STOCHASTIC CALCULUS OF RATES

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Basic Principles

- The price processes of tradeable assets are *martingales* under any *risk-neutral probability measure*.
- Risk-neutrality of a probability measure depends on the "perspective" which we will call the *numeraire*.
- Currencies are **not** themselves tradeable assets, there are interest-bearing accounts involved: *money market* accounts)

Exchange Rate Model

Let Y_t denote the exchange rate at time t between US Dollars \$ and UK Pounds Sterling £, i.e., the number of pounds that one dollar will buy. Our usual simple exponential brownian motion model is then

$$dY_t = \mu_t Y_t \, dt + \sigma_t Y_t \, dW_t$$

where W_t is a standard Wiener process under the risk neutral measure for £ investors, and μ and σ are often constants.

Interest Rates

In both US Dollar and UK Pound Sterling we will assume the money market is riskless. (See the crisis of 2008 for counterexamples).

Define A_t and B_t as the "share prices" of US Money Market and UK Money Market, respectively, and for simplicity assume that the time-zero share prices are both 1.

Interest Rates (continued)

Let us say that the riskless rates of return r_A, r_B in the two currencies are constant, but they are not (necessarily) equal to each other. Then

$$A_t = e^{r_A t} \text{ dollars}$$

$$B_t = e^{r_B t} \text{ pounds}$$

You may wonder how this could be consistent with traded values, and why anyone would own the currency with a lower interest rate. The answer is that *forward rates are not the same as* current rates.

Exchange and Interest Rates

The asset US Money Market is riskless to a Dollar investor, but it is not so to a Pound Sterling investor. As perceived in Pounds Sterling, the share price of the US Money Market asset is

$$A_t Y_t = Y_0 \exp\left(r_A t + \mu t - \sigma^2 t/2 + \sigma W_t\right)$$

where W_t is a standard Wiener Process under the risk neutral probability measure Q_B for Pound investors.

Exchange and Interest Rates: Main Result

We will now argue that

$$\mu = r_B - r_A$$

Proof

Since US Money Market is a tradeable asset, its share price Y_0 at time $t = 0$ must be the expected value of its discounted share price $A_t Y_t$ (in £) at time t, where the discount rate is r_B, and the expectation is taken under Q_B.

Proof (continued)

Thus

$$Y_0 = \mathbb{E}_{Q_B}\left[e^{-r_B t} A_t Y_t\right]$$

Proof (continued)

$$
\begin{aligned}
Y_0 &= \mathbb{E}_{Q_B}\left[e^{-r_B t} A_t Y_t\right] \\
&= \mathbb{E}_{Q_B}\left[e^{-r_B t} Y_0 \exp\left(r_A t + \mu t - \sigma^2 t + \sigma W_t\right)\right]
\end{aligned}
$$

Proof (conclusion)

$$
\begin{aligned}
Y_0 &= \mathbb{E}_{Q_B}\left[e^{-r_B t} A_t Y_t\right] \\
&= \mathbb{E}_{Q_B}\left[e^{-r_B t} Y_0 \exp\left(r_A t + \mu t - \sigma^2 t + \sigma W_t\right)\right] \\
Y_0 &= Y_0 \exp\left((r_A - r_B + \mu - \sigma^2/2)t\right) \mathbb{E}_{Q_B}\big[\exp(\sigma W_t)\big]
\end{aligned}
$$

but we know from Itô that

$$
\mathbb{E}_{Q_B}\big[\exp(\sigma W_t)\big] = \frac{1}{2}\sigma^2 t
$$

so

$$
Y_0 = Y_0 \exp\left\{(r_A - r_B + \mu)t\right\}
$$

Therefore $\mu = r_B - r_A$.

Currency Options

Consider a call option that gives the owner the right to buy \$1 for £K at time T. What is the arbitrage price at time 0?

Currency Options: Solution

Solution: The option is identical to a call on $e^{-r_A T}$ shares of the US Money Market. To a £ investor, the US Money Market is a risky asset with price process $e^{-r_A t} Y_t$. Thus, the call option may be priced using the Black-Scholes Formula.

Risk-Neutral Measure for $

Let Q_A be the risk-neutral probability measure for the US Dollar investor, and Q_B the risk-neutral measure for the UK Pound Sterling investor. Unless $\sigma = 0$ (that is, unless the exchange rate is purely deterministic), it must be the case that

$$Q_A \neq Q_B$$

This is a special case of a more general phenomenon:

Numeraire Change

Suppose that a market has tradeable assets A, B with share price processes S_t^A and S_t^B (evaluated in a common numeraire C). Let Q_A and Q_B be risk-neutral measures for numeraires A, B, respectively.

Numeraire Change: Main Theorem

Usually $Q_A \neq Q_B$. $Q_A = Q_B$ if , and only if, S_t^A / S_t^B is a constant "random" variable.

