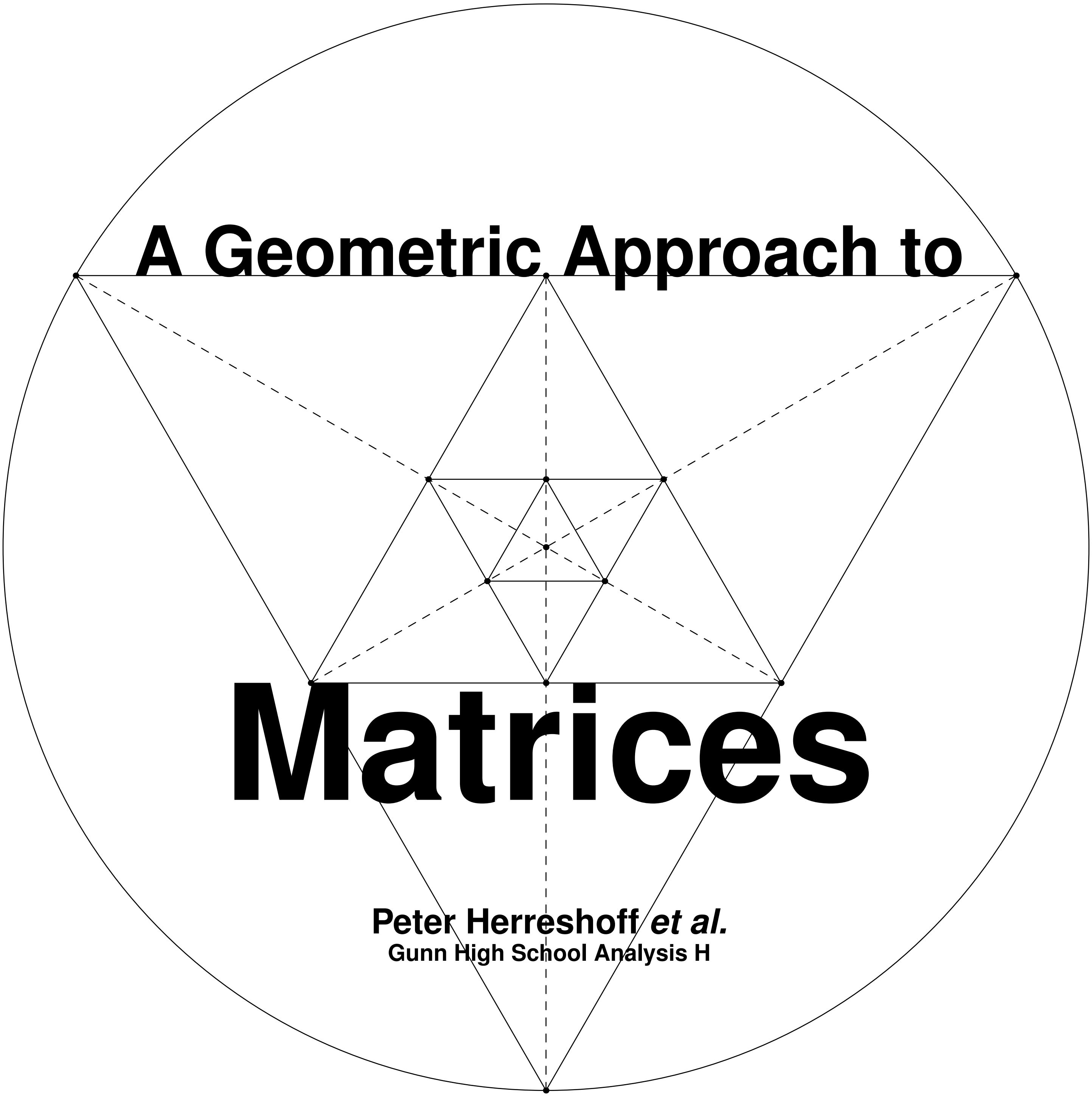
A Geometric Approach to

Matrices



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Gunn High School Analysis H

A Geometric Approach To Matrices

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Contents

1 It's a Snap

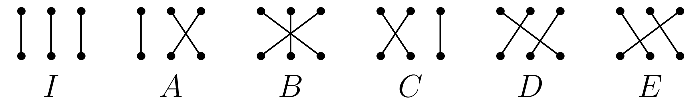

Figure 1: The six possibilities for connections between two rows of three posts.

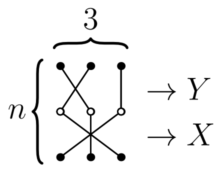

Figure 2: A grid with three strings.

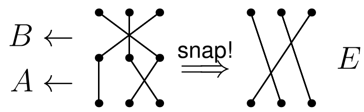

Figure 3: $A \bullet B = E$.

Let's start with a concept that ties together ideas from geometry, complex numbers, matrices, combinatorics, and group theory. You likely studied geometry in 9th grade and complex numbers in 10th grade, so you should have good start on this investigation. The other three topics, however, may be unfamiliar.

Consider a grid of posts with 3 rows and 3 columns. An elastic string is anchored to one post in the top row and one post in the bottom row, and loops around one post in the middle row. Two other strings are anchored and looped in the same way. Each post must have exactly one string touching it; no more, no less. An example is depicted in Figure 2.

