Math Camp Study Questions
MACSS 33000
Due September 13

1 Linear equations, notation, sets, and functions

1.1 Applied (short answer): Solve for x in the equation $6x^2 - 12x = 0$.

Solution: Factor the equation:

$$6x(x-2) = 0$$

The solutions are $x = 0$ and $x = 2$.

1.2 Applied (long answer): Solve the system of equations using substitution.

$$3x - 2y = 18 \quad (1)$$

$$5x + 10y = -10 \quad (2)$$

Solution: From equation (2), solve for x in terms of y:

$$x + 2y = -2 \quad \Rightarrow \quad x = -2y - 2$$

Substitute this into equation (1):

$$3(-2y-2) - 2y = 18$$

$$-6y - 6 - 2y = 18 \quad \Rightarrow \quad -8y = 24 \quad \Rightarrow \quad y = -3$$

Substitute $y = -3$ into $x = -2y - 2$:

$$x = 4$$

Thus, $x = 4, y = -3$.

1.3 6. Applied (long answer): Find the roots of the equation $9x^2 - 3x - 12 = 0$.

Solution: First, factor the equation:

$$9x^2 - 3x - 12 = 0$$

$$3(3x^2 - x - 4) = 0$$

Factoring further:

$$(3x-4)(x+1) = 0$$

The solutions are:

$$x = \frac{4}{3}, -1$$

1.4 Theoretical (short answer): Define a bijective function.

1.5 Answer:

A bijective function is a function that is both injective (one-to-one) and surjective (onto), meaning every element in the domain maps to a unique element in the codomain, and every element of the codomain is mapped to by some element of the domain.

1.6 Applied (short answer): Simplify the expression $(2a^2)(4a^4)$.

1.7 Solution:

$$(2a^2)(4a^4) = 8a^6$$

1.8 Theoretical (long answer): Explain the difference between a function being injective and surjective with examples.

1.9 Answer:

A function is injective (one-to-one) if no two different elements in the domain map to the same element in the codomain. For example, $f(x) = 2x$ is injective. A function is surjective (onto) if every element of the codomain is the image of at least one element from the domain. For example, $f(x) = x^3$ from $\mathbb{R} \to \mathbb{R}$ is surjective.

1.10 10. Applied (long answer): Solve the set equation $(A \cup B) \cap C$, where

$$A = \{2, 3, 7, 9, 13, 16\}, B = \{4, 5, 6, 7, 8\}, C = \{2, 3, 5, 7, 13\}$$

1.11 Solution:

First, find $A \cup B$:

$$A \cup B = \{2, 3, 4, 5, 6, 7, 8, 9, 13, 16\}$$

Now, find $(A \cup B) \cap C$:

$$(A \cup B) \cap C = \{2, 3, 5, 7, 13\}$$

1.12 Applied (long answer)

Category: Word Problem (system of equations)

A coffee shop sells two types of coffee blends. Blend A costs \$5 per pound, and Blend B costs \$8 per pound. The shop owner wants to make 100 pounds of a mixture that sells for \$6.50 per pound. How many pounds of each blend should be used?

Solution: Let x be the pounds of Blend A, and y be the pounds of Blend B. We can set up the following system of equations:

$$x + y = 100 \quad \text{(the total weight of the mixture)}$$

$$5x + 8y = 6.50 \times 100 \quad \text{(the total cost of the mixture)}$$

$$5x + 8y = 650$$

Solve the system of equations: From the first equation, solve for x:

$$x = 100 - y$$

Substitute this into the second equation:

$$5(100 - y) + 8y = 650$$

$$500 - 5y + 8y = 650$$

$$3y = 150$$

$$y = 50$$

Substitute $y = 50$ into $x = 100 - y$:

$$x = 100 - 50 = 50$$

Thus, 50 pounds of Blend A and 50 pounds of Blend B should be used.

1.13 Applied (long answer)

Category: Word Problem (system of equations)

A theater sold 500 tickets for a play. Adult tickets cost \$20 each, and child tickets cost \$12 each. The total revenue from ticket sales was \$7,600. How many adult tickets and child tickets were sold?

Solution: Let a be the number of adult tickets sold, and c be the number of child tickets sold. We can set up the following system of equations:

$$a + c = 500 \quad \text{(total number of tickets sold)}$$

$$20a + 12c = 7600 \quad \text{(total revenue from ticket sales)}$$

Solve the system of equations: From the first equation, solve for a:

$$a = 500 - c$$

Substitute this into the second equation:

$$20(500 - c) + 12c = 7600$$

$$10000 - 20c + 12c = 7600$$

$$-8c = -2400$$

$$c = 300$$

Substitute $c = 300$ into $a = 500 - c$:

$$a = 500 - 300 = 200$$

Thus, 200 adult tickets and 300 child tickets were sold

2 Logarithms, sequences, and limits

2.1 Theoretical (short answer): Define an arithmetic sequence.

Answer: An arithmetic sequence is a sequence in which the difference between consecutive terms is constant. The general form is $u_n = a + (n-1)d$, where a is the first term and d is the common difference.

2.2 Theoretical (short answer): What is a geometric sequence?

Answer: A geometric sequence is a sequence where each term after the first is found by multiplying the previous term by a constant called the common ratio. The general form is $u_n = ar^{n-1}$, where a is the first term and r is the common ratio.

2.3 Theoretical (long answer): Explain whether the sequence $u_n = n3^n$ is arithmetic, geometric, or neither.

Answer: The sequence $u_n = n3^n$ is neither arithmetic nor geometric. It doesn't have a constant difference between consecutive terms (as required for arithmetic) and the ratio between consecutive terms is not constant (as required for geometric sequences).

2.4 Applied (long answer): Find the limit of the sequence $u_n = \left(\frac{1}{2}\right)^n$ as $n \to \infty$.

Solution:

$$\lim_{n\to\infty} \left(\frac{1}{2}\right)^n = 0$$

As n increases, the term $\left(\frac{1}{2}\right)^n$ approaches 0. Hence, the limit of the sequence is 0.

2.5 Applied (short answer): Does the sequence $u_n = 1 + \frac{1}{2n}$ tend to a limit as $n \to \infty$? If yes, what is the limit?

Solution:

$$\lim_{n\to\infty} \left(1 + \frac{1}{2n}\right) = 1$$

As n increases, $\frac{1}{2n}$ approaches 0. Therefore, the sequence tends to 1.

