STOCHASTIC CALCULUS OF RATES

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Basic Principles

- The price processes of tradeable assets are *martingales* under any *risk-neutral probability measure*.
- Risk-neutrality of a probability measure depends on the "perspective" which we will call the *numeraire*.
- Currencies are **not** themselves tradeable assets, there are interest-bearing accounts involved: *money market* accounts)

Exchange Rate Model

Let Y_t denote the exchange rate at time t between US Dollars \$ and UK Pounds Sterling £, i.e., the number of pounds that one dollar will buy. Our usual simple exponential brownian motion model is then

$$dY_t = \mu_t Y_t \, dt + \sigma_t Y_t \, dW_t$$

where W_t is a standard Wiener process under the risk neutral measure for £ investors, and μ and σ are often constants.

Interest Rates

In both US Dollar and UK Pound Sterling we will assume the money market is riskless. (See the crisis of 2008 for counterexamples).

Define A_t and B_t as the "share prices" of US Money Market and UK Money Market, respectively, and for simplicity assume that the time-zero share prices are both 1.

Interest Rates (continued)

Let us say that the riskless rates of return r_A, r_B in the two currencies are constant, but they are not (necessarily) equal to each other. Then

$$A_t = e^{r_A t} \text{ dollars}$$

$$B_t = e^{r_B t} \text{ pounds}$$

You may wonder how this could be consistent with traded values, and why anyone would own the currency with a lower interest rate. The answer is that *forward rates are not the same as* current rates.

Exchange and Interest Rates

The asset US Money Market is riskless to a Dollar investor, but it is not so to a Pound Sterling investor. As perceived in Pounds Sterling, the share price of the US Money Market asset is

$$A_t Y_t = Y_0 \exp\left(r_A t + \mu t - \sigma^2 t/2 + \sigma W_t\right)$$

where W_t is a standard Wiener Process under the risk neutral probability measure Q_B for Pound investors.

Exchange and Interest Rates: Main Result

$\mu = r_B - r_A.$

Proof

Since US Money Market is a tradeable asset, its share price Y_0 at time $t = 0$ must be the expected value of its discounted share price $A_t Y_t$ (in £) at time t, where the discount rate is r_B, and the expectation is taken under Q_B.

Proof (continued)

Thus

$$Y_0 = E_{Q_B} e^{-r_B t} A_t Y_t$$

Proof (continued)

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\begin{aligned}
Y_0 &= E_{Q_B} e^{-r_B t} A_t Y_t \\
&= E_{Q_B} e^{-r_B t} Y_0 \exp\left(r_A t + \mu t - \sigma^2 t + \sigma W_t \right)
\end{aligned}
$$

Proof (continued)

$$Y_0 = Y_0 \exp\left((r_A - r_B + \mu - \sigma^2/2)t\right) E_{Q_B} \exp\{\sigma W_t$$

Proof (conclusion)

$$Y_0 = Y_0 \exp\left\{(r_A - r_B + \mu)t\right\}$$

Therefore $\mu = r_B - r_A$.

Currency Options

Consider a call option that gives the owner the right to buy \$1 for £K at time T. What is the arbitrage price at time 0?

Currency Options: Solution

Solution: The option is identical to a call on $e^{-r_A T}$ shares of the US Money Market. To a £ investor, the US Money Market is a risky asset with price process $e^{-r_A t} Y_t$. Thus, the call option may be priced using the Black-Scholes Formula.

Risk-Neutral Measure for $

Theorem 0.1. *Let Q_A be the risk-neutral probability measure for the US Dollar investor, and Q_B the risk-neutral measure for the UK Pound Sterling investor. Unless $\sigma = 0$ (that is, unless the exchange rate is purely deterministic), it must be the case that*

$$Q_A \neq Q_B$$

This is a special case of a more general phenomenon:

Numeraire Change

Suppose that a market has tradeable assets A, B with share price processes S_t^A and S_t^B (evaluated in a common numeraire C). Let Q_A and Q_B be risk-neutral measures for numeraires A, B, respectively.

