A NORMAL APPROXIMATION FOR BINOMIAL, F, BETA, AND OTHER COMMON, RELATED TAIL PROBABILITIES, I†

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This paper concerns a new Normal approximation to the beta distribution and its relatives, in particular, the binomial, Pascal, negative binomial, F, t, Poisson, gamma, and chi square distributions. The approximate Normal deviates are expressible in terms of algebraic functions and logarithms, but for desk calculation it is preferable in most cases to use an equivalent expression in terms of a function specially tabulated here.

Graphs of the error are provided. They show that the approximation is good even in the extreme tails except for beta distributions which are J or U shaped or nearly so, and they permit correction to obtain still more accuracy. For use beyond the range of the graphs, some standard recursive relations and some classical continued fractions are listed, with some properties of the latter which seem to be partly new.

Various Normal approximations are compared, with further graphs. The new approximation appears far more accurate than the others.

Everything an ordinary user of the approximation might want to know is included in this paper. The theory behind the approximation and most proofs are postponed to a second paper immediately following this one.

1. INTRODUCTION AND SUMMARY

THE binomial, Pascal, negative binomial, beta, and F distributions are all related in the sense that mathematical expressions exist relating their tail probabilities to one another. Furthermore, Poisson, chi square, and gamma tail probabilities are limiting cases of these. Consequently the problem of approximating these eight different distributions can be treated as a single mathematical problem, not eight different problems.

Direct tabulation of, for instance, the binomial distribution presents a severe interpolation problem because the function which must be tabulated varies rapidly in each of the three variables, p, n, and s, say. A Poisson or chi square approximation does not entirely eliminate this problem, but a Normal approximation, such as we are about to define, does, since it requires only single entry tables permitting a fine enough grid to make interpolation trivial. This is an advantage not only for large n and s, where even very large direct tables can include by no means every important integer value, but also for small, noninteger values, which arise, for instance, in many approximations to noncentral distributions.

Let P be a binomial, Pascal, negative binomial, beta, or F tail probability. Let Φ be the standard Normal left or right tail function according as P is a left or right tail probability, that is, for all z, let

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$$\Phi(z) = \int_{-\infty}^{z} (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}x^2} dx \quad \text{if} \quad P \text{ is a left tail probability,} \tag{1.1a}$$

$$\Phi(z) = \int_{z}^{\infty} (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}x^2} dx \quad \text{if} \quad P \text{ is a right tail probability.} \tag{1.1b}$$

Then P is closely approximated by $\Phi(z_1)$ or by a refinement $\Phi(z_2)$, where the approximate Normal deviates z_1 and z_2 are defined analytically by the first of the following formulas but more safely calculated by the second:

$$z_i = \frac{d_i}{|S - np|} \left\{ \frac{2}{1 + (6n)^{-1}} \left(S \ln \frac{S}{np} + T \ln \frac{T}{nq} \right) \right\}^{\frac{1}{2}}, \tag{1.2a}$$

$$= d_i \left\{ \frac{1 + qg\left(\frac{S}{np}\right) + pg\left(\frac{T}{nq}\right)}{(n + \frac{1}{6})pq} \right\}^{\frac{1}{2}}, \qquad (i = 1, 2) \tag{1.2b}$$

where ln denotes natural logarithms and g is a function tabulated in Table 1 and discussed further in Sections 3 and 10.

Precise definitions of n, p, q, S, T, d_1, and d_2 for each specific distribution are given in Section 2 and summarized by Table A of that section. For the binomial distribution, n is the number of trials, p and q are the probabilities of success and failure on each trial, S and T are the number of successes and the number of failures with $\frac{1}{2}$ corrections, and $d_1 = S - np + (1 - 2p)/6$ is a modification and d_2 a further modification of the difference between the number of successes observed and the number expected. For the beta distribution, q is the variable and $T + .5$ and $S + .5$ are the usual parameters. Thus J and U shaped beta distributions correspond to $T < .5$ or $S < .5$ or both.

Always $p + q = 1$ and $S + T = n$. The approximations are not defined if S or T is negative.

When limits are taken in such a way that P becomes a Poisson, gamma, or chi square tail probability, the approximations become $\Phi(z_1)$ and $\Phi(z_2)$ where

$$z_i = \frac{d_i}{|S - M|} \left(2S \ln \frac{S}{M} + 2(M - S) \right)^{\frac{1}{2}} \tag{1.3a}$$

$$= d_i \left\{ \frac{1 + g\left(\frac{S}{M}\right)}{M} \right\}^{\frac{1}{2}} \qquad (i = 1, 2). \tag{1.3b}$$

The definitions of M, S, d_1, and d_2 for these distributions are also given in Section 2. For the Poisson distribution, M is the mean. S, d_1 and d_2 have meanings similar to their previous ones, though the signs of d_1 and d_2 are reversed for the gamma and chi square distributions.

Following the definitions of Section 2, further calculating instructions and numerical examples are given in Section 3.

The approximation to the t distribution obtained by way of the standard relation between t and F is far less good than that obtained from another relation leading to equal degrees of freedom in numerator and denominator, or equivalently to $S=T$. Section 4 discusses briefly both the case $S=T$ and the t distribution, along with its relatives, the beta distribution with one parameter equal to $\frac{1}{2}$ and the F distribution with one degree of freedom in the numerator or denominator.

The accuracy of the approximation has been thoroughly investigated numerically. For the refinement z_2, graphs of the errors are given in Fig. 1 and explained in Section 5. Some over-all statements which can be made are:

$$|P-\Phi(z_2)| < \begin{cases} .001 & \text{if} \quad S, T \geq 1.5, \\ .01 & \text{if} \quad S, T \geq .5, \end{cases} \tag{1.4}$$

$$\frac{|P-\Phi(z_2)|}{P} < \begin{cases} .01 & \text{if} \quad S, T \geq 2.5 \quad \text{and} \quad .2 \leqq Sq/Tp \leqq 5, \\ .02 & \text{if} \quad S, T \geq 1.25 \quad \text{and} \quad .125 \leqq Sq/Tp \leqq 8, \\ .03 & \text{if} \quad S, T \geq 1 \quad \text{and} \quad .1 \leqq Sq/Tp \leqq 10. \end{cases} \tag{1.5}$$

(For the Poisson, gamma, and chi square distributions, take $T=\infty$ and $Sq/Tp=S/M$.) When these statements don't guarantee sufficient accuracy, the graphs just mentioned can often be used to check the accuracy and to correct the approximation if necessary, as exemplified in Section 3. Otherwise, when S or T is very small, see Section 7; when very far out in the tails, see Section 9.

The present approximation is considerably superior to several prominent Normal approximations (some of which are easier to compute, however, or have other advantages for certain purposes). For "small" samples, this is evident from Fig. 2, which gives graphs of the relative error of various approximations for $S=0.5$, 2.5, and 12.5. Section 6 defines these approximations, explains the graphs further, and describes certain asymptotic respects in which the present approximation is superior by an order of magnitude. Further details and derivations of these and other asymptotic results form the main topic of a second paper (Pratt, 1968), referred to hereafter as Part II.

Sections 7–10 could be considered appendices. Section 7 gives for reference some standard recursive relations. With these one can shift the value of S or T or both before using the approximation. This may occasionally be desirable if S or T is small, and essential if S or T is negative so that our approximation is undefined.

Section 8 gives a method of calculating small Normal tails for use when readily available tables are inadequate.

Section 9 lists some classical and some closely related continued fractions and gives some properties thereof which seem to be partly new. These can be used for desk calculation in the extreme tails. They appear useful for high-speed computers as well, but we are not directly concerned with high-speed computer methods in this paper.

The equality of the two forms of z_i in (1.2) and (1.3) follows from the analytical definition of the function g. This is given in Section 10, along with some properties of g.

The motivation for the approximation is discussed in Part II. Roughly, we started from the usual Normal approximation to the binomial distribution, $z_0=(S-np)/(npq)^{\frac{1}{2}}$, modified the numerator to d_1 and then to d_2 to improve the accuracy near the median, and then multiplied by a factor correcting the asymptotic behavior in the tails. This leads to the form (1.2b) of z_1 and z_2, where the contributions of g may be viewed as corrections since the values of g are ordinarily small.

2. DEFINITION OF SYMBOLS

This section defines the symbols we use, by means of two summary tables, Tables A and B, together with subsections giving further details for the binomial, Pascal, negative binomial, beta, F, Poisson, gamma, and chi square distributions, in that order. The subsections define these distributions in the notation to which Tables A and B apply, and give some formulas which are equivalent to and ordinarily more convenient for computation than more general formulas for the same quantities given elsewhere. However, the subsections do not repeat formulas given elsewhere and must be used in conjunction with Tables A and B.

In each case, P is defined for definiteness as a left tail probability. The complementary right tail probability has the same Normal deviate, and the approximate probabilities are just the Normal tail probabilities to the left and right, respectively, of the approximate Normal deviate. It is not necessary to convert right to left tails by interchanging p with q and S with T or the equivalent. To make the definitions of S and T apply to both tails, we adopt the following conventions for the discrete distributions. A binomial or Poisson *left* tail is a probability of *s or less;* a *right* tail is a probability of *more than s*. A Pascal *left* tail is a probability of *n or less;* a *right* tail is a probability of *more than n*. A negative binomial *left* tail is a probability of *f or less;* a *right* tail is a probability of *more than f*.

2.1 *Binomial distribution.* Here P is the probability of s or less under the binomial distribution with parameters p and n:

$$P = F_b(s; p, n) = \sum_{j=0}^{s} \binom{n}{j} p^j q^{n-j}, \qquad (0 \le p = 1 - q \le 1; s = 0, 1, \cdots, n; n = 1, 2, \cdots). \tag{2.1}$$

Some alternatives to the formulas of Table A are:

$$d_1 = s + \tfrac{2}{3} - (n + \tfrac{1}{3})p, \tag{2.2}$$

$$\begin{aligned} d_2 &= \left(s + \tfrac{2}{3} + \frac{.02}{s+1} + \frac{.01}{n+1}\right)q - \left(n - s - \tfrac{1}{3} + \frac{.02}{n-s} + \frac{.01}{n+1}\right)p \\ &= d_1 + .02\left(\frac{q}{s+1} - \frac{p}{n-s} + \frac{q - .5}{n+1}\right). \end{aligned} \tag{2.3}$$

For right tail probabilities, s is defined as the largest integer *not* in the right tail. In either tail S and T are the usual half-corrected values as given in the first line of Table A.

