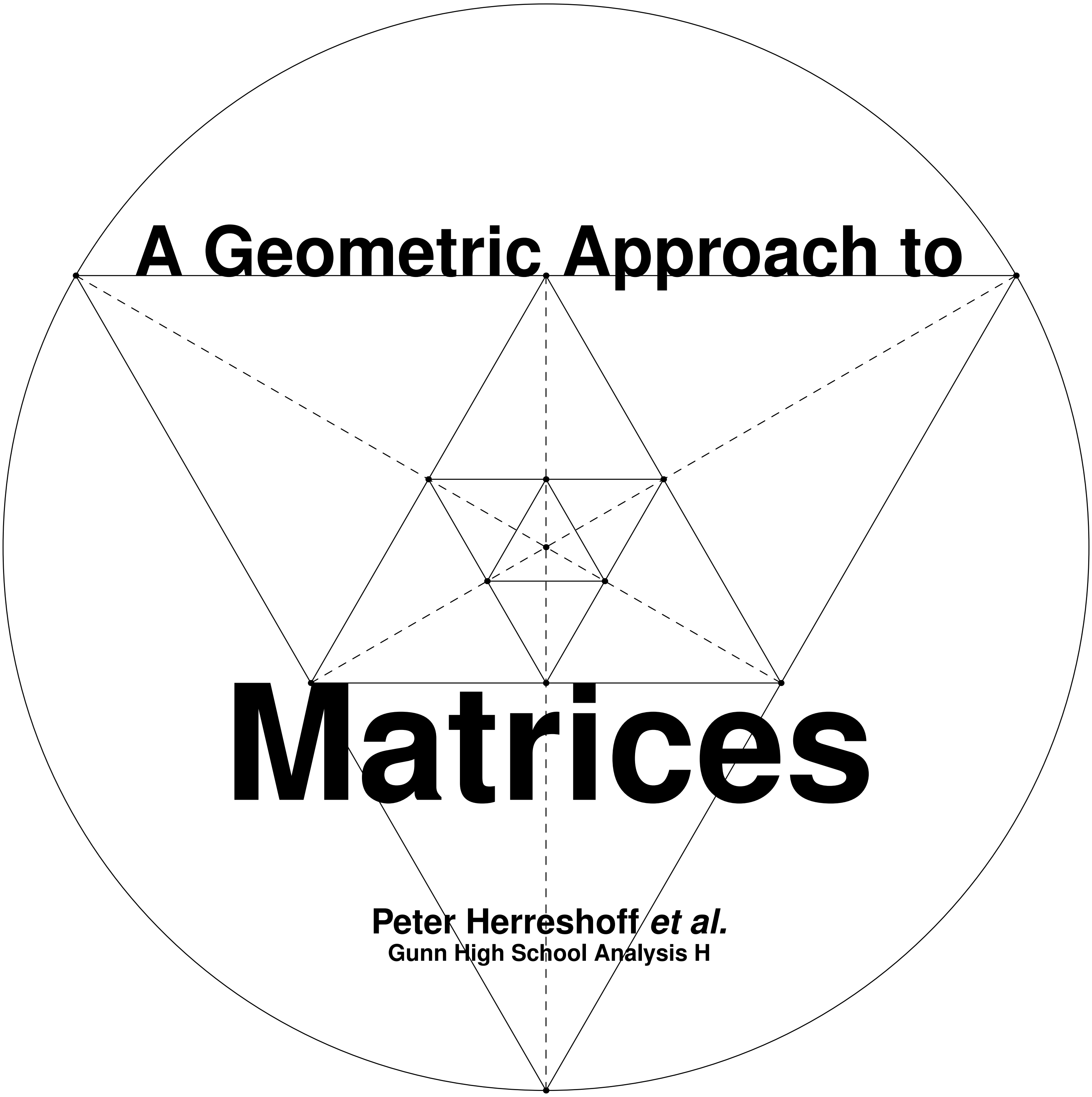
A Geometric Approach to

Matrices



Peter Herreshoff *et al.*

Gunn High School Analysis H

A Geometric Approach To Matrices

Peter Herreshoff

c/o Gunn High School

780 Arastradero Road

Palo Alto, CA, 94306 USA

pherreshoff@pausd.org

Thanks to Josh Zucker who wrote the first draft of several chapters (particularly The Geometry of Complex Numbers, Vitamin i, Matrix Multiplication, and Inverses part I).

