Counterfactual Worlds

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July 18, 2024

Abstract

This paper extends Kit Fine's [6, 7, 8] truthmaker framework to provide a novel semantics for tensed counterfactual conditionals. Rather than taking possible worlds to be primitive elements in a model, possible worlds will be defined in terms of *states*, *parthood*, *times*, and *tasks* where the latter encodes the possible transitions between states. Instead of invoking a primitive similarity or imposition relation, possible worlds will be compared at a time independent of that time's past and future where the comparison will be carried out in purely mereological and modal terms. After reviewing the motivations for this approach, I provide the hyperintensional semantics that I implemented in the `anonymous` software for counterfactual conditionals. I will then extend the language to include tense operators in order to analyze forwards, backwards, and backtracking counterfactual conditionals claims.

Keywords: Counterfactual Conditionals, Truthmaker Semantics, Hyperintensionality

1 Introduction

Drawing on Kripke's [15] semantics for the metaphysical modal operators, intensional semantics theories of counterfactual conditionals model *possible worlds* as structureless points. Although taking possible worlds to be primitive is a harmless idealization when interpreting modal claims, the same cannot be said for counterfactual claims of the form ⌜If it were the case that A, then it would be the case that B⌝, or in symbols, ⌜$A \boxright B$⌝. Rather, I will define possible worlds as histories of moments where each moment is rich in mereological structure. In order to motivate this approach, this section reviews Fine's [2] early criticisms of the *similarity theory* that Lewis [16] developed as well the *imposition theory* that Fine [4, 5] went on to provide. Despite their differences, these views are not equipped to interpret tensed counterfactuals and may be criticized for including primitives in the metalanguage that are at least as hard to understand as the object language that they aim to interpret.

Instead of positing primitive similarity or imposition relations, §2 defines Fine's imposition relation in purely modal and mereological terms. In order to also interpret tense operators, I will introduce a primitive *task relation* which encodes the possible transitions between *states*. Given this additional resource, §3 defines the *possible states* that Fine takes to be primitive as well as the temporally extended *world histories* needed to interpret tensed counterfactuals. By drawing on these resources, I will provide a unified semantics for tensed counterfactuals in §4, employing this framework to analyze forwards, backwards, and backtracking counterfactuals in §5. The paper will conclude by considering objections and extensions in §6 and presenting minor results in §7. The following subsections will motivate these developments by presenting the shortcomings that similarity and imposition theories of counterfactual conditionals face.

1.1 Totality

In addition to a primitive set of *possible worlds* W and *accessibility relation* R over W, Stalnaker [19] and Lewis [16] equip models with a *selection function* or *comparative similarity relation*, respectively. Abstracting from their differences, similarity theories take $A \boxright B$ to be true in a world w just in case B is true in the most similar accessible world(s) to w in which A is true.[1] Independent of how similarity is to be understood, modeling possible worlds as structureless points imposes the following limitation:

Totality: Similarity selects/relates possible worlds considered in their entirety.

Although in general the worlds in W do not include any internal structure, this does not stop similarity theorists from assuming otherwise while working over an intended model which includes the possible worlds themselves along with all of their spatiotemporal structure and other relevant features. However, the various features that one might take possible worlds to have in an intended model are not included in the definition of a model. For the purposes of logic, the models of a given object language only have the structure that they are given, and standard intensional models do not include any internal structure in the possible worlds that are used to interpret the language. Focusing on Lewis' account, W is any nonempty set where the similarity relation stipulates an ordering over the worlds in W by treating each world as a structureless point.

Insofar as possible worlds are taken to model complete histories of everything, *Totality* leads similarity theories to predict the wrong results. For instance, consider the present tense analogue of the counterexample Fine [2, p. 425] raised against Lewis' account:

`(N)` If Nixon were to press the button there would be a nuclear holocaust.

Imagine Nixon hovering over the detonator where one of his advisers asserts `N` to another. As Fine points out, Lewis' theory predicts that `N` is false since for any world in which Nixon pressed the button and a nuclear holocaust followed, there will be a much more similar world which includes, "a change that prevents the holocaust but that does not require such a great divergence from reality." For instance, perhaps the button is disconnected. Even so, `N` is true since the results of Nixon pressing the button are irrelevant to the moment we are considering in which Nixon did not press the button but could have. Rather than measuring the similarity of the actual world in its entirety against all of what would result if Nixon were to press the button, it is natural to consider the result of making minimal changes to the moment in which Nixon did not press the button to one in which he did. Given any such minimally changed moment, `N` makes a claim about the disaster to follow even if those futures end up diverging widely from the actual course of events. So long as the moments that result from changing the actual moment in which Nixon did not press the button to one in which he did do not include any other changes (e.g., a disconnected button), we may expect `N` to be true.

However natural it is to restrict consideration to the moment in which the antecedent is to be evaluated while ignoring later moments, standard intensional models do not provide the resources to do so.[2] If possible worlds are modeled by points without any

[1] Whereas Stalnaker [19] takes there to be a closest A-world, Lewis [16] permits there to be equally close A-worlds as well as unending sequences of ever closer A-worlds. For simplicity, I will often leave the world of evaluation as well as the restriction to accessible worlds implicit in what follows, referring simply to the most similar world(s) in which the antecedent is true.

[2] Although intervals could be considered, I will restrict attention to a single moment of comparison for the majority of what follows, postponing consideration of possible extensions of the semantics to §6.

temporal structure, a similarity relation can only order possible worlds considered in their entirety as stated by *Totality*. So long as possible worlds are taken to model temporally extended histories, *Totality* makes it easy to imagine possible worlds in which Nixon pushed the button that are much more similar to the actual world than to possible worlds in which he pushed the button and a nuclear holocaust took place.

Lewis [17] responded to Fine's challenge by dispensing with our pretheoretic similarity judgments, opting instead for weighted rules that constrain how the similarity of worlds is to be measured. Instead of precise rules which are easy to apply, Lewis' weighting system consists of open-ended suggestions for how to adjust our untutored intuitions about similarity while working over an intended model that includes the possible worlds themselves. In particular, Lewis sought to maximize the spatiotemporal regions of the counterfactual worlds which accord with actuality while reducing the extent to which those counterfactual worlds violate the actual laws of nature. Whether one adopts Lewis' rules or not, it is natural to worry that the metalanguage that Lewis used to articulate a semantics for counterfactuals is considerably harder to grasp than the object language that he aims to interpret.[3] Insofar as a semantics theory is to assign meaningful truth-conditions to sentences, we should hope to develop that semantics in a metalanguage that is understood at least as well as the object language under study. By contrast, admitting vague or context sensitive terms into the metalanguage undermines the ambition to provide a predicatively powerful semantics for the language.

Setting these issues aside, the assumption that worlds are structureless points despite modeling temporally extended histories raises a much more fundamental problem. To bring this out, suppose that Nixon had pushed the button before it was activated. Although it is natural to take `N` to be true if asserted after the button was activated, Lewis' semantics predicts the opposite result. After all, the actual world is a world in which Nixon pushed the button, making the actual world closer than any other world. Since the actual world does not include a nuclear holocaust, `N` is false. What is missing from the semantics is the time at which the antecedent is to be evaluated.

In addition to predicting the wrong results for counterfactuals in the present tense, evaluating sentences at possible worlds on their own makes the semantics incapable of interpreting tense operators. For instance, consider Fine's [2] original example in which a tense operator takes wide scope over the counterfactual claim `N` from before:

(N′) If Nixon had pressed the button there would have been a nuclear holocaust.

As above, it matters whether N′ is evaluated before or after the button has been activated. Since counterfactuals often occur together with tense operators, it is natural to develop a common framework in which to model their interaction. In order to both interpret tense operators and provide a semantics for counterfactuals that compares worlds for similarity at specific times, the following subsection will present a similarity theory which rejects *Totality*. Instead of taking possible worlds to be primitive, I will define possible worlds to be histories of *moments*, evaluating sentences at world-time pairs. Despite answering the objections raised above, the resulting semantic theory faces the same problems that Fine [2, 4, 5] raised against Lewis' original theories of counterfactuals. These considerations will motivate a departure not only from positing a primitive similarity relation, but from intensional semantic theories of counterfactual conditionals altogether. Nevertheless, it is by first defining worlds as histories of moments that we may later come to appreciate the mereological structure of each moment.

[3] Something similar may be said for Stalnaker's [19] selection function which is assumed to specify the most similar antecedent-world while ignoring irrelevant details.

1.2 Restriction

Given a primitive set of *possible moments* M, *times* T, and an *accessibility relation* R over M, a similarity theorist may avoid the problems brought out above by taking *similarity* to order moments instead of worlds. Whereas possible worlds model temporally extended histories of everything, possible moments are the time slices of those histories where each moment determines the truth-value of every sentence letter of the language. Rather than taking accessibility to encode the relative possibility of one complete possible history from another, a moment m is *accessible* to k just in case it is possible for m to transition to k. By taking the set of times T to be the integers for simplicity, a *possible world* may be defined as any function from times to moments where every moment is accessible from its predecessor.[4] Counterfactuals may then be interpreted at world-time pairs so that $A \boxright B$ is true in a world w at a time x just in case B is true in u at x whenever $u(x)$ is one of the most similar moments to $w(x)$ and A is true in u at x.[5] Letting '$\Diamond\!\!\!\!\downarrow A$' read 'It was the case that A', we may take $\Diamond\!\!\!\!\downarrow A$ to be true in a world w and time x just in case there is a time $y < x$ where A is true in w at y. Something similar may be said for '$\Diamond\!\!\!\!\uparrow A$' which reads 'It is going to be the case that A'. Eliminating temporal ambiguity from the framework in this way avoids the counterexamples brought out above. In particular, $\mathtt{N}$ is true in a world w and time x just in case there is a time $y > x$ in which a nuclear holocaust occurs in u at y whenever $u(x)$ is one of the most similar moments to $w(x)$ in which Nixon pushes the button. Moreover, $\mathtt{N}'$ is true in a world w and time x just in case there is a time $z < x$ where $\mathtt{N}$ is true in w at z.[6]

Whereas comparing possible worlds for similarity makes $\mathtt{N}$ and $\mathtt{N}'$ false, the same cannot be said for comparing moments for similarity. With respect to the actual moment in which Nixon did not press the button but could have, the most similar moments in which Nixon did push the button do not include any broken wires or blown fuses, and so are sure to lead to a nuclear holocaust. Thus $\mathtt{N}$ is predicted to come out true, where something similar may be said for $\mathtt{N}'$. Even so, this semantics invalidates *Simplification of Disjunctive Antecedents* for familiar reasons. For simplicity, we may restrict consideration to tenseless counterfactuals, replacing world-time pairs with moments for the time being. Accordingly, $A \boxright B$ is true in a moment m just in case B is true in the most similar moment(s) to m in which A is true. As a result, the following principle is invalid:

SDA $A \vee B \boxright C \ \vdash (A \boxright C) \wedge (B \boxright C)$.

Assuming the consequent is true in the most similar moments in which the antecedent is true does not require the consequent to be true in the most similar moments in which a disjunct of the antecedent is true.[7] For instance, suppose that $A \vee B \boxright C$ is true in m. Since the most similar $A \vee B$-moments to m may all be $A \wedge C$-moments without also all being B-moments, nothing requires $B \boxright C$ to be true in m. Adding color, suppose that although Philip is often the life of the party, Peter avoids parties since he always gets in fights. Assuming neither Philip nor Peter are at the party, the most similar moments in which either of them is at the party are moments in which only Philip is at the party. It follows that if either Philip or Peter were at the party, it would have been better, though it is false to claim that if Peter were at the party, it would have been better.

[4] I develop this framework in much greater detail elsewhere in order to provide a semantics for a bimodal language which includes both tense and modal operators. I develop the hyperintensional analogue in §3.

[5] Since it does not matter for present purposes whether we employ a selection function or similarity ordering over moments, I will leave 'most similar moments' undefined to accommodate both views.

[6] Such a theorist may regiment $\mathtt{N}$ as $B \boxright \Diamond\!\!\!\!\uparrow H$ and $\mathtt{N}'$ as $\Diamond\!\!\!\!\downarrow(B \boxright \Diamond\!\!\!\!\uparrow H)$. See §5 for further discussion.

[7] Fine [2, 4] makes this observation for Lewis' account.

The invalidity of **SDA** does not turn on anything peculiar to moments or similarity. For the sake of generality, let *points* be generic elements at which sentences are evaluated. We may then observe that any theory of counterfactual conditionals that conforms to the following principle will invalidate **SDA** for the same reason given above:

> *Restriction:* The consequent is only required to be true in a restricted subset of the points at which the antecedent is true (e.g., the most similar moments).

If **SDA** is to be maintained, *Restriction* must be given up. Taking this lesson to heart, *strict theories* of counterfactuals require the consequent to be true at every point at which the antecedent is true. Replacing points with moments for consistency with the above, $A \boxright B$ is true in the moment m just in case B is true in every moment accessible from m in which A is true.[8] It follows that **SDA** is valid for if C is true in every accessible moment in which $A \vee B$ is true, then C is also true in every accessible moment in which A (similarly B) is true. Even so, strict theories validate *Strengthening the Antecedent*:

STA $A \boxright C \ \vdash A \wedge B \boxright C$.

If C is true in every accessible moment in which A is true, C is also true in every accessible moment in which $A \wedge B$, thereby validating **STA** on a strict theory of counterfactuals. However, there are natural counterexamples to **STA**. Just because the match would light if it were struck, it does not follow that the match would light if it were struck and wet. In response, defenders of **STA** provide pragmatic explanations for why the counterexamples are only apparent by appealing to contextual shifts in R between the evaluation of the premise(s) and conclusion.[9] Nevertheless, strict theories admit that in any particular context, counterfactual conditionals have the same logic as the strict conditional, and so reject attempts to identify a distinct logic for counterfactual conditionals.

Rather than looking to pragmatics in order to provide an error theory for the apparent invalidity of **STA**, I will take the counterexamples to **STA** to show that the logic for the counterfactual conditional cannot be identified with the logic for the strict conditional. Additionally, I will maintain the validity of **SDA** in order to account for the apparent validity of simplifying disjunctive antecedents in counterfactual conditionals.[10] Although there would seem to be no space between invalidating **SDA** by accepting *Restriction* and validating **STA** by rejecting *Restriction*, this choice is not forced. Since **STA** follows from **SDA** in an intensional logic for a transparent language, the following section will motivate the hyperintensional alternative that Fine [4, 5] develops in order to validate **SDA** without also validating **STA**. Despite these advantages, Fine's semantics is not equipped to interpret tensed counterfactuals and posits a primitive imposition relation which has a counterfactual reading in the metalanguage. In order to address the latter concern, the following section will conclude by defining imposition in purely modal and mereological terms. By deriving the constraints that Fine requires the imposition relation to satisfy, the resulting semantic theory validates a logic for counterfactuals that is at least as strong as the logic that Fine defends while positing fewer assumptions. In addition to simplifying Fine's semantics conceptually, defining imposition streamlines the implementation of the semantics in the `anonymous` software that I developed for finding hyperintensional countermodels for counterfactual conditionals. Given these advantages, §3 will extend the framework to accommodate tense operators.

8 In order to accommodate tense, a strict theorist may take $A \boxright B$ to be true in a world w at time x just in case B is true in w' at x whenever $w'(x)$ is accessible from $w(x)$ and A is true in w' at x.

9 Warmbrōd [20] and Gillies [11, 12] provide strict conditional views in intensional frameworks.

10 See Fine [4] for related discussion.

2 Imposition Semantics

Intensional semantic theories model propositions as subsets of a set of points. Letting '$\Box$' express *metaphysical necessity*, '$\equiv$' express *propositional identity*, and $\ulcorner C_{[B/A]}\urcorner$ be the result of uniformly substituting B for A in C, the standard intensional semantic clauses validate **INT** where **LL** is unassailable in a language which excludes opaque operators:

INT $\Box(A \leftrightarrow B) \vdash A \equiv B$.[11] **LL** $A \equiv B,\ C \vdash C_{[B/A]}$.