2.2 *Pascal distribution.* Here P is the probability of n or less under the Pascal or binomial waiting time distribution with parameters y and s:

$$P = F_{Pa}(n;\ y,\ s) = \sum_{m=s}^{n} \binom{m-1}{s-1} y^s(1-y)^{m-s}, \tag{2.4}$$

$$(0 \le y \le 1; n = s, s+1, \cdots; s > 0).$$

SUMMARY TABLES

TABLE A: BINOMIAL, PASCAL, NEGATIVE BINOMIAL, BETA AND F DISTRIBUTIONS

	S	T	n	p	q
Binomial (2.1)	$s+\frac{1}{2}$	$n-s-\frac{1}{2}$	n	p	$1-p$
Pascal (2.4)	$n-s+\frac{1}{2}$	$s-\frac{1}{2}$	n	$1-y$	y
Negative binomial (2.7)	$f+\frac{1}{2}$	$s-\frac{1}{2}$	$f+s$	$1-y$	y
Beta (2.8)	$b-\frac{1}{2}$	$a-\frac{1}{2}$	$a+b-1$	$1-y$	y
F (2.11)	$\frac{1}{2}(\nu-1)$	$\frac{1}{2}(\mu-1)$	$\frac{\mu+\nu}{2}-1$	$\frac{\nu}{\mu F+\nu}$	$1-p$
t (4.6)	See Section 4				

$$d_1 = S + \tfrac{1}{6} - (n + \tfrac{1}{3})p$$

$$d_2 = \left(S + \frac{1}{6} + \frac{.02}{S+.5} + \frac{.01}{n+1}\right) q - \left(T + \frac{1}{6} + \frac{.02}{T+.5} + \frac{.01}{n+1}\right) p$$

$$= d_1 + .02\left(\frac{q}{S+.5} - \frac{p}{T+.5} + \frac{q-.5}{n+1}\right)$$

$$z_i = d_i \left\{ \frac{1 + qg\left(\frac{S}{np}\right) + pg\left(\frac{T}{nq}\right)}{(n+\frac{1}{6})pq} \right\}^{\frac{1}{2}} \qquad (i = 1, 2)$$

TABLE B. POISSON, GAMMA, AND CHI SQUARE DISTRIBUTIONS

	S	M	d_1	d_2
Poisson (2.21)	$s+\frac{1}{2}$	M	$s+\frac{2}{3}-M$	$d_1+\frac{.02}{s+1}$
Gamma (2.22)	$r-\frac{1}{2}$	y	$y+\frac{1}{3}-r$	$d_1-\frac{.02}{r}$
Chi square (2.23)	$\frac{1}{2}(\nu-1)$	$\frac{1}{2}X^2$	$\frac{1}{2}X^2-\frac{1}{2}\nu+\frac{1}{3}$	$d_1-\frac{.04}{\nu}$

$$z_i = d_i \left\{ \frac{1 + g\left(\frac{S}{M}\right)}{M} \right\}^{\frac{1}{2}} \qquad (i = 1, 2)$$

(If needed, $T = \infty, n = \infty, p = 0, q = 1, Tp = Np = M, Sq/Tp = S/M$.)

TABLE 1. $g(x) = (1-x)^{-2}(1-x^2+2x\ln x)$

	0	1	2	3	4	5	6	7	8	9	10	Proportional parts 1	2	3	4	5
.000	1.0000	9984	9970	9957	9945	9934	9923	9912	9902	9892	9882			All differences and proportional parts are negative		
		16	14	13	12	11	11	11	10	10	10					
.00	Do not interpolate; see previous line.	9882	9790	9710	9635	9565	9499	9437	9376	9318	9262					
		92	80	75	70	66	62	61	58	56						
.0		9262	8779	8382	8039	7733	7456	7201	6965	6744	6537					
			483	397	343	306	277	255	236	221	207					
.1	6537	6341	6156	5980	5812	5652	5498	5351	5209	5073	4941					
		196	185	176	168	160	154	147	142	136	132					
.2	4941	4814	4691	4572	4456	4344	4235	4129	4027	3926	3829					
		127	123	119	116	112	109	106	102	101	97					
.3	3829	3734	3641	3551 3551	3462	3376	3291 3291	3209	3128	3049	2971	9 9 8	19 17 16	28 26 24	37 35 .32	46 43 40
.4	2971	2895	2821	2748 2748	2677	2606	2538 2538	2470	2403	2338	2274	7 7 7	15 14 13	22 21 20	30 28 26	37 35 33
.5	2274	2211	2149	2088 2088	2028	1969	1911 1911	1854	1798	1743	1688	6 6 6	12 12 11	19 18 17	25 24 22	31 30 28
.6	1688	1634	1581	1529	1478	1427 1427	1377	1328	1279	1231	1184	5 5	10 10	16 15	21 19	26 24
.7	1184	1137	1091	1046	1001	0956	0912	0869	0827	0784	0743	4	9	13	18	22
.8	0743	0701	0661	0620	0581	0541	0502	0464	0426	0388	0351	4	8	12	16	20
.9	0351	0314	0278	0242	0206	0171	0136	0102	0067	0034	0000	4	7	11	14	18

Differences and proportional parts negative. Decimal omitted. $g(0) = 1$. $g(1) = 0$. For $x > 1$, use $g(x) = -g(1/x)$.
Example: $g(.625) = .1581 - .0026 = .1555$; $g(1.6) = -g(1/1.6) = -g(.625) = -.1555$.

Some alternatives to the formulas of Table A are:

$$d_1 = (f + \tfrac{2}{3})y - (s - \tfrac{1}{3})(1 - y), \tag{2.5}$$

$$\begin{aligned} d_2 &= \left(f + \tfrac{2}{3} + \frac{.02}{f+1} + \frac{.01}{n+1}\right)y - \left(s - \tfrac{1}{3} + \frac{.02}{s} + \frac{.01}{n+1}\right)(1 - y) \\ &= d_1 + .02\left(\frac{y}{f+1} - \frac{1-y}{s} + \frac{y - .5}{n+1}\right) \end{aligned} \tag{2.6}$$

where

$$f = n - s.$$

For right tail probabilities, n is defined as the largest integer *not* in the right tail.

2.3 *Negative binomial distribution.* Here P is the probability of f or less under the negative binomial distribution with parameters y and s:

$$P = F_{nb}(f; y, s) = \sum_{k=0}^{f} \binom{s+k-1}{s-1} y^s (1-y)^k, \tag{2.7}$$
$$(0 \leq y \leq 1; f = 0, 1, 2, \cdots; s > 0).$$

Some alternatives to the formulas of Table A are (2.5) and (2.6) above, with $n = f + s$. For right tail probabilities, f is defined as the largest integer *not* in the right tail.

2.4 *Beta distribution (incomplete beta function).* Here P is the cumulative beta distribution or incomplete beta function with parameters a and b at the value y:

$$P = F_\beta(y; a, b) = I_y(a, b) = \frac{1}{B(a, b)} \int_0^y x^{a-1}(1-x)^{b-1} dx, \tag{2.8}$$
$$(0 \leq y \leq 1; a > 0; b > 0).$$

Some alternatives to the formulas of Table A are:

$$d_1 = (n + \tfrac{1}{3})y - (a - \tfrac{1}{3}), \tag{2.9}$$

$$\begin{aligned} d_2 &= \left(b - \tfrac{1}{3} + \frac{.02}{b} + \frac{.01}{a+b}\right)q - \left(a - \tfrac{1}{3} + \frac{.02}{a} + \frac{.01}{a+b}\right)p \\ &= d_1 + .02\left(\frac{q}{b} - \frac{p}{a} + \frac{q - .5}{a + b}\right). \end{aligned} \tag{2.10}$$

2.5 *F distribution.* Here P is the probability of F or less under the F distribution with μ degrees of freedom in the numerator and ν degrees of freedom in the denominator:

$$P = F_F(F; \mu, \nu) = \frac{\mu^{\frac{1}{2}\mu}\nu^{\frac{1}{2}\nu}}{B(\frac{1}{2}\mu, \frac{1}{2}\nu)} \int_0^F x^{\frac{1}{2}\mu - 1}(\mu x + \nu)^{-\frac{1}{2}\mu - \frac{1}{2}\nu} dx, \tag{2.11}$$
$$(F \geq 0; \mu > 0; \nu > 0).$$

When F has arisen from a calculation of the form

$$F = \frac{\frac{1}{\mu} SS_\mu}{\frac{1}{\nu} SS_\nu} \tag{2.12}$$

then instead of computing p and q as in Table A, it is convenient to compute directly

$$q = \frac{SS_\mu}{SS_\mu + SS_\nu}, \qquad p = 1 - q. \tag{2.13}$$

Some alternatives to the formulas of Table A are:

$$d_1 = \frac{\nu}{2} - \tfrac{1}{3} - \left(\frac{\mu + \nu}{2} - \tfrac{2}{3}\right) p, \tag{2.14}$$

$$\begin{aligned} d_2 &= \left(\frac{\nu}{2} - \tfrac{1}{3} + \frac{.04}{\nu} + \frac{.02}{\mu + \nu}\right) q - \left(\frac{\mu}{2} - \tfrac{2}{3} + \frac{.04}{\mu} + \frac{.02}{\mu + \nu}\right) p \\ &= d_1 + .04\left(\frac{q}{\nu} - \frac{p}{\mu} + \frac{q - .5}{\mu + \nu}\right). \end{aligned} \tag{2.15}$$

Still another alternative is to use, instead of (1.2),

$$z_i = \frac{d_i^*}{|S^* - n^* p|} \left\{\frac{1}{1 + (3n^*)^{-1}} \left(S^* \ln \frac{S^*}{n^* p} + T^* \ln \frac{T^*}{n^* q}\right)\right\}^{\frac{1}{2}} \tag{2.16a}$$

$$= d_i^* \left\{\frac{1 + qg\left(\frac{S^*}{n^* p}\right) + pg\left(\frac{T^*}{n^* q}\right)}{2(n^* + \frac{1}{3}) pq}\right\}^{\frac{1}{2}} \qquad (i = 1, 2) \tag{2.16b}$$

where

$$S^* = 2S = \nu - 1, \qquad T^* = 2T = \mu - 1, \tag{2.17}$$

$$n^* = 2n = \mu + \nu - 2, \tag{2.18}$$

$$d_1^* = 2d_1 = \nu - \tfrac{2}{3} - (n^* + \tfrac{1}{3}) p, \tag{2.19}$$

$$\begin{aligned} d_2^* &= 2d_2 = \left(\nu - \tfrac{2}{3} + \frac{.08}{\nu} + \frac{.04}{\mu + \nu}\right) q - \left(\mu - \tfrac{2}{3} + \frac{.08}{\mu} + \frac{.04}{\mu + \nu}\right) p \\ &= d_1^* + .08\left(\frac{q}{\nu} - \frac{p}{\mu} + \frac{q - .5}{\mu + \nu}\right). \end{aligned} \tag{2.20}$$

Note that the relevant graph in Fig. 1 is labeled by $S = S^*/2$, not by S^*, and that $Sq/Tp = S^*q/T^*p$.

2.6 *Poisson distribution.* Here P is the probability of s or less under the Poisson distribution with mean M:

$$P = F_P(s; M) = \sum_{j=0}^{s} \frac{M^j}{j!} e^{-M}, \qquad (s = 0, 1, 2, \cdots; M \geq 0). \tag{2.21}$$

For right tail probabilities, s is defined as the largest integer *not* in the right tail. In either tail, S is the usual half-corrected value as given in the first line of Table B.

2.7 *Gamma distribution* (*incomplete gamma function*). Here P is the cumulative gamma distribution or incomplete gamma function (ratio) with parameter r at the value y:

$$P = F_\gamma(y; r) = \frac{1}{\Gamma(r)} \int_0^y x^{r-1} e^{-x} dx, \qquad (y \geq 0; r > 0). \tag{2.22}$$

2.8 *Chi square distribution*. Here P is the probability of X^2 or less under the chi square distribution with ν degrees of freedom:

$$P = F_{\chi^2}(X^2; \nu) = \frac{1}{2\Gamma(\frac{1}{2}\nu)} \int_0^{X^2} (\tfrac{1}{2}x)^{\frac{1}{2}\nu - 1} e^{-\frac{1}{2}x} dx, \qquad (X^2 \geq 0; \nu > 0). \tag{2.23}$$

An alternative is to use, instead of (1.3),

$$z_i = \frac{d_i^*}{|X^2 - S^*|} \left\{ S^* \ln \frac{S^*}{X^2} + X^2 - S^* \right\}^{\frac{1}{2}} \tag{2.24a}$$

$$= d_i^* \left\{ \frac{1 + g\left\{\frac{S^*}{X^2}\right\}}{2X^2} \right\}^{\frac{1}{2}} \qquad (i = 1, 2) \tag{2.24b}$$

where

$$S^* = 2S = \nu - 1, \tag{2.25}$$

$$d_1^* = 2d_1 = X^2 - \nu + \tfrac{2}{3}, \tag{2.26}$$

$$d_2^* = d_1^* - \frac{.08}{\nu} \cdot \tag{2.27}$$

Note that the relevant graph in Fig. 1a is labeled by $S = S^*/2$, not by S^*, and that $S/M = S^*/X^2$.