Numeraire Change: Main Theorem

Theorem 0.2. *$Q_A = Q_B$ if and only if S_t^A / S_t^B is a constant random variable. Furthermore, in general, for any finite time T,*

$$\left(\frac{dQ_B}{dQ_A}\right)_{F_T} = \left(\frac{S_T^B}{S_T^A}\right)\left(\frac{S_0^A}{S_0^B}\right)$$

Consequence

In the foreign exchange context, the riskless assets for the two numeraires are US Money Market and UK Money Market, with share prices (in \$)

$$A_t = \exp\{r_A t\}$$
$$B_t = \exp\{r_B t\}/Y_t$$

Consequence (continued)

Therefore, the likelihood ratio between the risk-neutral measures for £ and \$ investors is

$$\left(\frac{dQ_B}{dQ_A}\right)_{F_T} = \left(\frac{Y_T}{Y_0}\right)^{-1} \exp\{(r_B - r_A)T\}$$

Likelihood Ratio Identity

Let V_t^i be the time-t share price of any contingent claim in numeraire $i = A, B, C$. These share prices satisfy:

$$V_t^A = V_t^C / S_t^A$$
$$V_t^B = V_t^C / S_t^B$$

Likelihood Ratio Identity (continued)

The time-zero share price is the discounted expected value of the time-t share price for each of the numeraires A, B. The discount factors are 1, so

$$V_0^A = V_0^C / S_0^A = E_A V_t^C / S_t^A$$
$$V_0^B = V_0^C / S_0^B = E_B V_t^C / S_t^B$$

Likelihood Ratio Identity: Main Result

It follows that for *every* contingent claim V with share price V_t^C (in numeraire C),

$$S_0^A E_A(V_t^C / S_t^A) = S_0^B E_B(V_t^C / S_t^B)$$

Likelihood Ratio Identity: Application

Apply this to the contingent claim with payoff $V_T^C S_T^B$ at time T to obtain the following identity, valid for all nonnegative random variables V_T^C measurable F_T:

$$E_B V_T^C = E_A V_T^C \left(\frac{S_T^B S_0^A}{S_T^A S_0^B} \right)$$

This is the defining property of a likelihood ratio.

Exponential Martingales

Let W_t be a standard Wiener process, with Brownian filtration F_t, and let θ_t be a bounded, adapted process. Define

$$Z_t = \exp\left\{\int_0^t \theta_s\,dW_s - \int_0^t \theta_s^2\,ds/2\right\}$$

Exponential Martingales: Key Fact

Fact: Z_t is a positive martingale.
Proof: Itô!

$$\begin{aligned} dZ_t &= Z_t\theta_t\, dW_t - Z_t\theta_t^2\, dt/2 + Z_t\theta_t^2\, dt/2 \\ &= Z_t\theta_t\, dW_t \end{aligned}$$

$$\Rightarrow Z_t = Z_0 + \int_0^t Z_s\theta_s\, dW_s$$

Girsanov's Theorem

Because Z_t is a positive martingale under P with initial value $Z_0 = 1$, for every fixed time T the random variable Z_T is a likelihood ratio: that is,

$$Q(F) := E_P(I_F Z_T)$$

defines a new probability measure on the σ-algebra F_T of events F that are observable by time T.

Girsanov's Theorem: Main Result

Theorem 0.3. *Under the measure Q, the process*

$$\left\{ W_t - \int_0^t \theta_s \, ds \right\}_{0 \le t \le T}$$

is a standard Wiener process.

Exchange Rates

Consider again the \$ and £ currencies. Assume that each has a riskless Money Market, and that the rates of return r_A, r_B are constant. Assume that the exchange rate Y_t obeys

$$dY_t = (r_B - r_A)Y_t\,dt + \sigma Y_t\,dW_t$$

where W_t is a standard Wiener process under the risk-neutral probability Q_B for £ investors. Thus,

$$Y_t = Y_0 \exp\left\{(r_B - r_A - \sigma^2/2)t + \sigma W_t\right\}.$$

Exchange Rates: Likelihood Ratio

Since

$$\left(\frac{dQ_A}{dQ_B}\right)_{F_T} = \left(\frac{Y_T}{Y_0}\right) \exp\{-(r_B - r_A)T\}$$
$$= \exp\left\{\sigma W_T - \sigma^2 T/2\right\}$$

Exchange Rates: Girsanov Application

Girsanov implies that under Q_A the process W_t is a Wiener process with drift σ. Thus, to the \$ investor, it appears that the exchange rate obeys

$$dY_t = (r_B - r_A - \sigma^2)Y_t\,dt + \sigma Y_t\,d\tilde{W}_t$$

where $\tilde{W}_t$ is a standard Wiener process under Q_A.