2.6 Applied (long answer): Find the limit of the sequence $a_n = \frac{3+5n^2}{n+n^2}$ as $n \to \infty$.

Solution:

$$\lim_{n\to\infty} \frac{3+5n^2}{n+n^2} = \lim_{n\to\infty} \frac{n^2\left(\frac{3}{n^2}+5\right)}{n^2\left(\frac{1}{n}+1\right)} = \frac{5}{1} = 5$$

Thus, the limit is 5.

2.7 Theoretical (long answer): Explain why the sequence $a_n = (-1)^{n-1}\frac{n}{n^2+1}$ converges to 0.

Answer: The alternating term $(-1)^{n-1}$ does not affect the limit since its magnitude is always 1. As $n \to \infty$, the dominant term in the denominator $n^2 + 1$ grows much faster than the numerator n, so the fraction $\frac{n}{n^2+1}$ tends to 0. Therefore, the entire sequence converges to 0.

2.8 Applied (long answer): Solve for the limit $\lim_{x\to -4} \frac{x^2+5x+4}{x^2+3x-4}$.

Solution: Factor the numerator and denominator:

$$\frac{(x+4)(x+1)}{(x+4)(x-1)}$$

Cancel the common factor $(x + 4)$:

$$\lim_{x\to -4} \frac{x+1}{x-1} = \frac{-4+1}{-4-1} = \frac{-3}{-5} = \frac{3}{5}$$

Thus, the limit is $\frac{3}{5}$.

3 Differentiation

3.1 Theoretical / Definition (short answer): State the definition of a critical point.

Answer: A critical point of a function $f(x)$ is a point $x = c$ where $f'(c) = 0$ or $f'(c)$ does not exist.

3.2 Applied (short answer): Find the critical points of $f(x) = x^3 - 3x^2$.

Solution:

$$f'(x) = 3x^2 - 6x$$

Set $f'(x) = 0$:

$$3x^2 - 6x = 0 \Rightarrow 3x(x-2) = 0$$

Thus, the critical points are $x = 0$ and $x = 2$.

3.3 Applied (long answer): Determine whether the critical points of $f(x) = x^3 - 3x^2$ are local minima, maxima, or neither.

Solution: The second derivative is:

$$f''(x) = 6x - 6$$

Evaluate at the critical points:

$$f''(0) = 6(0) - 6 = -6 \quad \text{(local maximum at } x = 0\text{)}$$

$$f''(2) = 6(2) - 6 = 6 \quad \text{(local minimum at } x = 2\text{)}$$

Thus, $x = 0$ is a local maximum and $x = 2$ is a local minimum.

3.4 Theoretical / Definition (long answer): What is the Mean Value Theorem and how is it applied?

Answer: The Mean Value Theorem states that if a function $f(x)$ is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists at least one point $c \in (a, b)$ such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

This theorem guarantees the existence of a point where the instantaneous rate of change (derivative) equals the average rate of change over the interval.

3.5 Applied (long answer): Find the absolute minimum and maximum of $f(x) = 3x^2 - 12x + 5$ on the interval $[0, 3]$.

Solution: First, find the critical points by setting $f'(x) = 0$:

$$f'(x) = 6x - 12$$

$$6x - 12 = 0 \Rightarrow x = 2$$

Now evaluate $f(x)$ at the critical point and the endpoints:

$$f(0) = 3(0)^2 - 12(0) + 5 = 5$$

$$f(2) = 3(2)^2 - 12(2) + 5 = -7$$

$$f(3) = 3(3)^2 - 12(3) + 5 = 5$$

Thus, the absolute minimum is $f(2) = -7$, and the absolute maximum is $f(0) = f(3) = 5$.

3.6 Theoretical (short answer): Define the concept of concavity and explain how it is related to the second derivative.

Answer: A function is concave up on an interval if its second derivative $f''(x) > 0$ for all x in the interval, and concave down if $f''(x) < 0$. Concavity describes the direction in which the function curves.

3.7 Applied (long answer): Use Newton's method to approximate the root of $f(x) = x^3 - 6x + 4 = 0$ starting with $x_0 = 2$.

Solution: Newton's method formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

First, compute $f'(x)$:

$$f'(x) = 3x^2 - 6$$

Start with $x_0 = 2$:

$$f(2) = 2^3 - 6(2) + 4 = 0$$

Since $f(2) = 0$, the approximation is already exact. The root is $x = 2$.

3.8 Applied (short answer): Find the critical points of $f(x) = x\ln(x)$ and classify them.

Solution: First, find the derivative:

$$f'(x) = \ln(x) + 1$$

Set $f'(x) = 0$:

$$\ln(x) + 1 = 0 \Rightarrow \ln(x) = -1 \Rightarrow x = \frac{1}{e}$$

The critical point is $x = \frac{1}{e}$. To classify it, find the second derivative:

$$f''(x) = \frac{1}{x}$$

Since $f''\left(\frac{1}{e}\right) > 0$, $x = \frac{1}{e}$ is a local minimum.

3.9 Applied (long answer): Verify that $f(x) = x^5 + x^3 + x + 1$ has no local extrema.

Solution: First, find the derivative:

$$f'(x) = 5x^4 + 3x^2 + 1$$

Set $f'(x) = 0$:

$$5x^4 + 3x^2 + 1 = 0$$

This equation has no real solutions since $5x^4 + 3x^2 + 1 > 0$ for all $x \in \mathbb{R}$. Thus, there are no critical points, and $f(x)$ has no local extrema.

4 Critical points and approximation

4.1 Theoretical / Definition (long answer): Explain how to determine if a critical point is a local minimum, maximum, or a saddle point.

Answer: To classify a critical point $x = c$, we use the second derivative test: - If $f''(c) > 0$, then $f(x)$ has a local minimum at c. - If $f''(c) < 0$, then $f(x)$ has a local maximum at c. - If $f''(c) = 0$, the test is inconclusive, and the point could be a saddle point or a higher-order critical point.

4.2 Applied (long answer): Find the local extrema of $f(x) = x^4 - 4x^3 + 4x^2$.

Solution: First, find the derivative:

$$f'(x) = 4x^3 - 12x^2 + 8x$$

Set $f'(x) = 0$:

$$4x(x^2 - 3x + 2) = 0$$

$$4x(x - 1)(x - 2) = 0$$

The critical points are $x = 0$, $x = 1$, and $x = 2$.

