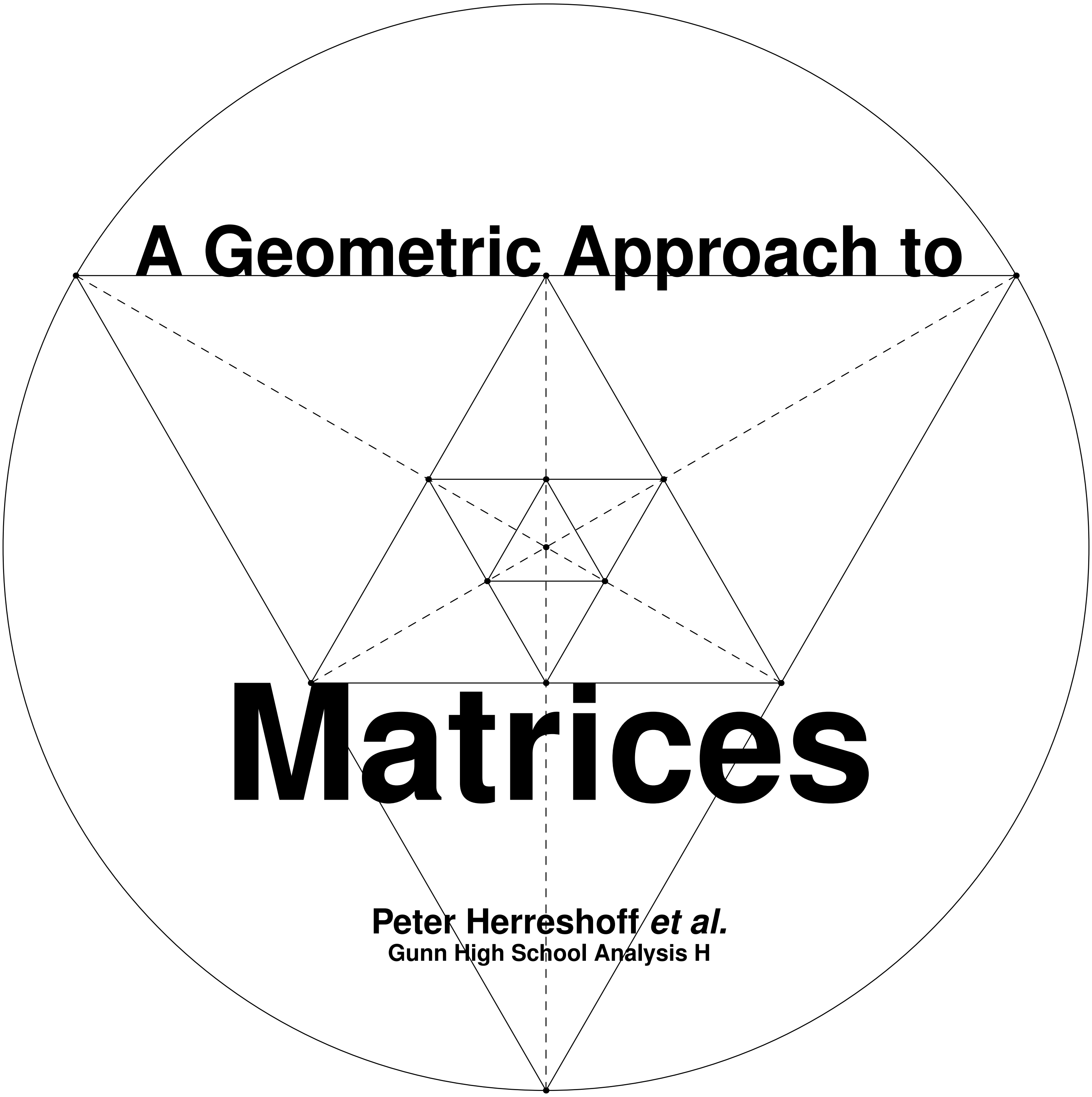

A Geometric Approach to Matrices

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Contents

1 It's a Snap

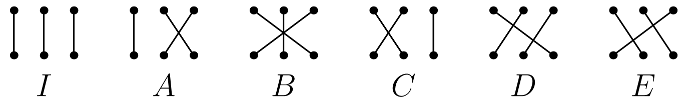

Figure 1: The six possibilities for connections between two rows of three posts.

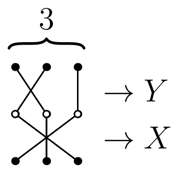

Figure 2: A grid with three strings.

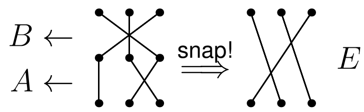

Figure 3: $A \bullet B = E$.

We begin with a problem that ties together ideas from geometry, complex numbers, matrices, combinatorics, and group theory. You likely studied geometry in 9$^{\text{th}}$ grade and complex numbers in 10$^{\text{th}}$ grade, so you should have a basis to start your investigation. The other three topics may be a bit unfamiliar at this point.

Consider a grid of posts with 3 rows and 3 columns. An elastic string is anchored to one post in the top row and one post in the bottom row. As the string descends from top to bottom, it loops around a post in the middle row. Two other strings are anchored and looped in the same way, with the condition that each post has exactly one string touching it. An example is depicted in Figure 2.

Figure 1 shows the six ways two rows can be connected with these rules – convince yourself that these are the only six. I have labeled them as I, A, B, C, D, and E.

Now, look at a 3×3 rectangle of posts, like the one in Figure 2. You should have two configurations X and Y, stacked so that Y is on top of X. When you remove the middle posts (the posts indicated by $\circ$), the elastic string will **snap** to one of the six configurations we drew initially. Let's call this operation "snap", or $\bullet$, so that $X \bullet Y$ reads "X snap Y." Keep in mind that when writing it this way, the bottom configuration X goes first, and the top configuration Y goes last. As an example, $A \bullet B = E$, as shown in Figure 3.

These six configurations form a mathematical **group** under the $\bullet$ operation, and we say that each configuration is an **element** of our group. We will call this the **snap group** of size 3, or S_3. The group is, unsurprisingly, the main concept studied in **group theory**, a topic mentioned at the beginning of this chapter. Let's study the snap group and characterize its properties.