Given that $\Box[A \leftrightarrow A \vee (A \wedge B)]$ is a theorem in any normal modal logic, we may derive $A \vee (A \wedge B) \Box\!\!\rightarrow C$ from the premise $A \Box\!\!\rightarrow C$ by **INT** and **LL**, and so $A \wedge B \Box\!\!\rightarrow C$ follows by **SDA**.[12] Thus **STA** is derivable from **SDA**. In order to avoid this inference, **INT** must be given up since **LL** is unassailable in a language that does not include opaque operators. In order to validate **SDA** without also validating **STA**, Fine [4, 5] provides a hyperintensional semantics which distinguishes necessarily equivalent propositions. After presenting the details of this account in the following subsection, §2.2 will define the imposition relation that Fine takes to be primitive in order to avoid positing counterfactual primitives, and §3.3 will present a general theory of hyperintensional propositions.

2.1 Outcomes

Instead of maintaining *Restriction*, Fine [4] introduces a primitive *imposition relation* where $t \rightarrow_w u$ indicates that, "u is a possible outcome of imposing the change t on the world [state] w" (p. 237).[13] Whereas w and u are *world states* where each, "either contains or is incompatible with any other state" (p. 236), t is a *possible state* which Fine glosses as a "fragment" of a world state.[14] Accordingly, Fine takes *parthood* $\sqsubseteq$ to partially order the space of all states, referring to the least upper bound of a set of states as a *fusion state*.[15] In general, states only settle the truth-values of those sentences that they *exactly verify* or *exactly falsify*, "tell[ing] us what it is *in* the world that makes the statement true if it is true or what it is *in* the world that makes it false if it is false" (p. 235). Building on Fine's [6, 7] more recent work, I will have more to say about states and world states in the following subsection. For the time being, it will be enough to consider the broad role that states play in Fine's semantics.

Given the addition of states together with the imposition relation, Fine takes $A \Box\!\!\rightarrow C$ to be true in a world state w just in case C is true in any world state u which is the outcome of imposing any exact verifier state t for A on w. Fine is clear that his *imposition theory* for counterfactuals is based on the following ideas:

> *Universal Realizability of the Antecedent:* There is no restriction on the exact verifiers for the antecedent which are to be imposed on the evaluation world state.
>
> *Universal Verifiability of the Consequent:* There is no restriction on the outcomes of imposing an exact verifier for the antecedent on the evaluation world state.

[11] *Intensionalists* typically assume $A \equiv B \coloneqq \Box(A \leftrightarrow B)$ instead of taking '$\equiv$' to be primitive, or else do not introduce '$\equiv$' at all. See Rayo [18] for a clear statement of intensionalism.

[12] See Fine [3, 4, 5] for the original argument.

[13] Whereas Fine [4] refers to world states simply as *worlds*, I will take worlds to be temporally extended histories for consistency throughout what follows.

[14] As brought out below, impossible states are not parts of world states, but states nonetheless.

[15] Whereas Fine [4] assumes that a set of states has a least upper bound if it has an upper bound, the next section will follow Fine [7] in taking the set of all states to form a complete lattice.

In this way, Fine's semantics rejects *Restriction* without identifying the exact verifier states for the antecedent with the outcome world states that they produce when imposed on the evaluation world state. An imposition theory of this kind is to be contrasted with strict theories of counterfactuals which require the consequent to be true at all accessible antecedent points themselves. Even so, the principles above preserve the unrestricted spirit of strict theories of counterfactuals while providing a means by which to validate **SDA** without also validating **STA**. Since the exact verifiers for $A \vee B$ include the exact verifiers for A as a subset, it follows that if C is true in every outcome that results from imposing an exact verifier for $A \vee B$ on a world state w, then C is also true in every outcome that results from imposing an exact verifier for A (similarly B) on w, and so **SDA** is valid. By contrast, the outcomes that result from imposing an exact verifier for A on w may differ considerably from the outcomes that result from imposing an exact verifier for $A \wedge B$ on w, and so even if C is true in all of the former outcomes, C may fail to be true in all of the latter outcomes, thereby invalidating **STA**.

Although Fine's imposition semantics validates **SDA** without also validating **STA**, a number of important questions remain to be answered. Postponing discussion of the precise relationship between states and world states to the following subsection, we may consider an objection that Fine addresses in defense of his account:

> It has been argued, in the second place, that the present semantics is relatively problematic in its conceptual commitments. For the transition relation must itself be understood in terms of counterfactuals. Thus to say that u is a possible outcome of t in w is just to say that, in w, u might obtain if t were to obtain (and also that u is maximal in this respect). (p. 241)

Fine responds by pointing out that Lewis [17] is guilty of something similar since counterfactual assumptions will be difficult to avoid in providing a "suitably doctored" notion of similarity. Instead of requiring a semantics to, "provide an analysis of the locutions with which it deals," Fine settles for the position that his imposition semantics provides, "a perspicuous account of how the truth-conditions of the sentences containing the locutions are to be determined." Although analysis is by no means the only standard for productive semantic theorizing, it is important to establish a solid understanding of the primitive terms used to articulate a semantics in order to provide an unambiguous specification of the truth-conditions for the locutions in question.

In order to clarify the interpretation of the imposition relation as well as validate a number of important counterfactual inferences, Fine [4] provides four constraints on the interpretation of the imposition relation where $t.v$ is the fusion of the states t and v:

Inclusion: If $t \rightarrowtriangle_w u$, then $t \sqsubseteq u$.

Actuality: If $t \sqsubseteq w$, then $t \rightarrowtriangle_w u$ for some $u \sqsubseteq w$.

Incorporation: If $t \rightarrowtriangle_w u$ and $v \sqsubseteq u$, then $t.v \rightarrowtriangle_w u$.

Completeness: If $t \rightarrowtriangle_w u$, then u is a world-state.

Although Fine [4] offers a plausible defense for the principles above, it is unnecessary to take imposition to be primitive. Instead of axiomatizing imposition, the following subsection defines imposition in purely modal and mereological terms. By deriving the constraints that Fine assumes above, I will avoid positing a counterfactual primitive in the metalanguage at no cost to the strength of the resulting logic. Moreover, simplifying the semantics in this way reduces the computational complexity of its implementation in the `anonymous`, expanding the range of inferences that the software can evaluate.

2.2 Defining Imposition

I will follow Fine [6, 7, 8] in taking a *state space* $\mathcal{S} = \langle S, \sqsubseteq \rangle$ to be any complete lattice where S is the set of *states* and $\sqsubseteq$ is the *parthood relation*. For instance, the state of Sanna sitting s includes a state of Sanna's legs being bent t as a proper part, where t is a *proper part* of s— i.e., $t \sqsubset s$— just in case $t \sqsubseteq s$ and $s \not\sqsubseteq t$. Since the least upper bound of any set of states $X \subseteq S$ is unique, we may refer to the least upper bound of the set of states X as the *fusion* $\bigsqcup X$ of X. In particular, $\bigsqcup \varnothing \coloneqq \square$ is the *null state* and $\bigsqcup S \coloneqq \blacksquare$ is the *full state*. When $X = \{s, t, \ldots\}$ is finite, it will be convenient to represent the fusion of the states in X as $s.t.\cdots \coloneqq \bigsqcup\{s, t, \ldots\}$. For instance, the fusion of a state s of Sanna sitting and a state k of Kevin cooking is the state $s.k$ of Sanna sitting and Kevin cooking. In general, a fusion will obtain just in case all of its parts obtain.

In order to encode modal structure, Fine goes on to take a *modalized state space* to be any ordered triple $\mathcal{S}^\Diamond = \langle S, P, \sqsubseteq \rangle$ where $\langle S, \sqsubseteq \rangle$ is a state space and $P \subseteq S$ is a primitive subset of *possible states* which satisfies the following constraints:

Nonempty: $P \neq \varnothing$.

Possibility: If $s \in P$ and $t \sqsubseteq s$, then $t \in P$.

Whereas every state in P can obtain, the states in S/P are *impossible* and so cannot obtain.[16] Given that every part of a possible state is possible, the null state $\square$ is possible. Fine takes the states s and t to be *compatible* just in case their fusion is possible, i.e., $s \circ t \coloneqq s.t \in P$. For instance, although any state of Sanna sitting is compatible with any state of Kevin cooking, the same cannot be said for the states of Sanna standing which are incompatible with the states of her sitting. Fine defines the *world states* as follows:

World States: $W \coloneqq \{w \in P : \forall s \circ w(s \sqsubseteq w)\}$.

World states are maximal possible states which include all compatible states as parts. Fine restricts attention to *world spaces* which are modalized state spaces that satisfy:

World Space: If $s \in P$, then $s \sqsubseteq w$ for some $w \in W$.

This principle makes possible states the parts of world states, thereby ruling out infinite sequences of ever bigger possible states which do not belong to a world state.

Given any world space, it is straightforward to define the imposition relation in terms of the primitive states S, possible states P, and parthood relation $\sqsubseteq$ that it includes. Letting '$s \sqsubseteq_t w$' read 's is a *t-compatible part* of w', I will assume the following definitions:

Compatible Part: $s \sqsubseteq_t w \coloneqq s \sqsubseteq w \wedge s \circ t$.

Maximal Compatible Parts: $w_t \coloneqq \{s \sqsubseteq_t w : \forall r \sqsubseteq_t w(s \sqsubseteq r \rightarrow r \sqsubseteq s)\}$.

Imposition: $t \rightarrowtriangle_w u \coloneqq u \in W \wedge \exists s \in w_t(s.t \sqsubseteq u)$.

Whereas a t-compatible part of w is a part of w that is compatible with t, a maximal t-compatible part of w is any t-compatible part of w that is not a proper part of any t-compatible part of w. Accordingly, an outcome u of imposing t on w is any world state which includes as parts both t as well as a maximal t-compatible part of w. Intuitively, the outcome world states are the results of minimally changing a given world state to include the imposed state, where outcomes need not be unique.

[16] Without admitting impossible states, the exact verifiers for a sentence could not be closed under fusion, and so $A \equiv A \wedge A$ may fail to hold.

Whereas Fine's primitive imposition relation has a counterfactual reading, a similar complaint cannot be raised against the primitives employed in the definitions above. Rather, the states S, possible states P, and parthood relation $\sqsubseteq$ have purely modal and mereological readings. Given the conceptual resources of a world space, the definitions above provide an analysis of the imposition relation, reducing the number of primitives included in the semantics. Since **P1 - P4** in §7 derive the constraints on imposition, we may validate a logic for counterfactual conditionals that is at least as strong as the logic that Fine defends. In addition to excluding counterfactual primitives from the metalanguage, simplifying the semantics in the way improves the efficiency of its implementation in the `anonymous` software that I developed for finding hyperintensional countermodels for counterfactual reasoning. In order to provide evidence that an inference is valid, we may use the `anonymous` to show that the inference does not have countermodels below a certain finite level of complexity, where the strength of that evidence is proportional to the number of models surveyed. Although the `anonymous` draws on Microsoft's state of the art SMT theorem prover Z3, every computational system has its limits. By simplifying the semantics implemented in Z3, the `anonymous` is able to evaluate inferences of greater complexity as well as survey countermodels with a greater number of atomic elements. As a result, simplifying the semantics strengthens the evidence that the `anonymous` can provide for a wider range of inferences.

Despite overcoming the problems that intensional similarity theories face without positing a counterfactual primitive in the metalanguage, the resulting semantics does not accommodate tense operators which often combine with counterfactual conditionals. In order provide a unified semantic framework for studying tensed counterfactuals, the following section will replace Fine's primitive set of possible states with a set of *times* and a two-place *task relation* which encodes the possible transitions between states. In addition to providing temporal structure, these resources provide the modal structure needed to define the possible states P that Fine takes to be primitive, deriving Nonempty, Possibility, and World Spaces. After defining both the instantaneous world states as well as the temporally extended possible worlds in these terms, the section will close by presenting a hyperintensional theory of propositions by which to interpret a language with extensional, modal, counterfactual conditional, and tense operators.

3 The Construction of Possible Worlds

Although states may be interpreted in any number of different ways, it will help to fix ideas by taking states to be *some things being a specific way*. Despite taking states to be primitive, we may draw conceptual connections between states and possible worlds by identifying the *possible states* with restrictions of possible worlds. Whereas possible worlds are complete histories of everything, states are *static* and typically *partial*, concerning some limited way for certain things to be at an instant. For instance, given any possible world in which Sanna is sitting at time x, restricting to the part of that world which occurs at time x and makes it true that Sanna is sitting is a state of Sanna sitting. In general, states are *wholly relevant* to the sentences that they exactly verify or falsify and, for this reason, much more discriminating than possible moments or worlds. Accordingly, most states will neither exactly verify nor falsify any given sentence.

States are also many. Just as the world-time pairs in which Sanna is sitting may be taken to cover all of the different ways for it to be true that Sanna is sitting, the states of Sanna sitting are the parts of the world time-slices which make it true that Sanna is sitting in all of those different ways. Moreover, states are *specific*: just as there is exactly one way for a possible world to be actual, there is exactly one way for a state to *obtain*. Put otherwise, neither possible worlds nor states are multiply realizable but rather model specific realizations. It is nevertheless important not to conflate the specificity of states with their possibility. Given a state s of Sanna sitting and a state t of her standing, the fusion state $s.t$ is just as specific as s and t considered on their own, amounting to a precise way for Sanna's body to be arranged despite it being impossible for $s.t$ to obtain. For instance, $s.t$ includes Sanna's legs being both bent as they are in her sitting state s and straight as they are in her standing state t. Insofar as t makes it false that Sanna is sitting, t makes it true that Sanna is not sitting, and so $s.t$ makes the conjunction true that Sanna is sitting and not sitting. Just as s obtaining is sufficient for it to be true that Sanna is sitting independent of whether s can obtain, $s.t$ obtaining is sufficient for it to be true that Sanna is sitting and not sitting despite the fact that $s.t$ cannot obtain.[17] Instead of taking an impossible proposition to have no exact verifiers, the present framework models impossibility by way of the impossible states themselves. It is for this reason that the proposition that Sanna is sitting and standing differs from the proposition that Kevin is cooking and sleeping, or any other impossible proposition: each impossible proposition will have its own range of impossible states, where each exact verifier state specifies an impossible way for that proposition to be true.

Rather than relying on a primitive set of possible states P to distinguish the possible and impossible states, the following section will introduce the *task relation* $\rightarrow$ which models the possible state transitions, defining the set of possible states in these terms. By also including a set of *times* T, I will demonstrate how to construct possible worlds. By contrast, merely specifying a primitive set of possible states P does not encode any temporal structure on its own. Insofar as states are static, they are entirely devoid of temporal structure. Even including a set of times T by which to index states into a time series, there is still no telling which time series count as possible worlds and which do not. It is this theoretical role which the task relation is intended to fill.

3.1 Task Space

Given any two states s and t, we may ask whether it is possible for s to transition to t. For instance, although it is possible for the state of Sanna sitting to transition to a state of Sanna standing, it is not possible for the state of Sanna sitting to transition to a state of Kevin cooking. In order to encode these differences, I will take any ordered pair of states $\langle s, t\rangle$ to represent a *state transition* from s obtaining to t obtaining. Since not all state transitions are possible, I will take there to be a *task* $s \rightarrow t$ just in case the transition $\langle s, t\rangle$ from s obtaining to t obtaining is possible. For any state s, we may consider the *trivial transition* $\langle s, s\rangle$ which leaves s unchanged. So long as it is possible for s to obtain, making no change to s amounts to a possible transition from s obtaining to s obtaining, and so there is a *trivial task* $s \rightarrow s$. Conversely, if s cannot obtain, then the transition from s obtaining to s obtaining is impossible even by making no change, and so $s \not\rightarrow s$. It follows that it is possible for s to obtain just in case there is a trivial task $s \rightarrow s$. Accordingly, I will define the set of *possible states* $P \coloneqq \{s \in S : s \rightarrow s\}$ to include all and only the states which have trivial tasks.

It remains to constrain the interpretation of $\rightarrow$ in order to support the intended reading of P as the set of possible states. Since $s \rightarrow t$ indicates that the transition from

[17] This gloss assumes that 'sufficient for' has a constitutive as opposed to merely modal reading.

s obtaining to t obtaining is possible, I will assume that it is possible for both s and t to obtain if $s \to t$. Accordingly, I will impose the following constraint:

Quasi-Reflexivity: If $s \to t$, then $s \to s$ and $t \to t$.

