Programmatic Truthmaker Semantics

ADVANCES IN TRUTHMAKER SEMANTICS: II

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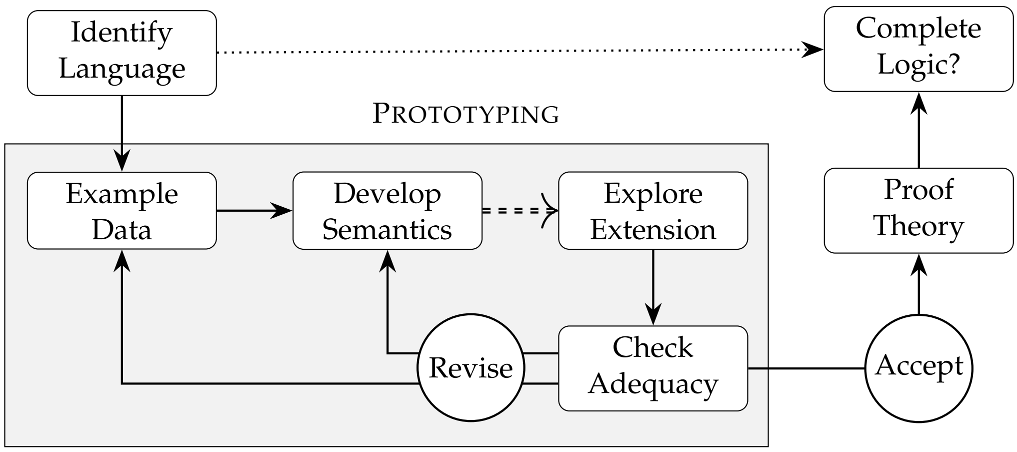
Broad Ambitions

Extend the standard methodology in semantics to:

- Prototype semantic theories by shifting computational load
- Develop semantic theories for languages with many operators
- Facilitate collaboration and increase accessibility

"Standard Methodology"



Difficulties

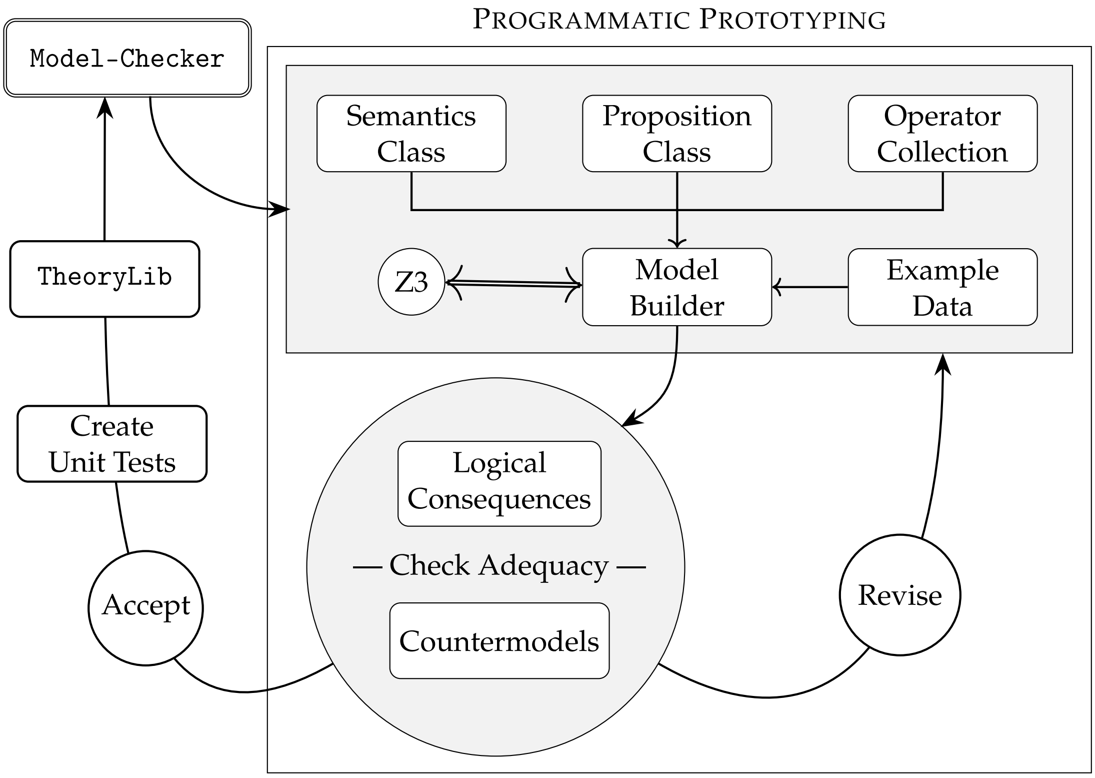
The standard methodology has the following drawbacks:

- Computationally grueling to prototype semantic theories
- Problems of accuracy, redundancy, and memory
- Limits the development of complex semantic theories
- Restricts which language fragments can be studied/combined

An Extended Methodology

Humans should not be carrying the computational load.

