# Contents

# 1 Trigonometry Review

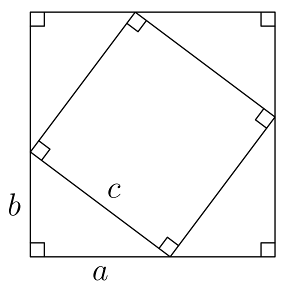


Figure 1: Scenario in Problem 1.

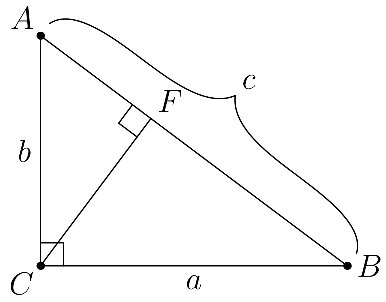


Figure 2: Scenario in Problem 2.

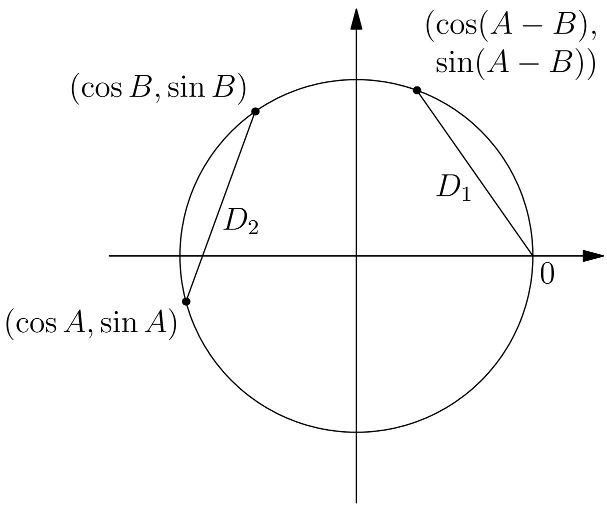


Figure 3: Scenario in Problem 3.

This is a review of material you learned last year which you will need as background knowledge for our upcoming study of linear algebra. If you don't know this material already, make sure to learn it.

1. Prove the Pythagorean theorem using "conservation of area." Start with Figure 1.
2. Prove the Pythagorean theorem using a right triangle with an altitude drawn to its hypotenuse, as shown in Figure 2, making use of similar right triangles.
3. We now prove the trigonometric identities.
   (a) Draw and label a right triangle and a unit circle, then write trig definitions for $\cos$, $\sin$, $\tan$, and $\sec$ in terms of your drawing.
   (b) Use a right triangle and the definitions of $\sin$ and $\cos$ to find and prove a value for $\sin^2\theta + \cos^2\theta$.
   (c) Use the picture of the unit circle in Figure 3 to find and prove a value for $\cos(A-B)$. Note that $D_1$ and $D_2$ are the same length because they subtend the same size arc of the circle. Set them equal and work through the algebra, using the distance formula and part (b) of this problem.
4. Write down as many trig identities as you can—no need to prove these.

$$\begin{array}{rrr}
\sin(A+B) = \qquad & \sin(A-B) = \qquad & \cos(A+B) = \\
\tan(A+B) = \qquad & \tan(A-B) = \qquad & \sin(2A) = \\
\cos(2A) = \qquad & \tan(2A) = \qquad & \sin\left(\frac{A}{2}\right) = \\
\cos\left(\frac{A}{2}\right) = \qquad & \tan\left(\frac{A}{2}\right) = \qquad &
\end{array}$$

5. Let's review complex numbers and DeMoivre's theorem.

(a) Recall that you can write a complex number both in Cartesian and polar forms. Let

$$a + bi = (a, b) = (r\cos\theta, r\sin\theta) = r\cos\theta + ir\sin\theta.$$

What is $r$ in terms of $a$ and $b$?

(b) Expand $(a + bi)(c + di)$ the usual way.

