STOCHASTIC CALCULUS OF RATES: JAMSHIDIAN'S TRICK

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Introduction

Jamshidian's trick an observation about combining increasing functions, and helps us translate pricing a swaption into the simpler problem of summing the prices of standard zero coupon bond (ZCB) options.

The Swaption Payoff

Consider a European defnPayer Swaption with expiry T_0. allowing us to enter into a swap starting at T_0 and ending at T_n, paying a fixed rate K and receiving the floating rate (e.g., SOFR).

This payoiff at expiry T_0 is mathematically equivalent to the payoff of a **Put Option** on a coupon bond. Specifically, it is an option to sell a bond with coupon K for a strike price of par (usually 1). And, later, we will view the coupon paying bond itself as a zum of ZCB cashflows.

Coupon Bond Value

Let the coupon bond pay cash flows c_i at times T_i for $i = 1, \ldots, n$. (They need not be all the same size)

The value of this underlying bond at time T_0, denoted $P_{CB}(T_0, r_{T_0})$, is the sum of the discounted cash flows:

$$P_{CB}(T_0, r_{T_0}) = \sum_{i=1}^{n} c_i P(T_0, T_i, r_{T_0})$$

where $P(T_0, T_i, r_{T_0})$ is the price at time T_0 of a zero-coupon bond maturing at T_i.

Payer Swaption

The payoff of a payer swaption at T_0 is:

$$V(T_0) = \max\big(1 - P_{CB}(T_0, r_{T_0}), 0\big)$$

Note that the strike price here is the principal $L = 1$. Substituting the definition of the bond:

$$V(T_0) = \max\left(1 - \sum_{i=1}^{n} c_i P(T_0, T_i, r_{T_0}), 0\right)$$

Jamshidian's Decomposition

The validity of Jamshidian's decomposition relies on the assumption that the bond prices $P(T_0, T_i, r)$ are strictly decreasing functions of the short rate r. (Works for most rate models, but watch out for "shifted" models)

Because $P(T_0, T_i, r)$ decreases as r increases, the entire coupon bond price $P_{CB}(r)$ is also a strictly decreasing function of r.

Finding the Critical Rate

We exercise the swaption (the put option on the bond) only when the equivalent bond price is roughly "low enough", which corresponds to the interest rate being "high enough."

Since $P_{CB}(r)$ is monotonic and continuous, there exists a unique critical short rate, denoted r^*, such that the bond price exactly equals the strike price (par, 1).

We find r^* by solving the following equation numerically (e.g., using Newton-Raphson):

$$\sum_{i=1}^{n} c_i P(T_0, T_i, r^*) = 1$$

The exercise condition $P_{CB}(r) < 1$ is therefore equivalent to $r > r^*$.

Decomposing the Strikes

Once r^* is determined, we can define individual strike prices K_i for each specific zero-coupon bond $P(T_0, T_i)$ corresponding to this critical rate level:

$$K_i = P(T_0, T_i, r^*)$$

By definition, summing these individual strikes weighted by the cash flows recovers the total strike of the swaption:

$$\sum_{i=1}^{n} c_i K_i = \sum_{i=1}^{n} c_i P(T_0, T_i, r^*) = P_{CB}(r^*) = 1$$

The Final Valuation Formula

We can now rewrite the swaption payoff. The option is exercised if $r > r^*$. In this region, $P(T_0, T_i, r) < K_i$.

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\begin{aligned}
V(T_0) &= \left(1 - \sum_{i=1}^{n} c_i P(T_0, T_i, r)\right) \cdot \mathbf{1}_{\{r > r^*\}} \\
&= \left(\sum_{i=1}^{n} c_i K_i - \sum_{i=1}^{n} c_i P(T_0, T_i, r)\right) \cdot \mathbf{1}_{\{r > r^*\}} \\
&= \sum_{i=1}^{n} c_i \big(K_i - P(T_0, T_i, r)\big) \cdot \mathbf{1}_{\{P(T_0, T_i, r) < K_i\}} \\
&= \sum_{i=1}^{n} c_i \max(K_i - P(T_0, T_i, r), 0)
\end{aligned}
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Taking the risk-neutral expectation at time $t = 0$, the value of the payer swaption is simply the sum of n put options on zero coupon bonds:

$$V_{swaption}(0) = \sum_{i=1}^{n} c_i \cdot \mathrm{ZBP}(0, T_0, T_i, K_i)$$

Where $\mathrm{ZBP}(0, T_0, T_i, K_i)$ is the price at time 0 of a European put option with expiry T_0, strike K_i, on a zero-coupon bond maturing at T_i.

Summary of the Algorithm

To value a swaption using a one-factor model:

(1) Calibrate the model parameters (e.g., mean reversion a, volatility σ).

(2) Solve numerically for r^* such that the coupon bond price equals the strike (here, 1).

(3) Calculate the individual strikes $K_i = P(T_0, T_i, r^*)$ using the analytical ZCB formula for the specific model.

(4) Value n individual ZCB options using a standard closed-form solution (e.g., the generalized Black-Scholes or specific Hull-White formula) with strikes K_i.

(5) Sum the option prices weighted by the cash flows c_i.