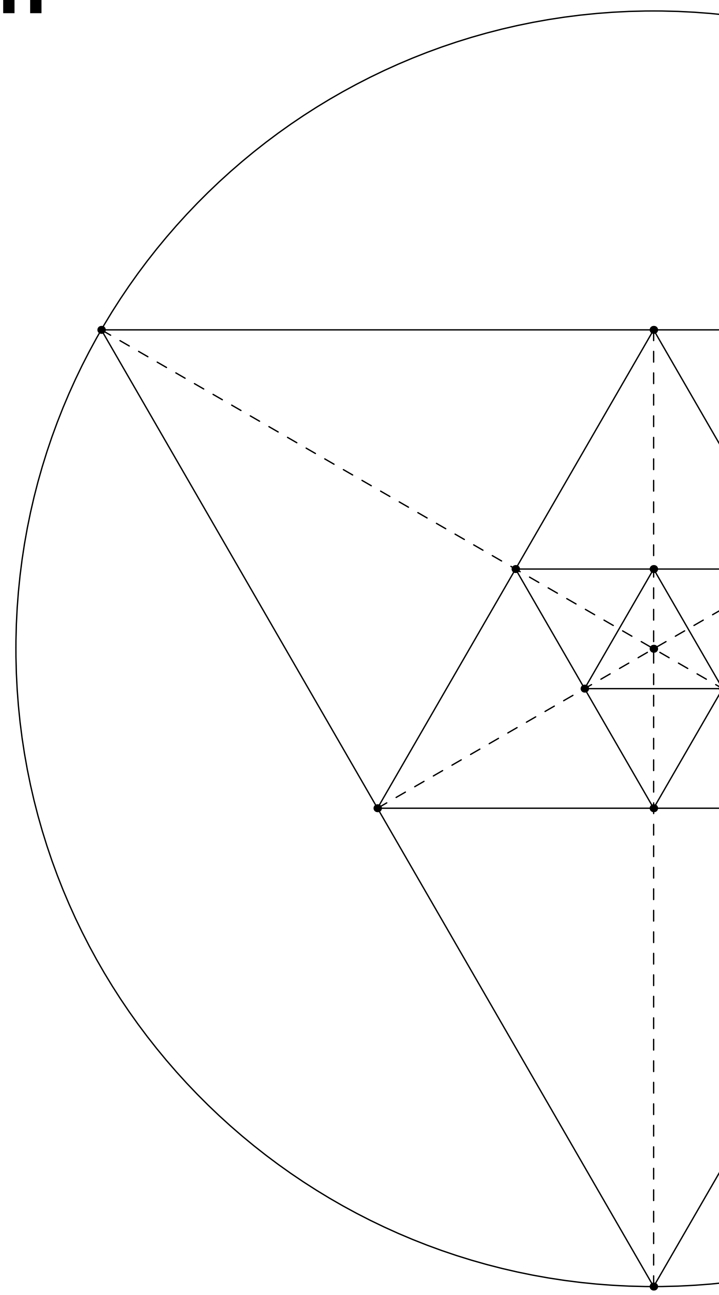
A Geometric Approach To Matrices

Answer Key

Timothy Herchen
Henry M. Gunn High School
Analysis Honors



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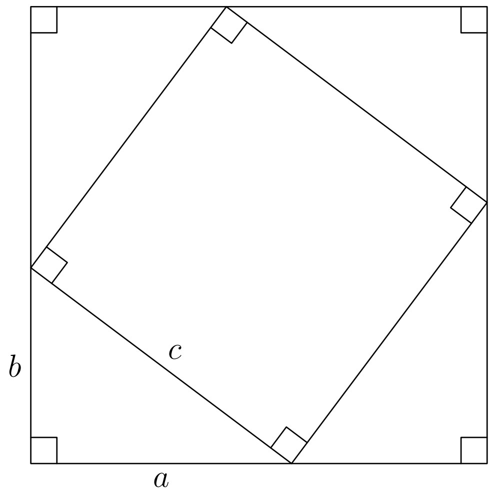

Figure 1: Scenario in Problem 1.

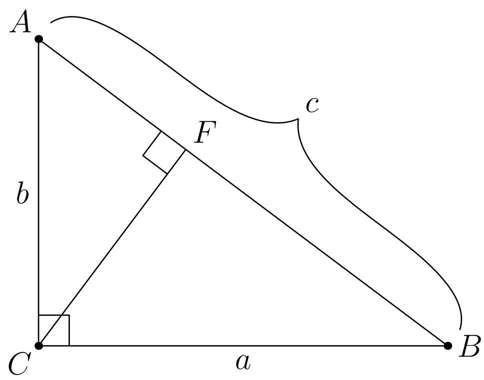

Figure 2: Scenario in Problem 2.

