A Geometric Approach To Matrices

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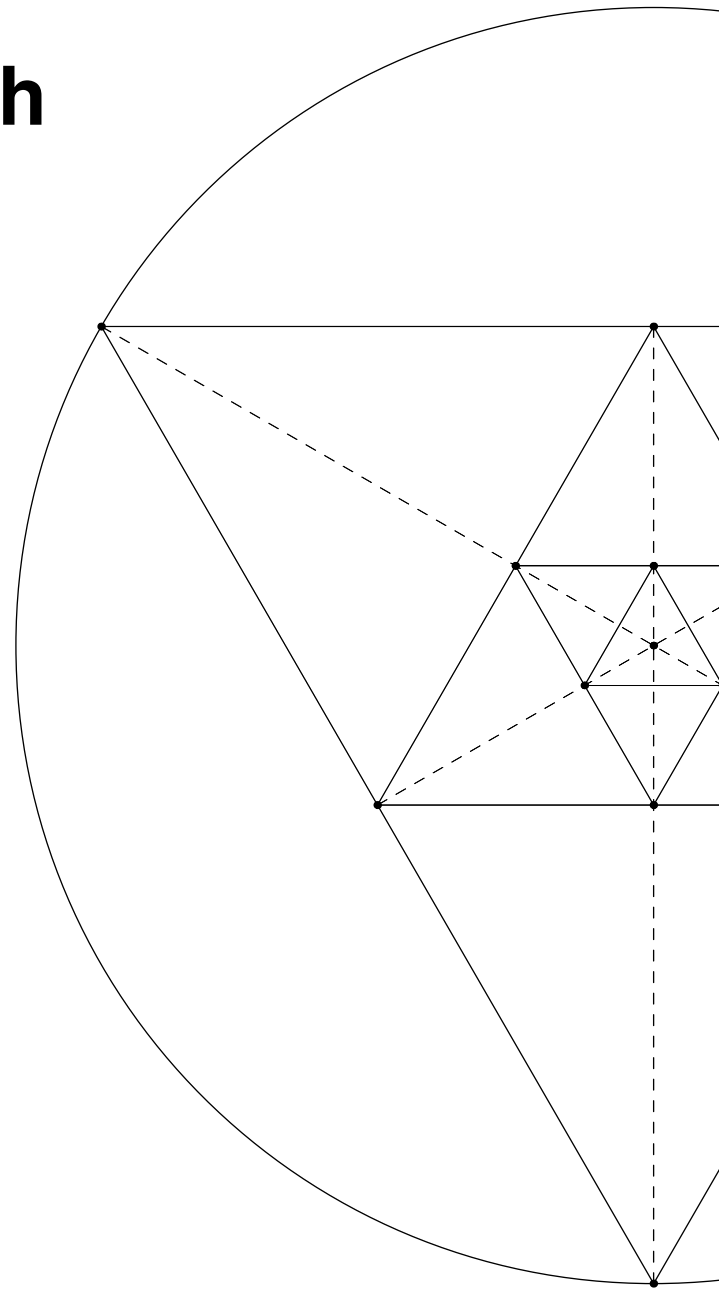
Analysis Honors



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1 It's a Snap

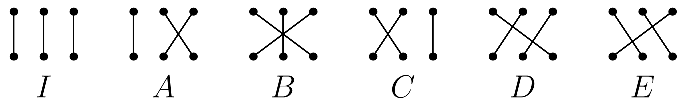


Figure 1: The six possibilities for connections between two rows of three posts.

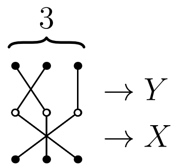


Figure 2: A grid with three strings.

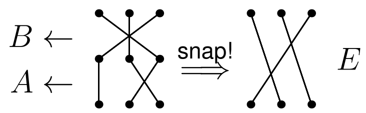


Figure 3: $A \bullet B = E$.

We begin with a problem that ties together ideas from geometry, complex numbers, matrices, combinatorics, and group theory. You likely studied geometry in 9th grade and complex numbers in 10th grade, so you should have a basis to start your investigation. The other three topics may be a bit unfamiliar at this point.

Consider a grid of posts with 3 rows and 3 columns. An elastic string is anchored to one post in the top row and one post in the bottom row. As the string descends from top to bottom, it loops around a post in the middle row. Two other strings are anchored and looped in the same way, with the condition that each post has exactly one string touching it. An example is depicted in Figure 2.

Figure 1 shows the six ways two rows can be connected with these rules – convince yourself that these are the only six. I have labeled them as I, A, B, C, D, and E.

Now, look at a 3×3 rectangle of posts, like the one in Figure 2. You should have two configurations X and Y, stacked so that Y is on top of X. When you remove the middle posts (the posts indicated by $\circ$), the elastic string will **snap** to one of the six configurations we drew initially. Let's call this operation "snap", or $\bullet$, so that $X \bullet Y$ reads "X snap Y." Keep in mind that when writing it this way, the bottom configuration X goes first, and the top configuration Y goes last. As an example, $A \bullet B = E$, as shown in Figure 3.

These six configurations form a mathematical **group** under the $\bullet$ operation, and we say that each configuration is an **element** of our group. We will call this the **snap group** of size 3, or S_3. The group is, unsurprisingly, the main concept studied in **group theory**, a topic mentioned at the beginning of this chapter. Let's study the snap group and characterize its properties.

1. Fill out a 6×6 table like the one in Figure 4, showing the results of each of the 36 $(6 \cdot 6)$ possible snaps, where $X \bullet Y$ is in X's row and Y's column.
2. Would this table look different if you wrote the elements A through E in a different order?
3. Which of the elements is the **identity element**?
4. Does every element have an inverse (can you get to the identity element from every element using only one snap)?

$\bullet$	I	A	B	C	D	E
I						
A			E			
B						
C						
D						
E						

Figure 4: Unfilled 3-post snap group table.

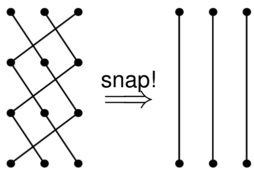


Figure 5: $E \bullet E \bullet E = I$; E has period 3.

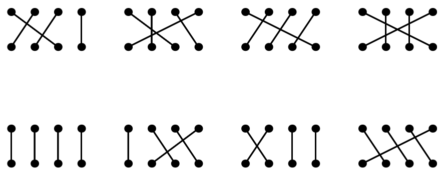


Figure 6: Some 4-post group elements.

5. Is the snap operation commutative (does $X \bullet Y = Y \bullet X$ for all X, Y)?

6. Is the snap operation associative (does $(X \bullet Y) \bullet Z = X \bullet (Y \bullet Z)$ for all X, Y, Z)?

7. (a) For any elements X, Y, is there always an element Z so that $X \bullet Z = Y$?
   (b) Is Z always unique?

8. What do you think is the definition of a mathematical group? (Hint: consider your answers to Problems 2–6.)

9. If you constructed a 5×5 table using only 5 of the snap elements, the table would not describe a group, because there would be entries in the table not in those 5. Therefore, a group must be **closed** under its operation; if $X, Y \in G$ ($\in$ means "is/are in"), then $X \bullet Y \in G$ for all X, Y. Some subsets, however, do happen to be closed. Write valid group tables using exactly 1, 2, and 3 elements from the snap group.

10. Notice that $E \bullet E \bullet E = I$. (See Figure **??**.) This means that E has a period of 3 when acting upon itself. Which elements have a period of 1, 2, and 3?

11. Answer the following with the 1, 2, and 4-post snap groups S_1, S_2 and S_4.
    (a) How many elements would there be?
    (b) Draw and name them systematically.
    (c) Make a group table of these elements. For 4 posts, instead of creating the massive table, give the number of entries that table would have.
    (d) What is the relationship of this new table to your original table?

12. Can you think of an easier way to generate a snap group table without drawing all the possible configurations?

13. (a) How many elements would there be in the 5-post snap group?
    (b) How many entries would its table have?
    (c) What possible periods would its elements have?
    (d) Extend your answers for (a)–(c) to M posts per row.

14. As we learned, a *permutation* of some things is an order they can be arranged in. What is the relationship between the set of permutations of M things and the M-post snap group?

2 From Snaps to Flips

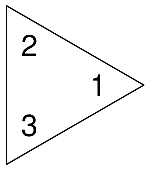


Figure 7: The paper triangle.

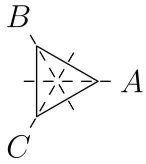


Figure 8: Its axes of reflection.

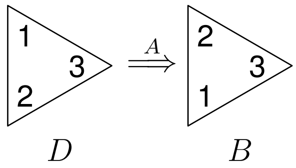


Figure 9: $AD = B$; Notice the RTL evaluation.

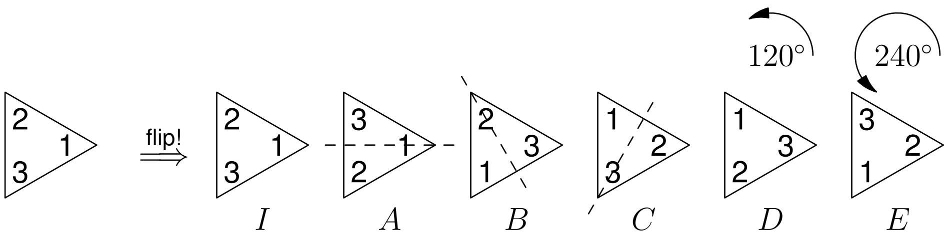


Figure 10: The six ending positions.

You can use a paper/cardboard triangle to help visualize the next concept: cut out an equilateral triangle, label its front vertices 1, 2, and 3 as shown in Figure **??**, and place it down in the shown orientation. Consider the possible ways to move this triangle From this starting position, you can reflect the triangle over one of three axes: A, B, or C, as shown in Figure **??**. You could also rotate the triangle $120°$ or $240°$ counterclockwise. The final possible positions are shown in Figure **??**.

Notice that each position corresponds to a different operation which preserves the triangle's location. For example, I means "leave the triangle alone," A means "flip the triangle about the A axis," and D means "rotate the triangle $120°$ counterclockwise." We can combine these operations to form other operations by writing them in sequence. Unlike most cases, however, we evaluate them right-to-left (RTL) rather than left-to-right (LTR). For example, $AD = B$, as shown in Figure **??**.

These six positions form another group: the **dihedral group** of order 3, or D_3. If we split "dihedral" into "di-" and "-hedral," we see it means "two faces"; these are the two faces of our paper triangle. Let's study the properties of this group.

1. The six positions or "operations" are considered to be isometries. Isometries are ways of mapping the triangle to itself, preserving shape and location. Are there any others on this triangle?

2. As with the snap group, we can make a group table for the flip group. Fill out a table like the one in Figure 4 in your notebook. Like the snap group table, the top row indicates what operation is done first and the left column indicates what's done second, so that XY is in the X^{th} row and Y^{th} column.

$\cdot$	I	A	B	C	D	E
I						
A					B	
B						
C						
D						
E						

Figure 11: Unfilled D_3 group table.

3. What is the relationship between the tables for the snap group S_3 and the flip group D_3?

4. S_3 and D_3 are said to be **isomorphic**. Groups A with operation $\bullet$ and B with operation $\star$ are isomorphic if you can find a one-to-one correspondence between the two groups' elements. This means we can find some pairing of elements between the two groups $A_1 \leftrightarrow B_1, A_2 \leftrightarrow B_2, \cdots, A_n \leftrightarrow B_n$ such that $A_j \bullet A_k = A_l \leftrightarrow B_j \star B_k = B_m$.

5. (a) Make a table for only the rotations of D_3, a subgroup of D_3.

$\cdot$	I	r	r^2	f	fr	fr^2
I						
r				fr^2		
r^2						
f						
fr						
fr^2						

Figure 12: Unfilled alternate D_3 table.

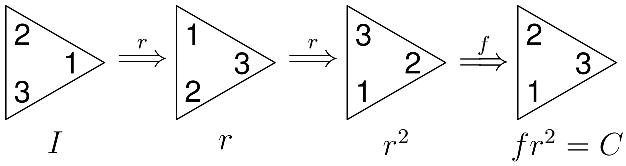


Figure 13: $fr^2 = C$. Again, notice the RTL evaluation.

(b) Which subgroup of the snap group S_3 is isomorphic to the subgroup in (a)?

6. What shape's dihedral (rotation and reflection) group is isomorphic to (a) the two post snap group S_2, (b) one post S_1, (c) four posts S_4 (hint: it's not a square), and (d) five posts S_5?
7. Find an combination of A and D that yields C.
8. We call A and D **generators** of the group because every element of the group is expressible as some combination of As and Ds. For convenience, let's call A "f" since it's a flip, and call D "r" meaning a $120°$ rotation counterclockwise. Then, for example, fr^2 is a rotation of $2 \cdot 120° = 240°$, followed by a flip across the A axis, equivalent to our original C. Make a new table using I, r, r^2, f, fr, and fr^2 as elements, like the one in Figure **??**. *Note that the element order is different!*
9. What other pairs of elements could you have used to generate that table?
10. You should notice the 3×3 table of a group you've already described in the top-left corner of your table. What is it, and what are the two possible generators of this three-element group?
11. Explain why each element of the flip group D_3 has the period it has.
12. Some pairs of elements of the flip group are two-element subgroups. What are they?
13. One of the elements forms a one-element subgroup. What is it?

A **group** G is a set of elements together with a **binary operation** that meets the following criteria:

(a) Identity: There is an element $I \in G$ such that for all $X \in G$, $X \bullet I = I \bullet X = X$.

(b) Closure: If X, Y are elements of the group, then $X \bullet Y$ is also an element of the group.

(c) Invertibility: Each element X has an inverse X^{-1} such that $X \bullet X^{-1} = X^{-1} \bullet X = I$.

(d) Associativity: For all elements X, Y, and Z, $X \bullet (Y \bullet Z) = (X \bullet Y) \bullet Z$.

14. Addition of two numbers is a **binary operation**, while addition of three numbers is not. In logic, $\wedge$ (and) and $\vee$ (or) are binary operations, but $\neg$ (not) is not. Define binary operation in your own words, and name some other binary operations.
15. In your original flip group table, what is

    (a) The identity element?

    (b) The inverse of A?

    (c) The inverse of E?

3 Rotation and Reflection Groups

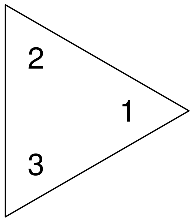


Figure 14: The paper triangle.

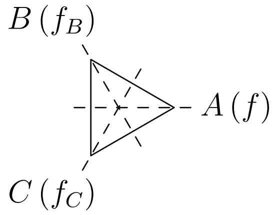


Figure 15: Its axes of reflection.

