrightarrow\; s \in |\varphi|^+ \text{ for some } s \sqsubseteq \alpha(x).
\end{aligned}$$

The equivalences given above follow immediately from the case assumption, induction hypothesis, and semantics for conjunction.

Case 3: Assume $\varphi = \psi \vee \chi$. We may then observe the following:

$$\begin{aligned}
\mathcal{M}, \alpha, x \vDash \varphi \;&\Leftrightarrow\; \mathcal{M}, \alpha, x \vDash \psi \text{ or } \mathcal{M}, \alpha, x \vDash \chi \\
&\Leftrightarrow\; d \in |\psi|^+ \text{ for some } d \sqsubseteq \alpha(x) \text{ or } t \in |\chi|^+ \text{ for some } t \sqsubseteq \alpha(x) \\
&\Leftrightarrow\; s \in |\psi|^+ \cup |\chi|^+ \cup |\psi \wedge \chi|^+ \text{ for some } s \sqsubseteq \alpha(x) \\
&\Leftrightarrow\; s \in |\varphi|^+ \text{ for some } s \sqsubseteq \alpha(x).
\end{aligned}$$

The equivalences given above follow immediately from the case assumption, induction hypothesis, and semantics for disjunction. □

R1 *If* $\Gamma \vDash C$*, then* $\varphi \,\Box\!\!\rightarrow \Gamma \vDash \varphi \,\Box\!\!\rightarrow C$.

Proof. Assume $\Gamma \vDash C$, letting $\mathcal{M}, \alpha, x \vDash \varphi \,\Box\!\!\rightarrow A_i$ for all $i \in I$ where $\Gamma = \{A_i \mid i \in I\}$. Given any $s \in |\varphi|^+$ and $\beta \in H_{\mathbb{Z}}$ where $s \rightarrow_{\alpha(x)} \beta(x)$, it follows by the semantics for counterfactuals that $\mathcal{M}, \beta, x \vDash A_i$ for all $i \in I$, and so $\mathcal{M}, \beta, x \vDash C$. □

L3 *If* $t \circ w$*, then* $w_t = \{w\}$.

Proof. Let $t \circ w$ and $r \sqsubseteq_t w$ where $w \sqsubseteq r$. Thus $w \sqsubseteq_t w$ since $w \sqsubseteq w$ and so $r \sqsubseteq w$. Generalizing on r, $w \in w_t$. Let $u \in w_t$. So $u \sqsubseteq_t w$ where $r \sqsubseteq u$ whenever $r \sqsubseteq_t w$ and $u \sqsubseteq r$. Since $w \sqsubseteq_t w$ and $u \sqsubseteq w$, we know $w \sqsubseteq u$, and so $w = u$. Hence $w_t = \{w\}$. □

L4 *If* $s \sqsubseteq \alpha(x)$ *for* $\alpha \in H_{\mathbb{Z}}$ *and* $x \in \mathbb{Z}$*, then* $s \rightarrow_{\alpha(x)} \alpha(x)$.

Proof. Let $t \sqsubseteq \alpha(x)$ where $\alpha \in H_{\mathbb{Z}}$ and $x \in \mathbb{Z}$. Thus $t.\alpha(x) \sqsubseteq \alpha(x)$ where $\alpha(x) \in W$. It follows by **L3** that $\alpha(x)_t = \{\alpha(x)\}$. Since there is some $s \in \alpha(x)_t$ where $s.t \sqsubseteq \alpha(x)$, namely where $s = \alpha(x)$, we may conclude that $t \rightarrow_{\alpha(x)} \alpha(x)$. □

C2 $\varphi,\ \varphi \,\Box\!\!\rightarrow A \vDash A$.

Proof. Let $\mathcal{M}, \alpha, x \vDash \varphi$ and $\mathcal{M}, \alpha, x \vDash \varphi \,\Box\!\!\rightarrow A$. By **P11**, $s \in |\varphi|^+$ for some $s \sqsubseteq \alpha(x)$, and so $s \rightarrow_{\alpha(x)} \alpha(x)$ by **L4**. By the semantics for counterfactuals, $\mathcal{M}, \alpha, x \vDash A$. □

C3 $\varphi \boxright \psi,\ \varphi \wedge \psi \boxright A \vDash \varphi \boxright A$.