- SAT solvers, SMT solvers, Z3
- Examples: inequalities, bitvectors as states
- Z3 constraints as truth-conditions



Conceptual Engineering

This methodology has the following advantages:

- Efficiently prototype new semantic theories
- Modular semantics, theory of propositions, operators
- Evaluate unified languages with many operators
- Compare different theories over large data sets

Give it a try at: `https://pypi.org/project/model-checker/`

Computational Complexity as a Theoretical Virtue

1. Z3 builds models with a mix of array-like and lambda-like objects.
2. This means Z3 saves every value (that it is forced to for a given countermodel) for every input combination, meaning that the (worst-case) space complexity of functions is proportional to the input space.
3. Computational complexity: an algorithm takes $O(n)$ runtime or space if it scales linearly with some input quantity n as n grows indefinitely large.
4. For our purposes, runtime complexity $\approx$ space complexity: larger space complexity means larger runtime complexity.
5. Definitions: $\mathbb{L}$ is the set of atomic sentences, and $\mathbb{B}$ is the set of all bitvectors.
6. `verify` : $\mathbb{L} \times \mathbb{B} \to \{0, 1\}$. (Worst-case) space complexity: $O(|\mathbb{L}| \cdot |\mathbb{B}|) = O(|\mathbb{L}| \cdot 2^N)$, N the size of the bitvectors.
7. `imposition` : $\mathbb{B}^3 \to \{0, 1\}$. (Worst-case) space complexity: $O(|\mathbb{B}|^3) = O(2^{3N})$.
8. Practically speaking: imposition theory approx. 10 times slower (i.e., takes 10 times as long to find countermodels or exhaust search space) than logos with $N = 4$, getting worse as N increases.

Bilateral Propositions

Following **???**, a *modalized state space* is any $\mathcal{S}^\Diamond = \langle S, P, \sqsubseteq \rangle$ where $\langle S, \sqsubseteq \rangle$ is a complete lattice of *states*, $P \subseteq S$ is a nonempty subset of *possible states*, and $\sqsubseteq$ is the *parthood relation* satisfying the following constraints:

NONEMPTY: $P \neq \varnothing$.

POSSIBILITY: If $s \in P$ and $t \sqsubseteq s$, then $t \in P$.

The world states may then be defined, and a further constraint imposed:

Compatible: $s \circ t \coloneqq s.t \in P$.

World States: $W \coloneqq \{w \in P \mid \forall s \circ w(s \sqsubseteq w)\}$.

WORLD SPACE: If $s \in P$, then $s \sqsubseteq w$ for some $w \in W$.

A *bilateral proposition* is any ordered tuple $\langle V, F \rangle \in \mathbb{P}$ where

Closure: $V, F \subseteq S$ are each closed under nonempty fusion.

Exclusive: The states in V are incompatible with the states in F.

Exhaustive: Every possible state in P is compatible with a state in V or F.

A *model* $\mathcal{M} = \langle S, P, \sqsubseteq, |\cdot| \rangle$ of $\mathcal{L}$ assigns each $|p_i| = \langle |p_i|^+, |p_i|^- \rangle \in \mathbb{P}$.

- $|\neg A| = \langle |A|^-, |A|^+ \rangle$.
- $|A \wedge B| = \langle |A|^+ \otimes |B|^+, |A|^- \oplus |B|^- \rangle$ where $X \otimes Y \coloneqq \{s.t \mid s \in X,\ t \in Y\}$.
- $|A \vee B| = \langle |A|^+ \oplus |B|^+, |A|^- \otimes |B|^- \rangle$ where $X \oplus Y \coloneqq X \cup Y \cup (X \otimes Y)$.

Minimal Countermodels

? originally introduced a primitive *imposition relation* $t \rightarrowtriangle_w u$ which indicates that "u is a possible outcome of imposing the change t on the world [state] w", and is subject to the following frame constraints on imposition:

INCLUSION: If $t \rightarrowtriangle_w u$, then $t \sqsubseteq u$.

ACTUALITY: If $t \sqsubseteq w$, then $t \rightarrowtriangle_w u$ for some $u \sqsubseteq w$.

INCORPORATION: If $t \rightarrowtriangle_w u$ and $v \sqsubseteq u$, then $t.v \rightarrowtriangle_w u$.

COMPLETENESS: If $t \rightarrowtriangle_w u$, then u is a world-state.

An abridged semantics for $\mathcal{L} = \langle \mathbb{L}, \neg, \wedge, \vee, \boxright \rangle$ may then be stated as:

- $\mathcal{M}, w \vDash p_i$ *iff* $s \in |p_i|^+$ for some $s \sqsubseteq w$.
- $\mathcal{M}, w \vDash A \boxright C$ *iff* $\mathcal{M}, u \vDash C$ whenever $t \in |A|^+$ and $t \rightarrowtriangle_w u$.