In the previous section, we started with the dihedral group of the equilateral triangle and discovered it had 6 elements: reflections about three different axes, rotations of $\pm 120°$, and the identity element. We identified a subgroup consisting of the identity I with two rotations r and r^2, and three other subgroups of just the identity and a single reflection.

1. Notice that the original dihedral group had twice as many elements as the rotation group. Why?
2. Make and justify a conjecture extending this observation to the dihedral groups of other shapes like rectangles, squares, hexagons, cubes, etc.
3. Write a table for the dihedral group of the rectangle, recalling that the allowed isometries are reflections and rotations. Let r be a $180°$ rotation, x a reflection over the x-axis, and y be a reflection over the y-axis. How does this table differ from the dihedral group of the equilateral triangle?
4. Write a table for the *rotation group* of the square, with 4 elements and 16 entries. Compare this table to problem 3.

We noticed that the rotation group for the equilateral triangle could be generated by just one of the elements, such as r—rotation by $120°$ counterclockwise (or ccw). Then r^2 is a rotation of $240°$ ccw, and $r^3 = I$, the identity (see Figure **??**). Since we can generate the entire rotation group with a single element r, a natural question to ask is whether we can do the same with the dihedral group D_3. Clearly we can't use the identity to do it, and a series of rotations always leaves us with a rotation, never a reflection. Also, a series of flips along one axis simply generates a two member group with elements I, f (see Figure **??**).

Let's try using two elements to generate our group, using the same definitions of f and r as in the previous section: a flip over the A axis and rotation by $120°$ ccw, respectively. As we found, fr is a flip over the B axis and rf is a flip over the C axis. Consecutive powers of r already got us the remaining elements, so r, f generates the group.

We can also generate the group using two reflections, say f and f_B (flip over the B axis, as shown in Figure **??**). Notice that an even number of reflections always results in a rotation—even the identity element I is just a rotation by 0. We can think of this as the existence of a "mirror world" and its unmirrored counterpart, and each reflection takes us into or out of the mirror world.

Moving to three dimensions, we can interpret D_3 as the set of rotations of an equilateral triangular prism. The rotation axes are coplanar with where the reflection axes used to be (see Figure **??**). Indeed, when you "flipped" your equilateral triangles, you were actually rotating a triangular prism in the third dimension. Flipping the triangular prism using a rotation requires a fourth-dimensional rotation—something we cannot easily visualize.

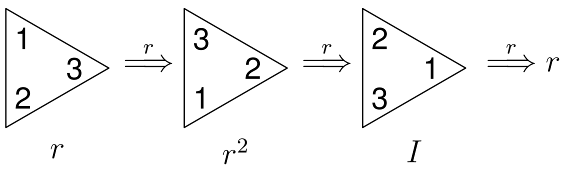


Figure 16: r generates a three member group.

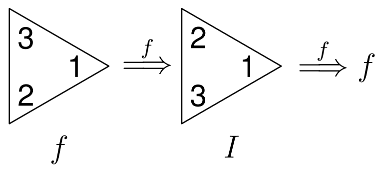


Figure 17: f generates a two member group.

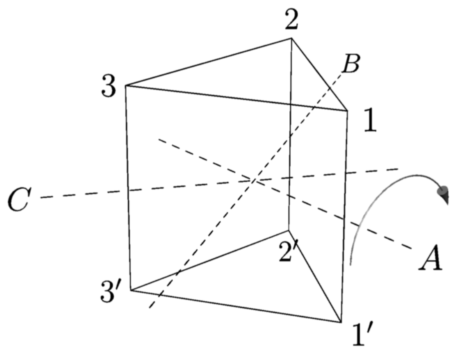


Figure 18: Triangular prism's corresponding axes of rotation.

As you complete the following problems, you can extend the above analysis to characterize group properties. You will notice that the more symmetries an object has, the larger its symmetry group is. Indeed, group theory is the mathematics of symmetry *par excellence*.

For each of the following groups, find the following:

(a) Number of elements (order)

(b) If order < 10, the set of elements; otherwise, an explanation of how you know the order

(c) A smallest possible generating set[1]

(d) Whether the group is commutative

If two problems have isomorphic groups, just write "isomorphic to Problem N" for the latter problem and move on.

5. Rectangle under rotation
6. Rectangle under reflection
7. Square under rotation
8. Square under reflection
9. Square prism under rotation
10. Square prism under reflection
11. Pentagon under rotation
12. Pentagon under reflection
13. Pentagonal prism under rotation
14. Pentagonal prism under reflection
15. Pentagonal pyramid under rotation
16. Pentagonal pyramid under reflection
17. Tetrahedron (triangular pyramid) under rotation
18. Tetrahedron under reflection
19. Cube under rotation
20. Cube under reflection

[1] There may be multiple generating sets of the same size.

4 Infinite Groups

All of the groups we've seen so far are finite in size. We can also construct groups of an infinite size.

A quick review: *iso-* means the same and *-morphic* means form. Two groups are said to be isomorphic if there is a mapping which takes each element of the first group to an element of the second group and vice versa, so that the products of the elements map in the same way.

1. Where have you come across the roots *iso-* and *-morphic* before?
2. Could two groups be isomorphic if they had different orders?
3. The rotation group for the hexagon H has six elements: the identity, and rotations of $\frac{\pi}{3}$, $\frac{2\pi}{3}$, π, $\frac{4\pi}{3}$, $\frac{5\pi}{3}$ radians. A rotation of $\frac{\pi}{3}$ generates the group. Which other rotation generates the group? What is the period of each element.
4. H has the same number of elements as the dihedral group D_3. Are the two groups isomorphic? How do you know? What is the period of each element of D_3, What can you say if the sets of the periods of the elements of each group are not the same? Which subgroups of C_6 and D_3 are isomorphic?
5. Could an infinite group be isomorphic to a finite group.
6. Do you think all infinite groups are isomorphic to each other. Find a counterexample if you can.

If an infinite group was somehow "bigger" than the other, they wouldn't be isomorphic. (Can you think of an example of two groups with the same "size" that also aren't isomorphic?) This raises the question: are all infinities equally big?

We can formalize the notion of sizes of infinity. Let's say that two infinite sets are of the same size if their elements can be put into a one-to-one correspondence with each other. For example, the positive numbers $1, 2, ... = \mathbb{N}$ and negative numbers $-1, -2, ... = -\mathbb{N}$ are of the same size, because we have the one-to-one correspondence $\mathbb{N} \ni n \leftrightarrow -n \in \mathbb{N}$. Every element of the positive numbers has exactly one "partner" in the negative numbers, and vice versa.

7. Make guesses to the relative sizes of the following pairs of sets. You may use shorthand like $a < b$, $a > b$, $a = b$. After you have made your guesses, we will analyze some of the cases and you can find out how good your intuition was.

(a)	natural numbers, $\mathbb{N}$	vs.	positive even numbers, $2\mathbb{N}$
(b)	natural numbers, $\mathbb{N}$	vs.	positive rational numbers, $\mathbb{Q}^+$
(c)	natural numbers, $\mathbb{N}$	vs.	real numbers between zero and one, $[0, 1)$
(d)	real numbers, $\mathbb{R}$	vs.	complex numbers, $\mathbb{C}$
(e)	real numbers, $\mathbb{R}$	vs.	points on a line
(f)	points on a line	vs.	points on a line segment
(g)	points on a line	vs.	points on a plane
(h)	rational numbers, $\mathbb{Q}$	vs.	Cantor set (look this up or ask your teacher)

It turns out that studying infinity involves some strange mathematics. For instance, even though it seems that there should be half as many positive even numbers as natural numbers (see 7a), we can construct a one-to-one correspondence between the two sets such that every positive even number is paired with a natural number and vice versa: $2\mathbb{N} \ni 2n \leftrightarrow n \in \mathbb{N}$. The existence of this correspondence means that the two sets are equal in size.

More surprisingly, we can establish a correspondence between the non-negative rational numbers ($\mathbb{Q}^+$) and the natural numbers. Draw the rational numbers in a table as shown in Figure **??**, and pair these numbers up with the numbers $1, 2, 3, 4....$ You can see that you will eventually list all of the non-negative rational numbers, multiple times, into a correspondence with the natural numbers. To make it one-to-one, only pair the rational numbers that are in simplest form. Here, we pair 2 with $\frac{1}{1}$ instead of $\frac{0}{2}$ – since $\frac{0}{1}$ is the same number, and is already paired with 1. This correspondence is depicted in Figure **??**. This prevents multiple natural numbers from being paired up with the same rational number: the correspondence is now one-to-one.

The real numbers between 0 and 1, however, cannot be put into a one-to-one correspondence with $\mathbb{N}$ (see 7c). We will prove this with contradiction. Suppose I told you that I have paired each real number $0 <= r_k < 1$

with a unique natural number k, and vice versa. Then, you can construct a real number ${r_\omega}^2$ whose 1^{st} digit (after the decimal point) differs from r_1's, whose 2^{nd} digit differs from r_2's, and so on. In other words, it differs from r_n in the n^{th} digit. We can make better sense of this construction by writing the numbers out in a table, as shown in Figure **??**. Your new number r_ω is at the bottom; it differs from all the previous numbers in at least one place, so it is a new real number. Therefore, my original list is incomplete, and the correspondence doesn't exist. This is called Cantor's diagonal argument, because the differing digits make a diagonal. (In the diagram, C_10 is the Champernowne constant, the result of joining all the natural numbers together to make a decimal.)

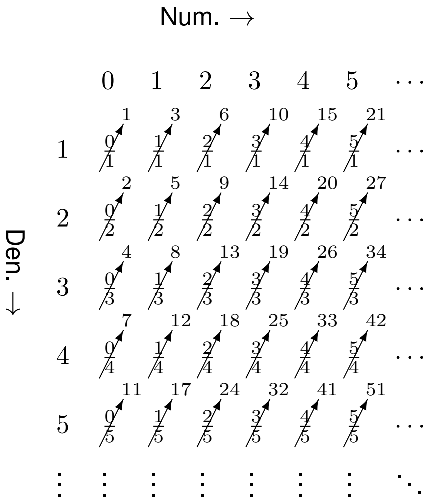


Figure 19: A correspondence between $\mathbb{Q}^+$ and $\mathbb{N}$, but not one-to-one.

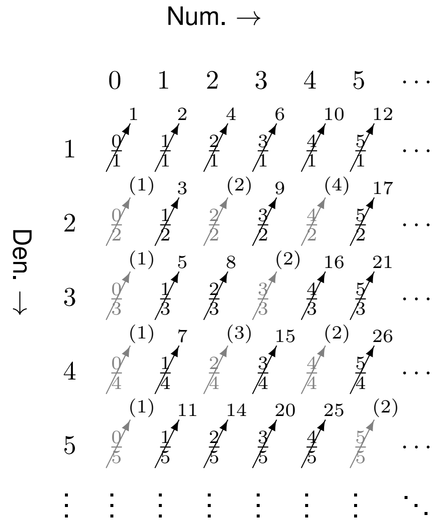


Figure 20: The previous correspondence, with duplicates parenthesized but not counted; it is now one-to-one.

Decimal Expansion →

r_1	0	0.	$\boxed{0}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$\cdots$
r_2	$\frac{1}{4}$	0.	2	$\boxed{5}$	0	0	0	0	0	0	0	0	0	0	0	0	0	$\cdots$
r_3	$\frac{\pi}{4}$	0.	7	8	$\boxed{5}$	3	9	8	1	6	3	3	9	7	4	4	8	3 $\cdots$
r_4	C_{10}	0.	1	2	3	$\boxed{4}$	5	6	7	8	9	1	0	1	1	1	2	1 $\cdots$
r_5	$e-2$	0.	7	1	8	2	$\boxed{8}$	1	8	2	8	4	5	9	0	4	5	2 $\cdots$
r_6	$\frac{\sqrt{2}}{10}$	0.	1	4	1	4	2	$\boxed{1}$	3	5	6	2	3	7	3	0	9	5 $\cdots$
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$		$\vdots$		$\vdots$		$\vdots$		$\vdots$		$\vdots$		$\ddots$
r_ω		0.	1	6	6	5	9	2	$\cdots$									

Figure 21: Cantor's diagonal argument. Notice how, by construction, r_ω differs from r_i in the circled digits.

8. Now, please return to problem 7 and revise your answers. Justify each answer by producing a one-to-one correspondence, or showing the impossibility of doing so. Part (h) is an optional challenge.

By the way, the infinities in problem 7 come in only two sizes: **countable** infinity—like the number of natural numbers—and **uncountable** infinity—like the number of real numbers. There are in fact an infinite number of sizes of infinity, but these two are the only ones we'll deal with in this class. Are there any infinities *between* the two we've discussed, uncountable and countable? This is a very deep mathematical question, known as the continuum hypothesis. It turns out that both the answer "yes" and "no" are consistent with the rest of our mathematics, so either can be taken as an axiom.

Two infinite groups can be the same, infinite size and still not be isomorphic, in the same way that two finite groups of the same size are sometimes not isomorphic (like $D_3 \neq S_6$). For example, the group of all rotations

[2]Pronounced "r omega."

of a rational number of degrees about the origin is countably infinite. So is the group of integers—positive and negative—under addition. But these two groups have completely different structures. For example, the former has two elements which are their own inverse: $0°$ and $180°$. The latter has only one such element: 0.

9. Here's a list of infinite sets, each with an operation. For each pair, answer: i. Does it form a group? ii. Which previous group(s) is it isomorphic to?
   (a) natural numbers, addition
   (b) integers, addition
   (c) even integers, addition
   (d) odd integers, addition
   (e) rational numbers, addition
   (f) real numbers, addition
   (g) complex numbers, addition
   (h) integers, multiplication
   (i) integer powers of 2, multiplication
   (j) rational numbers, multiplication
   (k) rational numbers excluding 0, multiplication
   (l) complex numbers, multiplication
   (m) rotation by a rational number of degrees
   (n) rotation by a rational number of radians
   (o) rotation by an integer number of radians

10. Can an irrational number taken to an irrational power ever be rational? Consider the potential example $a = \sqrt{2}^{\sqrt{2}}$. To help you answer this question, let $b = a^{\sqrt{2}}$. Simplify b, and explain why we don't need to know whether a is rational or irrational.

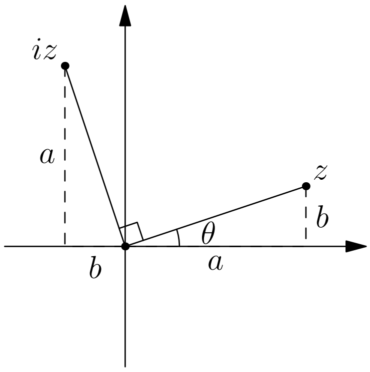


Figure 22: iz is perpendicular to z.