1.1 Problems

1. Fill out a 6×6 table like the one in Figure 4, showing the results of each of the 36 $(6 \cdot 6)$ possible snaps, where $X \bullet Y$ is in X's row and Y's column.

2. Would this table look different if you wrote the elements A through E in a different order?

3. Which of the elements is the **identity element**?

4. Does every element have an inverse (can you get to the identity element from every element using only one snap)?

$\bullet$	I	A	B	C	D	E
I						
A			E			
B						
C						
D						
E						

Figure 4: Unfilled 3-post snap group table.

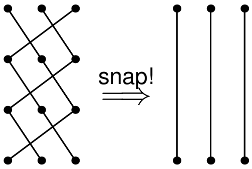

Figure 5: $E \bullet E \bullet E = I$; E has period 3.

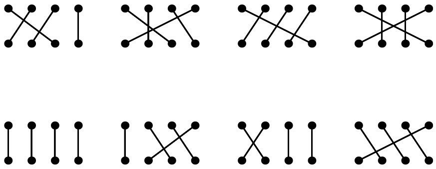

Figure 6: Some 4-post group elements.

5. Is the snap operation commutative (does $X \bullet Y = Y \bullet X$ for all X, Y)?
6. Is the snap operation associative (does $(X \bullet Y) \bullet Z = X \bullet (Y \bullet Z)$ for all X, Y, Z)?
7. (a) For any elements X, Y, is there always an element Z so that $X \bullet Z = Y$?
   (b) Is Z always unique?
8. What do you think is the definition of a mathematical group? (Hint: consider your answers to Problems 2–6.)
9. If you constructed a 5×5 table using only 5 of the snap elements, the table would not describe a group, because there would be entries in the table not in those 5. Therefore, a group must be **closed** under its operation; if $X, Y \in G$ ($\in$ means "is/are in"), then $X \bullet Y \in G$ for all X, Y. Some subsets, however, do happen to be closed. Write valid group tables using exactly 1, 2, and 3 elements from the snap group.
10. Notice that $E \bullet E \bullet E = I$. (See Figure **??**.) This means that E has a period of 3 when acting upon itself. Which elements have a period of 1, 2, and 3?
11. Answer the following with the 1, 2, and 4-post snap groups S_1, S_2 and S_4.
    (a) How many elements would there be?
    (b) Draw and name them systematically.
    (c) Make a group table of these elements. For 4 posts, instead of creating the massive table, give the number of entries that table would have.
    (d) What is the relationship of this new table to your original table?
12. Can you think of an easier way to generate a snap group table without drawing all the possible configurations?
13. (a) How many elements would there be in the 5-post snap group?
    (b) How many entries would its table have?
    (c) What possible periods would its elements have?
    (d) Extend your answers for (a)–(c) to M posts per row.
14. As we learned, a *permutation* of some things is an order they can be arranged in. What is the relationship between the set of permutations of M things and the M-post snap group?

2 From Snaps to Flips

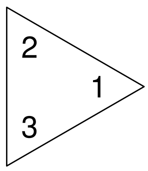

Figure 7: The paper triangle.

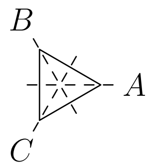

Figure 8: Its axes of reflection.

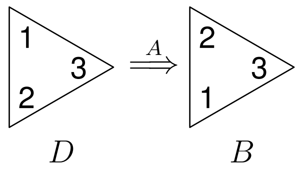

Figure 9: $AD = B$; Notice the RTL evaluation.

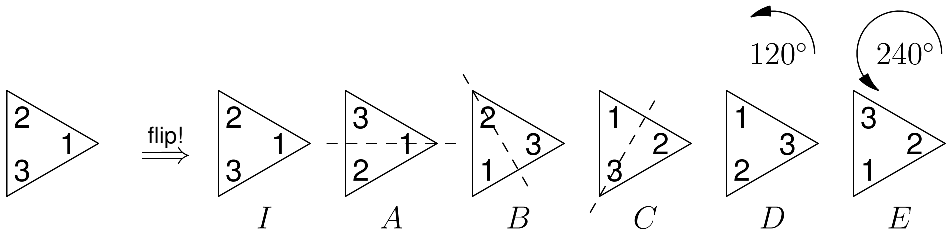

Figure 10: The six ending positions.

You can use a paper/cardboard triangle to help visualize the next concept: cut out an equilateral triangle, label its front vertices 1, 2, and 3 as shown in Figure **??**, and place it down in the shown orientation. Consider the possible ways to move this triangle From this starting position, you can reflect the triangle over one of three axes: A, B, or C, as shown in Figure **??**. You could also rotate the triangle 120° or 240° counterclockwise. The final possible positions are shown in Figure **??**.

Notice that each position corresponds to a different operation which preserves the triangle's location. For example, I means "leave the triangle alone," A means "flip the triangle about the A axis," and D means "rotate the triangle 120° counterclockwise." We can combine these operations to form other operations by writing them in sequence. Unlike most cases, however, we evaluate them right-to-left (RTL) rather than left-to-right (LTR). For example, $AD = B$, as shown in Figure **??**.

These six positions form another group: the **dihedral group** of order 3, or D_3. If we split "dihedral" into "di-" and "-hedral," we see it means "two faces"; these are the two faces of our paper triangle. Let's study the properties of this group.

2.1 Problems

1. The six positions or "operations" are considered to be isometries. Isometries are ways of mapping the triangle to itself, preserving shape and location. Are there any others on this triangle?

2. As with the snap group, we can make a group table for the flip group. Fill out a table like the one in Figure 4 in your notebook. Like the snap group table, the top row indicates what operation is done first and the left column indicates what's done second, so that XY is in the X^{th} row and Y^{th} column.

$\cdot$	I	A	B	C	D	E
I						
A					B	
B						
C						
D						
E						

Figure 11: Unfilled D_3 group table.