Next, find the second derivative:

$$f''(x) = 12x^2 - 24x + 8$$

Evaluate $f''(x)$ at the critical points:

$$f''(0) = 8 \quad \text{(local minimum at } x = 0\text{)}$$
$$f''(1) = -4 \quad \text{(local maximum at } x = 1\text{)}$$
$$f''(2) = 8 \quad \text{(local minimum at } x = 2\text{)}$$

Thus, there is a local maximum at $x = 1$, and local minima at $x = 0$ and $x = 2$.

4.3 Theoretical / Definition (short answer): What does it mean for a set of vectors to be linearly independent?

Answer: A set of vectors $\{v_1, v_2, \ldots, v_n\}$ is linearly independent if the only solution to $c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0$ is $c_1 = c_2 = \cdots = c_n = 0$.

4.4 Applied (short answer): Are the vectors $v_1 = [1, 2]$, $v_2 = [2, 4]$ linearly independent?

Solution: The vectors are multiples of each other:

$$v_2 = 2v_1$$

Thus, the vectors are linearly dependent.

4.5 Applied (long answer): Find the determinant of the following matrix and determine if it is invertible.

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 2 & 4 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

Solution: The determinant of A is:

$$\det(A) = 3\begin{vmatrix} 4 & 1 \\ 2 & 3 \end{vmatrix} - 1\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} + 2\begin{vmatrix} 2 & 4 \\ 1 & 2 \end{vmatrix}$$
$$= 3(4(3) - 1(2)) - 1(2(3) - 1(1)) + 2(2(2) - 4(1))$$
$$= 3(12 - 2) - 1(6 - 1) + 2(4 - 4)$$
$$= 3(10) - 1(5) + 2(0) = 30 - 5 = 25$$

Since the determinant is non-zero, A is invertible.

4.6 Theoretical / Definition (long answer): What is the relationship between a matrix being invertible and its determinant?

Answer: A matrix is invertible if and only if its determinant is non-zero. If the determinant is zero, the matrix is singular and not invertible.

4.7 Applied (short answer): Find the length of the vector $v = (3, 4, 5)$.

Solution: The length (magnitude) of the vector is:

$$||v|| = \sqrt{3^2 + 4^2 + 5^2} = \sqrt{9 + 16 + 25} = \sqrt{50} = 5\sqrt{2}$$

4.8 Applied (long answer): Solve the following system of linear equations using matrix inversion.

$$\begin{aligned} x + y + z &= 6 \\ 2x - y + 3z &= 14 \\ 3x + 4y - z &= 10 \end{aligned}$$

Solution: First, write the system as a matrix equation:

$$A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \\ 10 \end{bmatrix}$$

where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 3 & 4 & -1 \end{bmatrix}$$

Use the Gauss-Jordan elimination method to find

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}.$$

5 Matrix algebra

5.1 Applied (short answer): Is the matrix $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 5 \end{bmatrix}$ upper or lower triangular?

Solution: The matrix has non-zero entries only on or above the diagonal, so it is an **upper triangular** matrix.

5.2 Applied (short answer): Find the submatrix formed by deleting the second row and second column of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Solution: Deleting the second row and second column gives the submatrix:

$$\begin{bmatrix} 1 & 3 \\ 7 & 9 \end{bmatrix}$$

5.3 Applied (long answer): Find the inverse of the upper triangular matrix $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 4 \end{bmatrix}$.

Solution: To find the inverse of the upper triangular matrix A, use the following steps:

1. Start with the matrix equation $AA^{-1} = I$, where I is the identity matrix. 2. Begin by finding the inverse using the properties of triangular matrices.

Since A is upper triangular, the inverse will also be upper triangular. We compute it manually or using matrix inversion techniques, resulting in:

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{6} & 0 \\ 0 & \frac{1}{3} & -\frac{1}{6} \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$$

5.4 Theoretical / Definition (short answer): What is cosine similarity?

Answer: Cosine similarity is a measure of similarity between two vectors, defined as the cosine of the angle between them. It is computed using the formula:

$$\text{Cosine similarity} = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|}$$

where $\mathbf{u} \cdot \mathbf{v}$ is the dot product of the vectors and $\|\mathbf{u}\|$, $\|\mathbf{v}\|$ are the magnitudes of the vectors.

5.5 Applied (long answer): Calculate the cosine similarity between the vectors $\mathbf{u} = (1, 2, 3)$ and $\mathbf{v} = (4, 5, 6)$.

Solution: First, compute the dot product:

$$\mathbf{u} \cdot \mathbf{v} = 1(4) + 2(5) + 3(6) = 4 + 10 + 18 = 32$$

Next, find the magnitudes of $\mathbf{u}$ and $\mathbf{v}$:

$$\|\mathbf{u}\| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{1 + 4 + 9} = \sqrt{14}$$

$$\|\mathbf{v}\| = \sqrt{4^2 + 5^2 + 6^2} = \sqrt{16 + 25 + 36} = \sqrt{77}$$

Now, compute the cosine similarity:

$$\text{Cosine similarity} = \frac{32}{\sqrt{14} \times \sqrt{77}} = \frac{32}{\sqrt{1078}} \approx 0.973$$

5.6 Applied (long answer): Solve for x in the following matrix equation:

$$\begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \end{bmatrix}$$

Solution: The matrix equation can be written as:

$$3x_1 = 6 \quad \text{and} \quad x_1 + 2x_2 = 5$$

Solving the first equation:

$$x_1 = \frac{6}{3} = 2$$

Substitute $x_1 = 2$ into the second equation:

$$2 + 2x_2 = 5 \quad \Rightarrow \quad 2x_2 = 3 \quad \Rightarrow \quad x_2 = \frac{3}{2}$$

Thus, $x_1 = 2$ and $x_2 = \frac{3}{2}$.

5.7 Theoretical / Definition (short answer): What is a submatrix?

Answer: A submatrix is a matrix formed by deleting one or more rows and/or columns from a larger matrix.

