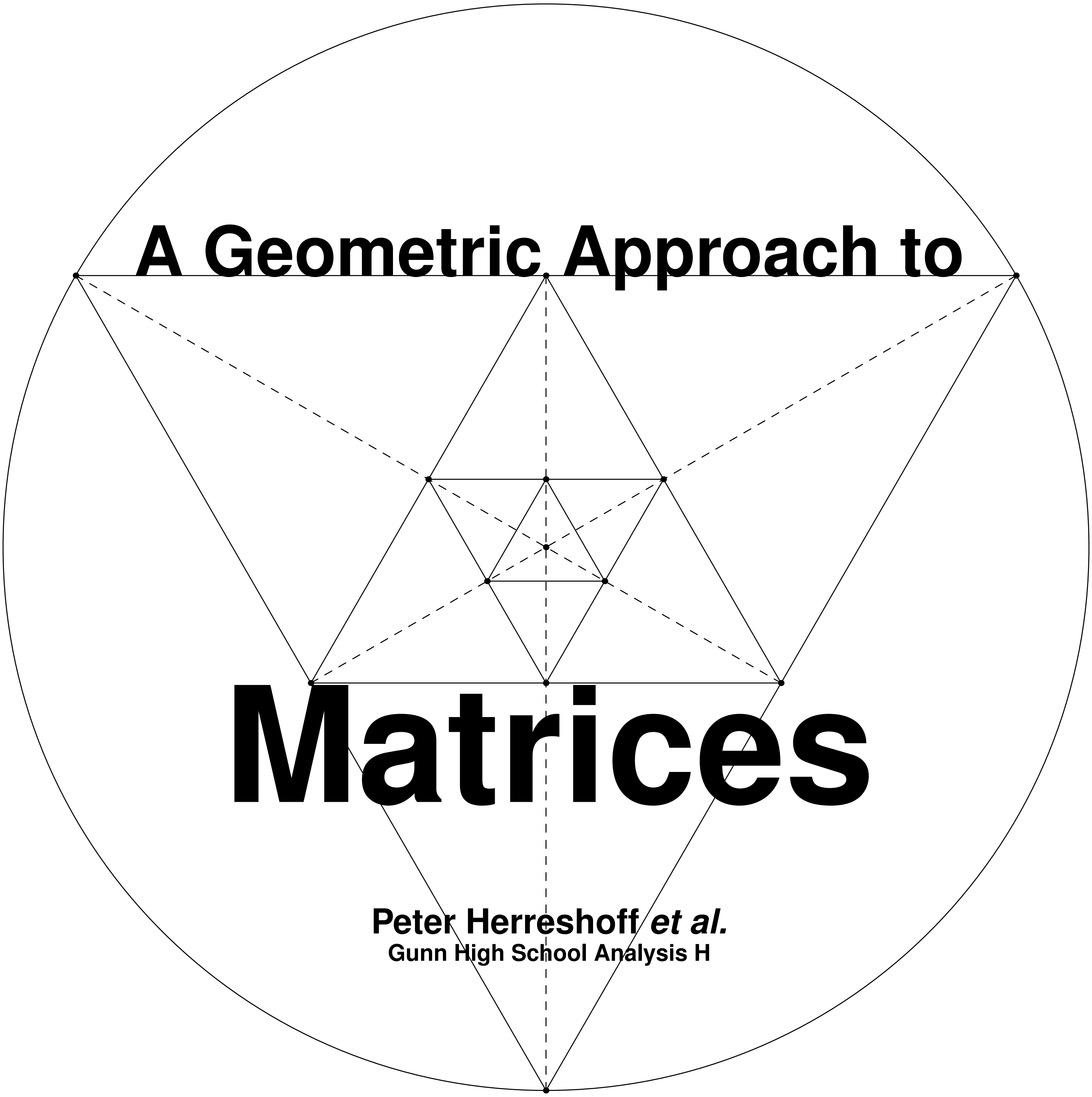
A Geometric Approach to

Matrices



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1 It's a Snap

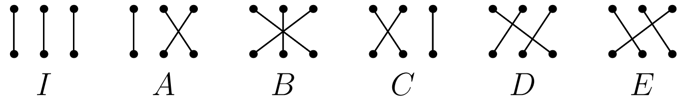

Figure 1: The six possibilities for connections between two rows of three posts.

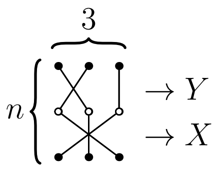

Figure 2: A grid with three strings.

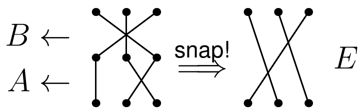

Figure 3: $A \bullet B = E$.

We begin with a problem that ties together ideas from geometry, complex numbers, matrices, combinatorics, and group theory. You likely studied geometry in 9th grade and complex numbers in 10th grade, so you should have a basis to start your investigation. The other three topics, however, may be unfamiliar.

Consider a rectangle of posts with 3 rows and 3 columns. An elastic string is anchored to one post in the topmost row and one post in the bottommost row. As the string descends top to bottom, it loops taut around one post in each row. Two other elastic strings are anchored and looped in the same way, with the condition that each post has exactly one string touching it; no more, no less. An example is depicted in Figure 2.

Figure 1 shows the six configurations in which two consecutive rows can be connected in this way. I have labeled them with letters I, A, B, C, D, and E.

