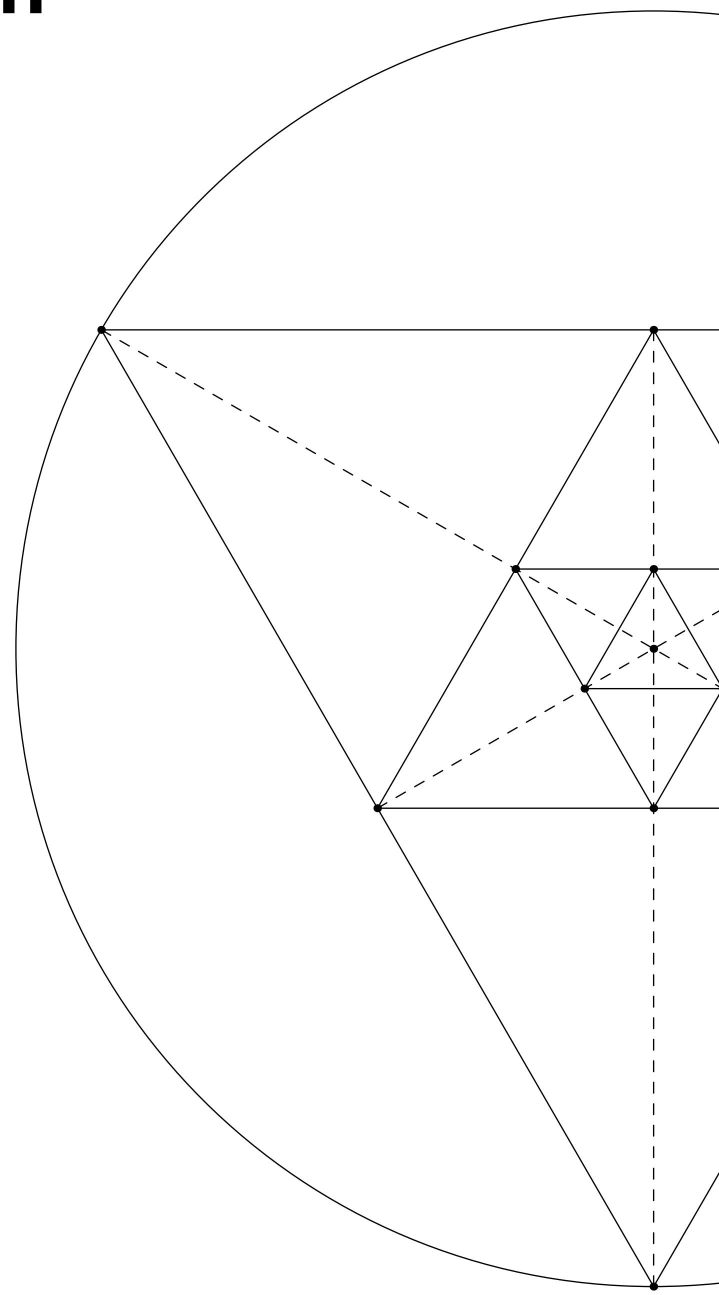
A Geometric Approach To Matrices

Answer Key

Timothy Herchen
Henry M. Gunn High School
Analysis Honors



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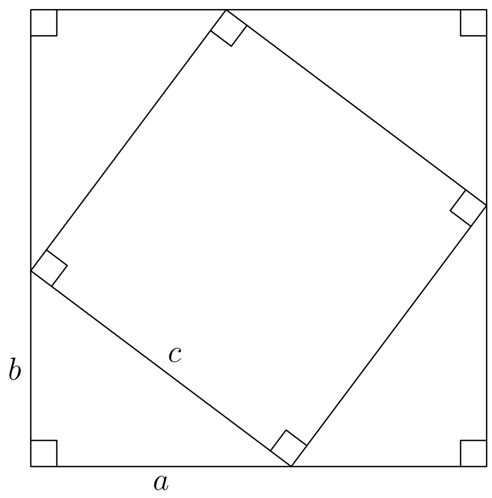

Figure 1: Scenario in Problem 1.

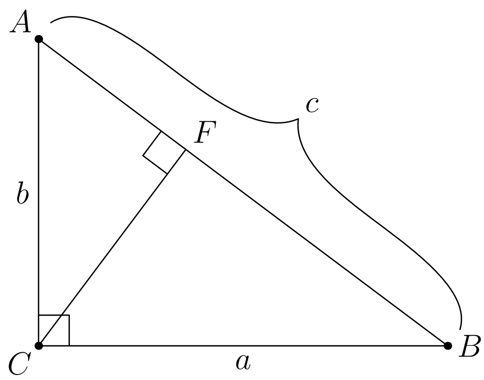

Figure 2: Scenario in Problem 2.

1 Trigonometry Review

1. Prove the Pythagorean theorem using “conservation of area.” Start with Figure 1.

In Figure 1, the larger square has side length $a + b$. The smaller, nested square has side length c. Four copies of the right triangle with side lengths a, b, c are placed around the square. We have

$$\begin{aligned}
A_{\text{triangles}} + A_{\text{small sq.}} &= A_{\text{big sq.}} && \text{[Conservation of area]}\\
4A_{\text{triangle}} + A_{\text{small sq.}} &= A_{\text{big sq.}} &&\\
4\left(\frac{1}{2}ab\right) + c^2 &= (a+b)^2 && \text{[Areas of triangle, square]}\\
2ab + c^2 &= a^2 + 2ab + b^2 && \text{[Expanding]}\\
c^2 &= a^2 + b^2. && \text{Q.E.D.}
\end{aligned}$$

2. Prove the Pythagorean theorem using a right triangle with an altitude drawn to its hypotenuse, making use of similar right triangles. This is shown in Figure 2.

Let $h = CF$, the length of the altitude to the hypotenuse. $\triangle ACF \sim \triangle ABC$ by AA Similarity because they share an angle and both have a right angle. Therefore, $\frac{AF}{AC} = \frac{AC}{AB}$. Substituting named variables for these lengths, we get

$$\frac{AF}{b} = \frac{b}{c} \Longrightarrow AF = \frac{b^2}{c}.$$

Applying the same logic to $\triangle CFB$, we get $\triangle CFB \sim \triangle ABC$, so $\frac{BF}{BC} = \frac{BC}{AB}$. Substituting, we get

$$\frac{BF}{a} = \frac{a}{c} \Longrightarrow BF = \frac{a^2}{c}.$$

Since F is between A and B, we have $AB = AF + FB$; substituting our found values for AF and FB, we get

$$\begin{aligned}
c = AB &= AF + FB\\
c &= \frac{b^2}{c} + \frac{a^2}{c}\\
c^2 &= b^2 + a^2. && \text{Q.E.D.}
\end{aligned}$$

3. Now you will prove the trig identities.

(a) Draw and label a right triangle and a unit circle, then write trig definitions for $\cos$, $\sin$, $\tan$, and $\sec$ in terms of your drawing.

