Assignment 10: Multivariate distribution
MACSS 33000 1
Due Friday, September 10

1 Calculating the conditional PDF

Let $f(x,y) = 15x^2y$ for $0 \le x \le y \le 1$. Find $f(x|y)$.

Solution.

$$
\begin{aligned}
f(x|y) &= \frac{f(x,y)}{\int_{-\infty}^{\infty} f(x,y)dx} \\
&= \frac{15x^2y}{\int_0^y 15x^2ydx} \\
&= \frac{15x^2y}{5x^3y\Big|_{x=0}^{y}} \\
&= \frac{15x^2y}{5y^4} \\
&= \frac{3x^2}{y^3}
\end{aligned}
$$

2 Properties of a joint PDF

Continuous random variables X and Y have the following PDF:

$$
f(x,y) = f_{XY}(x,y) = \begin{cases} kx^2y^3 & \text{where } 0 < x, y < 6 \\ 0 & \text{otherwise} \end{cases}
$$

1. Find k.

   Solution.

$$
\begin{aligned}
1 &= \int_0^6 \int_0^6 kx^2y^3dxdy \\
&= k\int_0^6 y^3 \left(\frac{1}{3}x^3 \Big|_{x=0}^{6} \right) dy \\
&= k\frac{6^3}{3} \int_0^6 y^3dy \\
&= k\frac{6^3}{3 \cdot 4} \left(y^4 \Big|_{y=0}^{6} \right) \\
\frac{12}{6^3 \cdot 6^4} &= k \\
k &= \frac{1}{23,328}
\end{aligned}
$$

2. Find the marginal PDF of X, $f_X(x)$.

   Solution.

$$\begin{aligned}
f_X(x) &= \int_0^\infty f_{XY}(x,y)dy \\
&= k\int_0^6 x^2y^3dy \\
&= \frac{k}{4}x^2\left(y^4\Big|_0^6\right) \\
&= \frac{6^4k}{4}x^2 \\
&= \frac{1}{72}x^2
\end{aligned}$$

3. Find the marginal PDF of Y, $f_Y(y)$.

   Solution.

$$\begin{aligned}
f_Y(y) &= \int_0^\infty f_{XY}(x,y)dx \\
&= k\int_0^6 x^2y^3dx \\
&= \frac{k}{3}y^3\left(x^3\Big|_0^6\right) \\
&= \frac{6^3k}{3}y^3 \\
&= \frac{1}{324}y^3
\end{aligned}$$

4. Find $E[X]$.

$$\begin{aligned}
E[X] &= \int_{-\infty}^\infty\int_{-\infty}^\infty xf_{XY}(x,y)dxdy \\
&= k\int_0^6\int_0^6 x^3y^3dxdy \\
&= \frac{6^4k}{4}\int_0^6 y^3dy \\
&= \frac{(6^4)^2}{4^2}k \\
&= 4.5
\end{aligned}$$

5. Find $E[Y]$.

Solution.

$$\begin{aligned}
E[Y] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y f_{XY}(x,y)dxdy \\
&= k\int_0^6\int_0^6 x^2y^4dxdy \\
&= \frac{6^3k}{3}\int_0^6 y^4dy \\
&= \frac{6^3\cdot 6^5}{3\cdot 5}k \\
&= 4.8
\end{aligned}$$

6. Find $Var[X]$.

   Solution.

$$\begin{aligned}
Var[X] &= E[X^2] - E[X]^2 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2 f_{XY}(x,y)dxdy - (4.5)^2 \\
&= k\int_0^6\int_0^6 x^4y^3dxdy - (4.5)^2 \\
&= \frac{6^5k}{5}\int_0^6 y^3dy - (4.5)^2 \\
&= \frac{6^5\cdot 6^4}{5\cdot 4}k - (4.5)^2 \\
&= 1.35
\end{aligned}$$

7. Find $Var[Y]$.

   Solution.

$$\begin{aligned}
Var[Y] &= E[Y^2] - E[Y]^2 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y^2 f_{XY}(x,y)dxdy - (4.8)^2 \\
&= k\int_0^6\int_0^6 x^2y^5dxdy - (4.8)^2 \\
&= \frac{6^3k}{3}\int_0^6 y^5dy - (4.8)^2 \\
&= \frac{6^3\cdot 6^6}{3\cdot 6}k - (4.8)^2 \\
&= 0.96
\end{aligned}$$

8. Find $Cov(X,Y)$.

Solution. First, we find $E[XY]$:

$$
\begin{aligned}
E[XY] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_{XY}(x,y)dxdy \\
&= k\int_0^6\int_0^6 x^3y^4dxdy \\
&= \frac{6^4\cdot 6^5}{4\cdot 5}k \\
&= 21.6.
\end{aligned}
$$

