A Geometric Approach To Matrices

Answer Key

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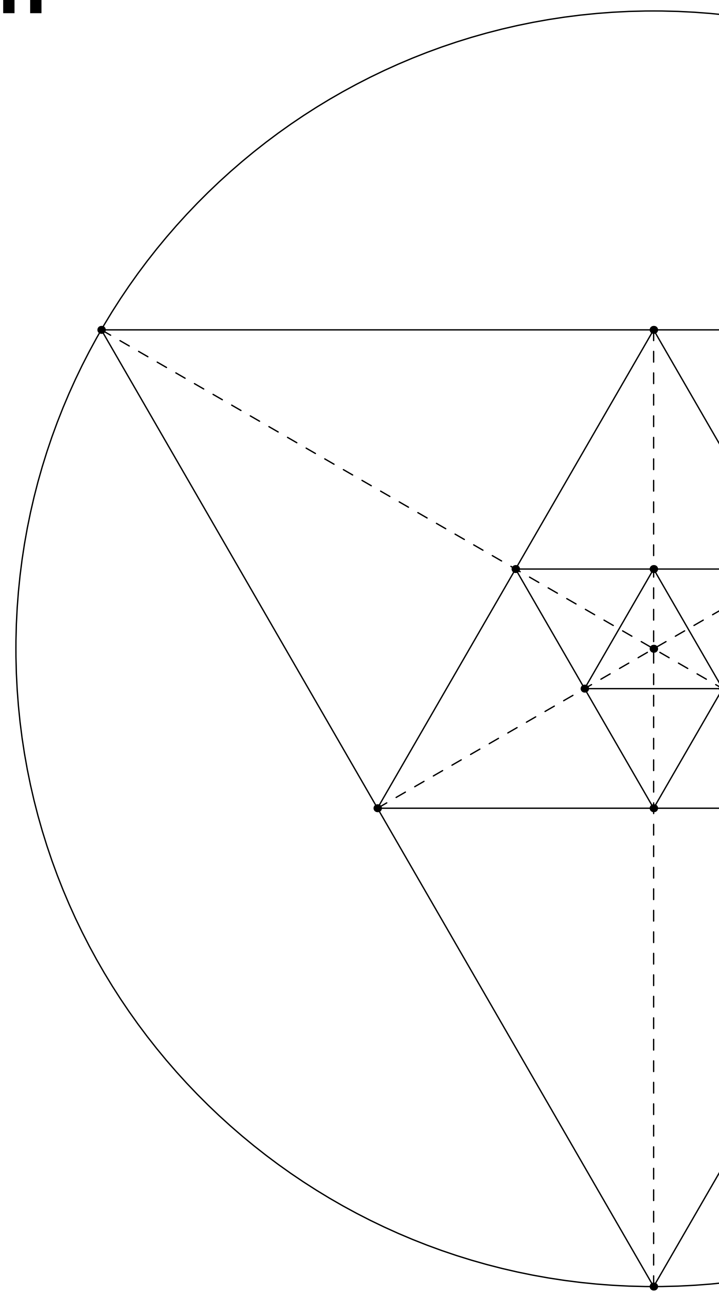

Contents

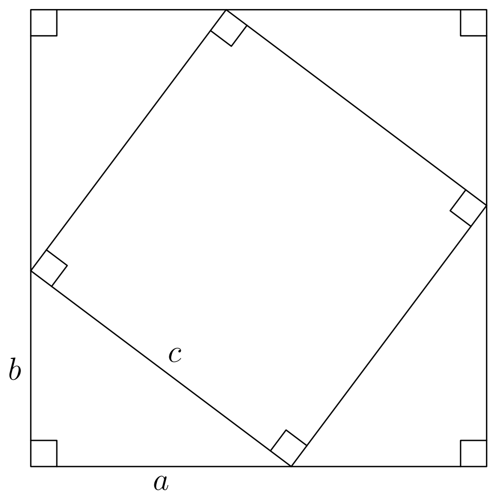

Figure 1: Scenario in Problem 1.

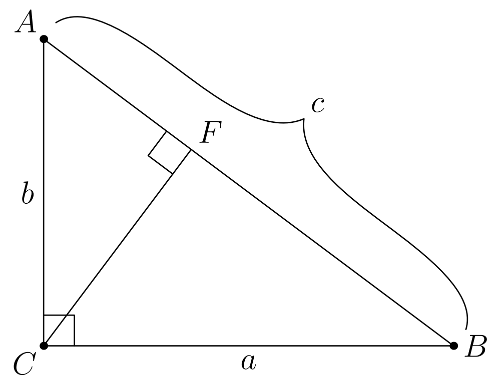

Figure 2: Scenario in Problem 2.

