Unified Approach for Hedging Impermanent Loss of Liquidity Provision

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Abstract

We develop static model-independent and dynamic model-dependent approaches for hedging of the impermanent loss (IL) of liquidity provision (LP) staked at Decentralised Exchanges (DEXes) which employ Uniswap V2 and V3 protocols. We provide detailed definitions and formulas for computing the IL to unify the different definitions occurring in the existing literature. We show that the IL can be seen as a contingent claim with a non-linear pay-off for a fixed maturity date. Thus, we introduce the contingent claim termed the IL protection claim which delivers the negative of IL pay-off at the maturity date. We apply arbitrage-based methods for the valuation and risk management of this claim. First, we develop the static model-independent replication method for the valuation of IL protection claim using traded European vanilla call and put options. We extend and generalise an existing method to show that the IL protection claim can be hedged perfectly with options if there is a liquid options market. Second, we develop the dynamic model-based approach for the valuation and hedging of IL protection claims under a risk-neutral measure. We derive analytic valuation formulas using a wide class of price dynamics for which the characteristic function is available under the risk-neutral measure. As base cases, we derive analytic valuation formulas for the IL protection claim under the Black-Scholes-Merton model and the log-normal stochastic volatility model. We finally discuss the estimation of the risk-reward of LP staking using our results.

Keywords: Automated Market Making, Liquidity Provision, Decentralised Finance, Uniswap, Cryptocurrencies, Impermanent Loss

JEL Classifications: C02, G12, G23

1 Introduction

Decentralised Exchanges (DEXes) play a fundamental role in the blockchain ecosystem by allowing users to swap digital assets. The functioning of DEXes requires liquidity providers to stake their liquidity in so-called pools, so that traders can use these pools to buy and sell tokens. Automated Market Making (AMM) protocol is a mechanism to settle buy and sell orders on DEXes. An AMM protocol is characterised by a constant function market maker (CFMM) which assigns buy and sell prices for given orders using order sizes and current liquidity of a pool. Uniswap V3 (Adams *et al.* (2021)) is the most widely employed CFMM which is adopted by many DEXes, in addition to Uniswap DEX itself. Uniswap V2 (Adams *et al.* (2020)) is an earlier AMM protocol that employs the constant product CFMM that is less capital efficient than the V3 CFMM. Uniswap V2 is still in use for old altcoin pools. For details of the design of various CFMMs, see, among others, Angeris *et al.* (2019), Lipton-Treccani (2021), Mohan (2022), Lipton-Hardjono (2022), Lipton-Sepp (2022), Milionis *et al.* (2022), Park (2023), Lehar-Parlour (2024). We note that in early 2025,

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the Uniswap development team launched the Uniswap V4 (Adams *et al.* (2023)) AMM, which has the same CFMM as Uniswap V3, so our results will be directly applicable to Uniswap V4 pools. Alexander *et al.* (2024) establish empirically that technological advances in AMM protocols, such as improvements in Uniswap V3 compared to V2, lead to significant improvements in the liquidity and price discovery of DEXes. In this respect, our results may help liquidity providers manage risk of their liquidity provisions and thus enhance the liquidity of DEXes.

Uniswap V3 protocol allows liquidity providers to concentrate liquidity in specified ranges. As a result, the liquidity of the pool can be increased in certain ranges (typically around the current price), and the potential to generate more trading fees from the LP is increased accordingly. We illustrate the dynamics of staked LP using the ETH/USDT pool as an example. A liquidity provider stakes liquidity in a specific range using an initial amount of ETH and USDT tokens as specified by Uniswap V3 CFMM. When the price of ETH falls, traders use the pool to swap USDT by depositing ETH, so that the LP accrues more units of ETH. Thus, when ETH falls persistently, the liquidity provider ends up holding more units of the depreciating asset, which is similar to being short a put option. In contrast, when the price of ETH increases, traders will deplete the reserves of ETH from the pool by depositing USDT tokens. Thus, the liquidity provider ends up holding less units of the appreciating asset, which is similar to being short a call option. The combined effect of increasing / decreasing the exposure to depreciating / appreciating assets leads to what is known as the impermanent loss in Decentralised Finance (DeFi) applications.

It is clear that the mechanism behind the impermanent loss of LPs is similar to being short a portfolio of call and put options. Consequently, we can apply well-developed methods from financial engineering for designing the valuation and risk management of LPs using static and dynamic methods for the valuation and risk management of derivative securities. Our contributions include static model-independent replication and dynamic model-dependent replication of IL.

1.1 Literature review and contributions

For the static replication of IL under Uniswap V2 protocol, Fukasawa *et al.* (2023) derive approximate hedging portfolio using variance and gamma swaps. In addition, Lipton (2024) extends Lipton-Sepp (2008), Lipton (2018), and obtains model-dependent costs of hedging portfolios using values of variance and gamma swaps under Heston model. Deng *et al.* (2023) and Maire-Wunsch (2024) develop the static replication for the Uniswap V3 protocol. We note that their result hinges specifically on the analytical formula of the IL under Uniswap V3 AMM. Also this method requires to hold both call and put options for in-the-money and out-of-the-money strikes, which could be very costly for practical implementation[1]. We provide a generic approach to designing a cost-effective replicating portfolio for IL hedging for general CFMM and illustrate our approach for the Uniswap V3 protocol.

We note that currently the liquid option market (on Deribit options exchanges and some other centralised exchanges) exists only for Bitcoin and Ether. Thus, hedging of LP stakes for altcoins requires a model-dependent approach for replication of IL by introducing the IL protection claim and applying dynamic delta-hedging of this claim. In the current ecosystem, some market-making and trading companies provide off-chain (over-the-counter) IL protection of claims for a wide range of digital assets. We develop a dynamic model-dependent approach for the valuation of IL protection claims, which appears to be new in the literature.

For valuation purposes under the Uniswap V3 protocol, we provide an original result with a decomposition of the IL into components including pay-offs of vanilla call and put options, digital options, and an exotic pay-off on the square root of the price. This decomposition allows for the

[1] At major options exchanges for digital and traditional markets, the liquidity is concentrated in call and put options with out-of-the-money and near at-the-money strikes. Options with in-the-money strikes are not liquid with high bid-ask spreads. As a result, a liquid and cost-efficient replicating portfolio should include only out-of-the-money and near-at-the-money strikes

valuation of the IL protection claims under a large class of price dynamics, which have solution for their characteristic functions, by utilising the Lipton-Lewis formula. As an important example, we derive an analytic solution for IL protecting claim under Black-Scholes-Merton model, which allows us to analyse the value of the IL protection claim using a single volatility parameter. As realistic price dynamics that includes stochastic volatility correlated with the price dynamics, we apply the log-normal stochastic volatility model developed in Sepp-Rakhmonov (2023).

Empirical analysis of the profitability of LP strategies is currently a very active area in the AMM-related literature. We refer, among others, to Heimbach *et al.* (2022), Cartea *et al.* (2023), Bergault *et al.* (2023), Cartea *et al.* (2024), Li *et al.* (2023), Li *et al.* (2024), Brönnimann *et al.* (2024). In particular, Cartea *et al.* (2024) consider a diffusion model with a stochastic mean and constant volatility for the dynamics of the swap rate and derive an optimal liquidity range assuming logarithmic utility of LP providers. Their results depend on the constant volatility of the swap dynamics. In such a setup, our results allow for mitigation of model risk by either using a static replication when liquid options are available for trading (in which case, the model risk is eliminated) or using realistic dynamics for the swap rate including jumps and stochastic volatility with dynamic replication of IL protection claim.

Our paper is organised as follows. In Section 2, we provide definitions and derivations of IL and pay-offs of IL protection claims. In Section 3, we apply these results for the Uniswap V2 and V3 protocols. Hereby, we derive the decomposition formula for IL in Uniswap V3 protocol into pay-offs of vanilla, digital, and square-root contracts, which we use further for model-dependent valuation. In Section 4, we develop a generic approach for static replication of IL using traded vanilla options. In Section 5, we develop the model-dependent approach for the valuation of protection claim against IL. We conclude in Section 7.

2 Impermanent Loss of Liquidity Provision

2.1 Liquidity Provision

We consider a liquidity pool on a pair of token 1 and token 2. Without loss of generality, we assume that token 1 is a volatile token and token 2 is a stable token with the spot price p of swapping one unit of token 1 to p units of token 2. For concreteness, we fix token 1 to be ETH and token 2 to be USDT with spot price p being ETH/USDT exchange price ($p_0 = 1600$ as of 19 April 2025).

We consider a liquidity provision (LP) provided for x_0 and y_0 units of token 1 (ETH) and token 2 (USDT), respectively. The initial value of the LP in the units of token 2 (USDT) is given by[2]

$$V_0^{(y)} = p_0 x_0 + y_0. \tag{1}$$

The value of the LP at time t is given by

$$V_t^{(y)} = p_t x_t + y_t, \tag{2}$$

where x_t and y_t are the current units of ETH and USDT (these units are the outputs from AMM protocol), respectively, in the staked LP and p_t is the current ETH/USDT spot price. We treat accrued LP fees separately in line with the Uniswap convention.

The value of the LP in units of token 1 is given by

$$V_t^{(x)} = x_t + p_t^{-1} y_t. \tag{3}$$

Our further results can be directly applied for pools with USDT/ETH type of conversion using corresponding inverse prices and ranges. We also note that in Uniswap V2 and V3, the price is

[2] We note that though the paper, the value of LP V and its allocated units x and y characterize the state of a stand-alone LP and not the state of the aggregate pool.

defined on the grid of price ticks which are functions of the pool fee tiers. Price ticks are dense for pools with small fee tiers, so we assume that price range for p_t is continuous (see Echenim *et al.* (2023) for the analysis using discrete ticks).

2.2 Profit-and-Loss of LP

We consider two types of LP strategies that exclude and include static delta hedging of the initial stake.

Definition 2.1 (USDT Funded LP position)**.** *Funded position is created by funding the initial allocation of x_0 and y_0 units with the total capital commitment of $V_0^{(y)}$ USDT. The stake in the funded position includes the purchase of x_0 units of ETH token at the price p_0.*

The value of the funded position equals the value of LP in Eq. (2): $V\ funded_t^{(y)} = V_t^{(y)}$. As a result, the Profit-and-Loss (P&L) of the funded position in token 2 (USDT) at time t is given by

$$\text{P\&L funded}^{(y)} \equiv V_t^{(y)} - V_0^{(y)} = (p_t x_t + y_t) - (p_0 x_0 + y_0)\,. \tag{4}$$

Definition 2.2 (Borrowed LP position)**.** *Borrowed position is created either by borrowing x_0 units of ETH or by purchasing x_0 units of ETH for staking and by simultaneously selling short the perpetual future for hedging the initial stake of x_0 units of ETH*[3].

For the borrowed position with hedging, we set the hedge position of selling short x_0 units of token 1 with strike/entry price p_0. The P&L of the hedge position in units of USDT token at time t is given by

$$\text{Hedge}_t^{(y)} = -\left(p_t - p_0\right) x_0. \tag{5}$$

We assume that the hedge can be implemented by short-selling the perpetual future and we treat the funding cost separately from the LP P&L. The initial value of the staking position is given in Eq. (2). The value of the borrowed LP position at time t is given by

$$\text{V borrowed}_t^{(y)} = p_t x_t + y_t - \left[p_t - p_0\right] x_0. \tag{6}$$

The P&L of the borrowed position at time t is given by

$$\begin{aligned}\text{P\&L borrowed}_t^{(y)} &= \left(p_t x_t + y_t - \left[p_t - p_0\right] x_0\right) - \left(p_0 x_0 + y_0\right) \\ &= \left(p_t x_t + y_t\right) - \left(p_t x_0 + y_0\right).\end{aligned} \tag{7}$$

2.3 Impermanent Loss

Using the two definitions of LPs in Eqs (4) and (7), we define the quantity known as impermanent loss in the following three ways.

Definition 2.3 (Impermanent Loss %)**.** *The nominal IL of funded LP is defined for the LP funded with USDT by*

$$\text{IL funded}^{(y)}(p_t) = \frac{(p_t x_t + y_t) - (p_0 x_0 + y_0)}{(p_0 x_0 + y_0)}. \tag{8}$$

The nominal IL of borrowed LP is defined for the LP with borrowed ETH by

$$\text{IL borrowed}^{(y)}(p_t) = \frac{(p_t x_t + y_t) - (p_t x_0 + y_0)}{(p_0 x_0 + y_0)}. \tag{9}$$

The relative IL of borrowed LP is defined for borrowed LP by

$$\text{Rel IL borrowed}^{(y)}(p_t) = \frac{(p_t x_t + y_t) - (p_t x_0 + y_0)}{(p_t x_0 + y_0)}. \tag{10}$$

[3]There is very liquid market for core cryptocurrencies on both on-chain exchanges (such as Hyperliquid, GMX, Aevo) and on off-chain exchanges (such as Binance, Bybit, Deribit), so that hedging of long exposures is possible.

In the literature, all three definitions are being used. Here, we clarify the meaning of each definition. The nominal IL of the funded LP is applicable when the LP provider funds the position by allocation V_0 USDT tokens and buys the initial stake of x_0 ETH tokens. The nominal IL for borrowed LP is common for LPs accompanied either by borrowing the initial stake of x_0 ETH tokens or by emplying static delta-hedging of the initial stake of x_0 tokens using perpetual futures. The relative IL of borrowed LP defines the P&L of the borrowed LP relative to the buy-and-hold position rather than the initial staked value of the LP.

The nominal IL can be easily interpreted because P&L of the LP in USDT is the nominal IL multiplied by the initial staked notional so that we obtain

$$\begin{aligned} \text{P\&L funded}^{(y)}(p_t) &= N^{(y)} \times \text{ IL funded}^{(y)}(p_t) \\ \text{P\&L borrowed}^{(y)}(p_t) &= N^{(y)} \times \text{ IL borrowed}^{(y)}(p_t), \end{aligned} \tag{11}$$

where $N^{(y)}$ is the initial notional in USDT token of the staked LP. In contrast, the relative IL lacks this interpretation. Thus, while relative IL appears in some of the literature to emphasize the IL relative to the buy-and-hold portfolio, its practical application for modelling of the realised P&L from a LP is not obvious. In this paper, we focus only on hedging of the nominal IL for funded and borrowed LPs in Eqs (8) and (9), respectively.

Since the tokens can be used interchangeably, our definitions are symmetric. For a position funded in x (ETH) tokens the corresponding P&L is obtained using $p_t^{-1} = 1/p_t$ and Eq. (11) becomes

$$\begin{aligned} \text{P\&L funded}^{(x)}(p_t^{-1}) &= N^{(x)} \times \text{IL funded}^{(x)}(p_t^{-1}), \\ \text{P\&L borrowed}^{(x)}(p_t^{-1}) &= N^{(x)} \times \text{IL borrowed}^{(x)}(p_t^{-1}), \end{aligned} \tag{12}$$

where $N^{(x)}$ is notional in x units and

$$\begin{aligned} \text{IL funded}^{(x)}(p_t^{-1}) &= \frac{\left(x_t + p_t^{-1} y_t\right) - \left(x_0 + p_0^{-1} y_0\right)}{\left(x_0 + p_0^{-1} y_0\right)} \\ \text{IL borrowed}^{(x)}(p_t^{-1}) &= \frac{\left(x_t + p_t^{-1} y_t\right) - \left(x_0 + p_t^{-1} y_0\right)}{\left(x_0 + p_0^{-1} y_0\right)}. \end{aligned} \tag{13}$$

2.4 Payoff of IL Protection Claim

We fix the maturity time T.

Definition 2.4 (Payoff of IL protection claim)**.** *We define the protection claim against the IL as a derivative security whose pay-off at time T equals the negative value of the IL. For the funded LP, the pay-off at time T is defined by*

$$\text{Payoff funded}(p_T) = -\text{IL funded}^{(y)}(p_T). \tag{14}$$

For the borrowed LP, the pay-off at time T is defined by

$$\text{Payoff borrowed}(p_T) = -\text{IL borrowed}^{(y)}(p_T). \tag{15}$$

A liquidity provider of staked LP with notional $N^{(y)}$ can buy the IL protection claim to eliminate impermanent loss of the staked LP. By Eq. (11), at time T the P&L of holder's LP will be matched by the pay-off of the IL protection claim in Eq. (14) or Eq. (15). As a result, the liquidity provider can perfectly hedge the IL at time T.

3 Applications to Uniswap AMM Protocol

We now derive explicit formulas for the IL of funded and borrowed LP stakes under the Uniswap V2 and V3 protocols.

3.1 Uniswap V2

In Uniswap V2 (Adams *et al.* (2020)), the CFMM is defined by the constant product rule as follows

$$x'y' = L'^2. \tag{16}$$

where x' and y' are pool reserves and L' is the pool liquidity parameter. In the Uniswap V2 white paper, the constant is defined by k. We use L'^2, $L' > 0$, according to the specification of the V3 protocol.

