

Counterfactual Worlds

Benjamin Brast-McKie

Abstract

This paper extends Kit Fine’s [1–5] truthmaker framework to provide a novel *task semantics* for tensed counterfactual conditionals. Instead of taking possible worlds to be primitive elements in a model, possible worlds will be defined in terms of *states*, *parthood*, *times*, and *tasks* where the latter encode the possible transitions between states. Rather than invoking primitive similarity or imposition relations, possible worlds will be compared at a time independent of that time’s past and future where the comparison will be carried out in mereological and modal terms. After reviewing motivations for this approach, I will provide the hyperintensional semantics for counterfactuals that is implemented in the `model-checker` software along with a unified logic for counterfactual, modal, and tense operators. I will then extend the language to include further tense operators in order to analyze forwards, backwards, and backtracking counterfactuals.

Keywords: Counterfactual Conditionals, Truthmaker Semantics, Hyperintensionality

1 Introduction

Drawing on Kripke’s [6] semantics for the metaphysical modal operators, intensional semantics theories of counterfactual conditionals model *possible worlds* as structureless points. Although taking possible worlds to be primitive is a harmless idealization when interpreting modal claims, the same cannot be said for counterfactual claims of the form ‘If it were the case that A , then it would be the case that B ’, or in symbols, ‘ $A \Box \rightarrow B$ ’. Rather, I will define possible worlds as histories of moments where each moment is rich in mereological structure. In order to motivate this approach, this section reviews Fine’s [7] early criticisms of the *similarity theory* that Lewis [8] developed as well the *imposition theory* that Fine [1, 2] went on to provide. Despite their differences, these views are not equipped to interpret tensed counterfactuals and may be criticized for including primitives in the metalanguage that are at least as hard to understand as the object language that they aim to interpret.

Instead of positing primitive similarity or imposition relations, §2 defines Fine’s imposition relation in purely modal and mereological terms. In order to also interpret tense operators, I will introduce a primitive *task relation* which encodes the possible

transitions between *states*. Given this addition, §3 defines the *possible states* that Fine takes to be primitive as well as the temporally extended *world histories* needed to interpret tensed counterfactuals. By drawing on these resources, §4 will provide a unified semantics and logic for tensed counterfactuals, employing this framework to analyze forwards, backwards, and backtracking counterfactuals in §5. The paper will conclude by considering objections and extensions in §6 and presenting minor results in §7. The following subsections will motivate these developments by presenting the shortcomings that similarity and imposition theories of counterfactual face.

1.1 Totality

In addition to a primitive set of *possible worlds* W and an *accessibility relation* R over W , Stalnaker [9] and Lewis [8] equip models with a *selection function* or *comparative similarity relation*, respectively. Abstracting from their differences, similarity theories take $A \Box \rightarrow B$ to be true in a world w just in case B is true in the most similar accessible world(s) to w in which A is true.¹ Independent of how similarity is to be understood, modeling possible worlds as structureless points imposes the following limitation:

Totality: Similarity selects/relates possible worlds considered in their entirety.

Although the worlds in W are simple points, this does not stop similarity theorists from assuming otherwise while working over an intended model which includes the possible worlds themselves along with all of their spatiotemporal structure and other relevant features. However, the various features that one might take possible worlds to have in an intended model are not included in the definition of a model. For the purposes of logic, the models of a given object language only have the structure that they are given, and standard intensional models do not include any internal structure in the possible worlds that are used to interpret the language. Focusing on Lewis' account, W is any nonempty set where the similarity relation stipulates an ordering over the worlds in W by treating each world as a structureless point.

Insofar as worlds are taken to model complete histories of everything, *Totality* leads similarity theories to predict the wrong results. For instance, consider the present tense analogue of the counterexample Fine [7, p. 425] raised against Lewis' account:

(N) If Nixon were to press the button there would be a nuclear holocaust.

Imagine that Nixon is hovering over the detonator where one of his advisers asserts N to another. As Fine points out, Lewis' theory predicts that N is false since for any world in which Nixon presses the button and a nuclear holocaust follows, there will be a much more similar world which includes, "a change that prevents the holocaust but that does not require such a great divergence from reality." For instance, perhaps the button is disconnected. Even so, it is natural to take N to be true since the results of Nixon pressing the button are irrelevant to the moment that we are considering in which Nixon did not press the button but easily could have. Rather than measuring

¹Whereas Stalnaker [9] assumes that there is always a closest A -world, Lewis [8] permits there to be equally close A -worlds as well as unending sequences of ever closer A -worlds. For simplicity, I will often leave the world of evaluation as well as the restriction to accessible worlds implicit in what follows.

the similarity of the actual world in its entirety against all of what would result if Nixon were to press the button, it is natural to consider the result of making minimal changes to the moment in which Nixon did not press the button to one in which he did. Given any such minimally changed moment, N makes a claim about the disaster to follow even if those futures end up diverging widely from the actual course of events. Since the moments that result from minimally changing the actual moment in which Nixon did not press the button to one in which he did do not include any other changes such as a disconnected button, etc., we may expect N to be true.

However natural it may be to restrict consideration to the moment in which the antecedent is to be evaluated while ignoring later moments, intensional models do not include the resources to do so.² If possible worlds are modeled by structureless points, a similarity relation can only order possible worlds considered in their entirety as articulated by *Totality*. So long as possible worlds are taken to model temporally extended histories, *Totality* makes it easy to imagine possible worlds in which Nixon pushed the button that are much more similar to the actual world than to any possible world in which he pushed the button and a nuclear holocaust took place.

Lewis [10] responded by dispensing with our pretheoretic similarity judgments, opting instead for weighted rules that constrain how the similarity of worlds is to be measured. Instead of precise rules which are easy to apply, Lewis' weighting system consists of open-ended suggestions for how to adjust our untutored intuitions about similarity while working over an intended model that includes the possible worlds themselves. In particular, Lewis sought to maximize the spatiotemporal regions of the counterfactual worlds which accord with actuality while reducing the number and size of the miracles that those counterfactual worlds include on account of violating the actual laws of nature. Whether one adopts Lewis' rules or not, it is natural to worry that the metalanguage that Lewis used to articulate his semantics is considerably harder to understand than the object language that he sought to interpret.³ Rather, a semantic theory that assigns meaningful truth-conditions to sentences ought to be developed in a metalanguage that is understood at least as well as the object language under study, providing a general account of the theoretical terms that it employs. By contrast, admitting sufficiently vague or context sensitive terms into the metalanguage undermines the ambition to provide a predictively powerful semantics for the object language. Although some amount of uncertainty about the extension of the primitives included in an intended model is to be expected, the similarity of temporally extended worlds makes for particularly unstable ground upon which to build a semantics. More generally, it is an advantage of any semantic theory to only posit primitives whose intended interpretation is not as vague as the object language under study.

In addition to these shortcomings, *Totality* poses a further problem for attempts to include tense in Lewis' semantics. For instance, consider Fine's [7] original example in which a tense operator takes wide scope over the counterfactual claim N given above:

(N') If Nixon had pressed the button there would have been a nuclear holocaust.

²Although intervals could be considered, I will restrict attention to a single moment of comparison for the majority of what follows, I will postpone consideration of intervals to §6.

³A similar complaint may be raised for Stalnaker's [9] selection function which is assumed to specify the most similar antecedent-world while ignoring irrelevant details in an intended model, as well as for theories that include causal primitives in the metalanguage such as [11] and [12].

Suppose that the button were activated for a ten minute period following a security meeting. After nearly pushing the button, Nixon decided to retire for the evening. Relieved, one of Nixon’s advisers asserts N' to another. Whereas N' is true if asserted after the activation period, N' is false if asserted before the button was activated. Although one might attempt to extend Lewis’ semantics to evaluate counterfactuals at world-time pairs, doing so is incompatible with taking similarity to be a relation between temporally extended possible worlds. Insofar as each similarity relation fixes an ordering of the possible worlds included in a model once and for all, shifting the time at which a sentence is evaluated has no effect on similarity.

Instead of giving up *Totality*, Lewis [8, p. 67] and [10, p. 465-6] takes similarity to relate possible worlds in their entirety where the similarity relation itself differs from context to context. It is for this reason that Lewis appeals to his weighted rules to determine which similarity relation to privilege in each context. Nevertheless, Lewis specifies his semantics [8, p. 119] in a general form in terms of *indices* which, “can be the set of worlds, the set of times, the set of things, or any other non-empty set whatever.” Given the present concern to evaluate tensed counterfactuals, one may take indices to be world-time pairs so that tense operators can shift the time of an index without also shifting the world of that index. By rejecting *Totality*, one may take similarity to relate indices rather than possible worlds, so that shifting the time in an index will effect which indices are closest, thereby changing which counterfactuals are true without changing the similarity relation itself. For instance, N' is true at an index $\langle w, x \rangle$ consisting of the actual world w and a time x shortly after the activation period just in case N is true at the index $\langle w, y \rangle$ which results from shifting to an earlier time $y < x$ in which the button is still activated. Insofar as similarity relates indices rather than possible worlds considered in their entirety, it does not matter that pushing the button will lead to a very different world since the set of closest indices to $\langle w, y \rangle$ is entirely determined by w and y independent of any other world or time.

Despite these advantages, evaluating sentence at world-time pairs while taking worlds to be primitive points makes times endogenous to the truth-conditions for the language, invalidating the principle that whatever is necessary is always the case.⁴ In the following section, I will extend the broad spirit of Lewis’ similarity semantics to accommodate tensed counterfactuals by defining possible worlds as histories of *moments*. Nevertheless, the resulting semantics faces a further problem that Fine [1, 2, 7] raised against Lewis’ theory of counterfactuals. These considerations will motivate a departure not only from positing a primitive similarity relation, but from employing an intensional semantics to study counterfactuals. Even so, it is by first defining worlds as histories of moments that we may appreciate the mereological and modal structure included in the hyperintensional semantics developed below.

1.2 Restriction

Given a primitive set of *possible moments* M , *times* T , and *accessibility relation* R over M , a similarity theorist may avoid the problems brought out above by taking *similarity* \leq to order moments instead of possible worlds. Whereas possible worlds

⁴Truth-conditions of the form $\{\langle w, x \rangle \mid w \in W\}$ which include every world at a fixed time x are necessary and yet temporary. Elsewhere, I present arguments that a semantics of this kind cannot easily be saved.

model temporally extended histories of a given system, possible moments are the time slices of those histories where each moment determines the truth-value of every sentence letter of the language, making times exogenous to the truth-conditions for the sentences of the language. Rather than taking accessibility to encode the relative possibility of one complete possible history from another, a moment m is *accessible* to k just in case it is possible for m to transition to k . By taking the set of times T to be the integers for simplicity, a *possible world* may be defined as any function from times to moments where every moment is accessible from its predecessor.⁵

By defining possible worlds, a similarity theorist may evaluate counterfactuals at world-time pairs so that $A \Box \rightarrow B$ is true in a world w at a time x just in case B is true in u at x whenever $u(x)$ is one of the most similar moments to $w(x)$ and A is true in u at x .⁶ Reading ' $\Diamond A$ ' as 'It has been the case that A ', we may take $\Diamond A$ to be true in a world w and time x just in case A is true in w at y for some time $y < x$. Something similar may be said for ' $\Diamond A$ ' which reads 'It is going to be the case that A '. Encoding temporality in this way avoids the counterexamples brought out above. In particular, N is true in a world w and time x just in case there is a time $y > x$ in which a nuclear holocaust occurs in u at y whenever $u(x)$ is one of the most similar moments to $w(x)$ in which Nixon pushes the button. Moreover, N' is true in a world w and time x just in case there is a time $z < x$ where N is true in w at z .⁷

Whereas comparing possible worlds for similarity makes both N and N' false, the same cannot be said for comparing moments for similarity. With respect to an actual moment in which the button is activated and Nixon did not press the button but easily could have, the most similar moments in which Nixon did push the button do not include any broken wires or blown fuses, and so are sure to lead to a nuclear holocaust. Thus N is predicted to come out true, where something similar may be said for N' . Even so, this semantics invalidates *Simplification of Disjunctive Antecedents* for familiar reasons. Restricting consideration to tenseless counterfactuals for simplicity, we may replace world-time pairs with moments for the time being. Accordingly, $A \Box \rightarrow B$ is true in a moment m just in case B is true in the most similar moment(s) to m in which A is true. As a result, the following principle is invalid:

$$\text{SDA} \quad A \vee B \Box \rightarrow C \vdash (A \Box \rightarrow C) \wedge (B \Box \rightarrow C).$$

Assuming the consequent is true in the most similar moments in which the antecedent is true does not require the consequent to be true in the most similar moments in which a disjunct of the antecedent is true.⁸ For instance, suppose that $A \vee B \Box \rightarrow C$ is true in m . Since the most similar $A \vee B$ -moments to m may all be $A \wedge C$ -moments without also all being B -moments, nothing requires $B \Box \rightarrow C$ to be true in m . Adding color, suppose that although Philip is typically the life of the party, Peter avoids parties since he often gets in fights. Assuming neither Philip nor Peter are at the party, the most similar moments in which either of them is at the party are moments in which

⁵I develop this framework in much greater detail elsewhere in order to provide a semantics for a bimodal language which includes both tense and modal operators. I develop a hyperintensional extension in §3.

⁶Although I will assume a Lewis-style ordering relation \leq which relates moments instead of worlds, these details can be adapted to employ a Stalnaker-style selection function for moments rather than worlds.

⁷Such a theorist may regiment N as $B \Box \rightarrow \Diamond H$ and N' as $\Diamond(B \Box \rightarrow \Diamond H)$. See §5 for further discussion.

⁸Fine [1, 7] makes this observation for Lewis' account. See also [13] for related observations and discussion.

only Philip is at the party. It follows that if either Philip or Peter were at the party, it would have been better, though it is false to claim that if Peter were at the party, it would have been better. Thus **SDA** is predicted to be invalid.

The invalidity of **SDA** does not turn on anything peculiar to similarity or moments. For the sake of generality, let *points* be generic elements in a sentence's truth-condition. We may then observe that any theory of counterfactuals that conforms to the following principle will invalidate **SDA** for the same reason given above:

Restriction: The consequent is only required to be true in a subset of the accessible points in the antecedent's truth-condition (e.g., the set of most similar moments).

If **SDA** is to be maintained, *Restriction* must be given up, undermining any attempt to provide a similarity semantics for counterfactuals. Rather than accepting these stark consequences, many theorists deny that **SDA** is valid, citing putative counterexamples to **SDA** in compensation for abandoning an otherwise natural account of the many instances in which **SDA** appears to hold. Nevertheless, theories of this kind indulge in significant complications, positing covert operators or else appealing to various pragmatic derivations in order to explain why **SDA** is so often appealing.⁹

By contrast with accounts which give up **SDA**, *strict theories* of counterfactuals ensure the validity of **SDA** by rejecting *Restriction*, requiring the consequent to be true at every point at which the antecedent is true. Replacing points with moments for consistency with the above, a strict theory of counterfactuals may take $A \Box \rightarrow B$ to be true in a moment m just in case B is true in every moment accessible from m in which A is true.¹⁰ It follows that **SDA** is valid for if C is true in every accessible moment in which $A \vee B$ is true, then C is also true in every accessible moment in which A (similarly B) is true. Even so, strict theories validate *Strengthening the Antecedent*:

STA $A \Box \rightarrow C \vdash A \wedge B \Box \rightarrow C$.

If C is true in all accessible moments in which A is true, C is also true in all accessible moments in which $A \wedge B$ is true. However, it has long been observed that there are natural counterexamples to **STA**. Just because the match would light if it were struck, the match would not have lit if it were struck while wet.¹¹ In response, strict theories defend **STA** by providing pragmatic explanations for why the counterexamples are only apparent by appealing to contextual shifts in R between the evaluation of the premise(s) and conclusion, though **STA** is valid in any particular context.¹²

Whereas similarly theories take **SDA** to be invalid and so must explain the data supporting **SDA**, strict theories assume that counterfactuals have the same logic as the strict conditional in any particular context, and so must provide an account of the counterexamples to both **STA** and **SDA**. Which if either of these strategies provides

⁹See [14], [15], [16], [17], [18], [19], and [20] for counterexamples to **SDA** in addition to providing theories that posit an ambiguity of one kind or another to explain why **SDA** often appears to be valid. Fine [1, p. 232] appeals to a principle of *Suppositional Accommodation* in order to account for such counterexamples.

¹⁰In order to accommodate tense, a strict theorist may take $A \Box \rightarrow B$ to be true in a world w at time x just in case B is true in u at x whenever $u(x)$ is accessible from $w(x)$ and A is true in u at x . Alternatively, a strict theorist could posit a separate relation for accessibility and the transitions between moments.

¹¹See Goodman's [21] discussion of this classic case.

¹²For instance, [22], [23], and [24] provide strict conditional theories in an intensional semantic framework.

the best account of the data that English speakers are reliably willing to recognize is an empirical question in the natural language semantics for English. By contrast, this paper is concerned to provide a logic for a counterfactual conditional operator that generalizes from the patterns of natural language data in a principled way. Instead of describing what is already expressible in a natural language, logic aims to describe theoretically valuable abstractions from the patterns of reasoning in natural language. In particular, I will present a logic for an unrestricted counterfactual operator that takes all ways for the antecedent of a counterfactual to be true into consideration, thereby validating **SDA**. In this right, the resulting counterfactual conditional bares a natural correspondence to the metaphysical modals which concern the broadest range of objective possibilities. As I will show in §4, the metaphysical modals are definable in terms of the counterfactual conditional that I will describe.

Rather than a regrettable consequence of the present theory, parting ways with natural language data is to be expected of any adequately developed logical operator. In addition to the metaphysical modal operators which are rarely if ever expressed in English, the Boolean operators provide a vivid example of the present ambition. For instance, conjunction in English is not commutative on account of often tracking temporal order. Even so, it is theoretically advantageous to include a commutative conjunction operator in formal languages. For instance, whereas proof systems which include a primitive conjunction operator may provide the standard introduction and elimination rules from which the commutativity of conjunction follows immediately, such rules must be rejected or heavily restricted if commutativity is taken to admit of exceptions. Similarly, the material conditional has a broad and systematic role to play throughout many formal disciplines despite the poor correspondence between the material conditional and any conditional naturally expressed in English.