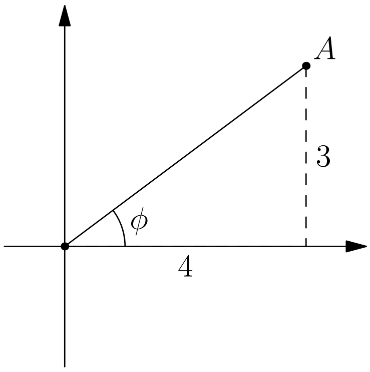


Figure 23: The complex number $A = 4 + 3i$.

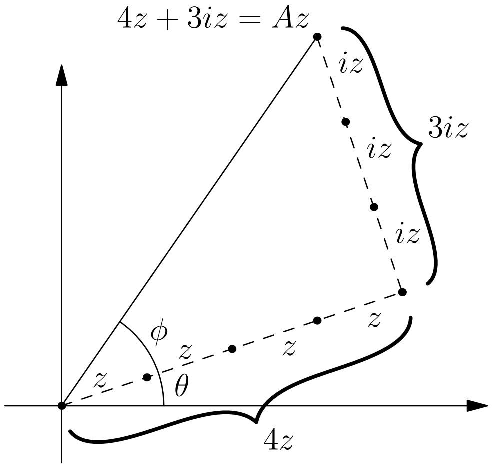


Figure 24: Breaking up Az into its components, we can observe the geometry of complex multiplication.

5 Geometry of Complex Numbers

Thanks to Tristan Needham's Visual Complex Analysis *for many of the problems/examples and to Josh Zucker for most of the text.*

Last year, you became masters of the art of manipulating complex numbers. In this section, we will build on that background. Throughout the rest of the book, you can reinforce your skills with a dose of Vitamin i.

You should know at least two ways to think about the equation $x^3 = 1$. One way is to see that $x = 1$ is a solution, so $x - 1$ is a factor of $x^3 - 1$, and then proceed with synthetic or long division. The other way is to use DeMoivre's beautiful theorem:

$$(r_1(\cos\theta + i\sin\theta))(r_2(\cos\phi + i\sin\phi)) = (r_1r_2)(\cos(\theta + \phi) + i\sin(\theta + \phi)).$$

Like last year, we will often write $\operatorname{cis}\theta = \cos\theta + i\sin\theta$. Recall that $z = r\operatorname{cis}\theta$ is a complex number r units away from the origin and making an angle of θ with the x-axis, taken counterclockwise. Let's rewrite DeMoivre's theorem using cis:

$$(r_1\operatorname{cis}\theta)(r_2\operatorname{cis}\theta) = (r_1r_2)(\operatorname{cis}(\theta + \phi)).$$

In words: when you multiply complex numbers, the magnitudes are multiplied and the angles are added. (What else is added when you multiply? Exponents! In fact, $\operatorname{cis}\theta = e^{i\theta}$, but that's another story.)

Through repeated application of this theorem, we know that $(r\operatorname{cis}\theta)^n = r^n\operatorname{cis} n * \theta$. If $x = r\operatorname{cis}\theta$, then $x^3 = r^3\operatorname{cis}3\theta$. Going back to our original $x^3 = 1$, since $1 = 1\operatorname{cis}(2\pi k)$ for some integer k, we find that $r = 1$ and $3\theta = 2\pi k$. This yields three solutions: $1\operatorname{cis}0$, $1\operatorname{cis}\frac{2\pi}{3}$, $1\operatorname{cis}\frac{3\pi}{3}$. These correspond to $k = 0, 1, 2$; other values of k produce angles with the same terminal sides as one of these three, and are therefore duplicates.

You can bashfully prove DeMoivre's theorem using the angle addition formulae for cos and sin. But you can also understand it through pure geometry. Consider a complex number $z = a + bi$, being multiplied by $A = 4 + 3i$. z forms an angle of θ with the x-axis, and A forms an angle of ϕ. In Figure 1, observe that iz is perpendicular to z for any z. In Figure 2 depicts the complex number A. Finally, in Figure 3, you see the multiplication carried out: $Az = (4 + 3i)z = 4z + 3iz$. These two components, $4z$ and $3iz$, are indicated.

Combining the observation in Figure 1 and knowledge from geometry, we know the triangles in Figure 2 and 3 are similar. Since the scaling is by a factor of $|z|$, multiplying A by z has the effect rotating z by the angle of A, and multiplying it by the length of A.

Some notation: The angle θ of a complex number $z = a+bi$ is often called the argument, written as $\operatorname{Arg} z$. The real part of z is written $\operatorname{Re}(z) = a$, and the imaginary part of z is written $\operatorname{Im}(z) = b$. Note that $\operatorname{Im}(z)$ is a

real number, **not** an imaginary number. In particular, $\mathrm{Im}(z) \neq bi$. Finally, the complex conjugate of z, where the imaginary part is negated, is written with a bar on top: $\overline{z} = a - bi$.

1. Explain why iz is perpendicular to z, without using DeMoivre's theorem.
2. How does $\operatorname{Arg}\overline{z}$ relate to $\operatorname{Arg} z$? (Hint: symmetry!)
3. Compute $z\overline{z}$ and relate it to the cis form of z.
4. Explain, using a picture, why $\tan(\operatorname{Arg} z) = \frac{\mathrm{Im}(z)}{\mathrm{Re}(z)}$.
5. Divide $\frac{a+bi}{c+di}$ by rationalizing the denominator.
6. Divide $\frac{r_1 \operatorname{cis}\theta}{r_2 \operatorname{cis}\phi}$ using DeMoivre's theorem.
7. Compare and contrast the methods of division in problems 5 and 6. Which is more convenient? Or does it depend on the circumstance?
8. If $z = r\operatorname{cis}\theta$, what is $\frac{1}{z}$? Explain how this shows $\frac{1}{a+bi} = \frac{a-bi}{a^2+b^2}$, without having to rationalize the denominator. (Hint: use problems 3, 4, and 7.)
9. Compute $(1+i)^{13}$; pencil, paper, and brains only. No calculators!
10. Compute $\frac{(1+i\sqrt{3})^3}{(1-i)^2}$ without a calculator.
11. Draw $\operatorname{cis}\left(\frac{\pi}{4}\right) + \operatorname{cis}\left(\frac{\pi}{2}\right)$. Use your picture to prove that $\tan\left(\frac{3\pi}{8}\right)$. (Hint: add them as vectors.)
12. Solve $z^3 = 1$, and show that its solutions under the operation of multiplication form a group, isomorphic to the rotation group of the equilateral triangle. Write a group table!
13. (a) Find multiplication groups of complex numbers which are isomorphic to the rotation groups for

       i. a non-square rectangle, and
       ii. a regular hexagon.

    (b) Make a table for each group.

    (c) Compare the regular hexagon's group to the dihedral group of the equilateral triangle, D_3. Consider: how are they the same? How are they different? Is the difference fundamental?
14. Which of the following sets is a group under (i) addition and (ii) multiplication?

    (a) $\{0\}$
    (b) $\{1\}$
    (c) $\{0, 1\}$
    (d) $\{-1, 1\}$
    (e) $\{1, -1, i, -i\}$
    (f) $\{$naturals$\}$
    (g) $\{$integers$\}$
    (h) $\{$rationals$\}, \mathbb{Q}$
    (i) $\{\mathbb{Q}$ without zero$\}$
    (j) $\{$complex numbers$\}, \mathbb{C}$
    (k) $\{\mathbb{C}$ without zero$\}$
15. For what values a and b does $\{a + b\sqrt{2}\}$ form a group under (a) addition and (b) multiplication? Look for several answers.

DeMoivre's theorem is the "universal" trig identity, in the sense that it can be used to calculate every other trig identity. For example, suppose you want an identity for $\cos 3\theta$. For convenience, let $c = \cos\theta$ and $s = \sin\theta$. Then we have

$$\begin{aligned}
\operatorname{cis} 3\theta &= (\operatorname{cis}\theta)^3 && \text{[DeMoivre's Theorem]} \\
&= (c + is)^3 && \text{[Definition of } \operatorname{cis}\text{]} \\
&= c^3 + 3c^2 si - 3cs^2 - s^3 i && \text{[Binomial expansion]} \\
\cos 3\theta + i\sin 3\theta &= (c^3 - 3c^2) + i(3c^2 s - s^3) && \text{[Combining like terms]}
\end{aligned}$$

Equating real parts on both sides, $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$.

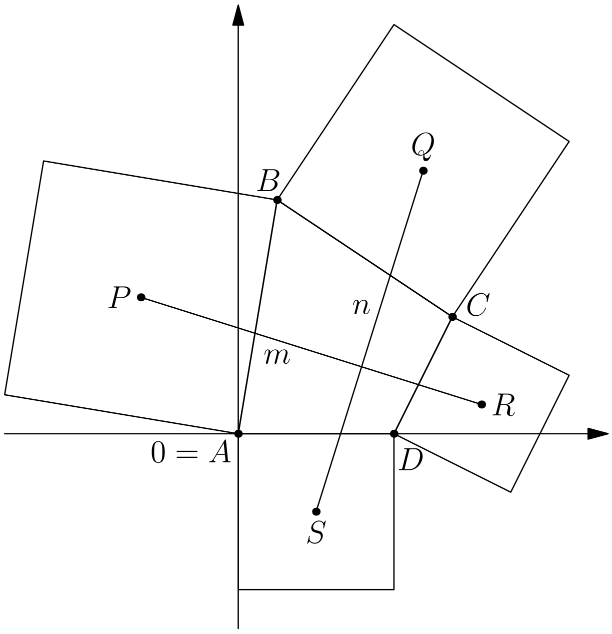


Figure 26: The quadrilateral with four squares.

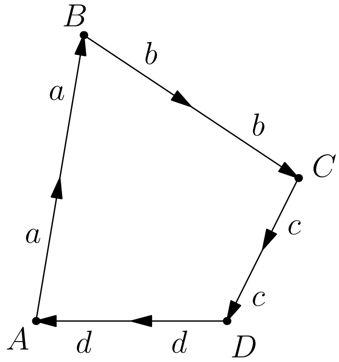


Figure 25: $2(a+b+c+d)=0$.

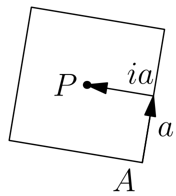


Figure 27: $P = a + ia$.

16. Prove that $(r_1 \operatorname{cis}\theta)(r_2 \operatorname{cis}\phi) = r_1 r_2 \operatorname{cis}(\theta+\phi)$ using brute force and the angle-sum trig identities for $\cos$ and $\sin$. Do you prefer this method or the one on the previous page? Which method gives you a better understanding of why DeMoivre's works?
17. Find an identity for $\sin 3\theta$ as we have done for $\cos$. (Hint: most of the work is already done for you!)
18. Your friend says that her textbook says $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$, different from our identity. Who's right?

Let's apply complex numbers to a geometry problem: Prove that if we construct squares with centers P, Q, R, S on the sides of any quadrilateral $ABCD$, as shown in Figure **??**, then (i) $\overline{PR} \cong \overline{QS}$ and (ii) $\overline{PR} \perp \overline{QS}$. In other words, segments joining centers of opposite squares are perpendicular and the same length.

We represent all points in the figure as complex numbers. For convenience, let $A = 0$ be the origin. The edges of the quadrilateral can be thought of as vectors in the form of complex numbers, and are found using subtraction; for example, the edge from A to B is $B - A$. Similarly, the edge from B to C is $C - B$. Now, define complex numbers

$$a = \frac{B-A}{2},\ b = \frac{C-B}{2},\ c = \frac{D-C}{2},\ d = \frac{A-D}{2}.$$

a is the vector halfway along $\overrightarrow{AB}$, b is halfway along $\overrightarrow{BC}$, etc.; this is shown in Figure **??**. We also have

$$a+b+c+d = \frac{B-A+C-B+D-C+A-D}{2} = \frac{0}{2} = 0.$$

More geometrically, this is because $2(a+b+c+d) = 2a+2b+2c+2d$ is the sum of the vectors $\overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{CD}, \overrightarrow{DA}$, which is just $\overrightarrow{AA} = \mathbf{0}$. This is shown in Figure **??**.

P, Q, R, S are also complex numbers. Also, let $m = R - P$ and $n = Q - S$, as the lengths of our two segments $\overline{PR}$ and $\overline{QS}$. To prove that they are perpendicular, recall that z is perpendicular to iz for any complex $z \neq 0$, so we just need to prove that $n = \pm im$.

We now need to relate P, Q, R, S back to a, b, c, d. It is easy to see that $p = a + ia$, remembering that a is the vector halfway along $\overrightarrow{AB}$. a takes you from the origin A to the midpoint of $\overline{AB}$, then ia takes you to P. This shown in Figure **??**. We can extend this logic to the other points, of course.

19. Now you can finish the rest of the proof.

    (a) Make a diagram of the approximate directions and magnitudes of a, b, c, d, m, n for the quadrilateral on the previous page.
    (b) Why is it sufficient to show that $n = \pm im$, to prove that the segments are (i) perpendicular and (ii) the same length?
    (c) Explain why $q = 2a + b + ib$.
    (d) Find formulae for r and s in terms of c and d.
    (e) Find m and n in terms of a, b, c, and d.
    (f) Check that $n + im = 0$, using the fact that $a + b + c + d = 0$.

20. In the previous problem, we drew squares outside a quadrilateral and connected their centers. Conjecture what happens if we draw equilateral triangles outside a triangle and connect their centers, as shown in . Prove your conjecture using complex numbers.

21. The hard way to find an identity for $\tan 3\theta$ is to divide the identity for $\sin$ and $\cos$ that we already found. Try this. Make sure your answer is in terms of $\tan$ only!

22. The easier way to get an identity for $\tan 3\theta$ starts with setting $z = 1 + i\tan\theta$.

    (a) Why is $\operatorname{Arg} z = \theta$?
    (b) Why is $\tan 3\theta = \frac{\operatorname{Im}(z^3)}{\operatorname{Re}(z^3)}$?
    (c) Use (b) to find an identity for $\tan 3\theta$.

23. Find multiplication groups of complex numbers isomorphic to rotation groups for the

    (a) regular octagon, and
    (b) regular pentagon.

24. Make tables for

    (a) the rotation group of the regular octagon, and
    (b) the dihedral group of the square.

    Is the difference between them fundamental?