6 Optimization with several variables

6.1 Theoretical / Definition (short answer): Define the gradient of a function.

Answer: The gradient of a function $f(x_1, x_2, \ldots, x_n)$, denoted by ∇f, is the vector of its first partial derivatives:

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right)$$

6.2 Applied (short answer): Find the gradient of $f(x, y) = 2x^2 + 3y^2$ and evaluate it at the point $(1, -2)$.

Solution: First, compute the partial derivatives:

$$\frac{\partial f}{\partial x} = 4x, \quad \frac{\partial f}{\partial y} = 6y$$

Thus, the gradient is:

$$\nabla f(x, y) = (4x, 6y)$$

At $(1, -2)$:

$$\nabla f(1, -2) = (4(1), 6(-2)) = (4, -12)$$

6.3 Applied (long answer): Find the Hessian matrix for the function $f(x, y) = x^3 + 3xy + y^2$.

Solution: First, compute the second-order partial derivatives:

$$\frac{\partial^2 f}{\partial x^2} = 6x, \quad \frac{\partial^2 f}{\partial x \partial y} = 3, \quad \frac{\partial^2 f}{\partial y^2} = 2$$

The Hessian matrix is:

$$H_f(x, y) = \begin{bmatrix} 6x & 3 \\ 3 & 2 \end{bmatrix}$$

6.4 Theoretical / Definition (long answer): Explain how to classify critical points using the Hessian matrix.

Answer: To classify critical points using the Hessian matrix:

1. Compute the Hessian matrix H_f at the critical point.
2. Compute the determinant $\det(H_f)$:
   (a) If $\det(H_f) > 0$ and $\frac{\partial^2 f}{\partial x^2} > 0$, the point is a local minimum.
   (b) If $\det(H_f) > 0$ and $\frac{\partial^2 f}{\partial x^2} < 0$, the point is a local maximum.
   (c) If $\det(H_f) < 0$, the point is a saddle point.
   (d) If $\det(H_f) = 0$, the test is inconclusive.

6.5 Applied (long answer): Find the critical points of $f(x, y) = x^4 + y^4 - 4xy + 2$, and classify them using the Hessian.

Solution: First, compute the gradient:

$$\nabla f(x, y) = \left(4x^3 - 4y, 4y^3 - 4x\right)$$

Set the gradient equal to zero:

$$\begin{aligned} 4x^3 - 4y = 0 \quad &\Rightarrow \quad x^3 = y \\ 4y^3 - 4x = 0 \quad &\Rightarrow \quad y^3 = x \end{aligned}$$

Thus, $x = y$. Substitute into either equation to get:

$$x^3 = x \quad \Rightarrow \quad x(x^2 - 1) = 0$$

So $x = 0, 1, -1$. Therefore, the critical points are $(0, 0)$, $(1, 1)$, and $(-1, -1)$.

Next, compute the Hessian:

$$H_f(x, y) = \begin{bmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{bmatrix}$$

At $(0, 0)$:

$$H_f(0, 0) = \begin{bmatrix} 0 & -4 \\ -4 & 0 \end{bmatrix}, \quad \det(H_f(0, 0)) = -16 \quad \text{(saddle point)}$$

At $(1, 1)$ and $(-1, -1)$:

$$H_f(1, 1) = H_f(-1, -1) = \begin{bmatrix} 12 & -4 \\ -4 & 12 \end{bmatrix}, \quad \det(H_f(1, 1)) = 128 \quad \text{(local minimum)}$$

6.6 Applied (short answer): Compute the first partial derivatives of $f(x, y, z) = x^2y + yz^2$.

Solution:

$$\frac{\partial f}{\partial x} = 2xy, \quad \frac{\partial f}{\partial y} = x^2 + z^2, \quad \frac{\partial f}{\partial z} = 2yz$$

6.7 Applied (long answer): Find the gradient of $f(x, y, z) = x^2 + y^2 + z^2$ and evaluate it at the point $(1, -1, 2)$.

Solution: First, compute the gradient:

$$\nabla f(x, y, z) = (2x, 2y, 2z)$$

At $(1, -1, 2)$:

$$\nabla f(1, -1, 2) = (2(1), 2(-1), 2(2)) = (2, -2, 4)$$

7 Integration and integral calculus

7.1 Theoretical / Definition (short answer): What is an improper integral?

Answer: An improper integral is an integral where either the limits of integration are infinite, or the integrand has an infinite discontinuity within the interval of integration.

7.2 Applied (long answer): Compute the definite integral $\int_1^4 \frac{1}{x}\, dx$.

Solution: The antiderivative of $\frac{1}{x}$ is $\ln |x|$. Therefore:

$$\int_1^4 \frac{1}{x}\, dx = [\ln |x|]_1^4 = \ln(4) - \ln(1) = \ln(4)$$

Since $\ln(1) = 0$, the final answer is $\ln(4)$.

7.3 Applied (short answer): Compute the indefinite integral $\int (2x - 3)\, dx$.

Solution: The antiderivative is:

$$\int (2x - 3)\, dx = x^2 - 3x + C$$

where C is the constant of integration.

7.4 Applied (long answer): Determine whether the improper integral $\int_1^\infty \frac{1}{x^2}\, dx$ converges or diverges, and if it converges, find its value.

Solution: The antiderivative of $\frac{1}{x^2}$ is $-\frac{1}{x}$. Therefore:

$$\int_1^\infty \frac{1}{x^2}\, dx = \lim_{b \to \infty} \left[-\frac{1}{x} \right]_1^b = \lim_{b \to \infty} \left(-\frac{1}{b} + 1 \right) = 1$$

Thus, the integral converges to 1.

7.5 Applied (long answer): Evaluate the double integral $\int_0^1 \int_0^2 (3x+2y)\, dx\, dy$.

Solution: First, compute the inner integral:

$$\int_0^2 (3x + 2y)\, dx = \left[\frac{3x^2}{2} + 2yx\right]_0^2 = \frac{12}{2} + 4y = 6 + 4y$$

Now, integrate with respect to y:

$$\int_0^1 (6 + 4y)\, dy = \left[6y + 2y^2\right]_0^1 = 6(1) + 2(1^2) = 6 + 2 = 8$$

Thus, the value of the double integral is 8.

7.6 Applied (long answer): Compute the indefinite integral $\int xe^{x^2}\, dx$ using substitution.

Solution: Let $u = x^2$, so that $du = 2x\, dx$. Therefore:

$$\int xe^{x^2}\, dx = \frac{1}{2}\int e^u\, du = \frac{1}{2}e^u + C$$

Substitute back $u = x^2$:

$$\int xe^{x^2}\, dx = \frac{1}{2}e^{x^2} + C$$

8 Sample space and probability

8.1 Theoretical / Definition (short answer): State the three axioms of probability.

Answer:

1. Non-negativity: For any event A, $P(A) \geq 0$.
2. Normalization: The probability of the sample space is 1, i.e., $P(S) = 1$.
3. Additivity: For any two mutually exclusive events A and B, $P(A \cup B) = P(A) + P(B)$.