Now, consider what happens when you look at a 3×3 rectangle of posts, like the one in Figure 2. You should have two configurations X and Y, stacked so that Y is on top of X. When you withdraw the middle posts (the posts indicated by $\circ$), the elastic string should **snap** to a configuration congruent to one of the six we drew initially. Let's call this operation snap, or $\bullet$, so that $X \bullet Y$ reads "X snap Y." Keep in mind that when writing it this way, the bottom configuration X goes first, and the top configuration Y goes last. As an example, $A \bullet B = E$, as shown in Figure 3.
[1]

These six configurations, along with the operation $\bullet$, form a mathematical *group*, and we say that each is an *element* of our group. In particular, we will call this the **snap group** of size 3, or S_3. Groups are the central objects studied in *group theory*, which I mentioned at the beginning of this problem. Let's study the snap group and characterize its properties.

1.1 Problems

1. Fill out a 6×6 table like the one in Figure 4, showing the results of each of the $6 \cdot 6 = 36$ possible snaps, where $X \bullet Y$ is in X's row and Y's column.

2. Would this table look different if you named the elements A through E in a different order?

[1] I'd like to put the definition of a group *here*, but I'm not sure if you want it after the problems as the problems seem to imply.

$\bullet$	I	A	B	C	D	E
I						
A			E			
B						
C						
D						
E						

Figure 4: Unfilled 3-post snap group table.

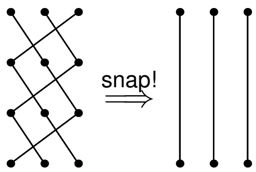

Figure 5: $E \bullet E \bullet E = I$; E has period 3.

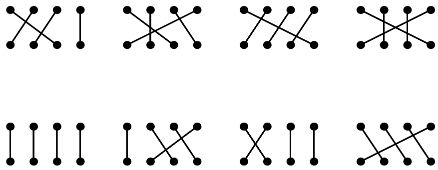

Figure 6: Some 4-post group elements.

3. Which of the elements is the *identity element*?
4. Does every element have an inverse? In other words, is there a way to get the identity element from every element using one snap?
5. Is the snap operation commutative? That is, does $X \bullet Y = Y \bullet X$ for all X, Y?
6. Is the snap operation associative? That is, does $(X \bullet Y) \bullet Z = X \bullet (Y \bullet Z)$ for all X, Y, Z?
7. (a) For given elements X, Y, is there always an element Z so that $X \bullet Z = Y$?
   (b) Is Z always unique?
8. What do you think is the definition of a mathematical group? (Hint: consider your answers to Problems 2–6.) [2]
9. If you constructed a 5×5 table using only 5 of the snap elements, the table would not describe a group, because there would be entries in the table requiring an element outside of these 5. Therefore, a group must be *closed* under its operation; if $X, Y \in G$, then $X \bullet Y \in G$ for all X, Y. Some subsets, however, are closed. Write valid group tables using exactly 1, 2, and 3 elements from the snap group. [3]
10. Notice that $E \bullet E \bullet E = I$. (See Figure 5.) This means that E has a **period** of 3 when acting upon itself. Which elements have a period of 1, 2, and 3?
11. (a) How many elements would there be if there were only 2 posts per row? [4]
    (b) Draw and name them systematically.
    (c) Make a group table of these elements.
    (d) What is the relationship of this new table to your original table?
    (e) Answer parts (a)–(c) if there was 1 post per row.
    (f) Answer parts (a), (b) and (d) if there was 4 posts per row. [5]
    (g) Would you like to make a table for the 4-post group? How many entries would a table have?
12. Can you think of an easier way to construct the a snap group table without drawing all the possible configurations?
13. (a) How many elements would there be in the 5-post snap group?
    (b) How many entries would its table have?
    (c) What possible periods would its elements have?
    (d) Extend your answers for (a)–(c) to M posts per row.
14. As we learned[6], a **permutation** of some things is an order they can be arranged in. What is the relationship between the set of permutations of M things and the M-post snap group? [7]

[2] This problem would be removed or replaced if group is defined earlier
[3] Very wordy problem, would be shortened if group is defined earlier
[4] I'd like to call it S_2 with your permission
[5] Ditto, S_4
[6] Talking about semester 1 probability unit. Maybe remove?
[7] Maybe call it S_M

2 From Snaps to Flips

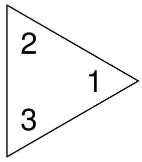

Figure 7: The paper triangle.

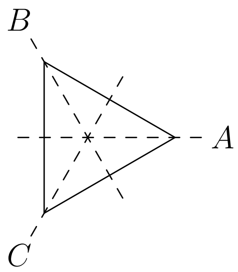

Figure 8: The triangles axes of reflection.

Cut out a paper equilateral triangle and label its vertices 1, 2, and 3 in permanent marker as shown in Figure 7, and place it down on a table in the shown orientation. This triangle has several *isometries* to itself, which are ways of mapping the triangle to itself. For example, you could *reflect* it around the three axes A, B and C, as shown in Figure 8. You could also rotate it by $120°$ or $240°$.

There are six isometries of the triangle to consider. These are shown in Figure 9.

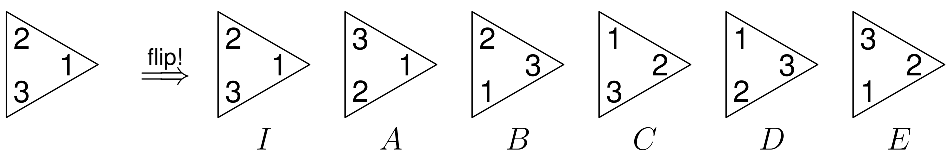

Figure 9: The six triangle isometries.