Programmatic Truthmaker Semantics

ADVANCES IN TRUTHMAKER SEMANTICS: II

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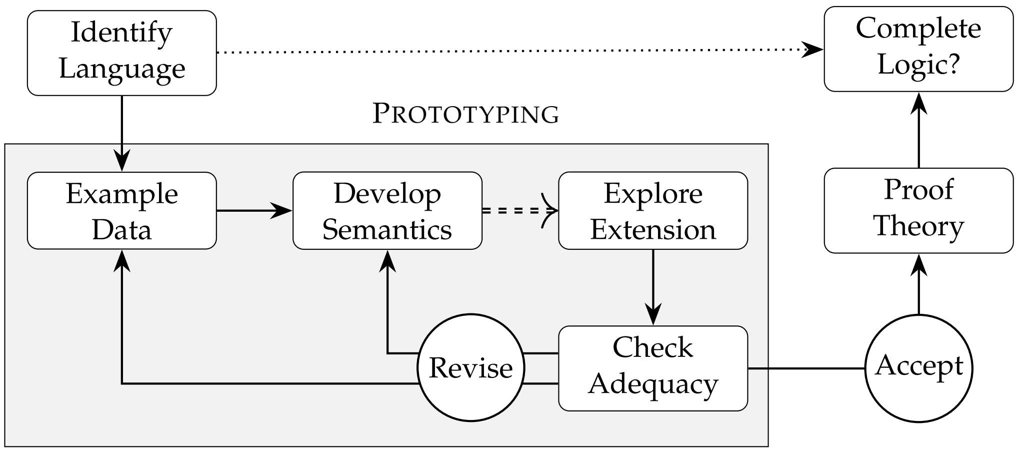
Broad Ambitions

Extend the standard methodology in semantics to:

- Rapidly prototype semantic theories by reducing cognitive load
- Facilitate collaboration and increase accessibility
- Support the maturity of the discipline

"Standard Methodology"



Difficulties

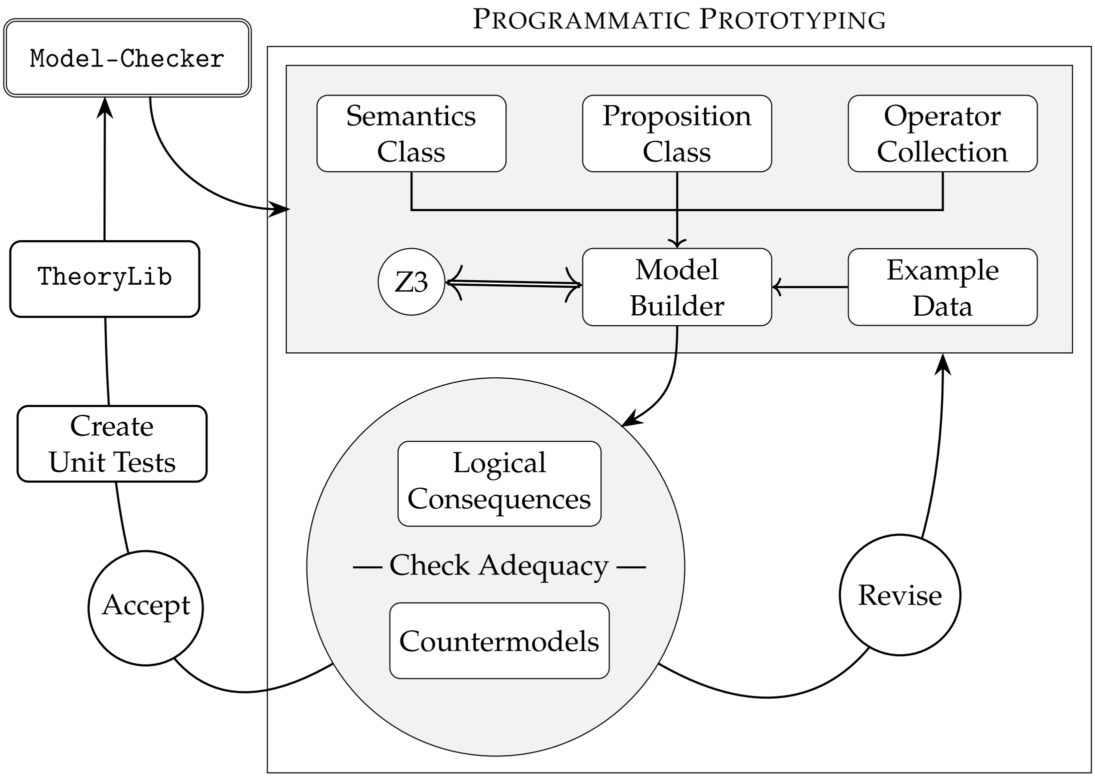
The standard methodology has the following drawbacks:

- Computationally grueling to prototype semantic theories
- Problems of accuracy, redundancy, and memory
- Limits the development of complex semantic theories
- Restricts which language fragments can be studied/combined

An Extended Methodology

Humans should not be carrying the computational load.