25. Which of the following tables defines a group? Why or why not?

(a)

\$	I	A	B	C	D
I	I	A	B	C	D
A	A	C	D	B	I
B	B	I	C	D	A
C	C	D	A	I	B
D	D	B	I	A	C

(b)

#	I	A	B	C	D
I	I	A	B	C	D
A	A	B	C	D	I
B	B	C	D	I	A
C	C	D	I	A	B
D	D	I	A	B	C

26. Which subsets of the complex numbers are groups under multiplication? After you list a few of each type, generalize.

27. Prove with a diagram that if $|z| = 1$, then $\operatorname{Im}\left(\frac{z}{(z+1)^2}\right) = 0$.

28. Prove geometrically that if $|z| = 1$, then $|1 - z| = 2\sin\left(\frac{\operatorname{Arg} z}{2}\right)$.

29. (a) Prove that if $(z-1)^{10} = z^{10}$, then $\operatorname{Re}(z) = \frac{1}{2}$. (Hint: if two numbers are equal, they have the same length.)
    (b) How many solutions does this equation have?

30. I claim that $e^{i\theta} = \cos\theta + i\sin\theta = \operatorname{cis}\theta$, for θ in radians.

(a) Find e^{-it}.

(b) Find $\frac{e^{i\theta}+e^{-i\theta}}{2}$.

(c) Find $\frac{e^{i\theta}-e^{-i\theta}}{2i}$.

Use your new, complex definitions for $\cos$ and $\sin$ to find:

(d) $\cos^2\theta + \sin^2\theta$

(e) $\tan\theta$

(f) $\cos 2\theta$

(g) $\sin 2\theta$

(h) What kind of group is generated by $e^{i\theta}$ under the operation of multiplication if θ is an integer? A rational multiple of π?

31. You've used the quadratic equation throughout high school, but there's also a cubic equation that finds the roots of any cubic. Let's derive it, starting with the cubic $x^3 + bx^2 + cx + d = 0$.

    (a) Make the substitution $x = y - \frac{b}{3}$. Combine like terms to create an equation of the form $y^3 - 3py - 2q = 0$, with p, q in terms of b, c, and d.

    (b) Rearrange this equation as $y^3 = 3py + 2q$.

    (c) Make the substitution $y = s + t$, and prove that y is a solution of the cubic in part (a) if $st = p$ and $s^3 + t^3 = 2q$.

    (d) Eliminate t between these two equations to get a quadratic in s^3.

    (e) Solve this quadratic to find s^3. By symmetry, what is t^3?

    (f) Find a formula for y in terms of p and q. What about a formula for x?

    (g) What if we started with $ax^3 + bx^2 + cx + d = 0$, with a coefficient in front of the x^3 term as well? Can you come up with a formula for x?

32. Starting with the same cubic as in problem 31b.

    (a) Let $c = \cos\theta$. Remember that $\cos 3\theta = 4c^3 - 3c$, as we proved. Substitute $y = 2c\sqrt{p}$ into $y^3 = 3py + 2q$ to obtain $4c^3 - 3c = \frac{q}{p^{3/2}}$.

    (b) Provided that $q^2 \le p^3$, show that $y = 2\sqrt{p}\cos\left(\frac{1}{3}(\phi + 4\pi n)\right)$, where n is an integer. Why does this yield all three solutions?

6 Your Daily Dose of Vitamin i

Attach one of these to each of your homework sets for the next eight days starting after the completion of "Geometry of Complex Numbers."

1. We will use complex numbers to find identities for $\cot$. Use Pascal's triangle to expand the following:

   (a) $(a+b)^3$ (b) $(a+b)^4$ (c) $(a+b)^5$

   Then substitute $b = i = \sqrt{-1}$ and expand:

   (d) $(a+i)^3$ (e) $(a+i)^4$ (f) $(a+i)^5$

   Finally, substitute $a = \cot\theta$ and expand:

   (g) $(\cot\theta + i)^3$ (h) $(\cot\theta + i)^4$ (i) $(\cot\theta + i)^5$

   3 Consider $z = i + \cot\theta$.

   (j) Use the above results to find identities for $\cot 3\theta$, $\cot 4\theta$, and $\cot 5\theta$.
   (k) Graph z, z^2, z^3, z^4, and z^5, with $\theta \approx 75°$. What is your solution method?

2. Compute $(1+i)^n$ for $n = 3, 4, 5, \ldots$. Can you find a general pattern?

3. Expand and graph $\operatorname{cis}^n \theta$ for $n = 2, 3, 4, \ldots$.

   (a) Why is the real part $\cos n\theta$ and the imaginary part $\sin n\theta$?
   (b) Use your results to write identities for $\cos n\theta$ and $\sin n\theta$ for $n = 2, 3, 4, 5$.

4. Compute $\cos 7° + \cos 79° + \cos 151° + \cos 223° + \cos 295$ without a calculator. (Hint: what does this have to do with complex numbers?)

5. Factor the following:

   (a) $x^6 - 1$ as a difference of squares
   (b) $x^6 - 1$ as a difference of cubes
   (c) $x^4 + x^2 + 1$ over the real numbers
   (d) $x^6 - 1$ completely
   (e) $x^4 + x^2 + 1$ completely

6. Let $f(z) = \frac{z+1}{z-1}$.

   (a) Without a calculator, compute $f^{2014}(z)$.
   (b) What if you replace 2014 with the current year?

7. Find $\operatorname{Im}\left((\operatorname{cis} 12° + \operatorname{cis} 48°)^6\right)$.

8. Let x satisfy the equation $x + \frac{1}{x} = 2\cos\theta$.

   (a) Compute $x^2 + \frac{1}{x^2}$ in terms of θ.
   (b) Compute $x^n + \frac{1}{x^n}$ in terms of n and θ.

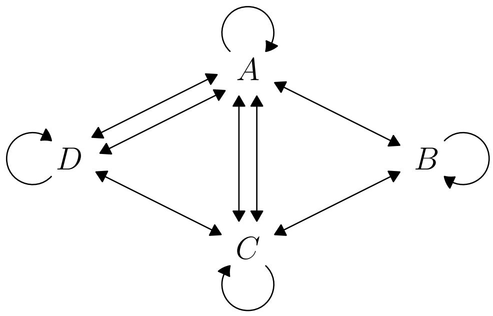


Figure 28: Four town transportation scenario.

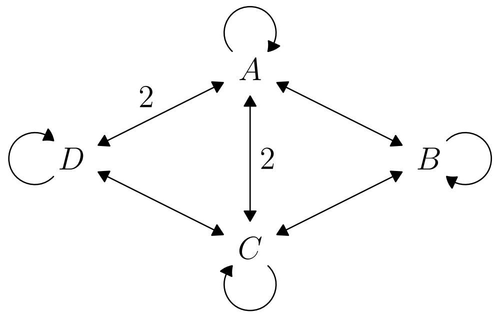


Figure 29: The scenario, with numbers instead of duplicate lines.

7 Matrix Multiplication

You've all seen a bunch of numbers organized in a table. Sometimes a table is just a table, but sometimes we will call it a matrix.

What makes a matrix different from a table? Although they encapsulate the same amount of information, we can meaningfully *multiply* matrices. This lesson's purpose is to explain why the matrix multiplication rule makes sense and when it is useful.

Consider a region with four towns, creatively named A, B, C, D. There are modes of transport between these towns; a path can go from a town to any town, including the same town. These paths are shown in Figure **??**. At each "step," you can take any path from one town to the next. For example, you might start on A, then take either of the two paths to D. But you cannot start on D and go directly to B. When there is more than one path between nodes, we could also just draw a line and label it with a number: this is shown in Figure **??**. Note how each town has a "path" going to itself. Taking this means you don't go anywhere.

Let us consider the **transportation matrix**, also known as an adjacency matrix, in this scenario. The number in the i^{th} row and j^{th} column of the matrix A, which we'll call a_{ij}, gives the number of ways to walk directly from town i to town j. This last fact is reflected in the matrix: $a_{42} = a_{24} = 0$.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = \text{from} \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \!\!\! \begin{array}{c} \text{to} \\ \begin{array}{cccc} A & B & C & D \end{array} \\ \left[\begin{array}{cccc} 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 0 \\ 2 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 \end{array}\right] \end{array}$$

This matrix is *symmetric*, meaning there are no one-way paths. In mathematical terms, we have $a_{ij} = a_{ji}$ for all valid i, j. In visual terms, the matrix is symmetric about the **main diagonal**, going from top left to bottom right. Furthermore, the main diagonal is all 1s, because we allow staying in the town you start in.

Suppose there's a shuttle bus that only goes one way, from town A to B to C to D and then back to A again. The transportation matrix for this scenario—again, allowing staying still—is shown in Figure **??**.

$$B = \text{from} \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \!\!\! \begin{array}{c} \text{to} \\ \begin{array}{cccc} A & B & C & D \end{array} \\ \left[\begin{array}{cccc} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{array}\right] \end{array}$$

Figure 30: Transportation matrix B.

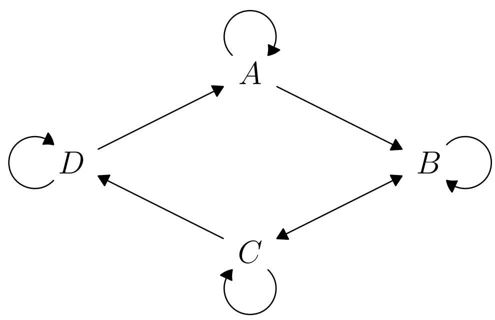


Figure 31: Graph of matrix B.

Because there are one-way connections, B is not symmetric. For example, $b_{12} = 1$, but $b_{21} = 0$. The graph for this matrix would therefore be directed; it would have arrows indicating the direction of each street. This is shown in Figure **??**.

Now, suppose you wanted to know the total number of ways to go from town to town in one step, by path or by bus. To find the total, you add the matrices in the obvious way: term by term, or $(a+b)_{ij} = a_{ij} + b_{ij}$. This is shown in Figure **??**.

$$A+B = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 0 \\ 2 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 & 2 \\ 1 & 2 & 2 & 0 \\ 2 & 1 & 2 & 2 \\ 3 & 0 & 1 & 2 \end{bmatrix}$$

Figure 32: Matrix addition of A and B.

$$A+B = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 1 & 1 & 2 & 0 \\ 2 & 1 & 1 & 2 \\ 3 & 0 & 1 & 1 \end{bmatrix}$$

Figure 33: $A+B$, with 1s on the diagonal.

Okay, so that's a little silly: we've counted two different ways to stay still, namely "not taking a path" and "not going anywhere on the bus." We should rewrite the matrix, putting ones on the diagonal, as in Figure **??**. Despite this minor issue, it's still true in general that this most naïve way of adding matrices is also the most convenient and useful. Just don't blindly follow a math recipe without considering its meaning!

But now comes a surprise: the most useful way to multiply matrices is not the obvious way. Why not? You'll see several different examples in the coming weeks. For now, think: what would it mean in terms of transportation if we just multiplied corresponding numbers like $a_{13}b_{13}$?[3] It would be meaningless, as far as I can tell.

Instead, we want multiplication of the two matrices B and A to represent taking one step by walking and then one step by bus. Similarly, multiplication of matrix A by itself will represent the number of ways to go from town to town in two steps by walking.

So, what rule of matrix multiplication will make that happen? To get from town A to C in two steps, for example, we have to go from town A to one of the four towns, then from that town to town C. The total number of ways to do this is

$$\underbrace{a_{11}}_{A\to A} \cdot \underbrace{a_{13}}_{A\to C} + \underbrace{a_{12}}_{A\to B} \cdot \underbrace{a_{23}}_{B\to C} + \underbrace{a_{13}}_{A\to C} \cdot \underbrace{a_{33}}_{C\to C} + \underbrace{a_{14}}_{A\to D} \cdot \underbrace{a_{43}}_{D\to C} = \sum_{j=1}^{4} a_{1j}a_{j3}.$$

And that's how we'll eventually define matrix multiplication. More formally, we can say that to determine the ij entry of the product XY of matrices X and Y, use the following formula:

$$(XY)_{ij} = x_{i1}y_{1j} + x_{i2}y_{2j} + \cdots + x_{in}y_{nj} = \sum_{k=1}^{n} x_{ik}y_{kj},$$

where X has n columns and Y has n rows. This is all fine and dandy if you're say, programming a computer to do matrix multiplication, but we should find a more intuitive way to interpret this definition.

Suppose we're multiplying two matrices X and Y. For convenience, let's make them 3×2[4] and 2×3:

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ \boxed{4 \quad 5 \quad 6} \end{bmatrix}}_{X} \underbrace{\begin{bmatrix} \boxed{\begin{matrix} 7 \\ 9 \\ 11 \end{matrix}} & \begin{matrix} 8 \\ 10 \\ 12 \end{matrix} \end{bmatrix}}_{Y} = \begin{bmatrix} 1\cdot 7 + 2\cdot 9 + 3\cdot 11 & 1\cdot 8 + 2\cdot 10 + 3\cdot 12 \\ 4\cdot 7 + 5\cdot 9 + 6\cdot 11 & 4\cdot 7 + 9\cdot 10 + 11\cdot 12 \end{bmatrix} = \begin{bmatrix} 58 & 64 \\ \boxed{139} & 250 \end{bmatrix}.$$

Observe the boxed numbers. To get 139, we multiplied the boxed rows and columns term by term. That is, we did $4\cdot 7 + 5\cdot 9 + 6\cdot 11$. We can think of this as the dot product of two vectors:

[3]This "obvious" product is actually known as the Hadamard product.

[4]Note that the first dimension is rows and the second is columns, as is the usual order.

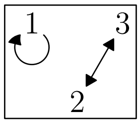
$$<4,5,6> \bullet <7,9,11> = 139.$$

In our example, to find the top left number a_{11}, we'd do $<1,2,3> \bullet <7,9,11>$. In general, to find $(XY)_{ij}$, we find the dot product of the ith row vector of X and jth column vector of Y.

With the ability to multiply matrices more easily, let's try some problems.

1. The three-post snap group can be represented by a set of graphs, each with three nodes. The posts are the towns and the elastic bands are the roads. For example,

$$A = \begin{array}{cc} & \text{to} \\ & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \text{from}\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{array} \longleftrightarrow$$



   (a) Draw the graphs and transportation matrices for this group.

   (b) Try a few multiplications and notice the isomorphism to the snap group.

2. Using A and B from this section, compute

   (a) $AA = A^2$ (b) AB (c) BA (d) B^2

   (e) Which one (AB and BA) represents taking a step by walking, then by bus?

   (f) Use your calculator to check your computations of A^2, AB, BA, and B^2.

3. Write a 3×3 matrix T that shows the following scenario: you can go from town B to C, C to D, and D to B by train, in exactly one way each, and not backwards.

   (a) Why can't you add this matrix to matrices A or B?

   (b) Rewrite matrix T so that it *can* be meaningfully added to matrices A and B. What did you do to its dimensions?