3. COMPUTATIONAL METHODS AND ILLUSTRATIONS

This section consists mainly of numerical examples, but starts with a few general remarks on desk calculation of the approximations. (We are not concerned with high-speed computer methods in this paper.)

For desk calculation we recommend using in most cases, and accordingly list in Summary Tables A and B, the expressions (1.2b) and (1.3b) in terms of the function g. They are not only convenient, but also avoid possibly serious loss of accuracy due to cancellation when the logarithms in the alternative expression (1.2a) and (1.3a) are evaluated at values near 1.

Table 1 gives $g(x)$ for $0 \leq x \leq 1$; for $x > 1$, it is necessary to use the property $g(x) = -g(1/x)$. When interpolating, remember that g is decreasing. It is useful to notice also that always one of the arguments of g in (1.2b) is less than 1 and the corresponding value of g positive while the other argument is greater than 1 and the corresponding value of g negative, except that both arguments of g may be exactly 1 in which case both values of g are 0. (For mathematical definition and discussion of g, see Section 10.)

Whenever two formulas are given for d_1 or d_2, as in Table A, the first tends to be preferable when a desk calculator is used for multiplication and division, the second when a slide rule is used. One can often judge before calculating d_2 whether it will give an appreciably different result from d_1.

Example 1: (a) What is the approximate probability of more than 3 under the binomial distribution with parameters $n=5$ and $p=.25$?

The quantities we need, according to Table A, are the usual half corrected values

$$S = 3 + .5 = 3.5 \quad \text{and} \quad T = 5 - 3.5 = 1.5$$

as well as

$$n = 5, \qquad p = .25, \text{ and } \quad q = 1 - p = .75.$$

The example is continued at (*) below.

(b) What is the approximate probability of 6 or more under the F distribution with $\mu=4$ degrees of freedom in the numerator and $\nu=8$ degrees of freedom in the denominator?

By Table A,

$$S = \tfrac{1}{2}(8 - 1) = 3.5, \qquad\qquad T = \tfrac{1}{2}(4 - 1) = 1.5,$$
$$n = \tfrac{1}{2}(8 + 4) - 1 = 5,$$
$$p = \frac{8}{4 \times 6 + 8} = .25, \quad \text{and} \quad q = 1 - p = .75.$$

(*) In both problems, the arguments of g in the formula for z_i are

$$\frac{S}{np} = \frac{3.5}{5 \times .25} = 2.8 \quad \text{and} \quad \frac{T}{nq} = \frac{1.5}{5 \times .75} = .4,$$

and their ratio is $(S/np)/(T/nq) = 2.8/.4 = 7$, which is within the .1 to 10 range of the graphs of Fig. 1, Section 5. The ratio is not near 1, however, so the part of the formula involving g can be important. Using Table 1, we find

$$1 + qg\left(\frac{S}{np}\right) + pg\left(\frac{T}{nq}\right) = 1 - .75g\left(\frac{1}{2.8}\right) + .25g(.4)$$
$$= 1 - .75 \times .3315 + .25 \times .2971 = .82565,$$

retaining an extra decimal place for intermediate calculations. Furthermore,

$$(n + \tfrac{1}{6})pq = (5 + \tfrac{1}{6}) \times .25 \times .75 = .96875,$$

and

$$\left\{\frac{.82565}{.96875}\right\}^{\frac{1}{2}} = .92319.$$

To compute z_1, we need

$$d_1 = S + \tfrac{1}{6} - (n + \tfrac{1}{3})p = 3.5 + \tfrac{1}{6} - (5 + \tfrac{1}{3}) \times .25 = 2.3333,$$

and we obtain

$$z_1 = 2.3333 \times .92319 = 2.1541.$$

To compute the refinement z_2, we need

$$d_2 = d_1 + .02\left(\frac{.75}{3.5 + .5} - \frac{.25}{1.5 + .5} + \frac{.25}{5 + 1}\right) = 2.3333 + .0021 = 2.3354,$$

and we obtain

$$z_2 = 2.3354 \times .92319 = 2.1560.$$

The desired approximations to the right tail probability are

$$\Phi(z_1) = .01562, \quad \text{and} \quad \Phi(z_2) = .01554.$$

The exact answer is .015625. In this example, z_1 was better, but z_2 is superior to z_1 on the whole.

The error of the approximation $\Phi(z_2) = .01554$ is guaranteed to be at most .001 by (1.4) and at most 2% (or .0003) by (1.5). A further refinement is possible using the graphs of Fig. 1, Section 5. Follow the instructions in Section 5, noting that T is smaller than S in this example. Find the page for $T/n = .3$. Find $Tp/Sq = .14$ on the horizontal scale. Read up to the curve for $T = 1.5$. The relative error in the approximation based on z_2 is read from the vertical scale. It is $\epsilon = +.005$. The corrected probability is, by formula (5.3),

$$P = .01554 + (+.005)(.01554) = .01562,$$

where the first $+$ is determined in accordance with the specific rules following the formula.

Example 2: (a) What is the probability of $s = 1$ or less under the Poisson distribution with parameter $M = 2$?

The quantities we need according to Table B are

$$S = s + \tfrac{1}{2} = 1.5, \qquad M = 2,$$

$$d_1 = s + \tfrac{2}{3} - M = 1 + \tfrac{2}{3} - 2 = -\ .3333,$$

$$d_2 = d_1 + \frac{.02}{s + 1} = -\ \tfrac{1}{3} + \frac{.02}{2} = -\ .3233.$$

The example is continued at (*) below.

(b) What is the probability of $X^2 = 4$ or more under the chi square distribution with $\nu = 4$ degrees of freedom?

By Table B,

$$S = \tfrac{1}{2}(\nu - 1) = \tfrac{1}{2}(4 - 1) = 1.5, \qquad M = \tfrac{1}{2}X^2 = 2,$$

$$d_1 = \tfrac{1}{2}X^2 - \tfrac{1}{2}\nu + \tfrac{1}{3} = 2 - 2 + \tfrac{1}{3} = .3333,$$

$$d_2 = d_1 - \frac{.04}{\nu} = .3233.$$

$$(*)\qquad z_i = d_i\left\{\frac{1+g\left(\frac{S}{M}\right)}{M}\right\}^{\frac{1}{2}} = d_i\left\{\frac{1+g(.75)}{2}\right\}^{\frac{1}{2}}$$
$$= d_i\left\{\frac{1+.0956}{2}\right\}^{\frac{1}{2}} = d_i\{.5478\}^{\frac{1}{2}} = d_i \times .7401.$$

In the case of the Poisson example, z_i is negative and corresponds to a left tail probability. In the case of the chi square example, z_i is positive and corresponds to the same right tail probability. Using the left tail, for example,

$$z_1 = d_1 \times .7401 = -.3333 \times .7401 = -.2467, \qquad \Phi(z_1) = .4026,$$
$$z_2 = d_2 \times .7401 = -.3233 \times .7401 = -.2393, \qquad \Phi(z_2) = .4054.$$

The exact answer is .40601 and z_2 was better in this example as is typical.

The error of the approximation $\Phi(z_2) = .4054$ is guaranteed to be at most .001 by (1.4) and at most 2% (.008) by (1.5). Section 5 and Figure 1, with $S=1.5$ and $S/M=.75$, give $\epsilon = +.0015$ and hence, by (5.3) and the specific rules following it, the corrected probability is

$$P = .4054 + (+.0015)(.4054) = .4060.$$

4. THE CASE $S=T$; THE t DISTRIBUTION AND ITS RELATIVES

In the case $S=T$, which arises, for instance, from the F distribution with equal degrees of freedom in the numerator and the denominator, (1.2a) simplifies to

$$z_1 = \pm(n + \tfrac{1}{3})\left\{\frac{-\ln(4pq)}{n+\frac{1}{6}}\right\}^{\frac{1}{2}} \tag{4.1a}$$

$$z_2 = \pm\left(n + \tfrac{1}{3} + \frac{0.1}{n+1}\right)\left\{\frac{-\ln(4pq)}{n+\frac{1}{6}}\right\}^{\frac{1}{2}}, \tag{4.1b}$$

where the sign is to be chosen to agree with the sign of $q-.5$. Special formulas in terms of the function g can be written, but one may as well use (4.1) directly provided the logarithms are evaluated to an appropriate number of significant places, not merely decimal places, when $4pq$ is close to 1.

Now let t have a t distribution with ν degrees of freedom. To approximate the t distribution, an approximation to F could be applied to t^2, which is F with 1 and ν degrees of freedom. Typical approximations to F will give far better accuracy, however, when applied to

$$F = 1 + \frac{2}{\nu}t^2 + \frac{2}{\nu}t(\nu + t^2)^{1/2}, \tag{4.2}$$

which is F with ν and ν degrees of freedom. The approximations to t and its relatives obtained thus from z_2 will be given below; z_1 would give the same results except that the fraction $1/10\nu$ would be omitted from (4.7), (4.9), (4.10), and (4.11) and $1/40b$ from (4.12). We shall not carry these approximations far, however, since Wallace (1959) discusses a number of approximations to t, including one which is almost equal to (4.7) for $\nu \geq 2$ and better for $\nu < 1.5$ or so, and a still better one which is extremely accurate over a very wide range and, as he says, amenable to slide rule calculation though somewhat complicated. See also Section 10.6 of Part II.

The relation (4.2) is given by Cacoullos (1965). The derivation of the formulas below proceeds a little more smoothly from the equivalent relation that

$$Y = \tfrac{1}{2} + \tfrac{1}{2}t(\nu + t^2)^{-\frac{1}{2}} \tag{4.3}$$

is beta with parameters $a = b = \frac{1}{2}\nu$. Both these relations are versions of the classical relation (Pearson, 1934):

$$F_\beta(y; \tfrac{1}{2}, b) = 2F_\beta(\tfrac{1}{2}[1 + y^{\frac{1}{2}}]; b, b) - 1. \tag{4.4}$$

Another equivalent relation is

$$F_F(F; 1, \nu) = 2F_\beta(\tfrac{1}{2}[1 + y^{\frac{1}{2}}]; \tfrac{1}{2}\nu, \tfrac{1}{2}\nu) - 1 \tag{4.5}$$

where $y = F/(\nu + F)$.

The t distribution. The probability of t or less under the t distribution with ν degrees of freedom

$$P = F_t(t; \nu) = \frac{\nu^{\frac{1}{2}\nu}}{B(\frac{1}{2}, \frac{1}{2}\nu)} \int_{-\infty}^{t} (\nu + x^2)^{-\frac{1}{2}\nu - \frac{1}{2}} dx, \qquad (\nu > 0) \tag{4.6}$$

is approximated by $\Phi(z_2)$ where

$$z_2 = \pm \left(\nu - \tfrac{2}{3} + \frac{1}{10\nu}\right) \left\{\frac{1}{\nu - \frac{5}{6}} \ln\left(1 + \frac{t^2}{\nu}\right)\right\}^{\frac{1}{2}}; \tag{4.7}$$

the sign is to be chosen to agree with the sign of t.