1 Trigonometry Review

1. Prove the Pythagorean theorem using "conservation of area." Start with Figure 1.

In Figure 1, the larger square has side length $a + b$. The smaller, nested square has side length c. Four copies of the right triangle with side lengths a, b, c are placed around the square. We have

$$\begin{aligned}
A_{\text{triangles}} + A_{\text{small sq.}} &= A_{\text{big sq.}} && \text{[Conservation of area]}\\
4A_{\text{triangle}} + A_{\text{small sq.}} &= A_{\text{big sq.}}\\
4\left(\frac{1}{2}ab\right) + c^2 &= (a+b)^2 && \text{[Areas of triangle, square]}\\
2ab + c^2 &= a^2 + 2ab + b^2 && \text{[Expanding]}\\
c^2 &= a^2 + b^2. && \text{Q.E.D.}
\end{aligned}$$

2. Prove the Pythagorean theorem using a right triangle with an altitude drawn to its hypotenuse, making use of similar right triangles. This is shown in Figure 2.

Let $h = CF$, the length of the altitude to the hypotenuse. $\triangle ACF \sim \triangle ABC$ by AA Similarity because they share an angle and both have a right angle. Therefore, $\frac{AF}{AC} = \frac{AC}{AB}$. Substituting named variables for these lengths, we get

$$\frac{AF}{b} = \frac{b}{c} \implies AF = \frac{b^2}{c}.$$

Applying the same logic to $\triangle CFB$, we get $\triangle CFB \sim \triangle ABC$, so $\frac{BF}{BC} = \frac{BC}{AB}$. Substituting, we get

$$\frac{BF}{a} = \frac{a}{c} \implies BF = \frac{a^2}{c}.$$

Since F is between A and B, we have $AB = AF + FB$; substituting our found values for AF and FB, we get

$$\begin{aligned}
c = AB &= AF + FB\\
c &= \frac{b^2}{c} + \frac{a^2}{c}\\
c^2 &= b^2 + a^2. && \text{Q.E.D.}
\end{aligned}$$

3. Now you will prove the trig identities.

(a) Draw and label a right triangle and a unit circle, then write trig definitions for $\cos$, $\sin$, $\tan$, and $\sec$ in terms of your drawing.

The scenario is depicted in Figure 3. By the definition of sine and cosine, we have $\sin\theta = AP$ and $\cos\theta = OA$. Since $\triangle OAP \sim \triangle OPT$ by AA Similarity, we have $\frac{TP}{OP} = \frac{AP}{OA}$. Substituting known values, we get

$$\frac{TP}{1} = \frac{\sin\theta}{\cos\theta} \Longrightarrow TP = \tan\theta.$$

Also, $\triangle OAP \sim \triangle OKS$ by AA, so $\frac{OS}{OK} = \frac{1}{\cos\theta}$. Similarly, we have

$$\frac{OS}{1} = \frac{1}{\cos\theta} \Longrightarrow OS = \sec\theta.$$

Finally, as an alternate interpretation of $\tan$, we have $\frac{KS}{OK} = \frac{AP}{OA}$, so

$$\frac{KS}{1} = \frac{\sin\theta}{\cos\theta} \Longrightarrow KS = \tan\theta.$$

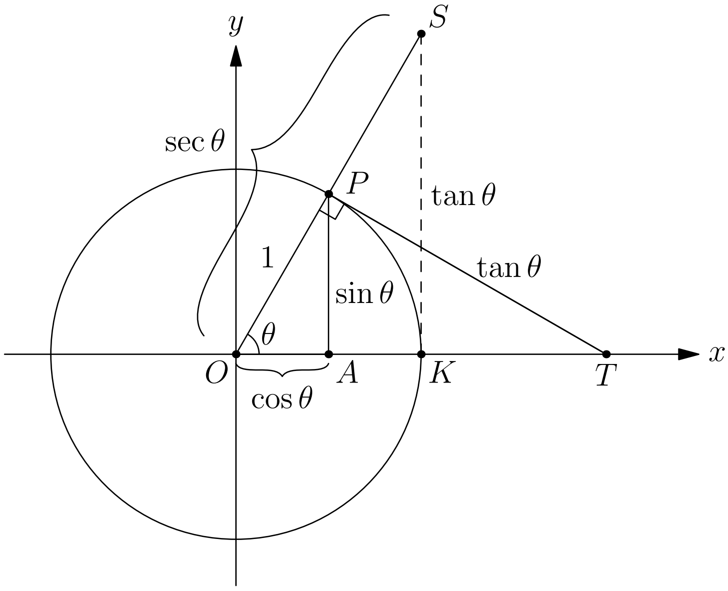

Figure 3: The right triangle and unit circle.

(b) Use a right triangle and the definitions of $\sin$ and $\cos$ to find and prove a value for $\sin^2\theta + \cos^2\theta$.

Referring back to Figure 3, focus on $\triangle OAP$. It is a right triangle with side lengths $a = \cos\theta$, $b = \sin\theta$, and $c = 1$. By the Pythagorean theorem, we have

$$\begin{aligned}
OA^2 + AP^2 &= OP^2 && \text{[Pythagorean theorem]}\\
\cos^2\theta + \sin^2\theta &= 1^2 && \text{[Substitution]}\\
\sin^2\theta + \cos^2\theta &= 1 && \text{[Rearrange]}
\end{aligned}$$

(c) Use the picture of the unit circle in Figure 4 to find and prove a value for $\cos(A - B)$. Note that D_1 and D_2 are the same length because they subtend the same size arc of the circle. Set them equal and work through the algebra, using the distance formula and part (b) of this problem.