Review of sigma notation! Sigma notation represents a sum. It is defined as

$$S = \sum_{k=m}^{n} f(k) = f(m) + f(m+1) + \cdots + f(n),$$

for function f and integers m, n. k is the index over which the summation is taking place. It takes on all integer values between m and n, inclusive. You might read the sum portion like so: "The summation of f of k from k equals m to n is equal to f equals S."

4. Evaluate the following:

   (a) $\sum_{k=1}^{4} k$ (b) $\sum_{k=0}^{5} k^2$ (c) $\sum_{k=1}^{1} 03$ (d) $\sum_{k=1}^{n} k$ (e) $\sum_{k=1}^{n} n$ (f) $\sum_{k=1}^{n} 1$

5. The matrix C^T whose rows are the same as the respective columns of matrix C is called the **transpose** of C. For example,

$$C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, C^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}.$$

   (a) Let the elements of C be c_{ij} and the elements of C^T be c'_{ij}. Write a formula for C^T in terms of these elements.

   (b) Write $\begin{bmatrix} 2 & 1 & 5 \\ 4 & -2 & 0 \end{bmatrix}^T$.

6. Fill in the blanks: Multiplying an $m \times n$ matrix by a(n) ___ $\times\, k$ matrix gives a(n) ___ $\times$ ___ matrix.
7. Dogs can eat cats, rats, or mice; cats can eat rats or mice; rats can eat mice.
   (a) Make a matrix E showing what can eat what.
   (b) Draw a directed graph.
   (c) Calculate and interpret E^2,E^3,E^4.

The following table shows the amount of each ingredient a bakery uses in making one batch of sourdough bread and biscuits. Of course the units vary depending on the ingredient. Let's call this matrix S for sourdough.

Sourdough	Flour	Starter	Yeast	Water	Salt	Soda	Sugar	Butter
Bread	5	1	0	$\frac{4}{3}$	1	1	0	0
Biscuits	5	1	1	$\frac{5}{4}$	$\frac{3}{4}$	0	$\frac{1}{3}$	2

$$\begin{array}{l} \text{Bread} \\ \text{Biscuits} \end{array} \begin{bmatrix} 5 & 1 & 0 & \frac{4}{3} & 1 & 1 & 0 & 0 \\ 5 & 1 & 1 & \frac{5}{4} & \frac{3}{4} & 0 & \frac{1}{3} & 2 \end{bmatrix} = S$$

The bakery wants to know how much the ingredients cost for one batch of bread, and how much for one batc of biscuits. The unit cost of the ingredients is given in the following table. Let's call this matrix C for cost.

	Flour	Starter	Yeast	Water	Salt	Soda	Sugar	Butter
\$ per unit	5	20	10	0	1	2	5	12

$$\text{\$ per unit} \begin{bmatrix} 5 & 20 & 10 & 0 & 1 & 2 & 5 & 12 \end{bmatrix} = C$$

8. (a) Unfortunately, if you try to multiply S and C as given it won't work. Why not?
   (b) What do you need to do to C so they can be multiplied? Explain the dimensions of each matrix.
   (c) Once you've fixed matrix C, do the multiplication. What are the dimensions of your answer?
9. Matrix multiplication is not necessarily commutative, even when the dimensions of a the matrices suggest it might be. How do we know? Be specific.
10. Matrix multiplication is associative, though. Prove that $(PX)T = P(XT)$ for

$$P = \begin{bmatrix} m & n \\ p & q \end{bmatrix}, X = \begin{bmatrix} x & y \\ z & w \end{bmatrix}, Y = \begin{bmatrix} r & s \\ t & u \end{bmatrix}.$$

11. Prove that matrix multiplication is distributive: $P(X + T) = PX + PT$.
12. When does $PX = XP$? Don't worry if you get some messy equations in your answer.
13. Cook's Seafood Restaurant in Menlo Park sells fish and chips. The Captain's order is two pieces of fish and one order of chips, while the Regular order is two pieces of fish and one order of chips.
    (a) Write a matrix representing these facts, with clear labels on your rows and columns.
    (b) The restaurant management estimates their cost at 0.75 for each piece of fish and 0.50 for each order of chips. Represent this as a matrix, then use matrix multiplication to calculate the cost of the two possible orders.
    (c) For a party, Cook's provides 10 Captain's orders and 5 Regular orders. Write this as a matrix and use matrix multiplication to find how many pieces of fish and orders of chips are provided.
    (d) Now use matrix multiplication to find out the cost of the party.
14. We will find matrices to be particularly useful for solving systems of linear equations. For instance,

$$\begin{cases} 3x + 4y &= 5 \\ 6x + 4y &= 8 \end{cases} \quad \longleftrightarrow \quad \begin{bmatrix} 3 & 4 \\ 6 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}.$$

Rewrite

$$\begin{cases} 2x + 3y + 4z &= 5 \\ 5x - 4y + 2z &= 2 \\ x + 2y &= 7 \end{cases}$$

in this way.

15. (a) What is the transpose of the matrix M from the previous problem?
    (b) Use M^T to rewrite the system in the previous problem.
    (c) What is the transpose of the transpose matrix, $(M^T)^T$?

$$\begin{array}{ccc} M & P & P' \\ \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix} & \begin{bmatrix} 2 \\ 1 \end{bmatrix} = & \begin{bmatrix} 7 \\ -1 \end{bmatrix} \end{array}$$

Figure 34: Matrix multiplication is a transformation.

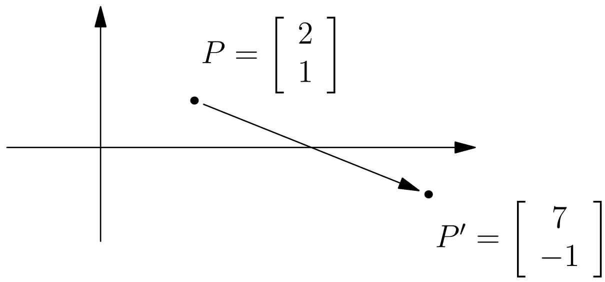


Figure 35: The transformation in the xy-plane.

8 Mapping the Plane with Matrices

By this time, you should be comfortable with the idea and process of matrix multiplication. You should know what dimensional relationship needs to be true of two matrices in order to be allowed to multiply them together.

Now we are going to consider 2×2 matrices as operators, which map each point in the plane to another point in the plane. This is done through matrix multiplication. Because 2 coordinates determine a point in the Cartesian plane, we have the option of representing each point by a 1×2 matrix—a row vector $\begin{bmatrix} a & b \end{bmatrix}$—or a 2×1 matrix—a column vector $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$. For consistency, we will use the second format, the column vector.

Figure **??** is an example of what happens when a randomly chosen matrix M operates on a point P, taking it to the point P'. The preimage $(2, 1)$ is mapped to the image $(7, -1)$ by the matrix; the geometric interpretation is shown in Figure **??**.

1. (a) Use the 2×2 matrix from Figure **??** to operate on the points $(0, 0)$, $(1, 0)$, and $(0, 1)$. What are their images? Graph them.
   (b) The preimage includes two perpendicular unit vectors, $(0, 1)$ and $(1, 0)$. What is the ratio of the lengths of their images? What is the angle of between the images?
   (c) You can conclude that multiplication by matrices does not, in general, preserve which two quantities between the image and preimage?

2. (a) Now, use the 2×2 matrix from Figure **??** to operate on each of these points: $(2, 1)$, $(1, 0)$, $(0, -1)$ and $(-1, -2)$. Do this by consolidating all the points into one matrix, with each point as a column vector, then performing a multiplication:
   $$\begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 & -1 \\ 1 & 0 & -1 & -2 \end{bmatrix} = \begin{bmatrix} & & & \\ & & & \end{bmatrix}.$$
   (b) Graph and label the preimage and the image of each point onto the same set of axes.
   (c) The points in the preimage are discontinuous, but they belong to a particular, infinite set of points. Write the equation of that set. (Hint: what is y in terms of x?)
   (d) Write an equation for the image of that set.
   (e) What other characteristic of the preimage points also applies to the image?
   (f) Name two things that seem to be conserved when mapping points with a matrix.

In Problem 2, you should have noticed that the points of the preimage were collinear, as were the points of the image. You should have also noticed that the points were equally spaced in the preimage and image. Was this a coincidence due to the particular matrix or set of points we picked, or is it generally true for all points and 2×2 matrices?

3. (a) Choose a different 2×2 matrix and a different set of three collinear, equally spaced unique points. Perform the appropriate matrix multiplication.
   (b) Graph and label the preimage points and the image points.
   (c) Have the collinearity and equal spacing been preserved?

(d) Make a conjecture about when a matrix will preserve collinearity and when a matrix will preserve equal spacing.

4. Now, we will check your conjecture.

   (a) Start with a general 2×2 matrix and three equally spaced points on a line, and multiply the two matrices:
   $$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x-h & x & x+h \\ m(x-h)+k & mx+k & m(x+h)+k \end{bmatrix} = \begin{bmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \end{bmatrix}.$$

   (b) How do you know that the second matrix indeed represents collinear and equally spaced points?

   (c) Are there any sets of collinear points that aren't representable by the 2×3 matrix?

   (d) Are the points in the image collinear? Show why or why not.

   (e) Can you find values for a, b, c, and d so that the image does not lie on a unique line? (Hint: all of the points in the image must lie on no line, or on multiple lines.)

   (f) Use the distance formula—or some other justification—to answer whether the points in the image are equally spaced.

5. There is a point which remains fixed—its image is the same as its preimage—when multiplied by the matrix $\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$. That is, $\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$.

   (a) Solve the above matrix equation for x and y to find the point.

   (b) There is a point Q that remains fixed no matter what matrix you multiply it by. Can you guess what point that is?

   (c) Prove your conjecture by plugging your point Q into $\begin{bmatrix} a & b \\ c & d \end{bmatrix} Q = Q$.

Matrix multiplication is a linear transformation. One way to think of a linear transformation is a transformation which takes lines to other lines, and keeps equally spaced points equally spaced. With this in mind, let's investigate what kinds of mappings we can do with matrices. You may remember transformations from 9$^{\text{th}}$ geometry: the identity, reflection across a line, rotation about a point, translation, and glide reflection preserve length, while others such as stretches and dilations change size. We will look for matrix representations of these, and if there are any matrix transformations new to us. We will also be investigating the case where multiple points in preimage are mapped to the same point in the image.

6. Begin with a triangle with vertices $(5, 0)$, $(10, 0)$, and $(5, 10)$, as shown in Figure **??**.

   (a) Map it with the following matrices.

   i. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ii. $\begin{bmatrix} .6 & -.8 \\ .8 & .6 \end{bmatrix}$ iii. $\begin{bmatrix} .6 & .8 \\ .8 & -.6 \end{bmatrix}$

   (b) Graph the preimage, then the image for each matrix on three separate sets of axes.

   (c) For each, describe the transformation as fully as you can. Try to classify them on the transformations we mentioned earlier, and quantify them if necessary (e.g. to describe the line of reflection or angle of rotation).

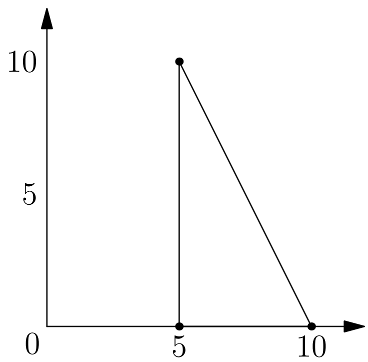


Figure 36: Problem 6's preimage.

7. Soon, we will map the unit square, which is shown in Figure **??**: it has vertices $(0, 0)$, $(1, 0)$, $(0, 1)$, and $(1, 1)$. We could actually get the entire image from the image of the unit vectors $(1, 0)$ and $(0, 1)$, which will later be useful.

   (a) How can we obtain the image of $(1, 1)$ from the images of $(1, 0)$ and $(0, 1)$?

   (b) Of $(0, 0)$?

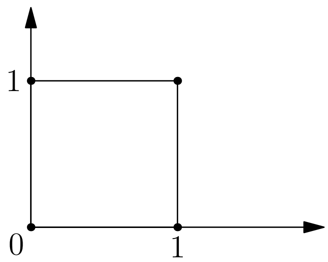


Figure 37: The unit square.

8. (a) Take the matrix $\left[\begin{smallmatrix}1 & 2\\ 0 & 1\end{smallmatrix}\right]$ and see what it does to the unit square. Please graph this, being careful to label each point and its image. The multiplication is done for you below.

$$\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix}0 & 1 & 1 & 0\\ 0 & 0 & 1 & 1\end{bmatrix}\end{array} = \begin{array}{c} \begin{array}{cccc} A' & B' & C' & D' \end{array} \\ \begin{bmatrix}0 & 1 & 3 & 2\\ 0 & 0 & 1 & 1\end{bmatrix}\end{array}.$$

This mapping is called a shear[5] in the direction of the x axis, perpendicular to the y axis. Quantitatively, the preimage is sheared horizontally by a factor of 2 of its height. In this case, the square is distorted into a parallelogram by "shoving" it along the x axis without increasing y. The 2 in the matrix could have been replaced by any other, nonzero[6] number and the matrix would still represent a shear in the x direction, just with a different magnitude.

(b) What happens to the area of image versus the preimage?

(c) We have $AB = BC$, but is $A'B'$ equal to $B'C'$? Should it?

9. (a) When is the ratio of distances between points in the image the same as the preimage?

(b) What is the image of the origin under any matrix mapping?

(c) What are the images of the points $(1, 0)$ and $(0, 1)$ under the mapping $\begin{bmatrix}a & b\\ c & d\end{bmatrix}$?

(d) Knowing the images of $(1, 0)$ and $(0, 1)$, where is $(1, 1)$ algebraically and geometrically?

10. How do these matrices map the plane? For each mapping, write a matrix for the images of the four corners of the unit square, then graph the preimage and image. Describe the mapping using words from geometry such as congruent, similar, rotate, reflect, shear, stretch, magnitude, and direction.

