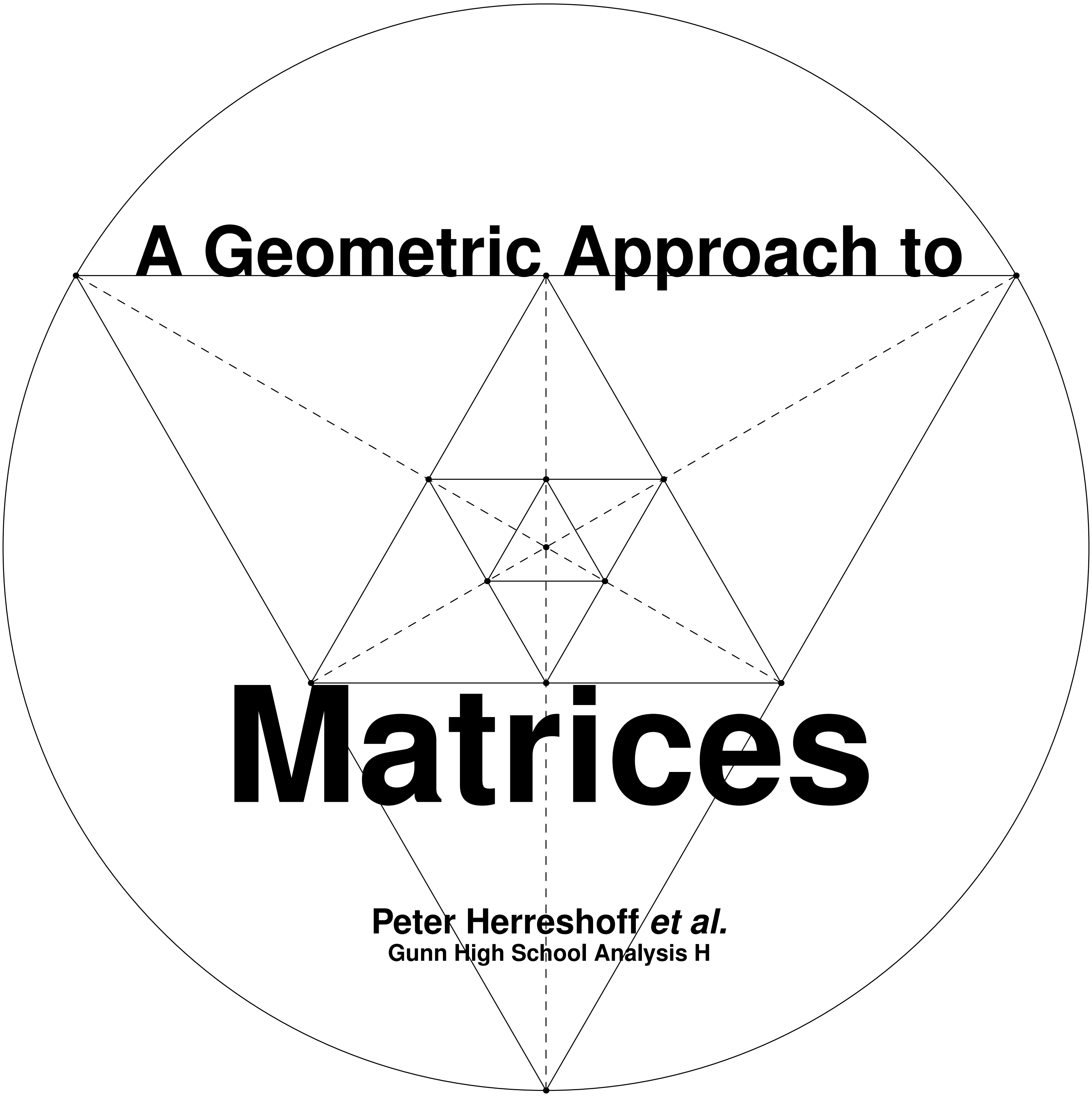
A Geometric Approach to

Matrices



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Gunn High School Analysis H

A Geometric Approach To Matrices

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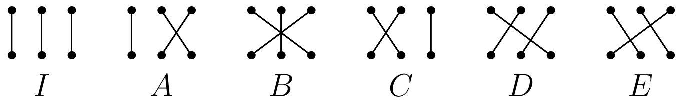

Figure 1: The six possibilities for connections between two rows of three posts.

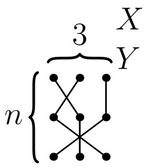

Figure 2: A grid with three strings.

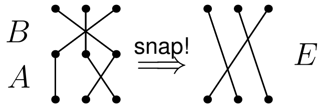

Figure 3: $A \bullet B = E$.

1 It's a Snap

We begin with a problem that ties together ideas from geometry, complex numbers, matrices, combinatorics, and group theory. You likely studied geometry in 9th grade and complex numbers in 10th grade, so you should have a basis to start your investigation. The other three words may be unfamiliar at this point. You will be coming back to this problem in a series of investigations over the coming weeks.

Consider a rectangle of posts with n rows and 3 columns. An elastic string is anchored to one post in the topmost row and one post in the bottommost row. As the string descends top to bottom, it loops taut around one post in each row. Two other elastic strings are anchored and looped in the same way, with the condition that each post has exactly one string on it; no more, no less. An example where $n = 3$ is depicted in Figure 2.

Figure 1 shows the six "ways" that two consecutive rows can be connected in this way — convince yourself these are the only six. I have labeled them with letters I, A, B, C, D, and E.

Now, consider what happens when you look at a 3×3 rectangle of posts, like the one in Figure 2. You should have two "ways," X and Y, stacked one on top of the other so that Y is on top of X. If you withdraw the middle posts, the elastic string should **snap** to a configuration identical to one of the six ways you drew initially. Let's call this operation snap, or $\bullet$, so that $X \bullet Y$ reads "X snap Y." For example, $A \bullet B = E$, as shown in Figure 3.

These six ways, along with the operation $\bullet$, form a mathematical *group*, and we say that each is an *element* of our group. In particular, we will call this the **snap group**. Groups are the central objects studied in *group theory*, which I mentioned at the beginning of this problem. Let's study the snap group and characterize its properties.