The pool price is determined by pool reserves as follows

$$p \equiv \frac{y'}{x'} \tag{17}$$

Thus, we need to solve Eq. (16) and (17) in the two unknowns x' and y'. Substituting $y' = px'$ from Eq. (17) into Eq. (16), we obtain that the LP stakes are given as follows

$$x' = \sqrt{\frac{L'^2}{p}},\ y' = \sqrt{pL'^2}. \tag{18}$$

We now consider the LP position with staked units x and y and the liquidity constant L. From Eq. (18), the value of LP position is given by[4]:

$$V_t^{(y)} = p_t x_t + y_t = 2L\sqrt{p_t}. \tag{19}$$

Proposition 3.1 (Funded LP)**.** *The funded P&L in Eq. (4) is computed by:*

$$\text{P\&L funded}^{(y)}(p_t) = 2L\sqrt{p_0}\left(\sqrt{\frac{p_t}{p_0}} - 1\right). \tag{20}$$

The nominal IL for funded LP in Eq. (8) is computed by

$$\text{Nom IL funded}^{(y)}(p_t) = \sqrt{\frac{p_t}{p_0}} - 1. \tag{21}$$

Proof. Using Eq. (19), we obtain

$$\text{P\&L funded}^{(y)}(p_t) = 2L\sqrt{p_t} - 2L\sqrt{p_0} = 2L\sqrt{p_0}\left(\sqrt{\frac{p_t}{p_0}} - 1\right). \tag{22}$$

Given an initial notional of the stake such as $N^y = V_0 = 2L\sqrt{p_0}$, we obtain the nominal IL. □

Proposition 3.2 (IL for Borrowed LP in Uniswap V2)**.** *The borrowed P&L in Eq. (7) is given by:*

$$\text{P\&L borrowed}^{(y)}(p_t) = -L\sqrt{p_0}\left(\sqrt{\frac{p_t}{p_0}} - 1\right)^2. \tag{23}$$

The nominal IL for borrowed LP in Eq. (9) is given by

$$\text{Nom IL borrowed}^{(y)} p_t = -\frac{1}{2}\left(\sqrt{\frac{p_t}{p_0}} - 1\right)^2. \tag{24}$$

Relative IL in Eq. (10) is given by

$$\text{Rel IL borrowed}^{(y)}(p_t) = -\frac{\left(\sqrt{\frac{p_t}{p_0}} - 1\right)^2}{\frac{p_t}{p_0} + 1}. \tag{25}$$

[4] This relationship follows from the condition that the internal price in Eq. (17) inferred by pool reserves follows an external price p_t, observed on other DEXes and centralised exchanges. In practice, the reserves of the liquidity pools are balanced so that the internal price in Eq. (17) follows the external price feeds within tight bands most of the time due to arbitrage operations of multiple arbitrators in the blockchain ecosystem. For details of such arbitrages see among others Milionis *et al.* (2022), Park (2023), Lehar-Parlour (2024), Cartea *et al.* (2024), Cartea *et al.* (2024). Same considerations apply for the internal price in Eq. (31) implied by pool reserves for Uniswap V3 pools.

Proof. From Eq. (18) we note that

$$p_t x_0 + y_0 = p_t \sqrt{\frac{L^2}{p_0}} + \sqrt{p_0 L^2} = \sqrt{p_0} L \left(\frac{p_t}{p_0} + 1 \right). \tag{26}$$

Thus, we obtain

$$\begin{aligned} \text{P\&L borrowed}^{(y)}(p_t) &= (p_t x_t + y_t) - (p_t x_0 + y_0) \\ &= 2\sqrt{p_t L^2} - \sqrt{p_0} L \left(\frac{p_t}{p_0} + 1 \right) \\ &= -L\sqrt{p_0} \left(\sqrt{\frac{p_t}{p_0}} - 1 \right)^2. \end{aligned} \tag{27}$$

□

It is clear that the minimum is 0 at $p_t = p_0$, and otherwise the borrowed P&L is negative for any value of p_1.

Corollary 3.1 (Payoff of claim for IL protection for Uniswap V2 AMM)**.** *Using definitions in Eq.* (14) *and Eq.* (15) *for pay-offs of protection claim against funded and borrowed LP, respectively, along with respective Eqs* (21) *and* (24)*, we obtain*

$$\begin{aligned} \text{Payoff funded}(p_T) &= 1 - \sqrt{\frac{p_t}{p_0}}, \\ \text{Payoff borrowed}(p_T) &= \frac{1}{2} \left(\sqrt{\frac{p_t}{p_0}} - 1 \right)^2. \end{aligned} \tag{28}$$

In subplot (A) of Figure 1, we show ETH units (left y-axis) and USDT units (right y-axis) for LP Uniswap V2 with $1m$ USDT notional and $p_0 = 2000$ ETH/USDT price. The initial LP units of (ETH, USDT) are $(250, 500000)$. The red bar at $p = 1500$ shows LP units of (289, 433013) with the higher allocation to ETH units as ETH/USDT price falls. The green bar at $p = 2500$ shows corresponding LP units of $(224, 559017)$ with the higher allocation to USDT units as ETH/USDT price rises. In subplot (B), we show USDT values of 50%/50% ETH/USDT portfolio, Funded LP position and Borrowed LP position. The funded LP underperforms the 50%/50% portfolio on both the upside (because LP position reduces ETH units) and on the downside (because LP position reduces ETH units). The value of the borrowed LP has zero first-order beta to ETH with negative quadratic convexity to ETH/USDT changes.

3.2 Application to Uniswap V3

In Uniswap V3 protocol (see Adams *et al.* (2021)), the staked liquidity (x, y) of LP satisfy the following equation:

$$x_v y_v = L^2, \tag{29}$$

where x_v and y_v are termed as virtual reserves:

$$x_v \equiv x + \frac{L}{\sqrt{p_b}}, \; y_v \equiv y + L\sqrt{p_a}, \tag{30}$$

with the liquidity amount L provided in the price range $[p_a, p_b]$.

For ETH/USDT pool, the price p is set by:

$$p \equiv \frac{y_v}{x_v} = \frac{y + L\sqrt{p_a}}{x + \frac{L}{\sqrt{p_b}}}. \tag{31}$$

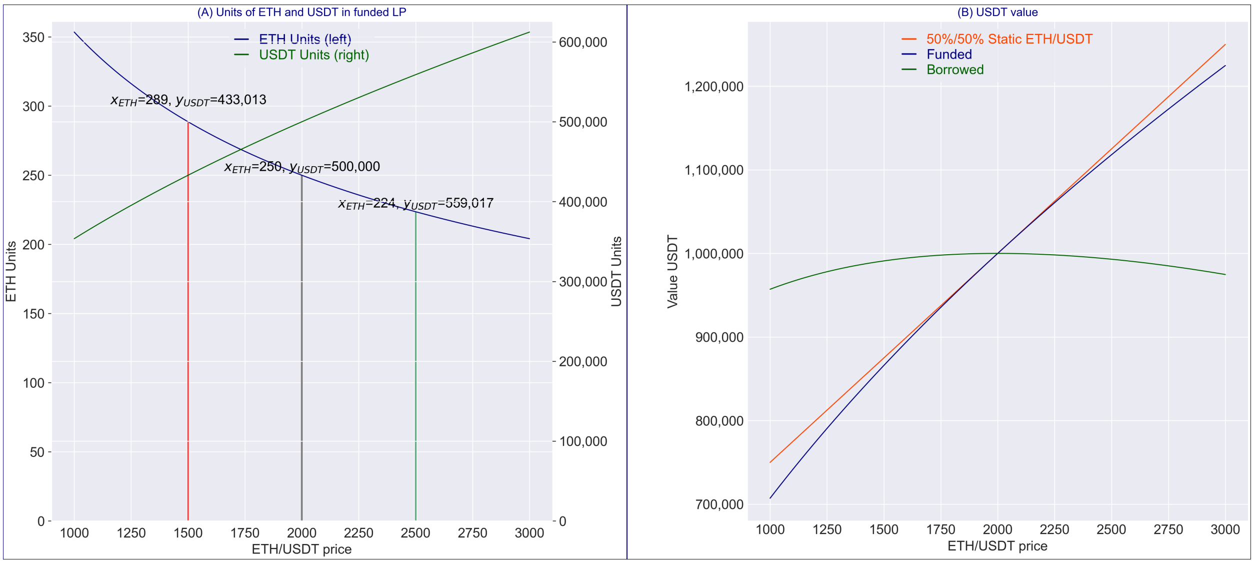

Figure 1: (A) ETH units (left y-axis) and USDT units (right y-axis) for LP Uniswap V2. (B) USDT value of 50%/50% ETH/USDT portfolio, Funded LP position and Borrowed LP position. Uniswap V2 LP position is constructed using $1m$ USDT notional with $p_0 = 2000$ ETH/USDT price.

Eqs (29) and (31) are viewed as two equations in four unknowns (x, y, L, p):

$$\begin{aligned}\left(x + \frac{L}{\sqrt{p_b}}\right)(y + L\sqrt{p_a}) &= L^2 \\ p &= \frac{y + L\sqrt{p_a}}{x + \frac{L}{\sqrt{p_b}}}.\end{aligned} \tag{32}$$

The above equations have to hold at any point in time, with internal price p tracking an internal price observed on other trading venues within tight bounds due to arbitragers adjust pool reserves accordingly to eliminate arbitrage opportunities. Adding (removing) liquidity amounts to increasing (decreasing) L through increasing x and y, while keeping p constant. Swapping (trading) tokens amounts to keeping changing x, y and p, while keeping L the same.

Next, we gather several technical results related to this "mechanics". Those follow from the whitepaper Adams *et al.* (2021) is a relatively straightforward manner. Similar derivations can be found in Elsts (2021). Firstly, we solve for x and y given L and p as independent variables. For a given position with L and p as external parameters, this solution provides how many units x and y are assigned to the LP position.

Lemma 3.1 (Solution for x and y)**.** *Thus for $p \in (p_a, p_b)$, the LP units x and y are given by:*

$$x = L\left(\frac{1}{\sqrt{p}} - \frac{1}{\sqrt{p_b}}\right), \;\; y = L\left(\sqrt{p} - \sqrt{p_a}\right). \tag{33}$$

For $p \le p_a$, the position is fully in token 1:

$$x = L\left(\frac{1}{\sqrt{p_a}} - \frac{1}{\sqrt{p_b}}\right), \;\; y = 0. \tag{34}$$

For $p \ge p_b$, the position is fully in token 2:

$$x = 0, \;\; y = L\left(\sqrt{p_b} - \sqrt{p_a}\right). \tag{35}$$

Proof. We substitute the second equation in Eq. (32)

$$(y + L\sqrt{p_a}) = p\left(x + \frac{L}{\sqrt{p_b}}\right) \tag{36}$$

into the first one to obtain

$$\left(x + \frac{L}{\sqrt{p_b}}\right)^2 = \frac{L^2}{p} \Rightarrow x = L\left(\frac{1}{\sqrt{p}} - \frac{1}{\sqrt{p_b}}\right) \tag{37}$$

and

$$(y + L\sqrt{p_a}) = L\sqrt{p} \Rightarrow y = L\left(\sqrt{p} - \sqrt{p_a}\right). \tag{38}$$

□

We obtain that the initial value of LP using Eq. (33) for $p_t \in (p_a, p_b)$ is given by

$$V_0 \equiv p_0 x_0 + y_0 = L\left(2\sqrt{p_0} - \frac{p_0}{\sqrt{p_b}} - \sqrt{p_a}\right). \tag{39}$$

Corollary 3.2 (Initial notional N^y). *Given an initial notional of the stake such as $N^y = V_0$, using Eq. (39) we obtain that the provided liquidity L is set by*

$$L = \frac{N^y}{2\sqrt{p_0} - \frac{p_0}{\sqrt{p_b}} - \sqrt{p_a}}. \tag{40}$$

3.2.1 Impermanent Loss

Proposition 3.3 (Funded P&L and IL in Uniswap V3). *The P&L of the funded position in Eq. (4) at time t with current price p_t is given by*

$$\text{P\&L funded}^{(y)} = \begin{cases} L\left[2\left(\sqrt{p_t} - \sqrt{p_0}\right) + \frac{p_0 - p_t}{\sqrt{p_b}}\right] & p_t \in (p_a, p_b) \\ L\left[p_t\left(\frac{1}{\sqrt{p_a}} - \frac{1}{\sqrt{p_b}}\right) + \frac{p_0}{\sqrt{p_b}} - 2\sqrt{p_0} + \sqrt{p_a}\right] & p_t \leq p_a \\ L\left[\sqrt{p_b} + \frac{p_0}{\sqrt{p_b}} - 2\sqrt{p_0}\right] & p_t \geq p_b, \end{cases} \tag{41}$$

where

$$L = \frac{N^y}{2\sqrt{p_0} - \frac{p_0}{\sqrt{p_b}} - \sqrt{p_a}} \tag{42}$$

and N^y is USDT notional.

The nominal IL for funded position defined in Eq. (8) is computed by

$$\text{Nom IL funded}^{(y)} = \frac{\text{P\&L funded}^{(y)}}{L\left(2\sqrt{p_0} - \frac{p_0}{\sqrt{p_b}} - \sqrt{p_a}\right)}. \tag{43}$$

Corollary 3.3 (The pay-off of IL protection claim for funded LP). *The pay-off of IL protection claim at maturity date T for funded LP in Eq. (14) is given by the following compact formula*

$$\text{Payoff funded}(p_t) = -\frac{\frac{p_t}{\sqrt{f(p_t;p_a,p_b)}} + \sqrt{f(p_t;p_a,p_b)} - \frac{p_t}{\sqrt{p_b}} - \sqrt{p_a}}{\frac{p_0}{\sqrt{f(p_0;p_a,p_b)}} + \sqrt{f(p_0;p_a,p_b)} - \frac{p_0}{\sqrt{p_b}} - \sqrt{p_a}} + 1, \tag{44}$$

where $f(x; p_a, p_b) = \max\left(\min\left(x, p_b\right), p_a\right)$.

Proof. See Appendix 7.1. □

Proposition 3.4 (P&L of borrowed LP)**.** *The P&L of the borrowed position in Eq. (7) is given by*

$$\text{P\&L borrowed}^{(y)}(p_t) = \begin{cases} -L\sqrt{p_0}\left(\sqrt{\frac{p_t}{p_0}}-1\right)^2 & p_t \in (p_a, p_b) \\ L\left[p_t\left(\frac{1}{\sqrt{p_a}}-\frac{1}{\sqrt{p_0}}\right)-\left(\sqrt{p_0}-\sqrt{p_a}\right)\right] & p_t \le p_a \\ L\left[\left(\sqrt{p_b}-\sqrt{p_0}\right)-p_t\left(\frac{1}{\sqrt{p_0}}-\frac{1}{\sqrt{p_b}}\right)\right] & p_t \ge p_b. \end{cases} \tag{45}$$

Nominal borrowed impermanent loss in Eq. (9) is given by

$$\text{Nom IL borrowed}^{(y)}(p_t) = \frac{\textit{P\&L borrowed}^{(y)}(p_t)}{L\left(2\sqrt{p_0}-\frac{p_0}{\sqrt{p_b}}-\sqrt{p_a}\right)}. \tag{46}$$

Proof. See Appendix 7.2. □

Corollary 3.4 (Payoff of IL protection claim for borrowed LP)**.** *The pay-off of IL protection claim against for funded LP in Eq.* (14) *is given by the following compact formula*

$$\text{Payoff borrowed}(p_t) = \frac{\frac{p_t}{\sqrt{f(p_t;p_a,p_b)}}+\sqrt{f(p_t;p_a,p_b)}-\frac{p_t}{\sqrt{f(p_0;p_a,p_b)}}-\sqrt{f(p_0;p_a,p_b)}}{\frac{p_0}{\sqrt{f(p_0;p_a,p_b)}}+\sqrt{f(p_0;p_a,p_b)}-\frac{p_0}{\sqrt{p_b}}-\sqrt{p_a}}, \tag{47}$$

where $f(x;p_a,p_b) = \max\left(\min\left(x,p_b\right),p_a\right)$*.*

In subplot (A) of Figure 2, we show ETH units (left y-axis) and USDT units (right y-axis) for LP on Uniswap V3 with $1m$ USDT notional and $p_0 = 2000$, $p_a = 1500$, $p_b = 2500$. The initial LP units of (ETH, USDT) are $(220, 559282)$. The red bar at $p = 1500$ shows LP units of (543, 0) with the LP fully in ETH units when the price falls below the lower threshold p_a. The green bar at $p = 2500$ shows the corresponding LP units of $(0, 1052020)$ with the LP fully in USDT units when the price rises above the upper threshold p_b. In subplot (B), we show USDT values of 50%/50% ETH/USDT portfolio, Funded LP positions and Borrowed LP positions. The value profile of the funded LP resembles the profile of a covered call option (long ETH and short out-of-the-money call). The value of the borrowed LP resembles the pay-off of a short straddle (short both at-the-money call and put).

In Figure 3, we show P&L profiles of borrowed and funded LPs as functions of ranges for Uniswap V3 and the full range for Uniswap V2. For borrowed LPs, narrow ranges result in higher losses at the same price levels. For funded LPs, narrower ranges result in higher downside losses and lower upside potential.