We have $D_1 = D_2$, so

$$\begin{aligned}
D_1^2 &= D_2^2\\
(\cos A - \cos B)^2 + (\sin A - \sin B)^2 &= (\cos(A-B) - 1)^2 + \sin^2(A-B)\\
\cos^2 A - 2\cos A\cos B + \cos^2 B + \sin^2 A - 2\sin A\sin B + \sin^2 B &= \cos^2(A-B) - 2\cos(A-B) +\\
&\quad 1 + \sin^2(A-B)
\end{aligned}$$

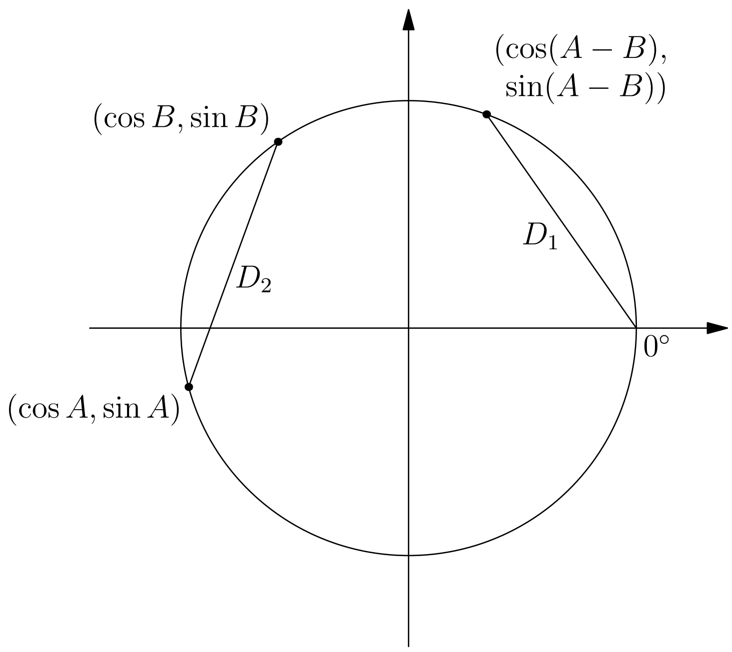

Figure 4: Scenario in Problem 3.

$$
\begin{aligned}
(\cos^2 A + \sin^2 A) + (\cos^2 B + \sin^2 A) - 2\sin A \sin B &= (\cos^2(A-B) + \sin^2(A-B)) + \\
&\quad 1 - 2\cos(A-B) \\
\cancelto{0}{1+1} - 2\sin A \sin B - 2\cos A \cos B &= \cancelto{0}{1+1} - 2\cos(A-B) \\
2\sin A \sin B + 2\cos A \cos B &= 2\cos(A-B) \\
\sin A \sin B + \cos A \cos B &= \cos(A-B).
\end{aligned}
$$

Q.E.D.

4. Write down as many trig identities as you can. There's no need to prove all of these right now.

$$
\begin{array}{rrr}
\sin(A+B) = & \sin(A-B) = & \cos(A+B) = \\
\tan(A+B) = & \tan(A-B) = & \sin(2A) = \\
\cos(2A) = & \tan(2A) = & \sin\left(\frac{A}{2}\right) = \\
\cos\left(\frac{A}{2}\right) = & \tan\left(\frac{A}{2}\right) = &
\end{array}
$$

You should probably memorize these for convenience.

$$
\begin{aligned}
\sin(A+B) &= \sin A \cos B + \cos A \sin A \\
\sin(A-B) &= \sin A \cos B - \cos A \sin B \\
\cos(A+B) &= \cos A \cos B - \sin A \sin B \\
\tan(A+B) &= \frac{\tan A + \tan B}{1 - \tan A \tan B} \\
\tan(A-B) &= \frac{\tan A - \tan B}{1 + \tan A \tan B} \\
\sin(2A) &= 2\sin A \cos A \\
\cos(2A) &= 2\cos^2 A - 1 = 1 - 2\sin^2 A = \cos^2 A - \sin^2 A \\
\tan(2A) &= \frac{2\tan A}{1 - \tan^2 A} \\
\sin\left(\frac{A}{2}\right) &= \pm\sqrt{\frac{1 - \cos A}{2}} \\
\cos\left(\frac{A}{2}\right) &= \pm\sqrt{\frac{1 + \cos A}{2}}
\end{aligned}
$$

$$\tan\left(\frac{A}{2}\right) = \frac{\sin A}{1+\cos A} = \frac{1-\cos A}{\sin A}$$

5. Let's review complex numbers and DeMoivre's theorem.

(a) Recall that you can write a complex number both in Cartesian and polar forms. Let

$$a + bi = (a, b) = (r\cos\theta, r\sin\theta) = r\cos\theta + ir\sin\theta.$$

What is r in terms of a and b?

r is just the distance to the origin from $a + bi$. Draw a right triangle as shown in Figure 5. By the pythagorean theorem, $r = \sqrt{a^2 + b^2}$.