Continuing with the example above, we may imagine that Sanna stands up, transitioning from sitting s to standing t. Insofar as this transition is possible— i.e., $s \to t$— we may conclude that it is also possible for Sanna to have continued sitting $s \to s$, and that once she is standing she may go on doing so $t \to t$. Despite being quasi-reflexive, we may observe that the task relation is not reflexive since impossible states do not have trivial tasks, and so are inaccessible even to themselves. For instance, consider the fusion state $s.t$ of Sanna sitting and standing: since $s.t$ cannot obtain, no transition to $s.t$ obtaining is possible, nor is any transition from $s.t$ obtaining possible.[18]

Recall that a state fusion is said to obtain just in case all of its parts obtain. Since a possible state can obtain, we ought to expect that each of its parts can obtain as a result. It is for this reason that Fine requires modalized state spaces to satisfy Possibility. Given that P is defined rather than primitive, I will derive Possibility by constraining the interaction between the task and parthood relations. In particular, tasks between fusions must be decomposable into subtasks between their respective parts, where $s \to t$ is a *subtask* of $s' \to t'$ just in case $s \sqsubseteq s'$ and $t \sqsubseteq t'$. More specifically, I will assume:

Parthood: If $d \sqsubseteq s$ and $s \to t$, then $d \to r$ for some $r \sqsubseteq t$.[19]
If $r \sqsubseteq t$ and $s \to t$, then $d \to r$ for some $d \sqsubseteq s$.

The constraints above ensure that every part is accounted for in any task between fusions. For example, given the red brake lights ahead (state r), we may take there to be a task from Nicky's driving state $d.r$ to a state $b.r'$ in which she brakes (state b) where r' is a similar arrangement of red brake lights. Since $d \sqsubseteq d.r$, it follows by Parthood that there is some suitable $t \sqsubseteq b.r'$ where $d \to t$. Moreover, since $b \sqsubseteq b.r'$, there is some $s \sqsubseteq d.r$ where $s \to b$. Although Parthood only specifies the existence of such states, we may expect that $s = d$ and $t = b$ so that in this case both $d \to b$ and $r \to r'$. Given Parthood and Quasi-Reflexivity, **P5** in §7 derives Possibility.[20]

Given that the null state $\square$ is a part of every state, Parthood is trivialized if $s \to \square$ and $\square \to s$ for any state s. In order to avoid instances of this scenario, it suffices to require the null state $\square$ to be necessary. I will define the set of necessary states by first letting s and t be *connected* $s \sim t$ just in case either $s \to t$ or $t \to s$. A state s is *contingent* just in case s is connected to a distinct state, i.e., $s \sim t$ for some $t \neq s$. Intuitively, the contingent states are those states which can change on account of possibly transitioning to or from a distinct state. Given Quasi-Reflexivity, contingent states are possible. By contrast, there are no tasks between impossible states. Rather, impossible states are non-contingent insofar as they cannot transition to or from any state at all, much less to or from distinct states. Despite being possible, necessary states are also non-contingent since they must obtain. Thus s is *necessary* just in case s is only connected to itself

[18] Although such states will not be needed below, one may admit *transient states* that are possible but do not have trivial tasks by defining $P \coloneqq \{s \in S : \exists t(s \to t \vee t \to s)\}$ and rejecting Quasi-Reflexivity.

[19] These principles are interderivable if $\to$ is symmetric. Although it is natural to take the task relation to be symmetric and transitive for certain applications, these constraints will not be required below.

[20] It is also natural to impose the following containment constraints: (F) if $s \to s$ where $d \sqsubseteq s$ and $d \to r$, then $s \to t$ for some t where $r \sqsubseteq t$; and (B) if $t \to t$ where $r \sqsubseteq t$ and $d \to r$, then $s \to t$ for some s where $d \sqsubseteq s$. However plausible these constraints may be, they will not be required for what follows.

where $N \coloneqq \{s \in S : \forall t \in S(s = t \Leftrightarrow s \sim t)\}$ is the set of all necessary states. We may then avoid trivializing instances of Parthood by assuming the following constraint:

Nullity: $\square = t$ just in case $\square \rightarrow t$ or $t \rightarrow \square$.

It follows that the null state $\square$ is necessary where this fits with its definition as the fusion of the empty set. Since a fusion obtains just in case all of its parts obtain, nothing is required for $\square$ to obtain, and so $\square$ obtains trivially and thus of necessity. In addition to avoiding trivializing instances of Parthood, it follows that every necessary state is possible, and so **P6** in §7 derives Nonempty from Nullity.[21]

It remains to show that every possible state is a part of a world state in accordance with Fine's World Space constraint. Given any state s and possible state t, we may consider the parts of s that are compatible with t. In order to prevent there from being bigger and bigger parts $r_1 \sqsubset r_2 \sqsubset \ldots$ of s without end which are all compatible with t, I will draw on the definitions given in §2.2 to provide the following constraint:

Maximality: If $s \in S$ and $t \in P$, there is some maximal t-compatible part $r \in s_t$.

Consider the total state s of the strategy room at the moment Nixon nearly pressed the button. We may imagine s to include parts that fix all the features of the objects which make up the room and its occupants. Although any state of Nixon pressing the button is incompatible with s, this is not true for the parts of s. For instance, the state a of Nixon's advisors sitting just as they are is a part of s and perfectly compatible with any state of Nixon pressing the button. Given a state b of Nixon pressing the button, Maximality requires there to be a maximal part of s which is compatible with b, ruling out unending chains of bigger and bigger b-compatible parts of s.[22]

Whereas Fine restricts attention to world spaces in which every possible state is part of a world state, **P7** in §7 derives World Space from Maximality. In accordance with Fine's [6] original observations, World Space ensures that every proposition has exactly one truth-value at every world state. Since I will identify possible worlds at a time with world states, it follows that the tautologies of classical logic are true in every model at every world at every time. This result provides a powerful reason to maintain World Space and given that World Space may be derived from Maximality, a corresponding reason to adopt Maximality. Moreover, Maximality is in keeping with an intuitive understanding of states, ruling out exotic cases in which there is no biggest part of a state s that is compatible with a possible state t.

Given these constraints, I will define a *task space* to be an ordered triple $\mathcal{T} = \langle S, \sqsubseteq, \rightarrow\rangle$ where $\langle S, \sqsubseteq\rangle$ is a state space and $\mathcal{T}$ satisfies Quasi-Reflexivity, Parthood, Nullity, and Maximality. Since Possibility, Nonempty, and World Space may be derived, every task space determines a world space. Whereas the world spaces that Fine defines are rich with mereological and modal structure, task spaces provide the primitive resources needed to encode the temporal structure that possible worlds include. By drawing on these elements, the following subsection will present the construction of possible worlds which will play a critical role in the semantics provided in the following section.

[21] It is also natural to assume Compatibility: if $s \in N$ and $t \in P$, then $s \circ t$. Additionally, one might impose Necessity: if $s \in N$ and $t \sqsubseteq s$, then $t \in N$. Neither constraint will be required for what follows.

[22] Recall that states are static. Since pushing the button is not static, one might deny that any state makes it true that Nixon pushed the button. I will return to this objection in the conclusion. For now, think of the state of Nixon pushing the button as the point of no return in which the circuit is closed.

3.2 Possible Worlds

Assuming a discrete theory of time for simplicity, I will model *times* by the set of integers $\mathbb{Z}$ where every time x will be interpreted as differing in time from $x+1$ by a unit value specified for the application at hand.[23] Letting $\mathcal{T}_{\mathbb{Z}} = \langle S, \sqsubseteq, \rightarrow, \mathbb{Z} \rangle$ be a *discrete task space*, we may define an *evolution* $\tau : \mathbb{Z} \rightarrow S$ to be any function from times to states where $\tau(x) \rightarrow \tau(x+1)$ for all times $x \in \mathbb{Z}$. I addition to being accessible from its predecessor, it follows by Quasi-Reflexivity that every state in an evolution is possible.

A *world history* is any evolution α where $\alpha(x) \in W$ for every time $x \in \mathbb{Z}$. Accordingly, we may take $H_{\mathbb{Z}}$ to be the set of all world histories parametrised by $\mathbb{Z}$. Given a history α, we may refer to the range of values that α occupies at different times as the *moments* of that history. Although officially the moments of a history α are its members which take the form $\langle x, \alpha(x) \rangle$, it is convenient to write them as $\alpha(x)$. For instance, suppose that α is a history which occupies the same world state $\alpha(x) = \alpha(y)$ at different times $x \neq y$. Even though the world state is the same at both times, the moments differ since they occur at different times in α's history. Letting $\alpha \approx \beta$ just in case there is some $z \in \mathbb{Z}$ where $\alpha(x) = \beta(x+z)$ for all $x \in \mathbb{Z}$, we may identify the *possible worlds* parametrised by $\mathbb{Z}$ with the set of equivalence classes $W_{\mathbb{Z}} = \{[\alpha]_{\mathbb{Z}} : \alpha \in H_{\mathbb{Z}}\}$ where $[\alpha]_{\mathbb{Z}} = \{\beta \in H_{\mathbb{Z}} : \alpha \approx \beta\}$. Whereas possible worlds represent genuinely distinct sequences of world states, world histories may be compared to coordinate planes which stipulate an origin while acknowledging that the choice of origin does not represent anything significant about space. Although the set of possible worlds may claim to hold a metaphysical standing that the set of histories cannot, it is the histories that will play an important role in the semantics presented below rather than possible worlds. Nevertheless, I will refer to histories as worlds, though officially they are to be distinguished.

In order to evaluate counterfactual conditionals in a language that includes tense operators, it will be important to consider the range of counterfactual worlds to result from making minimal changes to the world of evaluation at the time of evaluation. Roughly, the semantics for counterfactual conditionals presented in §4 will quantify over the worlds that result from imposing an exact verifier state for the antecedent on the world of evaluation at the time of evaluation, checking to see if the consequent is true. Before presetting these details, the following section will conclude by motivating the hyperintensional theory of proposition that I will use to interpret the language.

3.3 Propositions

Intuitively, a counterfactual is true at a moment just in case minimally changing that moment to make the antecedent true also makes the consequent true. If the antecedent is interpreted by a set of moments (or worlds), it is unclear what the result of making a minimal change would be. After all, a moment in which the antecedent is true may differ from the evaluation moment in many completely unrelated ways. The reason the exact verifier states for the antecedent are of greater use in this application than moments is that they amount to specific ways for the antecedent to be true without including anything irrelevant. Put otherwise, the exact verifier states for the antecedent belong to the *subject-matter* of the antecedent. By contrast, the moments in which the antecedent is true may contain elements reaching far beyond the subject-matter of the antecedent. This motivates a departure from intensional theories of propositions to a hyperintensional theory of propositions which is sensitive to differences in subject-matter in addition to

[23] I will consider an objection to the arbitrariness of the unit in §6 along with a continuous analogue.

modal profile. By taking moments to be world states which have mereological structure, we may make minimal changes to the evaluation moment without including irrelevant details, checking to see if the consequent is made true as a result.

In *Anonymous*, I draw on Fine's [6, 7] recent work in order to identify and defend a bilateral theory of propositions which distinguishes propositions that differ in either subject-matter or modal profile without positing any unnecessary distinctions between propositions. Given that the present application calls for a theory of propositions which is sensitive to both modal profile and subject-matter, I will employ that bilateral theory here without reproducing its defense.[24] The theory is said to be *bilateral* on account of taking propositions to consist of both a positive and negative content. For instance, 'John is sitting down' expresses a proposition that consists of a set of exact verifier states which make that sentence true along with a set of exact falsifier states which make that sentence false. Nevertheless, not any ordered pair of sets of states qualifies as a proposition. In order to define the space of propositions, let a set of states X be *closed under fusion* just in case the fusion of any non-empty subset $Y \subseteq X$ belongs to X. Given any sets of states V and F, the ordered pair $\langle V, F\rangle$ is *exclusive* just in case the states in V are incompatible with the states in F, and *exhaustive* just in case every possible state is compatible with some state in either V or F. An ordered pair $\langle V, F\rangle$ is a *bilateral proposition* just in case $\langle V, F\rangle$ is exclusive and exhaustive where V and F are both closed under fusion. Given any proposition $\langle V, F\rangle$, every world state includes a part which belongs to V or F but not both, thereby validating classical logic.[25] Letting $\mathbb{P}$ be the set of bilateral propositions, we may now turn to interpret a propositional language with extensional, modal, counterfactual conditional, and tense operators.

4 Counterfactual Logic

Reading '$\boxed{\scriptstyle P}A$' as 'It has always been the case that A' and '$\boxed{\scriptstyle F}A$' as 'It is always going to be the case that A', I will take $\mathcal{L} = \langle \mathbb{L}, \top, \bot, \neg, \boxed{\scriptstyle P}, \boxed{\scriptstyle F}, \wedge, \vee, \Box\!\!\rightarrow\rangle$ to be a propositional language where $\mathbb{L} := \{p_i : i \in \mathbb{N}\}$ is the set of *sentence letters*, and $\top$ and $\bot$ are designated *top* and *bottom* elements. We may then define the *extensional sentences* of $\mathcal{L}$ as follows:

$$\varphi \ ::= \ p_i \mid \top \mid \bot \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi.$$

Letting $\texttt{ext}(\mathcal{L})$ be the set of extensional sentences of $\mathcal{L}$, we may define the *well-formed sentences* of $\mathcal{L}$ by restricting the antecedent of a counterfactual to $\varphi \in \texttt{ext}(\mathcal{L})$:

$$A \ ::= \ p_i \mid \top \mid \bot \mid \neg A \mid A \wedge A \mid A \vee A \mid \boxed{\scriptstyle P}A \mid \boxed{\scriptstyle F}A \mid \varphi \Box\!\!\rightarrow A.$$

Although the consequent of a counterfactual may be any well-formed sentence of $\mathcal{L}$, only the extensional operators may occur in the antecedent. We may then take a *discrete model* $\mathcal{M} = \langle S, \sqsubseteq, \rightarrow, \mathbb{Z}, |\cdot|\rangle$ of $\mathcal{L}$ to be any ordered tuple where $\langle S, \sqsubseteq, \rightarrow, \mathbb{Z}\rangle$ is a discrete task space, $|p_i| \in \mathbb{P}$ for all $p_i \in \mathbb{L}$, and both $|\top| = \langle S, \{\blacksquare\}\rangle$ and $|\bot| = \langle \varnothing, \{\square\}\rangle$.[26] After presenting a compositional semantics in the following subsection, I will provide a theory of logical consequence in §4.2, discussing countermodels in §4.3. I will then apply these resources to analyze forwards, backwards, and backtracking counterfactuals in §5.

[24] See Fine [9] for a comparison of the present approach to Yablo's [21] theory of subject-matter.

[25] See **P9** below and Fine [6, p. 630] for the original observation and definitions.

[26] Although $\bot$ will not be needed below, I have included $\bot$ in addition to $\top$ since $|\bot| \neq |\neg\top|$. Rather, $\neg\bot$ and $\neg\top$ are the other two top and bottom elements making for a total of four *extremal elements*.

4.1 A Unified Semantics

In *Anonymous*, I argue that Fine's [4, 6] exact inclusive semantics for the extensional operators preserves differences in subject-matter and modal profile without drawing any unnecessary distinctions. Given the present aim to preserve differences in both subject-matter and modal profile, I will reproduce that semantics below:

Product: $X \otimes Y \coloneqq \{s.t : s \in X,\ t \in Y\}$.

Sum: $X \oplus Y \coloneqq X \cup Y \cup (X \otimes Y)$.

Conjunction: $\langle V, F\rangle \wedge \langle V', F'\rangle \coloneqq \langle V \otimes V',\ F \oplus F'\rangle$

Disjunction: $\langle V, F\rangle \vee \langle V', F'\rangle \coloneqq \langle V \oplus V',\ F \otimes F'\rangle$

Negation: $\neg\langle V, F\rangle \coloneqq \langle F, V\rangle$.

Whereas the product $X \otimes Y$ is the set of pairwise fusions of the states in X with the states in Y, the sum $X \oplus Y$ is the union of X, Y, and their product $X \otimes Y$. Given these definitions, conjunction takes the product of the verifiers for the conjuncts followed by the sum of the falsifiers for the conjuncts, and disjunction does the reverse. By contrast, negation exchanges the verifiers and falsifiers of the negated proposition.[27]

In §7, **P8** shows that $\mathbb{P}$ is closed under the propositional operators defined above. Accordingly, we may extend the interpretation provided by a model $\mathcal{M}$ to all extensional sentences with the following *exact inclusive semantic* clauses where $|\varphi| \in \mathbb{P}$ by **P9**:

[$\wedge$] $\quad |\varphi \wedge \psi| = |\varphi| \wedge |\psi|$.