Despite departing from the ambition to capture the full range of natural language data, there would seem to be no space in the dialectic between invalidating **SDA** by accepting *Restriction* and validating **STA** by rejecting *Restriction*. Nevertheless, this choice is not forced. Rather, the following section presents Fine’s [1, 2] hyperintensional alternative which validates **SDA** without also validating **STA**. Nevertheless, Fine’s semantics is not equipped to interpret tensed counterfactuals and may be faulted for positing a primitive imposition relation which has a counterfactual reading. In order to address the latter concern, I will define Fine’s imposition relation in terms of the purely modal and mereological primitives that he includes in his semantics. By deriving the constraints that Fine requires his primitive imposition relation to satisfy, the resulting semantics will be shown to validate a logic for counterfactuals that is at least as strong as the logic that Fine defends while positing strictly fewer primitives. In addition to simplifying Fine’s semantics conceptually, defining imposition increases the computability of the semantics as implemented in the `model-checker` software. Given these advantages, §3 will extend the framework to accommodate tense operators, providing a unified semantics and logic for counterfactual conditionals and tense.

2 Imposition Semantics

Intensional semantic theories identify truth-conditions with subsets of a set of points. Letting ‘ \Box ’ express *metaphysical necessity*, ‘ \equiv ’ express *propositional identity*, and ‘ $C_{(B/A)}$ ’ replace one or more occurrences of A with B in C , intensional theories validate **INT** where **LL** is valid in any language without opaque operators:

$$\mathbf{INT} \quad \Box(A \leftrightarrow B) \vdash A \equiv B. \qquad \mathbf{LL} \quad A \equiv B, C \vdash C_{(B/A)}.$$

Since $A \leftrightarrow A \vee (A \wedge B)$ is a classical tautology, $\Box[A \leftrightarrow A \vee (A \wedge B)]$ is a theorem of any normal modal logic. We may then derive $A \vee (A \wedge B) \Box \rightarrow C$ from the premise $A \Box \rightarrow C$ by **INT** and **LL**, and so $A \wedge B \Box \rightarrow C$ follows by **SDA**.¹³ Thus **STA** is derivable from **SDA** given **INT** and **LL**. In order to avoid this inference, **INT** must be given up since **LL** is unassailable in a language that does not include opaque operators.¹⁴ In order to validate **SDA** without also validating **STA**, Fine [1, 2] provides a hyperintensional semantics which distinguishes necessarily equivalent propositions. After presenting the details of this account in the following subsection, §2.2 will define the imposition relation that Fine introduces in order to avoid positing counterfactual primitives. I will then present a revised theory of hyperintensional propositions in §3 in order to provide a truth-conditional semantics for a language with counterfactual operators.

2.1 Outcomes

Instead of maintaining *Restriction*, Fine [1] introduces a primitive *imposition relation* where $t \rightarrow_w u$ indicates that, “ u is a possible outcome of imposing the change t on the world [state] w ” (p. 237).¹⁵ Whereas w and u are *world states* where each, “either contains or is incompatible with any other state” (p. 236), t is a *possible state* which Fine glosses as a “fragment” of a world state.¹⁶ Accordingly, Fine takes *parthood* \sqsubseteq to partially order the space of all states, referring to the least upper bound of a set of states as a *fusion state*.¹⁷ In general, states only determine the truth-values of those sentences that they *exactly verify* or *exactly falsify*, “tell[ing] us what it is *in* the world that makes the statement true if it is true or what it is *in* the world that makes it false if it is false” (p. 235). Building on Fine’s [3, 4] more recent work, I will have more to say about states and world states in the following subsection. For the time being, it will be enough to consider the broad role that states play in Fine’s semantics.

Given the addition of states and the imposition relation, Fine takes $A \Box \rightarrow C$ to be true in a world state w just in case C is true in any world state u which is the outcome of imposing any exact verifier state t for A on w . Fine makes it clear that his *imposition theory* for counterfactuals is based on the following ideas:

¹³See Fine [1, 2, 25], [13], and Ellis et al. [26] for closely related arguments.

¹⁴I define *transparency* in [27], where an operator is *opaque* otherwise. By taking a *transparent language* to exclude opaque operators, I show that a language is transparent just in case **LL** is a theorem.

¹⁵Whereas Fine [1] refers to world states simply as *worlds*, I will take *worlds* to be temporally extended histories and *world states* to be instantaneous time-slices for consistency throughout what follows.

¹⁶As brought out below, impossible states are not parts of world states, but states nonetheless.

¹⁷Whereas Fine [1] assumes that a set of states has a least upper bound if it has an upper bound, the next section will follow Fine [4] in taking the set of all states to form a complete lattice.

Universal Realizability of the Antecedent: There is no restriction on the exact verifiers for the antecedent which are to be imposed on the evaluation world state.

Universal Verifiability of the Consequent: There is no restriction on the outcomes of imposing an exact verifier for the antecedent on the evaluation world state.

Fine’s semantics rejects *Restriction* without identifying the exact verifier states for the antecedent with the outcome world states that they produce when imposed on the evaluation world state. An imposition theory of this kind is to be contrasted with strict theories of counterfactuals which require the consequent to be true at all accessible antecedent points themselves. Even so, the principles above preserve the unrestricted spirit of strict theories of counterfactuals while providing a means by which to validate **SDA** without validating **STA**. Since the exact verifiers for $A \vee B$ include the exact verifiers for A , it follows that if C is true in every outcome that results from imposing any exact verifier for $A \vee B$ on a world state w , then C is also true in every outcome that results from imposing any exact verifier for A (similarly B) on w , and so **SDA** is valid. Nevertheless, the outcomes that result from imposing an exact verifier for A on w may differ considerably from the outcomes that result from imposing an exact verifier for $A \wedge B$ on w , and so even if C is true in all of the former outcomes, C may fail to be true in all of the latter outcomes, thereby invalidating **STA**.

Although Fine’s imposition semantics validates **SDA** without validating **STA**, a number of important questions remain to be answered. Postponing discussion of the precise relationship between states and world states to the following subsection, we may consider an objection that Fine [1, p. 241] addresses in defense of his account:

It has been argued, in the second place, that the present semantics is relatively problematic in its conceptual commitments. For the transition relation must itself be understood in terms of counterfactuals. Thus to say that u is a possible outcome of t in w is just to say that, in w , u might obtain if t were to obtain (and also that u is maximal in this respect).

In response, Fine observes that Lewis is guilty of the same fault since counterfactual assumptions will be difficult to avoid in providing a suitable account of similarity. Instead of requiring a semantics to, “provide an analysis of the locutions with which it deals,” Fine settles for the position that his semantics provides, “a perspicuous account of how the truth-conditions of the sentences containing the locutions are to be determined.” Although analysis is by no means the only standard for productive semantic theorizing, it is nevertheless important to establish a solid understanding of the primitive terms used to articulate a semantics in order to provide an unambiguous specification of the truth-conditions for the locutions in question.

In order to clarify the interpretation of the imposition relation as well as to validate a number of important counterfactual inferences, Fine [1] provides four constraints on the imposition relation where $t.v$ is the fusion of the states t and v :

INCLUSION: If $t \rightarrow_w u$, then $t \sqsubseteq u$.

ACTUALITY: If $t \sqsubseteq w$, then $t \rightarrow_w u$ for some $u \sqsubseteq w$.

INCORPORATION: If $t \rightarrow_w u$ and $v \sqsubseteq u$, then $t.v \rightarrow_w u$.

COMPLETENESS: If $t \rightarrow_w u$, then u is a world-state.

Although Fine [1] offers a plausible defense for the principles above, it is unnecessary to take imposition to be primitive. Rather, the following subsection defines imposition in purely modal and mereological terms. By deriving the constraints that Fine assumes above I will avoid positing a counterfactual primitive in the metalanguage at no cost to the strength of the resulting logic. Moreover, simplifying the semantics reduces the computational complexity of its implementation in the `model-checker` software, expanding the range of inferences that the software can effectively evaluate.

2.2 Defining Imposition

I will follow Fine [3–5] in taking a *state space* $\mathcal{S} = \langle S, \sqsubseteq \rangle$ to be any complete lattice where S is the set of *states* and \sqsubseteq is the *parthood relation*. For instance, the state of Sanna sitting s includes a state of Sanna’s legs being bent t as a proper part, where t is a *proper part* of s — i.e., $t \sqsubset s$ — just in case $t \sqsubseteq s$ and $s \not\sqsubseteq t$. Since the least upper bound of any set of states $X \subseteq S$ is unique, we may refer to the least upper bound of the set of states X as the *fusion* $\bigsqcup X$ of X . In particular, $\bigsqcup \emptyset := \sqcap$ is the *null state* and $\bigsqcup S := \sqtop$ is the *full state*. When $X = \{s, t, \dots\}$ is finite, it will be convenient to represent the fusion of the states in X as $s.t.\dots := \bigsqcup \{s, t, \dots\}$. For instance, the fusion of a state s of Sanna sitting and a state k of Kevin cooking is the state $s.k$ of Sanna sitting and Kevin cooking. In general, a fusion obtains just in case all of its parts obtain, where this interpretation will play a critical role below.

In order to encode modal structure, Fine goes on to take a *modalized state space* to be any ordered triple $\mathcal{S}^\diamond = \langle S, P, \sqsubseteq \rangle$ where $\langle S, \sqsubseteq \rangle$ is a state space and $P \subseteq S$ is a primitive subset of *possible states* which satisfies the following constraints:

NONEMPTY: $P \neq \emptyset$.

POSSIBILITY: If $s \in P$ and $t \sqsubseteq s$, then $t \in P$.

Whereas every state in P can obtain, the states in S/P are *impossible* and so cannot obtain.¹⁸ Given that every part of a possible state is possible, the null state \sqcap is possible. Additionally, Fine takes the states s and t to be *compatible* just in case their fusion is possible, or in symbols, $s \circ t := s.t \in P$. For instance, although any state of Sanna sitting is compatible with any state of Kevin cooking, the same cannot be said for the states of Sanna standing which are incompatible with the states of her sitting. I will follow Fine in defining the *world states* as follows:

World States: $W := \{w \in P \mid \forall s \circ w (s \sqsubseteq w)\}$.

World states are maximal possible states which include all compatible states as parts. Fine restricts attention to *world spaces* which are modalized state spaces that satisfy:

WORLD SPACE: If $s \in P$, then $s \sqsubseteq w$ for some $w \in W$.

¹⁸Without admitting impossible states, the exact verifiers and falsifiers for a sentence could not be closed under fusion while preserving their intended interpretation as including all specific (typically incompatible) realizations, and so $A \equiv A \wedge A$ and $A \equiv A \vee A$ would admit counterexamples. See [27] for further discussion.

This principle amounts to identifying the possible states with the parts of world states in accordance with the gloss above, thereby ruling out infinite sequences of ever larger possible states which do not belong to a maximal possible world state.

Given any world space, it is straightforward to define Fine’s imposition relation in terms of the primitive states S , possible states P , and the parthood relation \sqsubseteq . Letting ‘ $s \sqsubseteq_t w$ ’ read ‘ s is a t -compatible part of w ’, I will define the following:

Compatible Part: $s \sqsubseteq_t w := s \sqsubseteq w \wedge s \circ t$.

Maximal Compatible Parts: $w_t := \{s \sqsubseteq_t w \mid \forall r \sqsubseteq_t w (s \sqsubseteq r \rightarrow r = s)\}$.

Imposition: $t \rightarrow_w u := u \in W \wedge \exists s \in w_t (s.t \sqsubseteq u)$.

Whereas a t -compatible part of w is a part of w that is compatible with t , a maximal t -compatible part of w is any t -compatible part of w that is not a proper part of any t -compatible part of w . Accordingly, an outcome u of imposing t on w is any world state which includes as parts both t as well as a maximal t -compatible part of w . Intuitively, an outcome world states is the results of minimally changing a given world state to include an imposed state, where outcomes need not be unique.

Whereas Fine admits that his primitive imposition relation has a counterfactual reading, this complaint cannot be raised against the primitive states S , possible states P , and parthood relation \sqsubseteq employed in the definitions above. Rather, the primitives included in a world space have purely modal and mereological readings. Given these resources, the definitions above provide an analysis of the imposition relation, reducing the number of primitives included in the semantics. Since **P1** - **P4** in §7 derive the constraints on imposition that Fine assumes, the resulting semantics validate a logic for counterfactuals that is at least as strong as the logic that Fine defends.

In addition to these merits, defining imposition increases the computability of the semantics as implemented in the `model-checker` that I developed for studying hyperintensional semantic theories.¹⁹ Rather than relying on intuitive estimations of the ideological complexity of a semantics, computability provides a practical measure. In order to produce evidence that an inference is valid, the `model-checker` shows that the inference does not have countermodels below a finite level of complexity, where the strength of the evidence is proportional to the number of models surveyed. Since every computational system has its limits, the computability of a semantics corresponds to the strength of the evidence that the `model-checker` is able to provide in support of a valid inference by surveying a wider range of countermodels.

Despite overcoming the problems that intensional similarity theories face without positing a counterfactual primitive in the metalanguage, the resulting semantics does not accommodate tense operators which often combine with counterfactuals. In order to present a unified semantic framework for studying tensed counterfactuals, I will replace Fine’s primitive set of possible states with a set of *times* and a two-place *task relation* in the following section. In addition to temporal structure, these resources provide the modal structure needed to define the set of possible states P that Fine takes to be primitive, deriving NONEMPTY, POSSIBILITY, and WORLD SPACES.

¹⁹I am grateful to Miguel Buitrago who helped me develop the software during his UROP at MIT. The `model-checker` software is free and open source: <https://pypi.org/project/model-checker/>

3 The Construction of Possible Worlds

Although states may be variously interpreted for different applications, it will help to fix ideas by taking states to be *some things being a specific way*. Despite taking states to be primitive, we may draw conceptual connections between states and possible worlds by identifying the *possible states* with restrictions of possible worlds. Whereas possible worlds are complete histories of everything, states are *static* and typically *partial*, concerning some limited way for certain things to be at an instant. For instance, given a world in which Sanna is sitting at time x , restricting to the part of that world which occurs at time x and makes it true that Sanna is sitting is a state of Sanna sitting. Since states are *wholly relevant* to the sentences that they exactly verify or falsify, most states will neither exactly verify nor falsify a given sentence.

States are also many. Just as the world-time pairs in which Sanna is sitting may be taken to cover all of the different ways for it to be true that Sanna is sitting, the states of Sanna sitting are the parts of the world time-slices which make it true that Sanna is sitting in all of those different ways. Moreover, states are *specific*: just as there is exactly one way for a possible world to be actual, there is exactly one way for a state to *obtain*. Put otherwise, neither possible worlds nor states are multiply realizable but rather model specific realizations. It is nevertheless important not to conflate the specificity of states with their possibility. Given a state s of Sanna sitting and a state t of her standing, the fusion state $s.t$ is just as specific as s and t considered on their own, amounting to a precise way for Sanna's body to be arranged despite it being impossible for $s.t$ to obtain. For instance, $s.t$ includes Sanna's legs being both bent as they are in her sitting state s and straight as they are in her standing state t . Insofar as t makes it false that Sanna is sitting, t makes it true that Sanna is not sitting, and so $s.t$ makes the conjunction true that Sanna is sitting and not sitting. Just as s obtaining is sufficient for it to be true that Sanna is sitting independent of whether s can obtain, $s.t$ obtaining is sufficient for it to be true that Sanna is sitting and not sitting despite the fact that $s.t$ cannot obtain. Instead of taking an impossible proposition to have no exact verifiers, the present framework models impossibility by the impossible states themselves.²⁰ The proposition that Sanna is sitting and standing differs from the proposition that Kevin is cooking and sleeping, or any other impossible proposition, by including distinct exact verifiers and falsifiers. In general, each impossible proposition will have its own range of impossible exact verifier states, where each exact verifier state specifies an impossible way for that proposition to be true.

Rather than relying on a primitive set of possible states P to distinguish the possible and impossible states, the following section will introduce the *task relation* \rightarrow to model the possible state transitions, defining the set of possible states in these terms. By also including a set of *times* T , I will demonstrate how to construct possible worlds. By contrast, merely specifying a primitive set of possible states P does not encode any temporal structure on its own. Insofar as states are static, they are entirely devoid of temporal structure. Even including a set of times T by which to index states into a time series, there is still no telling which time series are possible worlds and which are not. It is this theoretical role which the task relation is intended to fill.

²⁰See [3, 3, 5, 28] and [27] for discussion of the role that impossible states play in the semantics.

3.1 Task Space

Given any two states s and t , we may ask whether it is possible for s to transition to t . For instance, although it is possible for the state of Sanna sitting to transition to a state of Sanna standing, it is not possible for the state of Sanna sitting to transition to a state of Kevin cooking. In order to capture these differences, I will take any ordered pair of states $\langle s, t \rangle$ to represent a *state transition* from s obtaining to t obtaining. Since not all state transitions are possible, I will take there to be a *task* $s \rightarrow t$ just in case the transition $\langle s, t \rangle$ from s obtaining to t obtaining is possible.

For any state s , the *trivial transition* $\langle s, s \rangle$ leaves s unchanged. So long as it is possible for s to obtain, then making no change to s amounts to a possible transition from s obtaining to s obtaining, and so there is a *trivial task* $s \rightarrow s$. Letting the states s and t be *connected* $s \sim t$ just in case either $s \rightarrow t$ or $t \rightarrow s$, it follows that every possible state is connected to some state, in particular to itself. Conversely, if it is impossible for s to obtain, then there is no state t where either the transition from s obtaining to t obtaining nor the transition from t obtaining to s obtaining are possible, and so there is no state t where $s \sim t$. It follows that it is possible for s to obtain just in case s is connected to some state, and so I will define the *possible states* $P := \{s \in S \mid \exists t \in S (s \sim t)\}$ to include all states connected to some state.

Continuing with the example above, we may imagine that Sanna stands up, thereby transitioning from sitting s to standing t . Insofar as this transition is possible— i.e., $s \rightarrow t$ — it follows that that it is possible for Sanna to sit and also possible for her to stand, and so s and t are both possible states. More generally, $s, t \in P$ whenever $s \rightarrow t$. Although it follows that $s \in P$ whenever $s \rightarrow s$, the converse need not hold:

RESTRICTED REFLEXIVITY: If $s \in P$, then $s \rightarrow s$.

