Assignment 4: Linear Algebra
MACSS 33000 1
Due Tuesday, September 2

1 Basic matrix arithmetic

If

$$\mathbf{a} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Find:

1. $\mathbf{a} + \mathbf{b}$
2. $-4\mathbf{b}$
3. $3\mathbf{a} - 4\mathbf{b}$

Solution

1. $\mathbf{a} + \mathbf{b} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$
2. $-4\mathbf{b} = \begin{bmatrix} -4 \\ -12 \end{bmatrix}$
3. $3\mathbf{a} - 4\mathbf{b} = \begin{bmatrix} 2 \\ -6 \end{bmatrix}$

2 More complex matrix arithmetic

Suppose

$$\mathbf{x} = \begin{bmatrix} 3 \\ 2q \\ 6 \end{bmatrix} \text{ and } \begin{bmatrix} p+2 \\ -5 \\ 3r \end{bmatrix}$$

If $\mathbf{x} = 2\mathbf{y}$, find p, q, r

Solution

$$\begin{aligned}
\mathbf{x} &= 2\mathbf{y} \\
\begin{bmatrix} 3 \\ 2q \\ 6 \end{bmatrix} &= 2\begin{bmatrix} p+2 \\ -5 \\ 3r \end{bmatrix} \\
\begin{bmatrix} 3 \\ 2q \\ 6 \end{bmatrix} &= \begin{bmatrix} 2p+4 \\ -10 \\ 6r \end{bmatrix} \\
&\begin{cases} 3 &= 2p+4 \\ 2q &= -10 \\ 6 &= 6r \end{cases} \\
&\begin{cases} p &= -\frac{1}{2} \\ q &= -5 \\ r &= 1 \end{cases}
\end{aligned}$$

3 Check for linear dependence

Which of the following sets of vectors are linearly dependent? In each part, you can denote each vector as $\mathbf{a}, \mathbf{b}, \mathbf{c}$ respectively.

1. $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
2. $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$
3. $\begin{bmatrix} 13 \\ 7 \\ 9 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 5 \\ 8 \end{bmatrix}$
4. $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$

Solution

1. Linearly independent
2. Linearly dependent. Proof:

$$\begin{vmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{vmatrix} = 1\left((5)(9) + (6)(8)\right) - 4\left((2)(9) + (8)(3)\right) + 7\left((2)(6) + (3)(5)\right) = 0$$

3. Linearly dependent. $\mathbf{b} = 0 \cdot \mathbf{a}$

4. Linearly independent. Proof:

$$\begin{aligned} \begin{vmatrix} 1 & 2 & 2 \\ 2 & -2 & 1 \\ 1 & -1 & 3 \end{vmatrix} &= 1\left((-2)(3) + (-1)(1)\right) - 2\left((2)(3) + (1)(1)\right) + 2\left((2)(-1) + (-2)(1)\right) \\ &= -29 \neq 0 \end{aligned}$$

4 Vector length

Find the length of the following vectors:

1. $(3, 4)$
2. $(0, -3)$
3. $(1, 1, 1)$
4. $(1, 2, 3)$
5. $(1, 2, 3, 4)$
6. $(3, 0, 0, 0)$

Solution

1. $||(3, 4)|| = \sqrt{3^2 + 4^2} = 5$
2. $||(0, -3)|| = \sqrt{0^2 + (-3)^2} = 3$
3. $||(1, 1, 1)|| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$
4. $||(1, 2, 3)|| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$
5. $||(1, 2, 3, 4)|| = \sqrt{1^2 + 2^2 + 3^2 + 4^2} = \sqrt{30}$
6. $(3, 0, 0, 0) = \sqrt{3^2} = 3$

5 Law of Cosines

For each of the following pairs of vectors, calculate the angle between them. Report your answers in both radians and degrees. To convert between radians and degrees

1. $\mathbf{v} = (1, 0), \mathbf{w} = (2, 2)$
2. $\mathbf{v} = (4, 1), \mathbf{w} = (2, -8)$
3. $\mathbf{v} = (1, 1, 0), \mathbf{w} = (1, 2, 2)$