The scenario is depicted in Figure 3. By the definition of sine and cosine, we have $\sin\theta = AP$ and $\cos\theta = OA$. Since $\triangle OAP \sim \triangle OPT$ by AA Similarity, we have $\frac{TP}{OP} = \frac{AP}{OA}$. Substituting known values, we get

$$\frac{TP}{1} = \frac{\sin\theta}{\cos\theta} \Longrightarrow TP = \tan\theta.$$

Also, $\triangle OAP \sim \triangle OKS$ by AA, so $\frac{OS}{OK} = \frac{1}{\cos\theta}$. Similarly, we have

$$\frac{OS}{1} = \frac{1}{\cos\theta} \Longrightarrow OS = \sec\theta.$$

Finally, as an alternate interpretation of $\tan$, we have $\frac{KS}{OK} = \frac{AP}{OA}$, so

$$\frac{KS}{1} = \frac{\sin\theta}{\cos\theta} \Longrightarrow KS = \tan\theta.$$

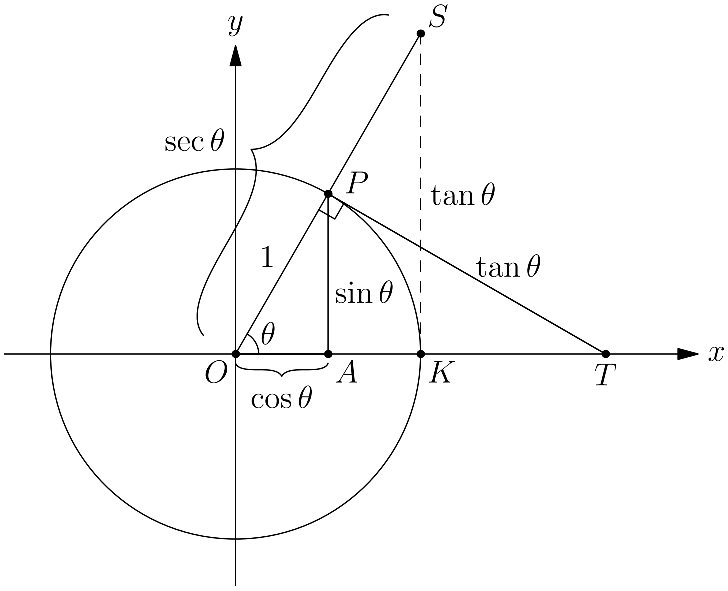

Figure 3: The right triangle and unit circle.

(b) Use a right triangle and the definitions of $\sin$ and $\cos$ to find and prove a value for $\sin^2\theta + \cos^2\theta$.

Referring back to Figure 3, focus on $\triangle OAP$. It is a right triangle with side lengths $a = \cos\theta$, $b = \sin\theta$, and $c = 1$. By the Pythagorean theorem, we have

$$\begin{aligned} OA^2 + AP^2 &= OP^2 && \text{[Pythagorean theorem]} \\ \cos^2\theta + \sin^2\theta &= 1^2 && \text{[Substitution]} \\ \sin^2\theta + \cos^2\theta &= 1 && \text{[Rearrange]} \end{aligned}$$

(c) Use the picture of the unit circle in Figure 4 to find and prove a value for $\cos(A - B)$. Note that D_1 and D_2 are the same length because they subtend the same size arc of the circle. Set them equal and work through the algebra, using the distance formula and part (b) of this problem.

We have $D_1 = D_2$, so

$$\begin{aligned} D_1^2 &= D_2^2 \\ (\cos A - \cos B)^2 + (\sin A - \sin B)^2 &= (\cos(A-B) - 1)^2 + \sin^2(A-B) \\ \cos^2 A - 2\cos A\cos B + \cos^2 B + \sin^2 A - 2\sin A\sin B + \sin^2 B &= \cos^2(A-B) - 2\cos(A-B) + \\ &\quad 1 + \sin^2(A-B) \end{aligned}$$

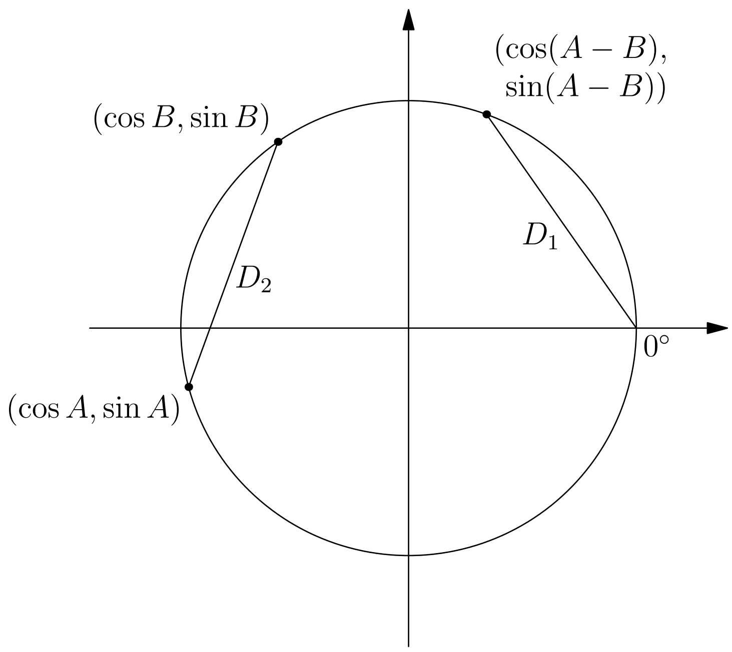

Figure 4: Scenario in Problem 3.

$$
\begin{aligned}
(\cos^2 A + \sin^2 A) + (\cos^2 B + \sin^2 A) - 2\sin A \sin B &= (\cos^2(A-B) + \sin^2(A-B)) + \\
&\quad 1 - 2\cos(A-B) \\
\cancelto{0}{1+1} - 2\sin A \sin B - 2\cos A \cos B &= \cancelto{0}{1+1} - 2\cos(A-B) \\
2\sin A \sin B + 2\cos A \cos B &= 2\cos(A-B) \\
\sin A \sin B + \cos A \cos B &= \cos(A-B).
\end{aligned}
$$

Q.E.D.

4. Write down as many trig identities as you can. There's no need to prove all of these right now.

$$
\begin{array}{rrr}
\sin(A+B) = & \sin(A-B) = & \cos(A+B) = \\
\tan(A+B) = & \tan(A-B) = & \sin(2A) = \\
\cos(2A) = & \tan(2A) = & \sin\left(\frac{A}{2}\right) = \\
\cos\left(\frac{A}{2}\right) = & \tan\left(\frac{A}{2}\right) = &
\end{array}
$$

You should probably memorize these for convenience.

$$
\begin{aligned}
\sin(A+B) &= \sin A \cos B + \cos A \sin A \\
\sin(A-B) &= \sin A \cos B - \cos A \sin B \\
\cos(A+B) &= \cos A \cos B - \sin A \sin B \\
\tan(A+B) &= \frac{\tan A + \tan B}{1 - \tan A \tan B} \\
\tan(A-B) &= \frac{\tan A - \tan B}{1 + \tan A \tan B} \\
\sin(2A) &= 2\sin A \cos A \\
\cos(2A) &= 2\cos^2 A - 1 = 1 - 2\sin^2 A = \cos^2 A - \sin^2 A \\
\tan(2A) &= \frac{2\tan A}{1 - \tan^2 A} \\
\sin\left(\frac{A}{2}\right) &= \pm\sqrt{\frac{1 - \cos A}{2}} \\
\cos\left(\frac{A}{2}\right) &= \pm\sqrt{\frac{1 + \cos A}{2}}
\end{aligned}
$$

$$\tan\left(\frac{A}{2}\right) = \frac{\sin A}{1+\cos A} = \frac{1-\cos A}{\sin A}$$

5. Let's review complex numbers and DeMoivre's theorem.

(a) Recall that you can write a complex number both in Cartesian and polar forms. Let

$$a + bi = (a, b) = (r\cos\theta, r\sin\theta) = r\cos\theta + ir\sin\theta.$$

What is r in terms of a and b?

r is just the distance to the origin from $a + bi$. Draw a right triangle as shown in Figure 5. By the pythagorean theorem, $r = \sqrt{a^2 + b^2}$.