A state is *transient* just in case it is possible but does not have a trivial task. Although such states will not be needed below, neither will they be ruled out by imposing the principle above. Nevertheless, we may observe that insofar as there are impossible states, the task relation is not reflexive since impossible states are not connected to any state, and so there is no task from an impossible state to itself. For instance, the fusion state $s.t$ of Sanna sitting and standing cannot obtain, and so there is no possible transition to $s.t$ obtaining, nor is there a possible transition from $s.t$ obtaining.

In order to constrain the interpretation of \rightarrow in support of the intended reading of P as the set of possible states, recall that a state fusion is said to obtain just in case all of its parts obtain. Since a possible state can obtain, we ought to expect that each of its parts can obtain as a result. It is for this reason that Fine requires modalized state spaces to satisfy POSSIBILITY. Given that P is defined rather than primitive, I will derive POSSIBILITY by constraining the interaction between the task and parthood relations. In particular, tasks between fusions must be decomposable into subtasks between their respective parts, where $s \rightarrow t$ is a *subtask* of $s' \rightarrow t'$ just in case $s \sqsubseteq s'$ and $t \sqsubseteq t'$. More specifically, I will assume the following interaction constraints:

PARTHOOD: (L) If $d \sqsubseteq s$ and $s \rightarrow t$, then $d \rightarrow r$ for some $r \sqsubseteq t$.²¹
(R) If $r \sqsubseteq t$ and $s \rightarrow t$, then $d \rightarrow r$ for some $d \sqsubseteq s$.

These constraints ensure that every part is accounted for in any task between fusions. For example, given the red brake lights ahead (state r), we may take there to be a task from Nicky's driving state $d.r$ to a state $b.r'$ in which she brakes (state b) where r' is a similar arrangement of red brake lights. Since $d \sqsubseteq d.r$, it follows by PARTHOOD that there is some suitable $t \sqsubseteq b.r'$ where $d \rightarrow t$. Moreover, since $b \sqsubseteq b.r'$, there is some $s \sqsubseteq d.r$ where $s \rightarrow b$. Although PARTHOOD only specifies the existence of such states, we may expect that $s = d$ and $t = b$ so that in this case both $d \rightarrow b$ and $r \rightarrow r'$. Assuming PARTHOOD, **P5** in §7 shows that it is easy to derive POSSIBILITY.

Given that the null state \square is a part of every state, PARTHOOD is trivialized if $s \rightarrow \square$ and $\square \rightarrow s$ for every state s . In order to avoid unwanted instances of this scenario, it suffices to require the null state \square to be necessary. A state s is *contingent* just in case s is connected to a distinct state, i.e., $s \sim t$ for some $t \neq s$. Intuitively, the contingent states are those states which can change on account of possibly transitioning to or from a distinct state. It follows immediately that all contingent states are possible. By contrast, there are no tasks between impossible states. Rather, impossible states are non-contingent insofar as they cannot transition to or from any state at all, much less to or from distinct states. Despite being possible, the necessary states are also non-contingent since they must obtain. Thus I will define s to be *necessary* just in case s is only connected to itself where $N := \{s \in S \mid \forall t \in S (s = t \Leftrightarrow s \sim t)\}$ is the set of all necessary states. We may then avoid trivializing instances of PARTHOOD by assuming:

NULLITY: $\square = t$ just in case $\square \sim t$.

It follows that the null state \square is necessary where this fits with its definition as the fusion of the empty set. Since a fusion obtains just in case all of its parts obtain, nothing is required for \square to obtain, and so \square obtains trivially and thus of necessity. In addition to protecting PARTHOOD from triviality, it follows that every necessary state is possible, and so **P6** in §7 derives NONEMPTY from NULLITY.²²

It remains to show that every possible state is a part of a world state in accordance with Fine's WORLD SPACE constraint. Given any state s and possible state t , we may consider the parts of s that are compatible with t . In order to prevent there from being ever larger parts $r_1 \sqsubset r_2 \sqsubset \dots$ of s without end which are all compatible with t , I will draw on the definitions from §2.2 to impose the following constraint:

MAXIMALITY: If $s \in S$ and $t \in P$, there is a maximal t -compatible part $r \in s_t$.

Consider the total state s of the strategy room at the moment Nixon nearly pressed the button. We may imagine s to include parts that fix all the features of the objects which make up the room and its occupants. Although any state of Nixon pressing the button is incompatible with s , this is not true for the parts of s . For instance, the state

²¹These principles are interderivable if \rightarrow is symmetric. Although it is natural to take the task relation to be symmetric and transitive for certain applications, these constraints will not be required below.

²²It is also natural to assume COMPATIBILITY: if $s \in N$ and $t \in P$, then $s \circ t$. Additionally, one might impose NECESSITY: if $s \in N$ and $t \sqsubseteq s$, then $t \in N$. Neither constraint will be required for what follows.

a of Nixon’s advisers sitting just as they are is a part of s and perfectly compatible with a state of Nixon pressing the button. Given a state b of Nixon pressing the button, MAXIMALITY requires there to be at least one maximal part of s which is compatible with b , ruling out unending chains of ever larger b -compatible parts of s .²³

Whereas Fine restricts attention to world spaces as a basic assumption, **P7** in §7 derives WORLD SPACE from MAXIMALITY. Nevertheless, WORLD SPACE ensures that every proposition has exactly one truth-value at every world state in accordance with Fine’s [3] original observations. Since I will identify possible worlds at a time with world states, it follows that the tautologies of classical logic are true in every model at every world at every time. This result provides a powerful reason to maintain WORLD SPACE and given that WORLD SPACE may be derived from MAXIMALITY, a corresponding reason to adopt MAXIMALITY. Moreover, MAXIMALITY is in keeping with an intuitive understanding of states, ruling out exotic cases in which there is no largest part of a state s that is compatible with a possible state t .²⁴

Given the above, I will define a *task space* to be any ordered triple $\mathcal{T} = \langle S, \sqsubseteq, \rightarrow \rangle$ where $\langle S, \sqsubseteq \rangle$ is a state space and \mathcal{T} satisfies PARTHOOD, NULLITY, and MAXIMALITY. Given **P5** - **P7**, it follows that every task space determines a world space. Whereas the world spaces that Fine defines are rich with mereological and modal structure, task spaces provide the additional resources needed to encode the temporal structure that I will take possible worlds to include. By drawing on these elements, the following subsection will present the construction of possible worlds which will play a critical role in the task semantics for tensed counterfactuals that I will present in §4.

3.2 Possible Worlds

Assuming a discrete theory of time for simplicity, I will model *times* by the set of integers \mathbb{Z} where every time x differs from its successor time $x + 1$ by a unit duration specified for the application at hand.²⁵ Letting $\mathcal{T}_{\mathbb{Z}} = \langle S, \sqsubseteq, \rightarrow, \mathbb{Z} \rangle$ be a *discrete task space*, a *discrete evolution* $\tau : \mathbb{Z} \rightarrow S$ is any function from times to states. A discrete evolution τ is *possible* just in case $\tau(x) \rightarrow \tau(x + 1)$ for all times $x \in \mathbb{Z}$. It follows immediately that every state in a possible discrete evolution is possible.

A *world history* parameterized by \mathbb{Z} is any possible evolution $\alpha : \mathbb{Z} \rightarrow W$ so that $\alpha(x) \in W$ for every time $x \in \mathbb{Z}$. We may then take $H_{\mathbb{Z}}$ to be the set of all world histories parameterized by \mathbb{Z} . Given a history $\alpha \in H_{\mathbb{Z}}$, we may refer to the range of values that α occupies at different times as the *moments* of that history. Although it is often convenient to write $\alpha(x)$, officially the moments of a history α are its members which take the form $\langle x, \alpha(x) \rangle$. For instance, suppose that α is a history which occupies the same world state $\alpha(x) = \alpha(y)$ at different times $x \neq y$. Even though the world state is the same at both times, the moments differ since they occur at different times in α ’s history. Letting $\alpha \approx \beta$ just in case there is some $z \in \mathbb{Z}$ where $\alpha(x) = \beta(x + z)$ for all $x \in \mathbb{Z}$, we may identify the *possible worlds* parameterized by \mathbb{Z} with the set of

²³Recall that states are static. Since pushing the button is not static, one might deny that any state makes it true that Nixon pushed the button. I will return to this objection in the conclusion. For now, think of the state of Nixon pushing the button as the point of no return in which the circuit is closed.

²⁴Certain applications may impose DETERMINISM: if $s \rightarrow t$ and $s \rightarrow r$, then $t = r$. Alternatively, one might reject DETERMINISM while maintaining WORLD DETERMINISM: if $s \rightarrow t$ and $s \rightarrow r$ where $s, t, r \in W$, then $t = r$. Neither of these constraints are required nor particularly natural for the present application.

²⁵I will consider an objection to the arbitrariness of the unit in §6, presenting a continuous analogue.

equivalence classes $W_{\mathbb{Z}} = \{[\alpha]_{\mathbb{Z}} \mid \alpha \in H_{\mathbb{Z}}\}$ where $[\alpha]_{\mathbb{Z}} = \{\beta \in H_{\mathbb{Z}} \mid \alpha \approx \beta\}$. Whereas possible worlds represent distinct sequences of world states, world histories may be compared to coordinate planes which stipulate an origin while acknowledging that the choice of origin does not encode anything significant about the space it represents. Although the set of possible worlds may claim to hold a metaphysical standing that the set of histories cannot, it is the histories that will play an important role in the semantics presented below rather than possible worlds. Nevertheless, I will refer to histories as worlds, though officially they are to be distinguished.

In order to evaluate counterfactual conditionals in a language that includes tense operators, it will be important to consider the range of counterfactual worlds to result from making minimal changes to the world of evaluation at the time of evaluation. Roughly, the semantics for counterfactual conditionals presented in §4 will quantify over the worlds that result from imposing an exact verifier state for the antecedent on the world of evaluation at the time of evaluation, checking to see if the consequent is true. Before presetting these details, the following section will conclude by motivating the hyperintensional theory of propositions that I will use to interpret the language.

3.3 Propositions

Intuitively, a counterfactual conditional is true at a moment of evaluation just in case the consequent is true in any counterfactual moment to result from minimally changing the evaluation moment so that the antecedent is true. However, if the antecedent is interpreted by a set of moments, it is unclear what the result of making a minimal change would be. After all, a counterfactual moment in which the antecedent is true may differ from the evaluation moment in unrelated ways. The reason the exact verifier states for the antecedent are of greater use in this application than moments is that they amount to specific ways for the antecedent to be true without including anything irrelevant. Put otherwise, the exact verifier states for the antecedent belong to the *subject-matter* of the antecedent. By contrast, the moments in which the antecedent is true contain elements reaching far beyond the subject-matter of the antecedent. This motivates a departure from intensional theories of propositions to a hyperintensional theory of propositions which is sensitive to differences in subject-matter in addition to modal profile. By taking moments to be world states which have mereological structure, we may make minimal changes to the evaluation moment without including irrelevant details, checking to see if the consequent is true as a result.

In [27], I draw on Fine's [3, 4] recent work in order to identify and defend a bilateral theory of propositions which distinguishes propositions that differ in either subject-matter or modal profile without positing any unnecessary distinctions between propositions. Given that the present application calls for a theory of propositions which is sensitive to both modal profile and subject-matter, I will employ that bilateral theory here without reproducing its defense.²⁶ The theory is said to be *bilateral* on account of taking propositions to consist of both a positive and negative content. For instance, 'John is sitting down' expresses a proposition that consists of a set of exact verifier states which make that sentence true along with a set of exact falsifier states which make that sentence false. However, not all ordered pairs of sets of states are

²⁶See Fine [29] for a comparison of a related approach to Yablo's [30] theory of subject-matter.

propositions. In order to define the space of propositions, let a set of states X be *closed* just in case the fusion of any nonempty subset $Y \subseteq X$ belongs to X . Given any sets of states V and F , the ordered pair $\langle V, F \rangle$ is *exclusive* just in case the states in V are incompatible with the states in F , and *exhaustive* just in case every possible state is compatible with some state in either V or F . An ordered pair $\langle V, F \rangle$ is a *bilateral proposition* just in case $\langle V, F \rangle$ is exclusive and exhaustive where V and F are both closed under fusion. Given any proposition $\langle V, F \rangle$, every world state includes a part which belongs to V or F but not both, thereby validating classical logic. Letting \mathbb{P} be the set of bilateral propositions, we may now turn to interpret a propositional language with extensional, modal, counterfactual conditional, and tense operators.

4 Counterfactual Logic

Reading ‘ $\Box A$ ’ as ‘It has always been the case that A ’ and ‘ $\Box A$ ’ as ‘It is always going to be the case that A ’, I will take $\mathcal{L} = \langle \mathbb{L}, \top, \perp, \neg, \Box, \Box, \wedge, \vee, \Box \rightarrow \rangle$ to be a propositional language where $\mathbb{L} := \{p_i \mid i \in \mathbb{N}\}$ is the set of *sentence letters*, and \top and \perp are the *top* and *bottom* constants. We may define the *extensional sentences* of \mathcal{L} as follows:

$$\varphi ::= p_i \mid \top \mid \perp \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi.$$

Letting $\mathbf{ext}(\mathcal{L})$ be the set of extensional sentences of \mathcal{L} , we may define the *well-formed sentences* of \mathcal{L} by restricting the antecedent of a counterfactual to $\varphi \in \mathbf{ext}(\mathcal{L})$:

$$A ::= p_i \mid \top \mid \perp \mid \neg A \mid A \wedge A \mid A \vee A \mid \Box A \mid \Box A \mid \varphi \Box \rightarrow A.$$

Although the consequent of a counterfactual may be any well-formed sentence of \mathcal{L} , only the extensional operators may occur in the antecedent. We may then define a *discrete model* $\mathcal{M} = \langle S, \sqsubseteq, \rightarrow, \mathbb{Z}, |\cdot| \rangle$ of \mathcal{L} to be any ordered tuple where $\langle S, \sqsubseteq, \rightarrow, \mathbb{Z} \rangle$ is a discrete task space, $|p_i| \in \mathbb{P}$ for all $p_i \in \mathbb{L}$, $|\top| = \langle S, \{\bullet\} \rangle$, and $|\perp| = \langle \emptyset, \{\square\} \rangle$. After presenting a compositional semantics in the following subsection, I will provide a theory of logical consequence in §4.2 and discuss countermodels in §4.3.

4.1 Task Semantics

In [27] I argue that Fine’s [1, 3] exact inclusive semantics for the extensional operators preserves differences in both subject-matter and modal profile without drawing any unnecessary distinctions. Given the present aim to preserve differences in the subject-matter and modal profile of a proposition, I will reproduce that semantics here:

$$\text{Product: } X \otimes Y := \{s.t \mid s \in X, t \in Y\}.$$

$$\text{Sum: } X \oplus Y := X \cup Y \cup (X \otimes Y).$$

$$\text{Conjunction: } \langle V, F \rangle \wedge \langle V', F' \rangle := \langle V \otimes V', F \oplus F' \rangle$$

$$\text{Disjunction: } \langle V, F \rangle \vee \langle V', F' \rangle := \langle V \oplus V', F \otimes F' \rangle$$

$$\text{Negation: } \neg \langle V, F \rangle := \langle F, V \rangle.$$

Whereas the product $X \otimes Y$ is the set of pairwise fusions of the states in X with the states in Y , the sum $X \oplus Y$ is the union of X , Y , and their product $X \otimes Y$. Thus conjunction takes the product of the verifiers for the conjuncts followed by the sum of the falsifiers for the conjuncts, and disjunction does the reverse. By contrast, negation exchanges the verifiers and falsifiers of the negated proposition.²⁷

In §7, **P9** shows that \mathbb{P} is closed under the propositional operators defined above. Thus we may extend the interpretation provided by a model \mathcal{M} to all extensional sentences with the following *exact inclusive semantic* clauses where $|\varphi| \in \mathbb{P}$ by **P10**:

$$\begin{aligned} [\neg] \quad & |\neg\varphi| = \neg|\varphi|.^{28} \\ [\wedge] \quad & |\varphi \wedge \psi| = |\varphi| \wedge |\psi|. \\ [\vee] \quad & |\varphi \vee \psi| = |\varphi| \vee |\psi|. \end{aligned}$$

Since there is no confusing ‘ φ ’ with ‘ $|\varphi|$ ’, the abuse of notation for ‘ \neg ’, ‘ \wedge ’, and ‘ \vee ’ may be forgiven. Having fixed the interpretation of the extensional sentences, we may interpret all well-formed sentences of \mathcal{L} where $\alpha, \beta \in H_{\mathbb{Z}}$ are worlds, $x, y \in T$ are times, $q \in \mathbb{L} \cup \{\top, \perp\}$ is either a sentence letter or extremal element, and $|\varphi| := \langle |\varphi|^+, |\varphi|^- \rangle$:

$$\begin{aligned} \mathcal{M}, \alpha, x \models q & \text{ iff there is some } s \sqsubseteq \alpha(x) \text{ where } s \in |q|^+. \\ \mathcal{M}, \alpha, x \models \neg A & \text{ iff } \mathcal{M}, \alpha, x \not\models A. \\ \mathcal{M}, \alpha, x \models A \wedge B & \text{ iff } \mathcal{M}, \alpha, x \models A \text{ and } \mathcal{M}, \alpha, x \models B. \\ \mathcal{M}, \alpha, x \models A \vee B & \text{ iff } \mathcal{M}, \alpha, x \models A \text{ or } \mathcal{M}, \alpha, x \models B. \\ \mathcal{M}, \alpha, x \models \Box A & \text{ iff } \mathcal{M}, \alpha, y \models A \text{ for all } y < x. \\ \mathcal{M}, \alpha, x \models \Box A & \text{ iff } \mathcal{M}, \alpha, y \models A \text{ for all } y > x. \\ \mathcal{M}, \alpha, x \models \varphi \Box \rightarrow C & \text{ iff } \mathcal{M}, \beta, x \models C \text{ whenever } t \in |\varphi|^+ \text{ and } t \rightarrow_{\alpha(x)} \beta(x). \end{aligned}$$

Whereas the exact semantics took a bilateral form in order to specify a propositional operation for negation that respects differences in subject-matter, evaluating sentences for truth does not require the same sensitivity to subject-matter. Suppressing reference to the model, **P11** in §7 shows that $\varphi \in \mathbf{ext}(\mathcal{L})$ is true at a world-time pair just in case there is a part of that world at that time which exactly verifies φ . Given the definition of a proposition in §3.3, it follows from WORLD SPACE that every extensional sentence has exactly one truth-value at every world-time pair, where the semantic clauses given above extend this property to all well-formed sentences of the language.²⁹

In addition to maintaining standard semantic clauses for the extensional and tense operators, the task semantics for the counterfactual conditional has been presented in terms of imposition in order to facilitate comparison with Fine’s account. However, given the definition of imposition provided in §2.2 above, we may restate the semantics for the counterfactual conditional in more basic terms:

²⁷The hyperintensional space of propositions \mathbb{P} forms a *non-interlaced bilattice* rather than a Boolean lattice as in intensional theories. Although \perp is not needed below, it is worth observing that $|\perp| \neq |\neg\top|$. Rather, $\neg\perp$ and $\neg\top$ are the other two top and bottom elements making for a total of four *extremal elements*. See [31] and [32] for definitions and [27] for further discussion.

