A Bivariate Lognormal Approximation for the Valuation of Options on Baskets Containing Both Positive and Negative Asset Weights

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ABSTRACT

The "Levy Basket Model" can be used to value options on baskets of lognormally distributed assets, as long as the weighting on each asset in the basket has the same sign. This paper presents an presents a two-dimensional extension of the Levy approach to handle options on baskets containing mixed positive and negative asset weights. The resulting option valuation formula requires a one-dimensional numerical quadrature, which can be computed very efficiently. The valuation formula has been implemented and verified against the corresponding two-dimensional Monte Carlo integration.

Introduction

In Black-Scholes option valuation, it is commonly assumed that the future distributions of assets (such as currencies) are lognormal. It is well known, however, that baskets of such assets, comprising weighted arithmetic sums of lognormal variables, are not lognormal. In the widely-used "Levy Basket Model", the basket distribution is approximated by a lognormal variable whose mean and variance exactly match the mean and variance of the (risk-neutral) basket distribution. The value of an option on the basket, then, is the discounted expected value of the option payoff, with respect to the distribution of the approximating lognormal variable. For European options, this leads to the famililar Black-Scholes analytic valuation formula. This approach is currently used at FCNBD for pricing options on currency baskets, when the weighting on each currency in the basket has the same sign.

For baskets containing both positive and negative asset weights (hereafter dubbed "mixed-weight baskets"), the above approximation breaks down. The problem is that, while the Levy Basket Model represents the basket distribution as a lognormal variable which is constrained to remain positive, the terminal distribution for a mixed-weight basket has positive probability of both positive and negative outcomes. The resulting error increases as the positive basket components ("positive sub-basket") and negative components ("negative sub-basket") approach equal magnitude.

Following in the spirit of the Levy Basket Model, we can approach the mixed-weight basket problem by modeling the positive sub-basket and the negative sub-basket, respectively, as two jointly lognormal variables. The mean and variance of the approximating lognormal variables are set to match the (risk-neutral) mean and variance of the corresponding sub-baskets. The correlation of the approximating lognormal distribution is set to match the correlation between the sub-baskets.

The value of an option on the mixed-weight basket, then, is just the discounted expected value of the option payoff, with respect to the approximating bivariate lognormal distribution. The resulting double integral can be integrated analytically with respect to one of the underlying variables, but must be integrated numerically with respect to the other. This one-dimensional numerical quadrature is very fast, and is practical for all production trading applications.

Development

The basket payoff depends on the terminal distribution of $i = 1 \ldots N$ assets, whose prices at time t are given by $S_i(t)$. The value at time t of the basket $B(t)$ is given by:

$$B(t) = \sum_{i=1}^{N} w_i S_i(t) \tag{1}$$

where w_i is the weight of asset i in the basket. The assets in the basket are assumed to follow the the following risk-neutral geometric Brownian motion, with correlated shocks:

$$dS_i = (r - r_i) S_i dt + \sigma_i S_i dz_i \qquad \forall i, i = 1, N \tag{2}$$

$$E[dz_i dz_j] = \rho_{ij} dt \qquad \forall ij, i = 1, N, j = 1, N \tag{3}$$

where:

$$\begin{gathered} r_i \equiv \text{Interest Rate on Asset } i \qquad r \equiv \text{Interest Rate on P\&L Currency} \\ \sigma_i \equiv \text{Volatility of Asset } i \\ \rho_{ij} \equiv \text{Correlation between Asset } i \text{ and Asset } j \end{gathered} \tag{4}$$

For the mixed-weight basket, the set of basket component indices can be mapped into a pair of non-empty subsets, I_+ and I_-, representing the positive and negative sub-basket components:

$$I_+ \equiv \{i \in \{1 \ldots N\};\ w_i > 0\} \qquad I_- \equiv \{i \in \{1 \ldots N\};\ w_i < 0\} \tag{5}$$

The basket value, $B(t)$, can then be represented as the difference between the positive and negative sub-baskets, $B_+(t)$ and $B_-(t)$:

$$B(t) = \sum_{i=1}^{N} w_i S_i(t) = B_+(t) - B_-(t) = \sum_{j \in I_+} w_j S_j(t) + \sum_{k \in I_-} w_k S_k(t) \tag{6}$$

Each sub-basket obeys the following risk-neutral geometric Brownian motion:

$$dB_{\pm} = (r - r_{\pm}) B_{\pm}\, dt + \sigma_{\pm} B_{\pm}\, dz_{\pm} \tag{7}$$

$$E[dz_+ dz_-] = \rho_{\pm}\, dt \tag{8}$$

where:

$$\begin{aligned} r_{\pm} &\equiv \text{Implied Interest Rate of Sub-Basket} \\ \sigma_{\pm} &\equiv \text{Volatility of Sub-Basket} \\ \rho_{\pm} &\equiv \text{Correlation between Positive and Negative Sub-Baskets} \end{aligned} \tag{9}$$