When t has arisen from a calculation of the form

$$t = r\left(\frac{\nu}{1 - r^2}\right)^{\frac{1}{2}}, \tag{4.8}$$

then it is convenient to compute directly

$$z_2 = \pm \left(\nu - \tfrac{2}{3} + \frac{1}{10\nu}\right) \left\{\frac{-\ln(1 - r^2)}{\nu - \frac{5}{6}}\right\}^{\frac{1}{2}} \tag{4.9}$$

where the sign is to be chosen to agree with the sign of r.

While (1.2) cannot generally be inverted in closed form to give approximate percent points, nor can the limiting case (1.3), the special case (4.1) and hence the approximations of this section can be. In particular, (4.7) yields as an approximate percent point of the t distribution

$$\pm \left\{\nu \exp\left[z^2(\nu - \tfrac{5}{6})\Big/\left(\nu - \tfrac{2}{3} + \frac{1}{10\nu}\right)^2\right] - \nu\right\}^{\frac{1}{2}} \tag{4.10}$$

where z is the corresponding Normal percent point and where the sign is to be chosen to agree with the sign of z.

The F distribution with one degree of freedom. Let $P = F_F(F; 1, \nu)$, the probability of F or less under an F distribution with 1 degree of freedom in the numerator and ν degrees of freedom in the denominator. Then P is approximately the probability *between* $-z_2$ and z_2 under the standard Normal distribution for

$$z_2 = \left(\nu - \tfrac{2}{3} + \frac{1}{10\nu}\right)\left\{\frac{1}{\nu - \frac{5}{6}} \ln\left(1 + \frac{F}{\nu}\right)\right\}^{\frac{1}{2}}. \tag{4.11}$$

The probability of F or more is approximately *twice* the standard Normal probability of z_2 or more. F tail probabilities with μ degrees of freedom in the numerator and 1 degree of freedom in the denominator may be obtained via $F_F(F; \mu, 1) = 1 - F_F(1/F; 1, \mu)$.

The beta distribution with a or $b = \frac{1}{2}$. Let $P = F_\beta(y; \frac{1}{2}, b) = I_y(\frac{1}{2}, b)$, the probability of y or less under a beta distribution with $a = \frac{1}{2}$. Then P is approximately the probability *between* $-z_2$ snd z_2 under the standard Normal distribution for

$$z_2 = \left(b - \tfrac{1}{3} + \frac{1}{40b}\right)\left\{\frac{-2\ln(1 - y)}{b - \dfrac{5}{12}}\right\}^{\frac{1}{2}}. \tag{4.12}$$

The probability of y or more is approximately *twice* the standard Normal probability of z_2 or more. Beta probabilities with $b = \frac{1}{2}$ may be obtained via $F_\beta(y; a, \frac{1}{2}) = 1 - F_\beta(1 - y; \frac{1}{2}, a)$.

5. GRAPHS OF RELATIVE ERROR

Figure 1 gives graphs of the relative error of $\Phi(z_2)$ covering most circumstances. These graphs can be used to check the accuracy of the approximation, and to make a correction if necessary. The individual curves are labeled by the smaller of S and T. On each curve, S, T, and n are fixed, while p and q vary. Thus, each curve belongs to a single beta or F distribution, or to a specified number of successes in a specified number of Bernoulli trials with unspecified p. The variable used on the horizontal scale (the abscissa) is not p or q, however, but Sq/Tp (or its reciprocal). This choice of variable has the following advantages: when it is 1, the tail probability is near $\frac{1}{2}$; if it is fixed and S and T approach infinity in a fixed ratio, the relative error approaches a limit; if a logarithmic scale is used, the curves for $S = T$ display the symmetry inherent in the situation. Writing

$$\frac{Sq}{Tp} = \frac{S}{np} \bigg/ \frac{T}{nq} = \frac{S}{T} \bigg/ \frac{p}{q} \tag{5.1}$$

we see that in the binomial case, this variable is first, the ratio of observed to expected successes (slightly adjusted) divided by the corresponding ratio for failures, and, second, the observed (adjusted) odds divided by the true odds.

The quantity graphed (the ordinate) is

$$\epsilon = \pm \frac{P - \Phi(z_2)}{\min\{\Phi(z_2),\, 1 - \Phi(z_2)\}} \tag{5.2}$$

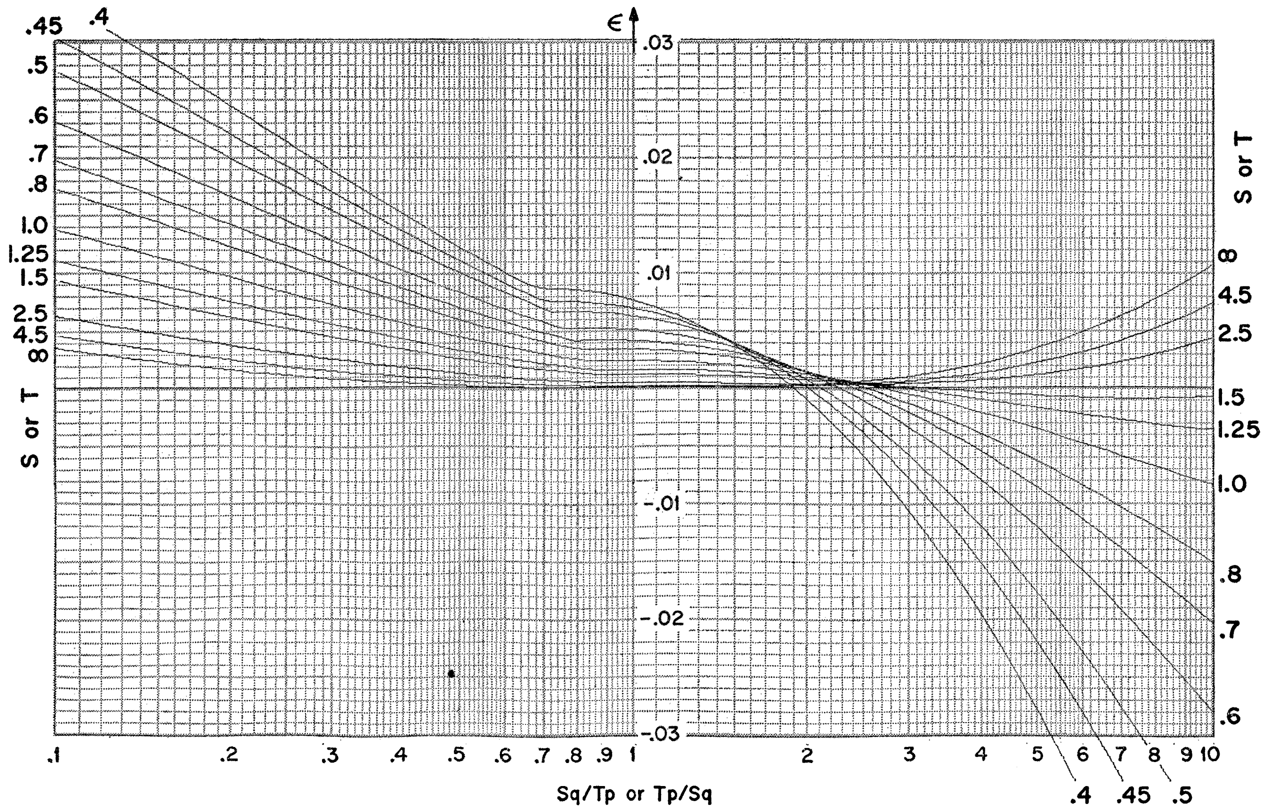

FIG. 1A. Relative error ϵ for S/n or T/n = 0 (and Poisson, gamma, and chi square). See Section 5. Vertical scale (in center) is ϵ. If $S<T$, curves are labeled by S at sides and horizontal scale is Sq/Tp. If $S>T$, curves are labeled by T at sides and horizontal scale is Tp/Sq. (For Poisson, gamma, and chi square, curves are labeled by S at sides and horizontal scale is S/M.)

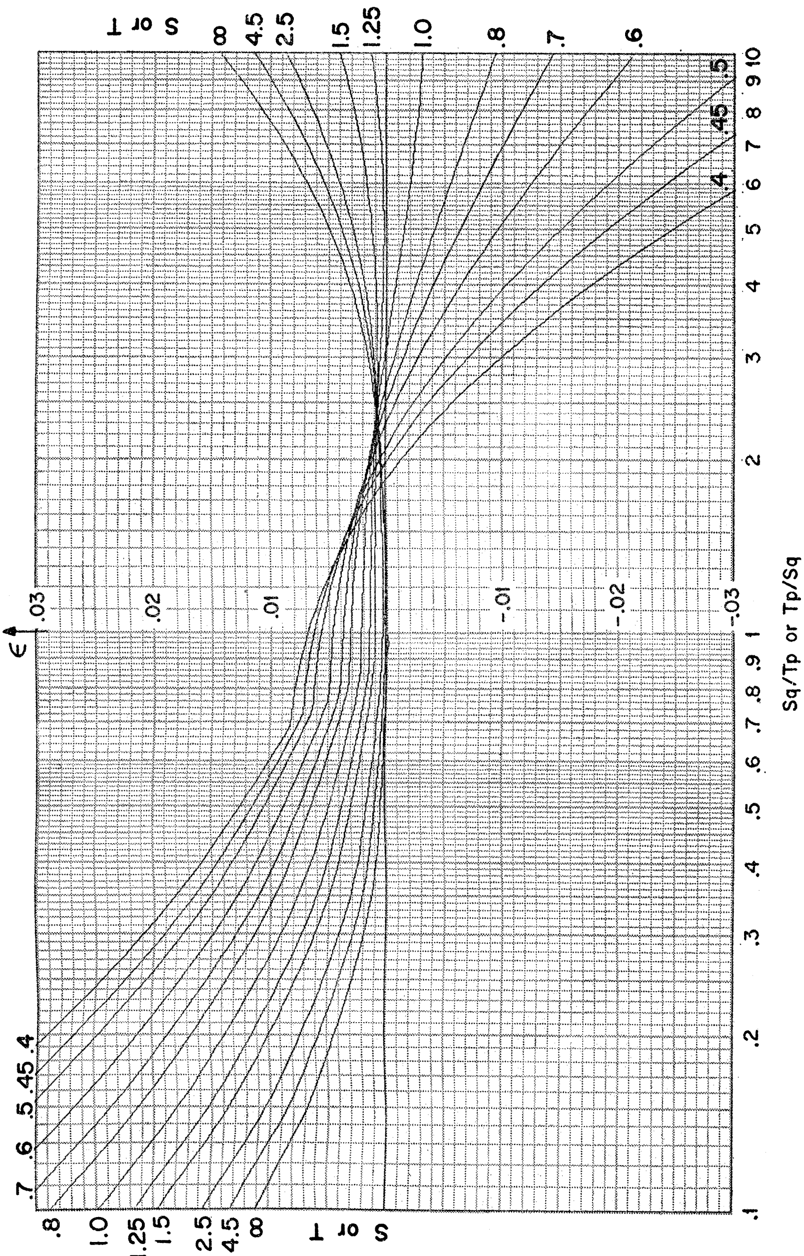

FIG. 1B. Relative error ϵ for S/n or $T/n = .05$. See Section 5. Vertical scale (in center), is ϵ. If $S < T$, curves are labeled by S at sides and horizontal scale is Sq/Tp. If $S > T$, curves are labeled by T at sides and horizontal scale is Tp/Sq.