²⁸See Fine [3, p. 630] for the original definitions and observation corresponding to **P10**.

²⁹See Fine [3, pp. 665-7] for related results.

$\mathcal{M}, \alpha, x \models \varphi \Box \rightarrow C$ iff for all $t \in |\varphi|^+$ and $\beta \in H_{\mathbb{Z}}$, if $s.t \sqsubseteq \beta(x)$ for some maximal t -compatible part $s \in \alpha(x)_t$, then $\mathcal{M}, \beta, x \models C$.

The clause above articulates the semantics for counterfactuals in purely modal and mereological terms. Intuitively, a counterfactual is true in a world α at a time x just in case the consequent is true in any world β at x where $\beta(x)$ is the result of minimally changing $\alpha(x)$ to make the antecedent true. This approach is to be contrasted with similarity theories which compare temporally extended worlds for similarity in their entirety, as well as with imposition theories which posit a counterfactual primitive in the metalanguage while restricting consideration to the present tense. Despite its definition in other terms, it will often be convenient to continue to speak of the outcome worlds to result from imposing a state on a world at a given time.

Having extended the interpretations provided by the models of \mathcal{L} to all well-formed sentences, we may present a theory of logical consequence for any $\Gamma \cup \{C\} \subseteq \mathbf{wfs}(\mathcal{L})$ in order to study the interactions between the operators included in the language:

Logical Consequence: $\Gamma \models C$ iff for any model \mathcal{M} of \mathcal{L} , world α , and time x , if $\mathcal{M}, \alpha, x \models A$ for all $A \in \Gamma$, then $\mathcal{M}, \alpha, x \models C$.

As usual, a rule schema is *valid* just in case for any instance, its conclusion is a logical consequence of its premises, and a metarule is *valid* just in case it preserves validity. Without attempting to provide a complete logic for the task semantics given above, the following subsection will present three extensions of classical propositional logic in order to study the interactions between the operators included in the language. In future work, I hope to establish the completeness of these systems.

4.2 Logics

Letting $\mathcal{L}^{\text{PL}} = \langle \mathbb{L}, \neg, \wedge, \vee \rangle$ be an extensional language without top and bottom, I will take the deduction relation of classical propositional logic \vdash_{PL} to be defined over a restriction to the well-formed sentences of \mathcal{L}^{PL} . By taking $\mathcal{L}^{\text{CL}} = \langle \mathbb{L}, \neg, \wedge, \vee, \Box \rightarrow \rangle$, I will assume that $\varphi, \psi, \chi, \dots \in \mathbf{ext}(\mathcal{L}^{\text{CL}})$ and $A, B, C, \dots \in \mathbf{wfs}(\mathcal{L}^{\text{CL}})$ as above, where $\varphi \Box \rightarrow \Gamma := \{\varphi \Box \rightarrow A \mid A \in \Gamma\}$. I will then define \vdash_{CL} to be the smallest extension of \vdash_{PL} to be closed under the standard structural rules and all instances of the following:

R1 If $\Gamma \vdash C$, then $\varphi \Box \rightarrow \Gamma \vdash \varphi \Box \rightarrow C$

C1 $\varphi \Box \rightarrow \varphi$

C2 $\varphi, \varphi \Box \rightarrow A \vdash A$

C3 $\varphi \Box \rightarrow \psi, \varphi \wedge \psi \Box \rightarrow A \vdash \varphi \Box \rightarrow A$

C4 $\varphi \vee \psi \Box \rightarrow A \vdash \varphi \wedge \psi \Box \rightarrow A$

C5 $\varphi \vee \psi \Box \rightarrow A \vdash \varphi \Box \rightarrow A$

C6 $\varphi \vee \psi \Box \rightarrow A \vdash \psi \Box \rightarrow A$

C7 $\varphi \Box \rightarrow A, \psi \Box \rightarrow A, \varphi \wedge \psi \Box \rightarrow A \vdash \varphi \vee \psi \Box \rightarrow A$

Whereas **R1** requires the consequences of a counterfactual hypothesis to be closed under deduction, **C1** is the *Identity* schema, **C2** is *Counterfactual Modus Ponens*, and **C3** is *Weakened Transitivity*.³⁰ Additionally, **C4** - **C6** eliminate a disjunction from

³⁰See Lewis [8, p. 35] and Fine [1] for discussion of **C3**.

the antecedent of a counterfactual, and **C7** introduces a disjunction to the antecedent of a counterfactual.³¹ It is easy to see that **SDA** follows from **C5** and **C6**. Referring to the resulting *Counterfactual Logic* as **CL**, §7 provides elements of the proof that **CL** is sound. Additionally, we may derive the following consequences from **R1**:

- D1** $\varphi \Box \rightarrow A, \varphi \Box \rightarrow B \vdash \varphi \Box \rightarrow A \wedge B$
D2 *If $A \vdash B$, then $\varphi \Box \rightarrow A \vdash \varphi \Box \rightarrow B$*

These rules correspond to Fine's [1, 2] *Finite Conjunction* and *Classical Weakening* rules respectively.³² Whereas the following section will present a number of inferences that are excluded from **CL** on account of admitting countermodels, the remainder of the present section extends **CL** to include modal and tense operators.

Restricting consideration to the expressive resources included in \mathcal{L}^{CL} , one might attempt to define metaphysical necessity in terms of the counterfactual conditional operator as $\Box\varphi := \neg\varphi \Box \rightarrow A \wedge \neg A$. However, given the limitation that $\varphi \in \text{ext}(\mathcal{L}^{\text{CL}})$, it follows that iterated modalities are not well-formed.³³ Rather, I will include the top and bottom elements in $\mathcal{L}^{\text{CML}} = \langle \mathbb{L}, \top, \perp, \neg, \wedge, \vee, \Box \rightarrow \rangle$ in order to define *metaphysical necessity* as $\Box A := \top \Box \rightarrow A$, letting \vdash_{CML} minimally extend \vdash_{CL} to accommodate:

- M1** \top **M2** $\neg\perp$
M3 $A \rightarrow \Box\Diamond A$ **M4** $\Box A \rightarrow \Box\Box A$
M5 $\Box(\varphi \rightarrow A) \vdash \varphi \Box \rightarrow A$

Having defined the modal operators in \mathcal{L}^{CML} instead of \mathcal{L}^{CL} , all instances of the axioms given above are well-formed. It is important to observe that **M5** together with **C2** entail that the counterfactual conditional is of intermediate strength between the strict conditional and material conditional. We may then derive the following:

- D3** *If $\vdash A$, then $\vdash \Box A$.* **D4** $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$
D5 $\Box A \rightarrow A$ **D6** $\neg A, \varphi \Box \rightarrow A \vdash \neg\varphi$
D7 $\Box A \leftrightarrow \neg\Diamond\neg A$ **D8** $\Box\varphi \leftrightarrow \neg\varphi \Box \rightarrow \perp$
D9 $\varphi \Box \rightarrow \top$ **D10** $\perp \Box \rightarrow A$

Given **D3** - **D5** and **M3** - **M4**, *Counterfactual Modal Logic* **CML** entails an S5 logic. Whereas S5 is only valid in an intensional semantics by imposing appropriate frame constraints, the validity of the schemata above does not depend on constraining the range of task spaces. Although one might add an accessibility relation between states to weaken the modal logic, I will assume that metaphysical modality is the strongest objective modality and has an S5 logic, undermining the motivation to extend the semantics in this way. We may then observe that **D6** is *Counterfactual Modus Tollens*,

³¹Fine's [1, 2] provides **C1** - **C7** but does not include **R1**.

³²Were infinite conjunction included in the language, we could also derive the infinite conjunction rule that Fine [2] assumes. I will restrict consideration to sentences with finite length for simplicity.

³³Although the `model-checker` implements an extension of the semantics above to permit the antecedent to be any well-formed sentence, developing and defending this extension is best presented independently.

D7 derives the standard duality between the metaphysical modals, **D8** entails the equivalence that we cannot use to define metaphysical necessity in \mathcal{L}^{CL} , and **D9** - **D10** derive the *Triviality* axioms that Fine [1] assumes to be basic.

In order to present a range of further axiom schemata within the unrestricted language \mathcal{L} , I will take $\Box\Gamma := \{\Box A \mid A \in \Gamma\}$ and $\Box_{\langle P|F \rangle}$ to be the result of exchanging ‘ \Box ’ and ‘ \Box ’ in A where $\Gamma_{\langle P|F \rangle} := \{A_{\langle P|F \rangle} \mid A \in \Gamma\}$. Defining $\Diamond A := \neg\Box\neg A$ which reads ‘It was the case that A ’ and $\Diamond A := \neg\Box\neg A$ which reads ‘It is going to be the case that A ’, I will take \vdash_{CTL} to minimally extend \vdash_{CML} to include all instances of the following:

- | | |
|--|---|
| R2 If $\Gamma \vdash C$, then $\Box\Gamma \vdash \Box C$ | R3 If $\Gamma \vdash C$, then $\Gamma_{\langle P F \rangle} \vdash C_{\langle P F \rangle}$ |
| T1 $A \rightarrow \Box\Diamond A$ | T2 $\Box A \rightarrow \Box\Box A$ |
| T3 $\Box A \rightarrow \Diamond A$ | T4 $\Diamond\Diamond A \rightarrow (\Diamond A \vee A \vee \Diamond A)$ |
| T5 $(\Box A \wedge A \wedge \Diamond\top) \rightarrow \Diamond\Box A$ | T6 $\Box\Box A \leftrightarrow \Box A$. |
| T7 $\Diamond\Box A \rightarrow \Box A$. | T8 $\Diamond\Diamond A \rightarrow \Diamond A$. |

Whereas **R2** entails analogues of **D3** - **D4** for the operator \Box , **R3** exchanges ‘ \Box ’ and ‘ \Box ’ throughout any inference, making the past and future have the same structure at every time in each world. Focusing on the future, **T1** captures the idea that every present moment is past in any future moment and **T2** - **T5** require that the temporal ordering be transitive, endless, linear, and discrete. The remaining axiom schemata describe the interactions between the metaphysical modal and tense operators. I will characterize this relationship by way of the derived schema **D11** $\Box A \rightarrow \nabla A$ in §7 where $\nabla A := \Box A \wedge A \wedge \Box A$ reads ‘It is always the case that A ’.

I will refer to the system including all axiom schemata and metarules defended above as *Counterfactual Tense Logic* **CTL** where I will derive **D1** - **D11** in §7. In addition to being sound over the discrete models of \mathcal{L} , **CTL** is strong enough to justify a counterfactual conditional reading of ‘ \Box ’, a metaphysical modal reading of ‘ \Box ’, as well as the temporal readings of ‘ \Box ’ and ‘ \Box ’ presented above. The following subsection will complement these findings by considering inferences excluded from **CTL**.

4.3 Countermodels

In order to get a better sense of the semantics for counterfactual conditionals, it will help to review a number of invalid schemata. Consider the following:

- | | |
|---|---|
| #1 $\varphi, A \vdash \varphi \Box \rightarrow A$. | #2 $\varphi \Box \rightarrow \psi, \psi \Box \rightarrow A \vdash \varphi \Box \rightarrow A$. |
| #3 $(\varphi \Box \rightarrow A) \vee (\varphi \Box \rightarrow \neg A)$. | #4 $\varphi \Box \rightarrow A \vee B \vdash (\varphi \Box \rightarrow A) \vee (\varphi \Box \rightarrow B)$. |
| #5 $\varphi \Box \rightarrow A, \psi \Box \rightarrow A \vdash \varphi \wedge \psi \Box \rightarrow A$. | #6 $\varphi \Box \rightarrow \psi, \varphi \Box \rightarrow A \vdash \varphi \wedge \psi \Box \rightarrow A$. |
| #7 $\varphi \wedge \psi \Box \rightarrow A \vdash \varphi \Box \rightarrow (\psi \Box \rightarrow A)$. | #8 $\varphi \Box \rightarrow \psi, \neg\varphi, \neg\psi \vdash \neg\psi \Box \rightarrow \neg\varphi$. |
| #9 $\varphi \Box \rightarrow A \vdash \varphi \wedge \psi \Box \rightarrow A$. | #10 $\varphi \Box \rightarrow (\psi \Box \rightarrow A) \vdash \varphi \wedge \psi \Box \rightarrow A$. |
| #11 $\varphi \Box \rightarrow \psi \vdash \neg\psi \Box \rightarrow \neg\varphi$. | #12 If $\Gamma, \varphi \vdash C$, then $\Gamma \vdash \varphi \Box \rightarrow C$. |

For brevity, I will focus on **#1**, **#8**, **#9**, **#11**, and **#12**, discussing a countermodel for each in order to shed light on the nature of counterfactual reasoning.

#1 $\varphi, A \vdash \varphi \Box \rightarrow A$

The ball is red and Mary likes it. Even so, it would be wrong to claim that if the ball were red Mary would like it since there are certain shades of red Mary does not like. Drawing on the semantics for \mathcal{L} , consider the countermodel to $p_1, p_2 \vdash p_1 \Box \rightarrow p_2$:

$$\begin{array}{lll} |p_1|^+ = \{a, c, a.c\} & \alpha(x) = a.b & W = \{a.b, c.d\} \\ |p_2| = \langle \{b\}, \{d\} \rangle & \beta(x) = c.d & \end{array}$$

The world state $a.b$ includes an exact verifier for both p_1 and p_2 , making both sentences true in α at x . However, c is also an exact verifier for p_1 and is incompatible with any exact verifier for p_2 . Thus the maximal c -part of $\alpha(x)$ is \Box , and so β is an outcome of imposing c on α at x which does not include a part that exactly verifies p_2 .

Adding color, we may take a to be a state of the ball being a shade of red that Mary likes, b to be the state of Mary liking the ball, c to be a state of the ball being a shade of red that Mary does not like, and d to be the state of Mary disliking the ball. It follows by the semantics for counterfactuals that $p_1 \Box \rightarrow p_2$ is not true in α at x .

#8 $\varphi \Box \rightarrow \psi, \neg\varphi, \neg\psi \vdash \neg\psi \Box \rightarrow \neg\varphi$

If Boris had gone to the party, Olga would have gone too.³⁴ Neither Olga nor Boris went in the end. Even so, it would be wrong to claim that if Olga were to not go to the party then Boris would not go to the party. For even though Olga likes to go to the parties Boris attends, Boris prefers to socialise without Olga. Drawing on the semantics for \mathcal{L} , we may present a countermodel to the instance $p_1 \Box \rightarrow p_2, \neg p_1, \neg p_2 \vdash \neg p_2 \Box \rightarrow \neg p_1$:

$$\begin{array}{lll} |p_1| = \langle \{a\}, \{c\} \rangle & \alpha(x) = c.d.f & \gamma(x) = a.e.g \\ |p_2| = \langle \{b\}, \{d, e, d.e\} \rangle & \beta(x) = a.b.f & W = \{a.b.f, c.d.f, e.a.g\} \end{array}$$

Since $\alpha(x)$ contains the exact falsifiers c for p_1 and d for p_2 , both $\neg p_1$ and $\neg p_2$ are true in α at x . Moreover, the only outcome of imposing an exact verifier for p_1 on $\alpha(x)$ is $\beta(x)$ which includes an exact verifier for p_2 , and so $p_1 \Box \rightarrow p_2$ is true in α at x . However, if the exact verifier e for $\neg p_2$ is imposed on $\alpha(x)$, the outcome $\gamma(x)$ does not include an exact verifier for $\neg p_1$, and so $\neg p_2 \Box \rightarrow \neg p_1$ is false in α at x .

Adding substance, we may take d and e to be states of Olga staying home where d is only compatible with the state f of Boris not knowing where Olga is and e is only compatible with the state g of Boris believing her to be home. Since the state of Boris being at the party a is compatible with him not knowing where Olga is f and the only world state to include both a and f as parts also includes the state of Olga going to the party b , imposing a on $\alpha(x)$ results in the world state $\beta(x)$ in which Olga goes to the party. Thus it is true at α at x that if Boris had gone to the party, then Olga would have gone. However, the state of Olga staying at home e is only compatible with Boris believing her to be home g , where the only world state $\gamma(x)$ to include both e and g as parts also includes Boris going to the party a . Thus it is false to claim that if Olga were to not go to the party, then Boris would not go either. All that is required is for

³⁴This example has been adapted from Lewis' [8, p. 35] case against counterfactual contraposition.

Olga to not go to the party in a way that is compatible with Boris believing her to be home. For instance, perhaps Boris can see that the lights are on in her house.