Solution

1. $\theta = \arccos\left(\frac{\mathbf{v}\cdot\mathbf{w}}{||\mathbf{v}||\ ||\mathbf{w}||}\right) = \arccos\left(\frac{(1,0)\cdot(2,2)}{||(1,0)||\ ||(2,2)||}\right) = \arccos\left(\frac{2}{(1)(2\sqrt{2})}\right) = \arccos\left(\frac{1}{\sqrt{2}}\right) = 45\,\text{deg},\ 0.785$ rad
2. $\theta = \arccos\left(\frac{\mathbf{v}\cdot\mathbf{w}}{||\mathbf{v}||\ ||\mathbf{w}||}\right) = \arccos\left(\frac{(4,1)\cdot(2,-8)}{||(4,1)||\ ||(2,-8)||}\right) = \arccos\left(\frac{8-8}{||(4,1)||\ ||(2,-8)||}\right) = \arccos(0) = 90\,\text{deg},$ 1.57 rad
3. $\theta = \arccos\left(\frac{\mathbf{v}\cdot\mathbf{w}}{||\mathbf{v}||\ ||\mathbf{w}||}\right) = \arccos\left(\frac{(1,1,0)\cdot(1,2,2)}{||(1,1,0)||\ ||(1,2,2)||}\right) = \arccos\left(\frac{3}{(\sqrt{2})(3)}\right) = \arccos\left(\frac{1}{\sqrt{2}}\right) = 45\,\text{deg},$ 0.785 rad

6 Matrix algebra

Using the matrices below, calculate the following. Some may not be defined; if that is the case, say so

$$\mathbf{A} = \begin{bmatrix} 3 \\ -2 \\ 9 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 8 \\ 0 \\ -1 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 7 & -1 & 5 \\ 0 & -2 & -4 \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} 3 & 1 \\ 3 & 4 \\ 3 & -7 \end{bmatrix} \quad \mathbf{E} = \begin{bmatrix} 5 & 2 & 3 \\ 1 & 0 & -4 \\ -2 & 1 & -6 \end{bmatrix}$$

$$\mathbf{F} = \begin{bmatrix} 4 & 1 & -5 \\ 0 & 7 & 7 \\ 2 & -3 & 0 \end{bmatrix} \quad \mathbf{G} = \begin{bmatrix} 2 & -8 & -5 \\ -3 & 7 & -4 \\ 1 & 0 & 3 \\ 1 & 2 & 6 \end{bmatrix} \quad \mathbf{K} = \begin{bmatrix} 9 \\ -2 \\ -1 \\ 0 \end{bmatrix}$$

$$\mathbf{L} = \begin{bmatrix} 5 & 0 & 3 & 1 \end{bmatrix} \quad \mathbf{M} = \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix}$$

1. $\mathbf{A}+\mathbf{B}$
2. $-\mathbf{G}$
3. $\mathbf{D}'$
4. $\mathbf{C}+\mathbf{D}$
5. $\mathbf{A}'\mathbf{B}$
6. $\mathbf{BC}$
7. $\mathbf{FB}$
8. $\mathbf{E}-5\mathbf{I}_3$
9. $\mathbf{M}^2$

Solution

1. $\mathbf{A}+\mathbf{B} = \begin{bmatrix} 3 \\ -2 \\ 9 \end{bmatrix} + \begin{bmatrix} 8 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 11 \\ -2 \\ 8 \end{bmatrix}$
2. $-\mathbf{G} = \begin{bmatrix} -2 & 8 & 5 \\ 3 & -7 & 4 \\ -1 & 0 & -3 \\ -1 & -2 & -6 \end{bmatrix}$

3. $\mathbf{D}' = \begin{bmatrix} 3 & 3 & 3 \\ 1 & 4 & -7 \end{bmatrix}$

4. $\mathbf{C} + \mathbf{D}$ is not defined since $\mathbf{C}, \mathbf{D}$ have different dimensions

5. $\mathbf{A}'\mathbf{B} = \begin{bmatrix} 3 & -2 & 9 \end{bmatrix} \begin{bmatrix} 8 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} (3)(8) + (-2)(0) + (9)(-1) \end{bmatrix} = \begin{bmatrix} 2 \end{bmatrix}$

6. $\mathbf{BC}$ is not defined since $\mathbf{B}_{col} \neq \mathbf{C}_{row}$

7. $\mathbf{FB} = \begin{bmatrix} 4 & 1 & -5 \\ 0 & 7 & 7 \\ 2 & -3 & 0 \end{bmatrix} \begin{bmatrix} 8 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} (4)(8) + (1)(0) + (-5)(-1) \\ (0)(8) + (7)(0) + (7)(-1) \\ (2)(8) + (-3)(0) + (0)(-1) \end{bmatrix} = \begin{bmatrix} 37 \\ -7 \\ 16 \end{bmatrix}$