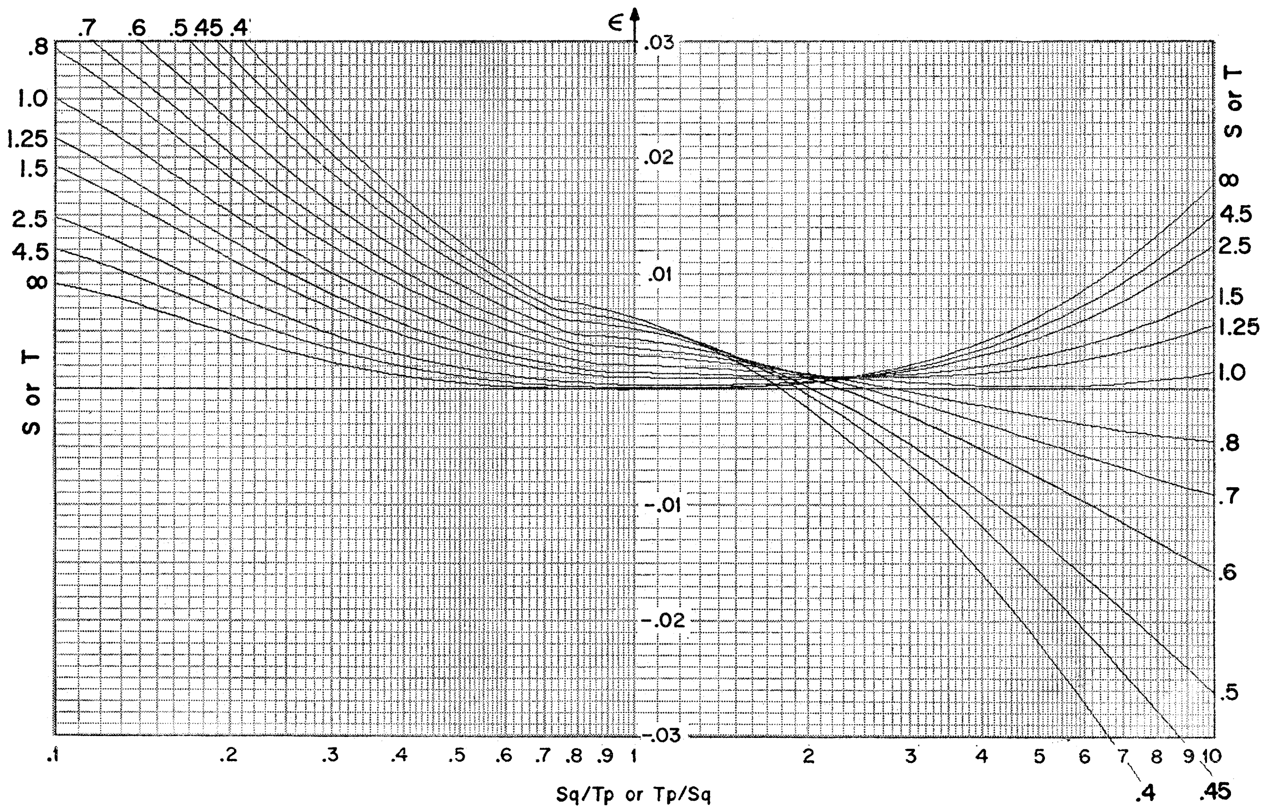

FIG. 1C. Relative error ε for S/n or T/n = .10. See Section 5. Vertical scale (in center) is ε. If S < T, curves are labeled by S at sides and horizontal scale is Sq/Tp. If S > T, curves are labeled by T at sides and horizontal scale is Tp/Sq.

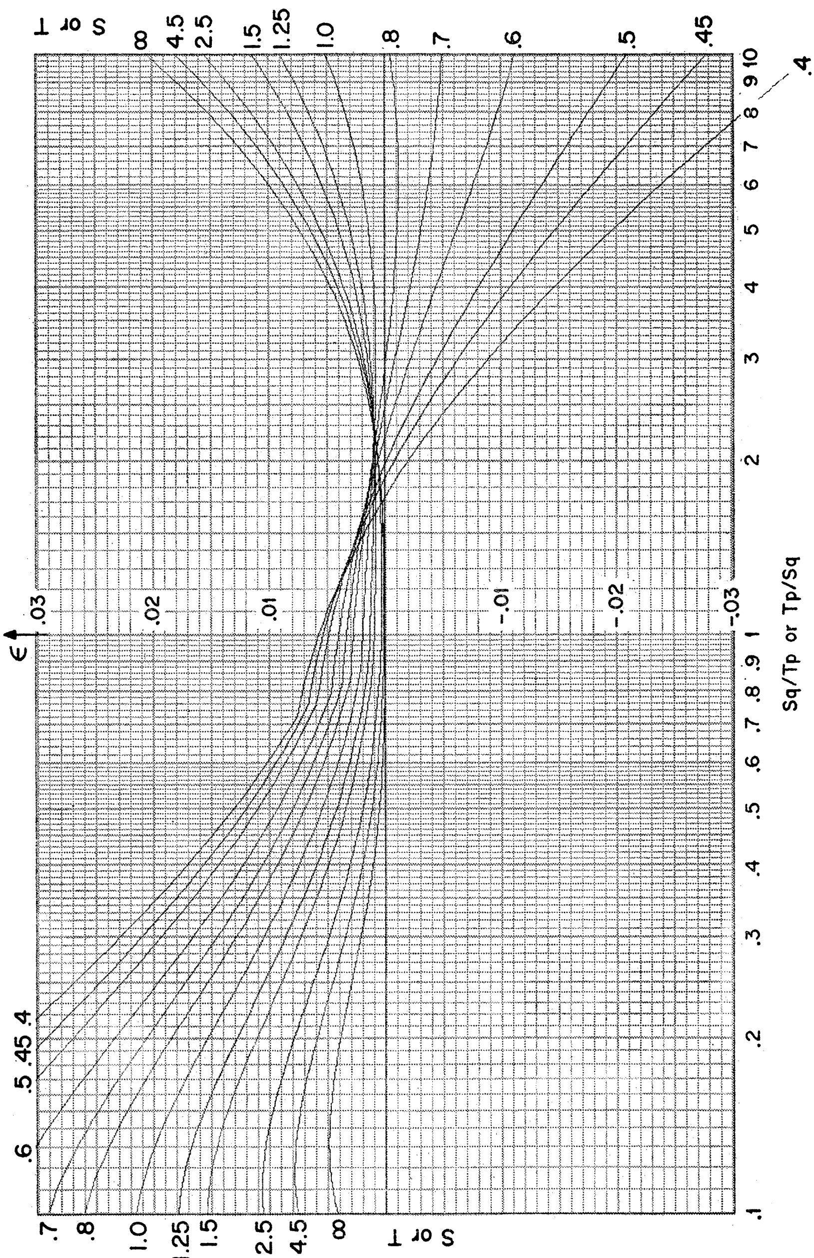

FIG. 1D. Relative error ϵ for $S/n = .15$. See Section 5. Vertical scale (in center) is ϵ. If $S < T$, curves are labeled by S at sides and horizontal scale is Sq/Tp. If $S > T$, curves are labeled by T at sides and horizontal scale is Tp/Sq.

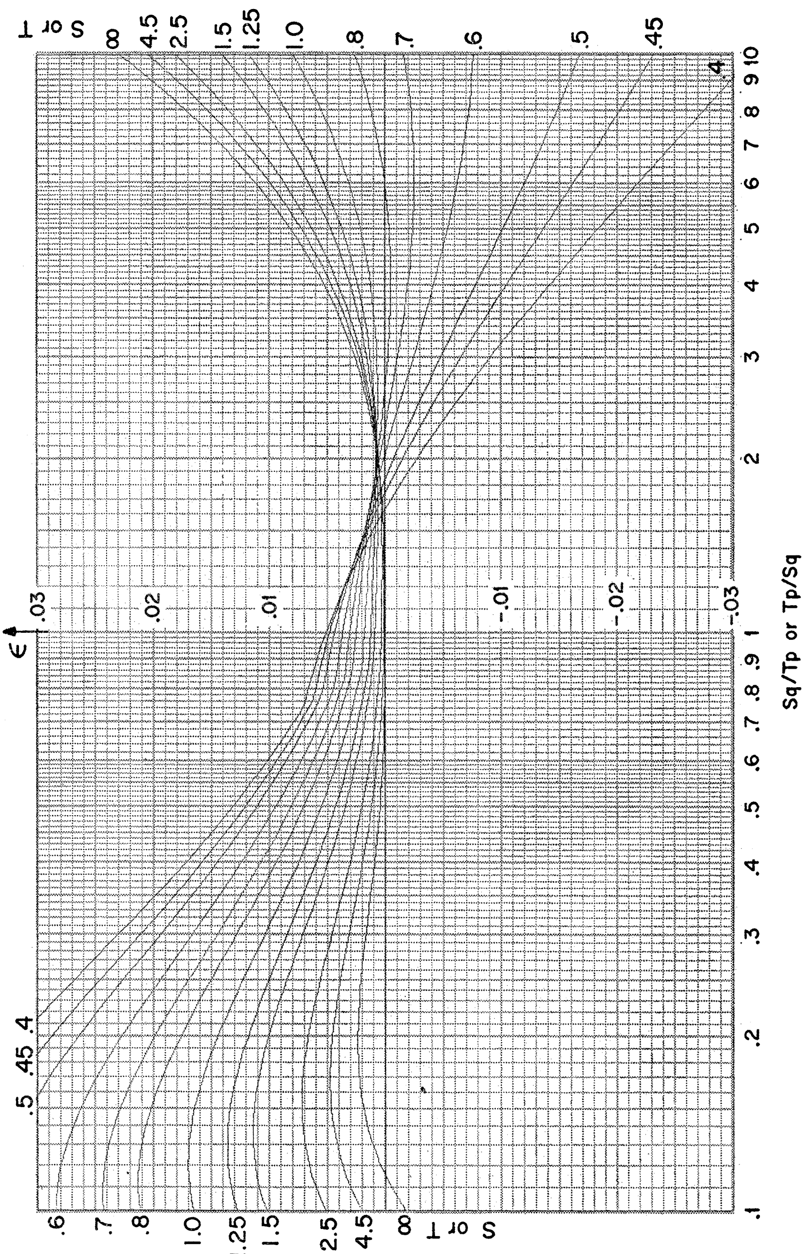

Fig. 1E. Relative error ϵ for S/n or T/n = .20. See Section 5. Vertical scale (in center) is ϵ. If $S<T$, curves are labeled by S at sides and horizontal scale is Sq/Tp. If $S>T$, curves are labeled by T at sides and horizontal scale is Tp/Sq.