Also, thanks to my fellow teachers at Gunn High School Dave Deggeller and Danny Hahn for resurrecting the use of this pamphlet in their 11th grade Analysis classes and to their Analysis classes of 2013-14 who took on the project of retyping the text after the loss of the soft copy in a computer theft.

In addition, thanks to Gautam Mittal for editing the text for the 2017-18 academic year.

Above all, thanks to my student assistant Maya Sankar for her work on editing the text, standardizing the text in LaTeX, and incorporating an answer key.

Contents

1 It's a Snap

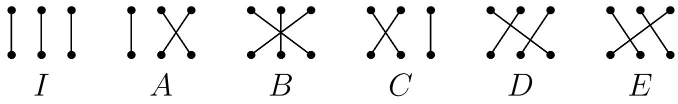


Figure 1: The six possibilities for connections between two rows of three posts.

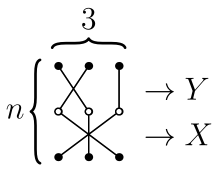


Figure 2: A grid with three strings.

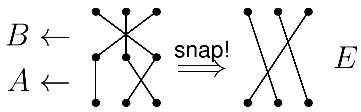


Figure 3: $A \bullet B = E$.

We begin with a problem that ties together ideas from geometry, complex numbers, matrices, combinatorics, and group theory. You likely studied geometry in 9$^{\text{th}}$ grade and complex numbers in 10$^{\text{th}}$ grade, so you should have a basis to start your investigation. The other three topics may be a bit unfamiliar at this point.

Consider a grid of posts with 3 rows and 3 columns. An elastic string is anchored to one post in the top row and one post in the bottom row. As the string descends from top to bottom, it loops around a post in the middle row. Two other strings are anchored and looped in the same way, with the condition that each post has exactly one string touching it. An example is depicted in Figure 2.

Figure 1 shows the six ways two rows can be connected with these rules – convince yourself that these are the only six. I have labeled them as I, A, B, C, D, and E.

Now, look at a 3×3 rectangle of posts, like the one in Figure 2. You should have two configurations X and Y, stacked so that Y is on top of X. When you remove the middle posts (the posts indicated by $\circ$), the elastic string will **snap** to one of the six configurations we drew initially. Let's call this operation "snap", or $\bullet$, so that $X \bullet Y$ reads "X snap Y." Keep in mind that when writing it this way, the bottom configuration X goes first, and the top configuration Y goes last. As an example, $A \bullet B = E$, as shown in Figure 3.

These six configurations form a mathematical **group** under the $\bullet$ operation, and we say that each configuration is an **element** of our group. We will call this the **snap group** of size 3, or S_3. The group is unsurprisingly the main concept studied in **group theory**, a topic mentioned at the beginning of this chapter. Let's study the snap group and its properties.

1.1 Problems

1. Fill out a 6×6 table like the one in Figure 11, showing the results of each of the 36 ($6 \cdot 6$) possible snaps, where $X \bullet Y$ is in X's row and Y's column.
2. Would this table look different if you wrote the elements A through E in a different order?

$\bullet$	I	A	B	C	D	E
I						
A			E			
B						
C						
D						
E						

Figure 4: Unfilled 3-post snap group table.

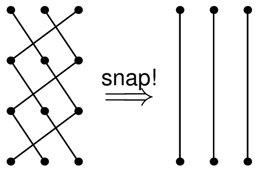


Figure 5: $E \bullet E \bullet E = I$; E has period 3.