3. What is the relationship between the tables for the snap group S_3 and the flip group D_3?

4. S_3 and D_3 are said to be **isomorphic**. Groups A with operation $\bullet$ and B with operation $\star$ are isomorphic if you can find a one-to-one correspondence between the two groups' elements. This means we can find some pairing of elements between the two groups $A_1 \leftrightarrow B_1, A_2 \leftrightarrow B_2, \cdots, A_n \leftrightarrow B_n$ such that $A_j \bullet A_k = A_l \leftrightarrow B_j \star B_k = B_m$.

$\cdot$	I	r	r^2	f	fr	fr^2
I						
r				fr^2		
r^2						
f						
fr						
fr^2						

Figure 12: Unfilled alternate D_3 table.

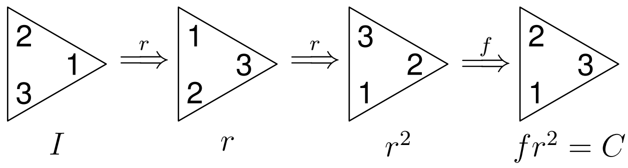

Figure 13: $fr^2 = C$. Again, notice the RTL evaluation.

5. (a) Make a table for only the rotations of D_3, a subgroup of D_3.
   (b) Which subgroup of the snap group S_3 is isomorphic to the subgroup in (a)?
6. What shape's dihedral (rotation and reflection) group is isomorphic to (a) the two post snap group S_2, (b) one post S_1, (c) four posts S_4 (hint: it's not a square), and (d) five posts S_5?
7. Find an combination of A and D that yields C.
8. We call A and D **generators** of the group because every element of the group is expressible as some combination of As and Ds. For convenience, let's call A "f" since it's a flip, and call D "r" meaning a $120°$ rotation counterclockwise. Then, for example, fr^2 is a rotation of $2 \cdot 120° = 240°$, followed by a flip across the A axis, equivalent to our original C. Make a new table using I, r, r^2, f, fr, and fr^2 as elements, like the one in Figure **??**. *Note that the element order is different!*
9. What other pairs of elements could you have used to generate that table?
10. You should notice the 3×3 table of a group you've already described in the top-left corner of your table. What is it, and what are the two possible generators of this three-element group?
11. Explain why each element of the flip group D_3 has the period it has.
12. Some pairs of elements of the flip group are two-element subgroups. What are they?
13. One of the elements forms a one-element subgroup. What is it?

A **group** G is a set of elements together with a **binary operation** that meets the following criteria:

(a) Identity: There is an element $I \in G$ such that for all $X \in G$, $X \bullet I = I \bullet X = X$.

(b) Closure: If X, Y are elements of the group, then $X \bullet Y$ is also an element of the group.

(c) Invertibility: Each element X has an inverse X^{-1} such that $X \bullet X^{-1} = X^{-1} \bullet X = I$.

(d) Associativity: For all elements X, Y, and Z, $X \bullet (Y \bullet Z) = (X \bullet Y) \bullet Z$.

14. Addition of two numbers is a **binary operation**, while addition of three numbers is not. In logic, $\wedge$ (and) and $\vee$ (or) are binary operations, but $\neg$ (not) is not. Define binary operation in your own words, and name some other binary operations.
15. In your original flip group table, what is
    (a) The identity element?
    (b) The inverse of A?
    (c) The inverse of E?

3 Rotation and Reflection Groups

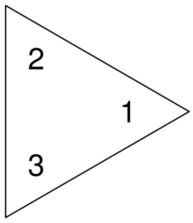

Figure 14: The paper triangle.

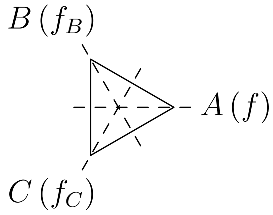

Figure 15: Its axes of reflection.

In the previous section, we started with the dihedral group of the equilateral triangle and discovered it had 6 elements: reflections about three different axes, rotations of $\pm 120°$, and the identity element. We identified a subgroup consisting of the identity I with two rotations r and r^2, and three other subgroups of just the identity and a single reflection.

1. Notice that the original dihedral group had twice as many elements as the rotation group. Why?
2. Make and justify a conjecture extending this observation to the dihedral groups of other shapes like rectangles, squares, hexagons, cubes, etc.
3. Write a table for the dihedral group of the rectangle, recalling that the allowed isometries are reflections and rotations. Let r be a $180°$ rotation, x a reflection over the x-axis, and y be a reflection over the y-axis. How does this table differ from the dihedral group of the equilateral triangle?
4. Write a table for the *rotation group* of the square, with 4 elements and 16 entries. Compare this table to problem 3.

We noticed that the rotation group for the equilateral triangle could be generated by just one of the elements, such as r—rotation by $120°$ counterclockwise (or ccw). Then r^2 is a rotation of $240°$ ccw, and $r^3 = I$, the identity (see Figure **??**). Since we can generate the entire rotation group with a single element r, a natural question to ask is whether we can do the same with the dihedral group D_3. Clearly we can't use the identity to do it, and a series of rotations always leaves us with a rotation, never a reflection. Also, a series of flips along one axis simply generates a two member group with elements I, f (see Figure **??**).

Let's try using two elements to generate our group, using the same definitions of f and r as in the previous section: a flip over the A axis and rotation by $120°$ ccw, respectively. As we found, fr is a flip over the B axis and rf is a flip over the C axis. Consecutive powers of r already got us the remaining elements, so r, f generates the group.

We can also generate the group using two reflections, say f and f_B (flip over the B axis, as shown in Figure **??**). Notice that an even number of reflections always results in a rotation—even the identity element I is just a rotation by 0. We can think of this as the existence of a "mirror world" and its unmirrored counterpart, and each reflection takes us into or out of the mirror world.

Moving to three dimensions, we can interpret D_3 as the set of rotations of an equilateral triangular prism. The rotation axes are coplanar with where the reflection axes used to be (see Figure **??**). Indeed, when you "flipped" your equilateral triangles, you were actually rotating a triangular prism in the third dimension. Flipping the triangular prism using a rotation requires a fourth-dimensional rotation—something we cannot easily visualize.