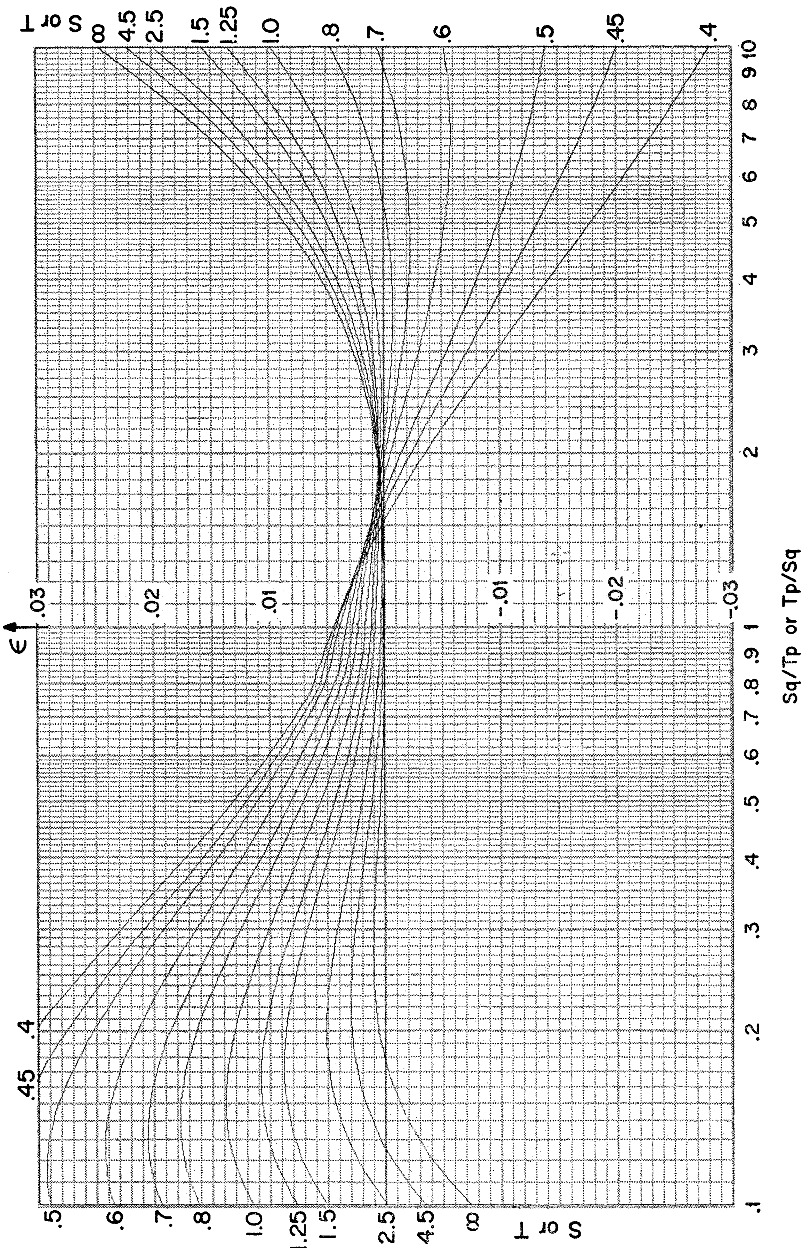

FIG. 1F. Relative error ε for S/n or T/n = .25. See Section 5. Vertical scale (in center) is ε. If S<T, curves are labeled by S at sides and horizontal scale is Sq/Tp. If S>T, curves are labeled by T at sides and horizontal scale is Tp/Sq.

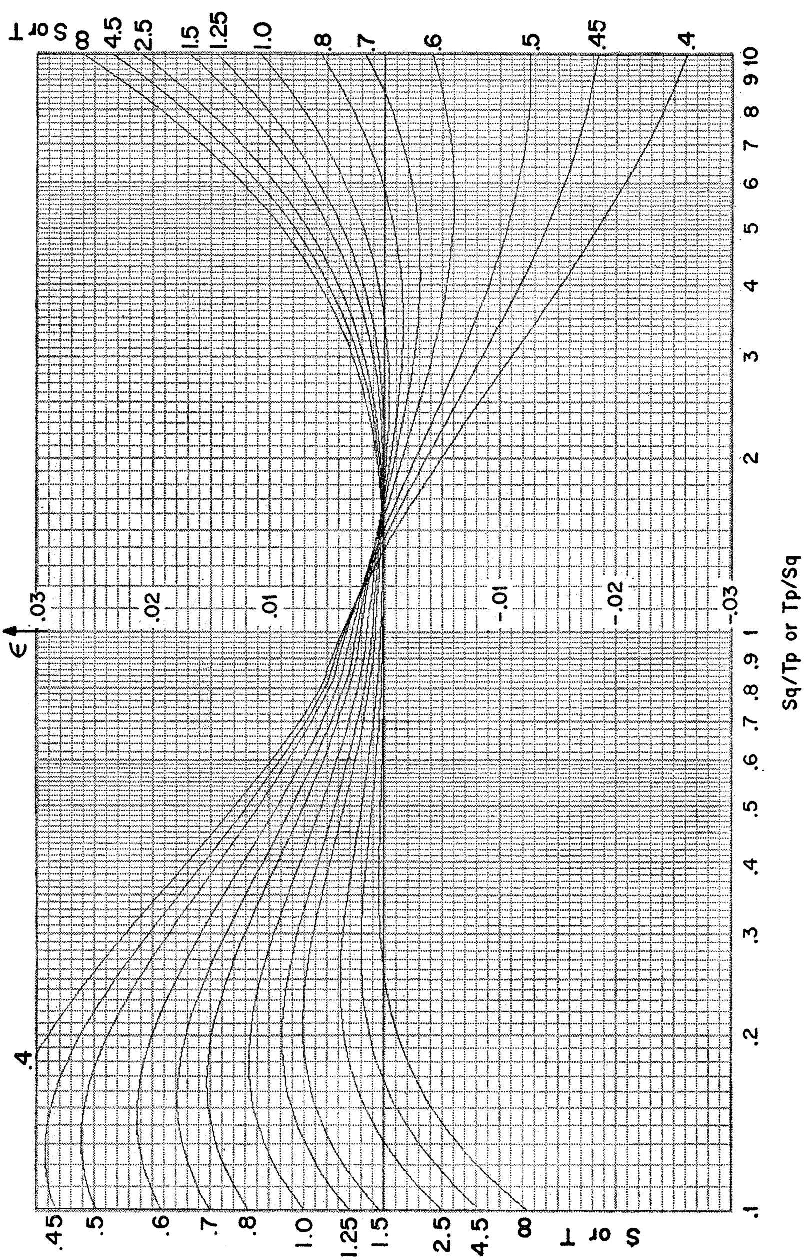

FIG. 1G. Relative error ϵ for S/n or T/n = .30. See Section 5. Vertical scale (in center) is ϵ. If S<T, curves are labeled by S at sides and horizontal scale is Sq/Tp. If S>T, curves are labeled by T at sides and horizontal scale is Tp/Sq.

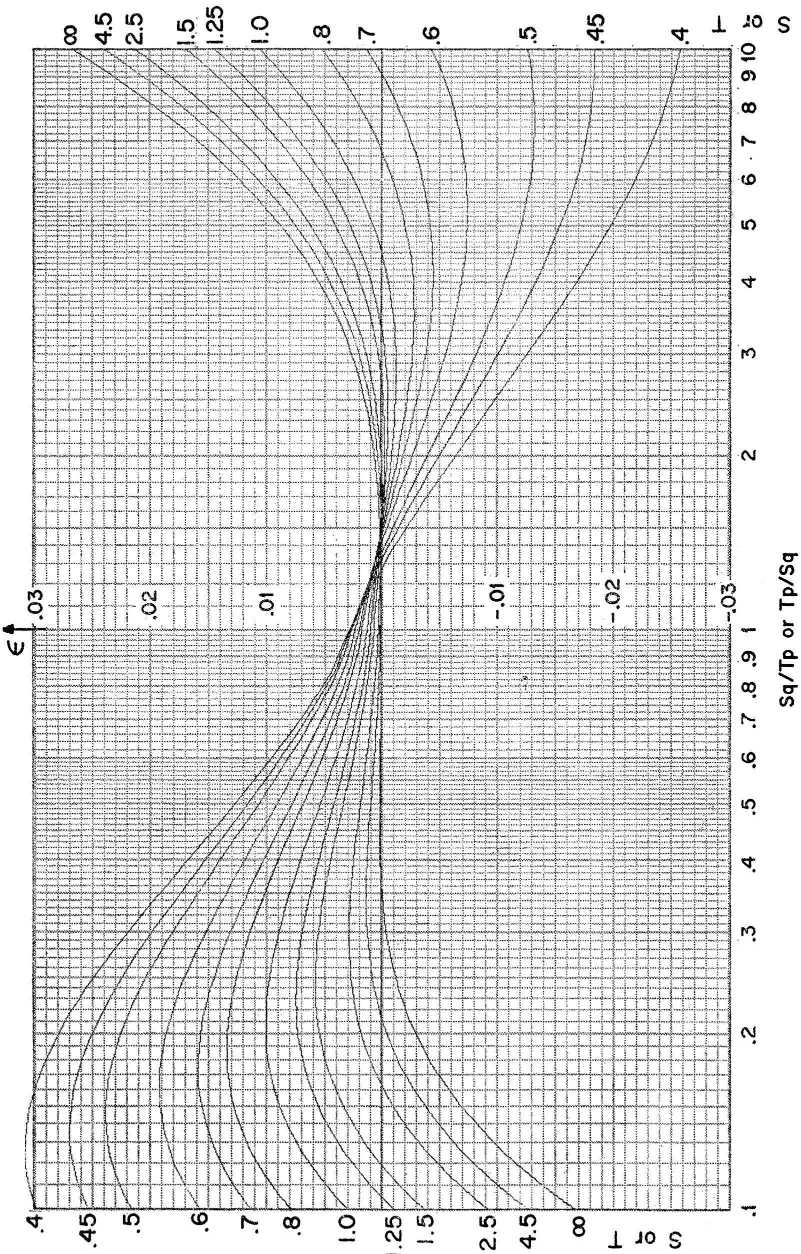

Fig. 1H. Relative error ϵ for S/n or T/n = .35. See Section 5. Vertical scale (in center) is ϵ. If S < T, curves are labeled by S at sides and horizontal scale is Sq/Tp. If S > T, curves are labeled by T at sides and horizontal scale is Tp/Sq.