[$\neg$] $\quad |\neg\varphi| = \neg|\varphi|$.

[$\vee$] $\quad |\varphi \vee \psi| = |\varphi| \vee |\psi|$.

Since there is no confusing 'φ' with '$|\varphi|$', the abuse of notion for '$\neg$', '$\wedge$', and '$\vee$' may be forgiven. Having fixed the interpretation of the extensional sentences, we may interpret all well-formed sentences of $\mathcal{L}$ where $\alpha, \beta \in H_{\mathbb{Z}}$ are worlds, $x, y \in T$ are times, and $q \in \mathbb{L} \cup \{\top, \bot\}$ is either a sentence letter, top element, or bottom element:

$\mathcal{M}, \alpha, x \vDash q$ *iff* there is some $s \sqsubseteq \alpha(x)$ where $s \in |q|^+$.

$\mathcal{M}, \alpha, x \vDash \neg A$ *iff* $\mathcal{M}, \alpha, x \nvDash A$.

$\mathcal{M}, \alpha, x \vDash A \wedge B$ *iff* $\mathcal{M}, \alpha, x \vDash A$ and $\mathcal{M}, \alpha, x \vDash B$.

$\mathcal{M}, \alpha, x \vDash A \vee B$ *iff* $\mathcal{M}, \alpha, x \vDash A$ or $\mathcal{M}, \alpha, x \vDash B$.

$\mathcal{M}, \alpha, x \vDash \boxed{\scriptstyle P}A$ *iff* $\mathcal{M}, \alpha, y \vDash A$ for all $y < x$.

$\mathcal{M}, \alpha, x \vDash \boxed{\scriptstyle F}A$ *iff* $\mathcal{M}, \alpha, y \vDash A$ for all $y > x$.

$\mathcal{M}, \alpha, x \vDash \varphi \mathrel{\Box\!\!\rightarrow} C$ *iff* $\mathcal{M}, \beta, x \vDash C$ whenever $t \in |\varphi|^+$ and $t \rightarrowtriangle_{\alpha(x)} \beta(x)$.

Whereas the exact semantics took a bilateral form in order to specify a propositional operation for negation that respects differences in subject-matter, evaluating sentences for truth does not require the same sensitivity to subject-matter. Suppressing reference to the model, **P10** in §7 shows that $\varphi \in \texttt{ext}(\mathcal{L})$ is true at a world-time pair just in case

[27] The hyperintensional space of propositions $\mathbb{P}$ may be shown to form a *non-interlaced bilattice*. See Ginsberg [13] and Fitting [10] for definitions.

there is a part of that world at that time which exactly verifies φ. Given the definition of a proposition in §3.3, it follows from World Space that every extensional sentence has exactly one truth-value at every world-time pair, where the remaining semantic clauses extend this property to all well-formed sentences of the language.[28]

In addition to maintaining standard semantic clauses for the extensional and tense operators, the semantics for the counterfactual conditional operator has been presented in terms of the imposition relation in order to facilitate comparison with Fine's account. However, given the definition of imposition provided in §2.2, we may restate the semantics for the counterfactual conditional in more basic terms:

$\mathcal{M}, \alpha, x \vDash \varphi \mathrel{\Box\!\!\rightarrow} C$ *iff* $\mathcal{M}, \beta, x \vDash C$ whenever $\beta \in H_{\mathbb{Z}}$, $t \in |\varphi|^+$, and there is a maximal t-compatible part $s \in \alpha(x)_t$ where $s.t \sqsubseteq \beta(x)$.

The clause given above articulates the semantics for counterfactual conditionals in purely modal and mereological terms. Intuitively, a counterfactual is true in a world α at a time x just in case the consequent is true in any world β at x where $\beta(x)$ is the result of minimally changing $\alpha(x)$ to make the antecedent true. This approach is to be contrasted with similarity theories which compare temporally extended worlds for similarity in their entirety, as well as with imposition theories that posit a counterfactual primitive in the metalanguage while restricting consideration to instantaneous world states.

Having extended the interpretation provided by a model to all well-formed sentences of the language, we may present a theory of logical consequence for $\mathcal{L}$ in order to study the interactions between the operators in the language. Letting $\Gamma \cup \{C\} \subseteq \texttt{wfs}(\mathcal{L})$ be a set of well-formed sentences of $\mathcal{L}$, consider the following definition:

Logical Consequence: $\Gamma \vDash C$ *iff* for any any model $\mathcal{M}$ of $\mathcal{L}$, world α, and time x, if $\mathcal{M}, \alpha, x \vDash A$ for all $A \in \Gamma$, then $\mathcal{M}, \alpha, x \vDash C$.

A rule schema is *valid* just in case for any instance, its conclusion is a logical consequence of its premises. Without attempting to provide a complete logic for the semantics above, the following subsection will present three extensions of classical propositional logic in order to study the interactions between the operators included in the language.

4.2 Logics

Letting $\mathcal{L}^{\text{PL}} = \langle \mathbb{L}, \neg, \wedge, \vee \rangle$ be an extensional language without top and bottom elements, I will take the deduction relation of classical propositional logic $\vdash_{\text{PL}}$ to be defined over a restriction to the well-formed sentences of $\mathcal{L}^{\text{PL}}$. By taking $\mathcal{L}^{\text{CL}} = \langle \mathbb{L}, \neg, \wedge, \vee, \mathrel{\Box\!\!\rightarrow} \rangle$, I will assume that $\varphi, \psi, \chi \ldots \in \texttt{ext}(\mathcal{L}^{\text{CL}})$ and $A, B, C, \ldots \in \texttt{wfs}(\mathcal{L}^{\text{CL}})$ as above, where $\varphi \mathrel{\Box\!\!\rightarrow} \Gamma \coloneqq \{\varphi \mathrel{\Box\!\!\rightarrow} A : A \in \Gamma\}$. I will then define $\vdash_{\text{CL}}$ to be the smallest extension of $\vdash_{\text{PL}}$ to be closed under the standard structural rules and all instances of the following:

R1 *If* $\Gamma \vdash C$, *then* $\varphi \mathrel{\Box\!\!\rightarrow} \Gamma \vdash \varphi \mathrel{\Box\!\!\rightarrow} C$

C1 $\varphi \mathrel{\Box\!\!\rightarrow} \varphi$

C2 $\varphi,\ \varphi \mathrel{\Box\!\!\rightarrow} A \vdash A$

C3 $\varphi \mathrel{\Box\!\!\rightarrow} \psi,\ \varphi \wedge \psi \mathrel{\Box\!\!\rightarrow} A \vdash \varphi \mathrel{\Box\!\!\rightarrow} A$

C4 $\varphi \vee \psi \mathrel{\Box\!\!\rightarrow} A \vdash \varphi \wedge \psi \mathrel{\Box\!\!\rightarrow} A$

C5 $\varphi \vee \psi \mathrel{\Box\!\!\rightarrow} A \vdash \varphi \mathrel{\Box\!\!\rightarrow} A$

C6 $\varphi \vee \psi \mathrel{\Box\!\!\rightarrow} A \vdash \psi \mathrel{\Box\!\!\rightarrow} A$

C7 $\varphi \mathrel{\Box\!\!\rightarrow} A,\ \psi \mathrel{\Box\!\!\rightarrow} A,\ \varphi \wedge \psi \mathrel{\Box\!\!\rightarrow} A \vdash \varphi \vee \psi \mathrel{\Box\!\!\rightarrow} A$

[28] See Fine [6, pp. 665-7] for related results.

Whereas **R1** guarantees that the consequences of a counterfactual hypothesis are closed under deduction, **C1** is the *Identity* schema, **C2** is *Counterfactual Modus Ponens*, and **C3** is *Weakened Transitivity*.[29] Additionally, **C4 - C6** eliminate a disjunction from the antecedent of a counterfactual, and **C7** introduces a disjunction to the antecedent of a counterfactual.[30] It is easy to see that **SDA** follows from **C5** and **C6**. Referring to the resulting *Counterfactual Logic* as **CL**, §7 provides elements of the proof that **CL** is sound. Additionally, we may derive the following consequences from **R1**:

D1 $\varphi \boxright A,\ \varphi \boxright B \vdash \varphi \boxright A \wedge B$

D2 *If* $A \vdash B$*, then* $\varphi \boxright A \vdash \varphi \boxright B$

These rules correspond to Fine's [4, 5] *Finite Conjunction* and *Classical Weakening* rules respectively.[31] Whereas the following section will present a number of countermodels in order to demonstrate which inferences are excluded from **CL**, the remainder of the present section extends **CL** to include modal and tense operators.

Provided with the expressive resources of $\mathcal{L}^{\text{CL}}$, metaphysical necessity may be defined in terms of the counterfactual conditional as $\Box\varphi \coloneqq \neg\varphi \boxright A \wedge \neg A$. However, given the limitation that $\varphi \in \texttt{ext}(\mathcal{L}^{\text{CL}})$, it follows that iterated modalities are not well-formed.[32] Rather, I will include the top and bottom elements in $\mathcal{L}^{\text{CML}} = \langle \mathbb{L}, \top, \bot, \neg, \wedge, \vee \rangle$, taking $\Box A \coloneqq \top \boxright A$ to define *metaphysical necessity*. By letting $\varphi \diamondright A \coloneqq \neg(\varphi \boxright \neg A)$ be the *might counterfactual conditional*, I will take $\Diamond A \coloneqq \top \diamondright A$ to define *metaphysical possibility* where $\vdash_{\text{CML}}$ minimally extends $\vdash_{\text{CL}}$ to include all instances of the following:

M1 $\top$

M2 $\neg\bot$

M3 $A \to \Box\Diamond A$

M4 $\Box A \to \Box\Box A$

M5 $\Box(\varphi \to A) \to (\varphi \boxright A)$

Having defined the modal operators in $\mathcal{L}^{\text{CML}}$ instead of $\mathcal{L}^{\text{CL}}$, all instances of the axioms given above are well-formed. It is important to observe that **M5** together with **C2** entail that the counterfactual conditional is of intermediate strength between the strict conditional and material conditional. We may then derive the following:

D3 *If* $\vdash A$*, then* $\vdash \Box A$*.*

D4 $\Box(A \to B) \to (\Box A \to \Box B)$

D5 $\Box A \to A$

D6 $\neg A,\ \varphi \boxright A \vdash \neg\varphi$

D7 $\Box A \leftrightarrow \neg\Diamond\neg A$

D8 $\Box\varphi \leftrightarrow \neg\varphi \boxright \bot$

D9 $\varphi \boxright \top$

D10 $\bot \boxright A$

Given **D3 - D5** and **M3 - M4**, *Counterfactual Modal Logic* (**CML**) entails an S5 logic. Whereas S5 is only valid given appropriate constraints on intensional Kripke frames, the validity of **M3 - M4** and **D5** does not depend on imposing constraints on the range of task spaces. Although a primitive accessibility relation between states could be added

[29] See Lewis [16, p. 35] and Fine [4] for discussion of **C3**.

[30] Fine's [4, 5] provides **C1 - C7** but does not include **R1**.

[31] Were infinite conjunction included in the language, we could also derive the infinite conjunction rule that Fine [5] assumes. I will restrict consideration to sentences with finite length for simplicity.

[32] Although it is possible to extend the semantics in order to permit the antecedent to be any well-formed sentence, interpreting counterfactuals with counterfactual antecedents remains unintuitive.

in order to weaken the logic, I will assume that metaphysical modality has an S5 logic, undermining the motivation to complicate the semantics in this way. Additionally, **D6** is *Counterfactual Modus Tollens*, **D7** derives the standard duality between the metaphysical modals, **D8** entails the equivalence that one might take to define metaphysical necessity in $\mathcal{L}^{\text{CL}}$, and **D9 - D10** derive the *Triviality* axioms that Fine [4] assumes.

Having considered the languages $\mathcal{L}^{\text{PL}}$, $\mathcal{L}^{\text{CL}}$, and $\mathcal{L}^{\text{CML}}$, it remains to include temporal temporal operators, presenting a range of further axiom schemata within the unrestricted language $\mathcal{L}$. In order to present these additions, I will take $\boxed{\scriptstyle\mathrm{F}}\Gamma \coloneqq \{\boxed{\scriptstyle\mathrm{F}}A : A \in \Gamma\}$ and $\ulcorner A_{\langle \mathrm{P}|\mathrm{F}\rangle}\urcorner$ to be the result of exchanging '$\boxed{\scriptstyle\mathrm{P}}$' and '$\boxed{\scriptstyle\mathrm{F}}$' in A where $\Gamma_{\langle \mathrm{P}|\mathrm{F}\rangle} \coloneqq \{A_{\langle \mathrm{P}|\mathrm{F}\rangle} : A \in \Gamma\}$. I will also assume the standard metalinguistic abbreviations $\Diamond_{\mathrm{P}} A \coloneqq \neg\boxed{\scriptstyle\mathrm{P}}\neg A$ which I will read 'It was the case that A' and $\Diamond_{\mathrm{F}} A \coloneqq \neg\boxed{\scriptstyle\mathrm{F}}\neg A$ for 'It is going to be the case that A'. We may then take $\vdash_{\text{CTL}}$ to minimally extend $\vdash_{\text{CML}}$ to include all instances of the following:

R2 *If* $\Gamma \vdash C$, *then* $\boxed{\scriptstyle\mathrm{F}}\Gamma \vdash \boxed{\scriptstyle\mathrm{F}}C$

R3 *If* $\Gamma \vdash C$, *then* $\Gamma_{\langle \mathrm{P}|\mathrm{F}\rangle} \vdash C_{\langle \mathrm{P}|\mathrm{F}\rangle}$

T1 $A \rightarrow \boxed{\scriptstyle\mathrm{F}}\Diamond_{\mathrm{P}} A$

T2 $\boxed{\scriptstyle\mathrm{F}}A \rightarrow \boxed{\scriptstyle\mathrm{F}}\boxed{\scriptstyle\mathrm{F}}A$

T3 $\boxed{\scriptstyle\mathrm{F}}A \rightarrow \Diamond_{\mathrm{F}} A$

T4 $\Diamond_{\mathrm{F}}\Diamond_{\mathrm{P}} A \rightarrow (\Diamond_{\mathrm{P}} A \vee A \vee \Diamond_{\mathrm{F}} A)$

T5 $(\boxed{\scriptstyle\mathrm{P}}A \wedge A \wedge \Diamond_{\mathrm{F}}\top) \rightarrow \Diamond_{\mathrm{F}}\boxed{\scriptstyle\mathrm{P}}A$

T6 $\Box\boxed{\scriptstyle\mathrm{F}}A \leftrightarrow \Box A.$

T7 $\Diamond_{\mathrm{F}}\Box A \rightarrow \Box A.$

T8 $\Diamond_{\mathrm{F}}\Diamond A \rightarrow \Diamond A.$

Whereas **R2** entails analogues of **D3 - D4** for the operator $\boxed{\scriptstyle\mathrm{F}}$, **R3** exchanges '$\boxed{\scriptstyle\mathrm{P}}$' and '$\boxed{\scriptstyle\mathrm{F}}$' throughout any inference, making the past and future have the same structure at every moment in each world. Focusing on just the future, **T1** captures the idea that every present moment is past in any future moment and **T2 - T5** require that the temporal ordering be transitive, endless, linear, and discrete, respectively. The remaining axiom schemata describe the interactions between the metaphysical modal and tense operators. I will characterize this relationship by way of the derived schema **D11** $\Box A \rightarrow \triangledown A$ in §7 where $\triangledown A \coloneqq \boxed{\scriptstyle\mathrm{P}}A \wedge A \wedge \boxed{\scriptstyle\mathrm{F}}A$ reads 'It is always the case that A'. Much more generally, we may assert the broad slogan: *modal truths are necessarily always the case.*

I will refer to the system including all axiom schemata and metarules defended above as *Counterfactual Tense Logic* (**CTL**), deriving **D1 - D11** in §7. In addition to being sound over the discrete models of $\mathcal{L}$, **CTL** is strong enough to justify a counterfactual conditional reading of '$\mathrel{\Box\!\!\rightarrow}$', a metaphysical modal reading of '$\Box$', as well as the temporal readings of '$\boxed{\scriptstyle\mathrm{P}}$' and '$\boxed{\scriptstyle\mathrm{F}}$' presented above. The following subsection will complement these findings by considering a number of invalid inferences that are excluded from **CTL** (as well as from **CL**) on account of admitting countermodels.