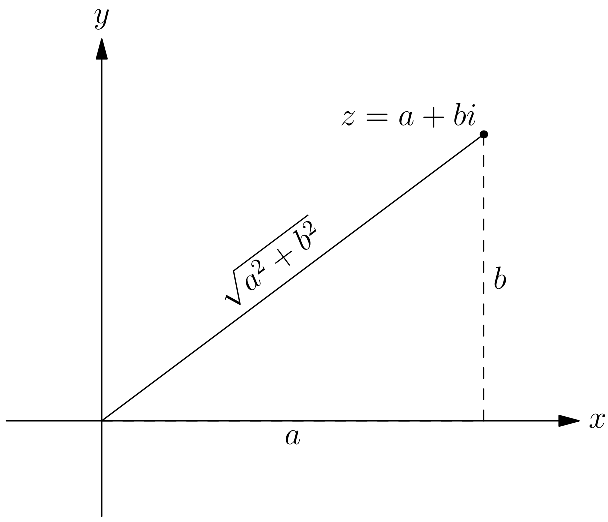

Figure 5: $a + bi$ in the complex plane.

(b) Multiply $(a + bi)(c + di)$ out using FOIL.

$$\begin{aligned}(a+bi)(c+di) &= ac + adi + bci + (bi)(di)\\ &= ac + (ad+bc)i - bd\\ &= ac - bd + (ad+bc)i.\end{aligned}$$

(c) Convert the two multiplicands[1] to polar form, noting that the two lengths and angles are different numbers. Call them $r_1(\cos\theta + i\sin\theta)$ and $r_2(\cos\phi + i\sin\phi)$.

We have $r_1 = \sqrt{a^2+b^2}$ and $\theta = \tan^{-1}\left(\frac{b}{a}\right)$ [2]; similarly, $r_2 = \sqrt{c^2+d^2}$ and $\phi = \tan^{-1}\left(\frac{d}{c}\right)$.

(d) Multiply them, and use your results from Problems 3c and 3d to show that multiplying two complex numbers involves multiplying their lengths and adding their angles. This is DeMoivre's theorem!

$$\begin{aligned}r_1(\cos\theta + i\sin\theta)r_2(\cos\phi + i\sin\phi) &= r_1r_2(\cos\theta\cos\phi - \sin\theta\sin\phi + i(\sin\theta\cos\phi + \cos\theta\sin\phi))\\ &= r_1r_2(\cos(\theta+\phi) + i\sin(\theta+\phi)).\end{aligned}$$

(e) Use part (d) to simplify $(\sqrt{3} + i)^{18}$.

[1] This is the word for parts of a multiplication! So for example, if $a \cdot b = c$, then a and b are the multiplicands.

[2] You can get this from drawing a right triangle.

We have $\sqrt{3}+i = r(\cos\theta + i\sin\theta) = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$.

$$\begin{aligned}
(2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right))^{18} &= 2^{18}\cdot\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^{18} \\
&= 2^{18}\cdot\underbrace{\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\cdots\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)}_{\text{18 copies}} \\
&= 2^{18}\cdot\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\underbrace{\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\cdots\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)}_{\text{16 copies}} \\
&= \vdots \\
&= 2^{18}\cdot(\cos 3\pi + i\sin 3\pi) \\
&= 2^{18}\cdot -1 \\
&= -2^{18}.
\end{aligned}$$

6. Here is a review of 2D rotation.

(a) Remember that we can graph complex numbers as 2D ordered pairs in the complex plane. Now, consider the complex number $z = \cos\theta + i\sin\theta$, where θ is fixed. What is the magnitude of z?

We have
$$|z| = \sqrt{\cos^2\theta + \sin^2\theta} = \sqrt{1} = 1.$$

(b) Multiplying $z\cdot(x+yi)$ yields a rotation of the point (x,y) counterclockwise by the angle θ around the origin. What if we wanted to rotate clockwise by θ instead?

We can multiply by the conjugate of z, since

$$\overline{z} = \cos\theta - i\sin\theta = \cos -\theta + i\sin -\theta.$$

Thus, the operation is $\overline{z}\cdot(x+yi)$ to rotate clockwise by θ.

7. Rotate the following conics by (i) $30°$, (ii) $45°$, and (iii) θ:

(a) $x^2 - y^2 = 1$

i. $30°$

We make the substitution $x' = x\cos 30° - y\sin 30° = \frac{\sqrt{3}}{2}x - \frac{y}{2}$ and $y' = x\sin 30° + y\cos 30° = \frac{x}{2} + \frac{\sqrt{3}}{2}y$:

$$\begin{aligned}
x'^2 - y'^2 &= 1 \\
\left(\frac{\sqrt{3}}{2}x - \frac{y}{2}\right)^2 - \left(\frac{x}{2} + \frac{\sqrt{3}}{2}y\right)^2 &= 1 \\
x^2/2 - \sqrt{3}xy - y^2/2 &= 1
\end{aligned}$$

ii. $45°$

We make the substitution $x' = x\cos 45° - y\sin 45° = \frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y$ and $y' = x\sin 45° + y\cos 45° = \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y$:

$$\begin{aligned}
x'^2 - y'^2 &= 1 \\
\left(\frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y\right)^2 - \left(\frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y\right)^2 &= 1 \\
-2xy &= 1.
\end{aligned}$$

iii. θ

We make the substitution $x' = x\cos\theta - y\sin\theta$ and $y' = x\sin\theta + y\cos\theta$:

$$\begin{aligned} x'^2 - y'^2 &= 1 \\ (x\cos\theta - y\sin\theta)^2 - (x\sin\theta + y\cos\theta)^2 &= 1. \end{aligned}$$