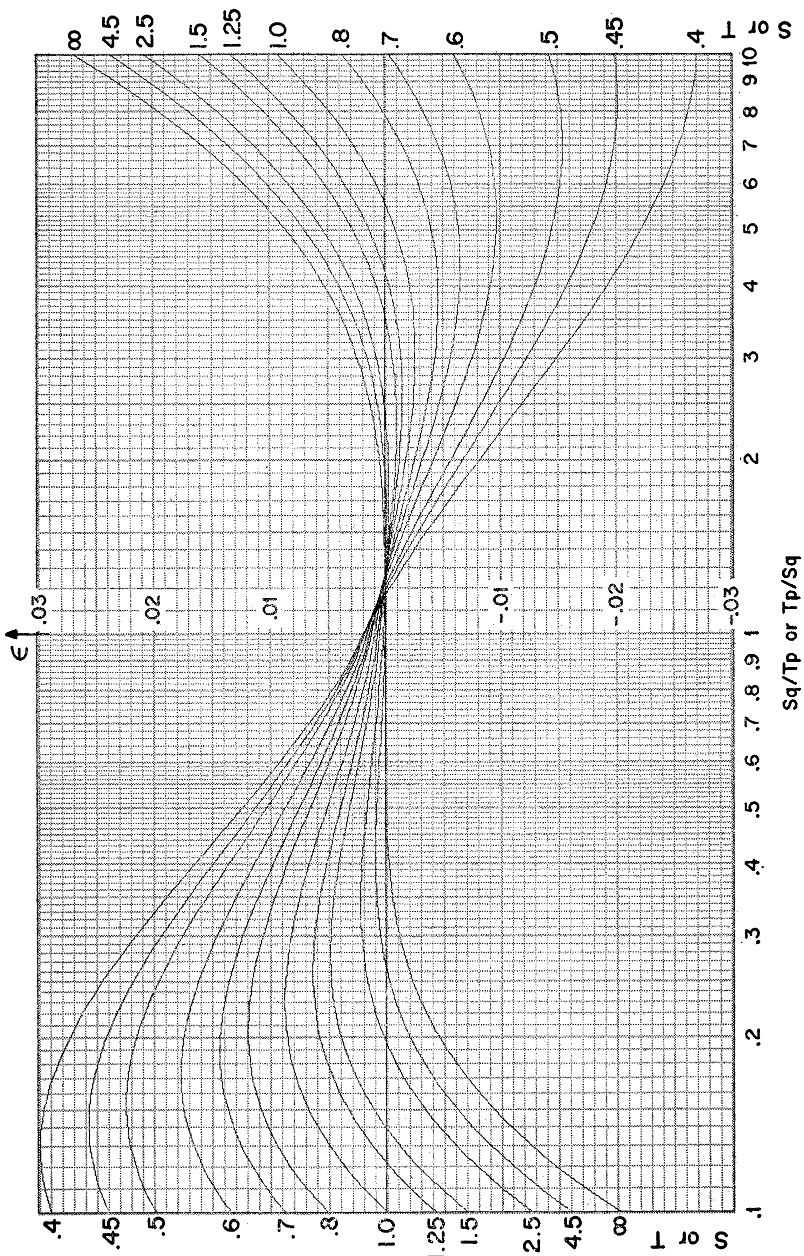

FIG. 1I. Relative error ϵ for S/n or T/n = .40. See Section 5. Vertical scale (in center) is ϵ. If $S<T$, curves are labeled by S at sides and horizontal scale is Sq/Tp. If $S>T$, curves are labeled by T at sides and horizontal scale is Tp/Sq.

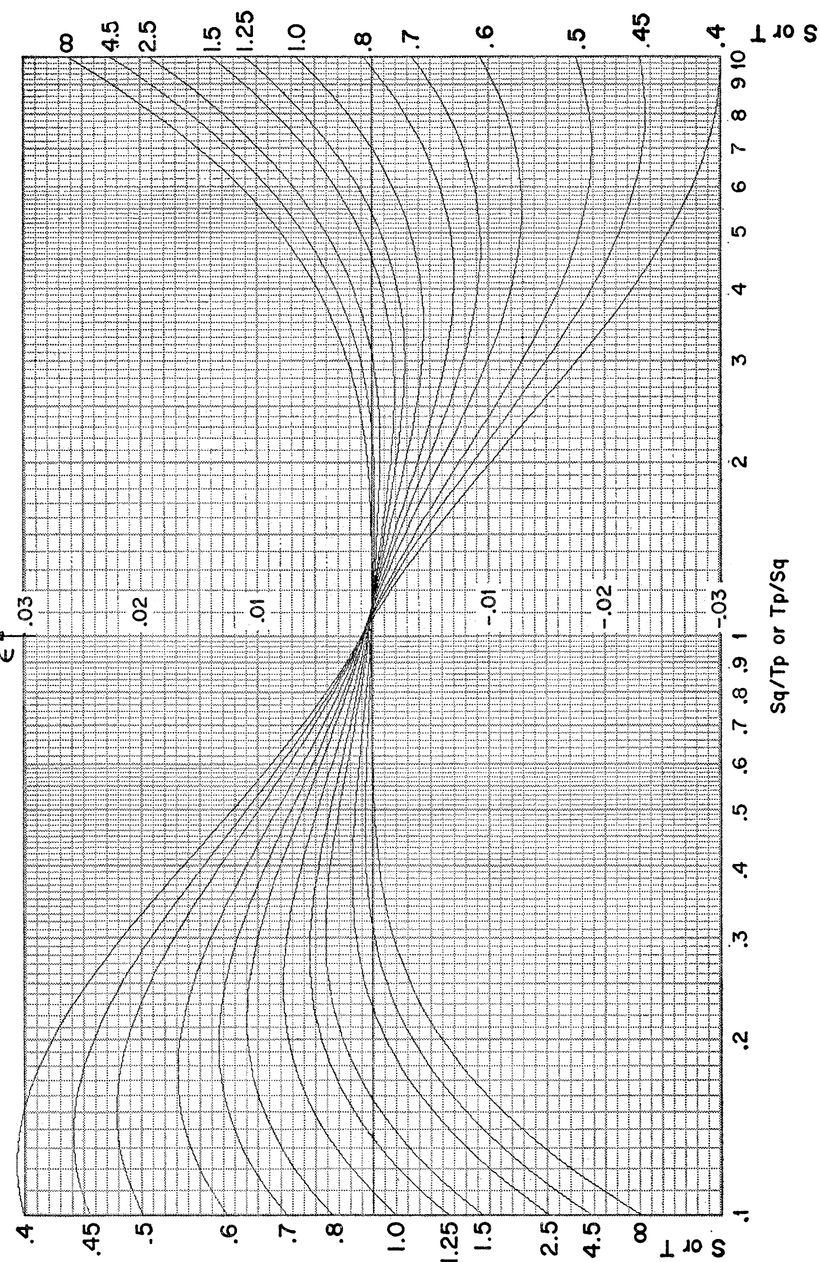

FIG. 1J. Relative error ε for S/n or T/n = .45. See Section 5. Vertical scale (in center) is ε. If S < T, curves are labeled by S at sides and horizontal scale is Sq/Tp. If S > T, curves are labeled by T at sides and horizontal scale is Tp/Sq.

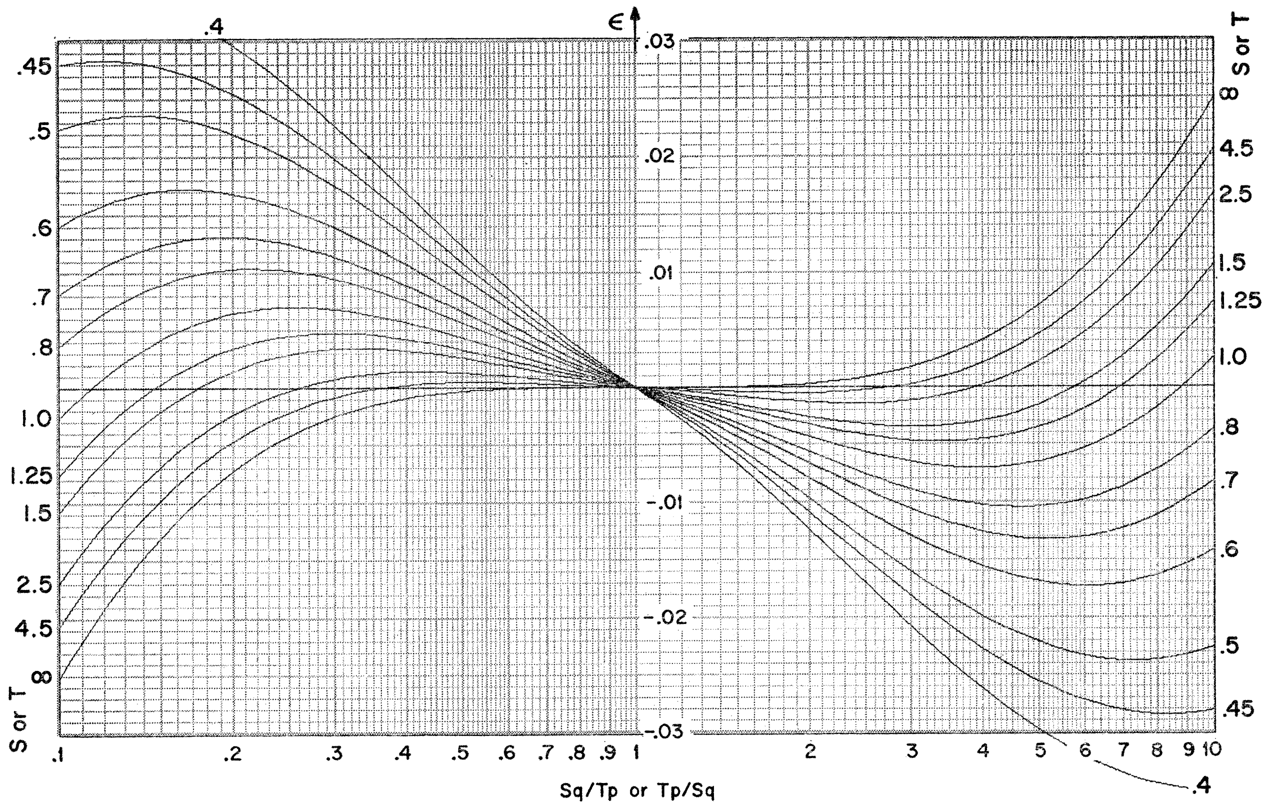

FIG. 1K. Relative error for $S/n=T/n=.50$ (and Student's t). See Section 5. Vertical scale (in center) is ϵ. If rules for $S \leq T$ are to be used (e.g., for interpolation toward $S<T$), take curves as labeled by S at sides and horizontal scale as Sq/Tp. If rules for $S \geq T$ are to be used, take curves as labeled by T at sides and horizontal scale as Tp/Sq. (For t, curves are labeled by $\frac{1}{2}(\nu-1)$ and horizontal scale is $(t+[\nu+t^2]^{1/2})^2/\nu$.)

where the choice of sign will be explained shortly. This quantity is the error in the tail probability relative to the approximate tail probability $\Phi(z_2)$ or to its complement $1-\Phi(z_2)$, whichever is smaller. The error in a left tail probability is of course the same, except for sign, as the error in the complementary right tail probability. The approximation $\Phi(z_2)$ can be corrected if necessary by reading ϵ from the graphs and computing

$$P = \Phi(z_2) \pm \epsilon \min\{\Phi(z_2), 1 - \Phi(z_2)\}. \tag{5.3}$$

The reason for measuring error relative to an approximate rather than a true probability is to simplify such correction.

Unfortunately the error ϵ is in places rather nonlinear across pages, i.e., across S/n (or T/n). Graphs are given for enough values so that it is adequate for most purposes to interpolate roughly across pages or even simply to use the page with nearest S/n or T/n. If interpolation is to be done at all accurately, however, nonlinearity should be checked, and taken into account if necessary, by reading values of ϵ from more than just the two pages with nearest S/n or T/n.

Outside the region covered by the graphs, see Section 7 if S or T is very small, Section 9 if the tail probability is very small.

Specific rules for the binomial, Pascal, negative binomial, beta, and F distributions. If $S \leq T$, the curves are labeled by values of S, the variable on the horizontal scale is Sq/Tp, and in both (5.2) and (5.3) the $+$ sign is to be used for left tails and the $-$ sign for right tails. If $S \geq T$ the curves are labeled by values of T, the variable on the horizontal scale is Tp/Sq, and in (5.2) and (5.3), the $-$ sign is to be used for left tails and the $+$ sign for right tails. For illustration, see Section 3, example 1.

Specific rules for the Poisson, gamma, and chi square distributions. Use the page for $S/n = 0$. The curves are labeled by values of S and the variable on the horizontal scale is S/M. In (5.2) and (5.3), the $+$ sign is to be used for left tail Poisson and right tail gamma and chi square probabilities and the $-$ sign for right tail Poisson and left tail gamma and chi square probabilities. For illustration, see Section 3, example 2.