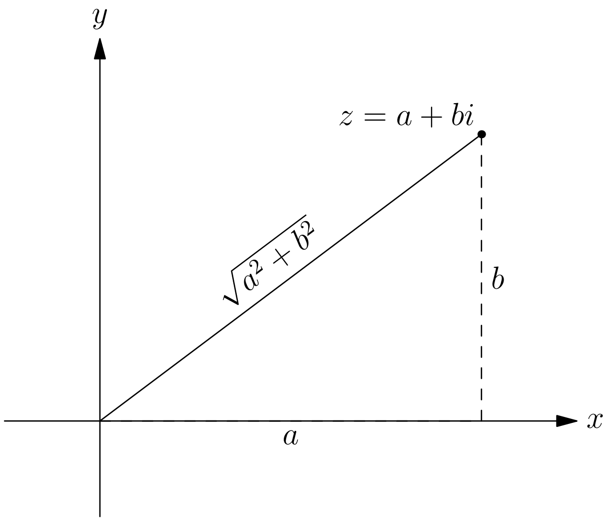

Figure 5: $a + bi$ in the complex plane.

(b) Multiply $(a + bi)(c + di)$ out using FOIL.

$$\begin{aligned}(a+bi)(c+di) &= ac + adi + bci + (bi)(di)\\ &= ac + (ad+bc)i - bd\\ &= ac - bd + (ad+bc)i.\end{aligned}$$

(c) Convert the two multiplicands[1] to polar form, noting that the two lengths and angles are different numbers. Call them $r_1(\cos\theta + i\sin\theta)$ and $r_2(\cos\phi + i\sin\phi)$.

We have $r_1 = \sqrt{a^2+b^2}$ and $\theta = \tan^{-1}\left(\frac{b}{a}\right)$ [2]; similarly, $r_2 = \sqrt{c^2+d^2}$ and $\phi = \tan^{-1}\left(\frac{d}{c}\right)$.

(d) Multiply them, and use your results from Problems 3c and 3d to show that multiplying two complex numbers involves multiplying their lengths and adding their angles. This is DeMoivre's theorem!

$$\begin{aligned}r_1(\cos\theta + i\sin\theta)r_2(\cos\phi + i\sin\phi) &= r_1r_2(\cos\theta\cos\phi - \sin\theta\sin\phi + i(\sin\theta\cos\phi + \cos\theta\sin\phi))\\ &= r_1r_2(\cos(\theta+\phi) + i\sin(\theta+\phi)).\end{aligned}$$

(e) Use part (d) to simplify $(\sqrt{3} + i)^{18}$.

[1] This is the word for parts of a multiplication! So for example, if $a \cdot b = c$, then a and b are the multiplicands.

[2] You can get this from drawing a right triangle.

We have $\sqrt{3}+i = r(\cos\theta + i\sin\theta) = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$.

$$\begin{aligned}
(2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right))^{18} &= 2^{18}\cdot\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^{18} \\
&= 2^{18}\cdot\underbrace{\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\cdots\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)}_{\text{18 copies}} \\
&= 2^{18}\cdot\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\underbrace{\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\cdots\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)}_{\text{16 copies}} \\
&= \vdots \\
&= 2^{18}\cdot(\cos 3\pi + i\sin 3\pi) \\
&= 2^{18}\cdot -1 \\
&= -2^{18}.
\end{aligned}$$

6. Here is a review of 2D rotation.

(a) Remember that we can graph complex numbers as 2D ordered pairs in the complex plane. Now, consider the complex number $z = \cos\theta + i\sin\theta$, where θ is fixed. What is the magnitude of z?

We have
$$|z| = \sqrt{\cos^2\theta + \sin^2\theta} = \sqrt{1} = 1.$$

(b) Multiplying $z\cdot(x+yi)$ yields a rotation of the point (x,y) counterclockwise by the angle θ around the origin. What if we wanted to rotate clockwise by θ instead?

We can multiply by the conjugate of z, since

$$\overline{z} = \cos\theta - i\sin\theta = \cos -\theta + i\sin -\theta.$$

Thus, the operation is $\overline{z}\cdot(x+yi)$ to rotate clockwise by θ.

7. Rotate the following conics by (i) $30°$, (ii) $45°$, and (iii) θ:

(a) $x^2 - y^2 = 1$

i. $30°$

We make the substitution $x' = x\cos 30° - y\sin 30° = \frac{\sqrt{3}}{2}x - \frac{y}{2}$ and $y' = x\sin 30° + y\cos 30° = \frac{x}{2} + \frac{\sqrt{3}}{2}y$:

$$\begin{aligned}
x'^2 - y'^2 &= 1 \\
\left(\frac{\sqrt{3}}{2}x - \frac{y}{2}\right)^2 - \left(\frac{x}{2} + \frac{\sqrt{3}}{2}y\right)^2 &= 1 \\
x^2/2 - \sqrt{3}xy - y^2/2 &= 1
\end{aligned}$$

ii. $45°$

We make the substitution $x' = x\cos 45° - y\sin 45° = \frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y$ and $y' = x\sin 45° + y\cos 45° = \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y$:

$$\begin{aligned}
x'^2 - y'^2 &= 1 \\
\left(\frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y\right)^2 - \left(\frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y\right)^2 &= 1 \\
-2xy &= 1.
\end{aligned}$$

iii. θ

We make the substitution $x' = x\cos\theta - y\sin\theta$ and $y' = x\sin\theta + y\cos\theta$:

$$\begin{aligned} x'^2 - y'^2 &= 1 \\ (x\cos\theta - y\sin\theta)^2 - (x\sin\theta + y\cos\theta)^2 &= 1. \end{aligned}$$

(b) $\frac{x^2}{16} - \frac{y^2}{9} = 1.$

i. $30°$

We make the substitution $x' = x\cos 30° - y\sin 30° = \frac{\sqrt{3}}{2}x - \frac{y}{2}$ and $y' = x\sin 30° + y\cos 30° = \frac{x}{2} + \frac{\sqrt{3}}{2}y$:

$$\begin{aligned} \frac{x'^2}{16} - \frac{y'^2}{9} &= 1 \\ \frac{\left(\frac{\sqrt{3}}{2}x - \frac{y}{2}\right)^2}{16} - \frac{\left(\frac{x}{2} + \frac{\sqrt{3}}{2}y\right)^2}{9} &= 1 \\ \frac{1}{576}(11x^2 - 50\sqrt{3}xy - 39y^2) &= 1. \end{aligned}$$

ii. $45°$

We make the substitution $x' = x\cos 45° - y\sin 45° = \frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y$ and $y' = x\sin 45° + y\cos 45° = \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y$:

$$\begin{aligned} \frac{x'^2}{16} - \frac{y'^2}{9} &= 1 \\ \frac{\left(\frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y\right)^2}{16} - \frac{\left(\frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y\right)^2}{9} &= 1 \\ \frac{1}{288}(-x - 7y)(7x + y) &= 1 \end{aligned}$$

iii. θ

We make the substitution $x' = x\cos\theta - y\sin\theta$ and $y' = x\sin\theta + y\cos\theta$:

$$\begin{aligned} \frac{x'^2}{16} - \frac{y'^2}{9} &= 1 \\ \frac{(x\cos\theta - y\sin\theta)^2}{16} - \frac{(x\sin\theta + y\cos\theta)^2}{9} &= 1. \end{aligned}$$

(c) $y^2 = 4Cx$

i. $30°$

We make the substitution $x' = x\cos 30° - y\sin 30° = \frac{\sqrt{3}}{2}x - \frac{y}{2}$ and $y' = x\sin 30° + y\cos 30° = \frac{x}{2} + \frac{\sqrt{3}}{2}y$:

$$\begin{aligned} y'^2 &= 4Cx' \\ \left(\frac{x}{2} + \frac{\sqrt{3}}{2}y\right)^2 &= 4C\left(\frac{\sqrt{3}}{2}x - \frac{y}{2}\right). \end{aligned}$$

ii. $45°$

We make the substitution $x' = x\cos 45° - y\sin 45° = \frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y$ and $y' = x\sin 45° + y\cos 45° = \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y$:

$$y'^2 = 4Cx'$$
$$\left(\frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y\right)^2 = 4C\left(\frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y\right)$$
$$\frac{1}{2}(x+y)^2 = 2C\sqrt{2}(x-y)$$

iii. θ

We make the substitution $x' = x\cos\theta - y\sin\theta$ and $y' = x\sin\theta + y\cos\theta$:

$$y'^2 = 4Cx'$$
$$(x\cos\theta - y\sin\theta)^2 = 4C\left(x\sin\theta + y\cos\theta\right).$$

$\bullet$	I	A	B	C	D	E
I						
A			E			
B						
C						
D						
E						

Figure 1: Unfilled 3-post snap group table.