Now we may compute:

$$
\begin{aligned}
Cov(X,Y) &= E[(X-E[X])(Y-E[Y])] \\
&= E[(X-4.5)(Y-4.8)] \\
&= E[XY] - 4.5E[Y] - 4.8E[X] + 21.6 \\
&= (21.6)^2 - 4.5(4.8) - 4.8(4.5) \\
&= 423.36.
\end{aligned}
$$

9. Are X, Y independent? Explain

   Solution. If X, Y are independent, then it must be true that

$$f_{XY}(x,y) = f_X(x)f_Y(y).$$

   Testing this, we have

$$
\begin{aligned}
kx^2y^3 &= \left(\frac{1}{72}x^2\right)\left(\frac{1}{324}y^3\right) \\
k &= \left(\frac{1}{72}\right)\left(\frac{1}{324}\right) \\
k &= \frac{1}{23,328}
\end{aligned}
$$

   which is true. Therefore, X, Y are independent.

10. What is the PDF of X conditional on Y, $f_{X|Y}(x|y)$?

    Solution.

$$
\begin{aligned}
f_{X|Y}(x|y) &= \frac{f_{XY}(x,y)}{f_Y(x,y)} \\
&= \frac{kx^2y^3}{\frac{1}{324}y^3} \\
&= \frac{1}{72}x^2
\end{aligned}
$$

11. What is the PDF of Y conditional on X, $f_{Y|X}(y|x)$?

Solution.

$$\begin{aligned} f_{Y|X}(y|x) &= \frac{f_{XY}(x,y)}{f_X(x,y)} \\ &= \frac{kx^2y^3}{\frac{1}{72}x^2} \\ &= \frac{1}{324}y^3 \end{aligned}$$

3 Properties of joint random variables

Suppose the following

- $E[D] = 10$
- $E[F] = 4$
- $E[DF] = 8$
- $Var(D) = 60$
- $Var(F) = 60$

1. What is $Cov(D, F)$?

   Solution.

$$\begin{aligned} Cov(D,F) &= E[(D - E[D])(F - E[F])] \\ &= E[(D - 10)(F - 4)] \\ &= E[DF] - 10E[F] - 4E[D] + 14 \\ &= -58 \end{aligned}$$

2. What is the correlation between D and F?

   Solution.

$$\begin{aligned} Corr(D,F) &= \frac{Cov(D,F)}{\sqrt{Var(D)Var(F)}} \\ &= \frac{-58}{\sqrt{(60)(60)}} \\ &= -\frac{29}{30} \end{aligned}$$

3. Suppose you multiplied F by 2 to generate a new variable H. What is $Cov(D, H)$

   Solution.

$$\begin{aligned} Cov(D,H) &= E[(D - E[D])(2F - 2E[F])] \\ &= E[(D - 10)(2)(F - 4)] \\ &= 2(E[DF] - 10E[F] - 4E[D] + 14) \\ &= -116 \\ &= 2Cov(D,F) \end{aligned}$$

4. What is $Corr(D, H)$? How does this compare to your answer in part (b) of this question

   Solution. Since $Cov(D, H) = 2Cov(D, F)$ and $Var(aX) = a^2Var(X)$ for any a, X, we have:

$$\begin{aligned} Corr(D, H) &= \frac{2Cov(D, F)}{\sqrt{Var(D)(4)Var(F)}} \\ &= \frac{-58(2)}{\sqrt{(60)(4)(60)}} \\ &= -\frac{29}{30} \\ &= Corr(D, F) \end{aligned}$$

5. Suppose instead that $Var(D) = 30$. How would this change $Corr(D, F)$?

   Solution.

$$\begin{aligned} Corr(D, F) &= \frac{Cov(D, F)}{\sqrt{Var(D)Var(F)}} \\ &= \frac{-58}{\sqrt{(30)(60)}} \\ &= -\frac{29\sqrt{2}}{30} \end{aligned}$$

4 Continuous Bayes' theorem

Prove the continuous Bayes' theorem.

$$f(\theta|X) = \frac{f(X|\theta)f(\theta)}{\int f(X|\theta)f(\theta)d\theta}$$

Solution.

$$\begin{aligned} f(\theta|X) &= \frac{f(\theta, X)}{f_X(x)} && \text{PDF of } \theta \text{ conditional on } X \\ &= \frac{f(\theta, X)}{\int f(\theta, X)d\theta} && \text{Definition of conditional probability} \\ &= \frac{f(X|\theta)f(\theta)}{\int f(X|\theta)f(\theta)d\theta} && f(x, y) = f(x|y)f(y) \end{aligned}$$

□