Proof. Let $\mathcal{M}, \alpha, x \vDash \varphi \boxright \psi$ and $\mathcal{M}, \alpha, x \vDash \varphi \wedge \psi \boxright A$. Assuming $s \in |\varphi|^+$ and $\beta \in H_{\mathbb{Z}}$ where $s \to_{\alpha(x)} \beta(x)$, it follows by the semantics for counterfactuals that $\mathcal{M}, \beta, x \vDash \psi$, and so $t \in |\psi|^+$ for some $t \sqsubseteq \beta(x)$ by **P11**. Since $s \sqsubseteq \beta(x)$ by **P1**, we know that $s.t \sqsubseteq \beta(x)$ where $s.t \in |\varphi \wedge \psi|^+$. Given that $s.t \to_{\alpha(x)} \beta(x)$ by **P3**, it follows that $\mathcal{M}, \alpha, x \vDash A$ again by the semantics for counterfactuals, and so $\mathcal{M}, \alpha, x \vDash \varphi \boxright A$. $\square$

C5 $\varphi \vee \psi \boxright A \vDash \varphi \boxright A$.

Proof. Let $\mathcal{M}, \alpha, x \vDash \varphi \vee \psi \boxright A$ where $s \in |\varphi|^+$ and $\beta \in H_{\mathbb{Z}}$ where $s \to_{\alpha(x)} \beta(x)$. It follows that $s \in |\varphi \vee \psi|^+$, and so $\mathcal{M}, \beta, x \vDash A$ by the semantics for the counterfactual. Thus we may conclude that $\mathcal{M}, \alpha, x \vDash \varphi \boxright A$. $\square$

L5 *If $w \in W$, then $w \to_s w$ for any $s \in S$.*

Proof. Assume $w \in W$ and let $s \in S$. It follows that (1): $q \sqsubseteq w$ whenever $q \circ w$. Since $w \in P$, there is some $r \in s_w$ by Maximality. By definition, $r \sqsubseteq s$ where $r \circ w$, and so $r \sqsubseteq w$ by (1). Thus we may conclude that $r.w \sqsubseteq w$, and so $w \to_s w$. $\square$

M3 $\vDash A \to \Box\Diamond A$.

Proof. Assume for contradiction that $\mathcal{M}, \alpha, x \nvDash A \to \Box\Diamond A$. Thus $\mathcal{M}, \alpha, x \vDash A$ and $\mathcal{M}, \alpha, x \nvDash \Box\Diamond A$, and so $\mathcal{M}, \alpha, x \nvDash \top \boxright \Diamond A$. It follows that $\mathcal{M}, \beta, x \nvDash \Diamond A$ for some $\beta \in H_{\mathbb{Z}}$ and $s \in |\top|^+$ where $s \to_{\alpha(x)} \beta(x)$. It follows that $\mathcal{M}, \beta, x \vDash \top \boxright \neg A$, and so $\mathcal{M}, \beta, x \nvDash \top \boxright A$. Since $\alpha(x) \in |\top|^+$, it follows by the semantics for counterfactuals that $\mathcal{M}, \gamma, x \nvDash A$ for any $\gamma \in H_{\mathbb{Z}}$ where $\alpha(x) \to_{\beta(x)} \gamma(x)$. By **L5**, $\alpha(x) \to_{\beta(x)} \alpha(x)$, and so $\mathcal{M}, \alpha, x \nvDash A$ in particular, thereby contradicting the above. $\square$

M4 $\vDash \Box A \to \Box\Box A$.

Proof. Assuming for contradiction that $\mathcal{M}, \alpha, x \nvDash \Box A \to \Box\Box A$. It follows that both $\mathcal{M}, \alpha, x \vDash \Box A$ and $\mathcal{M}, \alpha, x \nvDash \Box\Box A$, or equivalently, both: (1) $\mathcal{M}, \alpha, x \vDash \top \boxright A$; and (2) $\mathcal{M}, \alpha, x \nvDash \top \boxright (\top \boxright A)$.

