LINEAR GAUSSIAN MODELS

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Linear Gaussian Models

"A modern interest rate model consists of three parts: a numéraire, a set of random evolution equations in the risk neutral world, and the Martingale pricing formula" -Hagan

Characterization

Linear gaussian models are a mathematically cleaner expression of 1- and multi-factor Hull-White, which is essentially the multivariate gaussian model.

Definition

Say we have a risk-neutral measure P, and consider a p-dimensional standard Brownian motion Z. We take Y as the solution of the following Ornstein-Uhlenbeck SDE

$$Y_t = y + \int_0^t \kappa(\theta - Y_s)ds + \int_0^t \sqrt{V} dZ_s,$$

where $\kappa \in M_p(\mathbb{R})$ is a matrix of order p, V is a semidefinite positive matrix of order p and $\theta \in \mathbb{R}^p$. The LGM assumes that the spot rate is an affine function of the vector Y:

$$r_t = \varphi + \sum_{i=1}^{p} Y_t^i,$$

and the coordinates Y^i are usually called the *factors* of the model.

Stationarity

We assume that the parameters (here κ, θ and V) are essentially unchanging, while the factors (here, the vector Y) evolve and reflect the current state of the market.

These factors affect different time scales: a factor with a small (resp. large) mean-reversion will influence the long-term (resp. short-term) behavior of the interest rate. So we will take

$$\kappa = \mathrm{diag}(\kappa_1, \ldots, \kappa_p) \text{ with } 0 < \kappa_1 < \cdots < \kappa_p,$$

One-Factor LGM and Hull White

Let us reduce momentarily to a 1-factor version of the LGM, in the manner of Roberts (). Here our state is X_t, and we choose a generic numéraire $N(t, X_t)$ for its mathematical properties rather than its tradability.We take

$$dX_t = \alpha(t)\, dW_t,$$

where $X_0 = 0$

One-Factor LGM Numéraire

We choose our numéraire to be financially artificial but mathematically useful

$$N(t, X_t) = \frac{1}{P^M(0,t)} e^{H(t)X_t + \frac{1}{2}H(t)^2\zeta(t)}$$

$$\zeta(t) = \int_0^t \alpha(s)^2\, ds.$$

The $\alpha(t)$ and $H(t)$ are deterministic, time-varying parameters. This LGM model is equivalent to the HW model when we set

$$H(t) = e^{-at}, \quad \alpha(t) = \frac{\sigma}{a} e^{at}.$$

One-factor LGM ZCB Prices

Since our numéraire is "special", the one-factor LGM model automatically matches the initial zero-coupon bond curve, and zero-coupon bond prices are

$$P(t,T) = \frac{P^M(0,T)}{P^M(0,t)} e^{(H(t)-H(T))X_t + \frac{1}{2}(H^2(t)-H^2(T))\zeta(t)}.$$

Vanilla Interest Rate Swaps

Recall that a payer swap is one in which the fixed leg is paid in exchange for the floating leg, whereas in a receiver swap the floating leg is paid in exchange for the fixed leg.Take notional N, trade time T_0 and matuirity T_N, with fixing $T_0, T_1, \ldots, T_{N-1}$ on which the floating rate is determined, and a series of payment dates $T_1, T_2, \ldots, T_N$ on which cashflows are exchanged.

Valuation

We get the value of a payer swap contract at any $T_0 \leq t < T_N$ as:

$$V_{\mathrm{IRS}}(t)/N = (1 + L(T_{k-1}, T_k)\tau_k)P(t, T_k) - P(t, T_N) - \sum_{i=k}^{N} P(t, T_i)K\tau_i$$

where $k = \max\{i : T_{i-1} \leq t\}$ is the index of the most recent reset date before t. Here $L(T_{k-1}, T_k)$ is the most recent fixing of the simple floating rate at T_{k-1}, K is the fixed rate and $\tau_i = T_i - T_{i-1}$ is the accrual period between the reset date T_{i-1} and payment date T_i.