The countermodel to #8 also invalidates #1 and *Counterfactual Contraposition* #11 $\varphi \Box \rightarrow \psi \vdash \neg\psi \Box \rightarrow \neg\varphi$. Nevertheless, I show in §7 that C2 is valid where this entails *Counterfactual Modus Tollens* D6. It follows that #12 does not preserve validity for otherwise we may derive #11 from D6. However, #12 is sometimes defended, at least for certain restricted subject-matters.³⁵

#9 (STA) $\varphi \Box \rightarrow A \vdash \varphi \wedge \psi \Box \rightarrow A$

Judy and Joey are at the party. If Cam were to join, he would argue with Judy, making her angry. Even so, if both Cam and Casey were to go to the party, Casey would keep Cam occupied, making Joey jealous. Consider the following countermodel:

$$\begin{aligned} |p_1| &= \langle \{a\}, \{b\} \rangle & |p_2| &= \langle \{c\}, \{d\} \rangle & |p_3| &= \langle \{e\}, \{f\} \rangle \\ \alpha(x) &= b.d.f.g & \beta(x) &= a.d.e.g & \gamma(x) &= a.c.f.h \\ W &= \{b.d.f.g, a.d.e.g, a.c.f.h\} \end{aligned}$$

Letting $p_1 \Box \rightarrow p_3 \vdash p_1 \wedge p_2 \Box \rightarrow p_3$ be the instance of #9, we may take a to be a state of Cam being at the party, b to be a state of Cam being at home, c to be a state of Casey being at the party, d to be a state of Casey being at home, e to be a state of Judy being angry, f to be a state of Judy being happy, g to be a state of Joey being content, and h to be a state of Joey being jealous. Since $\beta(x)$ is the only world state to result from imposing an exact verifier for p_1 on $\alpha(x)$ and includes an exact verifier for p_3 , the claim $p_1 \Box \rightarrow p_3$ is true in α at x . Nevertheless, the world state $\gamma(x)$ results from imposing an exact verifier for $p_1 \wedge p_2$ on $\alpha(x)$ and does not include an exact verifier for p_3 , and so $p_1 \wedge p_2 \Box \rightarrow p_3$ is false in α at x .³⁶ Thus STA is invalid.

It is worth pausing to consider the role that states as opposed to moments play in invalidating STA. Although all $A \wedge B$ -moments are A -moments, imposing an exact verifier state for $A \wedge B$ on an evaluation moment $\alpha(x)$ need not be an outcome of imposing an exact verifier states for A on $\alpha(x)$. The sensitivity of the outcomes induced by imposing a state on a moment of evaluation is made possible not only by the mereological structure that each moment includes, but also by the requirement that the exact verifier states for the antecedent belong to the subject-matter of the antecedent, where this requirement constitutes the *exactness* of the state semantics. It is for this reason that it is appropriate to assume a hyperintensional theory of propositions which is sensitive to differences in subject-matter in addition to modal profile.

Although the `model-checker` software may be used to find simpler countermodels than those presented above, it is often helpful to add states to a countermodel to support an intended interpretation of the countermodel in question. Nevertheless, using the `model-checker` to find minimal countermodels can greatly ease the process of finding interpreted countermodels to counterfactual conditional reasoning.

³⁵See [33] for such a view and [34] for further discussion.

³⁶This model can be extended to make $p_1 \wedge p_2 \wedge p_4 \Box \rightarrow p_3$ true while still making $p_1 \Box \rightarrow p_3$ true and $p_1 \wedge p_2 \Box \rightarrow p_3$ false. To do so, let $|p_4| = \langle \{i\}, \{j\} \rangle$ where i is the state of Daniel attending the party, j is the state of Daniel being at home, and $\alpha(x) = b.d.f.g.j$, $\beta(x) = a.d.e.g.j$, $\gamma(x) = a.c.f.h.j$, and $\delta(x) = a.c.e.h.i$ are world states. Repeating this strategy generates a Sobel sequence.

5 Counterfactuals and Tense

Although many of the examples above were stated in simple present tense, this was by contrivance rather than a reflection of standard practice. Rather, counterfactuals are often stated in the past tense where Fine's [25] original example is one such case:

(N') If Nixon had pressed the button there would have been a nuclear holocaust.

Suppose that a year after the button was activated, one of Nixon's advisers asserts N' to another. Whereas it is natural to take this assertion to be true, regimenting N' as $B \Box \rightarrow H$ where B reads 'Nixon pushes the button' and H reads 'There is a nuclear holocaust' predicts that the assertion is false. Not only is the button disconnected at the time of the assertion, the nuclear holocaust would not occur immediately even if the button were connected and pushed. Rather, consider the following alternative:

(N) $\Diamond(B \Box \rightarrow \Diamond H)$.

If Nixon were to push the button during the activation period there would be a future time in which a nuclear holocaust occurs. Since there is a time that is both during the activation period and before the adviser's assertion, N is predicted to be true, and so may be taken to provide a better regimentation of N' than $B \Box \rightarrow H$.

Although there would have been a nuclear holocaust if Nixon had pressed the button after finishing the security meeting when the button was activated, there would not have been a nuclear holocaust had Nixon pressed the button before it was activated or anytime after it was deactivated. As a result, $\Diamond(B \Box \rightarrow \neg \Diamond H)$ is also true if it were asserted a year after the incident. Instead of taking N' to assert the existence of any past time where $B \Box \rightarrow \Diamond H$ is true, we may take N' to assert that $B \Box \rightarrow \Diamond H$ is true and $B \Box \rightarrow \neg \Diamond H$ is false at a specific time. In order to capture this temporally specific reading, the following subsection will further extend the language to include two additional tense operators for speaking about specific times. Given these resources, §5.2 will analyze forwards, backwards, and backtracking counterfactuals.

5.1 Temporal Operators

Letting \mathcal{L}^* extend \mathcal{L} to include the unary *store operator* \uparrow_i and unary *recall operator* \downarrow^i for all $i \in \mathbb{N}$, we may define $\mathbf{ext}(\mathcal{L}^*)$ and $\mathbf{wfs}(\mathcal{L}^*)$ as before. By including a vector $\vec{v} = \langle v_1, v_2, \dots \rangle$ of stored times in the point of evaluation, we may present the following:

$$\begin{aligned} (\uparrow) \quad \mathcal{M}, \alpha, x, \vec{v} &\models \uparrow^i A \text{ iff } \mathcal{M}, \alpha, x, \vec{v}_{[x/v_i]} \models A. \\ (\downarrow) \quad \mathcal{M}, \alpha, x, \vec{v} &\models \downarrow^i A \text{ iff } \mathcal{M}, \alpha, v_i, \vec{v} \models A. \end{aligned}$$

Whereas $\uparrow^i A$ replaces the i^{th} value of \vec{v} with the current time of evaluation, the sentence $\downarrow^i A$ shifts the time of evaluation to the i^{th} value stored in \vec{v} . Despite adding \vec{v} as a parameter to the point of evaluation, the semantics for \mathcal{L} may otherwise be maintained. Drawing on these expressive resources, we may regiment N' as follows:

(N') $\downarrow^1(B \Box \rightarrow \Diamond H)$.

Let v_1 store the time at which Nixon almost pressed the button. Although this time may be forever imprinted on the minds of Nixon's advisers, the time at which the nuclear holocaust would have taken place may be unknown. It might have taken days for the disaster to unfold, or perhaps only hours. Instead of assuming there is a specific time, we may take N' to assert that there is a future time at which the nuclear holocaust would have taken place. Given the semantics above, N' is true at a world α , time x , and vector \vec{v} just in case for every world β which results from imposing an exact verifier for B on α at v_1 , there is some future time $y > v_1$ at which H is true in β .

Letting α be the actual world and x be the time of the adviser's assertion a year after the incident, N' is true whenever v_1 is a time where the button is activated. Even so, we may observe that $\downarrow^1(B \Box \rightarrow \Diamond \neg H)$ is also true assuming there is a time shortly after v_1 which occurs before the nuclear holocaust has taken place. Although this may be admitted, in certain circumstances there may be a specific future time at which it is appropriate to evaluate the consequent. For instance, perhaps Nixon's adviser had intended to specify that a nuclear holocaust is underway at the time of his assertion in all of the counterfactual worlds in which Nixon pushed the button a year prior:

$$(N'') \uparrow^2 \downarrow^1(B \Box \rightarrow \downarrow^2 H).$$

By first storing the time of assertion and then reverting back to the time at which the button was engaged, N'' asserts that if Nixon were to have pressed the button at that time of engagement, then a nuclear holocaust would have occur at the stored time of assertion. In certain circumstances, it may be enough to assert the weaker claim that $\uparrow^2 \Diamond(B \Box \rightarrow \downarrow^2 H)$ which merely requires there to be some past time where if Nixon were to push the button at that time we would now be in a nuclear holocaust.

Although typical, first going back in time and then going forward is not the only way to interpret tensed counterfactual claims, and sometimes not the most natural. The following subsection will consider both backwards and backtracking counterfactuals, providing their analysis with the resources of the present framework.

5.2 Backtracking

Following tradition, I will focus on an example presented by Jackson [35] in which Smith is standing on the edge of a building threatening to jump. Standing behind him in safety, Beth and Bill watch in fear for their friend. Moments later, Smith steps back from the edge. We may then consider the following claims:

- (D) If Smith had jumped, he would have died.
- (U) If Smith had jumped, a net would have been installed beneath him.
- (L) If Smith had jumped, he would have lived.

Kicking things off, suppose that Bill asserts D. Beth responds by insisting on U. After all, they both know that Smith wants to live so the only way Smith would jump is if jumping would not kill him. For instance, perhaps there would have been a net there to catch him. Having made her case for U, Beth goes on to assert L. Unconvinced, Bill points out that nobody would install a net on the building, reasserting D. This

prompts Beth to enumerate further reasons why Smith would never jump without a net, insisting on U and L once more. This may be imagined to continue round after round. The challenge is to identify what they are disagreeing about and to provide a theory of counterfactuals that adequately models their disagreement.

Whereas D is a forward counterfactual and can be regimented in a manner similar to N' above, U and L express *backwards* and *backtracking* counterfactuals, respectively. Rather than evaluating the consequent after the time at which the antecedent is made true, backwards counterfactuals such as U evaluate the consequent at an earlier time than the time at which the antecedent is made true. Consider the regimentations:

- (D) $\downarrow^2(J \Box \rightarrow \downarrow^3 D)$.
- (U) $\downarrow^2(J \Box \rightarrow \downarrow^1 U)$.
- (L) $\downarrow^2(J \Box \rightarrow \downarrow^3 L)$.

Holding the world α , time x , and stored times \vec{v} fixed for the purposes of evaluating Bill and Beth's conversation, suppose Bill and Beth agree that $v_1 < v_2 < v_3$. Nevertheless, Bill claims that D is true since he takes any world to result from imposing a state in which Smith jumps in α at v_2 to be one in which Smith dies at v_3 . Beth disagrees, claiming that L is true since she takes any world to result from imposing a state in which Smith jumps in α at v_2 to be one in which Smith lives at v_3 . Since the states of Smith being alive are incompatible with the states of Smith being dead, the worlds that Beth predicts will result from imposing a state of Smith jumping at v_2 make it false that Smith dies at v_3 , and so Beth and Bill disagree.

In defense of her view, Beth asserts U which requires the worlds that result from imposing a state of Smith jumping in α at v_2 to include the placement of a net at v_1 prior to the time v_2 of Smith's jump. Bill disagrees, where Beth's claim U helps to bring the source of the disagreement to light. Whereas Beth assumes that minimally changing $\alpha(v_2)$ to include a state of Smith jumping will also include a net there beneath him, Bill does not. What they disagree about are the outcomes of imposing a state of Smith jumping. Perhaps Beth's optimism reflects the emphasis she places on Smith's love for life, assuming that more would have to change about the actual world at v_2 for Smith to be willing to die than including a net there to catch him. Whereas jumping without a net is incompatible with Smith's love for life, jumping with a net is perfectly compatible. By contrast, Bill's pessimism may be attributed to his willingness to see Smith change his outlook on life, taking this to be a smaller change to actuality. Alternatively, Beth may exclude consideration of the possibility of Smith jumping without a net entirely, where Bill may be more permissive in the possibilities that he is willing to countenance. Instead of disagreeing about which mereological change to actuality is more substantial, we may take Beth and Bill's disagreement to be about which possibilities the intended model should be taken to include in the first place. Assuming the dispute to be over which model is intended may then be explained by Fine's [1, p. 232] *Suppositional Accommodation* principle, "according to which we always attempt to interpret a counterfactual in such a way that its antecedent A represents a genuine counterfactual possibility." In this case, Beth may be said to resist Bill's presumption that Smith might jump without a net.

Although the task semantics can help to model counterfactual claims, the semantics does not make any assumptions about which states or tasks an intended model might include, nor does it presume that there is a uniquely privileged intended model with a special metaphysical status. In the disagreement between Bill and Beth, each believes different worlds are the outcomes of imposing a state of Smith jumping in the actual world α at v_2 when he was poised to do so. Nevertheless, the task semantics helps to shed light on the nature of backtracking counterfactuals by providing an account of what this commits each speaker to in any given model. Even though backtracking counterfactuals like L are forward counterfactuals just as much as D, Beth's proposed changes at the time of evaluation propagate backwards on account of requiring an altered past. Assuming that a net is present at v_2 requires that it was installed on the building at an earlier time v_1 . By contrast, agreeing with Bill that Smith would have died only requires that Smith give up his love for life in the moment of jumping without requiring any substantial changes to the past. Given that the task semantics only selects outcome worlds by requiring them to occupy moments that differ minimally from the moment of evaluation, there is no requirement that the worlds which result from imposing a state on the world of evaluation at a given time agree with the world of evaluation at earlier times. Indeed, small changes at the time of evaluation may require massive changes to the past leading up to that point of evaluation.

In order to emphasize this final point, I will conclude the present section with a backwards counterfactual of a much more dramatic kind. To begin with, suppose that there are no alien civilizations anywhere in the universe. Consider the following case:

Icosahedron: Deep inside an Egyptian tomb, Harry discovered a perfectly formed icosahedron that appeared to be made out of pure titanium. In fact, the icosahedron was only a crude iron alloy. Relieved, Harry couldn't help but say out loud, "If the icosahedron had been titanium, then advanced aliens put it there."

Although the icosahedron is not in fact made of titanium, supposing otherwise requires massive changes to the past. Even if aliens did not put it there, assuming humans had developed the metallurgy needed to refine pure titanium at the time of the pyramids still requires a significant departure from actuality. Nevertheless, changing the moment of Harry's discovery to include a titanium icosahedron instead of the crude alloy that he found does not require much change at all. It is worth comparing such a case to Fine's Nixon example given above. Whereas N' makes a small change to the moment of supposition that results in a drastically altered future, **Icosahedron** makes a small change to the moment of supposition which requires a drastically altered past. Despite these differences, the two cases are handled by the task semantics in the same way: any world which differs minimally from the actual world at the time of evaluation is to be considered, no matter how far that world may diverge in its past or future.

Whereas Lewis' semantics compares worlds in their entirety for similarity, Fine's semantics considers the world states that result from imposing an exact verifier state for the antecedent of a counterfactual on the world state at which the counterfactual is evaluated. In this respect, Fine's semantics restricts consideration to the moment of evaluation. Despite strengthening the logic for counterfactuals, no consideration is paid to the effects that changing the present may have on the past or future.

6 Conclusion

Instead of taking possible worlds to be primitive points, the task semantics constructs worlds from states, parthood, tasks, and times. Worlds may then be compared at a time where that comparison is carried out in purely mereological and modal terms. By contrast, Lewis takes the similarity relation to be primitive and Fine posits a primitive imposition relation. Although Fine’s semantics validates a stronger logic for counterfactuals than intensional semantic theories are capable of by validating **SDA** without also validating **STA**, taking the imposition relation to be primitive makes the semantics homophonic by including a counterfactual primitive in the metalanguage. Although a homophonic semantics may be tolerated when there is no better option, §2.2 shows that it is unnecessary to take imposition to be primitive by providing its definition in purely modal and mereological terms. In addition to simplifying the semantics conceptually, defining the imposition relation increases the computability of the semantics implemented in the **model-checker** which strengthens the evidence that the **model-checker** can provide by surveying a broader range of models.³⁷

In addition to taking imposition to be primitive rather than defined, Fine does not provide the temporal structure needed to interpret tensed counterfactuals. By contrast, I have taken the task relation to be primitive, defining the possible states in its terms. Intuitively, there is a task $s \rightarrow t$ just in case the transition from the state s obtaining to the state t obtaining is possible. Whereas the imposition relation has a counterfactual reading, the task relation has a purely modal reading. Given a set of times, I show how to construct possible worlds, drawing on these resources in order to provide a semantics for tensed counterfactuals. Although focusing attention on an intended model may help to keep track of the working parts of the theory, nothing about the semantics turns on this reading or its supposed metaphysics. Nevertheless, a number of difficulties remain, inspiring extensions of the present framework.

6.1 Events

It is easy to consider exact verifier states for sentences like ‘The coin is heads up’ and ‘The icosahedron is pure titanium’. Such states are specific, perfectly static ways for certain things to be at an instant. For instance, we may consider the arrangement of titanium atoms which make up the icosahedron, or the exact orientation of the coin relative to the table. By contrast, consider the antecedent of **N** from before:

(B) Nixon is pressing the button.

It is natural to deny that there is any individual state of Nixon pressing the button. Rather, we may model Nixon’s action by a sequence of states in which he moves his finger towards the button, presses it, and then pulls his finger back. Letting a *discrete process* $\bar{e} := \langle e_1, \dots, e_n \rangle$ be any sequence of states where \bar{e} is *possible* just in case $e_i \rightarrow e_{i+1}$ for all $1 \leq i < n$, we may take the *event* $\langle \bar{V}, \bar{F} \rangle$ of Nixon pressing the button to include the set of processes \bar{V} which make the sentence **B** true together with the set of processes \bar{F} which make **B** false. Given a counterfactual conditional whose

³⁷Elsewhere I discuss the **model-checker** and the programmatic methodology it provides for semantics.

antecedent is an event, one might hope to impose the exact verifier processes in \bar{V} on a world over a duration rather than imposing any individual state at a time.

Before attempting to provide a theory of process imposition, it is important to consider the merits of doing so. For instance, if the result is a satisfying metaphysics which distinguishes temporally extended events from static propositions but which does not otherwise affect the semantics for counterfactuals, one might take some states to represent processes as a simplifying idealization. Nevertheless, it is natural to worry that conflating states with processes is incompatible with the intended interpretation guiding the construction of the semantics as well as the resulting principles. After all, it is not clear that any part of a possible process is possible, where the analogous principle for states played an important role in the semantics above. For instance, perhaps the possibility of some processes depend on the presence and coordination of their parts. Moreover, exact verifier processes may have different lengths and so will not have unique fusions on account of admitting fusions in which the processes being fused overlap in different ways. At the very least, considerably more would have to be said in order to provide a coherent process-theoretic semantics.