4.3 Countermodels

In order to get a better sense of the semantics for counterfactual conditionals, it will help to review a number of invalid schemata. Consider the following:

#1 $\varphi, A \vdash \varphi \mathrel{\Box\!\!\rightarrow} A.$

#2 $\varphi \mathrel{\Box\!\!\rightarrow} \psi, \psi \mathrel{\Box\!\!\rightarrow} A \vdash \varphi \mathrel{\Box\!\!\rightarrow} A.$

#3 $(\varphi \mathrel{\Box\!\!\rightarrow} A) \vee (\varphi \mathrel{\Box\!\!\rightarrow} \neg A).$

#4 $\varphi \mathrel{\Box\!\!\rightarrow} A \vee B \vdash (\varphi \mathrel{\Box\!\!\rightarrow} A) \vee (\varphi \mathrel{\Box\!\!\rightarrow} B).$

#5 $\varphi \mathrel{\Box\!\!\rightarrow} A, \psi \mathrel{\Box\!\!\rightarrow} A \vdash \varphi \wedge \psi \mathrel{\Box\!\!\rightarrow} A.$

#6 $\varphi \mathrel{\Box\!\!\rightarrow} \psi, \neg\varphi, \neg\psi \vdash \neg\psi \mathrel{\Box\!\!\rightarrow} \neg\varphi.$

#7 $\varphi \wedge \psi \mathrel{\Box\!\!\rightarrow} A \vdash \varphi \mathrel{\Box\!\!\rightarrow} (\psi \mathrel{\Box\!\!\rightarrow} A).$

#8 $\varphi \mathrel{\Box\!\!\rightarrow} (\psi \mathrel{\Box\!\!\rightarrow} A) \vdash \varphi \wedge \psi \mathrel{\Box\!\!\rightarrow} A.$

#9 $\varphi \mathrel{\Box\!\!\rightarrow} A \vdash \varphi \wedge \psi \mathrel{\Box\!\!\rightarrow} A.$

#10 *If* $\Gamma, \varphi \vdash C$, *then* $\Gamma \vdash \varphi \mathrel{\Box\!\!\rightarrow} C.$

For brevity, I will focus attention on **#1**, **#6**, **#9**, and **#10**, discussing a countermodel for each in order to shed light on the nature of counterfactual reasoning.

#1 $\varphi, A \vdash \varphi \boxright A$

The ball is red and Mary likes it. Even so, it would be wrong to claim that if the ball were red Mary would like it since there are certain shades of red Mary does not like. In accordance with these assumptions, we may draw on the task semantics in order to provide a countermodel for the instance $p_1, p_2 \vdash p_1 \boxright p_2$:

$$|p_1|^+ = \{a, c, a.c\} \qquad \alpha(x) = a.b \qquad W = \{a.b, c.d\}$$

$$|p_2| = \langle\{b\}, \{d\}\rangle \qquad \beta(x) = c.d$$

The world state $a.b$ includes an exact verifier for both p_1 and p_2, making both sentences true in α at x. However, c is also an exact verifier for p_1 and is incompatible with any exact verifier for p_2. Thus the maximal c-part of $\alpha(x)$ is $\square$, and so β is an outcome of imposing c on α at x which does not include a part that exactly verifies p_2. Adding color, we may take a to be a state of the ball being a shade of red that Mary likes, b to be the state of Mary liking the ball, c to be a state of the ball being a shade of red that Mary does not like, and d to be the state of Mary disliking the ball. It follows by the semantics for counterfactuals that $p_1 \boxright p_2$ is not true in α at x. The rest of the details needed to complete the model do not matter and so will be omitted.

#6 $\varphi \boxright \psi, \neg\varphi, \neg\psi \vdash \neg\psi \boxright \neg\varphi$

If Boris had gone to the party, Olga would have gone too.[33] Neither Olga nor Boris went in the end. Even so, it would be wrong to claim that if Olga were to not go to the party then Boris would not go to the party. For even though Olga likes to go to the parties Boris attends, Boris prefers to socialise without Olga. Drawing on the task semantics, we may present a countermodel to the instance $p_1 \boxright p_2, \neg p_1, \neg p_2 \vdash \neg p_2 \boxright \neg p_1$:

$$|p_1| = \langle\{a\}, \{c\}\rangle \qquad \alpha(x) = c.d.f \qquad \gamma(x) = a.e.g$$

$$|p_2| = \langle\{b\}, \{d, e, d.e\}\rangle \qquad \beta(x) = a.b.f \qquad W = \{a.b.f, c.d.f, e.a.g\}$$

Since $\alpha(x)$ contains the exact falsifiers c and d for p_1 and p_2 respectively, we know that $\neg p_1$ and $\neg p_2$ are true in α at x. Moreover, the only outcome of imposing an exact verifier for p_1 on $\alpha(x)$ is $\beta(x)$ which includes an exact verifier for p_2, and so $p_1 \boxright p_2$ is true in α at x. However, if the exact verifier e for $\neg p_2$ is imposed on $\alpha(x)$, the outcome $\gamma(x)$ does not include an exact verifier for $\neg p_1$, and so $\neg p_2 \boxright \neg p_1$ is false in α at x.

Adding substance, we may take d and e to be states of Olga staying home where d is only compatible with the state f of Boris not knowing where Olga is and e is only compatible with the state g of Boris believing her to be home. Since the state of Boris being at the party a is compatible with him not knowing where Olga is f and the only world state to include both a and f as parts also includes the state of Olga going to the party b, imposing a on $\alpha(x)$ results in the world state $\beta(x)$ in which Olga goes to the party. Thus it is true at α at x that if Boris had gone to the party, then Olga would have gone. However, the state of Olga staying at home e is only compatible with Boris

[33] This example has been adapted from Lewis' [16, p. 35] case against counterfactual contraposition.

believing her to be home g, where the only world state $\gamma(x)$ to include both e and g as parts also includes Boris going to the party a. Thus it is false to claim that if Olga were to not go to the party, then Boris would not go either. All that is required is for Olga to not go to the party in a way that is compatible with Boris believing her to be home. For instance, perhaps Boris can see that the lights are on in her house.

The countermodel to **#6** also invalidates **#1** as well as the principle of *Counterfactual Contraposition* (**#11**) $\varphi \boxright \psi \vdash \neg\psi \boxright \neg\varphi$. Nevertheless, I show in §7 that **C2** is valid where this entails *Counterfactual Modus Tollens* **D6**. It follows that **#10** does not preserve validity for otherwise we may derive **#11** from **D6**. However, **#10** is sometimes defended, at least for certain restricted subject-matters.[34]

#9 (STA) $\varphi \boxright A \vdash \varphi \wedge \psi \boxright A$

Judy and Joey are at the party. If Matan were to go to the party, he would argue with Judy, making her angry. Even so, if both Matan and Stav were to go to the party, Stav would keep Matan occupied, making Joey jealous. Consider the following countermodel:

$$|p_1| = \langle\{a\},\{b\}\rangle \qquad |p_2| = \langle\{c\},\{d\}\rangle \qquad |p_3| = \langle\{e\},\{f\}\rangle$$

$$\alpha(x) = b.d.f.g \qquad \beta(x) = a.d.e.g \qquad \gamma(x) = a.c.f.h$$

$$W = \{b.d.f.g, a.d.e.g, a.c.f.h\}$$

Letting $p_1 \boxright p_3 \vdash p_1 \wedge p_2 \boxright p_3$ be the instance of **#9**, we may take a to be a state of Matan being at the party, b to be a state of Matan being at home, c to be a state of Stav being at the party, d to be a state of Stav being at home, e to be a state of Judy being angry, f to be a state of Judy being happy, g to be a state of Joey being content, and h to be a state of Joey being jealous. Since $\beta(x)$ is the only world state to result from imposing an exact verifier for p_1 on $\alpha(x)$ and includes an exact verifier for p_3, the counterfactual $p_1 \boxright p_3$ is true in α at x. Nevertheless, the world state $\gamma(x)$ which results from imposing an exact verifier for $p_1 \wedge p_2$ on $\alpha(x)$ does not include an exact verifier for p_3, and so $p_1 \wedge p_c \boxright p_3$ is false in α at x.[35] Thus **STA** is invalid.

It is worth pausing to consider the role that states as opposed to moments play in invalidating **STA**. Although all $A \wedge B$-moments are A-moments, imposing an $A \wedge B$-state on an evaluation moment $\alpha(x)$ need not be an outcome of imposing an A-state of $\alpha(x)$. The sensitivity of the outcomes induced by imposing a state on a moment of evaluation is made possible not only by the mereological structure of each moment, but by the requirement that the exact verifier states for the antecedent belong to the subject-matter of the antecedent, where this requirement constitutes the *exactness* of the state semantics. It is for this reason that it is appropriate to assume a hyperintensional theory of propositions which is sensitive to differences in subject-matter in addition to modal profile. Although the `anonymous` may be used to find simpler countermodels than those presented above, it is often helpful to add states to a countermodel in support of an intuitive interpretation of the countermodel in question. Nevertheless, using the `anonymous` to find minimal countermodels can vastly accelerate the process of finding interpreted countermodels to counterfactual conditional reasoning.

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[34] See Yli-Vakkuri and Hawthorne [22] for such a view and Elgin [1] for further discussion.

[35] This model can be extended to make $p_1 \wedge p_2 \wedge p_4 \boxright p_3$ true while still making $p_1 \boxright p_3$ true and $p_1 \wedge p_2 \boxright p_3$ false. To do so, let $|p_4| = \langle\{i\},\{j\}\rangle$ where i is the state of Daniel attending the party, j is the state of Daniel being at home, and $\alpha(x) = b.d.f.g.j$, $\beta(x) = a.d.e.g.j$, $\gamma(x) = a.c.f.h.j$, and $\delta(x) = a.c.e.h.i$ are world states. Repeating this strategy generates a Sobel sequence.

5 Counterfactuals and Tense

Although many of the examples discussed above were stated in simple present tense, this was by contrivance rather than a reflection of standard practice. By contrast, past tense counterfactuals are commonplace where Fine's [3] original example is one such case:

(N′) If Nixon had pressed the button there would have been a nuclear holocaust.

Suppose that the button were connected for ten minutes after a security meeting before being deactivated again. Shortly after the button was deactivated, one of Nixon's advisers asserts N′ to another. Whereas it is natural to take this assertion to be true, regimenting N′ as $B \boxright H$ where B reads 'Nixon pushes the button' and H reads 'There is a nuclear holocaust' predicts that the assertion is false. Not only is the button disconnected at the time of the assertion, the nuclear holocaust would not occur immediately even if the button were connected and pushed. Rather, consider the following alternative:

(N) $\Diamond(B \boxright \Diamond H)$.

If Nixon were to push the button during the activation period, then there would be a future time in which a nuclear holocaust occurs. Since there is a time that is both during the activation period and before the adviser's assertion, N is predicted to be true, and so may be taken to provide a better regimentation of N′ than $B \boxright H$.

Although there would have been a nuclear holocaust if Nixon had pressed the button after finishing the security meeting when the button was engaged, there would not have been a Nuclear holocaust had Nixon pressed the button before it was engaged or anytime after it was disengaged. As a result, $\Diamond(B \boxright \neg\Diamond H)$ is also true if asserted shortly after the button was deactivated as imagined above. Instead of taking N′ to assert the existence of any past time where $B \boxright \Diamond H$ is true, we may take N′ to concern a specific time at which $B \boxright \Diamond H$ is true and $B \boxright \neg\Diamond H$ is false. We may also motivate consideration of a specific time at which the consequent is evaluated by supposing that Nixon's adviser asserts N′ a year after the incident. Rather than asserting the existence of any future time in which there is a nuclear holocaust, we may take the adviser's assertion to concern the present time of the assertion. In order to capture both of these temporally specific reading, the following subsection will further extend the language to include two additional tense operators for speaking about specific times. Given these resources, I will analyze forwards, backwards, and backtracking counterfactuals in §5.2.

5.1 Temporal Operators

Letting $\mathcal{L}^*$ extend $\mathcal{L}$ to include the unary *store operator* $\uparrow_i$ and unary *recall operator* $\downarrow^i$ for all $i \in \mathbb{N}$, we may define `ext`$(\mathcal{L}^*)$ and `wfs`$(\mathcal{L}^*)$ as before. By including a vector $\vec{v} = \langle v_1, v_2, \ldots\rangle$ of stored times in the point of evaluation, we may present the following:

($\uparrow$) $\mathcal{M}, \alpha, x, \vec{v} \vDash \uparrow^i A$ *iff* $\mathcal{M}, \alpha, x, \vec{v}_{[x/v_i]} \vDash A$.

($\downarrow$) $\mathcal{M}, \alpha, x, \vec{v} \vDash \downarrow^i A$ *iff* $\mathcal{M}, \alpha, v_i, \vec{v} \vDash A$.

Whereas $\uparrow^i A$ stores the current time of evaluation in the i^{th} value of $\vec{v}$, the sentence $\downarrow^i A$ shifts the time of evaluation to the i^{th} value stored in $\vec{v}$. By adding $\vec{v}$ as a parameter to the point of evaluation, the semantic clauses given for $\mathcal{L}$ may otherwise be maintained. Given these expressive resources, we may regiment N′ as follows:

(N′) $\downarrow^1(B \boxright \Diamond H)$.

Let v_1 store the time at which Nixon almost pressed the button. Although this time may be forever imprinted on the minds of Nixon's advisers, the time at which the nuclear holocaust would have taken place may not be known. It might have taken days for the disaster to unfold, or perhaps only hours. Rather than assuming there is a specific time, we may take N′ to assert that there is a future time at which the nuclear holocaust would have taken place. Given the semantics above, N′ is true at a world α, time x, and vector $\vec{v}$ just in case for every world β which results from imposing an exact verifier for B on α at v_1, there is some future time $y > v_1$ at which H is true in β.

Letting α be the actual world and x be the time of the adviser's assertion moments after the button was disengaged, N′ is true whenever v_1 is a time at which the button is activated. Even so, we may observe that $\downarrow^1(B \boxright \Diamond\neg H)$ is also true assuming there is a time shortly after v_1 which occurs before the nuclear holocaust has taken place. Although this may be admitted, in certain circumstances there may be a specific future time at which it is appropriate to evaluate the consequent. For instance, in the case described above where Nixon's adviser asserts `N′` a year after the incident, the adviser may intend to specify that a nuclear holocaust is underway at the time of the assertion in all of the counterfactual worlds in which Nixon pushed the button a year prior:

(N″) $\uparrow^2 \downarrow^1(B \boxright \downarrow^2 H)$.

By first storing the time of assertion and then reverting back to the time at which the button was engaged, N″ asserts that if Nixon were to have pressed the button at that time of engagement, then a nuclear holocaust would have occur at the stored time of assertion. In certain circumstances, it may be enough to assert the weaker claim that $\uparrow^2 \Diamond(B \boxright \downarrow^2 H)$ which merely requires there to be some past time where if Nixon were to push the button at that time we would now be in a nuclear holocaust. Which regimentation is right may depend on what the speaker intends.

Although typical, first going back in time and then going forward is not the only way to interpret tensed counterfactual claims, and sometimes not the most natural. The following subsection will consider both backwards and backtracking counterfactuals, providing their analysis with the resources of the present framework.

5.2 Backtracking

Following tradition, I will focus on an example presented by Jackson [14] in which Smith is standing on the edge of a building threatening to jump. Standing behind him in safety, Beth and Bill watch in fear for their friend. Moments later, Smith steps back from the edge. We may then consider the following claims:

(J) If Smith had jumped, he would have died.

(U) If Smith had jumped, a net would have been installed beneath him.

(L) If Smith had jumped, he would have lived.

Kicking things off, suppose that Bill asserts J. Beth responds by insisting on U. After all, they both know that Smith wants to live so the only way Smith would jump is if jumping would not kill him. Having made her case for U, Beth goes on to assert L. Unconvinced, Bill points out that nobody would install a net on the building, reasserting J. This prompts Beth to enumerate further reasons why Smith would never jump without a net there to catch him, insisting on U and L once more. This may be imagined to continue round after round. The challenge is to identify what they are disagreeing about and to provide a theory of counterfactuals that adequately models their disagreement.