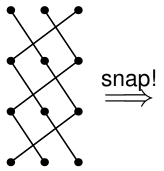

Figure 2: $E \bullet E \bullet E = I$; E has period 3.

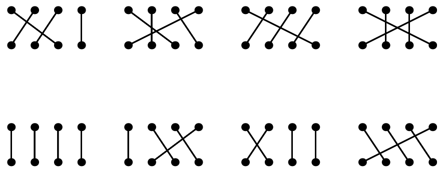

Figure 3: Some 4-post group elements.

2 It's a Snap

1. Fill out a 6×6 table like the one in Figure 1, showing the results of each of the 36 possible snaps, where $X \bullet Y$ is in X's row and Y's column. $A \bullet B = E$ is done for you.

$\bullet$	I	A	B	C	D	E
I	I	A	B	C	D	E
A	A	I	E	D	C	B
B	B	D	I	E	A	C
C	C	E	D	I	B	A
D	D	B	C	A	E	I
E	E	C	A	B	I	D

2. Would this table look different if you wrote the elements A through E in a different order?

Yes; here's an example:

$\bullet$	I	E	A	D	B	C
I	I	E	A	D	B	C
E	E	D	C	I	A	B
A	A	B	I	C	E	D
D	D	I	B	E	C	A
B	B	C	D	A	I	E
C	C	A	E	B	D	I

3. Which of the elements is the **identity element** K, such that $X \bullet K = K \bullet X = X$ for all X?

The identity element is I, since $I \bullet A = A \bullet I = A$, $I \bullet B = B \bullet I = B$, and so forth.

4. Does every element have an inverse; can you get to the identity element from every element using only one snap?

Yes you can. The inverses are shown below.

$$I \leftrightarrow I$$
$$A \leftrightarrow A$$
$$B \leftrightarrow B$$
$$C \leftrightarrow C$$
$$D \leftrightarrow E$$

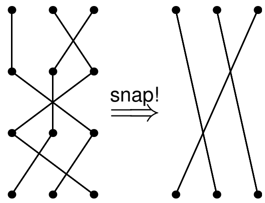

Figure 4: A 4×3 grid of posts has a unique result after the snap operation.

Note that the inverse of an element X is denoted X^{-1}.

5.

(a) Is the snap operation commutative (does $X \bullet Y = Y \bullet X$ for all X, Y)?

No, the snap operation is not commutative. For example, $A \bullet B = E$, but $B \bullet A = D$.

(b) Is the snap operation associative (does $(X \bullet Y) \bullet Z = X \bullet (Y \bullet Z)$ for all X, Y, Z)?'

Yes, the snap operation is associative. You can rationalize this as the fact that a 4×3 grid of posts is snapped to a single configuration, regardless of which middle row you remove first. This is shown in Figure 4.

6.

(a) For any elements X, Y, is there always an element Z so that $X \bullet Z = Y$?

Yes, there is always a way to get from one element to another in one snap. You can prove this by construction. If element X connects n_1 to n_1', n_2 to n_2', and n_3 to n_3', and element Y connects m_1 to m_1', m_2 to m_2', and m_3 to m_3', then the solution Z to $X \bullet Z = Y$ connects m_1 to $n_{m_1'}$, m_2 to $n_{m_2'}$, and m_3 to $n_{m_3'}$.

That's probably a bit hard to understand, but a more clever solution uses inverses. We multiply X by X^{-1}, then by Y:

$$X \bullet X^{-1} \bullet Y = Y.$$

But since every element has an inverse, and the snap operation is associative, we have

$$\begin{aligned} X \bullet (X^{-1} \bullet Y) &= Y \\ \implies Z &= X^{-1} \bullet Y. \end{aligned}$$

In this way, we have constructed the element Z.

(b) For (a), is Z always unique?

Yes. To show this, we use a proof by contradiction. Suppose we have two solutions Z_1 and Z_2 so that $Z_1 \neq Z_2$ and

$$\begin{aligned} X \bullet Z_1 &= Y \\ X \bullet Z_2 &= Y. \end{aligned}$$

We multiply to the left by Y^{-1}. Note that since the snap operation is not commutative, we need to multiply both sides on a specific side:

$$\begin{aligned} Y^{-1} \bullet X \bullet Z_1 &= Y^{-1} \bullet Y = I \\ Y^{-1} \bullet X \bullet Z_2 &= I \end{aligned}$$

So Z_1, Z_2 are the inverses of $Y^{-1} \bullet X$. But the inverse of an element is unique; we've showed this by listing them all out! Thus, $Z_1 = Z_2$, contradicting our assumption and proving that Z is unique in $X \bullet Z = Y$.

7. If you constructed a 5×5 table using only 5 of the snap elements, the table would not describe a group, because there would be entries in the table not in those 5. Therefore, a group must be **closed** under its operation; if $X, Y \in G$ ($\in$ means "is/are in"), then $X \bullet Y \in G$ for all X, Y. Some subsets, however, do happen to be closed.

   Write valid group tables using exactly 1, 2, and 3 elements from the snap group. These are known as **subgroups**.

Here are tables with 1, 2, and 3 elements:

$\bullet$	I
I	I

$\bullet$	I	A
I	I	A
A	A	I

$\bullet$	I	D	E
I	I	D	E
D	D	E	I
E	E	I	D

8. What do you guess is the complete definition of a mathematical group? (Hint: consider your answers to Problems 3–7.)

(Answers may vary.)

Definition of **group**: A group G is a set of elements together with a **binary operation** that meets the following criteria:

(a) Identity: There is an element $I \in G$ such that for all $X \in G$, $X \bullet I = I \bullet X = X$.

(b) Closure: If X, Y are elements of the group, then $X \bullet Y$ is also an element of the group.

(c) Invertibility: Each element X has an inverse X^{-1} such that $X \bullet X^{-1} = X^{-1} \bullet X = I$.

(d) Associativity: For all elements X, Y, and Z, $X \bullet (Y \bullet Z) = (X \bullet Y) \bullet Z$.

9. Notice that $E \bullet E \bullet E = I$. (See Figure 2.) This means that E has a **period** of 3 when acting upon itself. Which elements have a period of

(a) 1?

I is the only element with a period of 1, since $I = I$.

(b) 2?

A, B, and C have periods of 2, since for each $X \in A, B, C$ we have $X \bullet X = I$.

(c) 3?

D and E have periods of 3, since for each $Y \in D, E$ we have $Y \bullet Y \neq I$, but $Y \bullet Y \bullet Y = I$.

10. Answer the following with the 1, 2, and 4-post snap groups S_1, S_2 and S_4.

(a) How many elements would there be?

S_1 has $1! = 1$ elements. S_2 has $2! = 2$ elements. S_4 has $4! = 24$ elements.

(b) Systematically draw and name them.

Figure 5: Elements of S_1.

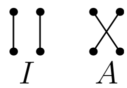

Figure 6: Elements of S_2.