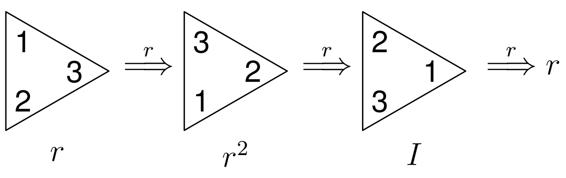

Figure 16: r generates a three member group.

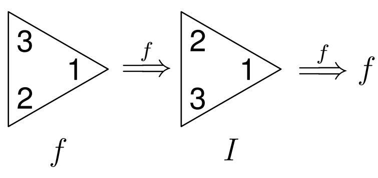

Figure 17: f generates a two member group.

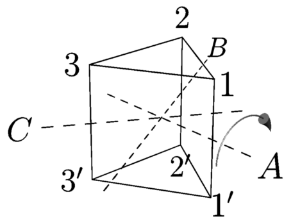

Figure 18: Triangular prism's corresponding axes of rotation.

As you complete the following problems, you can extend the above analysis to characterize group properties. You will notice that the more symmetries an object has, the larger its symmetry group is. Indeed, group theory is the mathematics of symmetry *par excellence*.

For each of the following groups, find the following:

(a) Number of elements (order)

(b) If order < 10, the set of elements; otherwise, an explanation of how you know the order

(c) A smallest possible generating set[1]

(d) Whether the group is commutative

If two problems have isomoprhic groups, just write "isomorphic to Problem N" for the latter problem and move on.

5. Rectangle under rotation
6. Rectangle under reflection
7. Square under rotation
8. Square under reflection
9. Square prism under rotation
10. Square prism under reflection
11. Pentagon under rotation
12. Pentagon under reflection
13. Pentagonal prism under rotation
14. Pentagonal prism under reflection
15. Pentagonal pyramid under rotation
16. Pentagonal pyramid under reflection
17. Tetrahedron (triangular pyramid) under rotation
18. Tetrahedron under reflection
19. Cube under rotation
20. Cube under reflection

[1] There may be multiple generating sets of the same size.

4 Infinite Groups

All of the groups we've seen so far are finite in size. We can also construct groups of an infinite size.

A quick review: *iso-* means the same and *-morphic* means form. Two groups are said to be isomorphic if there is a mapping which takes each element of the first group to an element of the second group and vice versa, so that the products of the elements map in the same way.

1. Where have you come across the roots *iso-* and *-morphic* before?

2. Could two groups be isomorphic if they had different orders?

3. The rotation group for the hexagon H has six elements: the identity, and rotations of $\frac{\pi}{3}$, $\frac{2\pi}{3}$, π, $\frac{4\pi}{3}$, $\frac{5\pi}{3}$ radians. A rotation of $\frac{\pi}{3}$ generates the group. Which other rotation generates the group? What is the period of each element.

4. H has the same number of elements as the dihedral group D_3. Are the two groups isomorphic? How do you know? What is the period of each element of D_3, What can you say if the sets of the periods of the elements of each group are not the same? Which subgroups of C_6 and D_3 are isomorphic?

5. Could an infinite group be isomorphic to a finite group.

6. Do you think all infinite groups are isomorphic to each other. Find a counterexample if you can.

If an infinite group was somehow "bigger" than the other, they wouldn't be isomorphic. (Can you think of an example of two groups with the same "size" that also aren't isomorphic?) This raises the question: are all infinities equally big?

We can formalize the notion of sizes of infinity. Let's say that two infinite sets are of the same size if their elements can be put into a one-to-one correspondence with each other. For example, the positive numbers $1, 2, ... = \mathbb{N}$ and negative numbers $-1, -2, ... = -\mathbb{N}$ are of the same size, because we have the one-to-one correspondence $\mathbb{N} \ni n \leftrightarrow -n \in \mathbb{N}$. Every element of the positive numbers has exactly one "partner" in the negative numbers, and vice versa.

7. Make guesses to the relative sizes of the following pairs of sets. You may use shorthand like $a < b$, $a > b$, $a = b$. After you have made your guesses, we will analyze some of the cases and you can find out how good your intuition was.

(a)	natural numbers, $\mathbb{N}$	vs.	positive even numbers, $2\mathbb{N}$
(b)	natural numbers, $\mathbb{N}$	vs.	positive rational numbers, $\mathbb{Q}^+$
(c)	natural numbers, $\mathbb{N}$	vs.	real numbers between zero and one, $[0, 1)$
(d)	real numbers, $\mathbb{R}$	vs.	complex numbers, $\mathbb{C}$
(e)	real numbers, $\mathbb{R}$	vs.	points on a line
(f)	points on a line	vs.	points on a line segment
(g)	points on a line	vs.	points on a plane
(h)	rational numbers, $\mathbb{Q}$	vs.	Cantor set (look this up or ask your teacher)

It turns out that studying infinity involves some strange mathematics. For instance, even though it seems that there should be half as many positive even numbers as natural numbers (see 7a), we can construct a one-to-one correspondence between the two sets such that every positive even number is paired with a natural number and vice versa: $2\mathbb{N} \ni 2n \leftrightarrow n \in \mathbb{N}$. The existence of this correspondence means that the two sets are equal in size.

More surprisingly, we can establish a correspondence between the non-negative rational numbers ($\mathbb{Q}^+$) and the natural numbers. Draw the rational numbers in a table as shown in Figure **??**, and pair these numbers up with the numbers $1, 2, 3, 4....$ You can see that you will eventually list all of the non-negative rational numbers, multiple times, into a correspondence with the natural numbers. To make it one-to-one, only pair the rational numbers that are in simplest form. Here, we pair 2 with $\frac{1}{1}$ instead of $\frac{0}{2}$ – since $\frac{0}{1}$ is the same number, and is already paired with 1. This correspondence is depicted in Figure **??**. This prevents multiple natural numbers from being paired up with the same rational number: the correspondence is now one-to-one.