3.3 Decomposition of the IL under Uniswap V3 into Simple Payoffs

First we derive the model-independent decomposition of IL into pay-offs of vanilla and "exotic" options. We will further apply these decompositions for model-based valuation of IL protection claims in Uniswap V3.

Proposition 3.5 (Decomposition of IL for Funded LP)**.** *IL of funded LP can be decomposed into the three parts as follows:*

$$\text{IL funded}^{(y)}(p_t) = u_0(p_t) + u_{1/2}(p_t) + u_1(p_t), \tag{48}$$

where $u_0(p_t)$*,* $u_{1/2}(p_t)$*, and* $u_1(p_t)$ *are linear part, (exotic) square root price, and (vanilla) option*

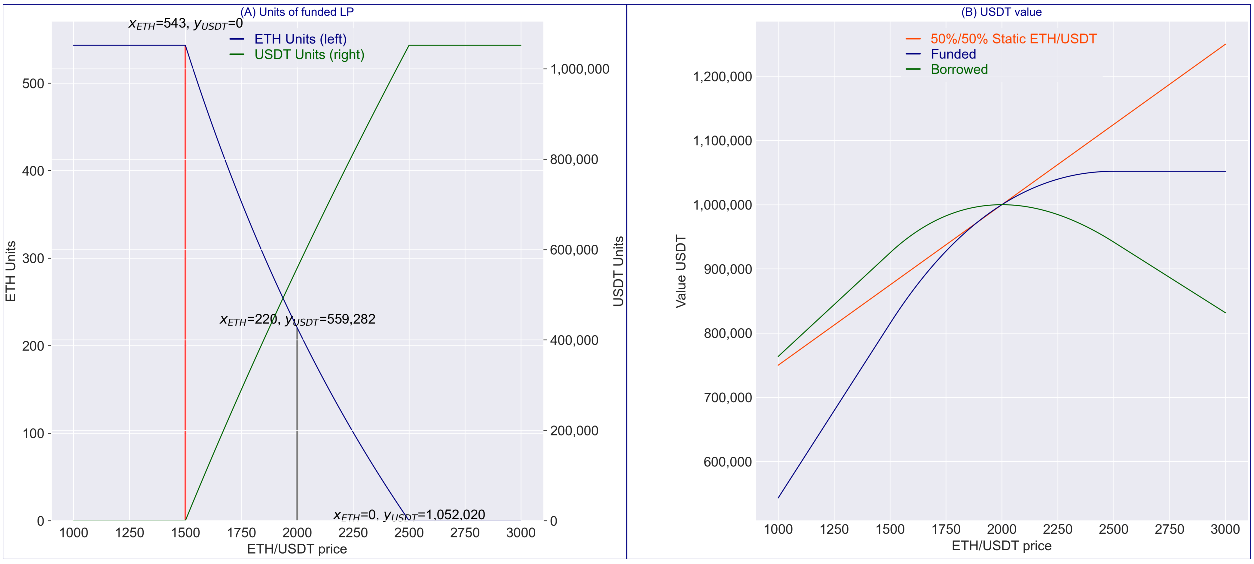

Figure 2: (A) ETH units (left y-axis) and USDT units (right y-axis) for LP on Uniswap V3. (B) USDT value of 50%/50% ETH/USDT portfolio, Funded LP position and Borrowed LP position. Uniswap V3 LP position is constructed using $1m$ USDT notional with $p_0 = 2000$, $p_a = 1500$, $p_b = 2500$.

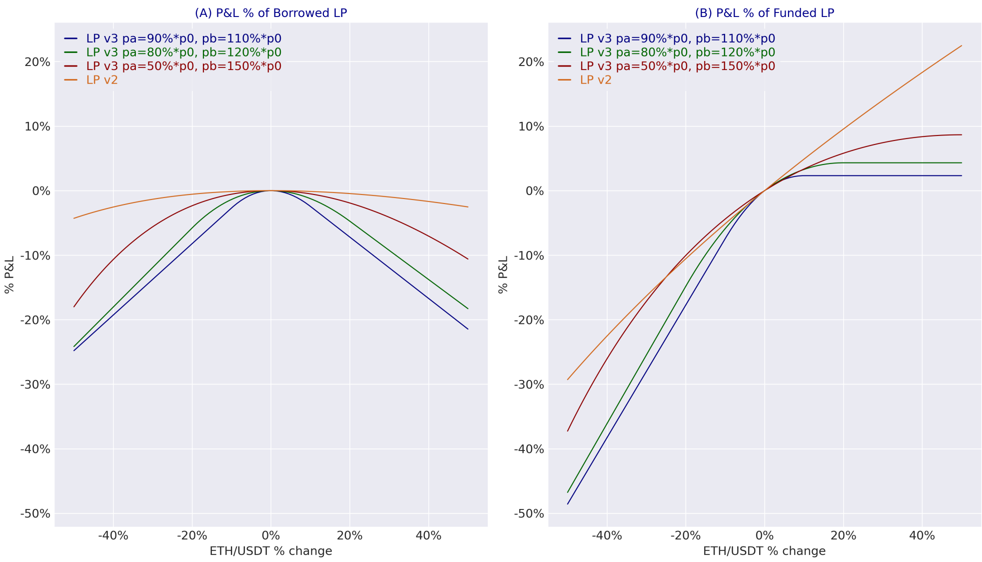

Figure 3: P&L % of hedged and funded LPs as functions of ranges for Uniswap V3 and full range for Uniswap V2. (A) P&L % of borrowed LP; (B) P&L % of funded LP. LP positions are constructed using $1m$ USDT notional with $p_0 = 2000$.

part defined as follows

$$\begin{aligned}
u_0(p_t) &= -\frac{1}{\sqrt{p_b}}p_t + \left(\frac{p_0}{\sqrt{p_b}} - 2\sqrt{p_0}\right)\\
u_{1/2}(p_t) &= \sqrt{p_t}\mathbb{1}\left\{p_a < p_t < p_b\right\}\\
u_1(p_t) &= -\frac{1}{\sqrt{p_a}}\max\left\{p_a - p_t, 0\right\} + \frac{1}{\sqrt{p_b}}\max\left\{p_t - p_b, 0\right\} + 2\sqrt{p_a}\mathbb{1}\left\{p_t \le p_a\right\} + 2\sqrt{p_b}\mathbb{1}\left\{p_t \ge p_b\right\}.
\end{aligned} \tag{49}$$

Proposition 3.6 (Decomposition of IL for Borrowed LP)**.** *IL of borrowed LP can be decomposed into the three parts as follows:*

$$\text{IL borrowed}^{(y)}(p_t) = u_0(p_t) + u_{1/2}(p_t) + u_1(p_t), \tag{50}$$

where $u_0(p_t)$, $u_{1/2}(p_t)$, and $u_1(p_t)$ are linear part, (exotic) square root price, and (vanilla) option part defined as follows

$$\begin{aligned}
u_0(p_t) &= -\frac{1}{\sqrt{p_0}}p_t - \sqrt{p_0}\\
u_{1/2}(p_t) &= \sqrt{p_t}\mathbb{1}\left\{p_a < p_t < p_b\right\}\\
u_1(p_t) &= -\frac{1}{\sqrt{p_a}}\max\left\{p_a - p_t, 0\right\} + \frac{1}{\sqrt{p_b}}\max\left\{p_t - p_b, 0\right\} + 2\sqrt{p_a}\mathbb{1}\left\{p_t \le p_a\right\} + 2\sqrt{p_b}\mathbb{1}\left\{p_t \ge p_b\right\}.
\end{aligned} \tag{51}$$

Proof. See Appendix 7.4. □

To summarise, we note that the only difference between the decomposition of IL of the funded LP in Eq. (48) and the borrowed LP in Eq. (50) is given by the linear term $u_0(p_t)$ with the square root and option terms being the same. As a result, we can unify our results for ILs of funded LP and borrowed LP as defined in Eqs.(8) and (9), respectively, for Uniswap V3 for the following.

Corollary 3.5 (Decomposition of IL for Funded and Borrowed LPs)**.** *Using Eq. (48) for IL of funded LP and Eq. (50) for IL of borrowed LP we obtain*

$$\begin{aligned}
\text{IL funded}^{(y)}(p_t) &= u_0^{funded}(p_t) + u_{1/2}(p_t) + u_1(p_t),\\
\text{IL borrowed}^{(y)}(p_t) &= u_0^{borrowed}(p_t) + u_{1/2}(p_t) + u_1(p_t),
\end{aligned} \tag{52}$$

where

$$\begin{aligned}
u_0^{funded}(p_t) &= -\frac{1}{\sqrt{p_b}}p_t + \left(\frac{p_0}{\sqrt{p_b}} - 2\sqrt{p_0}\right)\\
u_0^{borrowed}(p_t) &= -\frac{1}{\sqrt{p_0}}p_t - \sqrt{p_0}\\
u_{1/2}(p_t) &= \sqrt{p_t}\mathbb{1}\left\{p_a < p_t < p_b\right\}\\
u_1(p_t) &= -\frac{1}{\sqrt{p_a}}\max\left\{p_a - p_t, 0\right\} + \frac{1}{\sqrt{p_b}}\max\left\{p_t - p_b, 0\right\} + 2\sqrt{p_a}\mathbb{1}\left\{p_t \le p_a\right\} + 2\sqrt{p_b}\mathbb{1}\left\{p_t \ge p_b\right\}.
\end{aligned} \tag{53}$$

Corollary 3.6 (Payoff of IL protection claim for Uniswap V3 AMM)**.** *Using the definitions in Eq. (14) and Eq. (15) for pay-offs of IL protection claim of the funded and borrowed LP, respectively, at maturity time T along with the decomposition of IL in Eq. (52), we obtain*

$$\begin{aligned}
\text{Payoff funded}(p_T) &= -\left[u_0^{funded}(p_T) + u_{1/2}(p_T) + u_1(p_T)\right],\\
\text{Payoff borrowed}(p_T) &= -\left[u_0^{borrowed}(p_T) + u_{1/2}(p_T) + u_1(p_T)\right],
\end{aligned} \tag{54}$$

where functions u are defined in Eq. (53).

4 Static Replication of Impermanent Loss with Vanilla Options

We derive a static replication of the IL at fixed maturity time T using a portfolio of European call and put options. Consequently, the IL can be hedged by buying a portfolio of traded options.

We introduce pay-off functions of call and put options as follows[5]:

$$u^{call}(p,k) = (p-k)^{+},\ u^{put}(p,k) = (k-p)^{+}, \tag{55}$$

where k is the strike price k and p is the current price.

We note that Deng *et al.* (2023) and Maire-Wunsch (2024) obtain the following replication formula for the replication of funded P&L as defined in Eq. (4) and its analytic expression for Uniswap V3 defined in Eq. (41) (expressed using our notation)

$$\begin{aligned}\frac{1}{L}\ \text{P\&L funded}^{(y)}(p_t) + 1 = &-\frac{1}{4}\int_{p_a}^{p_b} k^{-3/2}\left(u^{put}(p_t,k) + u^{call}(p_t,k)\right)dk \\ &+ \frac{1}{2\sqrt{p_a}}\left(u^{call}(p_t,p_a) - u^{put}(p_t,p_a)\right) + \frac{1}{2\sqrt{p_b}}\left(u^{put}(p_t,p_a) - u^{call}(p_t,p_a)\right).\end{aligned} \tag{56}$$

Deng *et al.* (2023) derive the replication formula (56) using the following representation of IL under Uniswap V3

$$\begin{aligned}\frac{1}{L}\ \text{P\&L funded}^{(y)}(p_t) + 1 = &\ p_t\left(\frac{1}{\sqrt{p_t}} - \frac{1}{\sqrt{p_b}}\right)^{+} - p_t\left(\frac{1}{\sqrt{p_t}} - \frac{1}{\sqrt{p_a}}\right)^{+} \\ &+ \left(\sqrt{p_t} - \sqrt{p_a}\right)^{+} - \left(\sqrt{p_t} - \sqrt{p_b}\right)^{+}.\end{aligned} \tag{57}$$

where we apply our definition of nominal IL in Eq. (43) and Carr-Madan representation (Carr-Madan (2001)). We note that since the function on the left-hand side in (57) is not twice differentiable, strictly speaking, the Carr-Madan representation does not apply. For completeness, in Appendix 7.7 we provide a derivation that only relies on the existence of the generalised derivatives.

Decomposition in Eq. (57) includes four exotic pay-offs on the square root of the price. In contrast, we derive an alternative decomposition of IL of the funded LP in Eq. (48) which decomposes the IL into one exotic pay-off on the square root of the price, two pay-offs of vanilla call and put options, and two pay-offs of digital options. It is clear that the decomposition of the IL is not unique and can be done with a different base pay-off function. Our decomposition for IL of funded and borrowed LPs in Eq. (48) and (50), respectively, is most suitable for model-dependent valuation of IL protection claims.

We note that there are two complications with the replication formula. (56) for practical usage. First, formula (56) assumes that there are strikes corresponding to lower and upper levels p_a and p_b. In practice, on the Deribit exchange, the BTC and ETH options are traded with the strike width of 1000 and 50 USD, respectively. Therefore, the practical application of formula (56) is very limited because it only applicable to LPs with ranges contained in strikes of traded options. Second, the replication formula (56) requires one to buy both calls and puts at the same strikes. Thus, for strikes smaller than p_t, both out-the-money puts and in-the-money calls are purchased and, for strikes above p_t, both in-the-money puts and out-the-money calls are purchased. In practice on the Deribit exchange, the liquidity for in-the-money calls and puts is limited with much wider bid-ask spreads, so that implementation of formula (56) can be too cost-inefficient in reality.

Instead, we derive an alternative replication formula which only purchases out-out-the money puts for strikes below the current price and out-out-the money calls for strikes above the current price. We also note that our replication formula for a generic CFMM of AMM protocols in addition to CFMMs of Uniswap V2 or V3 protocols.

[5] We note that on crypto exchanges most options are the so-called inverse options with the pay-off paid in the underlying token. There is a direct arbitrage-based equivalence between vanilla and inverse options, see Alexander *et al.* (2023) and Lucic-Sepp (2024) for details, so that our analysis follows the same logic when using inverse options.

4.1 Replication of IL with Vanilla Options

We derive a replication portfolio for IL under a generic AMM protocol. We assume that IL $IL(p)$ is a function of the current price p and is specified by Eq. (4) for funded LP or by Eq. (7) for borrowed LP. In particular, the IL for funded and borrowed LPs in Uniswap V2 is obtained using Eq. (21) and (24), respectively. For Uniswap V3, the IL for funded and borrowed LPs is obtained using Eq. (43) and (46), respectively. The corresponding IL for all these specifications is denoted by $IL(p)$.

Proposition 4.1 (Replication portfolio for generic LP)**.** *We consider IL of a generic AMM as a function of price $IL(p)$. We fix the maturity time of the option by T and consider a set of call and put options traded for this maturity time.*

Put side*. We assume a discrete grid of strikes $\mathcal{K}^{put}$ and corresponding pay-offs of put options $\mathcal{U}^{put}$:*

$$\mathcal{K}^{put} = (k_1, k_2, ..., k_N)\,,\ \mathcal{U}^{put} = (u_1, u_2, ..., u_N)\,, \tag{58}$$

where $u_n = (k_n - p)^+$, $n = 1, .., N$, $k_{n-1} < k_n$ and $k_N \leq p_0$ with p_0 being T-forward price. We consider the replication portfolio of puts with weights w_n:

$$\Pi^{put} = \sum_{n=1}^{N} w_n u_n^{put}. \tag{59}$$

We define the first-order derivative of the IL function at discrete strike points as follows:

$$\delta IL(k_n) = \frac{IL(k_n) - IL(k_{n-1})}{k_n - k_{n-1}}. \tag{60}$$

Then the weights of put options for the replication portfolio in Eq. (59) are computed by:

$$w_{n-1} = -\left(\delta IL(k_n) - \delta IL(k_{n-1})\right), n = N, ..., 3 \tag{61}$$

with $w_N = \delta IL(k_N)$ and with $w_1 = 0$.

Call side*. We assume a discrete grid of strikes $\mathcal{K}$ and corresponding pay-offs of call options $\mathcal{U}^{call}$:*

$$\mathcal{K}^{call} = (k_1, k_2, ..., k_M)\,,\ \mathcal{U}^{call} = (u_1, u_2, ..., u_M)\,, \tag{62}$$

where $u_m = (p - k_m)^+$, $m = 1, .., M$, with $k_m > k_{m-1}$ and $k_1 \geq p_0$ being T-forward price.. We consider the hedging portfolio of calls with weights w_m:

$$\Pi^{call} = \sum_{m=1}^{M} w_m u_m^{call}. \tag{63}$$

We compute the first-order derivative of the IL function $\delta IL(k_m)$ as in Eq. (60).

Then the weights of call options for the replication portfolio in Eq. (63) are computed by:

$$w_m = -\left(\delta IL(k_m) - \delta IL(k_{m-1})\right), m = 2, ..., M - 1 \tag{64}$$

with $w_1 = \delta IL(k_1)$ and with $w_M = 0$.