(c) Let $a + bi = r_1(\cos\theta + i\sin\theta)$ and $c + di = r_2(\cos\phi + i\sin\phi)$. Multiply them, and use the angle addition formulas to show that multiplying two complex numbers involves multiplying their lengths and adding their angles. This is DeMoivre's theorem!

(d) Use part (c) to simplify $(\sqrt{3} + i)^{18}$.

6. Here is a review of 2D rotation.

(a) Recall that we can graph complex numbers as ordered pairs in the complex plane. Now, consider the complex number $z = \cos\theta + i\sin\theta$, where $\theta$ is fixed. What is the magnitude of $z$?

(b) Multiplying $z \cdot (x + yi)$ yields a rotation of the point $(x, y)$ counterclockwise around the origin by the angle $\theta$. Notice that rotating the graph counterclockwise around the origin has the same effect as rotating the coordinate axes clockwise around the origin by the same angle $\theta$. What if we wanted to rotate clockwise by $\theta$ instead?

7. Rotate the following conics by (i) $30°$, (ii) $45°$, and (iii) $\theta$:

(a) $x^2 - y^2 = 1$ (b) $\frac{x^2}{16} - \frac{y^2}{9} = 1$ (c) $y^2 = 4Cx$

You should have mastery of this material so that we can immediately investigate novel and interesting ideas. These often have surprising connections to the trigonometry and transformational geometry you learned last year. For example, we will soon find another convenient way to do a rotation of coordinates.

## 2 It's a Snap

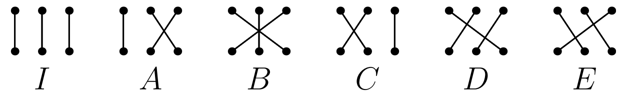


Figure 1: The six possibilities for connections between two rows of three posts.

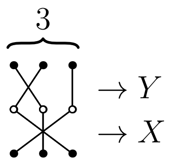


Figure 2: A grid with three strings.

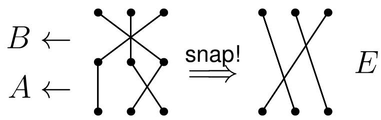


Figure 3: $A \bullet B = E$.

We begin with a problem that ties together ideas from geometry, complex numbers, matrices, combinatorics, and group theory. You have studied geometry, complex numbers, and combinatorics before, so you should have a basis to start this investigation, but the other two topics may be a bit unfamiliar at this point.

Figure 1 shows the six ways the posts in two rows, each row containing three posts, can be paired up. Convince yourself that these are the only six. Let's label them $I$, $A$, $B$, $C$, $D$, and $E$. For example, $I$ pairs every post in the top row with the post directly below it, while $A$ switches the pairings of the second and third posts.

Consider a grid of posts with three rows and three columns. An elastic string is anchored to one post in the top row, looping around a post in the middle row, and finally descending to a post in the bottom row. Two other strings are anchored and looped in the same way, with the condition that each post has exactly one string touching it. An example is depicted in Figure 2.

Notice that we can easily represent such a grid of posts with two of our six elements. You should have two stacked elements, $X$ and $Y$, with $Y$ on top of $X$, as shown in Figure 2. When you remove the middle posts (the posts indicated by $\circ$), the elastic string will **snap** to one of the six configurations we drew initially. Let's call this operation "snap," or $\bullet$, so that $X \bullet Y$ reads "$X$ snap $Y$." Keep in mind that when evaluating the snap operation, the *bottom* configuration $X$ goes first, and the *top* configuration $Y$ goes last. In Figure 2, $X = B$ and $Y = C$, and $X \bullet Y$ makes $E$, so $B \bullet C = E$. As another example, $A \bullet B = E$, as shown in Figure 3.

Why do we put the bottom configuration first and the top configuration last? This choice is somewhat arbitrary, but there is a reason. Remember that when we compose functions, we write $(f \circ g)(x) = f(g(x))$. The right function, $g$, is evaluated first and used as an input to the left function, $f$. Similarly, when we write $X \bullet Y$, the overall configuration (from top to bottom) first goes through $Y$, then through $X$. As we will see, these six elements often behave more like functions, rather than simply elements. Thus, it is natural to order them as if they were functions.

