STOCHASTIC CALCULUS OF RATES: 1-FACTOR EXAMPLE

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Girsanov Drift Equivalence

Assume various useful mathematical conditions. Recall the equivalence between adding/subtracting drift and changing probability measure via the Radon-Nikdym derivative, i.e. the quotient of probabilities.

Let's take $W_t^{\mathbb{Q}}$ as

$$dW_t^{\mathbb{Q}} = dW_t^{\mathbb{P}} + \lambda_t \, dt$$

Then, under the new measure $\mathbb{Q}$ defined via the Radon-Nikodym derivative

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_T} = \exp\left(-\int_0^T \lambda_s \, dW_s^{\mathbb{P}} - \frac{1}{2}\int_0^T \lambda_s^2 \, ds \right)$$

the process $W_t^{\mathbb{Q}}$ is a Brownian motion under $\mathbb{Q}$.

SDE Form

In SDE form: If X_t satisfies, under $\mathbb{P}$,

$$dX_t = \mu_t \, dt + \sigma_t \, dW_t^{\mathbb{P}}$$

then under $\mathbb{Q}$,

$$dX_t = [\mu_t - \sigma_t \lambda_t] \, dt + \sigma_t \, dW_t^{\mathbb{Q}}$$

Why Do We Like Girsanov?

Why do we care about this?

We like the ability to use Girsanov because, by adjusting drift terms so as to cancel out μ_t, we achieve stationary brownian motions. These have

- "Easy"-to-compute expectation integrals
- Fast Monte Carlo simulation

Interest Rates Model: Hull-White 1-Factor

$$\begin{aligned} \text{Real-world probabilities } (\mathbb{P}) \text{ SDE:} \quad & dr_t = [\theta(t) - ar_t]\,dt + \sigma\,dW_t^{\mathbb{P}} \\ \text{Risk-neutral probabilities } (\mathbb{Q}) \text{ SDE:} \quad & dr_t = [\phi(t) - ar_t]\,dt + \sigma\,dW_t^{\mathbb{Q}} \\ \text{Drift applied:} \quad & dW_t^{\mathbb{Q}} = dW_t^{\mathbb{P}} + \lambda_t\,dt \end{aligned}$$

Thus,

$$\phi(t) = \theta(t) - \sigma\lambda_t$$

Drift under $\mathbb{Q}$ Using Girsanov's Theorem

$$
\begin{aligned}
dr_t &= [\theta(t) - ar_t]\, dt + \sigma\, dW_t^{\mathbb{P}} \\
&= [\theta(t) - ar_t]\, dt + \sigma\big(dW_t^{\mathbb{Q}} - \lambda_t\, dt\big) \\
&= [\theta(t) - ar_t - \sigma\lambda_t]\, dt + \sigma\, dW_t^{\mathbb{Q}}
\end{aligned}
$$

Finding $\phi(t)$

We typically calibrate drift $\phi(t)$ under $\mathbb{Q}$ is such that the model fits the initial term structure. In the Hull-White model:

$$\phi(T) = \frac{\partial f(0,T)}{\partial t} + af(0,T) + \frac{\sigma^2}{2a}\left(1 - e^{-2aT}\right)$$

where $f(0,T)$ is the initial instantaneous forward rate at tenor T.

Calculating λ_t

From the previous equations,

$$\lambda_t = \frac{\theta(t) - \phi(t)}{\sigma}$$

Plugging in the expression for $\phi(t)$ gives:

$$\boxed{\lambda_t = \frac{1}{\sigma}\left(\theta(t) - \frac{\partial f(0,t)}{\partial t} - af(0,t) - \frac{\sigma^2}{2a}\big(1 - e^{-2at}\big)\right)}$$

Zero-coupon Bond Price

The zero-coupon bond price $P(t, T, r)$ can be solved in closed form as:

$$P(t, T, r) = A(t, T) \cdot \exp\Big(- B(t, T)\, r \Big)$$

where

$$B(t, T) = \frac{1 - e^{-a(T-t)}}{a}$$

$$A(t, T) = \exp\left(\int_t^T \left[\phi(s) - \frac{\sigma^2}{2a^2}(1 - e^{-a(T-s)})^2 \right] B(s, T) ds \right)$$

1FHW PDE

Let $V(t, r)$ denote the price at time t when the short rate is r.

Recall we ahve:

$$dr_t = [\phi(t) - ar_t]\, dt + \sigma\, dW_t^{\mathbb{Q}}$$

where $\phi(t)$ is the risk-neutral drift as given earlier.

Applying Itô's Lemma to $V(t, r_t)$ in 1FHW

$$
\begin{aligned}
dV &= \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial r} dr_t + \frac{1}{2} \frac{\partial^2 V}{\partial r^2} (dr_t)^2 \\
&= \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial r} ([\phi(t) - a r_t] dt + \sigma dW_t^{\mathbb{Q}}) + \frac{1}{2} \frac{\partial^2 V}{\partial r^2} \sigma^2 dt \\
&= \left(\frac{\partial V}{\partial t} + [\phi(t) - a r_t] \frac{\partial V}{\partial r} + \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial r^2} \right) dt + \sigma \frac{\partial V}{\partial r} dW_t^{\mathbb{Q}}
\end{aligned}
$$

Risk-Neutral Pricing in 1FHW

The value process discounted at r_t must be a $\mathbb{Q}$-martingale, so the drift must equal $r_t V$, i.e.,

$$dV = r_t V \, dt + \text{martingale terms}$$

Thus, the pricing PDE is:

$$\boxed{\frac{\partial V}{\partial t} + [\phi(t) - ar]\frac{\partial V}{\partial r} + \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial r^2} - rV = 0}$$

with appropriate terminal/boundary conditions depending on the claim being priced.