For a basket option with time-to-expiry T, the moment matching relationships for the two sub-baskets are as follows:

$$\begin{gathered} E[B_{\pm}(T) | B_{\pm}^0] = B_{\pm}^0 \exp((r - r_{\pm})T) = \\ \pm E\left[\sum_{i \in I_{\pm}} w_i S_i(T) | S_i^0\right] = \pm \sum_{i \in I_{\pm}} w_i S_i^0 \exp((r - r_i)T) = \pm \sum_{i \in I_{\pm}} w_i F_i(T) \end{gathered} \tag{10}$$

$$\begin{gathered} E[B_{\pm}(T)^2 | B_{\pm}^o] = (B_{\pm}^0 \exp((r - r_{\pm})T))^2 \exp(\sigma_{\pm}^2 T) = \\ E\left[\left(\sum_{i \in I_{\pm}} w_i S_i(T)\right)^2 \middle| S_i^0\right] = \sum_{j \in I_{\pm}} \sum_{k \in I_{\pm}} w_j w_k F_j(T) F_k(T) \exp(\rho_{jk} \sigma_j \sigma_k T) \end{gathered} \tag{11}$$

$$\begin{gathered} E[B_+(T) B_-(T) | B_+^0, B_-^0] = (B_+^0 \exp((r - r_+)T))(B_-^0 \exp((r - r_-)T)) \exp(\rho_{\pm} \sigma_+ \sigma_- T) \\ = E\left[\left(\sum_{i \in I_+} w_i S_i(T)\right)\left(-\sum_{j \in I_-} w_j S_j(T)\right) \middle| S_i^0, S_j^0\right] = -\sum_{i \in I_+} \sum_{j \in I_-} w_i w_j F_i(T) F_j(T) \exp(\rho_{ij} \sigma_i \sigma_j T) \end{gathered} \tag{12}$$

After using these relations to fix the parameters $B_{\pm}^0$, $r_{\pm}$, $\sigma_{\pm}$ and $\rho_{\pm}$, we can compute the terminal joint density function for the two sub-baskets, $B_{\pm}(T)$. It is easiest at this point to work in log-price variables. Defining:

$$\begin{aligned} X_{\pm} &\equiv \log(S_{\pm}(T)) \\ \bar{X}_{\pm} &\equiv E[\log(S_{\pm}(T)) | S_{\pm}^0] = \log(S_{\pm}^0) + \left(r - r_{\pm} - \frac{1}{2}\sigma_{\pm}^2\right)T \\ \hat{X}_{\pm} &\equiv X_{\pm} - \bar{X}_{\pm} \end{aligned} \tag{13}$$

then the terminal conditional joint density function, $\rho\,(\hat{X}_+, \hat{X}_-\,|\,\hat{X}^0_+, \hat{X}^0_-)$ can be written:

$$\rho\,(\hat{X}_+, \hat{X}_-\,|\,\hat{X}^0_+, \hat{X}^0_-) = \left((2\pi)^2\,(\sigma^2_+\sigma^2_-T^2)\,(1-\rho^2_\pm)\right)^{\frac{-1}{2}}$$
$$exp\left(\left(\frac{-1}{2\,(1-\rho^2_\pm)}\right)\left(\frac{\hat{X}^2_+}{\sigma^2_+T} - \frac{2\rho_\pm\,\hat{X}_+\hat{X}_-}{\sigma_+\sigma_-T} + \frac{\hat{X}^2_-}{\sigma^2_-}\right)\right) \tag{14}$$

Given the joint density function, the value of the European option on the basket can be written as:

$$W\,(X^0_+, X^0_-, T) = exp\,(-rT)\int_{-\infty}^{\infty} d\hat{X}_- \int_{-\infty}^{\infty} d\hat{X}_+\,\rho\,(\hat{X}_+, \hat{X}_-\,|\,\hat{X}^0_+, \hat{X}^0_-) \times$$
$$MAX\,(\phi \times (exp\,(\hat{X}_+ + \bar{X}_+) - exp\,(\hat{X}_- + \bar{X}_-) - K), 0) \tag{15}$$

where:

$$K \equiv \text{Strike Price in P\&L Currency}$$
$$\phi \in \{-1, 1\} \qquad -1 \Rightarrow \text{Put on Basket} \qquad 1 \Rightarrow \text{Call on Basket} \tag{16}$$