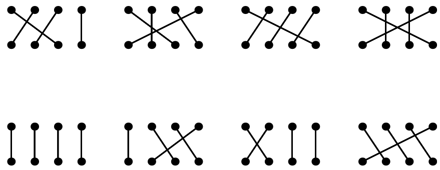


Figure 6: Some 4-post group elements.

3. Which of the elements is the **identity element**?
4. Does every element have an inverse (can you get to the identity element from every element using only one snap)?
5. Is the snap operation commutative (does $X \bullet Y = Y \bullet X$ for all X, Y)?
6. Is the snap operation associative (does $(X \bullet Y) \bullet Z = X \bullet (Y \bullet Z)$ for all X, Y, Z)?
7. (a) For any elements X, Y, is there always an element Z so that $X \bullet Z = Y$?
   (b) Is Z always unique?
8. What do you think is the definition of a mathematical group? (Hint: consider your answers to Problems 2–6.)
9. If you constructed a 5×5 table using only 5 of the snap elements, the table would not describe a group, because there would be entries in the table not in those 5. Therefore, a group must be **closed** under its operation; if $X, Y \in G$ ($\in$ means "is/are in"), then $X \bullet Y \in G$ for all X, Y. Some subsets, however, do happen to be closed. Write valid group tables using exactly 1, 2, and 3 elements from the snap group.
10. Notice that $E \bullet E \bullet E = I$. (See Figure 5.) This means that E has a period of 3 when acting upon itself. Which elements have a period of 1, 2, and 3?
11. Answer the following with the 1, 2, and 4-post snap groups S_1, S_2 and S_4.
    (a) How many elements would there be?
    (b) Draw and name them systematically.
    (c) Make a group table of these elements. For 4 posts, instead of creating the massive table, give the number of entries that table would have.
    (d) What is the relationship of this new table to your original table?
12. Can you think of an easier way to generate a snap group table without drawing all the possible configurations?
13. (a) How many elements would there be in the 5-post snap group?
    (b) How many entries would its table have?
    (c) What possible periods would its elements have?
    (d) Extend your answers for (a)–(c) to M posts per row.
14. As we learned, a *permutation* of some things is an order they can be arranged in. What is the relationship between the set of permutations of M things and the M-post snap group?

2 From Snaps to Flips

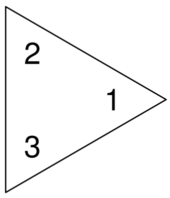


Figure 7: The paper triangle.

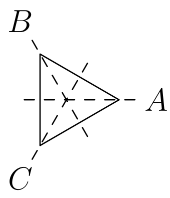


Figure 8: The triangles axes of reflection.

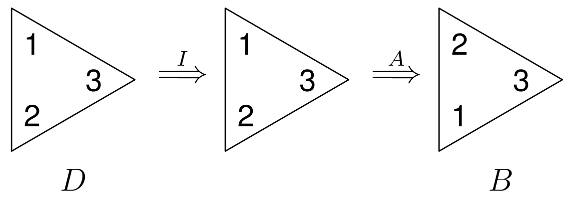


Figure 9: $AID = B$. Notice the RTL evaluation.

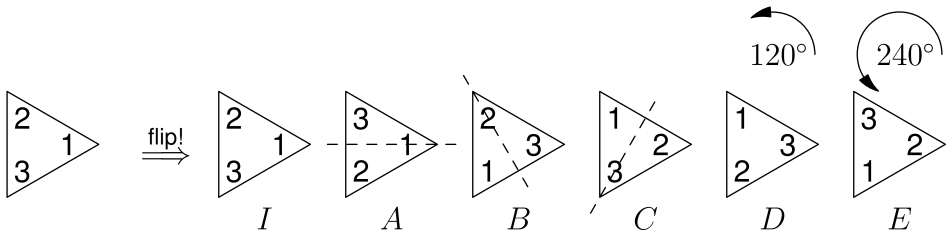


Figure 10: The six ending positions.