Hull White Added To Heston Extension To Black-Scholes

Let's consider what the math looks like when we combine the 1-factor Hull White model with Heston. In the Heston model,

$$\frac{dS(t)}{S(t)} = r\,dt + \sqrt{V(t)}\,dW_1(t), \quad S(t) > 0,$$
$$dV(t) = \kappa(\eta - V(t))\,dt + \gamma\sqrt{V(t)}\,dW_2(t), \quad V(t) > 0,$$

Linked SDEs

Combining with 1FHW we ahve

$$
\begin{aligned}
dS(t) &= R(t)S(t)\,dt + \sqrt{V(t)}S(t)\,dW_1(t),\\
dV(t) &= \kappa(\eta - V(t))\,dt + \sigma_V\sqrt{V(t)}\,dW_2(t),\\
dR(t) &= a(b(t) - R(t))\,dt + \sigma_R\,dW_3(t),
\end{aligned}
$$

Option Price

The risk-neutral value for a European option is:

$$\varphi(s,v,r,t) =$$

$$\mathbb{E}\left[\exp\left(-\int_t^T R(\varsigma)d\varsigma\right) F(S(T),V(T),R(T)) \,\middle|\, \{S,V,R\} = \{s,v,r\}\right]$$

with v as the variance and the other arguments obvious. $F(\cdot)$ is the payoff function.

Pricing PDE

Write our pricing function as

$$u(s, v, r, \tau) = \varphi(s, v, r, T - \tau).$$

The PDE we solve is:

$$\begin{aligned}
\frac{\partial u(s, v, r, \tau)}{\partial \tau} = & \frac{1}{2} s^2 v \frac{\partial^2 u}{\partial s^2} + \frac{1}{2} \sigma_V^2 v \frac{\partial^2 u}{\partial v^2} + \frac{1}{2} \sigma_R^2 \frac{\partial^2 u}{\partial r^2} \\
& + \rho_{SV} \sigma_V s v \frac{\partial^2 u}{\partial s \partial v} \\
& \quad + \rho_{SR} \sigma_R s \sqrt{v} \frac{\partial^2 u}{\partial s \partial r} + \rho_{VR} \sigma_V \sigma_R \sqrt{v} \frac{\partial^2 u}{\partial v \partial r} \\
& + r s \frac{\partial u}{\partial s} + \kappa(\eta - v) \frac{\partial u}{\partial v} + a(b(T - \tau) - r) \frac{\partial u}{\partial r} \\
& - r u
\end{aligned}$$

Formulas For European Swaptions

Let's consider a payer swaption, with notional N, which matures at some future time T_0 and has a strike rate K. Further suppose that the first reset of the underlying swap coincides with T_0, and that the underlying swap matures at T_n. This swaption has the following payoff at T_0:

$$N\left(1 - P(T_0, T_N) - \sum_{i=1}^{n} K\tau_i P(T_0, T_i)\right)^+ .$$

Recall Jamshidian's trick saying this swaption can be valued at $t < T_0$ as an option on a coupon bearing bond. Let $c_i = K\tau_i$ for $i = 1, 2, \ldots, n-1$ and $c_n = 1 + K\tau_n$.

First, an r^* must be found such that

$$\sum_{i=1}^{n} c_i A(T_0, T_i) e^{-B(T_0,T_i) r^*} = 1.$$

Set The Strikes

Then, setting $K_i = A(T_0, T_i)e^{-B(T_0,T_i)r^*}$, the value of the payer swaption at t is given by

$$V_{\text{PS}}(t) = N \sum_{i=1}^{n} c_i \text{ZPUT}(t; T_0, T_i, K_i),$$

where $\text{ZPUT}(t; T_0, T_i, K_i)$ is the time t value of a put maturing at T_0 on zero-coupon bond with maturity T_i, struck at K_i. Each zero-coupon bond put has value

$$\text{ZPUT}(t, T_0, T_i, K_i) = K_i P(t, T_0)\Phi(-h_i + \sigma_{p,i}) - P(t, T_i)\Phi(-h_i),$$

where Φ is the standard normal cumulative distribution function, and

$$\sigma_{p,i} = \sigma\sqrt{\frac{1 - e^{-2a(T_0 - t)}}{2a}} B(T_0, T_i)$$

$$h_i = \frac{1}{\sigma_{p,i}} \ln\left(\frac{P(t, T_i)}{P(t, T_0) K_i}\right) + \frac{\sigma_{p,i}}{2}.$$

Receiver

Similarly the value of a receiver swaption is

$$V_{\mathrm{RS}}(t) = N \sum_{i=1}^{n} c_i \mathrm{ZCALL}(t; T_0, T_i, K_i),$$

where $\mathrm{ZCALL}(t; T_0, T_i, K_i)$ is a zero-coupon bond call with value

$$\mathrm{ZCALL}(t, T_0, T_i, K_i) = P(t, T_i)\Phi(h_i) - K_i P(t, T_0)\Phi(h_i - \sigma_{p,i}),$$

and $\sigma_{p,i}$ and h_i are defined as before.