Given (2), it follows that $\mathcal{M}, \beta, x \nvDash \top \boxright A$ for some $\beta \in H_{\mathbb{Z}}$ and $s \in |\top|^+$ where $s \to_{\alpha(x)} \beta(x)$, and so we know that $\mathcal{M}, \gamma, x \nvDash A$ for some $\gamma \in H_{\mathbb{Z}}$ and $t \in |\top|^+$ where $t \to_{\beta(x)} \gamma(x)$. However, it follows from (1) that $\mathcal{M}, \delta, x \vDash A$ for any $\delta \in H_{\mathbb{Z}}$ and $r \in |\top|^+$ where $r \to_{\alpha(x)} \delta(x)$. Since $\gamma(x) \in |\top|^+$ where $\gamma(x) \to_{\alpha(x)} \gamma(x)$ by **L5**, it follows that $\mathcal{M}, \gamma, x \vDash A$, contradicting the above. $\square$

M5 $\vDash \Box(\varphi \to A) \to (\varphi \boxright A)$.

Proof. Assume for contradiction that $\mathcal{M}, \alpha, x \nvDash \Box(\varphi \to A) \to (\varphi \boxright A)$. It follows that (1) $\mathcal{M}, \alpha, x \vDash \top \boxright (\varphi \to A)$ and (2) $\mathcal{M}, \alpha, x \nvDash \varphi \boxright A$, and so $\mathcal{M}, \beta, x \nvDash A$ for some $\beta \in H_{\mathbb{Z}}$ and $s \in |\varphi|^+$ where $s \to_{\alpha(x)} \beta(x)$ by the latter.

Since $s \sqsubseteq \beta(x)$ by **P1**, we know that $\mathcal{M}, \beta, x \vDash \varphi$ by **P11**, and so $\mathcal{M}, \beta, x \nvDash \varphi \to A$ by the semantics. However, given (1), $\mathcal{M}, \gamma, x \vDash \varphi \to A$ for any $\gamma \in H_{\mathbb{Z}}$ and $r \in |\top|^+$ where $r \to_{\alpha(x)} \gamma(x)$. Since $\beta(x) \in |\top|^+$ where $\beta(x) \to_{\alpha(x)} \beta(x)$ by **L5**, it follows that $\mathcal{M}, \beta, x \vDash \varphi \to A$, thereby contradicting the above. $\square$

T6 $\models \Box\boxed{\scriptstyle F}A \leftrightarrow \Box A$.

Proof. Assume $\mathcal{M}, \alpha, x \not\models \Box\boxed{\scriptstyle F}A$, it follows that $\mathcal{M}, \beta, x \not\models \boxed{\scriptstyle F}A$ for some $\beta \in H_{\mathbb{Z}}$ and $s \in |\top|^+$ where $s \to_{\alpha(x)} \beta(x)$, and so $\mathcal{M}, \beta, y \not\models A$ for some $y > x$. Since $\beta(y) \in |\top|^+$ where $\beta(y) \to_{\alpha(x)} \beta(y)$ by **L5**, it follows that $\mathcal{M}, \alpha, x \not\models \Box A$.

Conversely, assuming $\mathcal{M}, \alpha, x \not\models \Box A$, it follows that $\mathcal{M}, \beta, x \not\models A$ for some $\beta \in H_{\mathbb{Z}}$ and $s \in |A|^+$ where $s \to_{\alpha(x)} \beta(x)$. Let $\gamma(z) = \beta(z-1)$ for all $z \in \mathbb{Z}$. It follows that $\gamma \in H_{\mathbb{Z}}$ where $\mathcal{M}, \gamma, x \not\models \boxed{\scriptstyle F}A$ since $\mathcal{M}, \gamma, x+1 \not\models A$ and $x + 1 > x$. Since $\gamma(x) \in |\top|^+$ where $\gamma(x) \to_{\alpha(x)} \gamma(x)$ by **L5**, it follows that $\mathcal{M}, \alpha, x \not\models \Box\boxed{\scriptstyle F}A$.

It follows that $\mathcal{M}, \alpha, x \not\models \Box\boxed{\scriptstyle F}A$ just in case $\mathcal{M}, \alpha, x \not\models \Box A$, and so equivalently $\mathcal{M}, \alpha, x \models \Box\boxed{\scriptstyle F}A$ just in case $\mathcal{M}, \alpha, x \models \Box A$. Thus $\models \Box\boxed{\scriptstyle F}A \leftrightarrow \Box A$. □

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