8. $\mathbf{E} - 5\mathbf{I}_3 = \begin{bmatrix} 5 & 2 & 3 \\ 1 & 0 & -4 \\ -2 & 1 & -6 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 3 \\ 1 & -5 & -4 \\ -2 & 1 & -11 \end{bmatrix}$

9. $\mathbf{M}^2 = \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} (1)(1) + (-1)(-1) & (1)(-1) + (-1)(3) \\ (-1)(1) + (3)(-1) & (-1)(-1) + (3)(3) \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ -4 & 10 \end{bmatrix}$

7 Matrix inversion

Invert each of the following matricies by hand (you can use a calculator or computer to check your solution, but be sure to show your work). Verify you have the correct inverse by calculating $\mathbf{XX}-1 = \mathbf{I}$. Not all of the matricies may be invertible - if not, show why.

1. $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

2. $\begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}$

3. $\begin{bmatrix} 2 & 4 & 0 \\ 4 & 6 & 3 \\ -6 & -10 & 0 \end{bmatrix}$

Solution

1. $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^{-1} = \frac{1}{(2)(1)-(1)(1)} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$

   Checking my work, we show $\mathbf{XX}^{-1} = \mathbf{I}$:

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} (2)(1) + (1)(-1) & (2)(-1) + (1)(2) \\ (1)(1) + (1)(-1) & (1)(-1) + (1)(2) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}_2$$

2. $\begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}^{-1} = \frac{1}{(2)(-2)-(-4)(1)} \begin{bmatrix} -2 & -1 \\ 4 & 2 \end{bmatrix} \rightarrow$ non-invertible since $ad = bc$.

3. First, we check the determinant of the matrix to check for invertibility.

$$\det \mathbf{X} = 2\left[(6)(0) - (3)(-10)\right] - 4\left[(4)(0) - (-6)(3)\right] + 0 = 60 - 72 = -12 \neq 0$$

Gauss-Jordan method:

$$\begin{bmatrix} 2 & 4 & 0 \\ 4 & 6 & 3 \\ -6 & -10 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 4 & 0 & | & 1 & 0 & 0 \\ 4 & 6 & 3 & | & 0 & 1 & 0 \\ -6 & -10 & 0 & | & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 14 & 0 & | & \frac{7}{2} & 0 & 0 \\ 4 & 6 & 3 & | & 0 & 1 & 0 \\ -6 & -10 & 0 & | & 0 & 0 & 1 \end{bmatrix} \begin{matrix} \leftarrow r_1 = \frac{7}{2}r_1 \\ \\ \\ \end{matrix}$$

$$= \begin{bmatrix} 1 & 4 & 0 & | & \frac{7}{2} & 0 & 1 \\ 4 & 6 & 3 & | & 0 & 1 & 0 \\ -6 & -10 & 0 & | & 0 & 0 & 1 \end{bmatrix} \begin{matrix} \leftarrow r_1 = r_1 + r_2 \\ \\ \\ \end{matrix}$$

$$= \begin{bmatrix} 1 & 4 & 0 & | & \frac{7}{2} & 0 & 1 \\ 4 & 6 & 3 & | & 0 & 1 & 0 \\ 0 & 1 & 0 & | & \frac{3}{2} & 0 & \frac{1}{2} \end{bmatrix} \begin{matrix} \\ \\ \leftarrow r_3 = \frac{1}{14}(r_3 + 6r_1) \end{matrix}$$

$$= \begin{bmatrix} 1 & 4 & 0 & | & \frac{7}{2} & 0 & 1 \\ 0 & 1 & 0 & | & \frac{3}{2} & 0 & \frac{1}{2} \\ 4 & 6 & 3 & | & 0 & 1 & 0 \end{bmatrix} \begin{matrix} \\ \leftarrow r_2 = r_3 \\ \leftarrow r_3 = r_2 \end{matrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & | & -\frac{5}{2} & 0 & -1 \\ 0 & 1 & 0 & | & \frac{3}{2} & 0 & \frac{1}{2} \\ 4 & 0 & 3 & | & -9 & 1 & -3 \end{bmatrix} \begin{matrix} \leftarrow r_1 = r_1 - 4r_2 \\ \\ \leftarrow r_3 = r_3 - 6r_2 \end{matrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & | & -\frac{5}{2} & 0 & -1 \\ 0 & 1 & 0 & | & \frac{3}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & | & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{matrix} \\ \\ \leftarrow r_3 = \frac{1}{3}(r_3 - 4r_1) \end{matrix}$$

$$\begin{bmatrix} 2 & 4 & 0 \\ 4 & 6 & 3 \\ -6 & -10 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{5}{2} & 0 & -1 \\ \frac{3}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