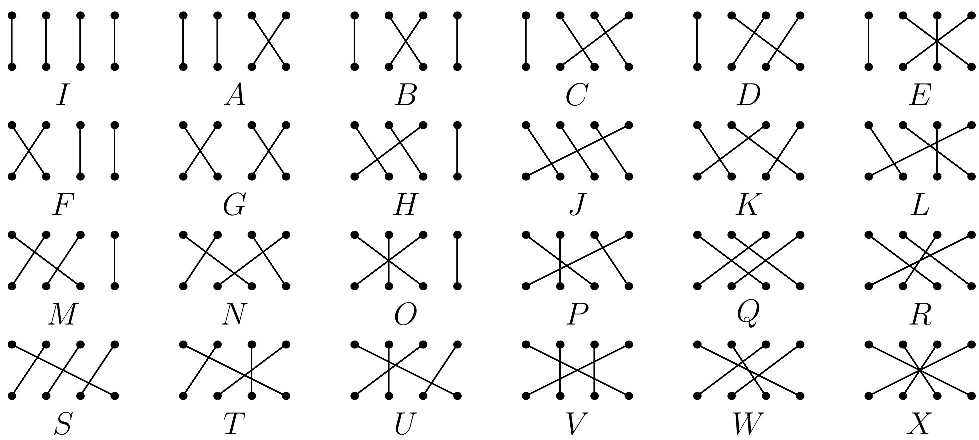

Figure 7: Elements of S_4.

(c) Make a group table of these elements. For 4 posts, instead of creating the massive table, give the number of entries that table would have.

Here are group tables for S_1 and S_2.

$\bullet$	I
I	I

Figure 8: Group table for S_1.

$\bullet$	I	A
I	I	A
A	A	I

Figure 9: Group table for S_2.

The table for S_4 is given at the end of the section in Figure 12 for the curious.

(d) What is the relationship between this new table and your original table?

Both S_1's and S_2's tables are subgroups of the original table for S_3. In turn, S_3 is a subgroup of S_4.

11. Can you think of an easier way to generate a snap group table without drawing all the possible configurations?

(Answers may vary.)

One way to do it is to treat each element as a list of indices. For example, I is the ordered triple $(1, 2, 3)$ because it takes column 1 to 1, 2 to 2, and 3 to 3. A is $(1, 3, 2)$, because it takes 1 to 1, 2 to 3, and 3 to 2.

This makes it a bit easier to calculate, because you can simply substitute indices for each configuration rather than make a drawing. It also makes it easy to write a program to calculate; this is actually how all the tables in this answer key were generated.

12.

(a) How many elements would there be in the 5-post snap group?

There would be $5! = 120$ elements in S_5.

(b) How many entries would its table have?

There would be $5!^2 = 14400$ entries in S_5's table.

(c) What possible periods would its elements have?

This is a more difficult question. We must ask what characteristics of an element determine its period.

If we observe the periodicity of an element with a pretty large period, say one from S_5 with a period of 6, you can see how a large period can arise. This is shown in Figure 11.

We can split up this element into two components: a component with period 3 and one with period 2. Let's call these components C_3 and C_2. After 2 steps, the C_3 has not completed one period, even though C_2. After 3 steps, C_3 has completed one period, but C_2 has gone through $\frac{3}{2}$. It takes $\operatorname{lcm}(2, 3) = 6$ steps before both components "line up!"

All elements can be split up into some number of these cyclic components, even if it doesn't look like it at first glance. For example, the element from S_8 shown in Figure 10 is actually two size 3 and size 2 components. It therefore has a period of $\operatorname{lcm}(2, 3, 3) = 6$. Note that it does *not* have a period of $2 \cdot 3 \cdot 3 = 18$!

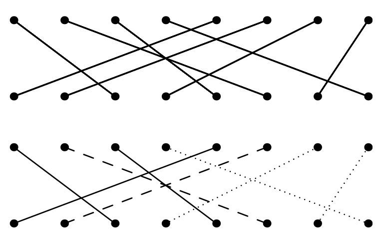

Figure 10: This element from S_8 has components of size $2, 3, 3$.

For S_5, we can split it up into components of size $1, 1, 1, 1, 1$, giving period 1; components of size $1, 1, 1, 2$, giving period 2; components of size $1, 1, 3$, giving period 3; components of size $1, 4$, giving period 4; a component of size 5, giving period 5; and component of size $1, 2, 3$, giving period 6. Thus, periods $1, 2, 3, 4, 5, 6$ are achievable.

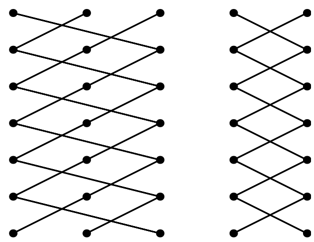

Figure 11: This element from S_5 has a period of 6.

(d) Extend your answers for a–c to M posts per row.

This is rather straightforward. There are (a) $M!$ elements in the M-post snap group, and thus (b) $M!^2$ elements in the corresponding group table. The possible periods are harder to calculate, but they can be generated like so:

Let integers $x_i > 0$ and $\sum_i x_i = M$. In other words, the sum of all x_i is M. Then $\operatorname{lcm}(x_1, x_2, \cdots, x_n)$ is a valid period; the least common multiple of all x_i is a possible period.

For fun: in set builder notation, we have the set of possible periods P_n for the n-post snap group as

$$P_n = \left\{ \operatorname{lcm}(x_1, x_2, \cdots, x_n) \;\middle|\; x_i \in \mathbb{Z}^+ \wedge \sum_i x_i = n \right\}.$$

The maximum such period (i.e. $\max P_n$) is actually known as Landau's function, $g(n)$.

13. As we learned, a **permutation** of some things is an order in which they can be arranged. What is the relationship between the set of permutations of m things and the m-post snap group?

We can make a pretty simple correspondence between a permutation of m things and an element of the m-post snap group. If we think back to the idea of treating each element of the group as a list of indices, the correspondence is obvious. For example, I is the ordered triple $(1, 2, 3)$ because it takes column 1 to 1, 2 to 2, and 3 to 3. A is $(1, 3, 2)$, because it takes 1 to 1, 2 to 3, and 3 to 2. But each ordered triple is a permutation of $1, 2, 3$! This extends to any m.