Before carrying out the indicated integrations, we would like to remove the $MAX(\ )$ by modifying the limits of integration. Fixing the value of $\hat{X}_-$, we can compute an "adjusted strike", $\hat{K}\,(\hat{X}_-)$, and it's logarithm, $\hat{X}_K\,(\hat{X}_-)$, defined by:

$$\hat{K}\,(\hat{X}_-) = exp\,(\hat{X}_- + \bar{X}_-) + K$$
$$\hat{X}_K\,(\hat{X}_-) = \log\,(\hat{K}\,(\hat{X}_-)) \tag{17}$$

In terms of these variables, then, we can rewrite the valuation integrals as follows:

$$W\,(X^0_+, X^0_-, T) = exp\,(-rT)\int_{-\infty}^{\infty} d\hat{X}_- \times \phi \int_{\hat{X}_K(\hat{X}_-) - \bar{X}_+}^{\phi \times \infty} d\hat{X}_+\,\rho\,(\hat{X}_+, \hat{X}_-\,|\,\hat{X}^0_+, \hat{X}^0_-) \times$$
$$\phi \times (exp\,(\hat{X}_+ + \bar{X}_+) - \hat{K}\,(\hat{X}_-)) \tag{18}$$

We can analytically integrate the inner integral with respect to $\hat{X}_+$, to get:

$$W(X_+^0, X_-^0, T) = exp(-rT) \int_{-\infty}^{\infty} d\hat{X}_- \times (\sigma_-^2 T)^{\frac{-1}{2}} n\left(\frac{\hat{X}_-}{\sigma_- T^{1/2}}\right) \times$$

$$\left[exp\left(X_+^0 + \left(r - r_+ - \frac{\rho_\pm^2 \sigma_+^2}{2}\right) T + \frac{\rho_\pm \hat{X}_- \sigma_+}{\sigma_-}\right) \times \right.$$

$$\left(N\left(\frac{X_+^0 + \left(r - r_+ - \frac{\sigma_+^2}{2}\right) T - \hat{X}_K(\hat{X}_-)}{((1-\rho_\pm^2)\sigma_+^2 T)^{1/2}} + ((1-\rho_\pm^2)\sigma_+^2 T)^{1/2} + \frac{\rho_\pm \hat{X}_-}{((1-\rho_\pm^2)\sigma_-^2 T)^{1/2}} \right) + \frac{\phi - 1}{2} \right) \tag{19}$$

$$\left. -\hat{K}(\hat{X}_-) \left(N\left(\frac{X_+^0 + \left(r - r_+ - \frac{\sigma_+^2}{2}\right) T - \hat{X}_K(\hat{X}_-)}{((1-\rho_\pm^2)\sigma_+^2 T)^{1/2}} + \frac{\rho_\pm \hat{X}_-}{((1-\rho_\pm^2)\sigma_-^2 T)^{1/2}} \right) + \frac{\phi - 1}{2} \right) \right]$$

where $n(\ )$ and $N(\ )$ are the normal density function, and normal distribution function, respectively. The remaining integration with respect to $\hat{X}_-$ cannot be done analytically, due to the complicated dependence of $\hat{X}_K(\hat{X}_-)$ and $\hat{K}(\hat{X}_-)$ on $\hat{X}_-$. However, this integral can be computed very quickly by a one-dimensional numerical integration.