Proof. See Appendix 7.5. □

Corollary 4.1 (Static portfolio for replication of the pay-off of IL Protection claim)**.** *The option replication portfolio for pay-off of IL protection claim is given by:*

$$\Pi \equiv \Pi^{put} + \Pi^{call} = \sum_{n=1}^{N} w_n U_n^{put} + \sum_{m=1}^{M} w_m U_m^{call}, \tag{65}$$

where the weights of put and call options are computed using Eq. (60) and Eq. (64), respectively, with the IL function $IL(p)$ specified using Eq. (28) and (54) for pay-offs under the Uniswap V2 and V3 protocols, respectively.

The cost of the replication portfolio is Π_0 computed using option prices observed at the inception of an IL protection claim. We use ask prices of options to calculate the initial cost Π_0. Option markets on Deribit are relatively liquid when traded nationals are below 1000 ETH, for less liquid options markets, we recommend to incorporate estimated slippage costs for computing IL hedging portfolio.

In Figure 4, we illustrate the application of formulas (59) and (63) for replicating of IL for borrowed Uniswap V3 LP using $1m$ USDT notional, $p_0 = 2000$ ETH/USDT with $p_a = 1500$ and $p_b = 2500$. We use strikes with widths of 50 USDT in alignment with ETH options traded on Deribit exchange (for options with maturity of less than 3 days, Deribit introduces new strikes with widths of 25). In subplot (A), we show the IL of the borrowed LP position and the pay-offs of replicating calls and puts portfolios (with negative signs to align with the P&L). In subplot (B), we show the residual computed as the difference between the IL and the pay-off of the replication portfolios. In Subplot (C), we show the number of put and call option contracts for the replication portfolios.

From Eq. (152) it is clear that the approximation error is zero at strikes in the grid, which is illustrated in subplot (B). The maximum value of the residual is 0.025% or 2.5 basis points, which is very small. A small approximation error with a similar magnitude will occur in case, p_0, p_a, p_b are not placed exactly at the strike grid.

We note that for ETH options on the Deribit exchange, the strike width of listed options with maturities up to one month is 100 USDT. For options with maturities up to one week and up to three days, Deribit adds additional strikes around at-the-money region with widths of 50 and 25 USDT, respectively. We notice that the magnitude of the residual error decreases when using narrow widths of strikes with 25. However, even for a wider width of 100, the residual error is less than 0.1% of the notional. For options on Bitcoin traded on Deribit, the strike width is varies between 500 for options close to the spot and 5000 for far out-of-the money strikes (where the absolute value of option delta is less than 0.1). We recommend setting the LP position in ranges where the deltas of the put and call options are less than -0.2 and greater than 0.2, respectively.

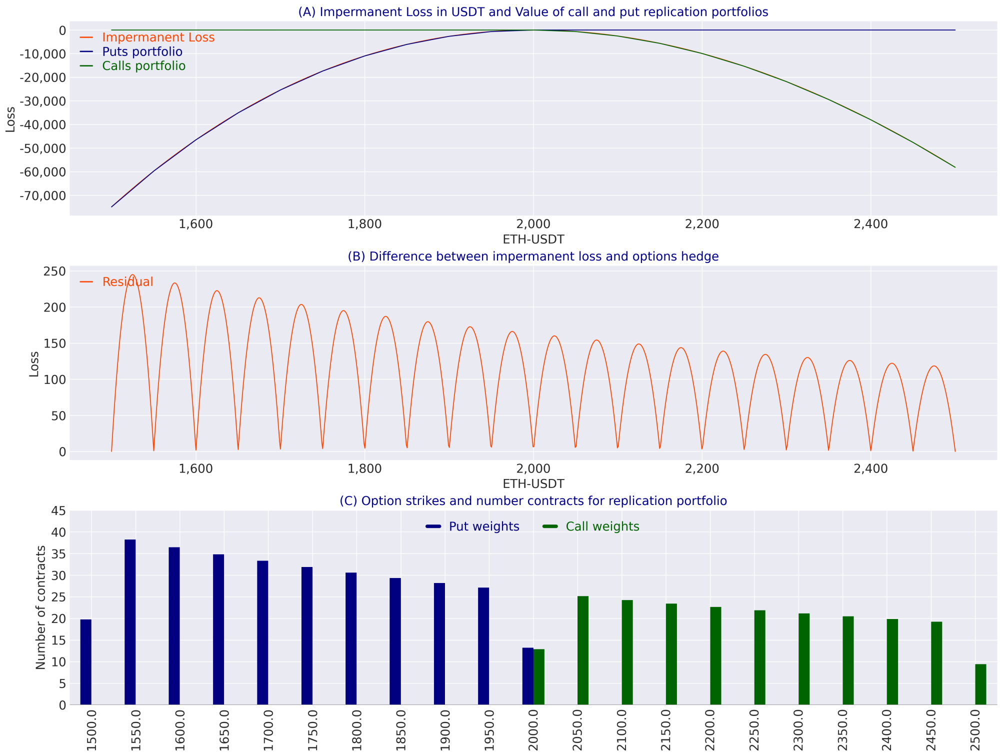

Figure 4: Residual hedging error (as % of LP notional) arising using options on the discrete strikes with the widths equal to 100, 50, and 25 arising from option-based replication of IL of borrowed Uniswap V3 LP for allocation of $1m$ USDT notional, $p_0 = 2000$ ETH/USDT with $p_a = 1500$ and $p_b = 2500$.

Figure 5: Replication of IL of borrowed Uniswap V3 LP for allocation of $1m$ USDT notional, $p_0 = 2000$ ETH/USDT with $p_a = 1500$ and $p_b = 2500$. (A) Impermanent loss in USDT and (negative) values of replicating puts and call portfolios; (B) Residual, which is the spread between IL and options replication portfolios; (C) Number of option contracts for put and calls portfolios.

5 Model-dependent Valuation of Protection Claims against IL

In this section, we develop the model-dependent valuation and dynamics hedging of IL protection claims for Uniswap V2 and V3 protocols.

5.1 Exponential Price Dynamics

We consider a continuous-time market with a fixed horizon date $T^* > 0$ and uncertainty modelled on the probability space $(\Omega, \mathbb{F}, \mathbb{P})$ equipped with filtration $\mathbb{F} = \{\mathcal{F}_t\}_{0 \le t \le T^*}$. We assume that $\mathbb{F}$ is right-continuous and satisfies the usual conditions.

We introduce exponential price dynamics for price p_t:

$$p_t = p_0 e^{x_t}, \ x_0 = 0, \tag{66}$$

where x_t is a stochastic process driving the log performance. We assume the existence of a risk-neutral measure $\mathbb{Q}$ such that[6]:

$$\mathbb{E}^{\mathbb{Q}}[p_T \mid \mathcal{F}_t] \equiv p_t \mathbb{E}^{\mathbb{Q}}[e^{x_T} \mid \mathcal{F}_t] = p_t e^{(r-q)(T-t)} \tag{67}$$

where r is the discount rate and q is the borrow rate.

For hedging an IL protection claim, a trader needs to sell short the underlying token. Short-selling can be readily executed either through a centralised exchange (CEX) using perpetual futures or through borrowing the token on a DeFi protocol using stablecoins as collateral. In the CEX case, the borrowing rate q is the negative of the funding rate reported by CEX (by the convention of crypto CEXes, the funding rate is the rate paid by traders with long positions). In the DeFi case, the borrowing rate q is the accrued borrow rate.

Regarding the discount rate r, most DEXes and CEXes (such as Deribit when marking their listed options) assume a zero discount rate. We would call r the low risk opportunity cost available in DeFi with the risk being a potential hack of blockchain technology when deposited and staked assets could be appropriated. Staking of high quality stablecoins in top DeFi protocols would yield $1\% - 2\%$ in the current environment, which is far less than rates on government short-term bonds in traditional markets ($4\% - 5\%$ as of June 2024).

When we value issued IL protection claims, we emphasize that the entry price p_0 is fixed at the time of initialising of an LP so that the entry price p_0 becomes a parameter of the IL formula in Eq. (50) along with the lower and upper ranges. Once the valuation time has advanced to time t and the new price is observed, we need to compute the expected IL using the price p_t. We introduce the following decomposition at time t for the total log-performance x_T

$$x_T = x_t + x_\tau, \tag{68}$$

where x_t is the realised log-performance over period $(0, t]$ and x_τ is the stochastic performance over the period $(t, T]$. Here τ, $\tau = T - t$, is the time to maturity. As a result, we model p_T using Eq. (66) with (68) as follows

$$p_T = p_0 e^{x_t + x_\tau}. \tag{69}$$

5.2 Model-dependent Valuation

We consider the valuation of claims for IL protection as defined in Eqs. (14) and (15). We focus on the valuation of these pay-offs for Uniswap V2 and V3 using the pay-off decomposition and by applying the exponential model in Eqs (66) and (69). Here, we fix the maturity time T.

[6]This assumption is valid for complete markets. For incomplete markets, e.g. dynamics including stochastic volatility or jumps, we fix a martingale measure using specific risk preferences (see for an example Lewis (2000)). Sepp-Rakhmonov (2023) consider the existence of equivalent risk-neutral measures for stochastic volatility models.

Corollary 5.1 (Representation of Payoff of IL protection claim in Uniswap V2 AMM under exponential model (66)). *Applying the exponential model in Eq.* (69) *to Eqs* (28)*, we obtain*

$$\begin{aligned} Payoff^{funded}(x_T) &= 1 - e^{\frac{1}{2}(x_t+x_\tau)}, \\ Payoff^{borrowed}(x_T) &= \frac{1}{2}\left(e^{\frac{1}{2}(x_t+x_\tau)} - 1\right)^2. \end{aligned} \tag{70}$$

Corollary 5.2 (Representation of Payoff of claim for IL protection for Uniswap V3 AMM under exponential model (66)). *Applying the exponential model in Eq.* (69) *to Eqs* (54)*, we obtain*

$$\begin{aligned} \text{Payofffunded}(x_\tau) &= -\left[u_0^{funded}(x_t+x_\tau) + u_{1/2}(x_t+x_\tau) + u_1(x_t+x_\tau)\right], \\ \text{Payoffborrowed}(x_\tau) &= -\left[u_0^{borrowed}(x_t+x_\tau) + u_{1/2}(x_t+x_\tau) + u_1(x_t+x_\tau)\right], \end{aligned} \tag{71}$$

where the linear part is computed by

$$\begin{aligned} u_0^{funded}(x_t+x_\tau) &= -\frac{p_0}{\sqrt{p_b}}e^{x_t+x_\tau} + \left(\frac{p_0}{\sqrt{p_b}} - 2\sqrt{p_0}\right) \\ u_0^{borrowed}(x_t+x_\tau) &= -\sqrt{p_0}\left(e^{x_t+x_\tau}+1\right), \end{aligned} \tag{72}$$

the square root part is computed by:

$$u_{1/2}(x_t+x_\tau) = \sqrt{p_0}\exp\left\{\frac{1}{2}(x_t+x_\tau)\right\}\mathbb{1}\left\{x_a < x_t+x_\tau < x_b\right\} \tag{73}$$

with $x_a = \ln(p_a/p_0)$ *and* $x_b = \ln(p_b/p_0)$*.*

The option part is computed by

$$\begin{aligned} u_1(x_t+x_\tau) =& \frac{1}{\sqrt{p_a}}\max\left\{p_a - p_0e^{x_t+x_\tau}, 0\right\} - \frac{1}{\sqrt{p_b}}\max\left\{p_0e^{x_t+x_\tau} - p_b, 0\right\} \\ &- 2\sqrt{p_a}\mathbb{1}\left\{x_t+x_\tau \le x_a\right\} - 2\sqrt{p_b}\mathbb{1}\left\{x_t+x_\tau \ge x_b\right\}. \end{aligned} \tag{74}$$

Given pay-off function in Eqs (70) and (71), the present value of the IL protection claim at time t is computed under the risk-neutral measure $\mathbb{Q}$ by:

$$PV(t, p_t; T) = e^{-r\tau}\mathbb{E}^{\mathbb{Q}}[Payoff(x_\tau) \mid \mathcal{F}_t]. \tag{75}$$

5.3 Valuation in Black-Scholes-Merton (BSM) model

We consider the BSM model with the price dynamics under the risk-neutral measure $\mathbb{Q}$ given by

$$dp_t = \mu p_t dt + \sigma p_t dw_t, \; p_0 = p, \tag{76}$$

where $\mu = r - q$ is the risk-neutral drift, w_t is a Brownian motion with $w_0 = 0$. Accordingly, the log-performance $x_t = \log p_t/p_0$ is driven by

$$dx_t = \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma dw_t, \; x_0 = 0, \tag{77}$$

and the distribution of x_τ is normal with mean $\left(\mu - \frac{1}{2}\sigma^2\right)\tau$ and volatility $\sigma\sqrt{\tau}$.

We note that the distribution of returns of cryptocurrencies deviates from normality (see, for an example, Chapter 10 in Lipton-Treccani (2021)). We propose to use BSM model as a simple quotation tool for marking the value of IL protection claim using log-normal volatility as the key input parameter. We also note that Alexander-Imeraj (2023) provide a comprehensive analysis that BSM model with correction for the implied volatility skew can be applied for delta-hedging of options on cryptocurrencies. Thus, we can also use BSM model with a corresponding adjustment for delta-hedging of IL protection claim.

Proposition 5.1 (BSM Value of IL protection claim under Uniswap V2)**.** *Applying model dynamics* (76) *to pay-off functions in Eq.* (70) *using valuation operator in Eq.* (75)*, we obtain*

$$
\begin{aligned}
PV^{funded}(t,p_t) &= e^{-r\tau}\left[1 - e^{\frac{1}{2}x_t} G\left(\tau; -\frac{1}{2}\right)\right], \\
PV^{borrowed}(t,p_t) &= \frac{1}{2}e^{-r\tau}\left[e^{(x_t+\mu\tau)} - 2e^{\frac{1}{2}x_t} G\left(\tau; -\frac{1}{2}\right) + 1\right],
\end{aligned}
\tag{78}
$$

where

$$
G\left(\tau; -\frac{1}{2}\right) = \exp\left\{\frac{1}{2}\left(\mu - \frac{1}{2}\sigma^2\right)\tau + \frac{1}{8}\sigma^2\tau\right\}. \tag{79}
$$

Proof. For the dynamics in Eq. (77), we apply the fact that the moment generation function $G(\tau;\phi)$ of x_τ is given by:

$$
G(\tau;\phi) \equiv \mathbb{E}^{\mathbb{Q}}[\exp\{-\phi x_\tau\} \mid \mathcal{F}_t] = \exp\left\{-\left(\mu - \frac{1}{2}\sigma^2\right)\phi\tau + \frac{1}{2}\phi^2\sigma^2\tau\right\}. \tag{80}
$$

□

Proposition 5.2 (BSM value of IL protection claim in Uniswap V3)**.** *The values of IL protection claims with pay-off functions in Eq.* (71) *under valuation operator in Eq. (75) and BSM dynamics* (77) *are given by*

$$
\begin{aligned}
PV^{funded}(t,p_t) &= -\left[U_0^{funded}(p_t) + U_{1/2}(p_t) + U_1(p_t)\right], \\
PV^{borrowed}(t,p_t) &= -\left[U_0^{borrowed}(p_t) + U_{1/2}(p_t) + U_1(p_t)\right],
\end{aligned}
\tag{81}
$$

where the linear part is computed by:

$$
\begin{aligned}
U_0^{funded}(t,p_t) &= e^{-q\tau}\frac{p_t}{\sqrt{p_b}} - e^{-r\tau}\left(\frac{p_0}{\sqrt{p_b}} - 2\sqrt{p_0}\right) \\
U_0^{borrowed}(t,p_t) &= \sqrt{p_0}\left(\frac{p_t}{p_0}e^{-q\tau} + e^{-r\tau}\right).
\end{aligned}
\tag{82}
$$

Here $U_{1/2}(p_t)$ is the BSM value of the square-root pay-off in Eq. (73) computed by:

$$
U_{1/2}(p_t) = 2e^{-r\tau}\sqrt{p_t}\exp\left\{\frac{1}{2}\mu\tau - \frac{1}{8}\sigma^2\tau\right\}\left(\mathbf{N}\left(\frac{\ln(p_b/p_t) - (r-q)\tau}{\sigma\sqrt{\tau}}\right) - \mathbf{N}\left(\frac{\ln(p_a/p_t) - (r-q)\tau}{\sigma\sqrt{\tau}}\right)\right), \tag{83}
$$

where $\mathbf{N}$ *is the cpdf of normal random variable.*

The option part is computed by

$$
\begin{aligned}
U_1(p_t) = &\frac{1}{\sqrt{p_a}}O^{BSM}(p_t;p_a,-1) - \frac{1}{\sqrt{p_b}}O^{BSM}(p_t;p_b,+1) \\
&- 2\sqrt{p_a}D^{BSM}(p_t;p_a,-1) - 2\sqrt{p_b}D^{BSM}(p_t;p_b,+1).
\end{aligned}
\tag{84}
$$