The real numbers between 0 and 1, however, cannot be put into a one-to-one correspondence with $\mathbb{N}$ (see 7c). We will prove this with contradiction. Suppose I told you that I have paired each real number $0 <= r_k < 1$ with a unique natural number k, and vice versa. Then, you can construct a real number r_ω[2] whose 1^{st} digit

[2] Pronounced "r omega."

(after the decimal point) differs from r_1's, whose 2^{nd} digit differs from r_2's, and so on. In other words, it differs from r_n in the n^{th} digit. We can make better sense of this construction by writing the numbers out in a table, as shown in Figure **??**. Your new number r_ω is at the bottom; it differs from all the previous numbers in at least one place, so it is a new real number. Therefore, my original list is incomplete, and the correspondence doesn't exist. This is called Cantor's diagonal argument, because the differing digits make a diagonal. (In the diagram, C_10 is the Champernowne constant, the result of joining all the natural numbers together to make a decimal.)

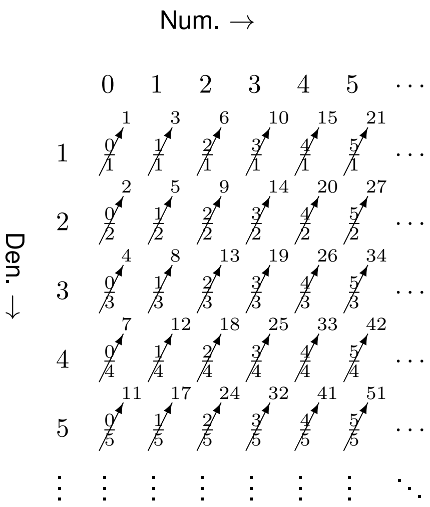

Figure 19: A correspondence between $\mathbb{Q}^+$ and $\mathbb{N}$, but not one-to-one.

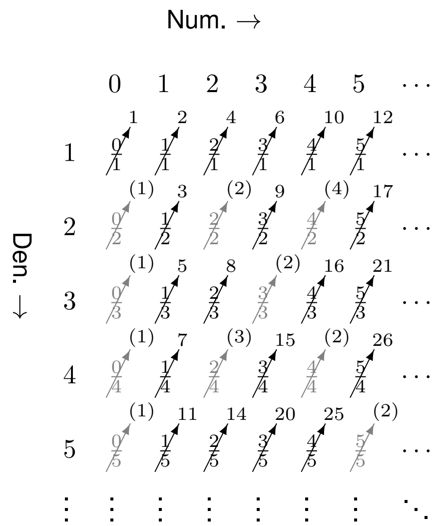

Figure 20: The previous correspondence, with duplicates parenthesized but not counted; it is now one-to-one.

Decimal Expansion →

r_1	0	$0.$	$\boxed{0}$	0	0	0	0	0	$0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \cdots$
r_2	$\frac{1}{4}$	$0.$	2	$\boxed{5}$	0	0	0	0	$0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \cdots$
r_3	$\frac{\pi}{4}$	$0.$	7	8	$\boxed{5}$	3	9	8	$1\ 6\ 3\ 3\ 9\ 7\ 4\ 4\ 8\ 3 \cdots$
r_4	C_{10}	$0.$	1	2	3	$\boxed{4}$	5	6	$7\ 8\ 9\ 1\ 0\ 1\ 1\ 1\ 2\ 1 \cdots$
r_5	$e-2$	$0.$	7	1	8	2	$\boxed{8}$	1	$8\ 2\ 8\ 4\ 5\ 9\ 0\ 4\ 5\ 2 \cdots$
r_6	$\frac{\sqrt{2}}{10}$	$0.$	1	4	1	4	2	$\boxed{1}$	$3\ 5\ 6\ 2\ 3\ 7\ 3\ 0\ 9\ 5 \cdots$
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$		$\vdots$	$\vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \ddots$
r_ω		$0.$	1	6	6	5	9	2	$\cdots$

Figure 21: Cantor's diagonal argument. Notice how, by construction, r_ω differs from r_i in the circled digits.

8. Now, please return to problem 7 and revise your answers. Justify each answer by producing a one-to-one correspondence, or showing the impossibility of doing so. Part (h) is an optional challenge.

By the way, the infinities in problem 7 come in only two sizes: **countable** infinity—like the number of natural numbers—and **uncountable** infinity—like the number of real numbers. There are in fact an infinite number of sizes of infinity, but these two are the only ones we'll deal with in this class. Are there any infinities *between* the two we've discussed, uncountable and countable? This is a very deep mathematical question, known as the continuum hypothesis. It turns out that both the answer "yes" and "no" are consistent with the rest of our mathematics, so either can be taken as an axiom.

Two infinite groups can be the same, infinite size and still not be isomorphic, in the same way that two finite groups of the same size are sometimes not isomorphic (like $D_3 \neq S_6$). For example, the group of all rotations of a rational number of degrees about the origin is countably infinite. So is the group of integers—positive and negative—under addition. But these two groups have completely different structures. For example, the former has two elements which are their own inverse: $0°$ and $180°$. The latter has only one such element: 0.

9. Here's a list of infinite sets, each with an operation. For each pair, answer: i. Does it form a group? ii. Which previous group(s) is it isomorphic to?
   (a) natural numbers, addition
   (b) integers, addition
   (c) even integers, addition
   (d) odd integers, addition
   (e) rational numbers, addition
   (f) real numbers, addition
   (g) complex numbers, addition
   (h) integers, multiplication
   (i) integer powers of 2, multiplication
   (j) rational numbers, multiplication
   (k) rational numbers excluding 0, multiplication
   (l) complex numbers, multiplication
   (m) rotation by a rational number of degrees
   (n) rotation by a rational number of radians
   (o) rotation by an integer number of radians
10. Can an irrational number taken to an irrational power ever be rational? Consider the potential example $a = \sqrt{2}^{\sqrt{2}}$. To help you answer this question, let $b = a^{\sqrt{2}}$. Simplify b, and explain why we don't need to know whether a is rational or irrational.