(b) $\frac{x^2}{16} - \frac{y^2}{9} = 1.$

i. $30°$

We make the substitution $x' = x\cos 30° - y\sin 30° = \frac{\sqrt{3}}{2}x - \frac{y}{2}$ and $y' = x\sin 30° + y\cos 30° = \frac{x}{2} + \frac{\sqrt{3}}{2}y$:

$$\begin{aligned} \frac{x'^2}{16} - \frac{y'^2}{9} &= 1 \\ \frac{\left(\frac{\sqrt{3}}{2}x - \frac{y}{2}\right)^2}{16} - \frac{\left(\frac{x}{2} + \frac{\sqrt{3}}{2}y\right)^2}{9} &= 1 \\ \frac{1}{576}(11x^2 - 50\sqrt{3}xy - 39y^2) &= 1. \end{aligned}$$

ii. $45°$

We make the substitution $x' = x\cos 45° - y\sin 45° = \frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y$ and $y' = x\sin 45° + y\cos 45° = \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y$:

$$\begin{aligned} \frac{x'^2}{16} - \frac{y'^2}{9} &= 1 \\ \frac{\left(\frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y\right)^2}{16} - \frac{\left(\frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y\right)^2}{9} &= 1 \\ \frac{1}{288}(-x - 7y)(7x + y) &= 1 \end{aligned}$$

iii. θ

We make the substitution $x' = x\cos\theta - y\sin\theta$ and $y' = x\sin\theta + y\cos\theta$:

$$\begin{aligned} \frac{x'^2}{16} - \frac{y'^2}{9} &= 1 \\ \frac{(x\cos\theta - y\sin\theta)^2}{16} - \frac{(x\sin\theta + y\cos\theta)^2}{9} &= 1. \end{aligned}$$

(c) $y^2 = 4Cx$

i. $30°$

We make the substitution $x' = x\cos 30° - y\sin 30° = \frac{\sqrt{3}}{2}x - \frac{y}{2}$ and $y' = x\sin 30° + y\cos 30° = \frac{x}{2} + \frac{\sqrt{3}}{2}y$:

$$\begin{aligned} y'^2 &= 4Cx' \\ \left(\frac{x}{2} + \frac{\sqrt{3}}{2}y\right)^2 &= 4C\left(\frac{\sqrt{3}}{2}x - \frac{y}{2}\right). \end{aligned}$$

ii. $45°$

We make the substitution $x' = x\cos 45° - y\sin 45° = \frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y$ and $y' = x\sin 45° + y\cos 45° = \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y$:

$$
\begin{aligned}
y'^2 &= 4Cx' \\
\left(\frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y\right)^2 &= 4C\left(\frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y\right) \\
\frac{1}{2}(x+y)^2 &= 2C\sqrt{2}(x-y)
\end{aligned}
$$

iii. θ

We make the substitution $x' = x\cos\theta - y\sin\theta$ and $y' = x\sin\theta + y\cos\theta$:

$$
\begin{aligned}
y'^2 &= 4Cx' \\
(x\cos\theta - y\sin\theta)^2 &= 4C\left(x\sin\theta + y\cos\theta\right).
\end{aligned}
$$

$\bullet$	I	A	B	C	D	E
I						
A			E			
B						
C						
D						
E						

Figure 1: Unfilled 3-post snap group table.

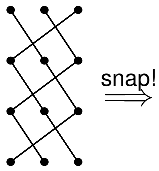

Figure 2: $E \bullet E \bullet E = I$; E has period 3.

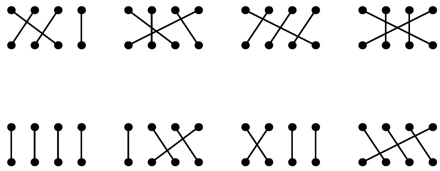

Figure 3: Some 4-post group elements.

2 It's a Snap

1. Fill out a 6×6 table like the one in Figure 1, showing the results of each of the 36 possible snaps, where $X \bullet Y$ is in X's row and Y's column. $A \bullet B = E$ is done for you.

