1 Calculating the conditional PDF

Let $f(x, y) = 15x^2y$ for $0 \leq x \leq y \leq 1$. Find $f(x|y)$.

Solution.

$$
\begin{aligned}
f(x|y) &= \frac{f(x, y)}{\int_{-\infty}^{\infty} f(x, y)dx} \\
&= \frac{15x^2y}{\int_0^y 15x^2y \, dx} \\
&= \frac{15x^2y}{5x^3y \Big|_{x=0}^{y}} \\
&= \frac{15x^2y}{5y^4} \\
&= \frac{3x^2}{y^3}
\end{aligned}
$$

2 Properties of a joint PDF

Continuous random variables X and Y have the following PDF:

$$
f(x, y) = f_{XY}(x, y) = \begin{cases} kx^2y^3 & \text{where } 0 < x, y < 6 \\ 0 & \text{otherwise} \end{cases}
$$

1. Find k.

   Solution.

$$
\begin{aligned}
1 &= \int_0^6 \int_0^6 kx^2y^3 \, dx \, dy \\
&= k \int_0^6 y^3 \left(\frac{1}{3}x^3 \Big|_{x=0}^6 \right) dy \\
&= k \frac{6^3}{3} \int_0^6 y^3 dy \\
&= k \frac{6^3}{3 \cdot 4} \left(y^4 \Big|_{y=0}^6 \right) \\
\frac{12}{6^3 \cdot 6^4} &= k \\
k &= \frac{1}{23,328}
\end{aligned}
$$

2. Find the marginal PDF of X, $f_X(x)$.

Solution.

$$f_X(x) = \int_0^\infty f_{XY}(x, y)dy$$
$$= k \int_0^6 x^2 y^3 dy$$
$$= \frac{k}{4}x^2 \left(y^4 \Big|_0^6 \right)$$
$$= \frac{6^4 k}{4}x^2$$
$$= \frac{1}{72}x^2$$

3. Find the marginal PDF of Y, $f_Y(y)$.

Solution.

$$f_Y(y) = \int_0^\infty f_{XY}(x, y)dx$$
$$= k \int_0^6 x^2 y^3 dx$$
$$= \frac{k}{3}y^3 \left(x^3 \Big|_0^6 \right)$$
$$= \frac{6^3 k}{3}y^3$$
$$= \frac{1}{324}y^3$$

4. Find $E[X]$.

$$E[X] = \int_{-\infty}^\infty \int_{-\infty}^\infty x f_{XY}(x, y)dxdy$$
$$= k \int_0^6 \int_0^6 x^3 y^3 dxdy$$
$$= \frac{6^4 k}{4} \int_0^6 y^3 dy$$
$$= \frac{(6^4)^2}{4^2}k$$
$$= 4.5$$

5. Find $E[Y]$.

Solution.

$$E[Y] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y f_{XY}(x,y) dx dy$$

$$= k \int_0^6 \int_0^6 x^2 y^4 dx dy$$

$$= \frac{6^3 k}{3} \int_0^6 y^4 dy$$

$$= \frac{6^3 \cdot 6^5}{3 \cdot 5} k$$

$$= 4.8$$

6. Find $Var[X]$.

   Solution.

$$Var[X] = E[X^2] - E[X]^2$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^2 f_{XY}(x,y) dx dy - (4.5)^2$$

$$= k \int_0^6 \int_0^6 x^4 y^3 dx dy - (4.5)^2$$

$$= \frac{6^5 k}{5} \int_0^6 y^3 dy - (4.5)^2$$

$$= \frac{6^5 \cdot 6^4}{5 \cdot 4} k - (4.5)^2$$

$$= 1.35$$

7. Find $Var[Y]$.

   Solution.

$$Var[Y] = E[Y^2] - E[Y]^2$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y^2 f_{XY}(x,y) dx dy - (4.8)^2$$

$$= k \int_0^6 \int_0^6 x^2 y^5 dx dy - (4.8)^2$$

$$= \frac{6^3 k}{3} \int_0^6 y^5 dy - (4.8)^2$$

$$= \frac{6^3 \cdot 6^6}{3 \cdot 6} k - (4.8)^2$$

$$= 0.96$$

8. Find $Cov(X,Y)$.

Solution. First, we find $E[XY]$:

$$E[XY] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f_{XY}(x,y)\,dx\,dy$$
$$= k \int_0^6 \int_0^6 x^3 y^4\,dx\,dy$$
$$= \frac{6^4 \cdot 6^5}{4 \cdot 5} k$$
$$= 21.6.$$