(a) $\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$

(b) $\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}$

(c) $\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}$

(d) $\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$

(e) $\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$

(f) $\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$

(g) $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

(h) $\begin{bmatrix}3 & 0\\ 0 & 1\end{bmatrix}$

(i) $\begin{bmatrix}1 & 0\\ -3 & 1\end{bmatrix}$

(j) $\begin{bmatrix}2 & 2\\ -3 & -3\end{bmatrix}$

(k) $\begin{bmatrix}3 & 2\\ 4 & -1\end{bmatrix}$

(l) $\begin{bmatrix}\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\end{bmatrix}$

(m) $\begin{bmatrix}\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\end{bmatrix}$

(n) $\begin{bmatrix}\frac{\sqrt{3}}{2} & \frac{1}{2}\\ \frac{1}{2} & -\frac{\sqrt{3}}{2}\end{bmatrix}$

(o) $\begin{bmatrix}\frac{\sqrt{3}}{2} & -\frac{1}{2}\\ \frac{1}{2} & \frac{\sqrt{3}}{2}\end{bmatrix}$

(p) $\begin{bmatrix}\frac{\sqrt{3}}{2} & \frac{1}{2}\\ -\frac{1}{2} & \frac{\sqrt{3}}{2}\end{bmatrix}$

One limitation with 2×2 matrix transformations is that they all involve a fixed point at the origin. A translation obviously takes the origin to a different point, so we can't represent it this way. One way around this problem is to do our mapping in three-dimensional space rather than the two-dimensional plane. We still keep the origin fixed, but put our preimages on the plane $z = 1$ and make sure that our images map to the same plane.

11. Carry out the following multiplications and convince yourself that the mappings are equivalent.

(a) $\begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}u\\ v\end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$

(b) $\begin{bmatrix}a & b & 0\\ c & d & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}u\\ v\\ 1\end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$

12. (a) Multiply these matrices: $\begin{bmatrix}1 & 0 & \alpha\\ 0 & 1 & \beta\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}u\\ v\\ 1\end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}.$

[5] You may have heard of wind shear, which is the change of velocity of the wind with altitude. Scissors exert a shearing action on paper to cut it.

[6] If it were 0, it would become the identity transformation, which we'll talk about later.

(b) Fill in the blanks: The result of the above multiplication is that the point $(u, v, 1)$ has been translated by ____ in the x direction, ____ in the y direction, and is still anchored to the plane $z =$ ___.

13. (a) Write a matrix which translates a point $(x, y, 1)$ 4 units in the x direction and 7 units, leaving z fixed at 1.

    (b) Check your work by applying your matrix to the point $(3, 5, 1)$.

14. Do these two multiplications. What does each represent?

(a) $\begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \alpha \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & \alpha \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix}$

15. What does each of these matrices represent?

(a) $\begin{bmatrix} a & b & \alpha \\ c & d & \beta \\ 0 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} \cos\theta & -\sin\theta & \alpha \\ \sin\theta & 1 & \cos\theta \\ 0 & 0 & 1 \end{bmatrix}$

16. (a) Rewrite your translation matrix from problem 12 and your preimage vector so that you do not restrict your translations to the plane $z = 1$, but can translate in the x, y, and z directions. (Hint: think four dimensions!)

    (b) Write a matrix product that translates the point $(2, 3, -5)$ by the vector $(4, -1, 2)$.

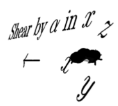


Figure 38: Translating a square by (α, β) in the plane $z = 1$.

9 Rotations of the Plane

You've learned about matrix multiplication and about complex numbers. You may have guessed at the relationship between them, particularly now that we've spent some time seeing how 2×2 matrices, certain 3×3 matrices, and complex numbers all relate to the geometry of the plane. We will now make that connection explicit and relate it to some ideas that you worked with last year, such as rotation of axes.

Recall that a matrix M acts on a column vector v by the multiplication Mv, not vM; a complex number z acts on the point (x, y) by multiplication $z(x + yi)$ or $(x + yi)z$, since complex multiplication is commutative. We'll call v or (x, y) the preimage and Mv or $z(x + yi)$ the image.

Some of the following problems are really trivial, so don't be alarmed if your answer takes only a few seconds. Some of them are fairly difficult and will take a bit of thought. Some of them are fairly tedious and will take some lengthy algebra but not much thought.

1. (a) Which matrix changes nothing, so that the image is the same as the preimage?
   (b) Which complex number changes nothing?
2. (a) Which matrix doubles the length of every vector but leaves angles unchanged?
   (b) Which complex number corresponds to the same transformation?
3. Based on your answers to the previous problems, which matrix corresponds to the real number r? Let's call this $\mathrm{M}(r)$ for short.
4. Explain why $\mathrm{M}(u) + \mathrm{M}(v) = \mathrm{M}(u + v)$.
5. Under a $90°$ counterclockwise rotation, what is the image of $(1, 0)$? What is the image of $(0, 1)$?
6. (a) Which matrix corresponds to a $90°$ rotation?
   (b) Which complex number corresponds to the same rotation?
7. Based on your answers to problems 1-6, what matrix corresponds to the complex number $x + yi$? Let's extend our function M and call this $\mathrm{M}(x + yi)$ for short.
8. Check that $\mathrm{M}(a+bi)+\mathrm{M}(c+di) = \mathrm{M}((a+bi)+(c+di))$. That is, prove that M has the same addition rules as complex numbers.
9. Check that $\mathrm{M}(a+bi)\,\mathrm{M}(c+di) = \mathrm{M}((a+bi)(c+di))$. That is, prove that M has the same multiplication rules as complex numbers.
10. Recall that multiplying by $\operatorname{cis}\theta$ rotates a complex number by θ radians.
    (a) Find $\mathrm{M}(\operatorname{cis}\theta)$.
    (b) To prove that this matrix really does rotate by θ:
        i. Check that the image and preimage have the same length;
        ii. Check that the angle of the image with the x axis is θ more than the preimage.
11. (a) Find $\mathrm{M}(r\operatorname{cis}\theta)$.
    (b) To prove that this matrix really does rotate by θ and stretch by r:
        i. Check that the length of the image is r times the length of the preimage;
        ii. Check that the angle of the image with the x axis is θ more than the preimage. (Hint: you may want to use the previous problem, or the tangent addition formulas.)

We've seen that there is a matrix for every complex number. These matrices have the same addition and multiplication rules as complex numbers. Furthermore, these matrices transform the plane in the same way as complex multiplication: a stretch by a factor of r and a rotation by θ. There are many matrices, however, that don't correspond to complex numbers.

1. (a) What matrix reflects over the x axis, taking (x, y) to $(x, -y)$?
   (b) What is the complex number operation equivalent to this transformation?

(c) Is there a complex number multiplication equivalent to this transformation? Justify your answer.

2. (a) What matrix reflects through the origin, taking (x, y) to $(-x, -y)$?
   (b) What is the complex number operation equivalent to this transformation?
   (c) Is there a complex number multiplication equivalent to this transformation? Justify your answer.

3. (a) Which of the 16 matrices on page ?? have corresponding complex numbers and which do not?
   (b) How can you tell algebraically?
   (c) How can you tell geometrically?

4. Make multiplication tables with the set of matrices which correspond to the elements of the rotation group for the square (a 4×4 table) and the equilateral triangle (a 3×3 table).

5. (a) Write a matrix for a rotation of θ around the origin followed by a translation by (a, b).
   (b) Write a matrix for a translation by (a, b) followed by a rotation of θ around the origin.

Now that we know how to rotate with matrices or with complex numbers, we can revisit the topic of rotation of axes that you studied toward the end of last year.

1. Use matrix multiplication to find the image (x', y') of a point (x, y) rotated by θ.

2. (a) Given the parabola $x = t, y = t^2$, use matrix multiplication to rotate it by $45°$.
   (b) Graph the new parametric equations on your calculator.
   (c) Does it look like a rotation clockwise or counterclockwise? Why?

10 Matrices Generate Groups

As in problem ?? in the previous section, the groups that we examined in the first couple sections of this class have representations with matrices under the operation of matrix multiplication.

Recall that the rotation group of the equilateral triangle could be generated by one element—repeatedly applying a rotation of $120°$. We call this the cyclic group of order 3, C_3 for short. It took two generators to produce the dihedral group of the equilateral triangle—either a rotation and a reflection or two reflections. We call this the dihedral group of order 6, D_3 for short.

In the following problems, you will be examining some of these groups, writing group tables, and determining which symmetry group each matrix group is isomorphic to. Look for patterns. Try to discover the characteristics of each matrix that tell you what it "does" geometrically.

For Problems 1-4:

(a) Specify the elements of the matrix group, unless they are all given.

(b) Describe what each matrix does to the plane

(c) Construct a group table; you can use a calculator.

(d) Decide which symmetry group your matrix is isomorphic to.

Let's see an example of this on the following matrices:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, C = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(a) They are given.

(b) I is the identity transformation.A rotates $180°$ (alternatively, it reflects through the origin). B reflects over the x axis. C reflects over the y axis.

(c) This group is isomorphic to the symmetry group for the rectangle, otherwise known as the dihedral group of the rectangle, D_2 for short.

(d)

	I	A	B	C
I	I	A	B	C
A	A	I	C	B
B	B	C	I	A
C	C	B	A	I

1. Analyze this group with the following elements, following the form of Example 1. What makes this group fundamentally different from the example?

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, C = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

2. The matrix $\begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$ generates a group of order 3. Enumerate the elements of this group and analyze per the example.

3. The matrices $\begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ generate a group of order 6, of which the group in problem 2 is a subgroup. Enumerate the elements of the group and analyze per the example.

   (f) What other two sets of two matrices could have generated this group?

4. The matrix $\begin{bmatrix} \frac{\sqrt{5}-1}{4} & -\frac{\sqrt{10+2\sqrt{5}}}{4} \\ \frac{\sqrt{10+2\sqrt{5}}}{4} & \frac{\sqrt{5}-1}{4} \end{bmatrix}$ generates a group of order 5! Enumerate the elements of the group and analyze per the example; you can use a calculator.

5. Let $A = \begin{bmatrix} \cos\frac{2\pi}{n} & -\sin\frac{2\pi}{n} \\ \sin\frac{2\pi}{n} & \cos\frac{2\pi}{n} \end{bmatrix}$, $B = \begin{bmatrix} \cos\frac{2\pi}{n} & \sin\frac{2\pi}{n} \\ \sin\frac{2\pi}{n} & -\cos\frac{2\pi}{n} \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, and n be an integer. What group is generated by the following sets of generators? Describe them geometrically.

   (a) $\{A\}$ (b) $\{B\}$ (c) $\{A, B\}$ (d) $\{B, C\}$

6. Do the previous problem, but replacing $\frac{2\pi}{n}$ with k, where k is an integer number of radians.

7. Given $C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$, what is the order of the group generated by the following sets of generators?

   (a) $\{C\}$ (b) $\{D\}$ (c) $\{C, D\}$

8. What matrix could generate the cyclic group of order n, C_n?

9. What two matrices could generate the dihedral group of order $2n$, D_n?

10. Look at problem ? on page ??. The adjacency matrices map to a subgroup of the full cube symmetry group. What rotations/reflections do they map to?

11. Given $P = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $Q = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, and $R = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, try understanding the groups generated by:

    (a) P (b) Q (c) R (d) P, Q (e) P, R (f) Q, R (g) P, Q, R.

12. The matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ produces a shear. What is its inverse; what undoes the shear?

13. The complex numbers, excluding zero, form a group under multiplication. What set of matrices is isomorphic to the same group under multiplication?

14. Does the set of all 2×2 matrices form a group under multiplication? Why or why not?

The following analyses are more in-depth than Problems 1-4.

10.1 Analysis 1

Analyze the group generated by $A = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$ under multiplication.

The elements are as follows:

$$A = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}, A^2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, A^3 = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}, A^4 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$A^5 = \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}, A^6 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, A^7 = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}, A^8 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

In order, these are rotations of 0, $\frac{\pi}{4}$, $\frac{\pi}{2}$, $\frac{3\pi}{4}$, π, $\frac{5\pi}{4}$, $\frac{3\pi}{2}$, and $\frac{7\pi}{4}$ radians counterclockwise. A, A^3, A^5, and A^7—A to any power relatively prime to 8—are all generators of the group[7].

This is the cyclic group of order 8, C_8. It is isomorphic to the rotation group of the regular octagon. $A^8 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the identity element.

[7] Can you figure out why?

$\cdot$	I	A	A^2	A^3	A^4	A^5	A^6	A^7
I	I	A	A^2	A^3	A^4	A^5	A^6	A^7
A	A	A^2	A^3	A^4	A^5	A^6	A^7	I
A^2	A^2	A^3	A^4	A^5	A^6	A^7	I	A
A^3	A^3	A^4	A^5	A^6	A^7	I	A	A^2
A^4	A^4	A^5	A^6	A^7	I	A	A^2	A^3
A^5	A^5	A^6	A^7	I	A	A^2	A^3	A^4
A^6	A^6	A^7	I	A	A^2	A^3	A^4	A^5
A^7	A^7	I	A	A^2	A^3	A^4	A^5	A^6

10.2 Analysis 2

Analyze the group generated by $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

The generated matrices are as follows. The third, last row is all duplicates.

$$C = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, C^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, C^3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, C^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$BC = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, BC^2 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, BC^3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$CB = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = BC^3, C^2B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = C^2, C^3B = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = BC, B^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

I is the identity. C rotates $90°$, C^2 rotates $180°$, and C^3 rotates $270°$. B reflects over the x axis, BC reflects over line $y = x$, BC^2 reflects over the y axis, and BC^3 reflects over the line $y = -x$. This group is D_4, the symmetry group of the square. It contains the subgroups C_2 and C_4 once each, and four copies of the subgroup D_2.

The group table is shown below. I've used B, BC, etc. instead of the representations CB, CB^2 etc.

	I	C	C^2	C^3	B	BC	BC^2	BC^3
I	I	C	C^2	C^3	B	BC	BC^2	BC^3
C	C	C^2	C^3	I	BC^3	B	BC	BC^2
C^2	C^2	C^3	I	C	BC^2	BC^3	B	BC
C^3	C^3	I	C	C^2	BC	BC^2	BC^3	B
B	B	BC	BC^2	BC^3	I	C	C^2	C^3
BC	BC	BC^2	BC^3	B	C^3	I	C	C^2
BC^2	BC^2	BC^3	B	BC	C^2	C^3	I	C
BC^3	BC^3	B	BC	BC^2	C	C^2	C^3	I

11 Composing Mappings of the Plane

So far we have identified matrices that result in some specific mappings of the plane. We have seen how matrices interact with each other in the context of groups. Now, let's see what happens if we combine two mappings of the plane. For example, let's see what a rotation of $-90°$ about the origin followed by a reflection across the x axis does to our unit vectors $(1, 0)$ and $(0, 1)$. Then, let's extend this to the (u, v).