Instead of facing this challenge, one might take events to be a convenient proxy for related static propositions which might otherwise be hard to express. For instance, one might take any process of Nixon pressing the button \bar{e} to include a critical state \hat{e} in which the electrodes in the button connect, completing the circuit. More generally, a critical state in an exact verifier process is a *point of no return*. So long as every exact verifier process for an event has a unique critical state, that event may be said to be a proxy for a static proposition consisting only of the corresponding critical states. Nevertheless, it is natural to worry that critical states can only be identified by appealing to counterfactuals, and so faces the same criticisms brought out before. Moreover, it is far from obvious what it is to be a critical state in an exact falsifier process, nor is it clear that the critical states will be closed under fusion. Given my purposes here, I will leave the development of a theory of events and critical states for another time, idealizing processes as states in the present semantics.

6.2 Containment

Given a discrete task space $\mathcal{T} = \langle S, \sqsubseteq, \rightarrow, \mathbb{Z} \rangle$, the world histories in $H_{\mathbb{Z}}$ were defined to be functions $\alpha : \mathbb{Z} \rightarrow S$ where $\alpha(x) \in W$ and $\alpha(x) \rightarrow \alpha(x+1)$ for all $x \in \mathbb{Z}$. Moreover, the space of world states W was defined in terms of the possible states P which in turn was defined in terms of the task relation. However, we may provide a much more direct definition by taking world histories to be maximal possible evolutions. Letting a discrete evolution $\tau : \mathbb{Z} \rightarrow S$ be *possible* just in case $\tau(x) \rightarrow \tau(x+1)$ for all $x \in \mathbb{Z}$ as before, we may define *evolution parthood* by taking $\tau \sqsubseteq \sigma$ just in case $\tau(x) \sqsubseteq \sigma(x)$ for all $x \in \mathbb{Z}$. We may then take a possible discrete evolution $\tau : \mathbb{Z} \rightarrow S$ to be *maximal* just in case $\sigma \sqsubseteq \tau$ for any possible evolution $\sigma : \mathbb{Z} \rightarrow S$ where $\tau \sqsubseteq \sigma$.

By contrast with the definition of world histories given before, the definition of the maximal possible discrete evolutions is stated in entirely task-theoretic terms. In addition to its conceptual simplicity, this definition does not rely on the possible states P defined before. Nevertheless, so far nothing requires the maximal possible discrete evolutions to be world histories. This motivates the following constraints:

- CONTAINMENT: (L) If $s \in P$, $d \sqsubseteq s$, and $d \rightarrow r$, then $s \rightarrow t.r$ for some $t \in S$.
(R) If $t \in P$, $r \sqsubseteq t$, and $d \rightarrow r$, then $s.d \rightarrow t$ for some $s \in S$.

Whereas the PARTHOOD constraints ensure that every part is accounted for in any task between fusions, the constraints above require that for any task to or from the parts of a possible fusion, there will be a corresponding task to or from that fusion. Given this constraint, **P8** in §7.3 shows that the maximal possible discrete evolutions are coextensive with the world histories parameterized by \mathbb{Z} . In addition to motivating CONTAINMENT, this result brings together two independent ways of thinking about world histories, shedding further light on the nature of possible worlds.

6.3 Continuous Time

Taking time to be discrete introduces a degree of arbitrariness since we must fix the unit in order to interpret a world history at all. Whereas some histories string together moments separated by mere seconds, others may take moments to be separated by minutes or even hours. Instead of making an arbitrary choice, we may take this decision to be practical in nature, choosing a temporal resolution which is as low as possible while preserving enough temporal features for the application at hand.

By contrast, taking time to be continuous requires amending the present theory. For instance, suppose that time has the structure of the real numbers \mathbb{R} . Given any state at a time, there is no next time which we may assign to an accessible state. As a result, the definition of an evolution cannot be maintained in its present form. Letting a *continuous evolution* by any function $\tau : \mathbb{R} \rightarrow S$ from the real numbers \mathbb{R} to states S , we may take τ to be *possible* at x just in case there is some $y > 0$ where $\tau(u) \rightarrow \tau(x)$ for all $u \in (x - y, x)$ and $\tau(x) \rightarrow \tau(v)$ for all $v \in (x, x + y)$. A continuous evolution $\tau : \mathbb{R} \rightarrow S$ is *possible* just in case τ is possible at all $x \in \mathbb{R}$, and a *continuous world history* is any continuous possible evolution α where $\alpha : \mathbb{R} \rightarrow W$ (or, alternatively, maximal as in §6.2). We may then replace **T5** with the density axiom:

$$\mathbf{T5.1} \quad \boxed{\mathbb{F}}\boxed{\mathbb{F}}A \rightarrow \boxed{\mathbb{F}}A.$$

Although time is modeled by the real numbers throughout the physical sciences, there is little evidence to suggest that time has the structure of the real numbers. For all we know, time could be discrete with a high resolution, and so well approximated by the real numbers.³⁸ Even for the purposes of metaphysics, there is no reason to privilege a *continuous task space* $\mathcal{T} = \langle S, \sqsubseteq, \rightarrow, \mathbb{R} \rangle$ over the discrete task spaces considered above. Moreover, I will assume that there is nothing to be gained for the purposes of semantics by restricting consideration to continuous task spaces. Until compelling motivation can be provided for taking time to be continuous, I will maintain the definitions provided for discrete task spaces on account of their simplicity.

³⁸See [36] for discussion of discrete theories of time in recent physics.

7 Appendix

The following subsection will derive a number of theorems within the various systems presented above. In §7.2 and §7.3, I will derive the constraints that Fine [1] assumes the imposition relation and set of possible states satisfy respectively, presenting parts of the soundness proof for **CLT** in §7.4. In future work, I hope to establish the completeness for appropriate extensions of these systems.

7.1 Theorems

Classical tautologies and applications of the standard structural rules will be assumed throughout the following proofs.

D1 $\varphi \Box \rightarrow A, \varphi \Box \rightarrow B \vdash_{\text{CL}} \varphi \Box \rightarrow A \wedge B.$

Proof. Follows immediately from $A, B \vdash_{\text{CL}} A \wedge B$ by **R1**. □

D2 *If $A \vdash_{\text{CL}} B$, then $\varphi \Box \rightarrow A \vdash_{\text{CL}} \varphi \Box \rightarrow B$.*

Proof. Immediate from **R1**. □

D3 *If $\vdash_{\text{CML}} A$, then $\vdash_{\text{CML}} \Box A$.*

Proof. Immediate from **R1**. □

D4 $\vdash_{\text{CML}} \Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$

Proof. Immediate from **C2** by **R1**. □

D5 $\vdash_{\text{CML}} \Box A \rightarrow A$

Proof. Since $\top, \top \Box \rightarrow A \vdash_{\text{CML}} A$ by **C2**, $\top \Box \rightarrow A \vdash_{\text{CML}} A$ by **M1**, so $\vdash_{\text{CML}} \Box A \rightarrow A$. □

D6 $\varphi \Box \rightarrow A, \neg A \vdash_{\text{CL}} \neg \varphi.$

Proof. Immediate from **C2**. □

D7 $\vdash_{\text{CML}} \Box \varphi \leftrightarrow \neg \varphi \Box \rightarrow \perp.$

Proof. Since $\top \Box \rightarrow \varphi, \neg \varphi \vdash_{\text{CML}} \neg \top$ by **D6** where $\neg \top \vdash_{\text{CML}} \perp$ by **M1**, it follows that $\top \Box \rightarrow \varphi, \neg \varphi \vdash_{\text{CML}} \perp$, and so $\top \Box \rightarrow \varphi \vdash_{\text{CML}} \neg \varphi \rightarrow \perp$. Equivalently, $\Box \varphi \vdash_{\text{CML}} \neg \varphi \rightarrow \perp$, and so $\Box \Box \varphi \vdash_{\text{CML}} \Box(\neg \varphi \rightarrow \perp)$ by **R1**. Thus $\Box \varphi \vdash_{\text{CML}} \Box(\neg \varphi \rightarrow \perp)$ given **M4**, and so $\Box \varphi \vdash_{\text{CML}} \neg \varphi \Box \rightarrow \perp$ follows by **M5**.

Conversely, $\neg \varphi \Box \rightarrow \perp, \neg \perp \vdash_{\text{CML}} \neg \neg \varphi$ by **D6**, and so $\neg \varphi \Box \rightarrow \perp \vdash_{\text{CML}} \varphi$ given **M2** and $\neg \neg \varphi \vdash_{\text{CML}} \varphi$. Thus $\neg \varphi \Box \rightarrow \perp, \top \vdash_{\text{CML}} \varphi$, and so $\neg \varphi \Box \rightarrow \perp \vdash_{\text{CML}} \top \rightarrow \varphi$, which is equivalent to $\neg \varphi \Box \rightarrow \perp \vdash_{\text{CML}} \Box \varphi$. Given the above, $\vdash_{\text{CML}} \Box \varphi \leftrightarrow \neg \varphi \Box \rightarrow \perp$. □

D8 $\vdash_{\text{CML}} \Box A \leftrightarrow \neg \Diamond \neg A$

Proof. Since $A \vdash_{\text{CML}} \neg \neg A$, we know that $\top \Box \rightarrow A \vdash_{\text{CML}} \top \Box \rightarrow \neg \neg A$ by **R1**, and so $\top \Box \rightarrow A \vdash_{\text{CML}} \neg \neg(\top \Box \rightarrow \neg \neg A)$. Equivalently, $\Box A \vdash_{\text{CML}} \neg \Diamond \neg A$. Running the same reasoning in reverse yields $\neg \Diamond \neg A \vdash_{\text{CML}} \Box A$, and so $\vdash_{\text{CML}} \Box A \leftrightarrow \neg \Diamond \neg A$. □

D9 $\vdash_{\text{CML}} \varphi \Box \rightarrow \top$

Proof. Immediate from **M1** by **R1**. \square

D10 $\vdash_{\text{CML}} \perp \Box \rightarrow A$

Proof. Since $\neg A \vdash_{\text{CML}} \neg \perp$ given **M2**, we know $\perp \rightarrow A$, and so $\perp \Box \rightarrow A$ by **M5**. \square

D11 $\vdash_{\text{CTL}} \Box A \rightarrow \nabla A$

Proof. Since $\Box A, \Box \Box A \leftrightarrow \Box A \vdash_{\text{CTL}} \Box \Box A$, it follows from **T6** that $\Box A \vdash_{\text{CTL}} \Box \Box A$ where $\Box \Box A \vdash_{\text{CTL}} \Box A$ by **D5**, and so $\Box A \vdash_{\text{CTL}} \Box A$. Thus $\Box A \vdash_{\text{CTL}} \Box A$ by **R3**, where $\Box A \vdash_{\text{CTL}} A$ again by **D5**. Given that $\Box A, A, \Box A \vdash_{\text{CTL}} \nabla A$, it follows that $\Box A \vdash_{\text{CTL}} \nabla A$, and so we may conclude that $\vdash_{\text{CTL}} \Box A \rightarrow \nabla A$. \square

7.2 Imposition

The proofs given below follow from the definition of imposition. For convenience, I have copied the relevant definitions here:

Compatible Part: $s \sqsubseteq_t w := s \sqsubseteq w \wedge s \circ t$.

Maximal Compatible Parts: $w_t := \{s \sqsubseteq_t w \mid \forall r \sqsubseteq_t w (s \sqsubseteq r \rightarrow r \sqsubseteq s)\}$.

Imposition: $t \rightarrow_w u := u \in W \wedge \exists s \in w_t (s.t \sqsubseteq u)$.

World States: $W := \{w \in P \mid \forall s \circ w (s \sqsubseteq w)\}$.

Instead of taking the imposition relation to be primitive and assuming the constraints that Fine [1] provides, the following results derive Fine's constraints from the definition of the imposition relation. Since these proofs do not concern the task relation, we may take the set of possible states P to be primitive for present purposes, working over the world spaces that Fine [3] introduces. Given that the constraints that Fine imposes on world spaces will also be derived in the following section, the results presented in this section will carry over to the task spaces that I define.

P1 (INCLUSION) *If $t \rightarrow_w u$, then $t \sqsubseteq u$.*

Proof. Assuming $t \rightarrow_w u$, it follows that $u \in W$ where $s.t \sqsubseteq u$ for some $s \in w_t$. Since $t \sqsubseteq s.t$, it follows that $t \sqsubseteq u$ as desired. \square

P2 (ACTUALITY) *If $t \sqsubseteq w$ and $w \in W$, then $t \rightarrow_w u$ for some $u \sqsubseteq w$.*

Proof. Assume $t \sqsubseteq w$ for $w \in W$. Thus $w \in P$ where $w.t = w$, and so $w \circ t$. Since $w \sqsubseteq w$, we know that $w \sqsubseteq_t w$. Letting $r \sqsubseteq_t w$ where $w \sqsubseteq r$, it follows that $r \sqsubseteq w$, and so $w \in w_t$. Since $w.t \sqsubseteq w$, we know that $t \rightarrow_w w$, and so $t \rightarrow_w u$ for some $u \sqsubseteq w$. \square

P3 (INCORPORATION) *If $t \rightarrow_w u$ and $v \sqsubseteq u$, then $t.v \rightarrow_w u$.*

Proof. Assuming $t \rightarrow_w u$ and $v \sqsubseteq u$, it follows that $u \in W$ where $s.t \sqsubseteq u$ for some $s \in w_t$. Thus $s.t.v \sqsubseteq u$ and $s \sqsubseteq_t w$ where (1): $r \sqsubseteq s$ whenever $r \sqsubseteq_t w$ and $s \sqsubseteq r$. It follows that $s \sqsubseteq w$. Since $u \in P$, we also know that $s \circ t.v$, and so $s \sqsubseteq_{t.v} w$.

Letting $q \sqsubseteq_{t.v} w$ where $s \sqsubseteq q$, it follows that $q \sqsubseteq w$ where $q \circ t.v$, and so $q.t.v \in P$. Thus $q.t \in P$, and so $q \circ t$. It follows that $q \sqsubseteq_t w$, and so $q \sqsubseteq s$ follows from (1). Generalizing on q , it follows that (2): $q \sqsubseteq s$ whenever $q \sqsubseteq_{t.v} w$ and $s \sqsubseteq q$.

Having already shown that $s \sqsubseteq_{t.v} w$, it follows from (2) that $s \in w_{t.v}$. Since $s.t.v \sqsubseteq u$ for $u \in W$, we may conclude that $t.v \rightarrow_w u$ as desired. \square

P4 (COMPLETENESS) *If $t \rightarrow_w u$, then u is a world-state.*

Proof. Immediate from the definition of *Imposition*. \square

7.3 World Space

The proofs given below follow from the constraints on the task relation along with the definitions copied here for convenience:

PARTHOOD: If $d \sqsubseteq s$ and $s \rightarrow t$, then $d \rightarrow r$ for some $r \sqsubseteq t$.

NULLITY: $\square = t$ just in case $\square \rightarrow t$ or $t \rightarrow \square$.

MAXIMALITY: If $s \in S$ and $t \in P$, there is some maximal t -compatible part $r \in s_t$.

CONTAINMENT: If $s \rightarrow s$, $d \sqsubseteq s$, and $d \rightarrow r$, then $s \rightarrow t.r$ for some $t \in S$.

If $t \rightarrow t$, $r \sqsubseteq t$, and $d \rightarrow r$, then $s.d \rightarrow t$ for some $s \in S$.

Possibility: $P := \{s \in S \mid \exists t(s \sim t)\}$.

Compatibility: $s \circ t$ iff $s.t \in P$.

Possible Evolutions: $E_{\mathbb{Z}}^{\diamond} := \{\tau : \mathbb{Z} \rightarrow S \mid \tau(x) \rightarrow \tau(x+1) \text{ for all } x \in \mathbb{Z}\}$.

Evolution Parthood: $\tau \sqsubseteq \sigma$ iff $\tau(x) \sqsubseteq \sigma(x)$ for all $x \in \mathbb{Z}$.

Maximal Possible Evolutions: $M_{\mathbb{Z}} := \{\tau \in E_{\mathbb{Z}}^{\diamond} \mid \sigma \sqsubseteq \tau \text{ for all } \sigma \in E_{\mathbb{Z}}^{\diamond} \text{ where } \tau \sqsubseteq \sigma\}$.

Given an arbitrary task space $\mathcal{T} = \langle S, \sqsubseteq, \rightarrow \rangle$, we may establish the following results in order to show that every task space determines a corresponding world space.

P5 (POSSIBILITY) *If $s \in P$ and $t \sqsubseteq s$, then $t \in P$.*

Proof. Letting $s \in P$ and $t \sqsubseteq s$, there is some $r \in S$ where $s \sim r$, and so either $s \rightarrow r$ or $r \rightarrow s$. If $s \rightarrow r$, then $t \rightarrow d$ for some $d \sqsubseteq r$ by PARTHOOD (L). Similarly, if $r \rightarrow s$, then $d \rightarrow t$ for some $d \sqsubseteq r$ by PARTHOOD (R). Thus $t \sim d$ for some $d \in S$, and so $t \in P$. \square

P6 (NONEMPTY) $P \neq \emptyset$.

Proof. Since $\square = \square$, it follows that $\square \rightarrow \square$ by NULLITY, and so $\square \in P$. Thus $P \neq \emptyset$. \square

P7 (WORLD SPACE) *If $t \in P$, then $t \sqsubseteq w$ for some $w \in W$.*

Proof. Let $t \in P$. Since $\blacksquare \in S$, MAXIMALITY requires there to be some $r \in \blacksquare_t$, and so $r \sqsubseteq_t \blacksquare$ where (1): $q \sqsubseteq r$ whenever $q \sqsubseteq_t \blacksquare$ and $r \sqsubseteq q$. Thus $r \circ t$, and so $r.t \in P$.