Whereas `J` is characteristic of forward counterfactuals and can be regimented in a manner similar to $\mathtt{N}'$ above, `U` and `L` express *backwards* and *backtracking* counterfactuals, respectively. Rather than changing the world of evaluation to make the antecedent true at one time and evaluating the consequent at a later time, backwards counterfactuals such as `U` evaluate the consequent at an earlier time than the time at which the antecedent is assumed to be true. Consider the following regimentations:

(D) $\downarrow^1(J \boxright \downarrow^2 D)$.

(U) $\downarrow^1(J \boxright \downarrow^3 U)$.

(L) $\downarrow^1(J \boxright \downarrow^2 L)$.

Holding the world α, time x, and vector of stored times $\vec{v}$ fixed for the purposes of evaluating Bill and Beth's conversation, I will assume that $v_3 < v_1 < v_2$, where Bill and Beth agree about this much. Nevertheless, Bill takes D to be true since any world to result from imposing a state in which Smith jumps in α at v_1 will be one in which Smith dies at v_2. Beth disagrees, claiming that L is true since any world to result from imposing a state in which Smith jumps in α at v_1 will be one in which Smith lives at v_2. Since the states of Smith being alive are incompatible with the states of Smith being dead, the worlds that Beth predicts will result from imposing a state of Smith jumping at v_1 make it false that Smith dies at v_2, and so Beth and Bill disagree.

In defence of her view, Beth asserts U which requires the worlds that result from imposing a state of Smith jumping in α at v_1 to include the placement of a net at v_3 prior to the time v_1 of Smith's jump. Bill disagrees. Beth's claim U helps to bring the source of the disagreement to light: whereas Beth assumes that making minimal changes to the actual moment $\alpha(v_1)$ to include a state of Smith jumping will also include a net having been placed there at an earlier time v_3, Bill does not. What they disagree about is which possible worlds are the outcome of imposing a state of Smith jumping on the actual world at the time v_1 that Beth and Bill are considering. Perhaps Beth's optimism reflects the emphasis she places on Smith's love for live, assuming that more would have to change about the actual world at v_1 for Smith to willingly die than merely including a net there to catch him. Whereas jumping without a net is incompatible with Smith's love for life, jumping with a net is perfectly compatible. By contrast, Bill's pessimism may be attributed to his willingness to see Smith change his outlook on life, taking this to be a smaller change than including a net there beneath him.

Although the task semantics can help model our counterfactual claims, the semantics cannot help us make up our minds about which world states are the result of making minimal changes to an actual moment. In the disagreement between Bill and Beth, each believes different worlds are the outcomes of imposing a state of Smith jumping in the actual world α at v_1 when he was poised to do so. Nevertheless, the task semantics helps to shed light on the nature of backtracking counterfactuals. Even though backtracking counterfactuals like L are forward counterfactuals just as much as D, the proposed changes at the time of evaluation propagate backwards on account of requiring an altered past. Assuming that a net is present at v_1 requires that it was installed on the building at an earlier time v_3. By contrast, agreeing with Bill that Smith would have died only requires that Smith give up his love for life in the moment of jumping without requiring any substantial changes to the past. Given that the task semantics only selects outcome worlds by requiring them to occupy moments that differ minimally from the moment of evaluation, there is no requirement whatsoever that the worlds which result from imposing some state on the world of evaluation at a given time agree with the world of evaluation at earlier times. Indeed, small changes at the time of evaluation may require massive changes to the past leading up to the point of evaluation.

In order to emphasise this final point, I will conclude the present section with another backwards counterfactual along similar lines. To begin with, suppose that there are no advanced alien civilizations anywhere in the universe. Consider the following case:

> `Icosahedron:` Deep inside an Egyptian tomb, Harry discovered a perfectly formed icosahedron that appeared to be made out of pure titanium. In fact, the icosahedron was only a crude iron alloy. Relieved, Harry couldn't help but say out loud, "If the icosahedron had been titanium, then advanced aliens put it there."

Although the icosahedron is not made of titanium, supposing otherwise requires massive changes to the past. Even if aliens did not put it there, supposing that humans had developed the metallurgy needed to refine pure titanium at the time of the pyramids still requires a radical shift from the past of the actual world. Nevertheless, changing the moment of Harry's discovery to include a titanium icosahedron instead of the crude alloy that he found does not require much change at all. It is worth comparing such a case to Fine's Nixon example given above. Whereas $\texttt{N}'$ makes a small change to the moment of supposition that results in a drastically altered future, `Icosahedron` makes a small change to the moment of supposition which requires a drastically altered past. Despite these differences, the two cases are handled by the task semantics in the same way: any world which differs minimally from the actual world at the time of evaluation is to be considered, no matter how far that world may diverge in its past or future.

6 Conclusion

Instead of taking possible worlds to be primitive elements, the task semantics constructs possible worlds from states, parthood, tasks, and times. Given the internal structure provided by their definition, worlds may be compared at a time where the comparison may be carried out in purely mereological and modal terms. By contrast, Lewis compares worlds for similarity by considering worlds in their entirety and Fine posits a primitive imposition relation while restricting consideration to instantaneous world states. Although Fine's semantics is able to strengthen the logic that Lewis provides for counterfactuals by validating **SDA** without validating **STA**, taking the imposition relation to be primitive makes the semantics homophonic by including a counterfactual primitive in the metalanguage. Although a homophonic semantics may be tolerated when there is no better option, §2.2 show that it is unnecessary to take imposition to be primitive by providing its definition in purely modal and mereological terms. In addition to simplifying the semantics conceptually, defining the imposition relation expands the range of counterfactual inferences that the `anonymous` software can evaluate in addition to strengthening the evidence that the `anonymous` can provide in support of the validity of a given inference by surveying a broader range of models.

In addition to taking imposition to be primitive rather than defined, Fine does not provide the temporal structure needed to interpret tensed counterfactuals. By contrast, I have taken the task relation to be primitive, defining the possible states in its terms. Intuitively, there is a task $s \rightarrow t$ just in case the transition from the state s obtaining to the state t obtaining is possible. Whereas the imposition relation has a counterfactual reading, the task relation has a purely modal reading. Given a set of times, I show how to construct possible worlds, drawing on these resources in order to provide a semantics for tensed counterfactuals. Although focusing attention on an intended model may help to keep track of the working parts of the theory, nothing about the semantics turns on this reading or its supposed metaphysics. Nevertheless, a number of difficulties remain, inspiring further extensions of the present framework. Accordingly, I will conclude by briefly mentioning two loose ends in the following subsections.

6.1 Events

It is easy to consider states which exactly verify sentences like 'The icosahedron is pure titanium' or 'The coin is heads up'. Exact verifier states for such sentences are specific, perfectly static ways for some things to be. For instance, we may consider the precise arrangement of titanium atoms which make up the icosahedron, or the exact orientation of the coin relative to the table. By contrast, consider the sentence B from before:

(B) Nixon is pressing the button.

Here one might deny that there is an individual state of Nixon pressing the button. Rather, we may model Nixon's action by a sequence of states in which he slowly depresses the button and then pulls his finger back. Letting a *discrete process* $\bar{e} \coloneqq \langle e_1, \ldots, e_n \rangle$ be any sequence of states where $\bar{e}$ is *possible* just in case $e_i \to e_{i+1}$ for all $1 \leqslant i < n$, we may take the *event* of Nixon pressing the button to be the set $\overline{V}$ of processes which make the sentence B true together with the set $\overline{F}$ of processes which make B false. Given a counterfactual whose antecedent is an event, one might hope to impose the exact verifier processes in $\overline{V}$ on a world rather than any individual state.

Before attempting to provide a theory of process imposition, it is important to consider the merits of doing so. For instance, if the result is a satisfying metaphysics which nicely distinguishes events from static propositions but which does not otherwise effect the semantics for counterfactuals, no harm will come in taking at least some states to represent processes as a simplifying idealization. Nevertheless, it is natural to worry that conflating states with processes is incompatible with the intended interpretation guiding the construction of the semantics as well as the resulting principles. After all, it is not clear than any part of a possible process will be possible, where the analogous principle for states played an important role in the semantics above. Moreover, exact verifier processes may have different lengths and will not have determinate fusions on account of being fused in different ways. At the very least, considerably more would have to be said to provide a coherent process-theoretic analogue of the semantics above.

Instead of facing this challenge, one might take events to be a convenient proxy for related static propositions which might otherwise be hard to express. For instance, one might take any process of Nixon pressing the button $\bar{e}$ to include a critical state $\hat{e}$ in which the electrodes in the button connect, completing the circuit. More generally, a critical state in an exact verifier process is a *point of no return.* So long as every exact verifier process for an event has a critical state, that event may be said to be a proxy for a static proposition consisting only of the corresponding critical states. Nevertheless, it is far from obvious what it is to be a critical state in an exact falsifier process, nor is it clear that the critical states will be closed under fusion. Given my purposes here, I will leave the development of a theory of events and critical states for another time, idealizing processes as states for the purposes of the present semantics.

6.2 Continuous Time

Taking time to be discrete introduces a degree of arbitrariness for in order to define a world history we must first fix the unit. Whereas some histories string together moments separated by mere seconds, others may take moments to be separated by minutes or even hours. Instead of taking the choice of unit to be arbitrary, we may take this decision to be practical in nature, choosing a temporal resolution which is as low as possible while nevertheless preserving enough temporal features for the application at hand.

By contrast, taking time to be continuous requires amending the present theory. For instance, suppose that time has the structure of the real numbers $\mathbb{R}$. Given any state at a time, there is no next time which we may assign to an accessible state. As a result, the definition of an evolution cannot be maintained in its present form. Letting an evolution $\tau : \mathbb{R} \to S$ be *continuous* at x just in case there is some $y > 0$ where $\tau(x) \sim \tau(z)$ for all $z \in (x - y, x + y)$, we may define a *continuous evolution* to be any evolution $\tau : \mathbb{R} \to S$ that is continuous at all $x \in \mathbb{R}$. A *continuous world* is any continuous evolution α where $\alpha : \mathbb{R} \to W$. We may then replace **T5** with the following density axiom:

T5* $\boxed{\scriptstyle F}\boxed{\scriptstyle F}A \to \boxed{\scriptstyle F}A$.

Although amending the present framework to accommodate continuous time in this way may make for an interesting project in metaphysics, little is to be gained for the purposes of semantics. It is for this reason that I have assumed time to be discrete throughout.

7 *Appendix*

The following subsection will prove a number of theorems within **CLT**. I will then derive the constraints that Fine [4] assumes that the imposition relation and set of possible states satisfy in §7.2 and §7.3, presenting parts of the soundness proof for **CLT** in §7.4.

7.1 Theorems

D1 $\varphi \mathrel{\Box\!\!\rightarrow} A,\ \varphi \mathrel{\Box\!\!\rightarrow} B \vdash \varphi \mathrel{\Box\!\!\rightarrow} A \wedge B$.

Proof. Follows immediately from $A, B \vdash A \wedge B$ by **R1**. □

D2 *If* $A \vdash B$, *then* $\varphi \mathrel{\Box\!\!\rightarrow} A \vdash \varphi \mathrel{\Box\!\!\rightarrow} B$.

Proof. Immediate from **R1**. □

D3 *If* $\vdash A$, *then* $\vdash \Box A$.

Proof. Immediate from **R1**. □

D4 $\Box(A \to B) \to (\Box A \to \Box B)$

Proof. Immediate from **C2** by **R1**. □

D5 $\Box A \to A$

Proof. Since $\top, \top \mathrel{\Box\!\!\rightarrow} A \vdash A$ by **C2**, $\top \mathrel{\Box\!\!\rightarrow} A \vdash A$ by **M1**, and so $\Box A \to A$. □

D6 $\varphi \mathrel{\Box\!\!\rightarrow} A, \neg A \vdash \neg\varphi$.

Proof. Immediate since $\varphi \mathrel{\Box\!\!\rightarrow} A \vdash \varphi \rightarrow A$ follows from **C2** and $\varphi \rightarrow A, \neg A \vdash \neg\varphi$. □

D7 $\Box\varphi \leftrightarrow \neg\varphi \mathrel{\Box\!\!\rightarrow} \bot$.

Proof. Since $\top \mathrel{\Box\!\!\rightarrow} \varphi, \neg\varphi \vdash \neg\top$ by **D6** where $\neg\top \vdash \bot$, we know that $\top \mathrel{\Box\!\!\rightarrow} \varphi, \neg\varphi \vdash \bot$, and so $\top \mathrel{\Box\!\!\rightarrow} \varphi \vdash \neg\varphi \rightarrow \bot$. Equivalently, $\Box\varphi \vdash \neg\varphi \rightarrow \bot$, and so $\Box\Box\varphi \vdash \Box(\neg\varphi \rightarrow \bot)$ by **R1**. Thus $\Box\varphi \vdash \Box(\neg\varphi \rightarrow \bot)$ given **M4**, and so $\Box\varphi \vdash \neg\varphi \mathrel{\Box\!\!\rightarrow} \bot$ follows by **M5**.

Conversely, $\neg\varphi \mathrel{\Box\!\!\rightarrow} \bot, \neg\bot \vdash \neg\neg\varphi$ by **D6**, and so $\neg\varphi \mathrel{\Box\!\!\rightarrow} \bot \vdash \varphi$ given **M2** and $\neg\neg\varphi \vdash \varphi$. Thus $\neg\varphi \mathrel{\Box\!\!\rightarrow} \bot, \top \vdash \varphi$, and so $\neg\varphi \mathrel{\Box\!\!\rightarrow} \bot \vdash \top \rightarrow \varphi$, which is equivalent to $\neg\varphi \mathrel{\Box\!\!\rightarrow} \bot \vdash \Box\varphi$. Given the above, it follows that $\Box\varphi \leftrightarrow \neg\varphi \mathrel{\Box\!\!\rightarrow} \bot$. □

D8 $\Box A \leftrightarrow \neg\Diamond\neg A$

Proof. Since $A \vdash \neg\neg A$, we know that $\top \mathrel{\Box\!\!\rightarrow} A \vdash \top \mathrel{\Box\!\!\rightarrow} \neg\neg A$ by **R1**, and so it follows that $\top \mathrel{\Box\!\!\rightarrow} A \vdash \neg\neg(\top \mathrel{\Box\!\!\rightarrow} \neg\neg A)$. Equivalently, $\Box A \vdash \neg\Diamond\neg A$. Running the same reasoning in reverse yields $\neg\Diamond\neg A \vdash \Box A$ from which the theorem follows. □

D9 $\varphi \mathrel{\Box\!\!\rightarrow} \top$

Proof. Immediate from **M1** by **R1**. □

D10 $\bot \mathrel{\Box\!\!\rightarrow} A$

Proof. Since $\neg A \vdash \neg\bot$ given **M2**, we know that $\bot \rightarrow A$, and so $\bot \mathrel{\Box\!\!\rightarrow} A$ by **M5**. □

D11 $\Box A \rightarrow \triangledown A$

Proof. Since $\Box A$, $\Box\boxed{\scriptstyle F}A \leftrightarrow \Box A \vdash \Box\boxed{\scriptstyle F}A$, it follows from **T6** that $\Box A \vdash \Box\boxed{\scriptstyle F}A$ where $\Box\boxed{\scriptstyle F}A \vdash \boxed{\scriptstyle F}A$ by **D5**, and so $\Box A \vdash \boxed{\scriptstyle F}A$. Thus $\Box A \vdash \boxed{\scriptstyle P}A$ by **R3**, where $\Box A \vdash A$ again by **D5**. Given that $\boxed{\scriptstyle P}A$, A, $\boxed{\scriptstyle F}A \vdash \triangledown A$, it follows that $\Box A \vdash \triangledown A$, and so $\Box A \rightarrow \triangledown A$. □

7.2 Imposition

The proofs given in this section follow from the definition of imposition. For convenience, I have copied the relevant definitions below:

Compatible Part: $s \sqsubseteq_t w \coloneqq s \sqsubseteq w \wedge s \circ t$.

Maximal Compatible Parts: $w_t \coloneqq \{s \sqsubseteq_t w : \forall r \sqsubseteq_t w(s \sqsubseteq r \rightarrow r \sqsubseteq s)\}$.