$\bullet$	I	A	B	C	D	E
I	I	A	B	C	D	E
A	A	I	E	D	C	B
B	B	D	I	E	A	C
C	C	E	D	I	B	A
D	D	B	C	A	E	I
E	E	C	A	B	I	D

2. Would this table look different if you wrote the elements A through E in a different order?

Yes; here's an example:

$\bullet$	I	E	A	D	B	C
I	I	E	A	D	B	C
E	E	D	C	I	A	B
A	A	B	I	C	E	D
D	D	I	B	E	C	A
B	B	C	D	A	I	E
C	C	A	E	B	D	I

3. Which of the elements is the **identity element** K, such that $X \bullet K = K \bullet X = X$ for all X?

The identity element is I, since $I \bullet A = A \bullet I = A$, $I \bullet B = B \bullet I = B$, and so forth.

4. Does every element have an inverse; can you get to the identity element from every element using only one snap?

Yes you can. The inverses are shown below.

$$
\begin{aligned}
I &\leftrightarrow I \\
A &\leftrightarrow A \\
B &\leftrightarrow B \\
C &\leftrightarrow C \\
D &\leftrightarrow E
\end{aligned}
$$

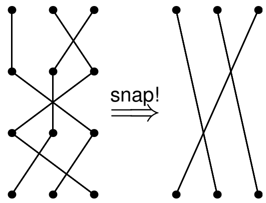

Figure 4: A 4×3 grid of posts has a unique result after the snap operation.

Note that the inverse of an element X is denoted X^{-1}.

5.

(a) Is the snap operation commutative (does $X \bullet Y = Y \bullet X$ for all X, Y)?

No, the snap operation is not commutative. For example, $A \bullet B = E$, but $B \bullet A = D$.

(b) Is the snap operation associative (does $(X \bullet Y) \bullet Z = X \bullet (Y \bullet Z)$ for all X, Y, Z)?'

Yes, the snap operation is associative. You can rationalize this as the fact that a 4×3 grid of posts is snapped to a single configuration, regardless of which middle row you remove first. This is shown in Figure 4.

6.

(a) For any elements X, Y, is there always an element Z so that $X \bullet Z = Y$?

Yes, there is always a way to get from one element to another in one snap. You can prove this by construction. If element X connects n_1 to n'_1, n_2 to n'_2, and n_3 to n'_3, and element Y connects m_1 to m'_1, m_2 to m'_2, and m_3 to m'_3, then the solution Z to $X \bullet Z = Y$ connects m_1 to $n_{m'_1}$, m_2 to $n_{m'_2}$, and m_3 to $n_{m'_3}$.

That's probably a bit hard to understand, but a more clever solution uses inverses. We multiply X by X^{-1}, then by Y:

$$X \bullet X^{-1} \bullet Y = Y.$$

But since every element has an inverse, and the snap operation is associative, we have

$$\begin{aligned} X \bullet (X^{-1} \bullet Y) &= Y \\ \implies Z &= X^{-1} \bullet Y. \end{aligned}$$

In this way, we have constructed the element Z.

(b) For (a), is Z always unique?

Yes. To show this, we use a proof by contradiction. Suppose we have two solutions Z_1 and Z_2 so that $Z_1 \neq Z_2$ and

$$\begin{aligned} X \bullet Z_1 &= Y \\ X \bullet Z_2 &= Y. \end{aligned}$$

We multiply to the left by Y^{-1}. Note that since the snap operation is not commutative, we need to multiply both sides on a specific side:

$$\begin{aligned} Y^{-1} \bullet X \bullet Z_1 &= Y^{-1} \bullet Y = I \\ Y^{-1} \bullet X \bullet Z_2 &= I \end{aligned}$$

So Z_1, Z_2 are the inverses of $Y^{-1} \bullet X$. But the inverse of an element is unique; we've showed this by listing them all out! Thus, $Z_1 = Z_2$, contradicting our assumption and proving that Z is unique in $X \bullet Z = Y$.

7. If you constructed a 5×5 table using only 5 of the snap elements, the table would not describe a group, because there would be entries in the table not in those 5. Therefore, a group must be **closed** under its operation; if $X, Y \in G$ ($\in$ means "is/are in"), then $X \bullet Y \in G$ for all X, Y. Some subsets, however, do happen to be closed.

   Write valid group tables using exactly 1, 2, and 3 elements from the snap group. These are known as **subgroups**.

Here are tables with 1, 2, and 3 elements:

$\bullet$	I
I	I

$\bullet$	I	A
I	I	A
A	A	I

$\bullet$	I	D	E
I	I	D	E
D	D	E	I
E	E	I	D

8. What do you guess is the complete definition of a mathematical group? (Hint: consider your answers to Problems 3–7.)

(Answers may vary.)

Definition of **group**: A group G is a set of elements together with a **binary operation** that meets the following criteria:

(a) Identity: There is an element $I \in G$ such that for all $X \in G$, $X \bullet I = I \bullet X = X$.

(b) Closure: If X, Y are elements of the group, then $X \bullet Y$ is also an element of the group.

(c) Invertibility: Each element X has an inverse X^{-1} such that $X \bullet X^{-1} = X^{-1} \bullet X = I$.

(d) Associativity: For all elements X, Y, and Z, $X \bullet (Y \bullet Z) = (X \bullet Y) \bullet Z$.

9. Notice that $E \bullet E \bullet E = I$. (See Figure 2.) This means that E has a **period** of 3 when acting upon itself. Which elements have a period of

(a) 1?