Defining Imposition

Definition: The frame constraints admit exceptions to $\Box A \coloneqq \top \boxright A$.

State Space: $P = \{\square,\ a,\ b,\ c,\ b.c\}, \quad W = \{a,\ b.c\}, \quad S/P = \{a.b,\ a.c,\ a.b.c\}$

Imposition: $\rightarrowtriangle\ = \{\ \langle a,\ a,\ a\rangle,\ \langle b,\ b.c,\ b.c\rangle,\ \langle c,\ b.c,\ b.c\rangle,$
$\langle \square,\ a,\ a\rangle,\ \langle \square,\ b.c,\ b.c\rangle,\ \langle \square,\ b.c,\ b.c\rangle\ \}$

Interpretation: $|A| = \langle \{a\},\ \{b.c\} \rangle$

Premise: $\mathcal{M}, a \vDash \top \boxright A$ since the set of $\top$-alternatives to $a = \{a\}$.

Conclusion: $\mathcal{M}, a \nvDash \Box A$ since $\mathcal{M}, b.c \nvDash A$.

The definition $\Box A \coloneqq \top \boxright A$ is preserved by the following definition of $\rightarrowtriangle$, where '$s \sqsubseteq_t w$' reads 's is a *t-compatible part* of w':

Compatible Part: $s \sqsubseteq_t w \coloneqq s \sqsubseteq w \wedge s \circ t$.

Maximal Compatible Parts: $w_t \coloneqq \{s \sqsubseteq_t w \mid \forall r \sqsubseteq_t w (s \sqsubseteq r \rightarrow r = s)\}$.

Imposition: $t \rightarrowtriangle_w u \coloneqq u \in W \wedge \exists s \in w_t (s.t \sqsubseteq u)$.

We may then derive rather than posit the frame constraints on imposition, making the logic for $\boxright$ both stronger and more computable.

Proofs

P1 (Inclusion) *If $t \mathrel{-\!\!\triangleright}_w u$, then $t \sqsubseteq u$.*

Proof. Assuming $t \mathrel{-\!\!\triangleright}_w u$, it follows that $u \in W$ where $s.t \sqsubseteq u$ for some $s \in w_t$. Since $t \sqsubseteq s.t$, it follows that $t \sqsubseteq u$ as desired. $\square$

P2 (Actuality) *If $t \sqsubseteq w$ and $w \in W$, then $t \mathrel{-\!\!\triangleright}_w u$ for some $u \sqsubseteq w$.*

Proof. Assume $t \sqsubseteq w$ for $w \in W$. Thus $w \in P$ where $w.t = w$, and so $w \circ t$. Since $w \sqsubseteq w$, we know $w \sqsubseteq_t w$. Let$r \sqsubseteq_t w$ where $w \sqsubseteq r$, and so $r \sqsubseteq w$, and so $w \in w_t$. Since $w.t \sqsubseteq w$, we know $t \mathrel{-\!\!\triangleright}_w w$, and so $t \mathrel{-\!\!\triangleright}_w u$ for some $u \sqsubseteq w$. $\square$

P3 (Incorporation) *If $t \mathrel{-\!\!\triangleright}_w u$ and $v \sqsubseteq u$, then $t.v \mathrel{-\!\!\triangleright}_w u$.*

Proof. Assuming $t \mathrel{-\!\!\triangleright}_w u$ and $v \sqsubseteq u$, we know $u \in W$ where $s.t \sqsubseteq u$ for some $s \in w_t$. Thus $s.t.v \sqsubseteq u$ and $s \sqsubseteq_t w$ where (1): $r \sqsubseteq s$ whenever $r \sqsubseteq_t w$ and $s \sqsubseteq r$. So $s \sqsubseteq w$. Since $u \in P$, we also know that $s \circ t.v$, and so $s \sqsubseteq_{t.v} w$.

Let $q \sqsubseteq_{t.v} w$ where $s \sqsubseteq q$, and so $q \sqsubseteq w$ where $q \circ t.v$, and so $q.t.v \in P$. Thus $q.t \in P$, and so $q \circ t$. It follows that $q \sqsubseteq_t w$, and so $q \sqsubseteq s$ follows from (1). Generalizing on q, we know (2): $q \sqsubseteq s$ whenever $q \sqsubseteq_{t.v} w$ and $s \sqsubseteq q$.

Having already shown that $s \sqsubseteq_{t.v} w$, it follows from (2) that $s \in w_{t.v}$. Since $s.t.v \sqsubseteq u$ for $u \in W$, we may conclude that $t.v \mathrel{-\!\!\triangleright}_w u$ as desired. $\square$