Letting $s = r.t$, it follows that $s.t \in P$, and so $s \circ t$. Since $s \sqsubseteq \blacksquare$ where $r \sqsubseteq s$, we know that $s \sqsubseteq_t \blacksquare$, and so $s \sqsubseteq r$ by (1). Hence $s = r$, and so $r = r.t$.

Let $k \in S$ where $k \circ r$. Thus $k.r \in P$, and so $k.r.t \in P$ given the above. It follows that $k.r \circ t$ where $k.r \sqsubseteq \blacksquare$, and so $k.r \sqsubseteq_t \blacksquare$. Since $r \sqsubseteq k.r$, it follows that $k.r \sqsubseteq r$ by (1), and so $k \sqsubseteq r$. Generalizing on k , we may conclude that $r \in W$ where $t \sqsubseteq r$. \square

P8 $M_{\mathbb{Z}} = H_{\mathbb{Z}}$.

Proof. Let $\tau \in M_{\mathbb{Z}}$, $x \in \mathbb{Z}$, and $s_0 \in S$ where $s_0 \circ \tau(x)$. It follows that $s_0.\tau(x) \in P$ and so $s_0.\tau(x) \sim q$ for some $q \in S$. Assume for induction that $s_{-n}, s_n \in P$. Since $\tau \in M_{\mathbb{Z}}$, we know that $\tau(x - (n+1)) \rightarrow \tau(x - n)$ and $\tau(x + n) \rightarrow \tau(x + (n+1))$. Given that $\tau(x - n) \sqsubseteq s_{-n}.\tau(x - n)$ and $\tau(x + n) \sqsubseteq s_n.\tau(x + n)$, it follows by CONTAINMENT that there are some $s_{-(n+1)}, s_{n+1} \in S$ where $s_{-(n+1)}.\tau(x - (n+1)) \rightarrow s_{-n}.\tau(x - n)$ and $s_n.\tau(x + n) \rightarrow s_{n+1}.\tau(x + (n+1))$. Letting $\alpha(y) = s_{x-y}.\tau(y)$ for all $y \in \mathbb{Z}$, it follows by induction that $\alpha(y) \rightarrow \alpha(y+1)$ for all $y \in \mathbb{Z}$, and so $\alpha \in E_{\mathbb{Z}}^{\diamond}$. Since $\tau(y) \sqsubseteq \alpha(y)$ for all $y \in \mathbb{Z}$, we know that $\tau \sqsubseteq \alpha$, and so $\alpha \sqsubseteq \tau$ where $\alpha(x) \sqsubseteq \tau(x)$ in particular. Given that $\alpha(x) = s_0.\tau(x)$, it follows that $s_0 \sqsubseteq \tau(x)$. By generalizing on $s_0 \in S$, we may conclude that $\tau(x) \in W$, and so $\tau \in H_{\mathbb{Z}}$ follows from generalizing on $x \in \mathbb{Z}$.

Assuming instead that $\tau \in H_{\mathbb{Z}}$ and letting $x \in \mathbb{Z}$, we know that $\tau(x) \in W$. Given any $\sigma \in E_{\mathbb{Z}}^{\diamond}$ where $\tau \sqsubseteq \sigma$, it follows that $\sigma(x) \rightarrow \sigma(x+1)$, and so $\sigma(x) \in P$. Since $\tau(x) \sqsubseteq \sigma(x)$, we know that $\sigma(x).\tau(x) = \sigma(x)$, and so $\sigma(x) \circ \tau(x)$. Given that $\tau(x) \in W$, we know that $\sigma(x) \sqsubseteq \tau(x)$, and so $\sigma \sqsubseteq \tau$ follows by generalizing on x . Thus $\tau \in M_{\mathbb{Z}}$. \square

7.4 Soundness

Letting \mathcal{M} be any model over a discrete task space $\mathcal{T}_{\mathbb{Z}} = \langle S, \sqsubseteq, \rightarrow, \mathbb{Z} \rangle$, I will assume the following semantic clauses where the others are standard, and so have been omitted:

$\mathcal{M}, \alpha, x \models q$ iff there is some $s \sqsubseteq \alpha(x)$ where $s \in |q|^+$.

$\mathcal{M}, \alpha, x \models \varphi \Box \rightarrow C$ iff $\mathcal{M}, \beta, x \models C$ whenever $t \in |\varphi|^+$ and $t \rightarrow_{\alpha(x)} \beta(x)$.

For brevity, I will establish the validity of a collection of characteristic axiom schemata for **CTL** since the others are similar. In order to ease the proofs presented below, it will help to begin by proving a number of supporting lemmas and propositions.

L1 $\sqcup\{\sqcup E_i \mid i \in I\} = \sqcup\sqcup\{E_i \mid i \in I\}$ where $E_i \subseteq S$ for all $i \in I$.

Proof. Letting $e \in E_i$, it follows that $e \sqsubseteq \sqcup E_i$ where $\sqcup E_i \sqsubseteq \sqcup\{\sqcup E_i \mid i \in I\}$, and so $\sqcup\{\sqcup E_i \mid i \in I\}$ is an upper bound of $\sqcup\{E_i \mid i \in I\}$. By definition, we may conclude that $\sqcup\sqcup\{E_i \mid i \in I\} \sqsubseteq \sqcup\{\sqcup E_i \mid i \in I\}$. Since $E_i \subseteq \sqcup\{E_i \mid i \in I\}$ for any $i \in I$, it follows that $\sqcup E_i \sqsubseteq \sqcup\sqcup\{E_i \mid i \in I\}$ for all $i \in I$, and so $\sqcup\sqcup\{E_i \mid i \in I\}$ is an upper bound of the set $\{\sqcup E_i \mid i \in I\}$. Thus $\sqcup\{\sqcup E_i \mid i \in I\} \sqsubseteq \sqcup\sqcup\{E_i \mid i \in I\}$. \square

L2 If $X, Y \subseteq S$ are closed under nonempty fusion, then so are $X \otimes Y$ and $X \oplus Y$.

Proof. Assume $X, Y \subseteq S$ are closed and $Z \subseteq X \otimes Y$ is nonempty. Let $Z = \{z_i \mid i \in I\}$ where $z_i = x_i.y_i$ for $x_i \in X$ and $y_i \in Y$ for each $i \in I$. Taking $Z_X = \{x_i \mid i \in I\}$ and $Z_Y = \{y_i \mid i \in I\}$, both $Z_X \subseteq X$ and $Z_Y \subseteq Y$ are nonempty, and so $\sqcup Z_X \in X$ and

$\sqcup Z_Y \in Y$ by assumption. Thus $\sqcup\{\sqcup Z_X, \sqcup Z_Y\} \in X \otimes Y$ where:

$$\begin{aligned} \sqcup\{\sqcup Z_X, \sqcup Z_Y\} &= \sqcup\{\sqcup\{x_i \mid i \in I\}, \sqcup\{y_i \mid i \in I\}\} \\ (*) &= \sqcup\bigcup\{\{x_i, y_i\} \mid i \in I\} \\ &= \sqcup\{\sqcup\{x_i, y_i\} \mid i \in I\} \\ &= \sqcup Z. \end{aligned}$$

The identities above are immediate with the exception of the starred line which follows by **L1**. Thus $\sqcup Z \in X \otimes Y$, and so $X \otimes Y$ is closed under nonempty fusion.

Assume instead that $Z \subseteq X \oplus Y$ is nonempty. By letting $Z^X = Z \cap X$, $Z^Y = Z \cap Y$, and $Z^{X \otimes Y} = Z \cap (X \otimes Y)$, it follows that (1): $Z = Z^X \cup Z^Y \cup Z^{X \otimes Y}$. If $X = \emptyset$, then $Z^X = Z^{X \otimes Y} = \emptyset$, and so $Z = Y$. By the same reasoning, $Z = X$ if $Y = \emptyset$. In either case, $\sqcup Z \in X \oplus Y$ since X and Y are both closed under nonempty fusion. Thus we may restrict attention to the case where both $X \neq \emptyset$ and $Y \neq \emptyset$.

Let $Z^{X \otimes Y} = \{z_j \mid j \in J\}$ where $z_j = x_j.y_j$ for $x_j \in X$ and $y_j \in Y$ for each $j \in J$. By setting $Z_X^{X \otimes Y} = \{x_j \mid j \in J\}$ and $Z_Y^{X \otimes Y} = \{y_j \mid j \in J\}$, we may observe as above that $\sqcup Z_X^{X \otimes Y} \in X$ and $\sqcup Z_Y^{X \otimes Y} \in Y$, and so $\sqcup\{\sqcup Z_X^{X \otimes Y}, \sqcup Z_Y^{X \otimes Y}\} \in X \otimes Y$. By the same reasoning above, $\sqcup\{\sqcup Z_X^{X \otimes Y}, \sqcup Z_Y^{X \otimes Y}\} = \sqcup Z^{X \otimes Y}$, and so $\sqcup Z^{X \otimes Y} \in X \otimes Y$.

Since $Z^X \subseteq X$ and $Z^Y \subseteq Y$ are both nonempty, it follows that $\sqcup Z^X \in X$ and $\sqcup Z^Y \in Y$ by assumption, and so $\sqcup\{\sqcup Z^X, \sqcup Z^Y\} \in X \otimes Y$. Having shown that $X \otimes Y$ is closed under nonempty fusion, $\sqcup\{\sqcup\{\sqcup Z^X, \sqcup Z^Y\}, \sqcup Z^{X \otimes Y}\} \in X \otimes Y$. Observe:

$$\begin{aligned} \sqcup\{\sqcup\{\sqcup Z^X, \sqcup Z^Y\}, \sqcup Z^{X \otimes Y}\} &= \sqcup\bigcup\{\{Z^X, Z^Y\}, Z^{X \otimes Y}\} \\ &= \sqcup\bigcup\{Z^X, Z^Y, Z^{X \otimes Y}\} \\ &= \sqcup Z. \end{aligned}$$

Whereas the first identity follows from **L1**, the second identity follows from (1) above. Thus we may conclude that $\sqcup Z \in X \otimes Y$, and so $\sqcup Z \in X \oplus Y$ since $X \otimes Y \subseteq X \oplus Y$. \square

P9 If $J, K \in \mathbb{P}$, then $\neg J, J \wedge K, J \vee K \in \mathbb{P}$.

Proof. Assuming $J, K \in \mathbb{P}$, both $J = \langle J^+, J^- \rangle$ and $K = \langle K^+, K^- \rangle$ are exclusive and exhaustive where J^+, J^-, K^+ , and K^- closed under nonempty fusion. By definition, $\neg J = \langle J^-, J^+ \rangle$, and so $\neg J$ satisfies the same properties. Thus $\neg J \in \mathbb{P}$.

In order to show that $J \wedge K \in \mathbb{P}$, we may recall that $J \wedge K = \langle J^+ \otimes K^+, J^- \oplus K^- \rangle$, assuming that $d \in J^+ \otimes K^+$ and $t \in J^- \oplus K^-$. It follows that $d = a.b$ where $a \in J^+$ and $b \in K^+$, and $t \in J^- \cup K^- \cup (J^- \otimes K^-)$. If $t \in J^-$, then $a.t \notin J$ by induction, and so $d.t \notin J$ by **P5** given that $a.t \sqsubseteq d.t$. By similar reasoning, $d.t \notin J$ if either $t \in K^-$ or $t \in J^- \otimes K^-$. Thus $J \wedge K$ satisfies exclusivity.

Assuming $s \in P$, it follows by **P7** that $s \sqsubseteq w$ for some $w \in W$, and so: (1) $w \in P$; and (2) $r \sqsubseteq w$ whenever $r \circ w$. By exhaustivity, there is some $j \in J^+ \cup J^-$ where $j \circ w$ and some $k \in K^+ \cup K^-$ where $k \circ w$, and so $j \sqsubseteq w$ and $k \sqsubseteq w$ by (2). Given that $s \sqsubseteq w$, we know that $s.j.k \sqsubseteq w$, and so $s.j, s.k, s.j.k \in P$ by **P5**. Thus $s \circ j, s \circ k$, and

$s \circ j.k$. If $j \in J^-$ or $k \in K^-$, then there is some $m \in J^- \oplus K^-$ where $s \circ m$, and so some $m \in (J^+ \otimes K^+) \cup (J^- \oplus K^-)$ where $s \circ m$. If $j \notin J^-$ and $k \notin K^-$, it follows that $j \in J^+$ and $k \in K^+$, and so $j.k \in J^+ \otimes K^+$. Thus there is some $m \in (J^+ \otimes K^+) \cup (J^- \oplus K^-)$ where $s \circ m$. Hence $J \wedge K$ satisfies exhaustivity.

Given **L2**, it follows that $J \wedge K \in \mathbb{P}$. Since disjunction inverts the order of the product and sum of the exact verifiers and falsifiers of the disjuncts as compared with conjunction, analogous reasoning shows that $J \vee K \in \mathbb{P}$. \square

P10 $|\varphi| \in \mathbb{P}$.

Proof. The proof goes by induction on the complexity of $\varphi \in \mathbf{ext}(\mathcal{L})$ where the base case is given by the definition of a model.

Assume that $|\varphi| \in \mathbb{P}$ whenever $\mathbf{comp}(\varphi) \leq n$. Letting $\mathbf{comp}(\varphi) = n + 1$, there are three cases to consider: (1) $\varphi = \neg\psi$; (2) $\varphi = \psi \wedge \chi$; and (3) $\varphi = \psi \vee \chi$.

Case 1: Assume $\varphi = \neg\psi$. Since $|\psi| \in \mathbb{P}$ by induction, $\neg|\psi| \in \mathbb{P}$ by **P9**. Given that $\neg|\psi| = |\neg\psi|$ by the exact inclusive semantics, $|\varphi| \in \mathbb{P}$ by the case assumption.

Case 2: Assume $\varphi = \psi \wedge \chi$. Since $|\psi|, |\chi| \in \mathbb{P}$ by induction, $|\psi| \wedge |\chi| \in \mathbb{P}$ by **P9**. Given that $|\psi| \wedge |\chi| = |\psi \wedge \chi|$ by the semantics, $|\varphi| \in \mathbb{P}$ by the case assumption.

Case 3: Similar to *Case 2*. \square

P11 $\mathcal{M}, \alpha, x \models \varphi$ just in case $s \in |\varphi|^+$ for some $s \sqsubseteq \alpha(x)$.

Proof. The proof goes by induction on the complexity of $\varphi \in \mathbf{ext}(\mathcal{L})$ where the base case is given by the semantics. Thus we may assume for induction that the proposition holds whenever $\mathbf{comp}(\varphi) \leq n$, letting $\mathbf{comp}(\varphi) = n + 1$.

Case 1: Assume $\varphi = \neg\psi$. The equivalences below follow from the case assumption, induction hypothesis, and semantics for negation with the exception of (*):

$$\begin{aligned} \mathcal{M}, \alpha, x \models \varphi &\Leftrightarrow \mathcal{M}, \alpha, x \not\models \psi \\ &\Leftrightarrow s \notin |\psi|^+ \text{ for any } s \sqsubseteq \alpha(x) \\ (*) &\Leftrightarrow t \in |\psi|^- \text{ for some } s \sqsubseteq \alpha(x) \\ &\Leftrightarrow t \in |\varphi|^+ \text{ for some } s \sqsubseteq \alpha(x). \end{aligned}$$

In support of (*), we may observe that $\alpha(x) \in W$, and so both: (1) $\alpha(x) \in P$; and (2) $r \sqsubseteq \alpha(x)$ whenever $r \circ \alpha(x)$. Additionally, $|\psi| \in \mathbb{P}$ follows by **P10**. By exhaustivity, it follows from (1) that there is some $t \in |\psi|^+ \cup |\psi|^-$ where $t \circ \alpha(x)$, and so $t \sqsubseteq \alpha(x)$ by (2). Given the \Rightarrow assumption, we may conclude that $t \in |\psi|^-$ for some $t \sqsubseteq \alpha(x)$. For the \Leftarrow direction, assume that there is some $t \in |\psi|^-$ where $s \sqsubseteq \alpha(x)$. Thus there cannot be any $s \in |\psi|^+$ where $s \sqsubseteq \alpha(x)$ since otherwise $s.t \sqsubseteq \alpha(x)$ where $s.t \notin P$ by exclusivity, and so $\alpha(x) \notin P$ by **P5**, thereby contradicting (1).

Case 2: Assume $\varphi = \psi \wedge \chi$. We may then observe the following:

$$\begin{aligned} \mathcal{M}, \alpha, x \models \varphi &\Leftrightarrow \mathcal{M}, \alpha, x \models \psi \text{ and } \mathcal{M}, \alpha, x \models \chi \\ &\Leftrightarrow d \in |\psi|^+ \text{ and } t \in |\chi|^+ \text{ for some } d, t \sqsubseteq \alpha(x) \\ &\Leftrightarrow s \in |\varphi|^+ \text{ for some } s \sqsubseteq \alpha(x). \end{aligned}$$

The equivalences given above follow immediately from the case assumption, induction hypothesis, and semantics for conjunction.

Case 3: Assume $\varphi = \psi \vee \chi$. We may then observe the following:

$$\begin{aligned} \mathcal{M}, \alpha, x \models \varphi &\Leftrightarrow \mathcal{M}, \alpha, x \models \psi \text{ or } \mathcal{M}, \alpha, x \models \chi \\ &\Leftrightarrow d \in |\psi|^+ \text{ for some } d \sqsubseteq \alpha(x) \text{ or } t \in |\chi|^+ \text{ for some } t \sqsubseteq \alpha(x) \\ &\Leftrightarrow s \in |\psi|^+ \cup |\chi|^+ \cup |\psi \wedge \chi|^+ \text{ for some } s \sqsubseteq \alpha(x) \\ &\Leftrightarrow s \in |\varphi|^+ \text{ for some } s \sqsubseteq \alpha(x). \end{aligned}$$

The equivalences given above follow immediately from the case assumption, induction hypothesis, and semantics for disjunction. \square

R1 If $\Gamma \models C$, then $\varphi \Box \rightarrow \Gamma \models \varphi \Box \rightarrow C$.