Fair Rate

The fair swap rate (FSR) is the fixed rate which sets the value of an IRS to zero at T_0. This is given by

$$\text{FSR} = \frac{1 - P(T_0, T_N)}{\sum_{i=1}^{N} P(T_0, T_i)\tau_i}.$$

European Swaptions in The 1-Factor LGM Model

Because we have a simple process, X_t, and special numéraire, $N(t, X_t)$, derivatives can be valued directly by integration. Given a contract with payoff $V(T, X_T)$ at maturity . We have value

$$V(t, X_t) = N(t, X_t) \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi(\zeta(T) - \zeta(t))}} e^{-\frac{(x - X_t)^2}{2(\zeta(T) - \zeta(t))}} \frac{V(T, x)}{N(T, x)}\, dx.$$

Swaption LGM

In the case of a payer swaption, struck at K and maturing at T_0, the following numéraire-deflated payoff must be integrated:

$$\Big[P(0,T_0)e^{-H(T_0)x-\frac{1}{2}H(T_0)^2\zeta(T_0)} - \sum_{i=1}^{n}\tau_i K P(0,T_i)e^{-H(T_i)x-\frac{1}{2}H(T_i)^2\zeta(T_0)} - P(0,T_n)e^{-H(T_n)x-\frac{1}{2}H(T_n)^2\zeta(T_0)}\Big]^+ .$$

Evaluation

To evaluate the integral one must integrate over the region where the payoff is non-zero. Because $H(t) = e^{-at}$ in the Hull-White equivalent formulation of the LGM model, $H(T_i) - H(T_0)$ is negative for each i and thus one can find a unique x^* such that

$$P(0, T_0) = \sum_{i=1}^{n} \tau_i K P(0, T_i) e^{-(H(T_i)-H(T_0))x^* - \frac{1}{2}(H(T_i)^2 - H(T_0)^2)\zeta(T_0)}$$
$$+ P(0, T_n) e^{-(H(T_n)-H(T_0))x^* - \frac{1}{2}(H(T_n)^2 - H(T_0)^2)\zeta(T_0)}$$

and for which the integrand is positive for $x < x^*$.

Sub In

Substituting the numéraire deflated payoff into the valuation formula and integrating yields

$$
\begin{aligned}
V_{\mathrm{PS}}(t, X_t) = {} & \frac{P(0,T_0)}{P(0,t)} e^{-(H(T_0)-H(t))X_t - \frac{1}{2}(H(T_0)^2 - H(t)^2)\zeta(t)} \Phi(d_0) \\
& - \sum_{i=1}^{n} \tau_i K \frac{P(0,T_i)}{P(0,t)} e^{-(H(T_i)-H(t))X_t - \frac{1}{2}(H(T_i)^2 - H(t)^2)\zeta(t)} \Phi(d \\
& + \frac{P(0,T_n)}{P(0,t)} e^{-(H(T_n)-H(t))X_t - \frac{1}{2}(H(T_n)^2 - H(t)^2)\zeta(t)} \Phi(d_n),
\end{aligned}
$$

where Φ is the standard normal cumulative density function and d_i is given by

$$
d_i = \frac{(x^* - X_t) + H(T_i)(\zeta(T_0) - \zeta(t))}{\sqrt{\zeta(T_0) - \zeta(t)}}.
$$

Receivers

The value of a receiver swaption can be determined by put-call parity to be

$$V_{\mathrm{RS}}(t, X_t) = V_{\mathrm{PS}}(t, X_t)$$

$$+ N(t, X_t) \left(\sum_{i=1}^{n} \tau_i K P(0, T_i) + P(0, T_n) - P(0, T_0) \right).$$

Multifactor LGM

Recall that our multifactor version has multi-dimensional stochasticity of Y with a covariance V.

ZCB

For $0 \leq t \leq T$, the price

$$P_{t,T} = \mathbb{E}\left[\exp\left(-\int_t^T r_s ds\right)|F_t\right]$$

at time t of the zero-coupon bond with maturity T is an exponential is given by:

$$P_{t,T} = \exp(E(T-t) + B(T-t)^\top Y_t),$$

where the *support function*

$$B(\tau) = -(\kappa^\top)^{-1}(I_p - e^{-\kappa^\top \tau})\mathbf{1}_p$$

and

$$\mathbb{E}(\tau) = -\varphi\tau + \int_0^\tau B(s)^\top \kappa\theta + \frac{B(s)^\top V B(s)}{2} ds$$

Tenor Effects

A factor Y^i associated with the larger κ_i has greater effect on the short term behavior of the yield curve while one associated with the smaller κ_i will drive the long term behavior.