8.2 Applied (short answer): A fair six-sided die is rolled once. What is the probability of rolling a number greater than 4?

Solution: The possible outcomes are $\{1, 2, 3, 4, 5, 6\}$. The favorable outcomes are $\{5, 6\}$.
Probability:

$$P(\text{rolling a number} > 4) = \frac{\text{Number of favorable outcomes}}{\text{Total number of outcomes}} = \frac{2}{6} = \frac{1}{3}$$

8.3 Applied (long answer): In a deck of 52 playing cards, what is the probability of drawing an ace or a heart?

Solution: Number of aces: 4
Number of hearts: 13
Number of aces that are hearts: 1
Using the inclusion-exclusion principle:

$$P(\text{Ace or Heart}) = P(\text{Ace}) + P(\text{Heart}) - P(\text{Ace and Heart}) = \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$$

8.4 Theoretical / Definition (short answer): Define independent events in probability.

Answer: Two events A and B are independent if the occurrence of one does not affect the probability of the occurrence of the other. Mathematically, $P(A \cap B) = P(A)P(B)$.

8.5 Applied (long answer): A box contains 5 red balls and 7 blue balls. Two balls are drawn at random without replacement. What is the probability that both balls are red?

Solution: First draw:

$$P(\text{Red on first draw}) = \frac{5}{12}$$

Second draw (without replacement):

$$P(\text{Red on second draw}|\text{Red on first draw}) = \frac{4}{11}$$

Combined probability:

$$P(\text{Both red}) = \frac{5}{12} \times \frac{4}{11} = \frac{20}{132} = \frac{5}{33}$$

8.6 Applied (short answer): If $P(A) = 0.6$, $P(B) = 0.5$, and $P(A \cap B) = 0.3$, find $P(A \cup B)$.

Solution: Using the formula:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.6 + 0.5 - 0.3 = 0.8$$

8.7 Applied (long answer): A coin is biased such that the probability of heads is $\frac{2}{3}$. If the coin is tossed three times, what is the probability of getting exactly two heads?

Solution: Number of ways to get exactly two heads in three tosses: 3
Probability of two heads and one tail:

$$P = \binom{3}{2} \left(\frac{2}{3}\right)^2 \left(\frac{1}{3}\right)^1 = 3 \times \frac{4}{9} \times \frac{1}{3} = \frac{4}{9}$$

8.8 Theoretical / Definition (short answer): What is conditional probability and how is it calculated?

Answer: Conditional probability is the probability of an event A occurring given that another event B has occurred. It is calculated using:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

provided $P(B) > 0$.

8.9 Applied (long answer): Given events A and B with $P(A) = 0.4$, $P(B) = 0.5$, and $P(A|B) = 0.6$, find $P(B|A)$.

Solution: First, find $P(A \cap B)$:

$$P(A \cap B) = P(A|B)P(B) = 0.6 \times 0.5 = 0.3$$

Then, compute $P(B|A)$:

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{0.3}{0.4} = 0.75$$

8.10 Applied (long answer): There are three boxes

- Box 1 contains 2 gold coins.
- Box 2 contains 1 gold and 1 silver coin.
- Box 3 contains 2 silver coins.

A box is chosen at random, and then a coin is randomly drawn from that box. If the coin is gold, what is the probability that it came from Box 1?

Solution: Let G be the event that a gold coin is drawn, and $B1$ be the event that Box 1 is chosen.
First, compute $P(B1) = \frac{1}{3}$.
Compute $P(G|B1) = 1$ (since both coins are gold in Box 1).
Compute $P(G|B2) = \frac{1}{2}$ (one gold coin out of two).
Compute $P(G|B3) = 0$ (no gold coins in Box 3).
Total probability of drawing a gold coin:

$$P(G) = P(B1)P(G|B1) + P(B2)P(G|B2) + P(B3)P(G|B3) = \frac{1}{3} \times 1 + \frac{1}{3} \times \frac{1}{2} + \frac{1}{3} \times 0 = \frac{1}{3} + \frac{1}{6} = \frac{1}{2}$$

Now, compute $P(B1|G)$ using Bayes' theorem:

$$P(B1|G) = \frac{P(B1)P(G|B1)}{P(G)} = \frac{\frac{1}{3} \times 1}{\frac{1}{2}} = \frac{\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}$$

9 Discrete random variables

9.1 Applied (short answer): Consider a discrete random variable X with the following PMF:

$$P(X = 0) = 0.2, \quad P(X = 1) = 0.5, \quad P(X = 2) = 0.3$$

What is the cumulative distribution function $F(x)$?

Solution:

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 0.2 & \text{if } 0 \leq x < 1 \\ 0.7 & \text{if } 1 \leq x < 2 \\ 1 & \text{if } x \geq 2 \end{cases}$$

9.2 Applied (long answer): A coin is biased such that the probability of heads is $p = 0.7$. If the coin is tossed 5 times, what is the probability of getting exactly 3 heads?

Solution: This is a binomial probability problem where $n = 5$ and $p = 0.7$. The probability of getting exactly 3 heads is:

$$P(X = 3) = \binom{5}{3}(0.7)^3(0.3)^2 = \frac{5!}{3!(5-3)!}(0.7)^3(0.3)^2$$

$$= 10 \times (0.343) \times (0.09) = 10 \times 0.03087 = 0.3087$$

Thus, the probability is 0.3087.

9.3 Theoretical / Definition (short answer): What is the expected value of a discrete random variable?

Answer: The expected value $E[X]$ of a discrete random variable X is the weighted average of all possible values of X, given by:

$$E[X] = \sum_x xP(X = x)$$

9.4 Applied (long answer): A discrete random variable X has the following PMF:

$$P(X = 0) = 0.1, \quad P(X = 1) = 0.2, \quad P(X = 2) = 0.3, \quad P(X = 3) = 0.4$$

Find the variance of X.

Solution: First, compute the expected value $E[X]$:

$$E[X] = 0(0.1) + 1(0.2) + 2(0.3) + 3(0.4) = 0 + 0.2 + 0.6 + 1.2 = 2$$

Next, compute $E[X^2]$:

$$E[X^2] = 0^2(0.1) + 1^2(0.2) + 2^2(0.3) + 3^2(0.4) = 0 + 0.2 + 1.2 + 3.6 = 5$$

Now, compute the variance:

$$\text{Var}(X) = E[X^2] - (E[X])^2 = 5 - 2^2 = 5 - 4 = 1$$

Thus, the variance of X is 1.

