1 It's a Snap

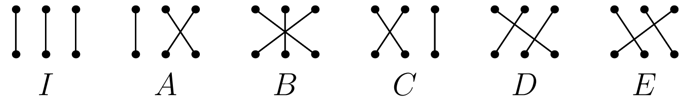

Figure 1: The six possibilities for connections between two rows of three posts.

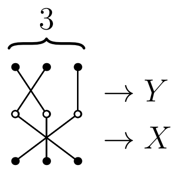

Figure 2: A grid with three strings.

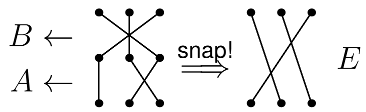

Figure 3: $A \bullet B = E$.

We begin with a problem that ties together ideas from geometry, complex numbers, matrices, combinatorics, and group theory. You likely studied geometry in 9th grade and complex numbers in 10th grade, so you should have a basis to start your investigation. The other three topics may be a bit unfamiliar at this point.

Consider a rectangle of posts with 3 rows and 3 columns. An elastic string is anchored to one post in the top row and one post in the bottom row. As the string descends top to bottom, it loops taut around a middle post. Two other elastic strings are anchored and looped in the same way, with the condition that each post has exactly one string touching it; no more, no less. An example is depicted in Figure 2.

Figure 1 shows the six configurations in which two consecutive rows can be connected in this way — convince yourself these are the only six. I have labeled them with letters I, A, B, C, D, and E.

Now, look at a 3×3 rectangle of posts, like the one in Figure 2. You should have two configurations X and Y, stacked so that Y is on top of X. When you remove the middle posts (the posts indicated by $\circ$), the elastic string will **snap** to one of the six configurations we drew initially. Let's call this operation snap, or $\bullet$, so that $X \bullet Y$ reads "X snap Y." Keep in mind that when writing it this way, the bottom configuration X goes first, and the top configuration Y goes last. As an example, $A \bullet B = E$, as shown in Figure 3.

These six configurations, along with the operation $\bullet$, form a mathematical **group**, and we say that each is an **element** of our group. In particular, we will call this the **snap group** of size 3, or S_3. Groups are the central objects studied in **group theory**, a topic mentioned at the beginning of this section. Let's study the snap group and characterize its properties.

1.1 Problems

1. Fill out a 6×6 table like the one in Figure 4, showing the results of each of the 36 $(6 \cdot 6)$ possible snaps, where $X \bullet Y$ is in X's row and Y's column.
2. Would this table look different if you named the elements A through E in a different order?
3. Which of the elements is the **identity element**?
4. Does every element have an inverse (is there a way to get the identity element from every element using only one snap)?

$\bullet$	I	A	B	C	D	E
I						
A			E			
B						
C						
D						
E						

Figure 4: Unfilled 3-post snap group table.

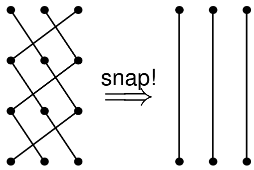

Figure 5: $E \bullet E \bullet E = I$; E has period 3.

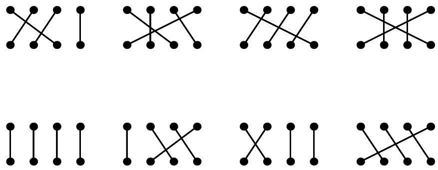

Figure 6: Some 4-post group elements.

5. Is the snap operation commutative (does $X \bullet Y = Y \bullet X$ for all X, Y)?
6. Is the snap operation associative (does $(X \bullet Y) \bullet Z = X \bullet (Y \bullet Z)$ for all X, Y, Z)?
7. (a) For any elements X, Y, is there always an element Z so that $X \bullet Z = Y$?
   (b) Is Z always unique?
8. What do you think is the definition of a mathematical group? (Hint: consider your answers to Problems 2–6.)
9. If you constructed a 5×5 table using only 5 of the snap elements, the table would not describe a group, because there would be entries in the table requiring an element outside of these 5. Therefore, a group must be **closed** under its operation; if $X, Y \in G$, then $X \bullet Y \in G$ for all X, Y. Some subsets, however, are closed. Write valid group tables using exactly 1, 2, and 3 elements from the snap group.
10. Notice that $E \bullet E \bullet E = I$. (See Figure 5.) This means that E has a period of 3 when acting upon itself. Which elements have a period of 1, 2, and 3?
11. Answer the following with the 1, 2, and 4-post snap groups S_1, S_2 and S_4.
    (a) How many elements would there be?
    (b) Draw and name them systematically.
    (c) Make a group table of these elements. For 4 posts, instead of creating the massive table, give the number of entries that table would have.
    (d) What is the relationship of this new table to your original table?
12. Can you think of an easier way to construct the a snap group table without drawing all the possible configurations?
13. (a) How many elements would there be in the 5-post snap group?
    (b) How many entries would its table have?
    (c) What possible periods would its elements have?
    (d) Extend your answers for (a)–(c) to M posts per row.
14. As we learned, a *permutation* of some things is an order they can be arranged in. What is the relationship between the set of permutations of M things and the M-post snap group?