- SAT solvers, SMT solvers, Z3
- Examples: inequalities, bitvectors as states
- Z3 constraints as truth-conditions



Conceptual Engineering

This methodology has the following advantages:

- Efficiently prototype new semantic theories
- Modular semantics, theory of propositions, operators
- Evaluate unified languages with many operators
- Compare rival theories over large data sets

Give it a try at: `https://pypi.org/project/model-checker/`

Bilateral Propositions

Following Fine (2017a,b,c), a *modalized state space* is any $\mathcal{S}^{\Diamond} = \langle S, P, \sqsubseteq \rangle$ where $\langle S, \sqsubseteq \rangle$ is a complete lattice of *states*, $P \subseteq S$ is a nonempty subset of *possible states*, and $\sqsubseteq$ is the *parthood relation* satisfying the following constraints:

NONEMPTY: $P \neq \varnothing$.

POSSIBILITY: If $s \in P$ and $t \sqsubseteq s$, then $t \in P$.

The world states may then be defined, and a further constraint imposed:

Compatible: $s \circ t := s.t \in P$.

World States: $W := \{w \in P \mid \forall s \circ w(s \sqsubseteq w)\}$.

WORLD SPACE: If $s \in P$, then $s \sqsubseteq w$ for some $w \in W$.

A *bilateral proposition* is any ordered tuple $\langle V, F \rangle \in \mathbb{P}$ where

Closure: $V, F \subseteq S$ are each closed under nonempty fusion.

Exclusive: The states in V are incompatible with the states in F.

Exhaustive: Every possible state in P is compatible with a state in V or F.

A *model* $\mathcal{M} = \langle S, P, \sqsubseteq, |\cdot| \rangle$ of $\mathcal{L}$ assigns each $|p_i| = \langle |p_i|^+, |p_i|^- \rangle \in \mathbb{P}$.

- $|\neg A| = \langle |A|^-, |A|^+ \rangle$.
- $|A \wedge B| = \langle |A|^+ \otimes |B|^+, |A|^- \oplus |B|^- \rangle$ where $X \otimes Y := \{s.t \mid s \in X,\ t \in Y\}$.
- $|A \vee B| = \langle |A|^+ \oplus |B|^+, |A|^- \otimes |B|^- \rangle$ where $X \oplus Y := X \cup Y \cup (X \otimes Y)$.

Minimal Countermodels

Fine (2012) originally introduced a primitive *imposition relation* $t \rightarrowtriangle_w u$ which indicates that "u is a possible outcome of imposing the change t on the world [state] w", and is subject to the following frame constraints on imposition:

INCLUSION: If $t \rightarrowtriangle_w u$, then $t \sqsubseteq u$.

ACTUALITY: If $t \sqsubseteq w$, then $t \rightarrowtriangle_w u$ for some $u \sqsubseteq w$.

INCORPORATION: If $t \rightarrowtriangle_w u$ and $v \sqsubseteq u$, then $t.v \rightarrowtriangle_w u$.

COMPLETENESS: If $t \rightarrowtriangle_w u$, then u is a world-state.

An abridged semantics for $\mathcal{L} = \langle \mathbb{L}, \neg, \wedge, \vee, \boxright \rangle$ may then be stated as:

- $\mathcal{M}, w \vDash p_i$ *iff* $s \in |p_i|^+$ for some $s \sqsubseteq w$.
- $\mathcal{M}, w \vDash A \boxright C$ *iff* $\mathcal{M}, u \vDash C$ whenever $t \in |A|^+$ and $t \rightarrowtriangle_w u$.

Defining Imposition

Definition: The frame constraints admit exceptions to $\Box A \coloneqq \top \,\Box\!\!\rightarrow A$.

State Space: $P = \{\Box, a, b, c, b.c\}, \quad W = \{a, b.c\}, \quad S/P = \{a.b, a.c, a.b.c\}$

Imposition: $\rightarrow\!\!\triangleright = \{\, \langle a, a, a\rangle, \langle b, b.c, b.c\rangle, \langle c, b.c, b.c\rangle,$
$\langle \Box, a, a\rangle, \langle \Box, b.c, b.c\rangle, \langle \Box, b.c, b.c\rangle \,\}$

Interpretation: $|A| = \langle \{a\}, \{b.c\}\rangle$

Premise: $\mathcal{M}, a \models \top \,\Box\!\!\rightarrow A$ since the set of $\top$-alternatives to $a = \{a\}$.