9.5 Applied (long answer): The average number of customer arrivals at a store per hour is 3. What is the probability that exactly 2 customers arrive in a given hour?

Solution: This is a Poisson distribution problem with $\lambda = 3$. The probability of exactly 2 arrivals is:

$$P(X = 2) = \frac{e^{-3}3^2}{2!} = \frac{e^{-3} \times 9}{2} = \frac{9e^{-3}}{2}$$

Using $e^{-3} \approx 0.0498$:

$$P(X = 2) = \frac{9 \times 0.0498}{2} \approx \frac{0.4482}{2} = 0.2241$$

Thus, the probability is 0.2241.

9.6 Distribution Interpretation (long answer): You roll a fair six-sided die 10 times. Let X be the number of times you roll a 6. Identify the type of distribution that X follows and compute the probability that you roll exactly 3 sixes.

Solution: The random variable X is binomially distributed, as there are 10 independent trials (rolling the die) with two possible outcomes (rolling a 6 or not). The probability of success (rolling a 6) is $p = \frac{1}{6}$. Therefore, $X \sim \text{Binomial}(n = 10, p = \frac{1}{6})$.

The probability of rolling exactly 3 sixes is:

$$\begin{aligned} P(X = 3) &= \binom{10}{3} \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^7 \\ &= \frac{10!}{3!(10-3)!} \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^7 \\ &= 120 \times \frac{1}{216} \times \frac{78125}{279936} \approx 0.155 \end{aligned}$$

Thus, the probability of rolling exactly 3 sixes is approximately 0.155.

9.7 Distribution Interpretation (long answer): A call center receives an average of 10 calls per hour. Let X represent the number of calls received in a 30-minute period. Identify the type of distribution for X, and find the probability that the center receives exactly 3 calls in 30 minutes.

Solution: The random variable X follows a Poisson distribution because the number of calls received is modeled as a Poisson process, where events (calls) occur independently over a continuous interval. The rate of calls per hour is 10, so the rate for 30 minutes is $\lambda = 5$.

Thus, $X \sim \text{Poisson}(\lambda = 5)$, and the probability of receiving exactly 3 calls is:

$$P(X = 3) = \frac{e^{-5}5^3}{3!} = \frac{e^{-5} \times 125}{6}$$

Using $e^{-5} \approx 0.0067$:

$$P(X = 3) = \frac{0.0067 \times 125}{6} \approx 0.14$$

Thus, the probability of receiving exactly 3 calls is approximately 0.14.

9.8 Distribution Interpretation (long answer): A factory produces light bulbs with a 2% defect rate. Let X be the number of defective light bulbs in a batch of 100 bulbs. Identify the type of distribution for X, and find the probability that exactly 5 bulbs are defective.

Solution: The random variable X follows a binomial distribution, as there are 100 independent trials (light bulbs) with two possible outcomes (defective or not defective). The probability of a defective bulb is $p = 0.02$.

Thus, $X \sim \text{Binomial}(n = 100, p = 0.02)$, and the probability of finding exactly 5 defective bulbs is:

$$P(X = 5) = \binom{100}{5}(0.02)^5(0.98)^{95}$$

$$= \frac{100!}{5!(100-5)!}(0.02)^5(0.98)^{95}$$

This can be calculated numerically to get approximately $P(X = 5) \approx 0.18$.

9.9 Betting Question (long answer): A lottery offers the following bet: You can buy a ticket for \$10. If your ticket wins, you receive \$1000; otherwise, you lose your \$10. The probability of winning the lottery is 0.001. Should you take this bet? Calculate the expected value.

Solution: Let X be the random variable representing your net gain or loss from the bet. The possible outcomes are: - Winning: Net gain of \$990 (since you spent \$10 and won \$1000). - Losing: Net loss of \$10.

The expected value of the bet is calculated as:

$$E[X] = (0.001)(990) + (0.999)(-10)$$

$$E[X] = 0.99 - 9.99 = -9$$

The expected value is -9, which means that, on average, you lose \$9 for every bet. Therefore, you should not take this bet.

10 General random variables

10.1 Theoretical / Definition (short answer): What is the cumulative distribution function (CDF) of a continuous random variable?

Answer: The cumulative distribution function (CDF), denoted $F(x)$, gives the probability that a continuous random variable X is less than or equal to x, i.e., $F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\,dt$, where $f(t)$ is the PDF.

10.2 Applied (short answer): Let $X \sim U(0, 5)$. Compute the expected value $E[X]$ and the variance Var(X).

Solution: For a uniform distribution $X \sim U(a, b)$, the expected value and variance are given by:

$$E[X] = \frac{a+b}{2}, \quad \mathrm{Var}(X) = \frac{(b-a)^2}{12}$$

Substituting $a = 0$ and $b = 5$:

$$E[X] = \frac{0+5}{2} = 2.5, \quad \mathrm{Var}(X) = \frac{(5-0)^2}{12} = \frac{25}{12}$$

10.3 Applied (long answer): The random variable X has the following PDF:

$$f_X(x) = \begin{cases} kx^3 & 0 \leq x \leq 5 \\ 0 & \text{otherwise} \end{cases}$$

1. Find the value of k. 2. Compute the CDF $F_X(x)$. 3. Compute $E[X]$.

Solution: 1. To find k, use the fact that the total probability must equal 1:

$$\int_0^5 kx^3\,dx = 1 \quad \Rightarrow \quad \left.\frac{kx^4}{4}\right|_0^5 = 1$$

$$\frac{k \times 625}{4} = 1 \quad \Rightarrow \quad k = \frac{4}{625}$$

2. The CDF is obtained by integrating the PDF:

$$F_X(x) = \int_0^x \frac{4}{625}t^3\,dt = \frac{4}{625} \times \frac{x^4}{4} = \frac{x^4}{625}$$

Thus, $F_X(x) = \frac{x^4}{625}$ for $0 \leq x \leq 5$.

3. The expected value is:

$$E[X] = \int_0^5 x \cdot \frac{4}{625}x^3\,dx = \frac{4}{625}\int_0^5 x^4\,dx = \frac{4}{625} \times \left.\frac{x^5}{5}\right|_0^5 = \frac{4 \times 3125}{3125} = 4$$

10.4 Applied (short answer): A random variable X has the PDF $f_X(x) = \frac{1}{10}$ for $0 \leq x \leq 10$. What is the probability that X is less than 4?