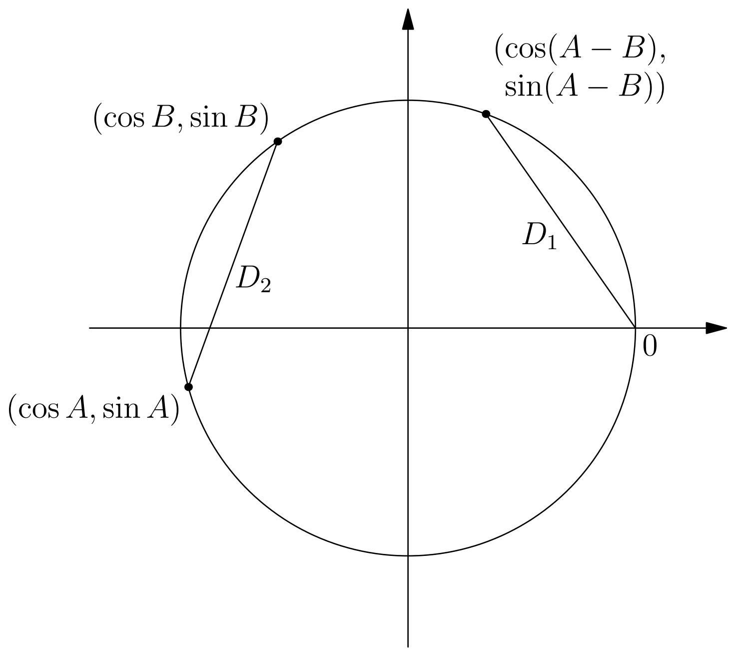

Figure 3: Scenario in Problem 3.

1 Trigonometry Review

1. Prove the Pythagorean theorem using "conservation of area." Start with Figure 1.

In Figure 1, the larger square has side length $a+b$. The smaller, nested square has side length c. Four copies of the right triangle with side lengths a, b, c are placed around the square. We have

$$\begin{aligned} A_{\text{triangles}} + A_{\text{small sq.}} &= A_{\text{big sq.}} && \text{[Conservation of area]} \\ 4A_{\text{triangle}} + A_{\text{small sq.}} &= A_{\text{big sq.}} \\ 4\left(\frac{1}{2}ab\right) + c^2 &= (a+b)^2 && \text{[Areas of triangle, square]} \\ 2ab + c^2 &= a^2 + 2ab + b^2 && \text{[Expanding]} \\ c^2 &= a^2 + b^2. \end{aligned}$$

2. Prove the Pythagorean theorem using a right triangle with an altitude drawn to its hypotenuse, making use of similar right triangles. This is shown in Figure 2.

3. Now you will prove the trig identities.

   (a) Draw and label a right triangle and a unit circle, then write trig definitions for $\cos$, $\sin$, $\tan$, and $\sec$ in terms of your drawing.

   (b) Use a right triangle and the definitions of $\sin$ and $\cos$ to find and prove a value for $\sin^2\theta + \cos^2\theta$.

   (c) Use the picture of the unit circle in Figure 3 to find and prove a value for $\cos(A - B)$. Note that D_1 and D_2 are the same length because they subtend the same size arc of the circle. Set them equal and work through the algebra, using the distance formula and part (b) of this problem.

4. Write down as many trig identities as you can. There's no need to prove all of these right now.

$$\begin{array}{rrr} \sin(A+B) = & \sin(A-B) = & \cos(A+B) = \\ \tan(A+B) = & \tan(A-B) = & \sin(2A) = \\ \cos(2A) = & \tan(2A) = & \sin(\frac{A}{2}) = \\ \cos\left(\frac{A}{2}\right) = & \tan\left(\frac{A}{2}\right) = & \end{array}$$

5. Let's review complex numbers and DeMoivre's theorem.

   (a) Recall that you can write a complex number both in Cartesian and polar forms. Let

   $$a + bi = (a, b) = (r\cos\theta, r\sin\theta) = r\cos\theta + ir\sin\theta.$$

   What is r in terms of a and b?

   (b) Multiply $(a + bi)(c + di)$ out using FOIL.

   (c) Convert the two multiplicands[1] to polar form, noting that the two lengths and angles are different numbers. Call them $r_1(\cos\theta + i\sin\theta)$ and $r_2(\cos\phi + i\sin\phi)$.

   (d) Multiply them, and use your results from Problems 3c and 3d to show that multiplying two complex numbers involves multiplying their lengths and adding their angles. This is DeMoivre's theorem!

   (e) Use part (c) to simplify $(\sqrt{3} + i)^{18}$.

6. Here is a review of 2D rotation.

   (a) Remember that we can graph complex numbers as 2D ordered pairs in the complex plane. Now, consider the complex number $z = \cos\theta + i\sin\theta$, where θ is fixed. What is the magnitude of z?

   (b) Multiplying $z \cdot (x + yi)$ yields a rotation of the point (x, y) counterclockwise by the angle θ around the origin. What if we wanted to rotate clockwise by θ instead?

7. Rotate the following conics by (i) $30°$, (ii) $45°$, and (iii) θ:

   (a) $x^2 - y^2 = 1$ (b) $\frac{x^2}{16} - \frac{y^2}{9} = 1$ (c) $y^2 = 4Cx$

[1] This is the word for parts of a multiplication! So for example, if $a \cdot b = c$, then a and b are the multiplicands.