$\cdot$	I	A	B	C	D	E	F	G	H	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X
I	I	A	B	C	D	E	F	G	H	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X
A	A	I	D	E	B	C	G	F	K	L	H	J	S	T	U	V	W	X	M	N	O	P	Q	R
B	B	C	I	A	E	D	M	N	O	P	Q	R	F	G	H	J	K	L	T	S	W	X	U	V
C	C	B	E	D	I	A	N	M	Q	R	O	P	T	S	W	X	U	V	F	G	H	J	K	L
D	D	E	A	I	C	B	S	T	U	V	W	X	G	F	K	L	H	J	N	M	Q	R	O	P
E	E	D	C	B	A	I	T	S	W	X	U	V	N	M	Q	R	O	P	G	F	K	L	H	J
F	F	G	H	J	K	L	I	A	B	C	D	E	O	P	M	N	R	Q	U	V	S	T	X	W
G	G	F	K	L	H	J	A	I	D	E	B	C	U	V	S	T	X	W	O	P	M	N	R	Q
H	H	J	F	G	L	K	O	P	M	N	R	Q	I	A	B	C	D	E	V	U	X	W	S	T
J	J	H	L	K	F	G	P	O	R	Q	M	N	V	U	X	W	S	T	I	A	B	C	D	E
K	K	L	G	F	J	H	U	V	S	T	X	W	A	I	D	E	B	C	P	O	R	Q	M	N
L	L	K	J	H	G	F	V	U	X	W	S	T	P	O	R	Q	M	N	A	I	D	E	B	C
M	M	N	O	P	Q	R	B	C	I	A	E	D	H	J	F	G	L	K	W	X	T	S	V	U
N	N	M	Q	R	O	P	C	B	E	D	I	A	W	X	T	S	V	U	H	J	F	G	L	K
O	O	P	M	N	R	Q	H	J	F	G	L	K	B	C	I	A	E	D	X	W	V	U	T	S
P	P	O	R	Q	M	N	J	H	L	K	F	G	X	W	V	U	T	S	B	C	I	A	E	D
Q	Q	R	N	M	P	O	W	X	T	S	V	U	C	B	E	D	I	A	J	H	L	K	F	G
R	R	Q	P	O	N	M	X	W	V	U	T	S	J	H	L	K	F	G	C	B	E	D	I	A
S	S	T	U	V	W	X	D	E	A	I	C	B	K	L	G	F	J	H	Q	R	N	M	P	O
T	T	S	W	X	U	V	E	D	C	B	A	I	Q	R	N	M	P	O	K	L	G	F	J	H
U	U	V	S	T	X	W	K	L	G	F	J	H	D	E	A	I	C	B	R	Q	P	O	N	M
V	V	U	X	W	S	T	L	K	J	H	G	F	R	Q	P	O	N	M	D	E	A	I	C	B
W	W	X	T	S	V	U	Q	R	N	M	P	O	E	D	C	B	A	I	L	K	J	H	G	F
X	X	W	V	U	T	S	R	Q	P	O	N	M	L	K	J	H	G	F	E	D	C	B	A	I

Figure 12: Group table for S_4.

3 From Snaps to Flips

1. The six "operations" are considered **isometries**. Isometries are ways of mapping the triangle to itself, preserving shape and location. Are there any others for this triangle?

There are no other isometries for this triangle; our list of operations is complete. To see why, note that the vertices must exchange places. At most there is $3! = 6$ ways to do this, so we have already achieved the maximum possible number of isometries.

$\cdot$	I	A	B	C	D	E
I						
A					B	
B						
C						
D						
E						

Figure 1: Unfilled D_3 group table.

2. As with the snap group, we can make a group table for the flip group. Fill out a table like the one in Figure 1 in your notebook. Like the snap group table, the top row indicates what operation is done first and the left column indicates what's done second, so that XY is in the X^{th} row and Y^{th} column. $AD = B$ is done for you.

The completed table is shown in Figure 2.

3. What is the relationship between the tables for the snap group S_3 and the flip group D_3?

D_3's table is S_3's table flipped over the top left–bottom right diagonal, and vice versa. Contrast D_3 from Figure 2 to S_3 in Figure 3. If these were matrices, one would be the transpose of the other: we'll get to that later.

$\cdot$	I	A	B	C	D	E
I	I	A	B	C	D	E
A	A	I	D	E	B	C
B	B	E	I	D	C	A
C	C	D	E	I	A	B
D	D	C	A	B	E	I
E	E	B	C	A	I	D

Figure 2: Completed D_3 group table.

$\bullet$	I	A	B	C	D	E
I	I	A	B	C	D	E
A	A	I	E	D	C	B
B	B	D	I	E	A	C
C	C	E	D	I	B	A
D	D	B	C	A	E	I
E	E	C	A	B	I	D

Figure 3: Completed S_3 group table from the last chapter.

4. Check your understanding by defining isomorphic in your own words.

(Answers may vary.)

Isomorphic means that two groups have the same structure. Isomorphic means that there is a correspondence between the elements of two groups so that the correspondence preserves the order. In the language of abstract algebra, an isomorphism between groups A and B exists if there is a homomorphism from A to B and from B to A.

5.

(a) Make a table for only the rotations of D_3, a subgroup of D_3.

The table is shown below. Note that the identity element I is a rotation of 0.

$\cdot$	I	D	E
I	I	D	E
D	D	E	I
E	E	I	D

Interestingly, this subgroup is a commutative group, also known as an abelian group.

(b) Which subgroup of the snap group S_3 is isomorphic to the subgroup in (a)?

The same elements (nominally) make the same subgroup:

$\bullet$	I	D	E
I	I	D	E
D	D	E	I
E	E	I	D

6. What shape's dihedral group is isomorphic to

(a) the two post snap group S_2?

The dihedral group of a line segment is isomorphic to S_2. After all, you can only reflect it over its midpoint, which is the other element of S_2 besides the identity. We can also think of this as permuting the two endpoints or vertices of a line segment.

(b) the one post snap group S_1?

The dihedral group of a point is isomorphic to S_1, because the only element is the identity element. This is permuting the one vertex of a point.

(c) the four post snap group S_4?

For this question we need to think 3 dimensions. There are four vertices to permute, but we can't do that on a square since diagonal points will remain on diagonals, as shown in Figure 4.

Instead, we choose the regular tetrahedron, so that there are no "diagonals"; every permutation is achievable. Note that rotations and reflections are now in 3 dimensional space, which is a bit difficult to visualize. A sample rotation is depicted in Figure 5.

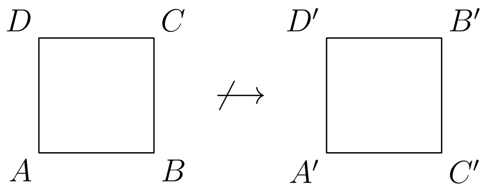

Figure 4: At right is a valid permutation of the vertices, but not a valid isometry of the square.

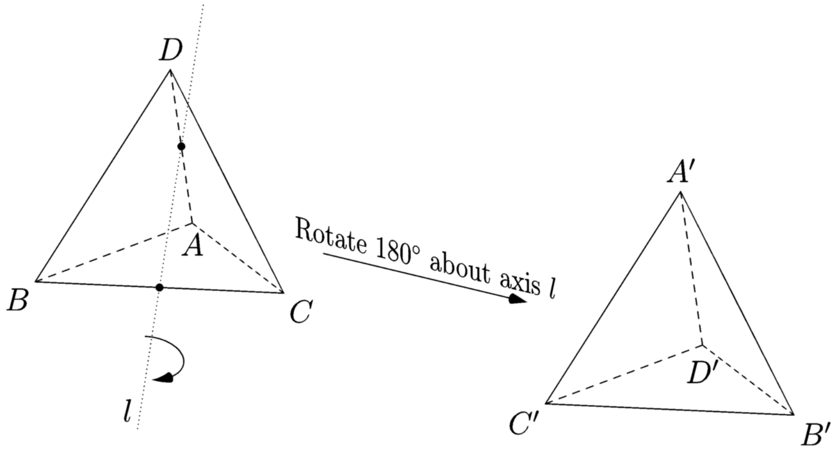

Figure 5: A rotation of the tetrahedron (orthographic view).

(d) the five post snap group S_5?

This is isomorphic to the dihedral group of the 4-dimensional equivalent of the tetrahedron, also known as the regular 4-simplex. A projection is shown in Figure 6, but it cannot be faithfully represented on this paper.

7. Find a combination of A and D that yields C.

8. We call A and D **generators** of the group because every element of the group is expressible as some combination of As and Ds. For convenience, let's call A "f" since it's a flip, and call D "r" meaning a $120°$ rotation counterclockwise. Then, for example, fr^2 is a rotation of $2 \cdot 120° = 240°$, followed by a flip across the A axis, equivalent to our original C. Make a new table using I, r, r^2, f, fr, and fr^2 as elements, like the one in Figure 7. *Note that the element order is different!*

$\cdot$	I	r	r^2	f	fr	fr^2
I						
r				fr^2		
r^2						
f						
fr						
fr^2						

Figure 7: Unfilled alternate D_3 table.