You can use a paper triangle to help visualize the next concept. Cut out an equilateral triangle, label its front vertices 1, 2, and 3 as shown in Figure 7, and place it down in the shown orientation. From this starting position, you can reflect the triangle over one of three axes: A, B, or C, as shown in Figure 8. Or you can rotate the triangle 120° or 240° counterclockwise. The final possible positions are shown in Figure 10.

Notice that each position corresponds to a different operation which preserves the triangle's location. For example, I means "leave the triangle alone," A means "flip the triangle about the A axis," and D means "rotate the triangle 120° counterclockwise." We can combine these operations to form other operations by writing them in sequence. Unlike most cases, however, we evaluate them right-to-left rather than left-to-right. For example, $AID = B$, as shown in Figure 9.

These six positions form another group: the **dihedral group** of order 3, or D_3. If we split "dihedral" into "di-" and "-hedral," we see it means "two faces," just like the two faces of our paper triangle. Let's study the properties of this group.

2.1 Problems

1. Figure 10 shows six **isometries** of the triangle. Isometries are ways of mapping the triangle to itself, preserving shape and location. Are there any others?

2. As with the snap group, we can make a group table for the flip group. Fill out a table like the one in Figure 11 in your notebook. Like the snap group table, the top row indicates what operation is done first and the left column indicates what's done second, so that XY is in the X^{th} row and Y^{th} column.

$\cdot$	I	A	B	C	D	E
I						
A					B	
B						
C						
D						
E						

Figure 11: Unfilled D_3 group table.

3. What is the relationship between the tables for the snap group S_3 and the flip group D_3?

4. S_3 and D_3 are said to be **isomorphic**. Two groups A with operation $\bullet$ and B with the operation $\star$ are isomorphic if you can find a bijection between the two groups' elements. This means $A_1 \leftrightarrow B_1, A_2 \leftrightarrow B_2, \cdots, A_n \leftrightarrow B_n$ and that $A_j \bullet A_k = A_l \leftrightarrow B_j \star B_k = B_m$.

$\cdot$	I	r	r^2	f	fr	fr^2
I						
r				fr^2		
r^2						
f						
fr						
fr^2						

Figure 12: Unfilled alternate D_3 table.

5. (a) Make a table for only the rotations of D_3, a subgroup of D_3.
   (b) Which subgroup of the snap group S_3 is isomorphic to the subgroup in (a)?
6. What shape's dihedral (rotation and reflection) group is isomorphic to (a) the two post snap group S_2, (b) one post S_1, (c) four posts S_4 (hint: it's not a square), and (d) five posts S_5?
7. Find an combination of A and D that yields C.
8. We call A and D **generators** of the group because every element of the group is expressible as some combination of As and Ds. For convenience, let's call A "f" since it's a flip, and call D "r" meaning a $120°$ rotation counterclockwise. Then, for example, fr^2 is a rotation of $2 \cdot 120° = 240°$, followed by a flip across the A axis, equivalent to our original C. Make a new table using I, r, r^2, f, fr, and fr^2 as elements, like the one in Figure 12. *Note that the element order is different!*
9. What other pairs of elements could you have used to generate that table?
10. You should notice the 3×3 table of a group you've already described in the top-left corner of your table. What is it, and what are the two possible generators of this three-element group?
11. Explain why each element of the flip group D_3 has the period it has.
12. Some pairs of elements of the flip group are two-element subgroups. What are they?
13. One of the elements forms a one-element subgroup. What is it?
14. Addition of two numbers is a **binary operation**, while addition of three numbers is not. In logic, $\wedge$ (and) and $\vee$ (or) are binary operations, but $\neg$ (not) is not. Define binary operation in your own words, and name some other binary operations.
15. In your original flip group table, what is
    (a) The identity element?
    (b) The inverse of A?
    (c) The inverse of E?