Solution: The CDF is obtained by integrating the PDF:

$$P(X < 4) = \int_0^4 \frac{1}{10}\, dx = \frac{4}{10} = 0.4$$

Thus, the probability is 0.4.

10.5 Applied (long answer): A random variable X has the following PDF:

$$f_X(x) = \lambda e^{-\lambda x} \quad \text{for } x \geq 0$$

Let $\lambda = 2$. Find the CDF $F_X(x)$, the expected value $E[X]$, and the variance $\text{Var}(X)$.

Solution: 1. The CDF is the integral of the PDF:

$$F_X(x) = \int_0^x 2e^{-2t}\, dt = 1 - e^{-2x}$$

Thus, $F_X(x) = 1 - e^{-2x}$.

2. The expected value of an exponential distribution is:

$$E[X] = \frac{1}{\lambda} = \frac{1}{2}$$

3. The variance of an exponential distribution is:

$$\text{Var}(X) = \frac{1}{\lambda^2} = \frac{1}{4}$$

10.6 Applied (short answer): The lifetime of a machine follows an exponential distribution with mean 5 years. What is the probability that the machine lasts more than 3 years?

Solution: Let X be the lifetime of the machine. The exponential distribution has $\lambda = \frac{1}{5}$, so:

$$P(X > 3) = 1 - F_X(3) = 1 - (1 - e^{-\frac{3}{5}}) = e^{-\frac{3}{5}} \approx 0.5488$$

Thus, the probability is approximately 0.5488.

10.7 Applied (long answer): The random variable X has the following PDF:

$$f_X(x) = \begin{cases} \frac{1}{25}x & 0 \leq x \leq 5 \\ 0 & \text{otherwise} \end{cases}$$

1. Find the CDF of X. 2. Compute $E[X]$. 3. Compute the median of X.

Solution: 1. The CDF is the integral of the PDF:

$$F_X(x) = \int_0^x \frac{1}{25} t\, dt = \frac{1}{50} x^2 \quad \text{for } 0 \leq x \leq 5$$

2. The expected value is:

$$E[X] = \int_0^5 x \cdot \frac{1}{25} x \, dx = \frac{1}{25} \int_0^5 x^2 \, dx = \frac{1}{25} \times \left. \frac{x^3}{3} \right|_0^5 = \frac{1}{25} \times \frac{125}{3} = \frac{5}{3}$$

3. The median m is the value where $F_X(m) = 0.5$:

$$\frac{1}{50} m^2 = 0.5 \quad \Rightarrow \quad m^2 = 25 \quad \Rightarrow \quad m = 5$$

Thus, the median is 5.

11 Multivariate distributions

11.1 Applied (short answer): Find the marginal PDF $f_X(x)$ for the joint PDF given by

$$f_{X,Y}(x, y) = 6x^2 y \quad \text{for } 0 \leq x \leq y \leq 1.$$

Solution: The marginal PDF $f_X(x)$ is obtained by integrating the joint PDF over y:

$$f_X(x) = \int_x^1 6x^2 y \, dy = 6x^2 \left(\frac{y^2}{2} \right) \Bigg|_x^1 = 6x^2 \left(\frac{1}{2} - \frac{x^2}{2} \right) = 3x^2(1 - x^2).$$

11.2 Applied (short answer): Find the conditional PDF $f_{X|Y}(x|y)$ for the joint PDF

$$f_{X,Y}(x, y) = kx^2 y^3 \quad \text{for } 0 < x, y < 6.$$

Solution: The conditional PDF $f_{X|Y}(x|y)$ is given by:

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x, y)}{f_Y(y)} = \frac{kx^2 y^3}{f_Y(y)}.$$

To find $f_Y(y)$, integrate $f_{X,Y}(x, y)$ over x:

$$f_Y(y) = \int_0^6 kx^2 y^3 \, dx = ky^3 \int_0^6 x^2 \, dx = ky^3 \times \frac{6^3}{3} = 72ky^3.$$

Thus,

$$f_{X|Y}(x|y) = \frac{x^2}{72}.$$

11.3 Theoretical / Definition (short answer): What is the covariance between two random variables X and Y?

Answer: The covariance between two random variables X and Y, denoted $\text{Cov}(X, Y)$, measures the degree to which X and Y change together. It is defined as:

$$\text{Cov}(X, Y) = E[(X - E[X])(Y - E[Y])] = E[XY] - E[X]E[Y].$$

11.4 Applied (long answer): Given the joint PDF

$$f_{X,Y}(x,y) = \frac{1}{8}(2x+y) \quad \text{for } 0 \leq x \leq 2,\, 0 \leq y \leq 2,$$

find the marginal PDFs $f_X(x)$ and $f_Y(y)$, and compute $E[X]$ and $E[Y]$.

Solution: 1. The marginal PDF $f_X(x)$ is:

$$f_X(x) = \int_0^2 \frac{1}{8}(2x+y)\,dy = \frac{1}{8}\left(2x \times 2 + \frac{y^2}{2}\bigg|_0^2\right) = \frac{1}{8}(4x+2) = \frac{1}{2}(x+\frac{1}{2}).$$

2. The marginal PDF $f_Y(y)$ is:

$$f_Y(y) = \int_0^2 \frac{1}{8}(2x+y)\,dx = \frac{1}{8}\left(2x^2\bigg|_0^2 + y \times 2\right) = \frac{1}{8}(8+2y) = \frac{1}{2}(1+\frac{y}{4}).$$

3. To compute $E[X]$, use the marginal PDF $f_X(x)$:

$$E[X] = \int_0^2 x \cdot \frac{1}{2}(x+\frac{1}{2})\,dx = \frac{1}{2}\left(\int_0^2 x^2\,dx + \frac{1}{2}\int_0^2 x\,dx\right) = 1.33.$$

4. To compute $E[Y]$, use the marginal PDF $f_Y(y)$:

$$E[Y] = \int_0^2 y \cdot \frac{1}{2}(1+\frac{y}{4})\,dy = 1.17.$$

11.5 Applied (long answer): Find the covariance Cov(X,Y) for the joint PDF

$$f_{X,Y}(x,y) = c(x+y) \quad \text{for } 0 \leq x, y \leq 1.$$

Also, determine whether X and Y are independent.

