Assignment 5-6: Functions of several variables and optimization with several variables
MACSS 33000 1
Due Tuesday, September 4

1 Find first partial derivatives

Find all of the first partial derivatives of each function.

1. $f(x,y) = 3x - 2y^4$

$$\frac{\partial}{\partial x} f(x,y) = 3$$
$$\frac{\partial}{\partial y} f(x,y) = -8y^3$$

2. $f(x,y) = x^5 + 3x^3y^2 + 3xy^4$

$$\frac{\partial}{\partial x} f(x,y) = 5x^4 + 9x^2y^2 + 3y^4$$
$$\frac{\partial}{\partial y} f(x,y) = 6x^3y + 12xy^3$$

3. $g(x,y) = xe^{3y}$

$$\frac{\partial}{\partial x} g(x,y) = e^{3y}$$
$$\frac{\partial}{\partial y} g(x,y) = 3xe^{3y}$$

4. $k(x,y) = \frac{x-y}{x+y}$

$$\frac{\partial}{\partial x} k(x,y) = \frac{2y}{(x+y)^2}$$
$$\frac{\partial}{\partial y} k(x,y) = \frac{-2x}{(x+y)^2}$$

5. $h(x,y,z) = x^2e^{yz}$

$$\frac{\partial}{\partial x} h(x,y,z) = 2xe^{yz}$$
$$\frac{\partial}{\partial y} h(x,y,z) = zx^2e^{yz}$$
$$\frac{\partial}{\partial z} h(x,y,z) = yx^2y^{yz}$$

2 Find the gradient

Find the gradient ∇f of the following functions and evaluate them at the given points

1. $f(x, y) = \sqrt{x^2 + y^2}, \quad (x, y) = (3, 4)$

$$\nabla f = \begin{bmatrix} x(x^2+y^2)^{-1/2} \\ y(x^2+y^2)^{-1/2} \end{bmatrix} \to \nabla f(x, y) = \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix}$$

2. $f(x, y, z) = (x + z)e^{x-y}, \quad (x, y, z) = (1, 1, 1)$

$$\nabla f = \begin{bmatrix} (x+z+1)e^{x-y} \\ -2(x+z)e^{x-y} \\ e^{x-y} \end{bmatrix} \to \nabla f(1, 1, 1) = \begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix}$$

3 Find the Hessian

Find the Hessian $\mathbb{H}$ for the following functions

1. $g(x, y) = x^4 - 3x^2y^3$

$$\begin{cases} \frac{\partial g}{\partial x} = 4x^3 - 6xy^3 \\ \frac{\partial g}{\partial y} = -9x^2y^2 \end{cases} \to \mathbb{H}f = \begin{bmatrix} \frac{\partial^2 g}{\partial x^2} & \frac{\partial^2 g}{\partial x \partial y} \\ \frac{\partial^2 g}{\partial x \partial y} & \frac{\partial^2 g}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 12x^2 - 6y^3 & -18xy^2 \\ -18xy^2 & -18x^2y \end{bmatrix}$$

2. $f(x, y, z) = xyz - x^2$

$$\begin{cases} \frac{\partial f}{\partial x} = yz - 2x \\ \frac{\partial f}{\partial y} = xz \\ \frac{\partial f}{\partial z} = xy \end{cases} \to \mathbb{H}f = \begin{bmatrix} -2 & z & y \\ z & 0 & x \\ y & x & 0 \end{bmatrix}$$

4 Find the critical points

Find the local minimum values, local maximum values, and saddle point(s) of the function. Remember the process we discussed in class: Calculate the gradient, set it equal to zero to solve the system of equations, calculate the Hessian, and assess the Hessian at critical values. Be sure to show your work on each of these steps.4

1. $f(x, y) = x^4 + y^4 - 4xy + 2$

$$\nabla f(x, y) = \begin{bmatrix} 4x^3 - 4y \\ 4y^3 - 4x \end{bmatrix} = \vec{0} \quad \to \quad \begin{cases} x^3 = y \\ y^3 = x \end{cases} \quad ; \quad \mathbb{H}f(x, y) = \begin{bmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{bmatrix}$$

(x_0, y_0)	$f_{xx}f(x_0, y_0)$	$\det \mathbb{H}f(x_0, y_0)$	Interpretation
$(0, 0)$	0	-16	Saddle point
$(1, 1)$	12	128	Local minimum
$(-1, -1)$	12	128	Local minimum

2. $k(x,y) = (1+xy)(x+y)$

$$\nabla k = \begin{bmatrix}(1+xy)+y(x+y)\\(1+xy)+x(x+y)\end{bmatrix} = \vec{0} \quad \rightarrow \quad \begin{cases}y^2+2xy+1=0\\x^2+2xy+1=0\end{cases} \quad ; \quad \mathbb{H}f(x,y) = \begin{bmatrix}2y & 2x+2y\\2x+2y & 2x\end{bmatrix}$$

(x_0, y_0)	$f_{xx}k(x_0,y_0)$	$\det \mathbb{H}k(x_0,y_0)$	Interpretation
$(-1,1)$	2	-4	Saddle point
$(1,-1)$	-2	-4	Saddle point

5 Definite integrals

Solve the following definite integrals using the antiderivative method.5 For all these problems, the basic approach to compute the definite integral of $f(x)$ from a to b is by using the formula $F(b)-F(a)$, where $F(x)$ is the antiderivative of f.