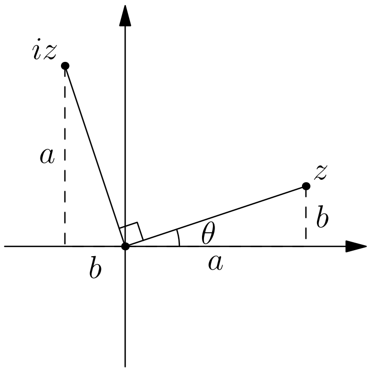

Figure 22: iz is perpendicular to z.

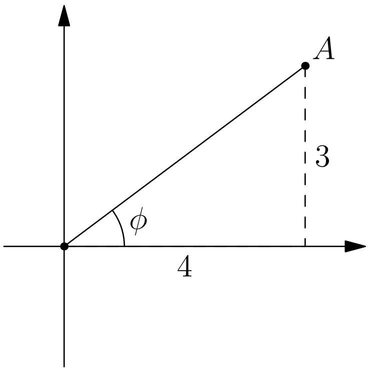

Figure 23: The complex number $A = 4 + 3i$.

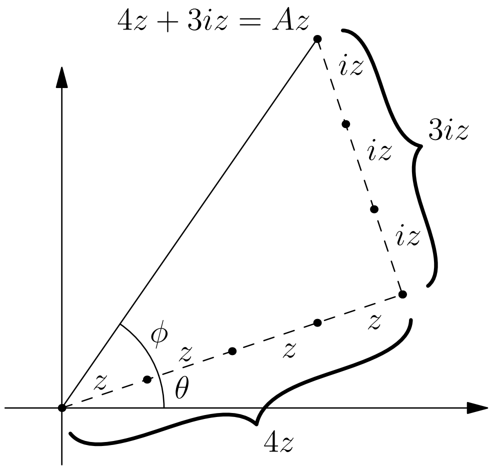

Figure 24: Breaking up Az into its components, we can observe the geometry of complex multiplication.

5 Geometry of Complex Numbers

Thanks to Tristan Needham's Visual Complex Analysis *for many of the problems/examples and to Josh Zucker for most of the text.*

Last year, you became masters of the art of manipulating complex numbers. In this section, we will build on that background. Throughout the rest of the book, you can reinforce your skills with a dose of Vitamin i.

You should know at least two ways to think about the equation $x^3 = 1$. One way is to see that $x = 1$ is a solution, so $x - 1$ is a factor of $x^3 - 1$, and then proceed with synthetic or long division. The other way is to use DeMoivre's beautiful theorem:

$$(r_1(\cos\theta + i\sin\theta))(r_2(\cos\phi + i\sin\phi)) = (r_1r_2)(\cos(\theta+\phi) + i\sin(\theta+\phi)).$$

Like last year, we will often write $\operatorname{cis}\theta = \cos\theta + i\sin\theta$. Recall that $z = r\operatorname{cis}\theta$ is a complex number r units away from the origin and making an angle of θ with the x-axis, taken counterclockwise. Let's rewrite DeMoivre's theorem using cis:

$$(r_1\operatorname{cis}\theta)(r_2\operatorname{cis}\theta) = (r_1r_2)(\operatorname{cis}(\theta+\phi)).$$

In words: when you multiply complex numbers, the magnitudes are multiplied and the angles are added. (What else is added when you multiply? Exponents! In fact, $\operatorname{cis}\theta = e^{i\theta}$, but that's another story.)

Through repeated application of this theorem, we know that $(r\operatorname{cis}\theta)^n = r^n \operatorname{cis} n * \theta$. If $x = r\operatorname{cis}\theta$, then $x^3 = r^3\operatorname{cis}3\theta$. Going back to our original $x^3 = 1$, since $1 = 1\operatorname{cis}(2\pi k)$ for some integer k, we find that $r = 1$ and $3\theta = 2\pi k$. This yields three solutions: $1\operatorname{cis}0$, $1\operatorname{cis}\frac{2\pi}{3}$, $1\operatorname{cis}\frac{3\pi}{3}$. These correspond to $k = 0, 1, 2$; other values of k produce angles with the same terminal sides as one of these three, and are therefore duplicates.

You can bashfully prove DeMoivre's theorem using the angle addition formulae for cos and sin. But you can also understand it through pure geometry. Consider a complex number $z = a + bi$, being multiplied by $A = 4 + 3i$. z forms an angle of θ with the x-axis, and A forms an angle of ϕ. In Figure 1, observe that iz is perpendicular to z for any z. In Figure 2 depicts the complex number A. Finally, in Figure 3, you see the multiplication carried out: $Az = (4 + 3i)z = 4z + 3iz$. These two components, $4z$ and $3iz$, are indicated.

Combining the observation in Figure 1 and knowledge from geometry, we know the triangles in Figure 2 and 3 are similar. Since the scaling is by a factor of $|z|$, multiplying A by z has the effect rotating z by the angle of A, and multiplying it by the length of A.

Some notation: The angle θ of a complex number $z = a+bi$ is often called the argument, written as $\operatorname{Arg} z$. The real part of z is written $\operatorname{Re}(z) = a$, and the imaginary part of z is written $\operatorname{Im}(z) = b$. Note that $\operatorname{Im}(z)$ is a **real** number, **not** an imaginary number. In particular, $\operatorname{Im}(z) \neq bi$. Finally, the complex conjugate of z, where the imaginary part is negated, is written with a bar on top: $\overline{z} = a - bi$.