8 Dummy encoding for categorical variables

Ordinary least squares regression is a common method for obtaining regression parameters relating a set of explanatory variables with a continuous outcome of interest. The vector $\hat{\mathbf{b}}$ that contains the intercept and the regression slope is calculated by the equation:

$$\hat{\mathbf{b}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$$

If an explanatory variable is nominal (i.e. ordering does not matter) with more than two classes (e.g. White, Black, Asian, Mixed, Other), the variable must be modified to include in the regression model. A common technique known as dummy encoding converts the column into a series of $n-1$ binary (0/1) columns where each column represents a single class and n is the total number of unique classes in the original column. Explain why this method converts the column into $n-1$ columns, rather than n columns, in terms of linear algebra. Reminder: $\mathbf{X}$ contains both the dummy encoded columns as well as a column of 1s representing the intercept.

Solution:

The method converts the column into $n-1$ columns, rather than n columns because the column of 1s in $\mathbf{X}$ representing the intercept will be dropped in order to compute the vector $\hat{\mathbf{b}}$. This is because this column is guaranteed to be linearly dependent on the other columns of binary entries, causing multicolinearity in $\mathbf{X}$. If this column is not dropped, $\mathbf{X}$ will not be invertible and $(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$ could not be computed.

9 Solve the system of equations

Solve the following systems of equations for x, y, z, either via matrix inversion or substitution

1. $\begin{cases} x + y + 2z &= 2 \\ 3x - 2y + z &= 1 \\ y - z &= 3 \end{cases}$

2. $\begin{cases} x - y + 2z &= 2 \\ 4x + y - 2z &= 10 \\ x + 3y + z &= 0 \end{cases}$

Solution

1. Let $\mathbf{Y} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$, $\mathbf{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $\beta = \begin{bmatrix} 1 & 1 & 2 \\ 3 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$ such that $\mathbf{Y} = \beta\mathbf{X}$. Using the known equations $\beta, \mathbf{Y}$ we compute $\beta^{-1}\mathbf{Y} = \mathbf{X}$.

   First, we check for invertibility of β:

$$\det\beta = (2-1) - (-3+0) + 2(3+0) = 10 \neq 0$$

   We now invert β using the Gauss-Jordan method:

$$\begin{aligned}
\begin{bmatrix} 1 & 1 & 2 \\ 3 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}^{-1} &\to \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 3 & -2 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \\
&\to \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{3}{10} & -\frac{1}{10} & \frac{1}{2} \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right] r_2 = -\tfrac{1}{10}(r_2 - 3r_1 - 5r_3) \\
&\to \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & \frac{7}{10} & \frac{1}{10} & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{3}{10} & -\frac{1}{10} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{3}{10} & -\frac{1}{10} & -\frac{1}{2} \end{array}\right] \begin{matrix} r_1 = r_1 - r_2 \\ \\ r_3 = -r_3 + r_2 \end{matrix} \\
&\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{10} & \frac{3}{10} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{3}{10} & -\frac{1}{10} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{3}{10} & -\frac{1}{10} & -\frac{1}{2} \end{array}\right] r_1 = r_1 - 2r_3
\end{aligned}$$

Now solving

$$\begin{aligned}
\mathbf{X} &= \beta^{-1}\mathbf{Y} \\
&= \begin{bmatrix} \frac{1}{10} & \frac{3}{10} & \frac{1}{2} \\ \frac{3}{10} & -\frac{1}{10} & \frac{1}{2} \\ \frac{3}{10} & -\frac{1}{10} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \\
\begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}
\end{aligned}$$

2. Let $\mathbf{Y} = \begin{bmatrix} 2 \\ 10 \\ 0 \end{bmatrix}$, $\mathbf{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $\beta = \begin{bmatrix} 1 & -1 & 2 \\ 4 & 1 & -2 \\ 1 & 3 & 1 \end{bmatrix}$ such that $\mathbf{Y} = \beta\mathbf{X}$. Using the known equations $\beta, \mathbf{Y}$ we compute $\beta^{-1}\mathbf{Y} = \mathbf{X}$.