1. For parts a through e, fill in the blank.
   (a) We could start by finding the images of our points under the $-90°$ rotation.
      i. Find the matrix R which results in a $-90°$ rotation.
      ii. Multiply R by our unit vectors and point (u, v):
      $$\begin{bmatrix} \quad & \quad \end{bmatrix}\begin{bmatrix} 1 & 0 & u \\ 0 & 1 & v \end{bmatrix} = \begin{bmatrix} \quad & \quad & \quad & \quad \end{bmatrix}.$$
   (b) Next, we reflect those intermediate image points over the line $y = 0$.
      i. Find the matrix S which does this.
      ii. Multiply S by the result of a, subpart ii.
   (c) You should notice that the net result of the two transformations taken together is a reflection over the line $y = x$. Which matrix that represents this transformation?
   (d) Notice that what we did to achieve this mapping was
   $$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & u \\ 0 & 1 & v \end{bmatrix} = \begin{bmatrix} \quad & \quad & \quad & \quad \end{bmatrix},$$
   where we multiplied the two rightmost matrices first but didn't use the commutative property to multiply the two leftmost matrices first. See what happens when you multiply the two left hand matrices together:
   $$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} \quad & \quad \end{bmatrix}.$$
   Look familiar?
   (e) See what happens when you reverse the order of multiplication:
   $$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} \quad & \quad \end{bmatrix}$$
   (f) i. What transformation does this new matrix result in?
      ii. How is a reflection followed by a rotation different from a rotation followed by a reflection? Visualize this by following what happens to a point under both sets of transformations.
   (g) Notice that we apply the transformations from right to left. If you wanted to read from left to right, what would you have to change about the way you wrote the mapping matrices, the vectors representing points, and the order of the matrices?
   (h) How does our convention for ordering transformation matrices compare...
      i. ... to the convention for writing composite functions?
      ii. ... to the "followed by" convention we used for "From Snaps to Flips?"
      iii. ... to the "from _ to _" convention for transportation matrices?
2. There are two, infinite classes of matrices which, taken together under multiplication, form a group isomorphic to the isometries of the plane. These are the rotation matrix and reflection matrix. Let's look first at the rotation matrix and make sure that it really always works the way it should.
   (a) What is the result of a rotation by an angle θ followed by one of ϕ?
   (b) Multiply their rotation matrices: $\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \quad & \quad \end{bmatrix}.$
   (c) Use angle addition formulae to simplify your answer.

(d) Should the result be the same if you reverse the order of rotation?

(e) What happens to the points $(1,0)$, $(0,1)$, and (x,y) when you operate on them with the rotation matrix?

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & x \\ 0 & 1 & y \end{bmatrix}.$$

3. Now let's check for the generalized reflection matrix.

(a) Take the matrix which results in a reflection over the line $y = x\tan\frac{\theta}{2}$ and reflect over that line twice:

$$\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}.$$

(b) Simplify your answer and explain the result.

(c) Now let's do a reflection over the line $y = \tan\frac{\theta}{2}$ followed by a reflection over the line $y = \tan\frac{\phi}{2}$:

$$\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{bmatrix}.$$

(d) Simplify your answer using the angle addition formulae and interpret.

(e) Does it make a difference which reflection comes first? Do the matrix multiplication to confirm your answer.

4. We've found specific matrices which map the plane in the following ways:

- identity
- rotation about the origin by θ
- reflection over a line $y = x\tan\frac{\theta}{2}$
- size change by some factor centered at the origin
- stretching along a specific line through the origin by some factor
- shearing perpendicular to a specific line through the origin by some factor

We want to generalize those ideas. What does each of the following matrices do? Be quantitative by specifying angle, equation of line, and/or factor:

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$

(c) $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

(d) $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

(e) $\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$

(f) $\begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}$

(g) $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$

(h) $\begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}$

(i) $\begin{bmatrix} a & b \\ ca & cb \end{bmatrix}$

(j) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

(k) $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

(l) $\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$

5. What matrix/transformation undoes each of Problem 4, parts a through l? For instance, j is a rotation of θ. It is undone by a rotation of $-\theta$, which is represented by matrix k.

6. In these exercises you will observe the effects of multiplying two or more matrices. Do the following matrix multiplications, graph the preimage and image, and identify the transformations and their order. Note the effect of order on the outcome! But first, an example:

$$\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} .8 & .6 \\ -.6 & .8 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ -3 & 4 \end{bmatrix}.$$

This is a rotation of about $\tan^{-1}\left(-\frac{3}{4}\right) = -37°$, followed by a size change by a factor of 5. Remember to read from right to left.

(a) $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} .6 & -.8 \\ .8 & .6 \end{bmatrix}$

(b) $\begin{bmatrix} .6 & -.8 \\ .8 & .6 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$

(d) $\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$

(e) $\begin{bmatrix} .8 & .6 \\ -.6 & .8 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$

(f) $\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} .8 & .6 \\ -.6 & .8 \end{bmatrix}$

(g) $\begin{bmatrix} .6 & -.8 \\ .8 & .6 \end{bmatrix} \begin{bmatrix} .8 & -.6 \\ .6 & .8 \end{bmatrix}$

(h) $\begin{bmatrix} .6 & .8 \\ .8 & -.6 \end{bmatrix} \begin{bmatrix} .6 & .8 \\ .8 & -.6 \end{bmatrix}$

(i) $\begin{bmatrix} .6 & .8 \\ .8 & -.6 \end{bmatrix} \begin{bmatrix} .8 & .6 \\ .6 & -.8 \end{bmatrix}$

7. A linear mapping f is one in which all lines are mapped to lines and the origin remains a fixed point. That is, $f\left(\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]\right) = xf\left(\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]\right) + yf\left(\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]\right)$. I claim that we can build any linear mapping of the plane by multiplying together some combination of the matrices from Problem 4. Only two classes of matrix, however, are necessary; all other matrices are products or examples of these. Which two classes of matrix do you think comprise the minimum set from which the others can be composed? Be able to justify your choice.

8. There are two isometries of the plane we have neglected; what are they?

9. Write matrix products that perform the following mappings. Do the indicated multiplication and graph the results.

   (a) Rotation by $-135°$ followed by a shear by a factor of $\frac{1}{2}$ perpendicular to the y-axis
   (b) Same transformations as in a, but reversed
   (c) Stretch in the y direction by a factor of 3 followed by a rotation of $60°$
   (d) Same transformations as in c, but reversed
   (e) Projection onto the line $y = 5x$
   (f) Reflection over $\theta = \frac{\pi}{12}$ followed by a stretch in the x direction by a factor of 2
   (g) Same transformations as in f, but reversed

10. Write a set of matrices which undoes Problem 8, parts a-g above. You will find one of them impossible to undo; explain why for that one.

11. (a) Find the height of the parallelogram in terms of b and a trig function in terms of T.
    (b) Find the area of the parallelogram in terms of a, b, and T.

12. At top right we have put our parallelogram onto a coordinate plane so that a makes an angle of θ with the x axis and b makes an angle of ϕ with the x axis. $T = \phi - theta$.

    (a) Rewrite the equation for the area of the parallelogram in terms of θ and ϕ.
    (b) Find the x and y coordinates of the endpoints of a and b in terms of a, b, ϕ, θ.
    (c) These coordinates form two column vectors which, taken together as a matrix, map the plane. Write the matrix so that the first column contains the coordinates of a and the second column contains the coordinates of b.
    (d) Your matrix has two diagonals. One rises from left to right and the other descends from left to right. Subtract the product of the entries of the ascending diagonal from the product of those of the descending diagonal.
    (e) Use angle addition formulas to simplify your answer.
    (f) You should find some relationship between your answers to problems 11a and 11d. What is it?
    (g) The difference of the products of the two diagonals of a two by two matrix is called the determinant of the matrix. What does it measure?
    (h) Find a matrix which produces a rotation. What is its determinant?
    (i) Find a matrix that produces a reflection.
        i. What is the absolute value of its determinant?
        ii. How does its determinant differ from that of a rotation matrix?
        iii. What property is not conserved under rotation?
        iv. What does the size of a determinant indicate?

13. Here is another way to think about the area of the image of the unit square under a linear transformation by the matrix $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$ and use it to transform the unit square into a parallelogram. We then graph the image:

    (a) There are three rectangles and four triangles in the figure. Find the dimensions and the area of each one. You can use this information to figure out the area of the parallelogram in terms of a, b, c, and d. Write a sentence or equation explaining how you can use the seven areas to find the area of the parallelogram.

    (b) Carry out the algebra to find the area.

    (c) Calculate the determinant of the matrix.

    (d) What is the relationship between the determinant of the matrix and the area of its associated parallelogram?

    (e) How would the area you calculated be different if (a, b) and (c, d) had switched places in the graph? What property would no longer be preserved by the transformation? What isometry would have been included in any composition yielding the mapping? What would be true of the determinant?

    (f) What does a reversal of the orientation of figure in its image say about the determinant of the transformation matrix? What does that same property of the determinant imply that a transformational matrix does? (What isometry reverses orientation?)

    (g) What would have happened to the parallelogram if $\left[\begin{smallmatrix} c \\ d \end{smallmatrix}\right] = k\left[\begin{smallmatrix} c \\ d \end{smallmatrix}\right]$? What would its area be? What would the determinant of the matrix be? What if $\left[\begin{smallmatrix} b & d \end{smallmatrix}\right] = r\left[\begin{smallmatrix} a & c \end{smallmatrix}\right]$?

Now that we are aware that the determinant of a matrix is a measure of size change and orientation change, we can decompose any linear mapping into a set of operations that we can visualize. Technically speaking, we can reduce all two dimensional transformational matrices into a combination of reflections and stretches along an axis. It is more intuitive, however, to include rotations, dilations, and shears along an axis in our repertoire of basic operations.

We will look at the image of the unit square under an arbitrary transformation and see how we can undo the transformation in steps until we are left with a unit square. Then we will retrace our steps, undoing each step until we have arrived at our original transformation through a set of mappings, each of which is easily visualized. We are looking for a recipe. Perhaps you can improve on the one that we will outline here! Start with $\left[\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right] = \left[\begin{smallmatrix} 2 & 3 \\ 4 & 1 \end{smallmatrix}\right]$.

1. (a) Start by checking the determinant. If it is nonzero, continue to step b. Otherwise, you are done. Why?

INSERT BETTER RECIPE

2. Look at Figure **??**, which is a visual representation of the previous problem, and describe what is going on in each step.

To reiterate, our process for undoing a matrix $M = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ with $a > 0$ and $\det M \neq 0$ is:

$$\begin{bmatrix} \text{shear} \\ \text{in x} \end{bmatrix} \begin{bmatrix} \text{stretch} \\ \text{in y} \end{bmatrix} \begin{bmatrix} \text{shear} \\ \text{in y} \end{bmatrix} \begin{bmatrix} \text{stretch} \\ \text{in x} \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Filling it in with numbers, we get

$$\begin{bmatrix} 1 & -\frac{c}{a} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{a}{ad-bc} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -b & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{a} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Don't forget to multiply from right to left.

Our ultimate goal is to build up the matrix from basic operations, not to just undo it. Fortunately, we can easily figure out how to undo each of these basic operations. Remember that matrix multiplication is associative, but not commutative.

1.

(a) How do you undo a shear in the x direction?

$$\begin{bmatrix} \; & \; \\ \; & \; \end{bmatrix} \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(b) How do you undo a shear in the y direction?

$$\begin{bmatrix} \; & \; \\ \; & \; \end{bmatrix} \begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(c) How do you undo a stretch along the x axis?

$$\begin{bmatrix} \; & \; \\ \; & \; \end{bmatrix} \begin{bmatrix} x & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(d) How do you undo a stretch along the y axis?

$$\begin{bmatrix} \; & \; \\ \; & \; \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

2. Now let's put this all together. Undo each of the operations in turn, until only matrix $\left[\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right]$ remains on the left side. Remember that what you do on the left side of the expression must also be done to the right side, so on the right side you will see the basic operations from which $\left[\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right]$ is composed. Order is important!

$$\begin{bmatrix} \; & \; \\ \; & \; \end{bmatrix}\begin{bmatrix} \; & \; \\ \; & \; \end{bmatrix}\begin{bmatrix} \; & \; \\ \; & \; \end{bmatrix}\begin{bmatrix} \; & \; \\ \; & \; \end{bmatrix}\begin{bmatrix} 1 & -\frac{c}{a} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \frac{a}{ad-bc} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -b & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{a} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} \; & \; \\ \; & \; \end{bmatrix}\begin{bmatrix} \; & \; \\ \; & \; \end{bmatrix}\begin{bmatrix} \; & \; \\ \; & \; \end{bmatrix}\begin{bmatrix} \; & \; \\ \; & \; \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

undoes

Now, let's see if you can apply this idea to find a set of basic transformations that is equivalent to some sample matrices.

1. Each step in the decomposition of $\begin{bmatrix} 3 & 4 \\ 2 & -5 \end{bmatrix}$ is explained below.

(i) Stretch along the x axis by factor of $\frac{1}{3}$.

$$\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 2 & -5 \end{bmatrix} = \begin{bmatrix} 1 & \frac{4}{3} \\ 2 & -5 \end{bmatrix}$$

(ii) Shear perpendicular to the x axis by -2

$$\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & \frac{4}{3} \\ 2 & -5 \end{bmatrix} = \begin{bmatrix} 1 & \frac{4}{3} \\ 0 & -\frac{23}{3} \end{bmatrix}$$

(iii) Stretch along y axis by $-\frac{3}{23}$

$$\begin{bmatrix} 1 & 0 \\ 0 & -\frac{3}{23} \end{bmatrix}\begin{bmatrix} 1 & \frac{4}{3} \\ 0 & -\frac{23}{3} \end{bmatrix} = \begin{bmatrix} 1 & \frac{4}{3} \\ 0 & 1 \end{bmatrix}$$

(iv) Shear perpendicular to the y axis by $-\frac{4}{3}$

$$\begin{bmatrix} 1 & -\frac{4}{3} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & \frac{4}{3} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Taken all together, the decomposition is:

$$\begin{bmatrix} 1 & -\frac{4}{3} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -\frac{3}{23} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 2 & -5 \end{bmatrix}$$

.

Therefore:

$$\begin{bmatrix} 3 & 4 \\ 2 & -5 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -\frac{23}{3} \end{bmatrix}\begin{bmatrix} 1 & \frac{4}{3} \\ 0 & 1 \end{bmatrix}.$$

What does each matrix do?

2. Here is another way that you could have decomposed the above matrix.

$$\overset{\text{i}}{\begin{bmatrix} 1 & 0 \\ 0 & \frac{13}{23} \end{bmatrix}}\overset{\text{ii}}{\begin{bmatrix} 1 & -\frac{2}{23} \\ 0 & 1 \end{bmatrix}}\overset{\text{iii}}{\begin{bmatrix} \frac{1}{\sqrt{13}} & 0 \\ 0 & \frac{1}{\sqrt{13}} \end{bmatrix}}\overset{\text{iv}}{\begin{bmatrix} \frac{3}{\sqrt{13}} & -\frac{2}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \end{bmatrix}}\overset{\text{v}}{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}\begin{bmatrix} 3 & 4 \\ 2 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(a) Identify what each matrix i through v does.