Proof. Assume $\Gamma \models C$, letting $\mathcal{M}, \alpha, x \models \varphi \Box \rightarrow A_i$ for all $i \in I$ where $\Gamma = \{A_i \mid i \in I\}$. Given any $s \in |\varphi|^+$ and $\beta \in H_{\mathbb{Z}}$ where $s \rightarrow_{\alpha(x)} \beta(x)$, it follows by the semantics for counterfactuals that $\mathcal{M}, \beta, x \models A_i$ for all $i \in I$, and so $\mathcal{M}, \beta, x \models C$. \square

L3 If $t \circ w$, then $w_t = \{w\}$.

Proof. Let $t \circ w$ and $r \sqsubseteq_t w$ where $w \sqsubseteq r$. Thus $w \sqsubseteq_t w$ since $w \sqsubseteq w$ and so $r \sqsubseteq w$. Generalizing on r , $w \in w_t$. Let $u \in w_t$. So $u \sqsubseteq_t w$ where $r \sqsubseteq u$ whenever $r \sqsubseteq_t w$ and $u \sqsubseteq r$. Since $w \sqsubseteq_t w$ and $u \sqsubseteq w$, we know $w \sqsubseteq u$, and so $w = u$. Hence $w_t = \{w\}$. \square

L4 If $s \sqsubseteq \alpha(x)$ for $\alpha \in H_{\mathbb{Z}}$ and $x \in \mathbb{Z}$, then $s \rightarrow_{\alpha(x)} \alpha(x)$.

Proof. Let $t \sqsubseteq \alpha(x)$ where $\alpha \in H_{\mathbb{Z}}$ and $x \in \mathbb{Z}$. Thus $t.\alpha(x) \sqsubseteq \alpha(x)$ where $\alpha(x) \in W$. It follows by **L3** that $\alpha(x)_t = \{\alpha(x)\}$. Since there is some $s \in \alpha(x)_t$ where $s.t \sqsubseteq \alpha(x)$, namely where $s = \alpha(x)$, we may conclude that $t \rightarrow_{\alpha(x)} \alpha(x)$. \square

C2 $\varphi, \varphi \Box \rightarrow A \models A$.

Proof. Let $\mathcal{M}, \alpha, x \models \varphi$ and $\mathcal{M}, \alpha, x \models \varphi \Box \rightarrow A$. By **P11**, $s \in |\varphi|^+$ for some $s \sqsubseteq \alpha(x)$, and so $s \rightarrow_{\alpha(x)} \alpha(x)$ by **L4**. By the semantics for counterfactuals, $\mathcal{M}, \alpha, x \models A$. \square

C3 $\varphi \Box \rightarrow \psi, \varphi \wedge \psi \Box \rightarrow A \models \varphi \Box \rightarrow A$.

Proof. Let $\mathcal{M}, \alpha, x \models \varphi \Box \rightarrow \psi$ and $\mathcal{M}, \alpha, x \models \varphi \wedge \psi \Box \rightarrow A$. Assuming $s \in |\varphi|^+$ and $\beta \in H_{\mathbb{Z}}$ where $s \rightarrow_{\alpha(x)} \beta(x)$, it follows by the semantics for counterfactuals that $\mathcal{M}, \beta, x \models \psi$, and so $t \in |\psi|^+$ for some $t \sqsubseteq \beta(x)$ by **P11**. Since $s \sqsubseteq \beta(x)$ by **P1**, we know that $s.t \sqsubseteq \beta(x)$ where $s.t \in |\varphi \wedge \psi|^+$. Given that $s.t \rightarrow_{\alpha(x)} \beta(x)$ by **P3**, it follows that $\mathcal{M}, \alpha, x \models A$ again by the semantics for counterfactuals, and so $\mathcal{M}, \alpha, x \models \varphi \Box \rightarrow A$. \square

C5 $\varphi \vee \psi \Box \rightarrow A \models \varphi \Box \rightarrow A$.

Proof. Let $\mathcal{M}, \alpha, x \models \varphi \vee \psi \Box \rightarrow A$ where $s \in |\varphi|^+$ and $\beta \in H_{\mathbb{Z}}$ where $s \rightarrow_{\alpha(x)} \beta(x)$. It follows that $s \in |\varphi \vee \psi|^+$, and so $\mathcal{M}, \beta, x \models A$ by the semantics for the counterfactual. Thus we may conclude that $\mathcal{M}, \alpha, x \models \varphi \Box \rightarrow A$. \square

L5 If $w \in W$, then $w \rightarrow_s w$ for any $s \in S$.

Proof. Assume $w \in W$ and let $s \in S$. It follows that (1): $q \sqsubseteq w$ whenever $q \circ w$. Since $w \in P$, there is some $r \in s_w$ by MAXIMALITY. By definition, $r \sqsubseteq s$ where $r \circ w$, and so $r \sqsubseteq w$ by (1). Thus we may conclude that $r.w \sqsubseteq w$, and so $w \rightarrow_s w$. \square

M3 $\models A \rightarrow \Box \Diamond A$.

Proof. Assume for contradiction that $\mathcal{M}, \alpha, x \not\models A \rightarrow \Box \Diamond A$. Thus $\mathcal{M}, \alpha, x \models A$ and $\mathcal{M}, \alpha, x \not\models \Box \Diamond A$, and so $\mathcal{M}, \alpha, x \not\models \top \Box \rightarrow \Diamond A$. It follows that $\mathcal{M}, \beta, x \not\models \Diamond A$ for some $\beta \in H_{\mathbb{Z}}$ and $s \in |\top|^+$ where $s \rightarrow_{\alpha(x)} \beta(x)$. It follows that $\mathcal{M}, \beta, x \models \top \Box \rightarrow \neg A$, and so $\mathcal{M}, \beta, x \not\models \top \Box \rightarrow A$. Since $\alpha(x) \in |\top|^+$, it follows by the semantics for counterfactuals that $\mathcal{M}, \gamma, x \not\models A$ for any $\gamma \in H_{\mathbb{Z}}$ where $\alpha(x) \rightarrow_{\beta(x)} \gamma(x)$. By **L5**, $\alpha(x) \rightarrow_{\beta(x)} \alpha(x)$, and so $\mathcal{M}, \alpha, x \not\models A$ in particular, thereby contradicting the above. \square

M4 $\models \Box A \rightarrow \Box \Box A$.

Proof. Assuming for contradiction that $\mathcal{M}, \alpha, x \not\models \Box A \rightarrow \Box \Box A$. It follows that both $\mathcal{M}, \alpha, x \models \Box A$ and $\mathcal{M}, \alpha, x \not\models \Box \Box A$, or equivalently, both: (1) $\mathcal{M}, \alpha, x \models \top \Box \rightarrow A$; and (2) $\mathcal{M}, \alpha, x \not\models \top \Box \rightarrow (\top \Box \rightarrow A)$.

Given (2), it follows that $\mathcal{M}, \beta, x \not\models \top \Box \rightarrow A$ for some $\beta \in H_{\mathbb{Z}}$ and $s \in |\top|^+$ where $s \rightarrow_{\alpha(x)} \beta(x)$, and so we know that $\mathcal{M}, \gamma, x \not\models A$ for some $\gamma \in H_{\mathbb{Z}}$ and $t \in |\top|^+$ where $t \rightarrow_{\beta(x)} \gamma(x)$. However, it follows from (1) that $\mathcal{M}, \delta, x \models A$ for any $\delta \in H_{\mathbb{Z}}$ and $r \in |\top|^+$ where $r \rightarrow_{\alpha(x)} \delta(x)$. Since $\gamma(x) \in |\top|^+$ where $\gamma(x) \rightarrow_{\alpha(x)} \gamma(x)$ by **L5**, it follows that $\mathcal{M}, \gamma, x \models A$, contradicting the above. \square

M5 $\models \Box(\varphi \rightarrow A) \rightarrow (\varphi \Box \rightarrow A)$.

Proof. Assume for contradiction that $\mathcal{M}, \alpha, x \not\models \Box(\varphi \rightarrow A) \rightarrow (\varphi \Box \rightarrow A)$. It follows that (1) $\mathcal{M}, \alpha, x \models \top \Box \rightarrow (\varphi \rightarrow A)$ and (2) $\mathcal{M}, \alpha, x \not\models \varphi \Box \rightarrow A$, and so $\mathcal{M}, \beta, x \not\models A$ for some $\beta \in H_{\mathbb{Z}}$ and $s \in |\varphi|^+$ where $s \rightarrow_{\alpha(x)} \beta(x)$ by the latter.

Since $s \sqsubseteq \beta(x)$ by **P1**, we know that $\mathcal{M}, \beta, x \models \varphi$ by **P11**, and so $\mathcal{M}, \beta, x \not\models \varphi \rightarrow A$ by the semantics. However, given (1), $\mathcal{M}, \gamma, x \models \varphi \rightarrow A$ for any $\gamma \in H_{\mathbb{Z}}$ and $r \in |\top|^+$ where $r \rightarrow_{\alpha(x)} \gamma(x)$. Since $\beta(x) \in |\top|^+$ where $\beta(x) \rightarrow_{\alpha(x)} \beta(x)$ by **L5**, it follows that $\mathcal{M}, \beta, x \models \varphi \rightarrow A$, thereby contradicting the above. \square

T6 $\models \Box \Box A \leftrightarrow \Box A$.

Proof. Assume $\mathcal{M}, \alpha, x \not\models \Box \Box A$, it follows that $\mathcal{M}, \beta, x \not\models \Box A$ for some $\beta \in H_{\mathbb{Z}}$ and $s \in |\top|^+$ where $s \rightarrow_{\alpha(x)} \beta(x)$, and so $\mathcal{M}, \beta, y \not\models A$ for some $y > x$. Since $\beta(y) \in |\top|^+$ where $\beta(y) \rightarrow_{\alpha(x)} \beta(y)$ by **L5**, it follows that $\mathcal{M}, \alpha, x \not\models \Box A$.

Conversely, assuming $\mathcal{M}, \alpha, x \not\models \Box A$, it follows that $\mathcal{M}, \beta, x \not\models A$ for some $\beta \in H_{\mathbb{Z}}$ and $s \in |A|^+$ where $s \rightarrow_{\alpha(x)} \beta(x)$. Let $\gamma(z) = \beta(z - 1)$ for all $z \in \mathbb{Z}$. It follows that $\gamma \in H_{\mathbb{Z}}$ where $\mathcal{M}, \gamma, x \not\models \Box A$ since $\mathcal{M}, \gamma, x + 1 \not\models A$ and $x + 1 > x$. Since $\gamma(x) \in |\top|^+$ where $\gamma(x) \rightarrow_{\alpha(x)} \gamma(x)$ by **L5**, it follows that $\mathcal{M}, \alpha, x \not\models \Box \Box A$.

It follows that $\mathcal{M}, \alpha, x \not\models \Box \Box A$ just in case $\mathcal{M}, \alpha, x \not\models \Box A$, and so equivalently $\mathcal{M}, \alpha, x \models \Box \Box A$ just in case $\mathcal{M}, \alpha, x \models \Box A$. Thus $\models \Box \Box A \leftrightarrow \Box A$. \square

References

- [1] Fine, K.: Counterfactuals Without Possible Worlds. *Journal of Philosophy* **109**(3), 221–246 (2012) <https://doi.org/10.5840/jphil201210938>
- [2] Fine, K.: A difficulty for the possible worlds analysis of counterfactuals. *Synthese* **189**(1), 29–57 (2012) <https://doi.org/10.1007/s11229-012-0094-y>
- [3] Fine, K.: A Theory of Truthmaker Content I: Conjunction, Disjunction and Negation. *Journal of Philosophical Logic* **46**(6), 625–674 (2017) <https://doi.org/10.1007/s10992-016-9413-y>
- [4] Fine, K.: A Theory of Truthmaker Content II: Subject-Matter, Common Content, Remainder and Ground. *Journal of Philosophical Logic* **46**(6), 675–702 (2017) <https://doi.org/10.1007/s10992-016-9419-5>
- [5] Fine, K.: Truthmaker Semantics. In: *A Companion to the Philosophy Of Language*, 2nd edition edn., pp. 556–577. Wiley-Blackwell, Malden, MA (2017). <https://doi.org/10.1002/9781118972090.ch22>
- [6] Kripke, S.A.: Semantical Analysis of Modal Logic I Normal Modal Propositional Calculi. *Mathematical Logic Quarterly* **9**(5-6), 67–96 (1963) <https://doi.org/10.1002/malq.19630090502>
- [7] Fine, K.: Critical Notice. *Mind* **84**(335), 451–458 (1975) [2253565](https://doi.org/10.1093/mind/84.335.451)
- [8] Lewis, D.: *Counterfactuals*. Harvard University Press, Harvard (1973)
- [9] Stalnaker, R.C.: A Theory of Conditionals. In: Rescher, N. (ed.) *Studies in Logical Theory*. American Philosophical Quarterly Monographs, vol. 2, pp. 98–112. Blackwell, Oxford (1968)
- [10] Lewis, D.: Counterfactual Dependence and Time’s Arrow. *Noûs* **13**(4), 455–476 (1979) <https://doi.org/10.2307/2215339>
- [11] Briggs, R.: Interventionist counterfactuals. *Philosophical Studies* **160**(1), 139–166 (2012) <https://doi.org/10.1007/s11098-012-9908-5>
- [12] Khoo, J.: Backtracking Counterfactuals Revisited. *Mind*, 005 (2016) <https://doi.org/10.1093/mind/fzw005>
- [13] Nute, D.: Counterfactuals and the Similarity of Words. *The Journal of Philosophy* **72**(21), 773–778 (1975) <https://doi.org/10.2307/2025340>
- [14] Loewer, B.: Counterfactuals with Disjunctive Antecedents. *The Journal of Philosophy* **73**(16), 531–537 (1976) <https://doi.org/10.2307/2025717>

- [15] McKay, T., Van Inwagen, P.: Counterfactuals with disjunctive antecedents. *Philosophical Studies* **31**(5), 353–356 (1977) <https://doi.org/10.1007/BF01873862>
- [16] Alonso-Ovalle, L.: Counterfactuals, correlatives, and disjunction. *Linguistics and Philosophy* **32**(2), 207–244 (2009) <https://doi.org/10.1007/s10988-009-9059-0>
- [17] Embry, B.: Counterfactuals Without Possible Worlds? A Difficulty for Fine’s Exact Semantics for Counterfactuals. *Journal of Philosophy* (5), 276–287 (2014) <https://doi.org/10.5840/jphil2014111522>
- [18] Ciardelli, I., Groenendijk, J., Roelofsen, F.: *Inquisitive Semantics*. Oxford Surveys in Semantics and Pragmatics. Oxford University Press, Oxford (2018). <https://doi.org/10.1093/oso/9780198814788.001.0001>
- [19] Lassiter, D.: Complex sentential operators refute unrestricted Simplification of Disjunctive Antecedents. *Semantics and Pragmatics* **11**, 9–19 (2018) <https://doi.org/10.3765/sp.11.9>
- [20] Khoo, J.: Disjunctive antecedent conditionals. *Synthese* **198**(8), 7401–7430 (2021) <https://doi.org/10.1007/s11229-018-1877-6>
- [21] Goodman, N.: The Problem of Counterfactual Conditionals. *The Journal of Philosophy* **44**(5), 113–128 (1947) <https://doi.org/10.2307/2019988>
- [22] Warmbröd, K.: An Indexical Theory of Conditionals. *Dialogue: Canadian Philosophical Review / Revue canadienne de philosophie* **20**(4), 644–664 (1981) <https://doi.org/10.1017/S0012217300021399>
- [23] Gillies, A.S.: Counterfactual scorekeeping. *Linguistics and Philosophy* **30**(3), 329–360 (2007) <https://doi.org/10.1007/s10988-007-9018-6>
- [24] Gillies, A.S.: On Truth-Conditions for If (but Not Quite Only If). *The Philosophical Review* **118**(3), 325–349 (2009) <https://doi.org/10.1215/00318108-2009-002>
- [25] Fine, K.: Vagueness, truth and logic. *Synthese* **30**(3-4), 265–300 (1975) <https://doi.org/10.1007/BF00485047>
- [26] Ellis, B., Jackson, F., Pargetter, R.: An objection to possible-world semantics for counterfactual logics. *Journal of Philosophical Logic* **6**(1), 355–357 (1977) <https://doi.org/10.1007/BF00262069>
- [27] Brast-McKie, B.: Identity and Aboutness. *Journal of Philosophical Logic* **50**(6), 1471–1503 (2021) <https://doi.org/10.1007/s10992-021-09612-w>
- [28] Fine, K.: Constructing the Impossible. In: Walters, L., Hawthorne, J. (eds.) *Conditionals, Paradox, and Probability: Themes from the Philosophy of Dorothy Edgington*, pp. 141–163. Oxford University Press, Oxford (2021). <https://doi.org/10.1093/oso/9780198712732.003.0009>

- [29] Fine, K.: Yablo on subject-matter. *Philosophical Studies* **177**(1), 129–171 (2020) <https://doi.org/10.1007/s11098-018-1183-7>
- [30] Yablo, S.: *Aboutness*. Princeton University Press, Berlin, Boston (2014)
- [31] Ginsberg, M.L.: Multivalued Logics: A Uniform Approach to Inference in Artificial Intelligence. *Computational Intelligence* **4**, 265–316 (1988)
- [32] Fitting, M.: Bilattices in logic programming. In: , *Proceedings of the Twentieth International Symposium on Multiple-Valued Logic*, 1990, pp. 238–246 (1990). <https://doi.org/10.1109/ISMVL.1990.122627>
- [33] Yli-Vakkuri, J., Hawthorne, J.: The Necessity of Mathematics. *Noûs* **54**(3), 549–577 (2020) <https://doi.org/10.1111/nous.12268>
- [34] Elgin, S.Z.: Counterfactual Logic and the Necessity of Mathematics. *Journal of Philosophical Logic* **50**(1), 97–115 (2021) <https://doi.org/10.1007/s10992-020-09563-8>
- [35] Jackson, F.: A causal theory of counterfactuals. *Australasian Journal of Philosophy* **55**(1), 3–21 (1977) <https://doi.org/10.1080/00048407712341001>
- [36] Garay, L.J.: Quantum gravity and minimum length. *International Journal of Modern Physics A* **10**(02), 145–165 (1995) <https://doi.org/10.1142/S0217751X95000085> [arXiv:gr-qc/9403008](https://arxiv.org/abs/gr-qc/9403008)