Imposition: $t \rightarrowtriangle_w u \coloneqq u \in W \wedge \exists s \in w_t(s.t \sqsubseteq u)$.

World States: $W \coloneqq \{w \in P : \forall s \circ w(s \sqsubseteq w)\}$.

Instead of taking the imposition relation to be primitive and assuming the constraints that Fine [4] provides, the following results derive the constraints from the definition of the imposition relation given above. Since these proofs do not concern the task relation, we may take the set of possible states P to be primitive for present purposes, working over the modalized state spaces that Fine [6] introduces.

P1 (Inclusion) *If $t \rightarrowtriangle_w u$, then $t \sqsubseteq u$.*

Proof. Assuming $t \rightarrowtriangle_w u$, it follows that $u \in W$ where $s.t \sqsubseteq u$ for some $s \in w_t$. Since $t \sqsubseteq s.t$, it follows that $t \sqsubseteq u$ as desired. □

P2 (Actuality) *If $t \sqsubseteq w$ and $w \in W$, then $t \rightarrowtriangle_w u$ for some $u \sqsubseteq w$.*

Proof. Assume $t \sqsubseteq w$ where $w \in W$. Thus $w \in P$ where $w.t = w$, and so $w \circ t$. Since $w \sqsubseteq w$, we know that $w \sqsubseteq_t w$. Letting $r \sqsubseteq_t w$ where $w \sqsubseteq r$, it follows that $r \sqsubseteq w$, and so $w \in w_t$. Given that $w.t \sqsubseteq w$, we know that $t \rightarrowtriangle_w w$, and so $t \rightarrowtriangle_w u$ for some $u \sqsubseteq w$. □

P3 (Incorporation) *If $t \rightarrowtriangle_w u$ and $v \sqsubseteq u$, then $t.v \rightarrowtriangle_w u$.*

Proof. Assuming $t \rightarrowtriangle_w u$ and $v \sqsubseteq u$, it follows that $u \in W$ where $s.t \sqsubseteq u$ for some $s \in w_t$. Thus $s.t.v \sqsubseteq u$ and $s \sqsubseteq_t w$ where (1): $r \sqsubseteq s$ whenever $r \sqsubseteq_t w$ and $s \sqsubseteq r$. It follows that $s \sqsubseteq w$. Since $u \in P$, we also know that $s \circ t.v$, and so $s \sqsubseteq_{t.v} w$.

Letting $q \sqsubseteq_{t.v} w$ where $s \sqsubseteq q$, it follows that $q \sqsubseteq w$ where $q \circ t.v$, and so $q.t.v \in P$. Thus $q.t \in P$, and so $q \circ t$. It follows that $q \sqsubseteq_t w$, and so $q \sqsubseteq s$ follows from (1). Generalizing on q, it follows that (2): $q \sqsubseteq s$ whenever $q \sqsubseteq_{t.v} w$ and $s \sqsubseteq q$.

Having already shown that $s \sqsubseteq_{t.v} w$, it follows from (2) that $s \in w_{t.v}$. Since $s.t.v \sqsubseteq u$ for $u \in W$, we may conclude that $t.v \rightarrowtriangle_w u$ as desired. □

P4 (Completeness) *If $t \rightarrowtriangle_w u$, then u is a world-state.*

Proof. Immediate from the definition of *Imposition*. □

7.3 World Space

The proofs given below follow from the constraints on the task relation along with the definition of the set of possible states copied here for convenience:

Quasi-Reflexivity: If $s \to t$, then $s \to s$ and $t \to t$.

Parthood: If $d \sqsubseteq s$ and $s \to t$, then $d \to r$ for some $r \sqsubseteq t$.

Nullity: $\square = t$ just in case $\square \to t$ or $t \to \square$.

Maximality: If $s \in S$ and $t \in P$, there is some maximal t-compatible part $r \in s_t$.

Possibility: $P \coloneqq \{s \in S : s \to s\}$.

Given an arbitrary task space $\mathcal{T} = \langle S, \sqsubseteq, \to\rangle$, we may establish the following results.

P5 (Possibility) *If $s \in P$ and $t \sqsubseteq s$, then $t \in P$.*

Proof. Assuming $s \in P$ and $t \sqsubseteq s$, it follows that $s \to s$, and so $t \to r$ for some $r \sqsubseteq s$ by Parthood. Thus $t \to t$ by Quasi-Reflexivity, and so $t \in P$. □

P6 (Nonempty) $P \neq \varnothing$.

Proof. Since $\square = \square$, it follows that $\square \to \square$ by Nullity, and so $\square \in P$. Thus $P \neq \varnothing$ □

P7 (World Space) *If $t \in P$, then $t \sqsubseteq w$ for some $w \in W$.*

Proof. Let $t \in P$. Since $\blacksquare \in S$, Maximality requires there to be some $r \in \blacksquare_t$, and so $r \sqsubseteq_t \blacksquare$ where (1): $q \sqsubseteq r$ whenever $q \sqsubseteq_t \blacksquare$ and $r \sqsubseteq q$. Thus $r \circ t$, and so $r.t \in P$.

Letting $s = r.t$, it follows that $s.t \in P$, and so $s \circ t$. Since $s \sqsubseteq \blacksquare$ where $r \sqsubseteq s$, we know that $s \sqsubseteq_t \blacksquare$, and so $s \sqsubseteq r$ by (1). Hence $s = r$, and so $r = r.t$.

Let $k \in S$ where $k \circ r$. Thus $k.r \in P$, and so $k.r.t \in P$ given the above. It follows that $k.r \circ t$ where $k.r \sqsubseteq \blacksquare$, and so $k.r \sqsubseteq_t \blacksquare$. Since $r \sqsubseteq k.r$, it follows that $k.r \sqsubseteq r$ by (1), and so $k \sqsubseteq r$. Generalizing on k, we may conclude that $r \in W$ where $t \sqsubseteq r$. □

7.4 Soundness

Letting $\mathcal{M}$ be any model over a discrete task space $\mathcal{T}_{\mathbb{Z}} = \langle S, \sqsubseteq, \rightarrow, \mathbb{Z} \rangle$, I will assume the following semantic clauses where the others are standard, and so have been omitted:

$\mathcal{M}, \alpha, x \vDash q$ *iff* there is some $s \sqsubseteq \alpha(x)$ where $s \in |q|^+$.

$\mathcal{M}, \alpha, x \vDash \varphi \mathrel{\Box\!\!\rightarrow} C$ *iff* $\mathcal{M}, \beta, x \vDash C$ whenever $t \in |\varphi|^+$ and $t \rightarrowtriangle_{\alpha(x)} \beta(x)$.

For brevity, I will establish the validity of a collection of characteristic axiom schemata for **CTL** since the others are similar. In order to ease the proofs presented below, it will help to begin by proving a number of supporting lemmas and propositions.

L1 $\bigsqcup\{\bigsqcup E_i : i \in I\} = \bigsqcup\bigcup\{E_i : i \in I\}$ *where $E_i \subseteq S$ for all $i \in I$.*

Proof. Letting $e \in E_i$, it follows that $e \sqsubseteq \bigsqcup E_i$ where $\bigsqcup E_i \sqsubseteq \bigsqcup\{\bigsqcup E_i : i \in I\}$, and so $\bigsqcup\{\bigsqcup E_i : i \in I\}$ is an upper bound of $\bigcup\{E_i : i \in I\}$. By definition, we may conclude that $\bigsqcup\bigcup\{E_i : i \in I\} \sqsubseteq \bigsqcup\{\bigsqcup E_i : i \in I\}$. Since $E_i \subseteq \bigcup\{E_i : i \in I\}$ for any $i \in I$, it follows that $\bigsqcup E_i \sqsubseteq \bigsqcup\bigcup\{E_i : i \in I\}$ for all $i \in I$, and so $\bigsqcup\bigcup\{E_i : i \in I\}$ is an upper bound of the set $\{\bigsqcup E_i : i \in I\}$. Thus $\bigsqcup\{\bigsqcup E_i : i \in I\} \sqsubseteq \bigsqcup\bigcup\{E_i : i \in I\}$. □

L2 *If $X, Y \subseteq S$ are closed under nonempty fusion, then so are $X \otimes Y$ and $X \oplus Y$.*

Proof. Assume $X, Y \subseteq S$ are closed under nonempty fusion and $Z \subseteq X \otimes Y$ is nonempty. Let $Z = \{z_i : i \in I\}$ where $z_i = x_i.y_i$ for $x_i \in X$ and $y_i \in Y$ for each $i \in I$. Taking $Z_X = \{x_i : i \in I\}$ and $Z_Y = \{y_i : i \in I\}$, both $Z_X \subseteq X$ and $Z_Y \subseteq Y$ are nonempty, and so $\bigsqcup Z_X \in X$ and $\bigsqcup Z_Y \in Y$ by assumption. Thus $\bigsqcup\{\bigsqcup Z_X, \bigsqcup Z_Y\} \in X \otimes Y$ where:

$$\begin{aligned}
\bigsqcup\{\bigsqcup Z_X, \bigsqcup Z_Y\} &= \bigsqcup\{\bigsqcup\{x_i : i \in I\}, \bigsqcup\{y_i : i \in I\}\} \\
(*) \quad &= \bigsqcup\bigcup\{\{x_i : i \in I\}, \{y_i : i \in I\}\} \\
(*) \quad &= \bigsqcup\bigcup\{\{x_i, y_i\} : i \in I\} \\
&= \bigsqcup\{\bigsqcup\{x_i, y_i\} : i \in I\} \\
&= \bigsqcup\{z_i : i \in I\} \\
&= \bigsqcup Z.
\end{aligned}$$

The identities above are immediate with the exception of the starred lines which follow by **L1**. Thus $\bigsqcup Z \in X \otimes Y$, and so $X \otimes Y$ is closed under nonempty fusion.

Assume instead that $Z \subseteq X \oplus Y$ is nonempty. By letting $Z^X = Z \cap X$, $Z^Y = Z \cap Y$, and $Z^{X \otimes Y} = Z \cap (X \otimes Y)$, it follows that (1): $Z = Z^X \cup Z^Y \cup Z^{X \otimes Y}$. If $X = \varnothing$, then

$Z^X = Z^{X \otimes Y} = \varnothing$, and so $Z = Y$. By the same reasoning, $Z = X$ if $Y = \varnothing$. In either case, $\bigsqcup Z \in X \oplus Y$ since X and Y are both closed under nonempty fusion. Thus we may restrict attention to the case where both $X \neq \varnothing$ and $Y \neq \varnothing$.

Let $Z^{X \otimes Y} = \{z_j : j \in J\}$ where $z_j = x_j.y_j$ for $x_j \in X$ and $y_j \in Y$ for each $j \in J$. By setting $Z^{X \otimes Y}_X = \{x_j : j \in J\}$ and $Z^{X \otimes Y}_Y = \{y_j : j \in J\}$, we may observe as above that $\bigsqcup Z^{X \otimes Y}_X \in X$ and $\bigsqcup Z^{X \otimes Y}_Y \in Y$, and so $\bigsqcup\{\bigsqcup Z^{X \otimes Y}_X, \bigsqcup Z^{X \otimes Y}_Y\} \in X \otimes Y$. By the same reasoning above, $\bigsqcup\{\bigsqcup Z^{X \otimes Y}_X, \bigsqcup Z^{X \otimes Y}_Y\} = \bigsqcup Z^{X \otimes Y}$, and so $\bigsqcup Z^{X \otimes Y} \in X \otimes Y$.

Since $Z^X \subseteq X$ and $Z^Y \subseteq Y$ are both nonempty, it follows that $\bigsqcup Z^X \in X$ and $\bigsqcup Z^Y \in Y$ by assumption, and so $\bigsqcup\{\bigsqcup Z^X, \bigsqcup Z^Y\} \in X \otimes Y$. Having shown that $X \otimes Y$ is closed under nonempty fusion, $\bigsqcup\{\bigsqcup\{\bigsqcup Z^X, \bigsqcup Z^Y\}, \bigsqcup Z^{X \otimes Y}\} \in X \otimes Y$. Observe:

$$\begin{aligned} \bigsqcup\{\bigsqcup\{\bigsqcup Z^X, \bigsqcup Z^Y\}, \bigsqcup Z^{X \otimes Y}\} &= \bigsqcup\bigcup\{Z^X, Z^Y, Z^{X \otimes Y}\} \\ &= \bigsqcup Z. \end{aligned}$$

Whereas the first identity follows from **L1**, the second identity follows from (1) above. Thus we may conclude that $\bigsqcup Z \in X \otimes Y$, and so $\bigsqcup Z \in X \oplus Y$ since $X \otimes Y \subseteq X \oplus Y$. □

P8 *If $J, K \in \mathbb{P}$, then $\neg J, J \wedge K, J \vee K \in \mathbb{P}$.*

Proof. Assume $J, K \in \mathbb{P}$, and so $J = \langle J^+, J^- \rangle$ and $K = \langle K^+, K^- \rangle$ are both exclusive and exhaustive where $J^+, J-, K^+$, and $K-$ closed under nonempty fusion. By definition, $\neg J = \langle J^-, J^+ \rangle$, and so $\neg J$ satisfies the same properties. Thus $\neg J \in \mathbb{P}$.

In order to show that $J \wedge K \in \mathbb{P}$, we may recall that $J \wedge K = \langle J^+ \otimes K^+, J^- \oplus K^- \rangle$, assuming that $d \in J^+ \otimes K^+$ and $t \in J^- \oplus K^-$. It follows that $d = a.b$ where $a \in J^+$ and $b \in K^+$, and $t \in J^- \cup K^- \cup (J^- \otimes K^-)$. If $t \in J^-$, then $a.t \notin J$ by induction, and so $d.t \notin J$ by **P5** given that $a.t \sqsubseteq d.t$. By similar reasoning, $d.t \notin J$ if either $t \in K^-$ or $t \in J^- \otimes K^-$. Thus $J \wedge K$ satisfies exclusivity.

Assuming $s \in P$, it follows by **P7** that $s \sqsubseteq w$ for some $w \in W$, and so: (1) $w \in P$; and (2) $r \sqsubseteq w$ whenever $r \circ w$. By exhaustivity, there is some $j \in J^+ \cup J^-$ where $j \circ w$ and some $k \in K^+ \cup K^-$ where $k \circ w$, and so $j \sqsubseteq w$ and $k \sqsubseteq w$ by (2). Given that $s \sqsubseteq w$, we know that $s.j.k \sqsubseteq w$, and so $s.j, s.k, s.j.k \in P$ by **P5**. Thus $s \circ j$, $s \circ k$, and $s \circ j.k$. If $j \in J^-$ or $k \in K^-$, then there is some $m \in J^- \oplus K^-$ where $s \circ m$, and so some $m \in (J^+ \otimes K^+) \cup (J^- \oplus K^-)$ where $s \circ m$. If $j \notin J^-$ and $k \notin K^-$, it follows that $j \in J^+$ and $k \in K^+$, and so $j.k \in J^+ \otimes K^+$. Thus there is some $m \in (J^+ \otimes K^+) \cup (J^- \oplus K^-)$ where $s \circ m$. Hence $J \wedge K$ satisfies exhaustivity.

Given **L2**, it follows that $J \wedge K \in \mathbb{P}$. Since disjunction inverts the order of the product and sum of the exact verifiers and falsifiers of the disjuncts as compared with conjunction, analogous reasoning shows that $J \vee K \in \mathbb{P}$. □

P9 $|\varphi| \in \mathbb{P}$.

Proof. The proof goes by induction on the complexity of $\varphi \in \texttt{ext}(\mathcal{L})$ where the base case is given by the definition of a model. Assume for induction that $|\varphi| \in \mathbb{P}$ whenever $\texttt{comp}(\varphi) \leqslant n$, letting $\texttt{comp}(\varphi) = n + 1$.

Case 1: Assume $\varphi = \neg\psi$. Since $|\psi| \in \mathbb{P}$ by induction, $\neg|\psi| \in \mathbb{P}$ by **P8**. Given that $\neg|\psi| = |\neg\psi|$ by the exact inclusive semantics, $|\varphi| \in \mathbb{P}$ by the case assumption.

Case 2: Assume $\varphi = \psi \wedge \chi$. Since $|\psi|, |\chi| \in \mathbb{P}$ by induction, $|\psi| \wedge |\chi| \in \mathbb{P}$ by **P8**. Given that $|\psi| \wedge |\chi| = |\psi \wedge \chi|$ by the semantics, $|\varphi| \in \mathbb{P}$ by the case assumption.