The filled table is shown in Figure 8 below.

$\cdot$	I	r	r^2	f	fr	fr^2
I	I	r	r^2	f	fr	fr^2
r	r	r^2	I	fr^2	f	fr
r^2	r^2	I	r	fr	fr^2	f
f	f	fr	fr^2	I	r	r^2
fr	fr	fr^2	f	r^2	I	r
fr^2	fr^2	f	fr	r	r^2	I

Figure 8: Completed alternate D_3 table.

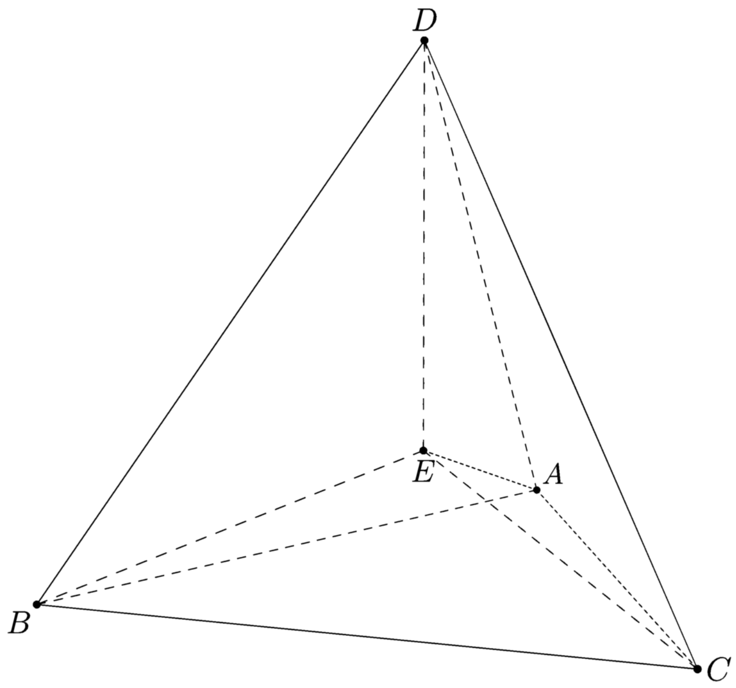

Figure 6: A 3D projection of the regular 4-simplex. In a true realization, every line segment here would be the same length.

Note that $I = I$, $A = f$, $B = fr$, $C = fr^2$, $D = r$, and $E = r^2$.

9. What other pairs of elements could you have used to generate the table?

You could also use any of the following pairs: A, E, B, D, B, E, C, D, C, E, A, B, B, C, A, C. In essence, you can generate it with any rotation element and any reflection element, or with any two reflection elements.

10. You should notice the 3×3 table of a group you've already described in the top-left corner of your table. What is it, and what are the two possible generators of this three-element group?

This is the cyclic group of order 3, C_3, also known as the rotation group of the equilateral triangle. The two possible generators are r and r^2.

11. Explain why each element of the flip group D_3 has the period it has.

I has a period of 1 because it is the identity. A, B, C have periods of 2 because they are reflections, so they are their own inverse transformation. D and E are rotations of a multiple of $1/3$ of a turn. Since 3 is a prime, they take 3 iterations to resolve, and thus have period 3.

12. Some pairs of elements of the flip group are two-element subgroups. Which pairs are they?

These would be the pairs I, A, I, B, and I, C, since $A \cdot A = B \cdot B = C \cdot C = I$ so the subgroup is closed. These are shown in Figure 9.

$\cdot$	I	A
I	I	A
A	A	I

$\cdot$	I	B
I	I	B
B	B	I

$\cdot$	I	C
I	I	C
C	C	I

Figure 9: The three two-element subgroups.

13. One of the elements forms a one-element subgroup. Which is it?

The element I forms the so-called trivial group, or the only group of order 1; this is shown in Figure 10. It is not very interesting.

$\cdot$	I
I	I

Figure 10: The trivial group.

14. Addition of two numbers is a binary operation, while addition of three numbers is not. In logic, $\wedge$ (and) and $\vee$ (or) are binary operations, but $\neg$ (not) is not. Define binary operation in your own words, and name some other binary operations.

(Answers may vary.)
A binary operation is an operation with two arguments.
Some binary operations:

1. multiplication
2. exponentiation
3. addition
4. subtraction
5. division
6. modulo operator
7. bitwise OR
8. bitwise AND
9. snap operation (●)
10. function convolution

15. In your original flip group table, what is

(a) The identity element?

The identity element is I.

(b) The inverse of A?

The inverse of A is also A, since it is a reflection.

(c) The inverse of E?

The inverse of E is D, since $240° + 120° \equiv 0°$.

4 Rotation and Reflection Groups

16. Notice that the original dihedral group had twice as many elements as the rotation group. Why?

(Answers may vary.)

There are a couple ways to think about this, but an intuitive way is to consider a “mirror world” of reflection and the “normal world” where the orientation in normal. Here, orientation is not absolute orientation, but the difference between clockwise and counterclockwise. For chemistry nerds, it is like chirality. Rotation preserves orientation, but reflection does not. Instead, it takes us between these two “worlds.” Thus, it allows twice the number of elements.

17. Make and justify a conjecture extending this observation to the dihedral groups of other shapes like rectangles, squares, hexagons, cubes, etc.

(Answers may vary.)

Conjecture: The dihedral groups of a shape has twice the order of its rotation group.

Informal Justification: A shape can be flipped or not, and it can have whatever rotational isometries applied to it whether it’s flipped or not. Thus, the dihedral group allows for twice the number of elements as the rotation group.

18. Let r be a $180°$ rotation, x be a reflection over the x-axis, and y be a reflection over the y-axis. Write a table for the dihedral group of the rectangle, recalling that the allowed isometries are reflections and rotations. How does this table differ from the dihedral group of the equilateral triangle?

$\cdot$	I	r	x	y
I	I	r	x	y
r	r	I	y	x
x	x	y	I	r
y	y	x	r	I

The table is shown above. The four elements are shown acting on a rectangle with “P” painted on it in Figure 1 to show the transformation a bit better.

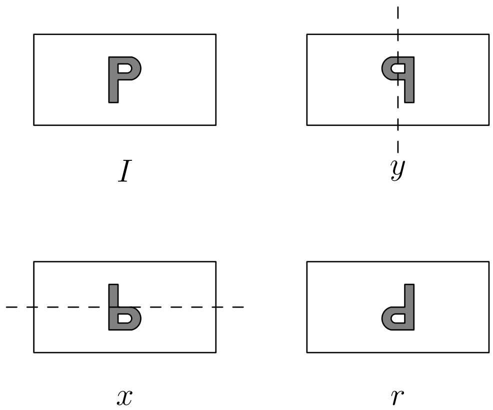

Figure 1: A rectangle AMBULATES and FLIPS around.

This differs from the dihedral group of the equilateral triangle, D_3, in several ways. The most obvious is that there are only 4 elements. Also, all elements besides I in this group have a period of 2, while D_3 has two elements with a period of 3.

19. Write a table for the rotation group of the square, with 4 elements and 16 entries. Compare this table to problem 3.

The elements are $I = r_0$, $r = r_{90}$, $r^2 = r_{180}$, and $r^3 = r_{270}$. The table is shown below.

$\cdot$	I	r	r^2	r^3
I	I	r	r^2	r^3
r	r	r^2	r^3	I
r^2	r^2	r^3	I	r
r^3	I	r	r^2	r^3

While this has the same order as the rectangle's dihedral group, it has a different structure. There are two elements with period 4 (r, r^3) and one element with period 2 (r^2).