First, we check for invertibility of β:

$$\det\beta = (1+6) + (4+2) + 2(12-1) = 35 \neq 0$$

We now invert β using the Gauss-Jordan method:

$$\begin{aligned}
\begin{bmatrix} 1 & -1 & 2 \\ 4 & 1 & -2 \\ 1 & 3 & 1 \end{bmatrix}^{-1} &\rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 4 & 1 & -2 & 0 & 1 & 0 \\ 1 & 3 & 1 & 0 & 0 & 1 \end{array}\right] \\
&= \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 5 & -10 & -4 & 1 & 0 \\ 0 & 4 & -1 & -1 & 0 & 1 \end{array}\right] \begin{matrix} \\ r_2 = r_2 - 4r_1 \\ r_3 = r_3 - r_1 \end{matrix} \\
&= \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -\frac{6}{35} & -\frac{1}{35} & \frac{2}{7} \\ 0 & 4 & -1 & -1 & 0 & 1 \end{array}\right] r_2 = -\tfrac{1}{35}(r_2 - 10r_3) \\
&= \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & \frac{29}{35} & -\frac{1}{35} & \frac{2}{7} \\ 0 & 1 & 0 & -\frac{6}{35} & -\frac{1}{35} & \frac{2}{7} \\ 0 & 0 & 1 & \frac{11}{35} & -\frac{4}{35} & \frac{1}{7} \end{array}\right] \begin{matrix} r_1 = r_1 + r_2 \\ \\ r_3 = -r_3 + 4r_2 \end{matrix} \\
&= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{13}{35} & \frac{8}{35} & \frac{-2}{7} \\ 0 & 1 & 0 & -\frac{6}{35} & -\frac{1}{35} & \frac{2}{7} \\ 0 & 0 & 1 & \frac{11}{35} & -\frac{4}{35} & \frac{1}{7} \end{array}\right] \begin{matrix} r_1 = r_1 - 2r_3 \\ \\ \\ \end{matrix} \\
\begin{bmatrix} 1 & -1 & 2 \\ 4 & 1 & -2 \\ 1 & 3 & 1 \end{bmatrix}^{-1} &= \begin{bmatrix} \frac{13}{35} & \frac{8}{35} & \frac{-2}{7} \\ -\frac{6}{35} & -\frac{1}{35} & \frac{2}{7} \\ \frac{11}{35} & -\frac{4}{35} & \frac{1}{7} \end{bmatrix} = \frac{1}{35}\begin{bmatrix} 13 & 8 & -10 \\ -6 & -1 & 10 \\ 11 & -4 & 5 \end{bmatrix}
\end{aligned}$$

Now solving

$$
\begin{aligned}
\mathbf{X} &= \beta^{-1}\mathbf{Y} \\
&= \frac{1}{35}\begin{bmatrix} 13 & 8 & -10 \\ -6 & -1 & 10 \\ 11 & -4 & 5 \end{bmatrix}\begin{bmatrix} 2 \\ 10 \\ 0 \end{bmatrix} \\
\begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} \frac{12}{7} \\ -\frac{26}{35} \\ -\frac{18}{35} \end{bmatrix}
\end{aligned}
$$

10 Multiplying by 0

When it comes to real numbers, we know that if $xy = 0$, then either $x = 0$ or $y = 0$ orboth. One might believe that a similar idea applies to matricies, but one would be wrong. Prove that if the matrix product $\mathbf{AB} = 0$ (by which we mean a matrix of appropriate dimensionality made up entirely of zeroes), then it is not necessarily true that either $\mathbf{A} = \mathbf{0}$ or $\mathbf{B} = \mathbf{0}$. Hint: in order to prove that something is not always true, simply identify one example where $\mathbf{AB} = \mathbf{0}$, $\mathbf{A}, \mathbf{B} \neq \mathbf{0}$.

Solution

Consider $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ such that $\mathbf{AB} = \mathbf{0}$. These meet the necessary conditions, yet $\mathbf{AB} = \mathbf{0}$. Hence, if $\mathbf{AB} = 0$, then it is not necessarily true that either $\mathbf{A} = \mathbf{0}$ or $\mathbf{B} = \mathbf{0}$