Now we may compute:

$$Cov(X,Y) = E[(X - E[X])(Y - E[Y])]$$
$$= E[(X - 4.5)(Y - 4.8)]$$
$$= E[XY] - 4.5E[Y] - 4.8E[X] + 21.6$$
$$= (21.6)^2 - 4.5(4.8) - 4.8(4.5)$$
$$= 423.36.$$

9. Are X, Y independent? Explain

   Solution. If X, Y are independent, then it must be true that

   $$f_{XY}(x,y) = f_X(x)f_Y(y).$$

   Testing this, we have

   $$kx^2 y^3 = \left(\frac{1}{72}x^2\right)\left(\frac{1}{324}y^3\right)$$
   $$k = \left(\frac{1}{72}\right)\left(\frac{1}{324}\right)$$
   $$k = \frac{1}{23,328}$$

   which is true. Therefore, X, Y are independent.

10. What is the PDF of X conditional on Y, $f_{X|Y}(x|y)$?

    Solution.

    $$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(x,y)}$$
    $$= \frac{kx^2 y^3}{\frac{1}{324}y^3}$$
    $$= \frac{1}{72}x^2$$

11. What is the PDF of Y conditional on X, $f_{Y|X}(y|x)$?

Solution.

$$f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x,y)}$$
$$= \frac{kx^2y^3}{\frac{1}{72}x^2}$$
$$= \frac{1}{324}y^3$$

3   Properties of joint random variables

Suppose the following

- $E[D] = 10$

- $E[F] = 4$

- $E[DF] = 8$

- $Var(D) = 60$

- $Var(F) = 60$

1. What is $Cov(D, F)$?

   Solution.

   $$Cov(D, F) = E[(D - E[D])(F - E[F])]$$
   $$= E[(D - 10)(F - 4)]$$
   $$= E[DF] - 10E[F] - 4E[D] + 40$$
   $$= -32$$

2. What is the correlation between D and F?

   Solution.

   $$Corr(D, F) = \frac{Cov(D, F)}{\sqrt{Var(D)Var(F)}}$$
   $$= \frac{-32}{\sqrt{(60)(60)}}$$
   $$= -\frac{8}{15}$$

3. Suppose you multiplied F by 2 to generate a new variable H. What is $Cov(D, H)$

   Solution.

   $$Cov(D, H) = E[(D - E[D])(2F - 2E[F])]$$
   $$= E[(D - 10)(2)(F - 4)]$$
   $$= 2(E[DF] - 10E[F] - 4E[D] + 14)$$
   $$= -64$$
   $$= 2Cov(D, F)$$

4. What is $Corr(D, H)$? How does this compare to your answer in part (b) of this question

   Solution. Since $Cov(D, H) = 2Cov(D, F)$ and $Var(aX) = a^2Var(X)$ for any a, X, we have:

   $$\begin{aligned}
   Corr(D, H) &= \frac{2Cov(D, F)}{\sqrt{Var(D)(4)Var(F)}} \\
   &= \frac{-32(2)}{\sqrt{(60)(4)(60)}} \\
   &= -\frac{8}{15} \\
   &= Corr(D, F)
   \end{aligned}$$

5. Suppose instead that $Var(D) = 30$. How would this change $Corr(D, F)$?

   Solution.

   $$\begin{aligned}
   Corr(D, F) &= \frac{Cov(D, F)}{\sqrt{Var(D)Var(F)}} \\
   &= \frac{-32}{\sqrt{(30)(60)}} \\
   &= -\frac{8\sqrt{2}}{15}
   \end{aligned}$$

4 Continuous Bayes' theorem

Prove the continuous Bayes' theorem.

$$f(\theta|X) = \frac{f(X|\theta)f(\theta)}{\int f(X|\theta)f(\theta)d\theta}$$

Solution.

$$\begin{aligned}
f(\theta|X) &= \frac{f(\theta, X)}{f_X(x)} && \text{PDF of } \theta \text{ conditional on } X \\
&= \frac{f(\theta, X)}{\int f(\theta, X)d\theta} && \text{Definition of conditional probability} \\
&= \frac{f(X|\theta)f(\theta)}{\int f(X|\theta)f(\theta)d\theta} && f(x, y) = f(x|y)f(y)
\end{aligned}$$

□