Next, we undo this sequence of operations by working backwards.

$$\overset{\text{i}}{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}} \overset{\text{ii}}{\begin{bmatrix} \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \\ -\frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \end{bmatrix}} \overset{\text{iii}}{\begin{bmatrix} \sqrt{13} & 0 \\ 0 & \sqrt{13} \end{bmatrix}} \overset{\text{iv}}{\begin{bmatrix} 1 & \frac{2}{23} \\ 0 & 1 \end{bmatrix}} \overset{\text{v}}{\begin{bmatrix} 1 & 0 \\ 0 & \frac{23}{13} \end{bmatrix}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & -5 \end{bmatrix}.$$

(a) Explain what happens at each matrix, i through v.

3. Find a set of basic transformations which is equivalent to each of the following matrices.

(a) $\begin{bmatrix} 12 & 8 \\ 5 & 15 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 24 \\ 4 & 7 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$

4. One of the matrices in problem ?? is a projection onto a line.

(a) Which matrix is it?

(b) What line does it project onto?

(c) If you try to decompose this matrix to the identity matrix, what happens? Why?

5. Onto what line does $\begin{bmatrix} a & b \\ 2a & 2b \end{bmatrix}$ project the plane? Solve for a and b such that the matrix projects orthogonally (at right angles) onto the line. You can do this because you know that a point on the line should not move under the projection and a point on a line perpendicular to the line has its image on the origin. Using this information you can set up two equations with two unknowns).

6. Use problem 21 to decompose $\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$ into a projection perpendicular to a line followed by a size change.

7. Decompose $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ into a projection perpendicular to a line followed by a size change.

8. Write matrices which project onto the following lines:

(a) $y = x$ (b) $y = 5x$ (c) $y = mx$

12 Inverses

Now that we've talked about matrix multiplication quite a bit, it's time to start talking about matrix "division." We've seen how matrix multiplication transforms the plane; since division is the opposite of multiplication, division will "untransform" the plane, putting all the points back where they started. First, we'll review division in a few more familiar contexts.

1. (a) With real numbers, one of the important purposes of division is that it lets you solve equations like $ax = b$ for x. Solve this by division (difficult!).
   (b) If division didn't exist, you could still solve this equation by multiplication. The number you'd multiply by is called the "multiplicative inverse" of a. What is the property that defines this special number?
   (c) The multiplicative inverse of a is often written a^{-1}. Why does this notation make sense?
2. (a) You might think that the equation $ax = b$ has only one solution, but sometimes it can have zero or infinitely many. Give an example of both cases.
   (b) How does the existence of a unique solution relate to the idea of multiplicative invertibility?
   (c) Are there any other possible numbers of solutions?
3. (a) Define "one-to-one" function.
   (b) Is $f(x) = ax$ a one-to-one function for all real a? (Hint: look for the silly exception(s)!)
4. Would your answers to the previous numbers change if you were talking about complex numbers instead of just real numbers? Why or why not?

Now, let's consider "clock arithmetic," in which we deal with integers 0 through 11 as hours on the clock—0 replaces 12 for mathematical convenience. In this world, numbers above or equal to 12 wrap around as the remainder, so $13 \to 1$ and $25 \to 1$. Instead of $=$ as in normal arithmetic, we use $\equiv$ to denote clock arithmetic.

As some basic examples, $7 + 7 \equiv 2$, because "7 hours after 7 o'clock is 2 o'clock." This is shown in Figure **??**. In addition $7 \times 7 \equiv 1$, because 49 has remainder 1 when divided by 12. In more formal language, this clock arithmetic is actually called "arithmetic modulo 12."

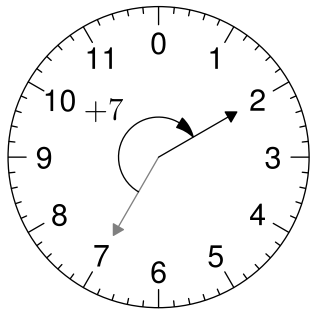


Figure 39: $7+7 \equiv 2$ on the clock.

1. (a) Find all solutions of $5x \equiv 7$ in clock arithmetic.
   (b) Find all solutions of $2x \equiv 6$ in clock arithmetic.
   (c) Find all solutions of $6x \equiv 6$ in clock arithmetic.
   (d) Find all solutions of $2x \equiv 7$ in clock arithmetic.
   (e) What are all possible numbers of solutions that $ax \equiv b$ can have in clock arithmetic?
2. How does the number of solutions to $ax \equiv b$ relate to the idea of multiplicative inverse? (Hint: You can try solving for $5, 2, 6, 3$ and $b = 1$. What numbers would be 5^{-1}, 2^{-1}, 6^{-1}, and 3^{-1} in clock arithmetic.)
3. (a) In clock arithmetic, for what values of a is $f(x) \equiv ax$ a one-to-one function?
   (b) How does this relate to whether $ax = 1$ has exactly one solution?
4. How does this all relate to groups?
   (a) The clock numbers are a group under clock addition. Name that group!
   (b) They are not a group under multiplication. Why?
   (c) A subset of four of the clock numbers form a group under the operation of multiplication. Find them, and write a group table.
   (d) Describe this group. What is the inverse of each element?
   (e) What symmetry group is it isomorphic to?
5. If the numbers on an advanced Mars clock went from 1 to 5,

(a) They would form a group under addition. Make a group table!

(b) What group is this isomorphic to?

(c) A subset of four of these numbers forms a group under multiplication. Find them and write a group table.

(d) Describe this multiplication group.

(e) What symmetry group is it isomorphic to?

Now we've seen how division (multiplicative inversion, or just inversion for short) works in a few more familiar situations, as well as the weird world of clock arithmetic. Let's see what happens with matrices. In the situation $AX = B$ we've been studying, X is a column vector representing a point in the plane, A is a 2×2 matrix describing a transformation, and B is the image of point X. Just like in the previous cases, we'll treat A and B as known and X as the unknown.

1. (a) Find all solutions (x, y) of $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$, by multiplying out the left side and rewriting this as a system of two equations with two unknowns.

   (b) Find all solutions (x, y) of $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$

   (c) Find all solutions (x, y) of $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$

   (d) What are all possible numbers of solutions that $AX = B$ can have in matrices? Use your knowledge of the properties of systems of equations.

2. Now, let's relate the two matrices from the previous problem to the transformations we know.

   (a) Contrast the mapping properties of $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$.

   (b) Find the determinants of these matrices. What do you notice?

   (c) When is $f(X) = AX$ a one-to-one function? That is, in mapping the plane, when does each point in the image have exactly one preimage?

   (d) Compare how you find the number of solutions of the real number equation $ax = b$ with how you find the number of solutions of the matrix equation $AX = B$.

3. Let $K = \begin{bmatrix} 5 & 7 \\ 8 & -3 \end{bmatrix}$.

   (a) Find all solutions to $K\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ 2 \end{bmatrix}$.

   (b) If we knew a matrix which was the inverse of K, which is written K^{-1}, we could write the following equation:

   $$K^{-1}K\begin{bmatrix} x \\ y \end{bmatrix} = K^{-1}\begin{bmatrix} 5 \\ 10 \end{bmatrix}.$$

   What would the left side reduce to?

Note that the right side would entail only very simple matrix multiplication. This would save you a bit of work. If you were dealing with a system of three equations and three unknowns, however, it would save a lot work. You wouldn't even want to touch a system with 6, let alone the systems of hundreds or thousands that appear in today's modern problems!

Our problem simplifies as follows: how do we find the inverse of a 2×2 matrix? We've actually already done this. On page ??, you read the following string of matrices:

$$\overbrace{\begin{bmatrix} 1 & -\frac{c}{a} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \frac{a}{ad-bc} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -b & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{a} & 0 \\ 0 & 1 \end{bmatrix}}\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The indicated matrices, to the left of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, together constitute $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}$.

1. (a) Now look at your results from problem ?? on page ??. Multiply the matrices that undid $\begin{bmatrix} 3 & 4 \\ 2 & -5 \end{bmatrix}$.

   Did you get $\frac{1}{23}\begin{bmatrix} -5 & -4 \\ -2 & 3 \end{bmatrix}$?

   (b) Do the same for problem ?? on page ?? and problems ?? and ?? on page ??.

   (c) You should have found a problem in inverting the matrix in ??. What is it? Answer geometrically, making reference to the matrix's mapping.

2. (a) Look for a pattern in the answers to problems ?? and ??.

   (b) Describe the inverse of an arbitrary matrix: $\begin{bmatrix} a & c \\ b & d \end{bmatrix}^{-1} = \frac{1}{\qquad} \begin{bmatrix} & \\ & \end{bmatrix}$. Use the word determinant in your answer.

   (c) We've been writing the inverse of matrix A as A^{-1}. Why does this notation make sense?

   (d) Find the product of the four matrices to the left of $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$ between problems ?? and ?? on page ?? and compare to problem ??.

3. Now, see what happens when you multiply the following matrices:

   (a) $-\frac{1}{2}\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 5 & -3 \\ -4 & 2 \end{bmatrix}$

   (b) $\frac{1}{71}\begin{bmatrix} 5 & 7 \\ 8 & -3 \end{bmatrix}\begin{bmatrix} 3 & 7 \\ 8 & -5 \end{bmatrix}$

   (c) $\begin{bmatrix} a & c \\ b & d \end{bmatrix}\frac{1}{ad-bc}\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$

   (d) $\frac{1}{ad-bc}\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}\begin{bmatrix} a & c \\ b & d \end{bmatrix}$

4. For another approach to finding the inverse of a matrix, solve the following for w, x, y, z in terms of a, b, c, d by converting the matrix equations into a set of four linear equations:

$$\begin{bmatrix} w & y \\ x & z \end{bmatrix}\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Gauss-Jordan elimination is an effective method we use to find inverses. If a matrix A is invertible, then as we've seen with matrix decomposition, there is a set of steps to reduce it to the identity matrix. Therefore, we have some set of elementary matrices E_i such that

$$E_n E_{n-1} \cdots E_2 E_1 A = I.$$

Multiplying by A^{-1} on the right, we get

$$E_n E_{n-1} \cdots E_2 E_1 I = A^{-1}.$$

Before, these elementary matrices were stretches, shears and the like. We will restrict ourselves to matrices like $\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$, which multiply a row by a, or matrices like $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, which adds the first row to the second. That's because instead of thinking "matrix so and so," we simply have to think "multiply this row by a" or "add these rows together," content that they are valid matrix multiplications. Some valid row operations are as follows:

- Swapping rows i and j
- Adding row i to j
- Multiplying row i by a constant $a \neq 0$
- Adding a times row i to j
- Adding multiple rows to another
- Taking multiple rows, multiplying them by any nonzero coefficients, and adding them to another

This last operation is the most powerful and encompasses most of the previous ones. To apply our multiplications easily, we **augment** a matrix A by juxtaposing it with the identity matrix. Also, we write what row operation we actually did on the left after each step. As shown in the first equation, we have multiplied all the elementary matrices once A has become I. But in that process, I will have become A^{-1}!

Choosing the steps is a bit of an art, which takes practice. Let's see Gauss-Jordan elimination in action on the matrix $A = \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}$.

We first augment the matrix like so:

$$[\ A \mid I\ ] = \left[\begin{array}{cc|cc} 3 & -4 & 1 & 0 \\ -2 & 3 & 0 & 1 \end{array}\right].$$

The line is there just so we remember this is no ordinary matrix, but two matrices joined together for convenience. Again, we want to turn the left side into the identity matrix.

We need a 1 in the bottom right corner, so we can add the top row to the bottom row to get -1:

$$\left[\begin{array}{cc|cc} 3 & -4 & 1 & 0 \\ -2 & 3 & 0 & 1 \end{array}\right] \implies R_2 = R_1 + R_2 \left[\begin{array}{cc|cc} 3 & -4 & 1 & 0 \\ 1 & -1 & 1 & 1 \end{array}\right].$$

We can then get 1 by multiplying the bottom row by -1. The rest of the steps are shown and justified as well:

$$\implies R_2 = -R_2 \left[\begin{array}{cc|cc} 3 & -4 & 1 & 0 \\ -1 & 1 & -1 & -1 \end{array}\right]$$

$$\text{Want } -1 \text{ in the top left corner} \implies R_1 = R_1 + 4R_2 \left[\begin{array}{cc|cc} -1 & 0 & -3 & -4 \\ -1 & 1 & -1 & -1 \end{array}\right]$$

$$\text{Turn } -1 \text{ into } 1 \implies R_1 = -R_1 \left[\begin{array}{cc|cc} 1 & 0 & 3 & 4 \\ -1 & 1 & -1 & -1 \end{array}\right]$$

$$\text{Get rid of } -1 \text{ in bottom left corner} \implies R_2 = R_1 + R_2 \left[\begin{array}{cc|cc} 1 & 0 & 3 & 4 \\ 0 & 1 & 2 & 3 \end{array}\right].$$

Observe! We have found that $A^{-1} = \left[\begin{smallmatrix} 3 & 4 \\ 2 & 3 \end{smallmatrix}\right]$. Let's try this new technique out on some systems of equations.

1. Rewrite each system of equations in matrix form. Use your calculator to calculate a matrix inverse, solve the system, and finally, check your answer. Remember to make clear in your work when you have used a calculator.

   (a) $\begin{cases} 2x + 3y &= 5 \\ 4x + 5y &= 7 \end{cases}$

   (b) $\begin{cases} 37x + 12y &= 65 \\ 93x + 40y &= 156 \end{cases}$

   (c) $\begin{cases} 2x + 5y + 3z &= 5 \\ 3x + 2y + 4z &= 7 \\ 13x + 16y + 18z &= 4 \end{cases}$

   (d) $\begin{cases} w + 2x + 3y + 4z &= 7 \\ 3w - x - 2y - 5z &= 5 \\ 5w + 3x - y - 4z &= 3 \\ 7w + 9x + 5y - 2z &= 2 \end{cases}$

   (e) $\begin{cases} 2x + 5y + 2z &= 1 \\ 3x + 2y + 4z &= 1 \\ 13x + 16y + 18z &= 5 \end{cases}$

   (f) When can you use matrix inverses to solve a system of equations?

2. You can fit a polynomial to any set of points in the plane, so long as the points pass the Vertical Line Test.

   (a) What is the least degree polynomial through

i. One point? ii. Two points? iii. Three points? iv. n points?

(b) Find a polynomial of least degree that passes through $(0, 3)$, $(1, 5)$, $(2, -3)$, $(3, 4)$, and $(4, 7)$.