Here $O^{BSM}(p_t;k,+1)$ and $O^{BSM}(p_t;k,-1)$ are the BSM prices of vanilla call and put with the strike price k and indicator $\omega \in \{+1,-1\}$ respectively:

$$
O^{BSM}(p_t;k,\omega) = e^{-q\tau}p_t\mathbf{N}(\omega d_+(p_t,k)) - ke^{-r\tau}\mathbf{N}(\omega d_-(p_t,k)), \tag{85}
$$

where

$$
d_\pm(p_t,k) = \frac{\ln(p_t/k) + (r-q)\tau}{\sigma\sqrt{\tau}} \pm \frac{1}{2}\sigma\sqrt{\tau}, \tag{86}
$$

and $D^{BSM}(p_t;k,+1)$ and $D^{BSM}(p_t;k,-1)$ are digital call and put options with strike price k, respectively:

$$
D^{BSM}(p_t;k,\omega) = e^{-r\tau}\mathbf{N}(\omega d_-(p_t,k)). \tag{87}
$$

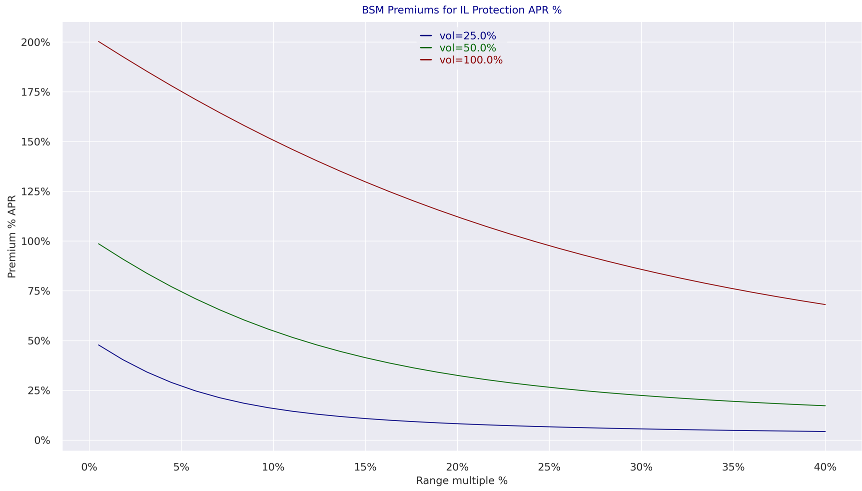

Figure 6: BSM premium annualised $(U^{borrower}(t, p_t)/T)$ for borrowed LP computed using Eq. (81) with time to maturity of two weeks $T = 14/365$ and notional of 1 USDT as function of the range multiple m such that $p_a(m) = e^{-m}p_0$ and $p_b(m) = e^m p_0$.

Proof. See Appendix 7.6. □

In Figure 6, we show the annualised cost (APR) % for the cost of BSM hedge for the borrowed LP as a function of the range multiple m such that $p_a(m) = e^{-m}p_0$ and $p_b(m) = e^m p_0$. We use two weeks to maturity $T = 14/365$ and different values of log-normal volatility σ. All else being equal, it is more expensive to hedge narrow ranges.

The protection seller must hedge the claim dynamically using option delta. As a result, we consider the option delta under BSM model.

Corollary 5.3 (The delta of IL protection claim in BSM model)**.**

$$\begin{aligned}
\Delta^{funded}(t, p_t) &= -\left[\partial_p U_0^{funded}(p_t) + \partial_p U_{1/2}(p_t) + \partial_p U_1(p_t)\right], \\
\Delta^{borrowed}(t, p_t) &= -\left[\partial_p U_0^{borrowed}(p_t) + \partial_p U_{1/2}(p_t) + \partial_p U_1(p_t)\right],
\end{aligned} \tag{88}$$

where

$$\partial_p U^{funded}(t, p_t) = e^{-q\tau}\frac{1}{\sqrt{p_b}}, \quad \partial_p U^{borrowed}(t, p_t) = e^{-q\tau}\frac{1}{\sqrt{p_0}}, \tag{89}$$

and

$$\begin{aligned}
\partial_p U_{1/2}(p_t) = \frac{1}{2\sqrt{p_t}} \exp\left\{\frac{1}{2}\mu\tau - \frac{1}{8}\sigma^2\tau\right\} \left(\mathbf{N}\left(\frac{\ln(p_b/p_t) - (r-q)\tau}{\sigma\sqrt{\tau}}\right) - \mathbf{N}\left(\frac{\ln(p_a/p_t) - (r-q)\tau}{\sigma\sqrt{\tau}}\right)\right) \\
- \frac{1}{\sigma\sqrt{\tau}\sqrt{p_t}} \exp\left\{\frac{1}{2}\mu\tau - \frac{1}{8}\sigma^2\tau\right\} \left(\mathbf{n}\left(\frac{\ln(p_b/p_t) - (r-q)\tau}{\sigma\sqrt{\tau}}\right) - \mathbf{n}\left(\frac{\ln(p_a/p_t) - (r-q)\tau}{\sigma\sqrt{\tau}}\right)\right).
\end{aligned} \tag{90}$$

Finally

$$\begin{aligned}
\partial_p U_1(p_t) = \frac{1}{\sqrt{p_a}}\Delta_O(p_t; p_a, -1) - \frac{1}{\sqrt{p_b}}\Delta_O(p_t; p_b, +1) \\
- 2\sqrt{p_a}\Delta_D(p_t; p_a, -1) - 2\sqrt{p_b}\Delta_{DG}(p_t; p_b, +1)
\end{aligned} \tag{91}$$

where $\Delta_O(p_t; k, +1)$ *and* $\Delta_D(p_t; k, -1)$ *are Black-Scholes-Merton deltas of vanilla option and digital option, respectively, given by:*

$$\Delta_O(p_t; k, \omega) = \mathbf{N}(\omega d_1), \quad \Delta_D(p_t; k, \omega) = \frac{\omega e^{-r\tau}}{p_t\sigma\sqrt{\tau}}\mathbf{n}(d_2). \tag{92}$$

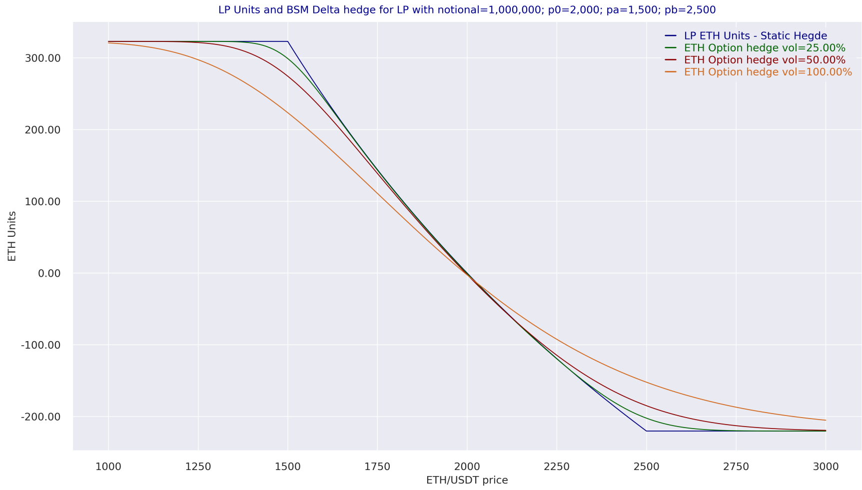

Figure 7: BSM delta for borrowed LP computed using Eq. (88) with time to maturity of two weeks $T = 14/365$ for Uniswap V3 LP with $p_0 = 2000$, $p_a = 1500$, $p_b = 2500$, and notional of 1000000 USDT.

Proof. Taking the partial derivative wrt p in Eq. (81). □

In Figure 7, we show the BSM delta for the borrowed LP calculated using Eq. (88) with time to maturity of two weeks $T = 14/365$ for borrowed LP in Uniswap V3 with $p_0 = 2000$, $p_a = 1500$, $p_b = 2500$, and notional of 1000000 USD. The initial units in the LP is $(220.36, 559282.18)$ units of ETH and USDT, respectively. The static hedge is constructed by shorting 220.36 units of ETH. The first line labelled LP ETH units corresponds to the excess ETH units of the borrowed LP with zero units at $p = p_0$ and being underhedged on the downside and overhedged on the upside. ETH option hedge shows the BSM delta computed using Eq. (88) (the hedge for borrower LP is implemented using the negative sign of BSM delta). For high volatilities or large maturity times, BSM delta under-hedges near the range.

5.4 Valuation using Moment Generating Function

We consider a wide class of Markovian exponential dynamics in Eq. (66) for which the moment generating function (MGF) for log-return x_τ is available in closed form. The closed-form solution for the MGF is available under a wide class of models including jump-diffusions and diffusions with stochastic volatility. Thus, we can develop an analytic solution for model-dependent valuation of IL protection claims under various models with analytic MGF.

In particular, Matic *et al.* (2023) analyse the calibration and hedging of cryptocurrency options under the Heston stochastic volatility model with simultaneous jumps in returns and variance processes and find strong evidence for the presence of stochastic volatility and jumps in the dynamics of implied volatilities of cryptocurrencies. Our MGF-based solution allows to evaluate and hedge IL protection claims under such models with stochastic volatility and jumps.

We denote the MGF by $G(\tau; \phi)$, where ϕ is a complex-valued transform variable such that $\phi = \phi_r + i\phi_i$, $i = \sqrt{-1}$. Formally, the MGF solves the following problem:

$$G(\tau; \phi) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\phi x_\tau} \mid \mathcal{F}_t\right], \tag{93}$$

where the expectation is computed using model dynamics under the risk-neutral measure $\mathbb{Q}$ and $\mathcal{F}_t$ is information set.

Using the MGF, the density of $x' = x_\tau$ denoted by $P(\tau, x; x')$ is computed by the Fourier inversion

$$P(\tau, x; x') = \frac{1}{\pi}\Re\left[\int_0^\infty \exp\left\{\phi x'\right\} G(\tau; \phi)\, d\phi\right] \equiv \frac{1}{\pi}\Re\left[\int_0^\infty \exp\left\{\phi(x' - x)\right\} E(\tau; \phi)\, d\phi\right], \tag{94}$$

with $d\phi \equiv d\phi_i$ and $G(\tau;\phi) \equiv e^{x\phi}E(\tau;\phi)$.

For the valuation of the IL protection claim using MGF $G(\tau;\phi)$, we need to evaluate the vanilla and digital options, and the square root pay-off in Eq. (71). First, we derive a generic valuation method for pay-off $u(x_\tau)$. We evaluate the present value $U(\tau, x_t)$ at time t of the pay-off function $u(x_\tau)$ which is given similarly to Eq. (75) by

$$U(\tau, x_t) = e^{-r\tau}\mathbb{E}^{\mathbb{Q}}[u(x_\tau) \mid \mathcal{F}_t]. \tag{95}$$

Using Eq.(94) we obtain

$$\begin{aligned} U(\tau, x) &= e^{-r\tau}\int_{-\infty}^{\infty} u(x')P(\tau, x; x')dx' \\ &= e^{-r\tau}\Re\left[\int_{-\infty}^{\infty} u(x')\left[\frac{1}{\pi}\int_0^{\infty} e^{\phi(x'-x)}E(\tau,\phi)d\phi\right]dx'\right] \\ &= \frac{1}{\pi}e^{-r\tau}\Re\left[\int_0^{\infty}\widehat{u}(\phi)E(\tau,\phi)d\phi\right], \end{aligned} \tag{96}$$

where we assume that the inner integrals are finite to exchange the order of the integration. Here $\widehat{u}(\phi)$ is the transformed pay-off function defined by

$$\widehat{u}(\phi) = e^{-\phi x}\int_{-\infty}^{\infty} e^{\phi x'}u(x')dx'. \tag{97}$$

Proposition 5.3 (Value of IL protection claim in Uniswap V2 under exponential model dynamics (76) using the MGF)**.** *Applying dynamics (76) to pay-off functions in Eq. (70) under expectation operator (75), we obtain*

$$\begin{aligned} PV^{funded}(t, p_t) &= e^{-r\tau}\left[1 - e^{\frac{1}{2}x_t}G\left(\tau; -\frac{1}{2}\right)\right], \\ PV^{borrowed}(t, p_t) &= \frac{1}{2}e^{-r\tau}\left[e^{(x_t+\mu\tau)} - 2e^{\frac{1}{2}x_t}G\left(\tau, -\frac{1}{2}\right) + 1\right]. \end{aligned} \tag{98}$$

Proof. We apply the definition of the MGF in Eq. (93) and the martingale condition in Eq. (67). ∎

Accordingly, the IL under Uniswap V2 can be solved analytically for a wide class of models which is first concluded in Lipton (2024).

Proposition 5.4 (Valuation of IL protection claim in Uniswap V3 using MGF)**.** *Given the MGF $G(\tau;\phi)$ defined in Eq. (94) for log-price x_τ in the exponential model in Eq. (66) and pay-off functions in Eq. (71), we obtain the following valuation formula*

$$\begin{aligned} U^{funded}(t, p_t) &= -\left[U_0^{funded}(p_t) + U_{1/2}(p_t) + U_1(p_t)\right], \\ U^{borrowed}(t, p_t) &= -\left[U_0^{borrowed}(p_t) + U_{1/2}(p_t) + U_1(p_t)\right], \end{aligned} \tag{99}$$

where $U_0^{funded}(t, p_t)$ and $U_0^{borrowed}(t, p_t)$ are model independent linear parts computed as in BSM model in Eq. (82)

$$\begin{aligned} U_0^{funded}(t, p_t) &= e^{-q\tau}\frac{p_t}{\sqrt{p_b}} - e^{-r\tau}\left(\frac{p_0}{\sqrt{p_b}} - 2\sqrt{p_0}\right) \\ U_0^{borrowed}(t, p_t) &= \sqrt{p_0}\left(\frac{p_t}{p_0}e^{-q\tau} + e^{-r\tau}\right). \end{aligned} \tag{100}$$

$U_{1/2}(p_t)$ is the square root claim computed using valuation formula (96) with transform in Eq. (113).

The option part is computed by

$$\begin{aligned} u_1(p_t) =& \frac{1}{\sqrt{p_a}} O(p_t; p_a, -1) - \frac{1}{\sqrt{p_b}} O(p_t; p_b, +1) \\ & - 2\sqrt{p_a} D(p_t; p_a, -1) - 2\sqrt{p_b} D(p_t; p_b, +1), \end{aligned} \tag{101}$$

where $O(p_t; k, +1)$ and $O(p_t, k, -1)$ are model values of call and put options, respectively, computed in Eq. (103); $D(p_\tau; k, +1)$ and $D(p_\tau; k, -1)$ are model values of digital options computed in Eq. (107).

We note that for the computation of the pay-off transform $\widehat{u}(\phi)$ of capped option in Eq. (106) we use the transform variable as $\phi = iy - 1/2$, where $y \in (-\infty, +\infty)$. Then for call and put digitals, we use Eq. (108) and (110) also set ϕ as $\phi = iy - 1/2$. Finally, the transform of the square root pay-off in Eq. (113) can be also evaluated using ϕ set by $\phi = iy - 1/2$. As a result, for the numerical implementation we fix the grid of real-valued transform variable $\{y\}$ and set $\phi = iy - 1/2$. We then compute MGF $E(\tau; \phi)$ and Fourier transforms of 5 pay-off functions defined with the transform variable $\phi = iy - 1/2$, and compute 5 option values using Eq. (96). Accordingly, the numerical implementation of the pricing formula in Eq. (99) is efficient.

Proof. **Vanilla call and put options**

We represent the put and call pay-off functions using capped pay-offs as

$$\begin{aligned} c(p_\tau, k) &= \max\{e^{x_\tau} - k, 0\} = p_\tau - \min\{e^{x_\tau}, k\}, \\ p(p_\tau, k) &= \max\{k - e^{x_\tau}, 0\} = k - \min\{e^{x_\tau}, k\}. \end{aligned} \tag{102}$$

Accordingly, we need to evaluate option on the capped pay-off using Eq. (96) so that we can value vanilla calls and puts using

$$\begin{aligned} O(p_t; k, +1) &= e^{-q\tau} p_t - U(p_t, k) \\ O(p_t; k, -1) &= e^{-r\tau} k - U(p_t, k), \end{aligned} \tag{103}$$

where $U(p_t; k, \omega)$ is the value of the capped pay-off.