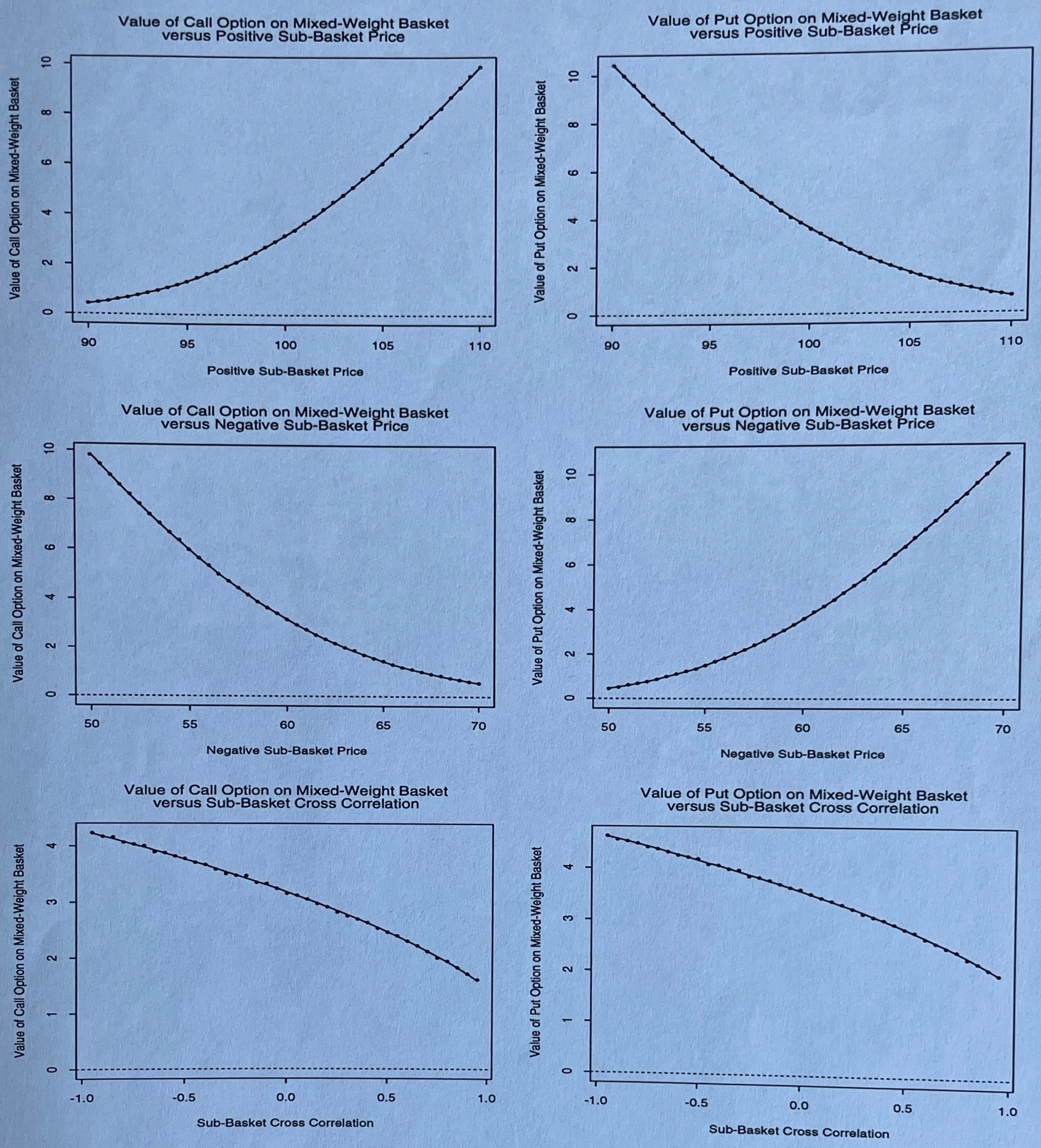
Results

A simple example is presented to demonstrate the behavior of the mixed-weight basket option model. In the following plots, the following "base-case" model parameters were assumed:

$$\begin{array}{ccc} B_+^0 = 100 & B_-^0 = 60 & K = 40 \\ \sigma_+ = 0.08 & \sigma_- = 0.06 & \rho_\pm = 0.0 \\ r_+ = 0.04 & r_- = 0.02 & r = 0.06 \\ & T = 1 \text{ year} & \end{array} \tag{20}$$

On the following page, the option value for puts and calls on the basket are presented, plotted versus: (1) the positive sub-basket spot price, (2) the negative sub-basket spot price, and (3) the correlation between the positive and negative sub-baskets. The model outputs are plotted as solid lines.

To validate the model, an independent two-dimensional Monte Carlo integration was also performed. The Monte Carlo integration results are plotted as superimposed dots.



As we would expect, increasing the positive sub-basket price increases the values of calls and reduces the value of puts. Increasing the negative sub-basket price decreases the value of calls, and increases the value of puts.

The effect of cross-correlation between the positive and negative sub-baskets is more interesting. As the correlation increases, the values of the both calls and puts decrease dramatically. When the positive and negative sub-baskets are positively correlated, variations in the sub-baskets tend to cancel each other out. This reduces the mixed-weight basket volatility, and therefore reduces the value of options on the basket.

This could have significant practical implications. For many of the basket options we might be asked to price, such as basket options struck in USD, most elements of the correlation matrix tend to be positive. When all basket weights have the same sign, this positive correlation always tends to increase the basket volatility, and the corresponding option premium.

For mixed-weight baskets the situation is not so simple. Within each sub-basket, correlation increases the sub-basket volatility, and the option premium. However, correlation between the components of the positive and negative sub-baskets reduces the overall basket volatility, and the option premium. Other things being equal, it appears that in a highly correlated environment, options on mixed-weight baskets might be more attractively priced than options on unmixed baskets.

Conclusions

A simple bivariate lognormal approximation has been developed for the valuation of options on baskets containing both positive and negative asset weights. The resulting valuation formula involves a fast, one-dimensional numerical integration. The model has been implemented, and has been verified against an independent two-dimensional Monte Carlo integration.

Preliminary examination of the model indicates the possibility of reduced premiums for options on mixed-weight baskets versus un-mixed baskets, when the underlying basket assets are positively correlated. This is often the case for options on currency baskets.