I is the only element with a period of 1, since $I = I$.

(b) 2?

A, B, and C have periods of 2, since for each $X \in A, B, C$ we have $X \bullet X = I$.

(c) 3?

D and E have periods of 3, since for each $Y \in D, E$ we have $Y \bullet Y \neq I$, but $Y \bullet Y \bullet Y = I$.

10. Answer the following with the 1, 2, and 4-post snap groups S_1, S_2 and S_4.

(a) How many elements would there be?

S_1 has $1! = 1$ elements. S_2 has $2! = 2$ elements. S_4 has $4! = 24$ elements.

(b) Systematically draw and name them.

Figure 5: Elements of S_1.

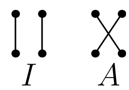

Figure 6: Elements of S_2.

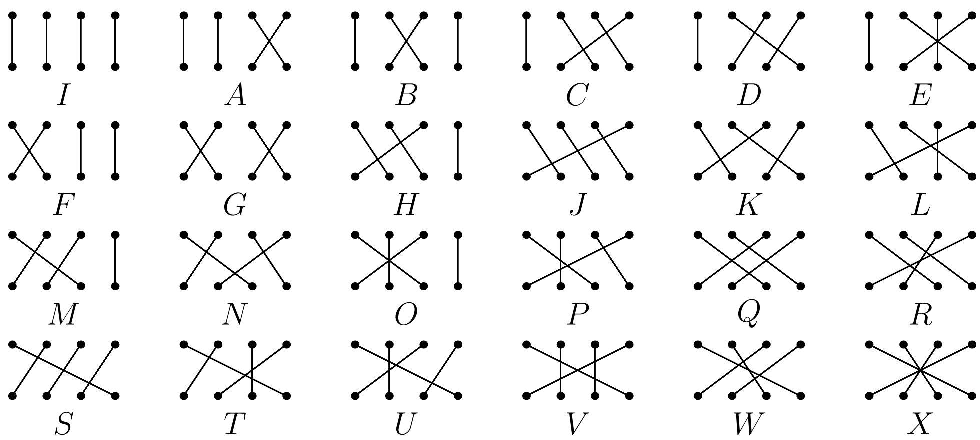

Figure 7: Elements of S_4.

(c) Make a group table of these elements. For 4 posts, instead of creating the massive table, give the number of entries that table would have.

Here are group tables for S_1 and S_2.

$\bullet$	I
I	I

Figure 8: Group table for S_1.

$\bullet$	I	A
I	I	A
A	A	I

Figure 9: Group table for S_2.

The table for S_4 is given at the end of the section in Figure 12 for the curious.

(d) What is the relationship between this new table and your original table?

Both S_1's and S_2's tables are subgroups of the original table for S_3. In turn, S_3 is a subgroup of S_4.

11. Can you think of an easier way to generate a snap group table without drawing all the possible configurations?

(Answers may vary.)

One way to do it is to treat each element as a list of indices. For example, I is the ordered triple $(1, 2, 3)$ because it takes column 1 to 1, 2 to 2, and 3 to 3. A is $(1, 3, 2)$, because it takes 1 to 1, 2 to 3, and 3 to 2.

This makes it a bit easier to calculate, because you can simply substitute indices for each configuration rather than make a drawing. It also makes it easy to write a program to calculate; this is actually how all the tables in this answer key were generated.

12.

(a) How many elements would there be in the 5-post snap group?

There would be $5! = 120$ elements in S_5.

(b) How many entries would its table have?

There would be $5!^2 = 14400$ entries in S_5's table.

(c) What possible periods would its elements have?

This is a more difficult question. We must ask what characteristics of an element determine its period.

If we observe the periodicity of an element with a pretty large period, say one from S_5 with a period of 6, you can see how a large period can arise. This is shown in Figure 11.

We can split up this element into two components: a component with period 3 and one with period 2. Let's call these components C_3 and C_2. After 2 steps, the C_3 has not completed one period, even though C_2. After 3 steps, C_3 has completed one period, but C_2 has gone through $\frac{3}{2}$. It takes $\operatorname{lcm}(2, 3) = 6$ steps before both components "line up!"

All elements can be split up into some number of these cyclic components, even if it doesn't look like it at first glance. For example, the element from S_8 shown in Figure 10 is actually two size 3 and size 2 components. It therefore has a period of $\operatorname{lcm}(2, 3, 3) = 6$. Note that it does *not* have a period of $2 \cdot 3 \cdot 3 = 18$!

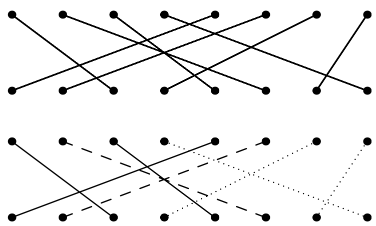

Figure 10: This element from S_8 has components of size $2, 3, 3$.

For S_5, we can split it up into components of size $1, 1, 1, 1, 1$, giving period 1; components of size $1, 1, 1, 2$, giving period 2; components of size $1, 1, 3$, giving period 3; components of size $1, 4$, giving period 4; a component of size 5, giving period 5; and component of size $1, 2, 3$, giving period 6. Thus, periods $1, 2, 3, 4, 5, 6$ are achievable.