SOFR and Swap

Now let's make formulas for swaptions and caplets. They are respectively expressed with respect to the forward SOFR rate L and the forward swap rate, which satisfy:

$$L_t(T,\delta) = \frac{1}{\delta}\left(\frac{P_{t,T}}{P_{t,T+\delta}} - 1\right)$$

$$S_t(T,m) = \frac{P_{t,T} - P_{t,T+m\delta}}{\delta\sum_{i=1}^{m} P_{t,T+i\delta}}.$$

Caplet and Swaption Prices

The prices of caplets and swaptions are respectively given by

$$C_t(T,\delta,K) = \mathbb{E}\left[e^{-\int_t^{T+\delta} r_s ds} (L_T(T,\delta) - K)^+ \middle| F_t \right]$$

$$\text{Swaption}_t(T,m,\delta,K) = \mathbb{E}\left[e^{-\int_t^{T} r_s ds} \sum_{i=1}^{m} \delta P_{T,T+i\delta} (S_T(T,m) - K)^+ \right]$$

Make It Nicer

Changing numeraire measure to $T+\delta$-forward neutral (caplet) or annuity neutral (swaption), i.e. the numeraire $P_{t,T+\delta}$ (caplet) or $\sum_{i=1}^{m} \delta P_{t,T+i\delta}$ (swaption) yields

$$C_t(T,\delta,K) = P_{t,T+\delta} E^{T+\delta}[(L_T(T,\delta) - K)^+]$$

$$\text{Swaption}_t(T,m,\delta,K) = \left(\sum_{i=1}^{m} \delta P_{t,T+i\delta} \right) E^A[(S_T(T,m) - K)^+]$$

Direct Lognormal Volatility

Within the LGM model, we directly get the log-normal implied volatility of the caplet is given by

$$\int_t^T [B(T-u)-B(T+\delta-u)]^\top V[B(T-u)-B(T+\delta-u)]du,$$

This implied volatility does not depend on the strike. We get no skew

Covariance Parameterization

We parameterize diagonal and off-diagonals of the covariance matrix V by the support functions

$$m_{ii}(\tau,\delta) = \left(\frac{1-e^{-\kappa_i\delta}}{\kappa_i}\right)^2 \frac{1-e^{-2\kappa_i\tau}}{2\kappa_i\tau}$$

.

$$m_{ij}(\tau,\delta) = \frac{1-e^{-\kappa_i\delta}}{\kappa_i}\frac{1-e^{-\kappa_j\delta}}{\kappa_j}\frac{1-e^{-(\kappa_i+\kappa_j)\tau}}{(\kappa_i+\kappa_j)\tau}$$

Direct Bachelier Volatility

We also have an approxzimation formula for normal implied volatility of the swaptions:

$$\int_t^T [B_S(u)]^\top V B_S(u) du,$$

with

$$B_S(u) = \omega_0^0 B(T-u) - \omega_m^0 B(T+m\delta-u) - S_0(T,m,\delta) \sum_{k=1}^{m} \omega_k^0 B(T+k\delta-u)$$

and

$$\omega_k^0 = \frac{P_{0,T+k\delta}}{\sum_{i=1}^{m} P_{0,T+i\delta}}$$

.

This is not time homogeneous but still lacks skew. For this reason, people combine with stochastic volatility.

SABR

We can think of the stochastic-alpha-beta-rho, or SABR, models as a variation on these ideas. In particular we use a stochastic model

$$d\vec{F} = \sigma(\vec{F} + \vec{c})^{\beta} d\vec{W}$$

with volatilities

$$d\vec{\sigma} = \nu\vec{\sigma} d\vec{\mathcal{W}}$$

where products are taken elementwise, and a correlation matrix Σ between the stochastic elements, which reduces to a single parameter ρ in the one dimensional case.

Variants introduce mean reversion in the volatility, which helps them handle term structures to a degree.