Applying Eq. (97) for capped pay-off $u(x') = \min\left\{e^{x'}, k\right\}$, we obtain

$$\widehat{u}(\phi) = e^{-\phi x}\left(\frac{e^{(\phi+1)k^*}}{\phi+1} - e^{k^*}\frac{1}{\Phi}e^{\phi k^*}\right) = e^{-\phi x}\left(-e^{(\phi+1)k^*}\frac{1}{(\phi+1)\phi}\right) = -ke^{-\phi x^*}\frac{1}{(\phi+1)\phi}, \tag{104}$$

where $x^* = \ln(p_t/k) + \mu\tau$ is the log-moneyness, $k^* = \ln k - \mu\tau$ with the first integral being finite for $\Re[\phi] > -1$ and the second integral being finite for $\Re[\phi] < 0$. The integral (104) is finite for $-1 < \phi_r < 0$. Setting $\phi = iy - 1/2$, we derive:

$$\widehat{u}(\phi = iy - 1/2) = -ke^{-(iy-1/2)x^*}\frac{1}{(1/2 + iy)(-1/2 + iy)} = ke^{-(iy-1/2)x^*}\frac{1}{y^2 + 1/4}. \tag{105}$$

Finally we obtain the valuation formula for capped pay-off known as Lipton-Lewis formula (Lipton (2001), Lewis (2000)) as follows

$$U(p_t, k) = \frac{ke^{-r\tau}}{\pi}\Re\left[\int_0^\infty e^{-(iy-1/2)x^*}\frac{1}{y^2 + 1/4}E(\tau; \phi = iy - 1/2)dy\right], \tag{106}$$

where $x^* = \ln(p_t/k) + \mu\tau$ is log-moneyness.

Digital options

We represent the value of digital calls and puts as follows

$$\begin{aligned} D(p_\tau, k = x_b, +1) &= \mathbb{1}\left\{x_\tau \geq k\right\} = 1 - \mathbb{1}\left\{x_\tau < k\right\}, \\ D(p_\tau, k = x_a, -1) &= \mathbb{1}\left\{x_\tau \leq k\right\} = 1 - \mathbb{1}\left\{x_\tau > k\right\}. \end{aligned} \tag{107}$$

We compute the transform of the pay-off function in (96) for digital call as follows

$$\widehat{u}^c(\phi) = e^{-\phi x}\int_{-\infty}^{\infty} e^{\phi x'}U^c(x')dx' = e^{-\phi x}\int_{x_b}^{\infty}\exp\left\{\phi x'\right\}dx' = -e^{-\phi x}\frac{1}{\phi}\exp\left\{\phi x_b\right\}, \tag{108}$$

where the integral converges if $\phi_r < 0$.

We compute the transform of the pay-off function in (96) for digital put as follows

$$\widehat{u}^p(\phi) = e^{-\phi x}\int_{-\infty}^{\infty} e^{\phi x'}U^p(x')dx' = e^{-\phi x}\int_{-\infty}^{x_a}\exp\left\{\phi x'\right\}dx' = e^{-\phi x}\frac{1}{\phi}\exp\left\{\phi x_a\right\}, \tag{109}$$

where the integral converges if $\phi_r > 0$.

We note that using (107), we can evaluate digital put as

$$D(p_\tau, k = x_a, \omega = -1) = 1 - D(p_\tau, k = x_a, \omega = +1), \tag{110}$$

so that we can use call transform in Eq. (108) to evaluate both call and put digital with $\phi_r < 0$.

Square root pay-off

We evaluate the value function in Eq. (95) corresponding to the square root pay-off in Eq. (73) as follows

$$U_{1/2}(\tau, p_t) = e^{-r\tau}\sqrt{p_t}\mathbb{E}\left[u(x)\right], \tag{111}$$

where

$$u(x) = \exp\left\{\frac{1}{2}x\right\}\mathbb{1}\left\{x_a < x_\tau < x_b\right\} \tag{112}$$

and $x_a = \ln(p_a/p_t)$ and $x_b = \ln(p_b/p_t)$.

We compute the transform of the pay-off function in (96) as follows

$$\begin{aligned} \widehat{u}(\phi) &= e^{-\phi x}\int_{-\infty}^{\infty} e^{\phi x'}u(x')dx' \\ &= e^{-\phi x}\int_{x_a}^{x_b}\exp\left\{\left(\phi + \frac{1}{2}\right)x'\right\}dx' \\ &= e^{-\phi x}\frac{1}{(\phi + \frac{1}{2})}\left[\exp\left\{\left(\phi + \frac{1}{2}\right)x_b\right\} - \exp\left\{\left(\phi + \frac{1}{2}\right)x_a\right\}\right], \end{aligned} \tag{113}$$

where the integral converges for $\phi_r \in \mathbf{R}$. For $x_b \to \infty$ we set $\phi_r < 0$, and for $x_a \to -\infty$ we set $\phi_r > 0$.

□

5.5 Application of Log-normal SV Model

We apply the log-normal SV model which can handle positive correlation between returns and volatility observed in price-volatility dynamics of digital assets (see Sepp-Rakhmonov (2023) for details). We consider price dynamics under the risk-neutral measure $\mathbb{Q}$ for the spot price S_t and the instantaneous volatility σ_t as follows

$$\begin{aligned} dS_t &= r(t)S_t dt + \sigma_t S_t dw_t^{(0)},\ S_0 = S, \\ d\sigma_t &= \left(\kappa_1 + \kappa_2\sigma_t\right)\left(\theta - \sigma_t\right)dt + \beta\sigma_t dw_t^{(0)} + \varepsilon\sigma_t dw_t^{(1)},\ \sigma_0 = \sigma, \end{aligned} \tag{114}$$

where $w^{(0)}$, $w^{(1)}$ are uncorrelated Brownian motions, $\kappa_1 > 0$ and $\kappa_2 \geq 0$ are linear and quadratic mean-reversion rates respectively, $\theta > 0$ is the mean of the volatility, $\beta \in \mathbb{R}$ is the volatility beta which measures the sensitivity of the volatility to changes in the spot price, and $\varepsilon > 0$ is the volatility of residual volatility.

Sepp-Rakhmonov (2023) find the first-order solution to MGF defined in Eq. (93) is given as follows

$$G(\tau;\phi) = \exp\{-\phi x\} E^{[1]}(\tau,\phi), \tag{115}$$

where $E^{[1]}$ is the exponential-affine function

$$E^{[1]}(\tau,\phi) = \exp\left\{\sum_{k=0}^{2} A^{(k)}(\tau;\phi)(\sigma-\theta)^k\right\}, \tag{116}$$

where $\vartheta^2 = \beta^2 + \varepsilon^2$ and vector function $\mathbf{A}(\tau) = \left\{A^{(k)}(\tau,\Phi)\right\}$, $k = 0,1,2$, solve the quadratic differential system as a function of τ:

$$A_\tau^{(k)} = \mathbf{A}^\top M^{(k)} \mathbf{A} + \left(L^{(k)}\right)^\top \mathbf{A} + H^{(k)}, \tag{117}$$

$$\begin{aligned}
M^{(k)} &= \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{\theta^2\vartheta^2}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \theta\vartheta^2 & \theta^2\vartheta^2 \\ 0 & \theta^2\vartheta^2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{\vartheta^2}{2} & 2\theta\vartheta^2 \\ 0 & 2\theta\vartheta^2 & 2\theta^2\vartheta^2 \end{pmatrix} \right\}, \\
L^{(k)} &= \left\{ \begin{pmatrix} 0 \\ -\theta^2\beta\phi \\ \theta^2\vartheta^2 \end{pmatrix}, \begin{pmatrix} 0 \\ -(\kappa_1+\kappa_2\theta) - 2\theta\beta\phi \\ 2(\theta\vartheta^2 - \theta^2\beta\phi) \end{pmatrix}, \begin{pmatrix} 0 \\ -\beta\phi - \kappa_2 \\ \vartheta - 2(\kappa_1+\kappa_2\theta) - 4\theta\beta\phi \end{pmatrix} \right\}, \\
H^{(k)} &= \left\{ \frac{1}{2}\theta^2\left(\phi^2+\phi\right),\ \theta\left(\phi^2+\phi-2\psi\right),\ \frac{1}{2}\left(\phi^2+\phi\right) \right\},
\end{aligned}$$

with the initial condition $\mathbf{A}(0) = (0,\ 0,\ 0)^\top$. The second-order solution is provided in Theorem 4.6. Sepp-Rakhmonov (2023).

In Subplot (A) of Figure 8[7], we show the implied volatilities of the log-normal SV model for a range of volatility of residual volatility ε with $\beta = 0$. In Subplot (B), we show the premium APR for IL protection as a function of the multiple range for a range of ε. We see that the model-value of IL protection is is not very sensitive to tails of implied distribution (or, equivalently, to the convexity of the implied volatility). The reason is that the most of the value of IL protection is derived from the center of returns distribution.

6 Practical Considerations

Hereby we will discuss our experience of implementing the proposed approach for hedging impermanent loss in practice and address some frequently raised questions related to the process.

Funded vs borrowed LP. The funded position is typically chosen by an LP provider with existing holdings of cryptocurrencies. In this case, the funded position is seen as a way to generate yield. The LP provider of the funded position buys the protection to hedge the downside risk. Borrowed LP is typically implemented by a LP provider seeking absolute return, in which case the LP provider borrows an amount of cryptocurrency to stake into the LP and buys the LP protection to hedge against both downside and upside potentials.

Hedging and short-selling. The borrowed LP typically required the two transactions. First, the LP provider purchases the necessary amount of volatile cryptocurrency (Bitcoin, ETH, etc.)

[7] Github project `https://github.com/ArturSepp/StochVolModels` provides Python code for this computations using the log-normal SV model.

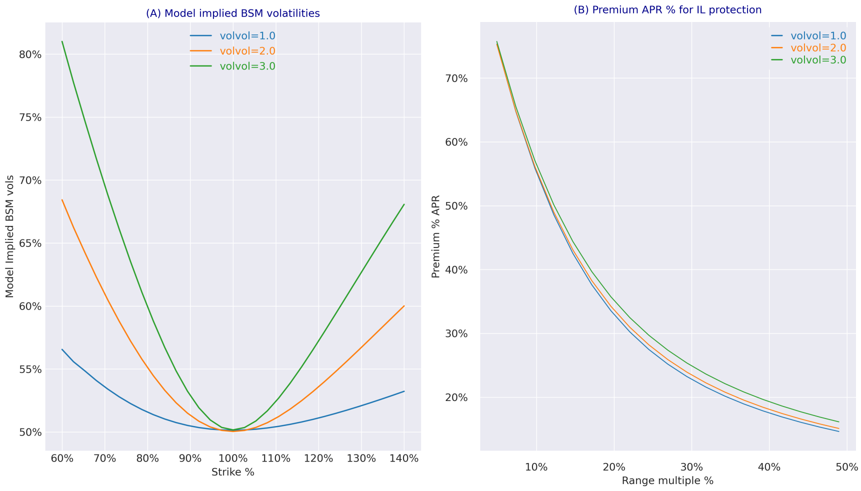

Figure 8: (A) BSM volatilities implied by log-normal SV model as function of volatility-of-volatility parameter ε for $\tau = 14/365$; (B) Premiums APR computed using log-normal SV model for borrowed LP as function of the range multiple m such that $p_a(m) = e^{-m}p_0$ and $p_b(m) = e^{m}p_0$. Other model parameters include $\sigma_0 = \theta = 0.50$, $\kappa_1 = 2.21$, $\kappa_2 = 2.18$, $\beta = 0.0$.

on-chain[8] and simultaneously sells short the same amount through perpetual or termed futures in a centralised crypto exchange (short-selling of most altcoins in moderate amounts can also be implemented in decentralised exchanges). We recommend that the capital efficiency is about 1.3 meaning that for the 1.0 ETH staked, the LP provider funds the margin account for the short future position by 0.3 ETH. Second, the LP provider buys the necessary number of protection options. It is efficient to do delta-hedging with option buying on the same exchange (such as Deribit) because the portfolio margin of the combined option and futures positions is smaller.

Hedging using lending and borrowing protocols. The LP provider can also obtain a short position by using lending protocols. The typical loan-to-value (LTV) LTV is 0.75, which means that the LP provider must commit the amount if stablecoins worth 1.3 ETH to borrow 1.0 ETH. Thus, the capital efficiency is about the same when short-selling through a centralised exchange. In our experience, borrowing altcoins from a lending platform provides less capacity and it is subject to higher liquidation risk.

Latency risk. LP staking and hedging required a few transactions for the borrowed LP including (1) buying underlying cryptocurrency on-chain, (2) hedging the exposure by short-selling term futures on a centralised exchange, (3) staking the cryptocurrency to a DEX on-chain, and (4) buying the option protection on a centralised exchange. In our experience, these steps cannot be fully automated because of the exposure both on-chain and off-chain protocols, different validation and execution times, and very fragmented liquidity of both centralised and decentralised exchanges for relatively modest staking amounts of (any position worth above 100 ETH can be considered as large in the current environment). Consequently, the latency risk for LP staking is present and must be addressed with a robust operational framework.

BTC and ETH vs altcoins. We note that the option-based hedging of IL using vanilla options is currently only possible for BTC and ETH cryptocurrencies. In addition, Deribit exchange currently lists options on XRP, BNB, and SOL, however the liquidity of these options is poor. The IL protection for (top 50) liquid altcoins can be bought from a handful of designated market-makers over-the-counter (OTC) who offer such products.

Overall profitability. Purchasing options is expensive in the long-term, so the LP provider must estimate the volume of a liquidity pool and the contribution of provided liquidity to the

[8]For large amount, it might be cost-efficient to purchase coining on a centralised exchange and to do a transfer to an onchain wallet.

liquidity profile of the pool. Ultimately, this decision comes down to the analysis of the risk-reward problem. If the liquidity provider regularly executes LP staking with option-based hedging, the realised P&L is driven by the difference between realised staking fees and premiums paid for purchasing options for LP IL hedging. We leave a quantitative approach and an empirical analysis of this topic for future research.

7 Conclusions

We have developed a unified approach for hedging of Impermanent Loss (IL) which arises when providing liquidity to Automated Market Making (AMM) pools in blockchain ecosystem. We have introduced the two ways to create a liquidity provision (LP) that includes a funded LP (with the long initial exposure to the underlying token) and a borrowed LP (with the zero initial exposure to the underlying token). We have applied Uniswap V2 and V3 protocols, which are the constant function market maker (CFMM) most commonly employed by most decentralised exchanges. We have shown that the IL can be represented with a non-linear function of the current spot price. As a result, using traditional methods of financial engineering, we can handle the valuation and risk management of the IL protection claim which delivers the negative of the IL at a fixed maturity date.

First, we have derived a static replication approach for the IL arising from a generic constant function market maker (CFMM) using a portfolio of traded call and put options at a fixed maturity date. This approach allows for model-free replication of the IL when there is a liquid option market, which is the case for core digital assets, including Bitcoin and Ethereum.

Second, for digital assets without a liquid options market, we have developed a model-based approach using the decomposition of the IL function into vanilla options, digital options, and an exotic square root pay-off. We have derived a closed-form valuation formula for a wide class of price dynamics with tractable characteristic and moment generating functions (MGF) by means of Fourier transform.

Model-based valuation can be employed by a few crypto-trading companies that currently sell over-the-counter IL protection claims. When using model-based dynamics delta-hedging for the replication of the pay-off of the IL protection claim, the profit-and-loss (P&L) of the dynamic delta-hedging strategy will be primarily driven by the realised variance of the price process. Thus, the total P&L of a trading desk will be the difference between the premiums received (from selling IL protection claims) and the variance realised through delta hedging. The trading desk can employ our results for the analysis of price dynamics and hedging strategies that optimise their total P&L.

For liquidity providers who buy IL protection claims for their LP position, the total P&L will be driven by the difference between the accrued fees from LP positions and the costs of IL protection claims. The cost of the IL protection claim can be estimated beforehand using either the cost of static options replicating portfolio or costs of buying IL protection from a trading desk. As a result, liquidity providers can focus on selecting DEX pools and liquidity ranges where expected fees could exceed hedging costs. Thus, liquidity providers can apply our analysis optimal allocation to LP pools and to create static replication portfolios using traded options or assessing costs quoted by providers of IL protection.

We leave the application of our model-free and model-dependent results for an optimal liquidity provision and an optimal design of LP pools for future research.