1. Explain why iz is perpendicular to z, without using DeMoivre's theorem.

2. How does $\operatorname{Arg}\overline{z}$ relate to $\operatorname{Arg} z$? (Hint: symmetry!)

3. Compute $z\overline{z}$ and relate it to the cis form of z.
4. Explain, using a picture, why $\tan(\operatorname{Arg} z) = \frac{\operatorname{Im}(z)}{\operatorname{Re}(z)}$.
5. Divide $\frac{a+bi}{c+di}$ by rationalizing the denominator.
6. Divide $\frac{r_1 \operatorname{cis}\theta}{r_2 \operatorname{cis}\phi}$ using DeMoivre's theorem.
7. Compare and contrast the methods of division in problems 5 and 6. Which is more convenient? Or does it depend on the circumstance?
8. If $z = r\operatorname{cis}\theta$, what is $\frac{1}{z}$? Explain how this shows $\frac{1}{a+bi} = \frac{a-bi}{a^2+b^2}$, without having to rationalize the denominator. (Hint: use problems 3, 4, and 7.)
9. Compute $(1+i)^{13}$; pencil, paper, and brains only. No calculators!
10. Compute $\frac{(1+i\sqrt{3})^3}{(1-i)^2}$ without a calculator.
11. Draw $\operatorname{cis}\left(\frac{\pi}{4}\right) + \operatorname{cis}\left(\frac{\pi}{2}\right)$. Use your picture to prove that $\tan\left(\frac{3\pi}{8}\right)$. (Hint: add them as vectors.)
12. Solve $z^3 = 1$, and show that its solutions under the operation of multiplication form a group, isomorphic to the rotation group of the equilateral triangle. Write a group table!
13. (a) Find multiplication groups of complex numbers which are isomorphic to the rotation groups for

    i. a non-square rectangle, and

    ii. a regular hexagon.

    (b) Make a table for each group.

    (c) Compare the regular hexagon's group to the dihedral group of the equilateral triangle, D_3. Consider: how are they the same? How are they different? Is the difference fundamental?
14. Which of the following sets is a group under (i) addition and (ii) multiplication?

    (a) $\{0\}$
    (b) $\{1\}$
    (c) $\{0,1\}$
    (d) $\{-1,1\}$
    (e) $\{1,-1,i,-i\}$
    (f) {naturals}
    (g) {integers}
    (h) {rationals}, $\mathbb{Q}$
    (i) {$\mathbb{Q}$ without zero}
    (j) {complex numbers}, $\mathbb{C}$
    (k) {$\mathbb{C}$ without zero}
15. For what values a and b does $\{a + b\sqrt{2}\}$ form a group under (a) addition and (b) multiplication? Look for several answers.

DeMoivre's theorem is the "universal" trig identity, in the sense that it can be used to calculate every other trig identity. For example, suppose you want an identity for $\cos 3\theta$. For convenience, let $c = \cos\theta$ and $s = \sin\theta$. Then we have

$$\begin{aligned}
\operatorname{cis} 3\theta &= (\operatorname{cis}\theta)^3 && \text{[DeMoivre's Theorem]}\\
&= (c+is)^3 && \text{[Definition of } \operatorname{cis}\text{]}\\
&= c^3 + 3c^2si - 3cs^2 - s^3 i && \text{[Binomial expansion]}\\
\cos 3\theta + i\sin 3\theta &= (c^3 - 3c^2) + i(3c^2 s - s^3) && \text{[Combining like terms]}
\end{aligned}$$

Equating real parts on both sides, $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$.

16. Prove that $(r_1 \operatorname{cis}\theta)(r_2 \operatorname{cis}\phi) = r_1 r_2 \operatorname{cis}(\theta+\phi)$ using brute force and the angle-sum trig identities for $\cos$ and $\sin$. Do you prefer this method or the one on the previous page? Which method gives you a better understanding of why DeMoivre's works?
17. Find an identity for $\sin 3\theta$ as we have done for $\cos$. (Hint: most of the work is already done for you!)
18. Your friend says that her textbook says $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$, different from our identity. Who's right?

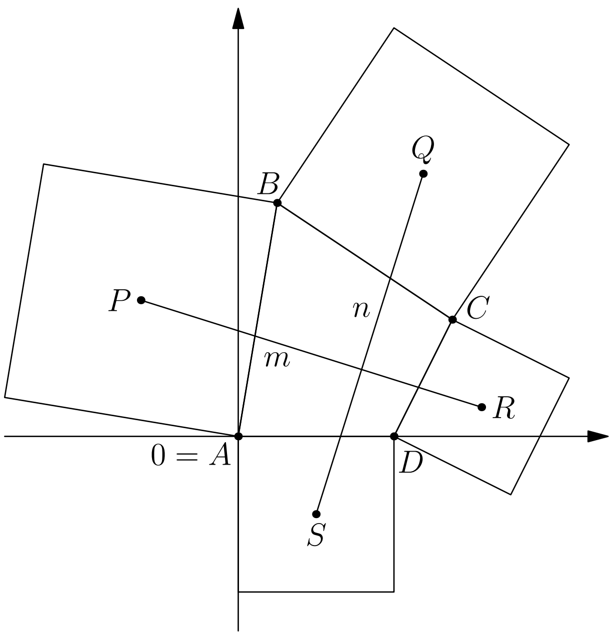

Figure 26: The quadrilateral with four squares.

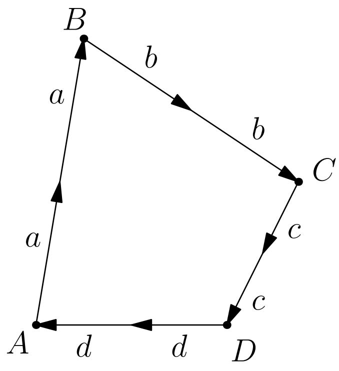

Figure 25: $2(a+b+c+d) = 0$.

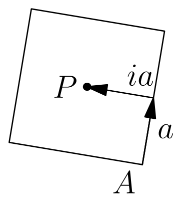

Figure 27: $P = a + ia$.

Let's apply complex numbers to a geometry problem: Prove that if we construct squares with centers P, Q, R, S on the sides of any quadrilateral $ABCD$, as shown in Figure **??**, then (i) $\overline{PR} \cong \overline{QS}$ and (ii) $\overline{PR} \perp \overline{QS}$. In other words, segments joining centers of opposite squares are perpendicular and the same length.