Some important terminology: these six configurations form a mathematical **group** under the operation $\bullet$, and we say that each configuration is an **element** of our group. More specifically, we will call this group the three-post **snap group**, or $S_3$. There are countless other mathematical groups, so we must be precise when we talk about a specific group. Note that this snap group is denoted $S_3$, not $S_6$, because the subscript $3$ is the number of posts in each row, *not* the size of the group. Groups are, unsurprisingly, the main objects studied in **group theory**. Let's study this snap group and characterize its properties.

1. Fill out a $6 \times 6$ table like the one in Figure 4, showing the results of each of the $36$ possible snaps, where $X \bullet Y$ is in $X$'s row and $Y$'s column. $A \bullet B = E$ is done for you.

2. Which of the elements is the **identity element** $K$, such that $X \bullet K = K \bullet X = X$ for all $X$?

3. Does every element have an inverse? In other words, can you get to the identity element from every element using only one snap?

4. (a) Is the snap operation commutative (does $X \bullet Y = Y \bullet X$ for all $X, Y$)?

   (b) Is the snap operation associative (does $(X \bullet Y) \bullet Z = X \bullet (Y \bullet Z)$ for all $X, Y, Z$)?

5. (a) For any elements $X, Y$, is there always an element $Z$ so that $X \bullet Z = Y$?

   (b) For (a), is $Z$ always unique?

| $\bullet$ | $I$ | $A$ | $B$ | $C$ | $D$ | $E$ |
|---|---|---|---|---|---|---|
| $I$ | | | | | | |
| $A$ | | | $E$ | | | |
| $B$ | | | | | | |
| $C$ | | | | | | |
| $D$ | | | | | | |
| $E$ | | | | | | |

Figure 4: Unfilled $3$-post snap group table.

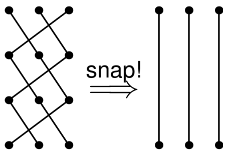


Figure 5: $E \bullet E \bullet E = I$; $E$ has period $3$.

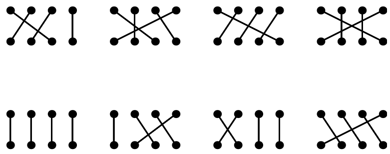

Figure 6: Some $4$-post group elements.

6. If you constructed a $5 \times 5$ table using only five of the snap elements, the table would not describe a group, because there would be entries in the table outside of those $5$. Therefore, a group must be **closed** under its operation. *Some* subsets of our six elements, however, do happen to be closed. Write valid group tables using exactly one, two, and three elements from the snap group. These are known as **subgroups**.

7. What do you guess is a good definition of a mathematical group? (Hint: consider your answers to Problems 2–6.)

8. Notice that $E \bullet E \bullet E = I$ (see Figure 5). This means that $E$ has a **period** of $3$ when acting upon itself. Which elements have a period of

   (a) $1$? (b) $2$? (c) $3$?

9. Answer the following with the one, two, and four-post snap groups $S_1$, $S_2$ and $S_4$. These are just the analogous groups for connections between one, two, and four posts.

   (a) How many elements would there be?
   (b) Systematically draw and name them.
   (c) Make a group table of these elements. For four posts, instead of creating the massive table, give the number of elements that the table would have.
   (d) What is the relationship of your original table to this new table?

10. Can you think of a shortcut to generate a snap group table without drawing every possible configuration?

11. (a) How many elements would there be in the five-post snap group?
    (b) How many entries would its table have?
    (c) What possible periods would its elements have? Make sure you include a period of six!
    (d) Extend your answers for Problems 11a through 11c to $M$ posts per row.

12. A **permutation** of a set of things is an order in which they can be arranged. What is the relationship between the set of permutations of $m$ things and the $m$-post snap group?