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Appendices

7.1 Proof of Proposition 3.3

Proof. Using Eq.(4) with Eq.(33) for $p_t \in (p_a, p_b)$:

$$
\begin{aligned}
\text{P\&L funded} &= (p_t x_t + y_t) - (p_0 x_0 + y_0) \\
&= L\left(p_t\left(\frac{1}{\sqrt{p_t}} - \frac{1}{\sqrt{p_b}}\right) + (\sqrt{p_t} - \sqrt{p_a})\right) - L\left(p_0\left(\frac{1}{\sqrt{p_0}} - \frac{1}{\sqrt{p_b}}\right) + (\sqrt{p_0} - \sqrt{p_a})\right) \\
&= L\left[\left(\sqrt{p_t} - \frac{p_t}{\sqrt{p_b}}\right) + (\sqrt{p_t} - \sqrt{p_a}) - \left(\sqrt{p_0} - \frac{p_0}{\sqrt{p_b}}\right) - (\sqrt{p_0} - \sqrt{p_a})\right] \\
&= L\left[\left(\sqrt{p_t} - \frac{p_t}{\sqrt{p_b}}\right) + \sqrt{p_t} - \left(\sqrt{p_0} - \frac{p_0}{\sqrt{p_b}}\right) - \sqrt{p_0}\right] \\
&= L\left[2\left(\sqrt{p_t} - \sqrt{p_0}\right) + \frac{p_0 - p_t}{\sqrt{p_b}}\right].
\end{aligned}
\tag{118}
$$

For $p_t \leq p_a$, using (34):

$$
\begin{aligned}
\text{P\&L funded} &= (p_t x_t + y_t) - (p_0 x_0 + y_0) \\
&= L\left[\left(p_t\left(\frac{1}{\sqrt{p_a}} - \frac{1}{\sqrt{p_b}}\right) + 0\right) - \left(p_0\left(\frac{1}{\sqrt{p_0}} - \frac{1}{\sqrt{p_b}}\right) + (\sqrt{p_0} - \sqrt{p_a})\right)\right] \\
&= L\left[p_t\left(\frac{1}{\sqrt{p_a}} - \frac{1}{\sqrt{p_b}}\right) - \left(p_0\left(\frac{1}{\sqrt{p_0}} - \frac{1}{\sqrt{p_b}}\right) + (\sqrt{p_0} - \sqrt{p_a})\right)\right] \\
&= L\left[p_t\left(\frac{1}{\sqrt{p_a}} - \frac{1}{\sqrt{p_b}}\right) + \frac{p_0}{\sqrt{p_b}} - 2\sqrt{p_0} + \sqrt{p_a}\right].
\end{aligned}
\tag{119}
$$

For $p_t \geq p_b$, using Eq. (35):

$$
\begin{aligned}
\text{P\&L funded} &= (p_t x_t + y_t) - (p_0 x_0 + y_0) \\
&= L\left[(0 + (\sqrt{p_b} - \sqrt{p_a})) - \left(p_0\left(\frac{1}{\sqrt{p_0}} - \frac{1}{\sqrt{p_b}}\right) + (\sqrt{p_0} - \sqrt{p_a})\right)\right] \\
&= L\left[(\sqrt{p_b} - \sqrt{p_a}) - \left(p_0\left(\frac{1}{\sqrt{p_0}} - \frac{1}{\sqrt{p_b}}\right) + (\sqrt{p_0} - \sqrt{p_a})\right)\right] \\
&= L\left[(\sqrt{p_b} - \sqrt{p_a}) + \frac{p_0}{\sqrt{p_b}} - 2\sqrt{p_0} + \sqrt{p_a}\right] \\
&= L\left[\sqrt{p_b} + \frac{p_0}{\sqrt{p_b}} - 2\sqrt{p_0}\right].
\end{aligned}
\tag{120}
$$

□

7.2 Proof to Proposition 3.4

Proposition 3.4. Using Eq.(7) with Eq.(33) for $p \in (p_a, p_b)$:

$$\begin{aligned}
\text{P\&L borrowed} &= (p_t x_t + y_t) - (p_t x_0 + y_0) \\
&= L\left[\left(p_t\left(\frac{1}{\sqrt{p_t}} - \frac{1}{\sqrt{p_b}}\right) + (\sqrt{p_t} - \sqrt{p_a})\right) - \left(p_t\left(\frac{1}{\sqrt{p_0}} - \frac{1}{\sqrt{p_b}}\right) + (\sqrt{p_0} - \sqrt{p_a})\right)\right] \\
&= L\left[p_t\left(\frac{1}{\sqrt{p_t}} - \frac{1}{\sqrt{p_0}}\right) + (\sqrt{p_t} - \sqrt{p_0})\right] \\
&= L\left[p_t\left(\frac{\sqrt{p_0} - \sqrt{p_t}}{\sqrt{p_0}\sqrt{p_t}}\right) + (\sqrt{p_t} - \sqrt{p_0})\right] \\
&= L\left[\frac{\sqrt{p_0 p_t} - p_t}{\sqrt{p_0}} + (\sqrt{p_t} - \sqrt{p_0})\right] \\
&= L\left[\frac{2\sqrt{p_0 p_t} - p_t - p_0}{\sqrt{p_0}}\right] \\
&= -\frac{L}{\sqrt{p_0}}(\sqrt{p_t} - \sqrt{p_0})^2 = -L\sqrt{p_0}\left(\sqrt{\frac{p_t}{p_0}} - 1\right)^2.
\end{aligned} \tag{121}$$

For $p \le p_a$, using (34):

$$\begin{aligned}
\text{P\&L borrowed} &= (p_t x_t + y_t) - (p_t x_0 + y_0) \\
&= L\left[\left(p_t\left(\frac{1}{\sqrt{p_a}} - \frac{1}{\sqrt{p_b}}\right) + 0\right) - \left(p_t\left(\frac{1}{\sqrt{p_0}} - \frac{1}{\sqrt{p_b}}\right) + (\sqrt{p_0} - \sqrt{p_a})\right)\right] \\
&= L\left[p_t\left(\frac{1}{\sqrt{p_a}} - \frac{1}{\sqrt{p_0}}\right) - (\sqrt{p_0} - \sqrt{p_a})\right].
\end{aligned} \tag{122}$$

For $p \ge p_b$, using Eq. (35):

$$\begin{aligned}
\text{P\&L borrowed} &= (p_t x_t + y_t) - (p_t x_0 + y_0) \\
&= L\left[(0 + (\sqrt{p_b} - \sqrt{p_a})) - \left(p_t\left(\frac{1}{\sqrt{p_0}} - \frac{1}{\sqrt{p_b}}\right) + (\sqrt{p_0} - \sqrt{p_a})\right)\right] \\
&= L\left[(\sqrt{p_b} - \sqrt{p_0}) - p_t\left(\frac{1}{\sqrt{p_0}} - \frac{1}{\sqrt{p_b}}\right)\right].
\end{aligned} \tag{123}$$

□

7.3 Proof to Proposition 3.5

Proof. We apply P&L of Funded LP in Eq. (41) as follows.

$$\text{P\&L funded}^{(y)}(p_t) = \begin{cases} 2\left(\sqrt{p_t} - \sqrt{p_0}\right) + \frac{p_0 - p_t}{\sqrt{p_b}} & p_t \in (p_a, p_b) \\ p_t\left(\frac{1}{\sqrt{p_a}} - \frac{1}{\sqrt{p_b}}\right) + \frac{p_0}{\sqrt{p_b}} - 2\sqrt{p_0} + \sqrt{p_a} & p_t \le p_a \\ \sqrt{p_b} + \frac{p_0}{\sqrt{p_b}} - 2\sqrt{p_0} & p_t \ge p_b. \end{cases} \tag{124}$$

We consider the range part in the interval $p_t \in (p_a, p_b)$ as follows.

$$\begin{aligned}
\text{Range}(p_t) &= 2(\sqrt{p_t} - \sqrt{p_0}) + \frac{p_0 - p_t}{\sqrt{p_b}} \\
&= -\frac{p_t}{\sqrt{p_b}} + 2\sqrt{p_t} + \left(\frac{p_0}{\sqrt{p_b}} - 2\sqrt{p_0}\right).
\end{aligned} \tag{125}$$

We extend the range part to the interval $p_t \in (0, +\infty)$ as follows

$$\mathrm{Range}(p_t) \equiv -\frac{1}{\sqrt{p_b}} u_1(p_t) + 2u_{1/2}(p_t) + \left(\frac{p_0}{\sqrt{p_b}} - 2\sqrt{p_0}\right) u_0(p_t), \tag{126}$$

where

$$\begin{aligned} u_1(p_t) &= p_t \mathbb{1}\left\{p_a < p_t < p_b\right\} \\ u_{1/2}(p_t) &= \sqrt{p_t}\mathbb{1}\left\{p_a < p_t < p_b\right\} \\ u_0(p_t) &= \mathbb{1}\left\{p_a < p_t < p_b\right\} \end{aligned} \tag{127}$$

are extended on $p_t \in (0, +\infty)$ with zero values outside $p_t \in (p_a, p_b)$.

It is clear that $u_1(p_t)$ can be decomposed on $p_t \in (0, \infty)$ as a collar option position along with put and call digital:

$$u_1(p_t) = p_t + \max\left\{p_a - p_t, 0\right\} - \max\left\{p_t - p_b, 0\right\} - p_a \mathbb{1}\left\{p_t \le p_a\right\} - p_b \mathbb{1}\left\{p_t \ge p_b\right\}. \tag{128}$$

We evaluate $u_0(p_t)$ term as follows

$$u_0(p_t) = 1 - \mathbb{1}\left\{p_t \ge p_b\right\} - \mathbb{1}\left\{p_t \le p_a\right\}. \tag{129}$$

Then $u_{1/2}(p_t)$ is the only non-linear claim which needs model valuation

$$u_{1/2}(p_t) = \sqrt{p_t}\mathbb{1}\left\{p_a < p_t < p_b\right\}. \tag{130}$$

Thus the range part on $p_t \in (0, \infty)$ becomes

$$\begin{aligned} &\mathrm{Range}(p_t) \equiv -\frac{1}{\sqrt{p_b}} u_1(p_t) + 2u_{1/2}(p_t) + \left(\frac{p_0}{\sqrt{p_b}} - 2\sqrt{p_0}\right) u_0(p_t) \\ &= -\frac{1}{\sqrt{p_b}}\left[p_t + \max\left\{p_a - p_t, 0\right\} - \max\left\{p_t - p_b, 0\right\} - p_a \mathbb{1}\left\{p_t \le p_a\right\} - p_b \mathbb{1}\left\{p_t \ge p_b\right\}\right] + 2u_{1/2}(p_t) \\ &+ \left(\frac{p_0}{\sqrt{p_b}} - 2\sqrt{p_0}\right)\left[1 - \mathbb{1}\left\{p_t \ge p_b\right\} - \mathbb{1}\left\{p_t \le p_a\right\}\right] \\ &= -\frac{1}{\sqrt{p_b}} p_t + 2u_{1/2}(p_t) + \left(\frac{p_0}{\sqrt{p_b}} - 2\sqrt{p_0}\right) \\ &- \frac{1}{\sqrt{p_b}} \max\left\{p_a - p_t, 0\right\} + \frac{1}{\sqrt{p_b}} \max\left\{p_t - p_b, 0\right\} + \frac{1}{\sqrt{p_b}} p_a \mathbb{1}\left\{p_t \le p_a\right\} + \sqrt{p_b}\mathbb{1}\left\{p_t \ge p_b\right\} \\ &- \left(\frac{p_0}{\sqrt{p_b}} - 2\sqrt{p_0}\right)\left[\mathbb{1}\left\{p_t \ge p_b\right\} + \mathbb{1}\left\{p_t \le p_a\right\}\right] \\ &= -\frac{1}{\sqrt{p_b}} p_t + 2u_{1/2}(p_t) + \left(\frac{p_0}{\sqrt{p_b}} - 2\sqrt{p_0}\right) - \frac{1}{\sqrt{p_b}} \max\left\{p_a - p_t, 0\right\} + \frac{1}{\sqrt{p_b}} \max\left\{p_t - p_b, 0\right\} \\ &+ \left[\frac{1}{\sqrt{p_b}} p_a - \left(\frac{p_0}{\sqrt{p_b}} - 2\sqrt{p_0}\right)\right] \mathbb{1}\left\{p_t \le p_a\right\} + \left[\sqrt{p_b} - \left(\frac{p_0}{\sqrt{p_b}} - 2\sqrt{p_0}\right)\right] \mathbb{1}\left\{p_t \ge p_b\right\} \\ &= -\frac{1}{\sqrt{p_b}} p_t + 2u_{1/2}(p_t) + \left(\frac{p_0}{\sqrt{p_b}} - 2\sqrt{p_0}\right) - \frac{1}{\sqrt{p_b}} \max\left\{p_a - p_t, 0\right\} + \frac{1}{\sqrt{p_b}} \max\left\{p_t - p_b, 0\right\} \\ &+ \left[\frac{p_a - p_0}{\sqrt{p_b}} + 2\sqrt{p_0}\right] \mathbb{1}\left\{p_t \le p_a\right\} + \left[\frac{p_b - p_0}{\sqrt{p_b}} + 2\sqrt{p_0}\right] \mathbb{1}\left\{p_t \ge p_b\right\}. \end{aligned} \tag{131}$$

Next we evaluate the put side for $p_t \le p_a$:

$$\begin{aligned}
Put(p_t) &= p_t\left(\frac{1}{\sqrt{p_a}} - \frac{1}{\sqrt{p_b}}\right) + \frac{p_0}{\sqrt{p_b}} - 2\sqrt{p_0} + \sqrt{p_a} \\
&= \left[\frac{\sqrt{p_b}-\sqrt{p_a}}{\sqrt{p_a}\sqrt{p_b}}\right](p_t \pm p_a) + \left(\frac{p_0}{\sqrt{p_b}} - 2\sqrt{p_0} + \sqrt{p_a}\right) \\
&= \left[\frac{\sqrt{p_b}-\sqrt{p_a}}{\sqrt{p_a}\sqrt{p_b}}\right](-\{p_a - p_t\} + p_a) + \left(\frac{p_0}{\sqrt{p_b}} - 2\sqrt{p_0} + \sqrt{p_a}\right) \\
&= -\left[\frac{\sqrt{p_b}-\sqrt{p_a}}{\sqrt{p_a}\sqrt{p_b}}\right]\{p_a - p_t\} + \left[\frac{\sqrt{p_b}-\sqrt{p_a}}{\sqrt{p_a}\sqrt{p_b}}\right]p_a + \left(\frac{p_0}{\sqrt{p_b}} - 2\sqrt{p_0} + \sqrt{p_a}\right) \\
&= -\left[\frac{\sqrt{p_b}-\sqrt{p_a}}{\sqrt{p_a}\sqrt{p_b}}\right]\{p_a - p_t\} + \frac{\sqrt{p_b}\sqrt{p_a} - p_a}{\sqrt{p_b}} + \frac{p_0 - 2\sqrt{p_0}\sqrt{p_b} + \sqrt{p_b}\sqrt{p_a}}{\sqrt{p_b}} \\
&= -\left[\frac{\sqrt{p_b}-\sqrt{p_a}}{\sqrt{p_a}\sqrt{p_b}}\right]\{p_a - p_t\} + \frac{(p_0 - p_a) - 2\sqrt{p_0}\sqrt{p_b} + 2\sqrt{p_b}\sqrt{p_a}}{\sqrt{p_b}} \\
&= -\left[\frac{\sqrt{p_b}-\sqrt{p_a}}{\sqrt{p_a}\sqrt{p_b}}\right]\{p_a - p_t\} + \frac{p_0 - p_a}{\sqrt{p_b}} - 2(\sqrt{p_0} - \sqrt{p_a}).
\end{aligned} \tag{132}$$

We further extend the last expression on $p_t \in (0, +\infty)$

$$Put(p_t) = -\left[\frac{\sqrt{p_b}-\sqrt{p_a}}{\sqrt{p_a}\sqrt{p_b}}\right]\max\{p_a - p_t, 0\} + \left[\frac{p_0 - p_a}{\sqrt{p_b}} - 2(\sqrt{p_0} - \sqrt{p_a})\right]\mathbb{1}\{p_t \le p_a\}. \tag{133}$$

Next we extend the call side for $p_t \in (0, +\infty)$ as follows

$$Call(p_t) = \left[\sqrt{p_b} + \frac{p_0}{\sqrt{p_b}} - 2\sqrt{p_0}\right]\mathbb{1}\{p_t \ge p_b\}. \tag{134}$$

Finally we sum up the three parts on $p_t \in (0, +\infty)$ as

$$\begin{aligned}
&\mathrm{Range}(p_t) + Put(p_t) + Call(p_t) \\
&= -\frac{1}{\sqrt{p_b}}p_t + 2u_{1/2}(p_t) + \left(\frac{p_0}{\sqrt{p_b}} - 2\sqrt{p_0}\right) - \frac{1}{\sqrt{p_b}}\max\{p_a - p_t, 0\} + \frac{1}{\sqrt{p_b}}\max\{p_t - p_b, 0\} \\
&+ \left[\frac{p_a - p_0}{\sqrt{p_b}} + 2\sqrt{p_0}\right]\mathbb{1}\{p_t \le p_a\} + \left[\frac{p_b - p_0}{\sqrt{p_b}} + 2\sqrt{p_0}\right]\mathbb{1}\{p_t \ge p_b\} \\
&- \left[\frac{\sqrt{p_b}-\sqrt{p_a}}{\sqrt{p_a}\sqrt{p_b}}\right]\max\{p_a - p_t, 0\} + \left[\frac{p_0 - p_a}{\sqrt{p_b}} - 2(\sqrt{p_0} - \sqrt{p_a})\right]\mathbb{1}\{p_t \le p_a\} \\
&+ \left[\sqrt{p_b} + \frac{p_0}{\sqrt{p_b}} - 2\sqrt{p_0}\right]\mathbb{1}\{p_t \ge p_b\} \\
&= -\frac{1}{\sqrt{p_b}}p_t + 2u_{1/2}(p_t) + \left(\frac{p_0}{\sqrt{p_b}} - 2\sqrt{p_0}\right) \\
&- \left[\frac{1}{\sqrt{p_b}} + \frac{\sqrt{p_b}-\sqrt{p_a}}{\sqrt{p_a}\sqrt{p_b}}\right]\max\{p_a - p_t, 0\} + \frac{1}{\sqrt{p_b}}\max\{p_t - p_b, 0\} \\
&+ \left[\frac{p_a - p_0}{\sqrt{p_b}} + 2\sqrt{p_0} + \frac{p_0 - p_a}{\sqrt{p_b}} - 2(\sqrt{p_0} - \sqrt{p_a})\right]\mathbb{1}\{p_t \le p_a\} \\
&+ \left[\frac{p_b - p_0}{\sqrt{p_b}} + 2\sqrt{p_0} + \sqrt{p_b} + \frac{p_0}{\sqrt{p_b}} - 2\sqrt{p_0}\right]\mathbb{1}\{p_t \ge p_b\} \\
&= -\frac{1}{\sqrt{p_b}}p_t + 2u_{1/2}(p_t) + \left(\frac{p_0}{\sqrt{p_b}} - 2\sqrt{p_0}\right) \\
&- \frac{1}{\sqrt{p_a}}\max\{p_a - p_t, 0\} + \frac{1}{\sqrt{p_b}}\max\{p_t - p_b, 0\} + 2\sqrt{p_a}\mathbb{1}\{p_t \le p_a\} + 2\sqrt{p_b}\mathbb{1}\{p_t \ge p_b\}.
\end{aligned} \tag{135}$$

□

7.4 Proof of Proposition 7.4

Proof. Using P&L of borrowed LP in Eq. (45) we obtain the following.

$$\text{P\&L borrowed}^{(y)}(p_t) = \begin{cases} -\sqrt{p_0}\left(\sqrt{\frac{p_t}{p_0}}-1\right)^2 & p_t \in (p_a, p_b) \\ p_t\left(\frac{1}{\sqrt{p_a}}-\frac{1}{\sqrt{p_0}}\right)-\left(\sqrt{p_0}-\sqrt{p_a}\right) & p_t \le p_a \\ \left(\sqrt{p_b}-\sqrt{p_0}\right)-p_t\left(\frac{1}{\sqrt{p_0}}-\frac{1}{\sqrt{p_b}}\right) & p_t \ge p_b \\ . \end{cases} \tag{136}$$

where $L = 1$

We split the pay-off in the three parts.