For each of the following problems, find the following:

(a) Number of elements; this is known as the **order**

(b) If order < 10, the set of elements; otherwise, an explanation of how you know the order

(c) A smallest possible **generating set**; in other words, the list of elements which generate a group[3]

(d) Whether the group is **commutative**; in other words, whether its operation $X \cdot Y$ doesn't care about the order of its operands (X and Y)

20. Rectangle under rotation

(a) Number of elements

This group has two elements, the identity and the rotation of $180°$.

(b) If order < 10, the set of elements; otherwise, an explanation of how you know the order

As stated, they are the identity I and the rotation r of $180°$, as shown in Figure 2.

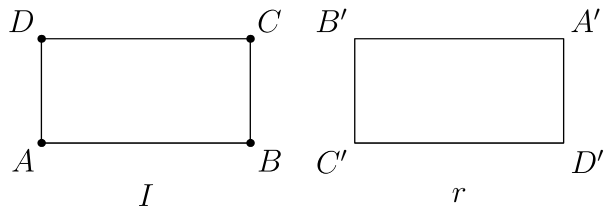

Figure 2: Rectangle under rotation.

(c) A smallest possible **generating set**

The smallest possible generating set is the singleton $\{r\}$.

(d) Whether the group is **commutative**

The group is commutative, since it's only comprised of rotations, which commute.

21. Rectangle under reflection

We already considered this in problem 3.

(a) Number of elements

There are 4 elements in this group.

(b) If order < 10, the set of elements; otherwise, an explanation of how you know the order

The elements are the identity I, rotation r by $180°$, reflection x over the x-axis, and reflection y over the y-axis.

(c) A smallest possible **generating set**

(Answers may vary.)
$\{r, x\}$, $\{r, y\}$, and $\{x, y\}$ all generate the group. No single element, however, can generate the group.

(d) Whether the group is **commutative**

This group is commutative.

22. Square under rotation

[3] There may be multiple generating sets of the same size.

Again, we have considered this group before.

(a) Number of elements

There are 4 elements.

(b) If order < 10, the set of elements; otherwise, an explanation of how you know the order

The elements are rotations $I = r_0$, $r = r_{90}$, $r^2 = r_{180}$, and $r^3 = r_{270}$.

(c) A smallest possible **generating set**

(Answers may vary.)
Both $\{r\}$ and $\{r^3\}$ generate the group, because $1, 3$ are coprime to 4.

(d) Whether the group is **commutative**

The group is commutative, since it consists of all rotations.

23. Square under reflection

(a) Number of elements

There are 8 elements in this group. We can quickly see this by noting that it is the dihedral group of the square, which has twice the order of the rotation group of the square. We just found that had 4 elements, and $2 \cdot 4 = 8$.

(b) If order < 10, the set of elements; otherwise, an explanation of how you know the order

The elements are as follows:
Rotations $I = r_0$, $r = r_{90}$, $r^2 = r_{180}$, and $r^3 = r_{270}$; reflections $f =$ flip over the x-axis, $fr = r$ then f, fr^2 and fr^3.
Recall that rotations can be generated by a sequence of two reflections.
Each of these elements is shown in Figure 3.

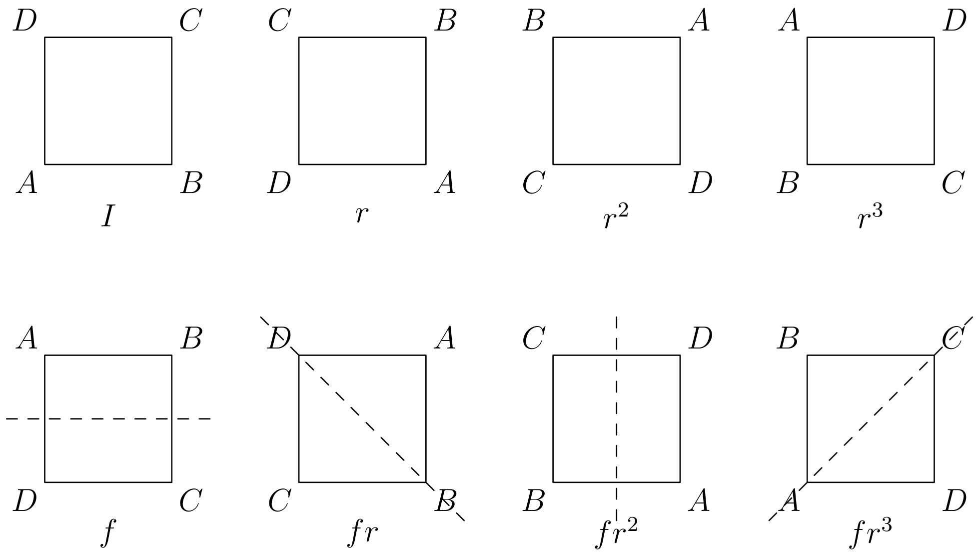

Figure 3: Reflections of a square.

(c) A smallest possible **generating set**

(Answers may vary.)
Any pair of a rotation and flip will generate the set, except for $\{r^2, fr^2\}$ and $\{r^2, f\}$; these will produce the rectangle group instead. Any pair of two flips will also work. As an example of both of these, both $\{r^2, fr^3\}$ and $\{f, fr\}$ will generate the group.

(d) Whether the group is **commutative**

This group is not commutative. For example, $fr = fr$, but $rf = fr^3$.

24. Square prism under rotation

(a) Number of elements

(b) If order < 10, the set of elements; otherwise, an explanation of how you know the order

(c) A smallest possible **generating set**

(d) Whether the group is **commutative**

25. Square prism under reflection

(a) Number of elements

(b) If order < 10, the set of elements; otherwise, an explanation of how you know the order

(c) A smallest possible **generating set**

(d) Whether the group is **commutative**

26. Pentagon under rotation

(a) Number of elements

(b) If order < 10, the set of elements; otherwise, an explanation of how you know the order

(c) A smallest possible **generating set**

(d) Whether the group is **commutative**

27. Pentagon under reflection

(a) Number of elements

(b) If order < 10, the set of elements; otherwise, an explanation of how you know the order

(c) A smallest possible **generating set**

(d) Whether the group is **commutative**

28. Pentagonal prism under rotation

(a) Number of elements

(b) If order < 10, the set of elements; otherwise, an explanation of how you know the order

(c) A smallest possible **generating set**

(d) Whether the group is **commutative**

29. Pentagonal prism under reflection

(a) Number of elements

(b) If order < 10, the set of elements; otherwise, an explanation of how you know the order

(c) A smallest possible **generating set**

(d) Whether the group is **commutative**

30. Pentagonal pyramid under rotation

(a) Number of elements

(b) If order < 10, the set of elements; otherwise, an explanation of how you know the order

(c) A smallest possible **generating set**

(d) Whether the group is **commutative**

31. Pentagonal pyramid under reflection

(a) Number of elements

(b) If order < 10, the set of elements; otherwise, an explanation of how you know the order

(c) A smallest possible **generating set**

(d) Whether the group is **commutative**

32. Tetrahedron (triangular pyramid) under rotation

(a) Number of elements

(b) If order < 10, the set of elements; otherwise, an explanation of how you know the order

(c) A smallest possible **generating set**

(d) Whether the group is **commutative**

33. Tetrahedron under reflection

(a) Number of elements

(b) If order < 10, the set of elements; otherwise, an explanation of how you know the order

(c) A smallest possible **generating set**

(d) Whether the group is **commutative**

34. Cube under rotation

(a) Number of elements

(b) If order < 10, the set of elements; otherwise, an explanation of how you know the order

(c) A smallest possible **generating set**

(d) Whether the group is **commutative**

35. Cube under reflection

(a) Number of elements

(b) If order < 10, the set of elements; otherwise, an explanation of how you know the order

(c) A smallest possible **generating set**

(d) Whether the group is **commutative**