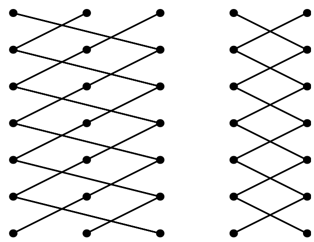

Figure 11: This element from S_5 has a period of 6.

(d) Extend your answers for a–c to M posts per row.

This is rather straightforward. There are (a) $M!$ elements in the M-post snap group, and thus (b) $M!^2$ elements in the corresponding group table. The possible periods are harder to calculate, but they can be generated like so:

Let integers $x_i > 0$ and $\sum_i x_i = M$. In other words, the sum of all x_i is M. Then $\operatorname{lcm}(x_1, x_2, \cdots, x_n)$ is a valid period; the least common multiple of all x_i is a possible period.

For fun: in set builder notation, we have the set of possible periods P_n for the n-post snap group as

$$P_n = \left\{ \operatorname{lcm}(x_1, x_2, \cdots, x_n) \;\middle|\; x_i \in \mathbb{Z}^+ \wedge \sum_i x_i = n \right\}.$$

The maximum such period (i.e. $\max P_n$) is actually known as Landau's function, $g(n)$.

13. As we learned, a **permutation** of some things is an order in which they can be arranged. What is the relationship between the set of permutations of m things and the m-post snap group?

We can make a pretty simple correspondence between a permutation of m things and an element of the m-post snap group. If we think back to the idea of treating each element of the group as a list of indices, the correspondence is obvious. For example, I is the ordered triple $(1, 2, 3)$ because it takes column 1 to 1, 2 to 2, and 3 to 3. A is $(1, 3, 2)$, because it takes 1 to 1, 2 to 3, and 3 to 2. But each ordered triple is a permutation of $1, 2, 3$! This extends to any m.

·	I	A	B	C	D	E	F	G	H	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X
I	I	A	B	C	D	E	F	G	H	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X
A	A	I	D	E	B	C	G	F	K	L	H	J	S	T	U	V	W	X	M	N	O	P	Q	R
B	B	C	I	A	E	D	M	N	O	P	Q	R	F	G	H	J	K	L	T	S	W	X	U	V
C	C	B	E	D	I	A	N	M	Q	R	O	P	T	S	W	X	U	V	F	G	H	J	K	L
D	D	E	A	I	C	B	S	T	U	V	W	X	G	F	K	L	H	J	N	M	Q	R	O	P
E	E	D	C	B	A	I	T	S	W	X	U	V	N	M	Q	R	O	P	G	F	K	L	H	J
F	F	G	H	J	K	L	I	A	B	C	D	E	O	P	M	N	R	Q	U	V	S	T	X	W
G	G	F	K	L	H	J	A	I	D	E	B	C	U	V	S	T	X	W	O	P	M	N	R	Q
H	H	J	F	G	L	K	O	P	M	N	R	Q	I	A	B	C	D	E	V	U	X	W	S	T
J	J	H	L	K	F	G	P	O	R	Q	M	N	V	U	X	W	S	T	I	A	B	C	D	E
K	K	L	G	F	J	H	U	V	S	T	X	W	A	I	D	E	B	C	P	O	R	Q	M	N
L	L	K	J	H	G	F	V	U	X	W	S	T	P	O	R	Q	M	N	A	I	D	E	B	C
M	M	N	O	P	Q	R	B	C	I	A	E	D	H	J	F	G	L	K	W	X	T	S	V	U
N	N	M	Q	R	O	P	C	B	E	D	I	A	W	X	T	S	V	U	H	J	F	G	L	K
O	O	P	M	N	R	Q	H	J	F	G	L	K	B	C	I	A	E	D	X	W	V	U	T	S
P	P	O	R	Q	M	N	J	H	L	K	F	G	X	W	V	U	T	S	B	C	I	A	E	D
Q	Q	R	N	M	P	O	W	X	T	S	V	U	C	B	E	D	I	A	J	H	L	K	F	G
R	R	Q	P	O	N	M	X	W	V	U	T	S	J	H	L	K	F	G	C	B	E	D	I	A
S	S	T	U	V	W	X	D	E	A	I	C	B	K	L	G	F	J	H	Q	R	N	M	P	O
T	T	S	W	X	U	V	E	D	C	B	A	I	Q	R	N	M	P	O	K	L	G	F	J	H
U	U	V	S	T	X	W	K	L	G	F	J	H	D	E	A	I	C	B	R	Q	P	O	N	M
V	V	U	X	W	S	T	L	K	J	H	G	F	R	Q	P	O	N	M	D	E	A	I	C	B
W	W	X	T	S	V	U	Q	R	N	M	P	O	E	D	C	B	A	I	L	K	J	H	G	F
X	X	W	V	U	T	S	R	Q	P	O	N	M	L	K	J	H	G	F	E	D	C	B	A	I

Figure 12: Group table for S_4.