We represent all points in the figure as complex numbers. For convenience, let $A = 0$ be the origin. The edges of the quadrilateral can be thought of as vectors in the form of complex numbers, and are found using subtraction; for example, the edge from A to B is $B - A$. Similarly, the edge from B to C is $C - B$. Now, define complex numbers

$$a = \frac{B-A}{2},\ b = \frac{C-B}{2},\ c = \frac{D-C}{2},\ d = \frac{A-D}{2}.$$

a is the vector halfway along $\overrightarrow{AB}$, b is halfway along $\overrightarrow{BC}$, etc.; this is shown in Figure **??**. We also have

$$a+b+c+d = \frac{B-A+C-B+D-C+A-D}{2} = \frac{0}{2} = 0.$$

More geometrically, this is because $2(a+b+c+d) = 2a+2b+2c+2d$ is the sum of the vectors $\overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{CD}, \overrightarrow{DA}$, which is just $\overrightarrow{AA} = \mathbf{0}$. This is shown in Figure **??**.

P, Q, R, S are also complex numbers. Also, let $m = R - P$ and $n = Q - S$, as the lengths of our two segments $\overline{PR}$ and $\overline{QS}$. To prove that they are perpendicular, recall that z is perpendicular to iz for any complex $z \neq 0$, so we just need to prove that $n = \pm im$.

We now need to relate P, Q, R, S back to a, b, c, d. It is easy to see that $p = a + ia$, remembering that a is the vector halfway along $\overrightarrow{AB}$. a takes you from the origin A to the midpoint of $\overline{AB}$, then ia takes you to P. This shown in Figure **??**. We can extend this logic to the other points, of course.

19. Now you can finish the rest of the proof.
    (a) Make a diagram of the approximate directions and magnitudes of a, b, c, d, m, n for the quadrilateral on the previous page.
    (b) Why is it sufficient to show that $n = \pm im$, to prove that the segments are (i) perpendicular and (ii) the same length?
    (c) Explain why $q = 2a + b + ib$.
    (d) Find formulae for r and s in terms of c and d.
    (e) Find m and n in terms of a, b, c, and d.
    (f) Check that $n + im = 0$, using the fact that $a + b + c + d = 0$.

20. In the previous problem, we drew squares outside a quadrilateral and connected their centers. Conjecture what happens if we draw equilateral triangles outside a triangle and connect their centers, as shown in . Prove your conjecture using complex numbers.

21. The hard way to find an identity for $\tan 3\theta$ is to divide the identity for $\sin$ and $\cos$ that we already found. Try this. Make sure your answer is in terms of $\tan$ only!

22. The easier way to get an identity for $\tan 3\theta$ starts with setting $z = 1 + i\tan\theta$.

    (a) Why is $\operatorname{Arg} z = \theta$?

    (b) Why is $\tan 3\theta = \frac{\operatorname{Im}(z^3)}{\operatorname{Re}(z^3)}$?

    (c) Use (b) to find an identity for $\tan 3\theta$.

23. Find multiplication groups of complex numbers isomorphic to rotation groups for the

    (a) regular octagon, and

    (b) regular pentagon.

24. Make tables for

    (a) the rotation group of the regular octagon, and

    (b) the dihedral group of the square.

    Is the difference between them fundamental?

25. Which of the following tables defines a group? Why or why not?

    (a)

$	I	A	B	C	D
I	I	A	B	C	D
A	A	C	D	B	I
B	B	I	C	D	A
C	C	D	A	I	B
D	D	B	I	A	C

    (b)

#	I	A	B	C	D
I	I	A	B	C	D
A	A	B	C	D	I
B	B	C	D	I	A
C	C	D	I	A	B
D	D	I	A	B	C

26. Which subsets of the complex numbers are groups under multiplication? After you list a few of each type, generalize.

27. Prove with a diagram that if $|z| = 1$, then $\operatorname{Im}\left(\frac{z}{(z+1)^2}\right) = 0$.

28. Prove geometrically that if $|z| = 1$, then $|1 - z| = 2\sin\left(\frac{\operatorname{Arg} z}{2}\right)$.

29. (a) Prove that if $(z-1)^{10} = z^{10}$, then $\operatorname{Re}(z) = \frac{1}{2}$. (Hint: if two numbers are equal, they have the same length.)

    (b) How many solutions does this equation have?

30. I claim that $e^{i\theta} = \cos\theta + i\sin\theta = \operatorname{cis}\theta$, for θ in radians.

    (a) Find e^{-it}.

    (b) Find $\frac{e^{i\theta}+e^{-i\theta}}{2}$.

    (c) Find $\frac{e^{i\theta}-e^{-i\theta}}{2i}$.

    Use your new, complex definitions for $\cos$ and $\sin$ to find:

    (d) $\cos^2\theta + \sin^2\theta$

    (e) $\tan\theta$

    (f) $\cos 2\theta$

    (g) $\sin 2\theta$

    (h) What kind of group is generated by $e^{i\theta}$ under the operation of multiplication if θ is an integer? A rational multiple of π?

31. You've used the quadratic equation throughout high school, but there's also a cubic equation that finds the roots of any cubic. Let's derive it, starting with the cubic $x^3 + bx^2 + cx + d = 0$.

    (a) Make the substitution $x = y - \frac{b}{3}$. Combine like terms to create an equation of the form $y^3 - 3py - 2q = 0$, with p, q in terms of b, c, and d.

(b) Rearrange this equation as $y^3 = 3py + 2q$.

(c) Make the substitution $y = s + t$, and prove that y is a solution of the cubic in part (a) if $st = p$ and $s^3 + t^3 = 2q$.

(d) Eliminate t between these two equations to get a quadratic in s^3.

(e) Solve this quadratic to find s^3. By symmetry, what is t^3?

(f) Find a formula for y in terms of p and q. What about a formula for x?

(g) What if we started with $ax^3 + bx^2 + cx + d = 0$, with a coefficient in front of the x^3 term as well? Can you come up with a formula for x?

32. Starting with the same cubic as in problem 31b.

(a) Let $c = \cos\theta$. Remember that $\cos 3\theta = 4c^3 - 3c$, as we proved. Substitute $y = 2c\sqrt{p}$ into $y^3 = 3py + 2q$ to obtain $4c^3 - 3c = \frac{q}{p^{3/2}}$.

(b) Provided that $q^2 \leq p^3$, show that $y = 2\sqrt{p}\cos\left(\frac{1}{3}(\phi + 4\pi n)\right)$, where n is an integer. Why does this yield all three solutions?