The range part we evaluate for $p_t \in (p_a, p_b)$ as follows

$$\begin{aligned} \text{Range}(p_t) &= -\sqrt{p_0}\left(\frac{p_t}{p_0} - 2\sqrt{\frac{p_t}{p_0}} + 1\right) \\ &\equiv \left(\frac{1}{\sqrt{p_0}}u_1(p_t) - 2u_{1/2}(p_t) + \sqrt{p_0}u_0(p_t)\right). \end{aligned} \tag{137}$$

It is clear that the first term can be decomposed for $p_t \in (0, \infty)$ as a collar option position :

$$u_1(p_t) = p_t + \max\{p_a - p_t, 0\} - \max\{p_t - p_b, 0\} - p_a\mathbb{1}\{p_t \le p_a\} - p_b\mathbb{1}\{p_t \ge p_b\}. \tag{138}$$

where $\mathbb{1}\{x\}$ is the indicator function.

We evaluate $u_0(p_t)$ term is as follows:

$$u_0(p_t) = 1 - \mathbb{1}\{p_t \ge p_b\} - \mathbb{1}\{p_t \le p_a\}. \tag{139}$$

Then $u_{1/2}(p_t)$ is the only non-linear claim which needs model valuation

$$u_{1/2}(p_t) = \sqrt{p_t}\mathbb{1}\{p_a < p_t < p_b\} \tag{140}$$

We evaluate the put side for $p_t \le p_a$ as follows

$$\begin{aligned} Put(p_t) &= (\sqrt{p_0} - \sqrt{p_a}) - p_t\left(\frac{\sqrt{p_0}-\sqrt{p_a}}{\sqrt{p_0p_a}}\right) \\ &= \left(\frac{\sqrt{p_0}-\sqrt{p_a}}{\sqrt{p_0p_a}}\right)[\pm p_a + \sqrt{p_0p_a} - p_t] \\ &= \left(\frac{\sqrt{p_0}-\sqrt{p_a}}{\sqrt{p_0p_a}}\right)[(p_a - p_t) + (\sqrt{p_0p_a} - p_a)] \\ &= \frac{\sqrt{p_0}-\sqrt{p_a}}{\sqrt{p_0p_a}}[p_a - p_t] + \frac{\sqrt{p_0}-\sqrt{p_a}}{\sqrt{p_0p_a}}(\sqrt{p_0p_a} - p_a) \\ &= \frac{\sqrt{p_0}-\sqrt{p_a}}{\sqrt{p_0p_a}}[p_a - p_t] + \frac{\left(\sqrt{p_0}-\sqrt{p_a}\right)^2}{\sqrt{p_0}}. \end{aligned} \tag{141}$$

As a result, the pay-off on the put side can be written for $p_t \in (0, \infty)$ as follows

$$Put(p_t) = \frac{\sqrt{p_0}-\sqrt{p_a}}{\sqrt{p_0p_a}}\max\{p_a - p_t, 0\} + \frac{\left(\sqrt{p_0}-\sqrt{p_a}\right)^2}{\sqrt{p_0}}\mathbb{1}\{p_t \le p_a\}. \tag{142}$$

Second, we evaluate the call side for $p_t \geq p_b$ as follows

$$\begin{aligned} Call(p_t) &= p_t \left(\frac{\sqrt{p_b} - \sqrt{p_0}}{\sqrt{p_0 p_b}} \right) - (\sqrt{p_b} - \sqrt{p_0}) \\ &= \left(\frac{\sqrt{p_b} - \sqrt{p_0}}{\sqrt{p_0 p_b}} \right) [p_t - \sqrt{p_0 p_b} \pm p_b] \\ &= \left(\frac{\sqrt{p_b} - \sqrt{p_0}}{\sqrt{p_0 p_b}} \right) [p_t - p_b] + \left(\frac{\sqrt{p_b} - \sqrt{p_0}}{\sqrt{p_0 p_b}} \right) [p_b - \sqrt{p_0 p_b}] \\ &= \left(\frac{\sqrt{p_b} - \sqrt{p_0}}{\sqrt{p_0 p_b}} \right) [p_t - p_b] + \frac{\left(\sqrt{p_b} - \sqrt{p_0}\right)^2}{\sqrt{p_0}}. \end{aligned} \tag{143}$$

As a result, the pay-off on the call side can be written for $p_t \in (0, \infty)$ as follows

$$Call(p_t) = \left(\frac{\sqrt{p_b} - \sqrt{p_0}}{\sqrt{p_0 p_b}} \right) \max\{p_t - p_b, 0\} + \frac{\left(\sqrt{p_b} - \sqrt{p_0}\right)^2}{\sqrt{p_0}} \mathbb{1}\{p_t \geq p_b\}. \tag{144}$$

Summing all together:

$$\begin{aligned} &\text{Range}(p_t) + Put(p_t) + Call(p_t) \\ &= -2u_{1/2}(p_t) + \frac{1}{\sqrt{p_0}} [p_t + \max\{p_a - p_t, 0\} - \max\{p_t - p_b, 0\} - p_a \mathbb{1}\{p_t \leq p_a\} - p_b \mathbb{1}\{p_t \geq p_b\}] \\ &+ \sqrt{p_0} [1 - \mathbb{1}\{p_t \geq p_b\} - \mathbb{1}\{p_t \leq p_a\}] \\ &+ \frac{\sqrt{p_0} - \sqrt{p_a}}{\sqrt{p_0 p_a}} \max\{p_a - p_t, 0\} + \frac{\left(\sqrt{p_0} - \sqrt{p_a}\right)^2}{\sqrt{p_0}} \mathbb{1}\{p_t \leq p_a\} \\ &+ \left(\frac{\sqrt{p_b} - \sqrt{p_0}}{\sqrt{p_0 p_b}} \right) \max\{p_t - p_b, 0\} + \frac{\left(\sqrt{p_b} - \sqrt{p_0}\right)^2}{\sqrt{p_0}} \mathbb{1}\{p_t \geq p_b\} \\ &= -2u_{1/2}(p_t) + \frac{1}{\sqrt{p_0}} p_t + \sqrt{p_0} \\ &+ \left[\frac{\sqrt{p_0} - \sqrt{p_a}}{\sqrt{p_0 p_a}} + \frac{1}{\sqrt{p_0}} \right] \max\{p_a - p_t, 0\} \\ &+ \left[\frac{\sqrt{p_b} - \sqrt{p_0}}{\sqrt{p_0 p_b}} - \frac{1}{\sqrt{p_0}} \right] \max\{p_t - p_b, 0\} \\ &+ \left[\frac{\left(\sqrt{p_0} - \sqrt{p_a}\right)^2}{\sqrt{p_0}} - \frac{p_a}{\sqrt{p_0}} - \sqrt{p_0} \right] \mathbb{1}\{p_t \leq p_a\} \\ &+ \left[\frac{\left(\sqrt{p_b} - \sqrt{p_0}\right)^2}{\sqrt{p_0}} - \frac{p_b}{\sqrt{p_0}} - \sqrt{p_0} \right] \mathbb{1}\{p_t \geq p_b\} \\ &= -2r_1(p_t) + \frac{1}{\sqrt{p_0}} p_t + \sqrt{p_0} \\ &+ \frac{1}{\sqrt{p_a}} \max\{p_a - p_t, 0\} \\ &- \frac{1}{\sqrt{p_b}} \max\{p_t - p_b, 0\} \\ &- 2\sqrt{p_a} \mathbb{1}\{p_t \leq p_a\} \\ &- 2\sqrt{p_b} \mathbb{1}\{p_t \geq p_b\}. \end{aligned} \tag{145}$$

The final result follows by collecting pay-offs of vanilla puts and calls and digitals.

□

7.5 Proof to Proposition 4.1

Proof. For the put side, it is clear that

$$\Pi(K) = \sum_{n=1}^{N} w_n P_n(K) = \sum_{n=1}^{N} w_n \max\left(K_n - K, 0\right). \tag{146}$$

We define the first-order derivatives at discrete strike points as follows:

$$\begin{aligned} \delta IL(K_n) &= \frac{IL(K_n) - IL(K_{n-1})}{K_n - K_{n-1}} \\ \delta\Pi(K_n) &= \frac{\Pi(K_n) - \Pi(K_{n-1})}{K_n - K_{n-1}}. \end{aligned} \tag{147}$$

In particular for $K_n \in \mathcal{K}$:

$$\begin{aligned} \Pi(K_n) &= \sum_{n' \geq n}^{N} w_{n'} P_{n'}(K_n) = \sum_{n' \geq n}^{N} w_{n'}\left(K_{n'} - K_n\right) \\ \Pi(K_{n-1}) &= \sum_{n' \geq n-1}^{N} w_{n'}\left(K_{n'} - K_{n-1}\right) = w_{n-1}\left(K_{n-1} - K_{n-1}\right) + \sum_{n' \geq n}^{N} w_{n'}\left(K_{n'} - K_{n-1}\right) = \sum_{n' \geq n}^{N} w_{n'}\left(K_{n'} - K_{n-1}\right) \end{aligned} \tag{148}$$

As a result:

$$\begin{aligned} \delta\Pi(K_n) &= \frac{1}{K_n - K_{n-1}}\left(\Pi(K_n) - \Pi(K_{n-1})\right) \\ &= \frac{1}{K_n - K_{n-1}}\left(\sum_{n' \geq n}^{N} w_{n'}\left(K_{n'} - K_n\right) - \sum_{n' \geq n}^{N} w_{n'}\left(K_{n'} - K_{n-1}\right)\right) \\ &= -\sum_{n' \geq n}^{N} w_{n'}, \end{aligned} \tag{149}$$

and

$$\delta\Pi(K_n) - \delta\Pi(K_{n-1}) = -w_{n-1}. \tag{150}$$

By piece-wise approximation over the interval $x \in (K_{n-1} - K_n)$:

$$\delta IL(x) = \delta L(K_n) - \frac{K_n - x}{K_n - K_{n-1}}\left(\delta IL(K_n) - \delta IL(K_{n-1})\right), \tag{151}$$

and

$$\begin{aligned} \delta\Pi(x) &= \delta\Pi(K_n) - \frac{K_n - x}{K_n - K_{n-1}}\left(\delta\Pi(K_n) - \delta\Pi(K_{n-1})\right) \\ &= \delta\Pi(K_n) - \frac{x - K_n}{K_{n-1} - K_n}\left(-w_{n-1}\right). \end{aligned} \tag{152}$$

The proof for the call side follows by analogy.

□

7.6 Proof of Proposition 5.2

Proof. We use Eq. (71) as follows

$$U^{funded}(t,p_t) = -e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[u_0^{funded}(x_t+x_\tau) + u_{1/2}(x_t+x_\tau) + u_1(x_t+x_\tau)\right],$$
$$U^{borrowed}(t,p_t) = -e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[u_0^{borrowed}(x_t+x_\tau) + u_{1/2}(x_t+x_\tau) + u_1(x_t+x_\tau)\right]. \tag{153}$$

The square pay-off in Eq. (73) is computed using:

$$\begin{aligned}
U_{1/2}(t,p_t) &= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[\sqrt{p_0 e^{x_t+x_\tau}}\mathbb{1}\left\{x_a < x_t + x_\tau < x_b\right\}\right] \\
&= e^{-r(T-t)}\sqrt{p_t}\int_{-\infty}^{\infty}\exp\left\{\frac{1}{2}\left(\mu-\frac{1}{2}\sigma^2\right)\tau + \frac{1}{2}\sigma\sqrt{\tau}x\right\}\mathbb{1}\left\{x_a - x_t < x < p_b - x_t\right\}\mathbf{n}(x)dx \\
&= e^{-r(T-t)}\sqrt{p_t}\int_{-d_-(p_t,p_a)}^{-d_-(p_t,p_b)}\exp\left\{\frac{1}{2}\left(\mu-\frac{1}{2}\sigma^2\right)\tau + \frac{1}{2}\sigma\sqrt{\tau}x\right\}\mathbf{n}(x)dx \\
&= e^{-r(T-t)}\sqrt{p_t}\exp\left\{\frac{1}{2}\left(\mu-\frac{1}{2}\sigma^2\right)\tau\right\}\int_{-d_-(p_t,p_a)}^{-d_-(p_t,p_b)}\exp\left\{\frac{1}{2}\sigma\sqrt{\tau}x\right\}\mathbf{n}(x)dx \\
&= e^{-r(T-t)}\sqrt{p_t}\exp\left\{\frac{1}{2}\left(\mu-\frac{1}{2}\sigma^2\right)\tau\right\}\left(m(-d_-(p_t,p_b)) - m(-d_-(p_t,p_a))\right) \\
&= e^{-r(T-t)}\sqrt{p_t}\exp\left\{\frac{1}{2}\mu\tau - \frac{1}{8}\sigma^2\tau\right\}\left(\mathbf{N}\left(\frac{\ln(p_b/p_t)-(r-q)\tau}{\sigma\sqrt{\tau}}\right) - \mathbf{N}\left(\frac{\ln(p_a/p_t)-(r-q)\tau}{\sigma\sqrt{\tau}}\right)\right),
\end{aligned} \tag{154}$$

where $\mathbf{n}$ is normal pdf and

$$m(x) = \exp\left\{\frac{1}{8}\sigma^2\tau\right\}\mathbf{N}(x - \frac{1}{2}\sigma\sqrt{\tau}). \tag{155}$$

The option part is computed using the option pricing formulas for BSM model.

□

7.7 Carr-Madan representation

In Carr-Madan (2001) a representation of an arbitrary, twice-differentiable function (pay-off) in terms of put and call pay-offs was given. Here we derive the same representation relaxing the smoothness assumption, only requiring that its first derivative possesses the generalized derivative everywhere.

To this end, assume that $f : \mathbb{R} \mapsto \mathbb{R}$, is such that f' has generalized derivative at every point, and fix arbitrary S, $F \geq 0$. Then we have

$$\begin{aligned}
f(S) &= f(F) + I_{\{S>F\}}\int_F^S f'(u)\,du - I_{\{S<F\}}\int_S^F f'(u)\,du \\
&= f(F) + I_{\{S>F\}}\int_F^S\left[f'(F) + \int_F^u f''(v)\,dv\right]du - I_{\{S<F\}}\int_S^F\left[f'(F) - \int_u^F f''(v)\,dv\right]du \\
&= f(F) + f'(F)(S-F) + I_{\{S>F\}}\int_F^S\int_F^u f''(v)\,dv\,du + I_{\{S<F\}}\int_S^F\int_u^F f''(v)\,dv\,du \\
&= f(F) + f'(F)(S-F) + \int_F^S f''^{+}dv + \int_0^F f''^{+}dv,
\end{aligned} \tag{156}$$

where in the last step we used Fubini's theorem. (Note that the upper limit in the first integral in (156) can be replaced by infinity.)