1. $\int_6^8 x^3 dx = \frac{1}{4}x^4\Big|_6^8 = \frac{1}{4}(8^4-6^4) = 700$
2. $\int_{-1}^0 (3x^2-1)dx = x^3 - x\Big|_{-1}^0 = 0-(-1-(-1)) = 0$
3. $\int_0^1 x^{\frac{3}{7}} dx = \frac{7}{10}x^{\frac{10}{7}}\Big|_0^1 = \frac{7}{10}$
4. $\int_1^2 t^{-2}dt = -t^{-1}\Big|_1^2 = -(\frac{1}{2}-1) = \frac{1}{2}$
5. $\int_2^4 e^y dy = e^y\Big|_2^4 = e^4 - e^2$
6. $\int_8^9 2^x dx = \int_8^9 e^{x\log(2)}dx = \frac{1}{\log(2)}\int_{8\log(2)}^{9\log(2)} e^u du = \frac{2^x}{\log(2)}\Big|_8^9 = \frac{2^8}{\log(2)}$
7. $\int_3^3 \sqrt{x^5+2}dx = 0$ by the Fundamental Theorem of Calculus

6 Applied integration

Who traveled the farthest? The least far?

$$d_A(t) = \int_{t_0}^{t_1} v_A(t)dt = \int_0^2 (2t^4+t)dt = \frac{2}{5}t^5 + \frac{1}{2}t^2\Big|_0^2 = \frac{74}{5}$$
$$d_B(t) = \int_{t_0}^{t_1} v_B(t)dt = \int_0^4 4\sqrt{t}dt = 4\cdot\frac{2}{3}\cdot t^{\frac{3}{2}}\Big|_0^4 = \frac{64}{3}$$
$$d_C(t) = \int_{t_0}^{t_1} v_C(t)dt = \int_0^{20} 2e^{-t}dt = -2e^{-t}\Big|_0^{20} = -2e^{-20}+2$$

Student B travelled the farthest and Student C travelled the least far.

7 Indefinite integrals

Calculate the following indefinite integrals

1. $\int (x^2 - x^{-\frac{1}{2}})dx = \frac{1}{3}x^3 - 2x^{\frac{1}{2}} + C$
2. $\int 360t^6 dt = \frac{360}{7}t^7 + C$
3. $\int 2x\log(x^2)dx = \int 4x\log(x)dx$

$$\begin{cases} u = \log(x) & du = \frac{1}{x}dx \\ dv = 4xdx & v = \frac{4}{5}x^5 + C \end{cases}$$

$$\begin{aligned} \int 2x\log\left(x^2\right)dx &= \frac{4}{5}x^5\log(x) - \frac{4}{5}\int x^4 dx \\ &= \frac{4}{5}x^5\log(x) - \frac{4}{25}x^5 + C \end{aligned}$$

8 Determining convergence

Determine whether each integral is convergent or divergent. Evaluate those that are convergent.

1. $\int_1^\infty \frac{1}{9x^2}dx = \lim\limits_{b\to\infty}\int_1^b \frac{1}{9x^2}dx = -\frac{1}{27}\lim\limits_{b\to\infty}(b^{-3} - 1) = \frac{1}{27} \to$ converges
2. $\int_0^\infty \cos(x)dx = \lim\limits_{b\to\infty}\int_0^b \cos(x)dx = \lim\limits_{b\to\infty}\sin(x) \to$ does not converge
3. $\int_0^\infty e^{-x}dx = \lim\limits_{b\to\infty}\int_0^b e^{-x}dx = \lim\limits_{b\to\infty}(-e^{-b} + 1) = 1 \to$ converges
4. $\int_{-\infty}^0 x^3 dx = \lim\limits_{b\to-\infty}\int_b^0 x^3 dx = \lim\limits_{b\to-\infty} -\frac{1}{4}b^4 \to$ does not converge

9 More integrals

Calculate the following integrals

1. $\int_0^1\int_2^3 x^2y^3 dxdy = \int_0^1\left(\frac{19}{3}\right)y^3 dy = \frac{19}{12}$
2. $\int_2^3\int_0^1 x^2y^3 dydx = \frac{1}{4}\int_2^3 x^2 = \frac{5}{12}$
3. $\int_0^1\int_0^{\sqrt{1-x^2}} 2x^3y dydx = \int_0^1 x^3(1-x^2)dx = \frac{1}{4}x^4 - \frac{1}{6}x^6\Big|_0^1 = -\frac{1}{12}$