1 It’s a Snap

// Custom command for the "snap" operation

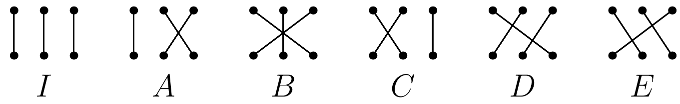

Figure 1: The six possibilities for connections between two rows of three posts.

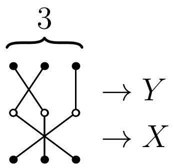

Figure 2: A grid with three strings.

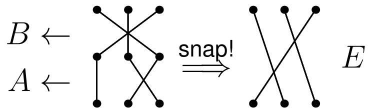

Figure 3: $A \bullet B = E$.

We begin with a problem that ties together ideas from geometry, complex numbers, matrices, combinatorics, and group theory. You likely studied geometry in 9^{th} grade and complex numbers in 10^{th} grade, so you should have a basis to start this investigation. The other three topics may be a bit unfamiliar at this point.

Consider a grid of posts with 3 rows and 3 columns. An elastic string is anchored to one post in the top row and one post in the bottom row. As the string descends from top to bottom, it loops around a post in the middle row. Two other strings are anchored and looped in the same way, with the condition that each post has exactly one string touching it. An example is depicted in Figure 2.

Figure 1 shows the six ways two rows can be connected with these rules – convince yourself that these are the only six. I have labeled them as I, A, B, C, D, and E.

Now, look at a 3×3 grid of posts, like the one in Figure 2. You should have two stacked configurations X and Y, with Y is on top of X. When you remove the middle posts (the posts indicated by $\circ$), the elastic string will **snap** to one of the six configurations we drew initially. Let’s call this operation “snap”, or $\bullet$, so that $X \bullet Y$ reads “X snap Y.” Keep in mind that when writing it this way, the bottom configuration X goes first, and the top configuration Y goes last. As an example, $A \bullet B = E$, as shown in Figure 3.

These six configurations form a mathematical **group** under the $\bullet$ operation, and we say that each configuration is an **element** of our group. More specifically, we will call this the **snap group** of size 3, or S_3; there are many other groups to name and we must be precise about which we are talking about. The group is, unsurprisingly, the main concept studied in **group theory**, a topic mentioned at the beginning of this chapter. Let’s study the snap group and characterize its properties.

1. Fill out a 6×6 table like the one in Figure 4, showing the results of each of the 36 possible snaps, where $X \bullet Y$ is in X’s row and Y’s column. $A \bullet B = E$ is done for you.

2. Would this table look different if you wrote the elements A through E in a different order?

3. Which of the elements is the **identity element** K, such that $X \bullet K = K \bullet X = X$ for all X?

4. Does every element have an inverse; can you get to the identity element from every element using only one snap?

$\bullet$	I	A	B	C	D	E
I						
A			E			
B						
C						
D						
E						

Figure 4: Unfilled 3-post snap group table.

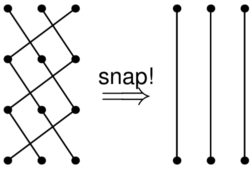

Figure 5: $E \bullet E \bullet E = I$; E has period 3.

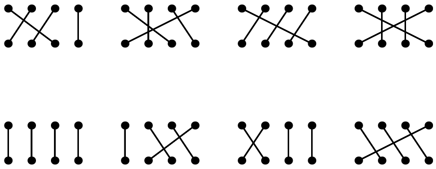

Figure 6: Some 4-post group elements.

5. (a) Is the snap operation commutative (does $X \bullet Y = Y \bullet X$ for all X, Y)?
   (b) Is the snap operation associative (does $(X \bullet Y) \bullet Z = X \bullet (Y \bullet Z)$ for all X, Y, Z)?
6. (a) For any elements X, Y, is there always an element Z so that $X \bullet Z = Y$?
   (b) For (a), is Z always unique?
7. If you constructed a 5×5 table using only 5 of the snap elements, the table would not describe a group, because there would be entries in the table not in those 5. Therefore, a group must be **closed** under its operation; if $X, Y \in G$ ($\in$ means "is/are in"), then $X \bullet Y \in G$ for all X, Y. Some subsets, however, do happen to be closed.

   Write valid group tables using exactly 1, 2, and 3 elements from the snap group. These are known as **subgroups**.
8. What do you guess is the complete definition of a mathematical group? (Hint: consider your answers to Problems 2–7.)
9. Notice that $E \bullet E \bullet E = I$. (See Figure 5.) This means that E has a **period** of 3 when acting upon itself. Which elements have a period of

   (a) 1? (b) 2? (c) 3?
10. Answer the following with the 1, 2, and 4-post snap groups S_1, S_2 and S_4.
    (a) How many elements would there be?
    (b) Systematically draw and name them.
    (c) Make a group table of these elements. For 4 posts, instead of creating the massive table, give the number of entries that table would have.
    (d) What is the relationship between this new table and your original table?
11. Can you think of an easier way to generate a snap group table without drawing all the possible configurations?
12. (a) How many elements would there be in the 5-post snap group?
    (b) How many entries would its table have?
    (c) What possible periods would its elements have?
    (d) Extend your answers for 12a–12c to M posts per row.
13. As we learned, a **permutation** of some things is an order in which they can be arranged. What is the relationship between the set of permutations of m things and the m-post snap group?