Solution: 1. To find c, integrate over the support to get the total probability:

$$\int_0^1 \int_0^1 c(x+y)\,dx\,dy = 1.$$

Thus,

$$c \times \left(\frac{x^2}{2} + yx\bigg|_0^1\right) = 1 \quad \Rightarrow \quad c = 2.$$

2. The covariance is given by:

$$\text{Cov}(X,Y) = E[XY] - E[X]E[Y].$$

Compute $E[XY]$, $E[X]$, and $E[Y]$ from the marginal PDFs. After computing these, you will find that $\text{Cov}(X,Y) = 0$, which suggests that X and Y are uncorrelated, but they are not necessarily independent.

11.6 Theoretical / Definition (short answer): When are two continuous random variables independent?

Answer: Two continuous random variables X and Y are independent if their joint PDF $f_{X,Y}(x, y)$ factors into the product of their marginal PDFs, i.e., $f_{X,Y}(x, y) = f_X(x)f_Y(y)$.

11.7 Applied (long answer): For the joint PDF given by

$$f_{X,Y}(x, y) = kx^2y^3 \quad \text{for } 0 \leq x, y \leq 1,$$

find k, the marginal PDFs, and check if X and Y are independent.

Solution: 1. To find k, integrate the joint PDF over the entire range:

$$\int_0^1 \int_0^1 kx^2y^3 \, dx \, dy = 1 \quad \Rightarrow \quad k = 20.$$

2. The marginal PDFs are:

$$f_X(x) = \int_0^1 20x^2y^3 \, dy = 5x^2, \quad f_Y(y) = \int_0^1 20x^2y^3 \, dx = 4y^3.$$

3. Since $f_{X,Y}(x, y) \neq f_X(x)f_Y(y)$, X and Y are not independent.

12 Classical statistical inference

12.1 Theoretical / Definition (short answer): What is an unbiased estimator?

Answer: An estimator $\hat{\theta}$ of a parameter θ is unbiased if $E[\hat{\theta}] = \theta$, meaning the expected value of the estimator is equal to the true value of the parameter.

12.2 Applied (short answer): A sample of 200 people is surveyed, and 120 people say they support a new policy. Estimate the population proportion of supporters and compute the standard error.

Solution: The point estimate for the population proportion $\hat{p}$ is:

$$\hat{p} = \frac{120}{200} = 0.6$$

The standard error for the proportion is:

$$\text{SE}(\hat{p}) = \sqrt{\frac{\hat{p}(1 - \hat{p})}{n}} = \sqrt{\frac{0.6(0.4)}{200}} = 0.03464$$

12.3 Applied (long answer): A sample of 100 undergraduates had a mean GPA of 3.2 with a standard deviation of 0.5. Construct a 95% confidence interval for the population mean GPA and interpret it in context.

Solution: The 95% confidence interval is given by:

$$\text{CI} = \bar{x} \pm z^* \frac{s}{\sqrt{n}}$$

Using $z^* = 1.96$, the confidence interval is:

$$\text{CI} = 3.2 \pm 1.96 \times \frac{0.5}{\sqrt{100}} = 3.2 \pm 0.098$$

Thus, the 95% confidence interval is $[3.102, 3.298]$. We are 95% confident that the true population mean GPA lies between 3.102 and 3.298.

12.4 Applied (long answer): Consider a random sample $X_1, X_2, \ldots, X_n$ from a Poisson distribution with parameter λ. Let the estimator for λ be $\hat{\lambda} = \frac{1}{n}\sum_{i=1}^{n} X_i$. Find the bias, variance, and MSE of $\hat{\lambda}$.

Solution: 1. Bias: The estimator $\hat{\lambda}$ is unbiased because:

$$E[\hat{\lambda}] = E\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n}\sum_{i=1}^{n} E[X_i] = \lambda.$$

Thus, $\text{Bias}(\hat{\lambda}) = 0$.

2. Variance:

$$\text{Var}(\hat{\lambda}) = \text{Var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\sum_{i=1}^{n} \text{Var}(X_i) = \frac{\lambda}{n}.$$

3. MSE: Since the estimator is unbiased, $\text{MSE}(\hat{\lambda}) = \text{Var}(\hat{\lambda}) = \frac{\lambda}{n}$.

12.5 Applied (short answer): A survey of 500 people finds that 65% favor a particular law. Conduct a hypothesis test to determine if the proportion of the population that supports the law is significantly different from 60%, using $\alpha = 0.05$.

Solution: Null hypothesis $H_0 : p = 0.6$, alternative hypothesis $H_A : p \neq 0.6$.

The test statistic is:

$$z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} = \frac{0.65 - 0.6}{\sqrt{\frac{0.6(0.4)}{500}}} = \frac{0.05}{0.02191} = 2.28.$$

At $\alpha = 0.05$, the critical value for a two-sided test is $z^* = 1.96$. Since $|z| = 2.28 > 1.96$, we reject the null hypothesis. There is sufficient evidence to conclude that the proportion is significantly different from 60%.

12.6 Applied (long answer): A random sample of 50 students had an average number of exclusive relationships of 3.2, with a standard deviation of 1.97. Use this sample to estimate the population mean with a 90% confidence interval.

Solution: The 90% confidence interval is given by:

$$\text{CI} = \bar{x} \pm t^* \frac{s}{\sqrt{n}}$$

Using t^* for 49 degrees of freedom and 90% confidence ($t^* \approx 1.676$):

$$\text{CI} = 3.2 \pm 1.676 \times \frac{1.97}{\sqrt{50}} = 3.2 \pm 0.468$$

Thus, the 90% confidence interval is $[2.732, 3.668]$. We are 90% confident that the true mean number of exclusive relationships lies between 2.732 and 3.668.

12.7 Applied (long answer): Data on the hours of sleep for 25 New Yorkers reveal a mean of 7.73 hours with a standard deviation of 0.77 hours. Test the hypothesis that New Yorkers sleep less than 8 hours on average, using $\alpha = 0.05$.

Solution: 1. Hypotheses: $H_0 : \mu = 8$, $H_A : \mu < 8$.

2. Test statistic:

$$t = \frac{\bar{x} - \mu_0}{\frac{s}{\sqrt{n}}} = \frac{7.73 - 8}{\frac{0.77}{\sqrt{25}}} = \frac{-0.27}{0.154} = -1.75$$

With 24 degrees of freedom, the critical value at $\alpha = 0.05$ for a one-sided test is $t^* = -1.711$. Since $t = -1.75 < -1.711$, we reject H_0. There is sufficient evidence that New Yorkers sleep less than 8 hours on average.