Conclusion: $\mathcal{M}, a \not\models \Box A$ since $\mathcal{M}, b.c \not\models A$.

The definition $\Box A \coloneqq \top \,\Box\!\!\rightarrow A$ is preserved by the following definition of $\rightarrow\!\!\triangleright$, where '$s \sqsubseteq_t w$' reads '$s$ is a *t-compatible part* of w':

Compatible Part: $s \sqsubseteq_t w \coloneqq s \sqsubseteq w \wedge s \circ t.$

Maximal Compatible Parts: $w_t \coloneqq \{s \sqsubseteq_t w \mid \forall r \sqsubseteq_t w(s \sqsubseteq r \rightarrow r = s)\}.$

Imposition: $t \rightarrow\!\!\triangleright_w u \coloneqq u \in W \wedge \exists s \in w_t(s.t \sqsubseteq u).$

We may then derive rather than posit the frame constraints on imposition, making the logic for $\Box\!\!\rightarrow$ both stronger and more computable.

Computational Complexity as a Theoretical Virtue

1. Z3 saves its 'Function' objects as a mix of array-like and lambda-like objects.
2. This means Z3 saves every value (that it is forced to for a given countermodel) for every input combination, meaning that the (worst-case) space complexity of functions is proportional to the input space.
3. Defining computational complexity: an algorithm takes $O(n)$ runtime or space if it scales linearly with some quantity n as n grows indefinitely large.
4. With inputs in $A \times B$, A being the space of atomic sentences and B the space of bitvectors, 'verify' has a worst-case complexity of $O(|A||B|) = O(|A|2^N)$, N the size of the bitvectors.
5. With inputs in B^3, 'imposition' has a complexity of $O(2^{3N})$.
6. In practice, this means much slower runtimes for the imposition semantics: imposition semantics takes about 10 times as long as logos to run for $N = 4$.

7. Since the complexity of 'imposition' is exponential with N, this is only more marked for larger values of N, which can be useful for finding easily interpretable models.
8. As a result, we now have methodological reason to favor theories that keep the arity of primitives low.
9. This reasoning is not too different from familiar questions of theoretical simplicity: this is just a notion of simplicity with regards to the computer

Proofs

P1 (INCLUSION) *If $t \rightarrowtriangle_w u$, then $t \sqsubseteq u$.*

Proof. Assuming $t \rightarrowtriangle_w u$, it follows that $u \in W$ where $s.t \sqsubseteq u$ for some $s \in w_t$. Since $t \sqsubseteq s.t$, it follows that $t \sqsubseteq u$ as desired. □

P2 (ACTUALITY) *If $t \sqsubseteq w$ and $w \in W$, then $t \rightarrowtriangle_w u$ for some $u \sqsubseteq w$.*

Proof. Assume $t \sqsubseteq w$ for $w \in W$. Thus $w \in P$ where $w.t = w$, and so $w \circ t$. Since $w \sqsubseteq w$, we know $w \sqsubseteq_t w$. Let$r \sqsubseteq_t w$ where $w \sqsubseteq r$, and so $r \sqsubseteq w$, and so $w \in w_t$. Since $w.t \sqsubseteq w$, we know $t \rightarrowtriangle_w w$, and so $t \rightarrowtriangle_w u$ for some $u \sqsubseteq w$. □

P3 (INCORPORATION) *If $t \rightarrowtriangle_w u$ and $v \sqsubseteq u$, then $t.v \rightarrowtriangle_w u$.*

Proof. Assuming $t \rightarrowtriangle_w u$ and $v \sqsubseteq u$, we know $u \in W$ where $s.t \sqsubseteq u$ for some $s \in w_t$. Thus $s.t.v \sqsubseteq u$ and $s \sqsubseteq_t w$ where (1): $r \sqsubseteq s$ whenever $r \sqsubseteq_t w$ and $s \sqsubseteq r$. So $s \sqsubseteq w$. Since $u \in P$, we also know that $s \circ t.v$, and so $s \sqsubseteq_{t.v} w$.

Let $q \sqsubseteq_{t.v} w$ where $s \sqsubseteq q$, and so $q \sqsubseteq w$ where $q \circ t.v$, and so $q.t.v \in P$. Thus $q.t \in P$, and so $q \circ t$. It follows that $q \sqsubseteq_t w$, and so $q \sqsubseteq s$ follows from (1). Generalizing on q, we know (2): $q \sqsubseteq s$ whenever $q \sqsubseteq_{t.v} w$ and $s \sqsubseteq q$.

Having already shown that $s \sqsubseteq_{t.v} w$, it follows from (2) that $s \in w_{t.v}$. Since $s.t.v \sqsubseteq u$ for $u \in W$, we may conclude that $t.v \rightarrowtriangle_w u$ as desired. □