Case 3: Similar to *Case 2*. □

P10 *$\mathcal{M}, \alpha, x \vDash \varphi$ just in case $s \in |\varphi|^+$ for some $s \sqsubseteq \alpha(x)$.*

Proof. The proof goes by induction on the complexity of $\varphi \in \texttt{ext}(\mathcal{L})$ where the base case is given by the semantics. Thus we may assume for induction that the proposition holds whenever $\texttt{comp}(\varphi) \leqslant n$, letting $\texttt{comp}(\varphi) = n + 1$.

Case 1: Assume $\varphi = \neg\psi$. The equivalences below follow from the case assumption, induction hypothesis, and semantics for negation with the exception of $(*)$:

$$\begin{aligned} \mathcal{M}, \alpha, x \vDash \varphi &\Leftrightarrow \mathcal{M}, \alpha, x \nvDash \psi \\ &\Leftrightarrow s \notin |\psi|^+ \text{ for any } s \sqsubseteq \alpha(x) \\ (*) &\Leftrightarrow t \in |\psi|^- \text{ for some } s \sqsubseteq \alpha(x) \\ &\Leftrightarrow t \in |\varphi|^+ \text{ for some } s \sqsubseteq \alpha(x). \end{aligned}$$

In support of $(*)$, we may begin by observing that $\alpha(x) \in W$, and so both: (1) $\alpha(x) \in P$; and (2) $r \sqsubseteq \alpha(x)$ whenever $r \circ \alpha(x)$. Additionally, $|\psi| \in \mathbb{P}$ follows by **P9**. By exhaustivity, it follows from (1) that there is some $t \in |\psi|^+ \cup |\psi|^-$ where $t \circ \alpha(x)$, and so $t \sqsubseteq \alpha(x)$ by (2). Given the $\Rightarrow$ assumption, we may conclude that $t \in |\psi|^-$ for some $t \sqsubseteq \alpha(x)$. For the $\Leftarrow$ direction, assume that there is some $t \in |\psi|^-$ where $s \sqsubseteq \alpha(x)$. Thus there cannot be any $s \in |\psi|^+$ where $s \sqsubseteq \alpha(x)$ since otherwise $s.t \sqsubseteq \alpha(x)$ where $s.t \notin P$ by exclusivity, and so $\alpha(x) \notin P$ by **P5**, thereby contradicting (1).

Case 2: Assume $\varphi = \psi \wedge \chi$. We may then observe the following:

$$\begin{aligned} \mathcal{M}, \alpha, x \vDash \varphi &\Leftrightarrow \mathcal{M}, \alpha, x \vDash \psi \text{ and } \mathcal{M}, \alpha, x \vDash \chi \\ &\Leftrightarrow d \in |\psi|^+ \text{ and } t \in |\chi|^+ \text{ for some } d, t \sqsubseteq \alpha(x) \\ &\Leftrightarrow s \in |\varphi|^+ \text{ for some } s \sqsubseteq \alpha(x). \end{aligned}$$

The equivalences given above follow immediately from the case assumption, induction hypothesis, and semantics for conjunction.

Case 3: Assume $\varphi = \psi \vee \chi$. We may then observe the following:

$$\begin{aligned} \mathcal{M}, \alpha, x \vDash \varphi &\Leftrightarrow \mathcal{M}, \alpha, x \vDash \psi \text{ or } \mathcal{M}, \alpha, x \vDash \chi \\ &\Leftrightarrow d \in |\psi|^+ \text{ for some } d \sqsubseteq \alpha(x) \text{ or } t \in |\chi|^+ \text{ for some } t \sqsubseteq \alpha(x) \\ &\Leftrightarrow s \in |\psi|^+ \cup |\chi|^+ \cup |\psi \wedge \chi|^+ \text{ for some } s \sqsubseteq \alpha(x) \\ &\Leftrightarrow s \in |\varphi|^+ \text{ for some } s \sqsubseteq \alpha(x). \end{aligned}$$

The equivalences given above follow immediately from the case assumption, induction hypothesis, and semantics for disjunction. □

R1 *If $\Gamma \vDash C$, then $\varphi \,\Box\!\!\rightarrow \Gamma \vDash \varphi \,\Box\!\!\rightarrow C$.*

Proof. Assume $\Gamma \vDash C$, letting $\mathcal{M}, \alpha, x \vDash \varphi \,\Box\!\!\rightarrow A_i$ for all $i \in I$ where $\Gamma = \{A_i : i \in I\}$. Given any $s \in |\varphi|^+$ and $\beta \in H_{\mathbb{Z}}$ where $s \rightarrow_{\alpha(x)} \beta(x)$, it follows by the semantics for counterfactuals that $\mathcal{M}, \beta, x \vDash A_i$ for all $i \in I$, and so $\mathcal{M}, \beta, x \vDash C$. □

L3 *If $t \circ w$, then $w_t = \{w\}$.*

Proof. Letting $t \circ w$, it follows that $w \sqsubseteq_t w$ since $w \sqsubseteq w$. Assuming $r \sqsubseteq_t w$ where $w \sqsubseteq r$, it follows that $r \sqsubseteq w$. By generalizing on r, we know that $w \in w_t$.

Let $u \in w_t$. So $u \sqsubseteq_t w$ where (1): $r \sqsubseteq u$ whenever $r \sqsubseteq_t w$ and $u \sqsubseteq r$. Since $w \sqsubseteq_t w$ and $u \sqsubseteq w$, it follows from (1) that $w \sqsubseteq u$, and so $w = u$. Hence $w_t = \{w\}$. □

L4 *If $s \sqsubseteq \alpha(x)$ for $\alpha \in H_{\mathbb{Z}}$ and $x \in \mathbb{Z}$, then $s \rightarrow_{\alpha(x)} \alpha(x)$.*

Proof. Let $t \sqsubseteq \alpha(x)$ where $\alpha \in H_{\mathbb{Z}}$ and $x \in \mathbb{Z}$. Thus $t.\alpha(x) \sqsubseteq \alpha(x)$ where $\alpha(x) \in W$. It follows by **L3** that $\alpha(x)_t = \{\alpha(x)\}$. Since there is some $s \in \alpha(x)_t$ where $s.t \sqsubseteq \alpha(x)$, namely where $s = \alpha(x)$, we may conclude that $t \rightarrow_{\alpha(x)} \alpha(x)$. □

C2 $\varphi,\ \varphi \mathrel{\Box\!\!\rightarrow} A \vDash A$.

Proof. Let $\mathcal{M}, \alpha, x \vDash \varphi$ and $\mathcal{M}, \alpha, x \vDash \varphi \mathrel{\Box\!\!\rightarrow} A$. By **P10**, $s \in |\varphi|^+$ for some $s \sqsubseteq \alpha(x)$, and so $s \rightarrow_{\alpha(x)} \alpha(x)$ by **L4**. By the semantics for counterfactuals, $\mathcal{M}, \alpha, x \vDash A$. □

C3 $\varphi \mathrel{\Box\!\!\rightarrow} \psi,\ \varphi \wedge \psi \mathrel{\Box\!\!\rightarrow} A \vDash \varphi \mathrel{\Box\!\!\rightarrow} A$.

Proof. Let $\mathcal{M}, \alpha, x \vDash \varphi \mathrel{\Box\!\!\rightarrow} \psi$ and $\mathcal{M}, \alpha, x \vDash \varphi \wedge \psi \mathrel{\Box\!\!\rightarrow} A$. Assuming $s \in |\varphi|^+$ and $\beta \in H_{\mathbb{Z}}$ where $s \rightarrow_{\alpha(x)} \beta(x)$, it follows by the semantics for counterfactuals that $\mathcal{M}, \beta, x \vDash \psi$, and so $t \in |\psi|^+$ for some $t \sqsubseteq \beta(x)$ by **P10**. Since $s \sqsubseteq \beta(x)$ by **P1**, we know that $s.t \sqsubseteq \beta(x)$ where $s.t \in |\varphi \wedge \psi|^+$. Given that $s.t \rightarrow_{\alpha(x)} \beta(x)$ by **P3**, it follows that $\mathcal{M}, \alpha, x \vDash A$ again by the semantics for counterfactuals, and so $\mathcal{M}, \alpha, x \vDash \varphi \mathrel{\Box\!\!\rightarrow} A$. □

C5 $\varphi \vee \psi \mathrel{\Box\!\!\rightarrow} A \vDash \varphi \mathrel{\Box\!\!\rightarrow} A$.

Proof. Let $\mathcal{M}, \alpha, x \vDash \varphi \vee \psi \mathrel{\Box\!\!\rightarrow} A$ where $s \in |\varphi|^+$ and $\beta \in H_{\mathbb{Z}}$ where $s \rightarrow_{\alpha(x)} \beta(x)$. It follows that $s \in |\varphi \vee \psi|^+$, and so $\mathcal{M}, \beta, x \vDash A$ by the semantics for the counterfactual. Thus we may conclude that $\mathcal{M}, \alpha, x \vDash \varphi \mathrel{\Box\!\!\rightarrow} A$. □

L5 *If $w \in W$, then $w \rightarrow_s w$ for any $s \in S$.*

Proof. Assume $w \in W$ and let $s \in S$. It follows that (1): $q \sqsubseteq w$ whenever $q \circ w$. Since $w \in P$, there is some $r \in s_w$ by Maximality. By definition, $r \sqsubseteq s$ where $r \circ w$, and so $r \sqsubseteq w$ by (1). Thus we may conclude that $r.w \sqsubseteq w$, and so $w \rightarrow_s w$. □

M3 $\vDash A \rightarrow \Box\Diamond A$.

Proof. Assume for contradiction that $\mathcal{M}, \alpha, x \nvDash A \rightarrow \Box\Diamond A$. It follows that $\mathcal{M}, \alpha, x \vDash A$ and $\mathcal{M}, \alpha, x \nvDash \Box\Diamond A$, and so $\mathcal{M}, \alpha, x \nvDash \top \mathrel{\Box\!\!\rightarrow} \Diamond A$. Thus $\mathcal{M}, \beta, x \nvDash \Diamond A$ for some $\beta \in H_{\mathbb{Z}}$ and $s \in |\top|^+$ where $s \rightarrow_{\alpha(x)} \beta(x)$. It follows that $\mathcal{M}, \beta, x \vDash \top \mathrel{\Box\!\!\rightarrow} \neg A$, and so $\mathcal{M}, \beta, x \nvDash \top \mathrel{\Box\!\!\rightarrow} A$. Since $\alpha(x) \in |\top|^+$, it follows by the semantics for counterfactuals that $\mathcal{M}, \gamma, x \nvDash A$ for any $\gamma \in H_{\mathbb{Z}}$ where $\alpha(x) \rightarrow_{\beta(x)} \gamma(x)$. By **L5**, $\alpha(x) \rightarrow_{\beta(x)} \alpha(x)$, and so $\mathcal{M}, \alpha, x \nvDash A$ in particular, thereby contradicting the above. □

M4 $\vDash \Box A \rightarrow \Box\Box A$.

Proof. Assuming for contradiction that $\mathcal{M}, \alpha, x \nvDash \Box A \rightarrow \Box\Box A$, both $\mathcal{M}, \alpha, x \vDash \Box A$ and $\mathcal{M}, \alpha, x \nvDash \Box\Box A$, and so: (1) $\mathcal{M}, \alpha, x \vDash \top \mathrel{\Box\!\!\rightarrow} A$; and (2) $\mathcal{M}, \alpha, x \nvDash \top \mathrel{\Box\!\!\rightarrow} (\top \mathrel{\Box\!\!\rightarrow} A)$. By (2), $\mathcal{M}, \beta, x \nvDash \top \mathrel{\Box\!\!\rightarrow} A$ for some $\beta \in H_{\mathbb{Z}}$ and $s \in |\top|^+$ where $s \rightarrow_{\alpha(x)} \beta(x)$, and so $\mathcal{M}, \gamma, x \nvDash A$ for some $\gamma \in H_{\mathbb{Z}}$ and $t \in |\top|^+$ where $t \rightarrow_{\beta(x)} \gamma(x)$. However, it follows from (1) that $\mathcal{M}, \delta, x \vDash A$ for any $\delta \in H_{\mathbb{Z}}$ and $r \in |\top|^+$ where $r \rightarrow_{\alpha(x)} \delta(x)$. Since $\gamma(x) \in |\top|^+$ where $\gamma(x) \rightarrow_{\alpha(x)} \gamma(x)$ by **L5**, it follows that $\mathcal{M}, \gamma, x \vDash A$, contradicting the above. □

M5 $\models \Box(\varphi \rightarrow A) \rightarrow (\varphi \mathrel{\Box\!\!\rightarrow} A)$.

Proof. Assume for contradiction that $\mathcal{M}, \alpha, x \not\models \Box(\varphi \rightarrow A) \rightarrow (\varphi \mathrel{\Box\!\!\rightarrow} A)$. It follows that (1) $\mathcal{M}, \alpha, x \models \top \mathrel{\Box\!\!\rightarrow} (\varphi \rightarrow A)$ and (2) $\mathcal{M}, \alpha, x \not\models \varphi \mathrel{\Box\!\!\rightarrow} A$, and so $\mathcal{M}, \beta, x \not\models A$ for some $\beta \in H_{\mathbb{Z}}$ and $s \in |\varphi|^+$ where $s \rightarrow_{\alpha(x)} \beta(x)$ by the latter. Since $s \sqsubseteq \beta(x)$ by **P1**, we know that $\mathcal{M}, \beta, x \models \varphi$ by **P10**, and so $\mathcal{M}, \beta, x \not\models \varphi \rightarrow A$ by the semantics. However, given (1), $\mathcal{M}, \gamma, x \models \varphi \rightarrow A$ for any $\gamma \in H_{\mathbb{Z}}$ and $r \in |\top|^+$ where $r \rightarrow_{\alpha(x)} \gamma(x)$. Since $\beta(x) \in |\top|^+$ where $\beta(x) \rightarrow_{\alpha(x)} \beta(x)$ by **L5**, it follows that $\mathcal{M}, \beta, x \models \varphi \rightarrow A$, thereby contradicting the above. $\square$

T6 $\models \Box\boxed{\scriptstyle F}A \leftrightarrow \Box A$.

Proof. Assume $\mathcal{M}, \alpha, x \not\models \Box\boxed{\scriptstyle F}A$, it follows that $\mathcal{M}, \beta, x \not\models \boxed{\scriptstyle F}A$ for some $\beta \in H_{\mathbb{Z}}$ and $s \in |\top|^+$ where $s \rightarrow_{\alpha(x)} \beta(x)$, and so $\mathcal{M}, \beta, y \not\models A$ for some $y > x$. Since $\beta(y) \in |\top|^+$ where $\beta(y) \rightarrow_{\alpha(x)} \beta(y)$ by **L5**, it follows that $\mathcal{M}, \alpha, x \not\models \Box A$.

Conversely, assuming $\mathcal{M}, \alpha, x \not\models \Box A$, it follows that $\mathcal{M}, \beta, x \not\models A$ for some $\beta \in H_{\mathbb{Z}}$ and $s \in |A|^+$ where $s \rightarrow_{\alpha(x)} \beta(x)$. Let $\gamma(z) = \beta(z - 1)$ for all $z \in \mathbb{Z}$. It follows that $\gamma \in H_{\mathbb{Z}}$ where $\mathcal{M}, \gamma, x \not\models \boxed{\scriptstyle F}A$ since $\mathcal{M}, \gamma, x + 1 \not\models A$ and $x + 1 > x$. Since $\gamma(x) \in |\top|^+$ where $\gamma(x) \rightarrow_{\alpha(x)} \gamma(x)$ by **L5**, it follows that $\mathcal{M}, \alpha, x \not\models \Box\boxed{\scriptstyle F}A$.

It follows that $\mathcal{M}, \alpha, x \not\models \Box\boxed{\scriptstyle F}A$ just in case $\mathcal{M}, \alpha, x \not\models \Box A$, and so equivalently $\mathcal{M}, \alpha, x \models \Box\boxed{\scriptstyle F}A$ just in case $\mathcal{M}, \alpha, x \models \Box A$. Thus $\models \Box\boxed{\scriptstyle F}A \leftrightarrow \Box A$. $\square$

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