SABR Skew

Because it has stochastic volatility, SABR can be fit to skews, unlike plain HW/LGM/LMM/BSM models.

SABR Formulas

This functional form allows us to ignore the intermediate stochastics (assuming we care only about terminal distributions). We come up with what is essentially a nice fitting formula for either Bachelier or lognormal models. LeF'loch gives useful formulas for this.

LGM For Collateralized Counterparty Risk

Another use of easy-to-simulate models such as one-factor LGM is in risk computations. This becomes important in correcting derivatives prices for counterparty risk, which we call *credit value adjustment* or CVA. These corrections are mandatory for some (especially European) risk computations.

Recall the martingale pricing formula is

$$\frac{V(t)}{N(t)} = \mathbb{E}_t^N \left[\frac{V(T)}{N(T)} \right].$$

In LGM

Reduced values $\tilde{V}(t) = V(t)/N(t)$ in terms of the Gaussian transition density of $z_T - z_t$ are then

$$\tilde{V}(t, z_t) = \frac{1}{\sqrt{2\pi\Delta_\zeta}} \int_{-\infty}^{\infty} e^{-(z_T - z_t)^2/2\Delta_\zeta} \tilde{V}(T, z_T) dz_T$$

where $\Delta_\zeta = \zeta(T) - \zeta(t)$.

ZCB

For a zero coupon bond recall

$$P(t,T,z_t) = \frac{P(0,T)}{P(0,t)} \exp\left\{-(H_T - H_t)z_t - \frac{1}{2}\left(H_T^2 - H_t^2\right)\zeta_t\right\}.$$

with

$$f(t,T) = -\frac{\partial \ln P(t,T)}{\partial T} = f(0,T) + z(t)H'(T) + \zeta(t)H'(T)H(T)$$
$$r(t) = f(t,t) = f(0,t) + z(t)H'(t) + \zeta(t)H'(t)H(t).$$

CVA Intro

With default time τ and loss given default LGD, the adjustment formula for a payoff is mathematically

$$CVA(t) = \mathbb{E}^{\mathbb{Q}}\left[\mathbf{1}_{t<\tau\leq T} \cdot LGD(\tau) \cdot D(t,\tau) \cdot NPV^{+}(\tau) \mid \mathcal{G}_t\right].$$

Simpler Math

Let us assume:

- Only the counterparty can default
- With a constant hazard process $\lambda(t)$ independent of everything else,
- Recovery rate / loss given default (LGD) are known constants
- There is (here) no collateral

Factor Out

We can now factor out LGD, We get a product of expectations involving survival S

$$\mathbb{E}_t^{\mathbb{Q}}[S(t, t_{i-1}) - S(t, t_i)] \times \mathbb{E}_t^{\mathbb{Q}}[D(t, \tilde{t}_i) \cdot NPV^+(\tilde{t}_i)].$$

$$S^M(t, T) = \mathbb{E}_t^{\mathbb{Q}}[S(t, T)] = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T \lambda(s)ds}\right],$$

so that

U-CVA$(t) =$

$$\mathbf{1}_{\tau > t} LGD \sum_{i=1}^{n} \left[S^M(t, t_{i-1}) - S^M(t, t_i)\right] \times \mathbb{E}_t^{\mathbb{Q}}\left[D(t, \tilde{t}_i) \cdot NPV^+(\tilde{t}_i)\right].$$

Options

Thus we really want to know the terms

$$\mathbb{E}_t^{\mathbb{Q}}\left[D(t,t_i)\cdot NPV^+(t_i)\right].$$

These, however, are simply European option prices on the underlying derivative, also known as *compound options* with expiry in time t_i . Sometimes these are easy to price, and sometimes not. For example if the derivative is a swap, it is common for us to be using rates models where swaptions have been set up to have quick closed form formulas for (approximate or exact) swaption price formulas.

Vanilla Interest Rate Swaps and Swaptions

So for a swap

$$\text{U-CVA}(t) =$$

$$\mathbf{1}_{\tau>t} LGD \sum_{i=1}^{n} \left[S^M(t, t_{i-1}) - S^M(t, t_i) \right] \times \text{Swaption}(t; t_j, t_n)$$