Figure 1 shows the six configurations in which two rows of three posts can be connected with these rules. We'll call them I, A, B, C, D, and E.

Now, look at a 3×3 rectangle of posts, like the one in Figure 2. You should have two configurations X and Y, stacked so that Y is on top of X. When you remove the middle posts (the posts indicated by $\circ$), the elastic string will **snap** to one of the six configurations we drew initially. Let's call this operation snap, or $\bullet$, so that $X \bullet Y$ reads "X snap Y." Keep in mind that when writing it this way, the bottom configuration X goes first, and the top configuration Y goes last. As an example, $A \bullet B = E$, as shown in Figure 3.

These six configurations, along with the operation $\bullet$, form a mathematical **group**, and we say that each is an **element** of our group. In particular, we will call this the **snap group** of size 3, or S_3. Groups are the central objects studied in **group theory**, a topic mentioned at the beginning of this chapter. Let's study the snap group and characterize its properties.

1.1 Problems

1. Fill out a 6×6 table like the one in Figure 4, showing the results of each of the 36 $(6 \cdot 6)$ possible snaps, where $X \bullet Y$ is in X's row and Y's column.

2. Would this table look different if you named the elements A through E in a different order?

3. Which of the elements is the **identity element**?

4. Does every element have an inverse (is there a way to get the identity element from every element using only one snap)?

$\bullet$	I	A	B	C	D	E
I						
A			E			
B						
C						
D						
E						

Figure 4: Unfilled 3-post snap group table.

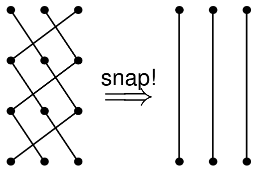

Figure 5: $E \bullet E \bullet E = I$; E has period 3.

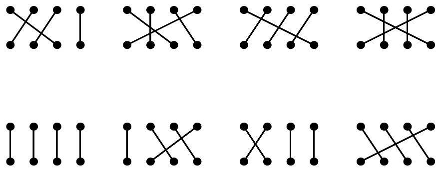

Figure 6: Some 4-post group elements.

5. Is the snap operation commutative (does $X \bullet Y = Y \bullet X$ for all X, Y)?
6. Is the snap operation associative (does $(X \bullet Y) \bullet Z = X \bullet (Y \bullet Z)$ for all X, Y, Z)?
7. (a) For any elements X, Y, is there always an element Z so that $X \bullet Z = Y$?
   (b) Is Z always unique?
8. What do you think is the definition of a mathematical group? (Hint: consider your answers to Problems 2–6.)
9. If you constructed a 5×5 table using only 5 of the snap elements, the table would not describe a group, because there would be entries in the table in those 5. Therefore, a group must be **closed** under its operation; if $X, Y \in G$, then $X \bullet Y \in G$ for all X, Y. Some subsets, however, are closed. Write valid group tables using exactly 1, 2, and 3 elements from the snap group.
10. Notice that $E \bullet E \bullet E = I$. (See Figure 5.) This means that E has a period of 3 when acting upon itself. Which elements have a period of 1, 2, and 3?
11. Answer the following with 1, 2, and 4 posts.
    (a) How many elements would there be?
    (b) Draw and name them systematically.
    (c) Make a group table of these elements. For 4 posts, instead of creating the massive table, give the number of entries that table would have.
    (d) What is the relationship of this new table to your original table?
12. Can you think of an easier way to construct the a snap group table without drawing all the possible configurations?
13. (a) How many elements would there be in the 5-post snap group?
    (b) How many entries would its table have?
    (c) What possible periods would its elements have?
    (d) Extend your answers for (a)–(c) to M posts per row.
14. As we learned, a *permutation* of some things is an order they can be arranged in. What is the relationship between the set of permutations of M things and the M-post snap group?

2 From Snaps to Flips

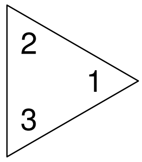

Figure 7: The paper triangle.

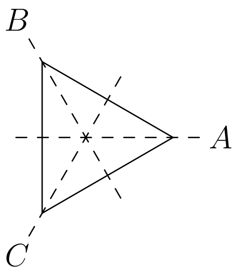

Figure 8: The triangles axes of reflection.

Cut out a paper equilateral triangle and label its vertices 1, 2, and 3 in permanent marker as shown in Figure 7, and place it down on a table in the shown orientation. This triangle has several *isometries* to itself, which are ways of mapping the triangle to itself. For example, you could *reflect* it around the three axes A, B and C, as shown in Figure 8. You could also rotate it by $120°$ or $240°$.

There are six isometries of the triangle to consider. These are shown in Figure 9.

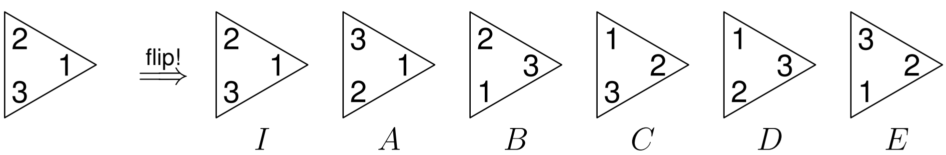

Figure 9: The six triangle isometries.