Furthermore, in general, for any finite time T, our theorem is that the numeraire change satisfies

$$\left(\frac{dQ_B}{dQ_A} \right)_{F_T} = \left(\frac{S_T^B}{S_T^A} \right) \left(\frac{S_0^A}{S_0^B} \right)$$

Which is to say, the correction term depends on differing drift rates.

Result for FX

In the foreign exchange context, the riskless assets for the two numeraires are US Money Market and UK Money Market, with share prices (in \$)

$$A_t = \exp(r_A t)$$
$$B_t = \exp(r_B t)/Y_t$$

Consequence (continued)

The Radon-Nikodym derivative, or likelihood ratio, between the risk-neutral measures for £ and \$ investors is

$$\left(\frac{dQ_B}{dQ_A}\right)_{F_T} = \left(\frac{Y_T}{Y_0}\right)^{-1} \exp((r_B - r_A)T)$$

Note that we saw likelihood ratios before, when doing importance sampling. This one is looks little simpler, having no quadratic terms. But they are still representing different measures.

Likelihood Ratio Identity

Let V_t^i be the time-t share price of any contingent claim in numeraire $i = A, B, C$. These share prices satisfy:

$$V_t^A = V_t^C / S_t^A$$
$$V_t^B = V_t^C / S_t^B$$

Likelihood Ratio Identity (continued)

The time-zero share price is the discounted expected value of the time-t share price for each of the numeraires A, B. The discount factors are 1, so

$$V_0^A = V_0^C / S_0^A = \mathbb{E}_A\Big[V_t^C / S_t^A\Big]$$

$$V_0^B = V_0^C / S_0^B = \mathbb{E}_B\Big[V_t^C / S_t^B\Big]$$

Likelihood Ratio Identity: Main Result

It follows that for *every* contingent claim V with share price V_t^C (in numeraire C),

$$S_0^A \mathbb{E}_A(V_t^C/S_t^A) = S_0^B \mathbb{E}_B\Big[V_t^C/S_t^B\Big]$$

Likelihood Ratio Identity: Application

Apply this to the contingent claim with payoff $V_T^C S_T^B$ at time T to obtain the following identity, valid for all nonnegative random variables V_T^C measurable F_T:

$$\mathbb{E}_B V_T^C = \mathbb{E}_A \left[V_T^C \left(\frac{S_T^B S_0^A}{S_T^A S_0^B} \right) \right]$$

This is the defining property of a Radon-Nikodym derivative / likelihood ratio.

Exponential Martingales

Let W_t be a standard Wiener process, with Brownian filtration F_t, and let θ_t be a bounded, adapted process. Define

$$Z_t = \exp\left(\int_0^t \theta_s \, dW_s - \int_0^t \theta_s^2 \, ds/2\right)$$

Exponential Martingales: Key Fact

Fact: Z_t is a positive martingale.
Proof: Itô!

$$\begin{aligned} dZ_t &= Z_t\theta_t \, dW_t - Z_t\theta_t^2 \, dt/2 + Z_t\theta_t^2 \, dt/2 \\ &= Z_t\theta_t \, dW_t \end{aligned}$$

$$\Rightarrow Z_t = Z_0 + \int_0^t Z_s\theta_s \, dW_s$$

Girsanov's Theorem

Because Z_t is a positive martingale under P with initial value $Z_0 = 1$, for every fixed time T the random variable Z_T is a likelihood ratio: that is,

$$Q(F) := \mathbb{E}_P[I_F Z_T]$$

defines a new probability measure on possible events observable by time T.

Girsanov's Theorem: Main Result

Girsanov's theorem states that under the measure Q, the process

$$\left\{ W_t - \int_0^t \theta_s \, ds \right\}_{0 \le t \le T}$$

is a *standard Wiener process.*

Exchange Rates

Consider again the \$ and £ currencies. Assume that each has a riskless Money Market, and that the rates of return r_A, r_B *are constant. Assume that the exchange rate* Y_t *obeys*

$$dY_t = (r_B - r_A)Y_t\,dt + \sigma Y_t\,dW_t$$

where W_t *is a standard Wiener process under the risk-neutral probability* Q_B *for £ investors. Thus,*

$$Y_t = Y_0 \exp\left((r_B - r_A - \sigma^2/2)t + \sigma W_t\right).$$

Exchange Rates: Likelihood Ratio

Since

$$\left(\frac{dQ_A}{dQ_B}\right)_{F_T} = \left(\frac{Y_T}{Y_0}\right) \exp\big(-(r_B - r_A)T\big)$$
$$= \exp\Big(\sigma W_T - \sigma^2 T/2\Big)$$

Exchange Rates: Girsanov Application

Girsanov implies that under Q_A the process W_t is a Wiener process with drift σ. Thus, to the \$ investor, it appears that the exchange rate obeys

$$dY_t = (r_B - r_A - \sigma^2/2) Y_t \, dt + \sigma Y_t \, d\tilde{W}_t$$

where $\tilde{W}_t$ is a standard Wiener process under Q_A.

Note the similarity to importance sampling, which also had a quadratic term in the correction.