Specific rules for the t distribution as approximated by (4.7). Use the page for $S/n = T/n = .5$. The curves are labeled by values of $\frac{1}{2}(\nu-1) = S$, say, and the variable on the horizontal scale is $(t + [\nu + t^2]^{\frac{1}{2}})^2/\nu$. In (5.2) and (5.3), the $+$ sign is to be used for left tails and the $-$ sign for right tails.

The graphs, were of course, all obtained empirically except those for $S = \infty$, which are shown in Part II to be given by

$$\epsilon = \mp G^{\frac{1}{2}} \exp\left\{\left(\frac{q-p}{6} - \frac{S-np}{12n}\right)\frac{S-np}{npq}G\right\} \pm 1 \tag{5.4}$$

where the lower signs apply for $S < np$ and the upper signs for $S > np$, and

$$G = 1 + qg\left(\frac{S}{np}\right) + pg\left(\frac{T}{nq}\right). \tag{5.5}$$

6. COMPARISON WITH OTHER NORMAL APPROXIMATIONS

Various Normal approximations, including the ordinary one and some improvements based on square roots and cube roots, are compared graphically in Fig. 2. We shall now define these approximations and then discuss their comparison.

By the "ordinary Normal approximation" we mean

$$z_0 = (S - np)(npq)^{-\frac{1}{2}}. \tag{6.1}$$

The square root approximation used in Fig. 2 reduces to Fisher's (1925) approximation in the chi square case and in general is a modification of Freeman and Tukey's (1950) approximation suggested by Pinkham (1957) after investigation of the first four moments. It is given by

$$Z_{\frac{1}{2}}' = [(4b - 1)q]^{\frac{1}{2}} - [(4a - 1)p]^{\frac{1}{2}} \tag{6.2}$$

where $a = T + \frac{1}{2}$ and $b = S + \frac{1}{2}$. (Always ap is to be replaced by M and q by 1 for the Poisson, gamma, and chi square distributions and the sign reversed for gamma and chi square.) Freeman and Tukey's approximation, namely

$$Z_{\frac{1}{2}}'' = 2[(bq)^{\frac{1}{2}} - (ap)^{\frac{1}{2}}], \tag{6.3}$$

is not included in Fig. 2, but it is somewhat less good in the chi square case and no better over all than (6.2), so the remarks below apply to it also. The cube root approximation used in Fig. 2 was given by Wilson and Hilferty (1931) for chi square, Paulson (1942) for F, and Camp (1951) for the binomial distribution, and in our notation can be written

$$Z_{\frac{1}{3}}' = \frac{\left(9 - \frac{1}{b}\right)\left(\frac{bq}{ap}\right)^{\frac{1}{3}} - 9 + \frac{1}{a}}{3\left[\frac{1}{b}\left(\frac{bq}{ap}\right)^{\frac{2}{3}} + \frac{1}{a}\right]^{\frac{1}{2}}}. \tag{6.4}$$

Also included in Fig. 2 is a Cornish-Fisher expansion with three adjustments (through terms of order $n^{-3/2}$), but it is so complicated for desk calculation that it has little direct interest here, and it is therefore omitted from the main discussion which begins after the next paragraph. There is some interest, nevertheless, in noting that it is less accurate than z_2 from the point of view of Fig. 2, although in general it must ultimately improve more rapidly and become better than all the other approximations discussed here as $S \to \infty$ with the tail probability fixed. (This is shown in Part II.) In graphs like those of Fig. 2 but not shown here, the Cornish-Fisher expansion with only two adjustments looks nearly as accurate over all as that with three adjustments; it it also much easier to calculate and avoids the unpleasant property, typical of an odd number of adjustments, that the approximate Normal deviate approaches the same infinite value in both tails. (The Cornish-Fisher expansion used, (8.1) of Part II, was obtained from the beta distribution. A different expansion would be obtained from each related distribution, and the choice is somewhat arbitrary. Fisher and Cornish (1960) use $z = \frac{1}{2} \ln F$, which may well

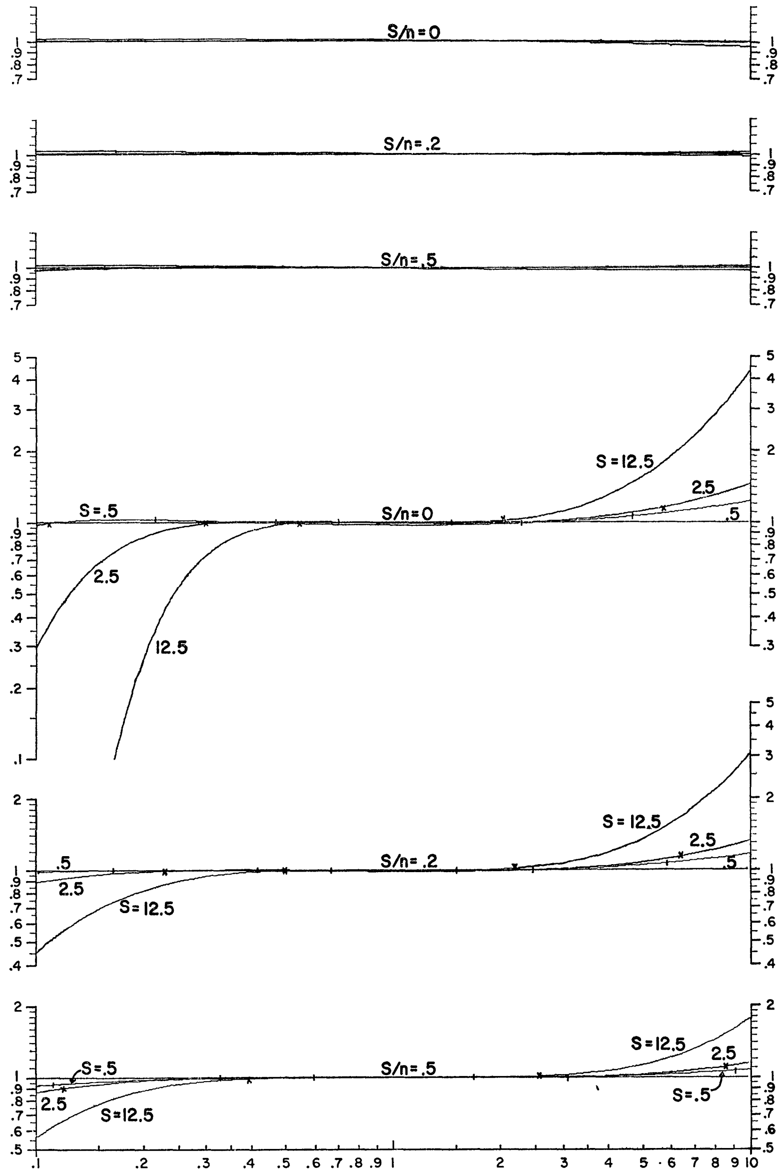

FIG. 2. Ratio of true and approximate P/(1−P) for various approximations. Vertical scale gives

$$\frac{\text{true left tail}}{\text{approx. left tail}} \times \frac{\text{approx. right tail}}{\text{true right tail}}$$

except for the gamma and chi square distributions, where it gives the reciprocal of this. Horizontal scale gives Sq/Tp as in Fig. 1. Bars and crosses on the curves indicate true tails of .1 and .01 respectively.

New approximation z_2 (upper graphs) and cube root approximation (lower graphs) on this page. Other approximations on subsequent pages.

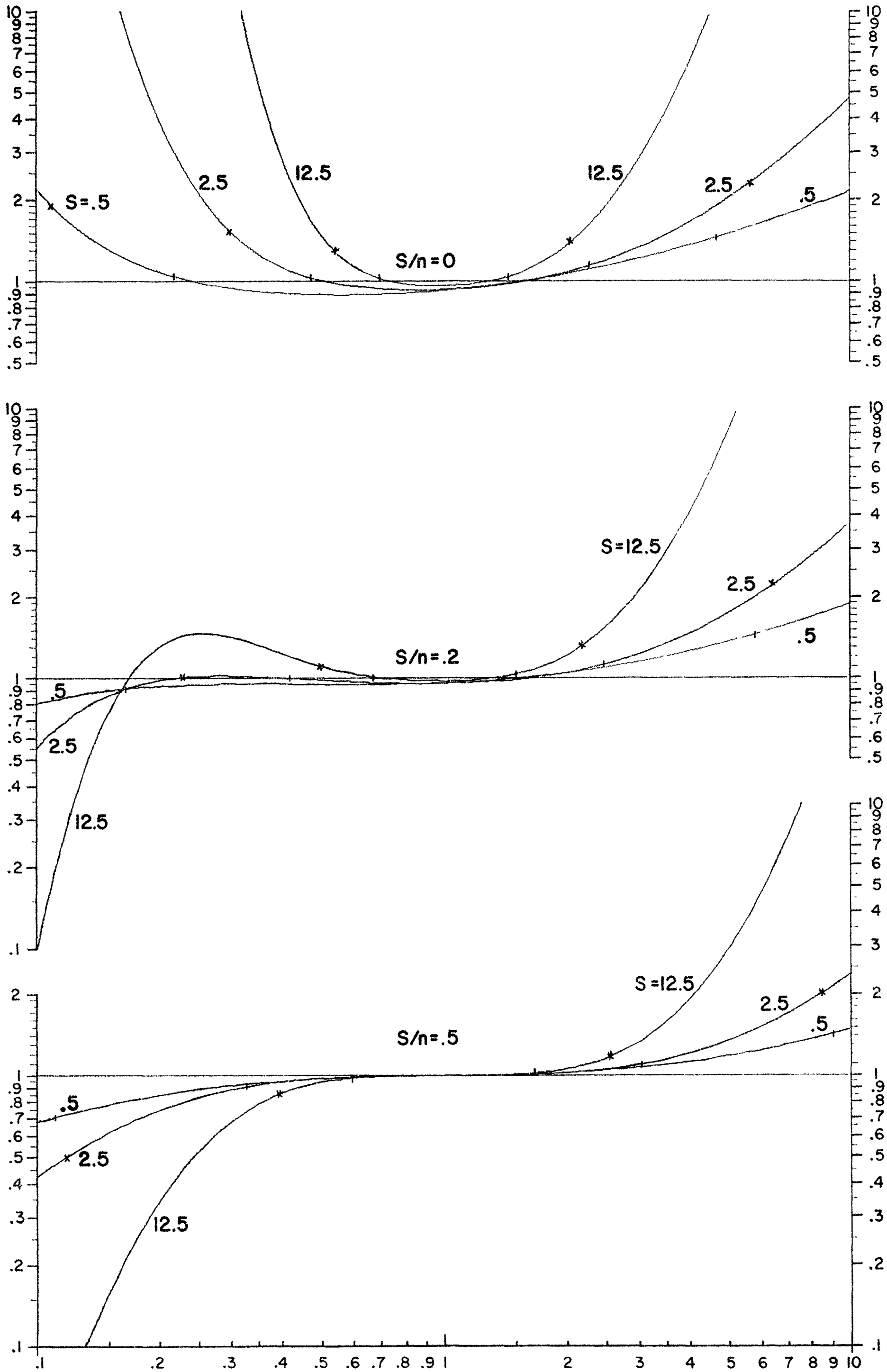

FIG. 2. (continued). Square root approximation.

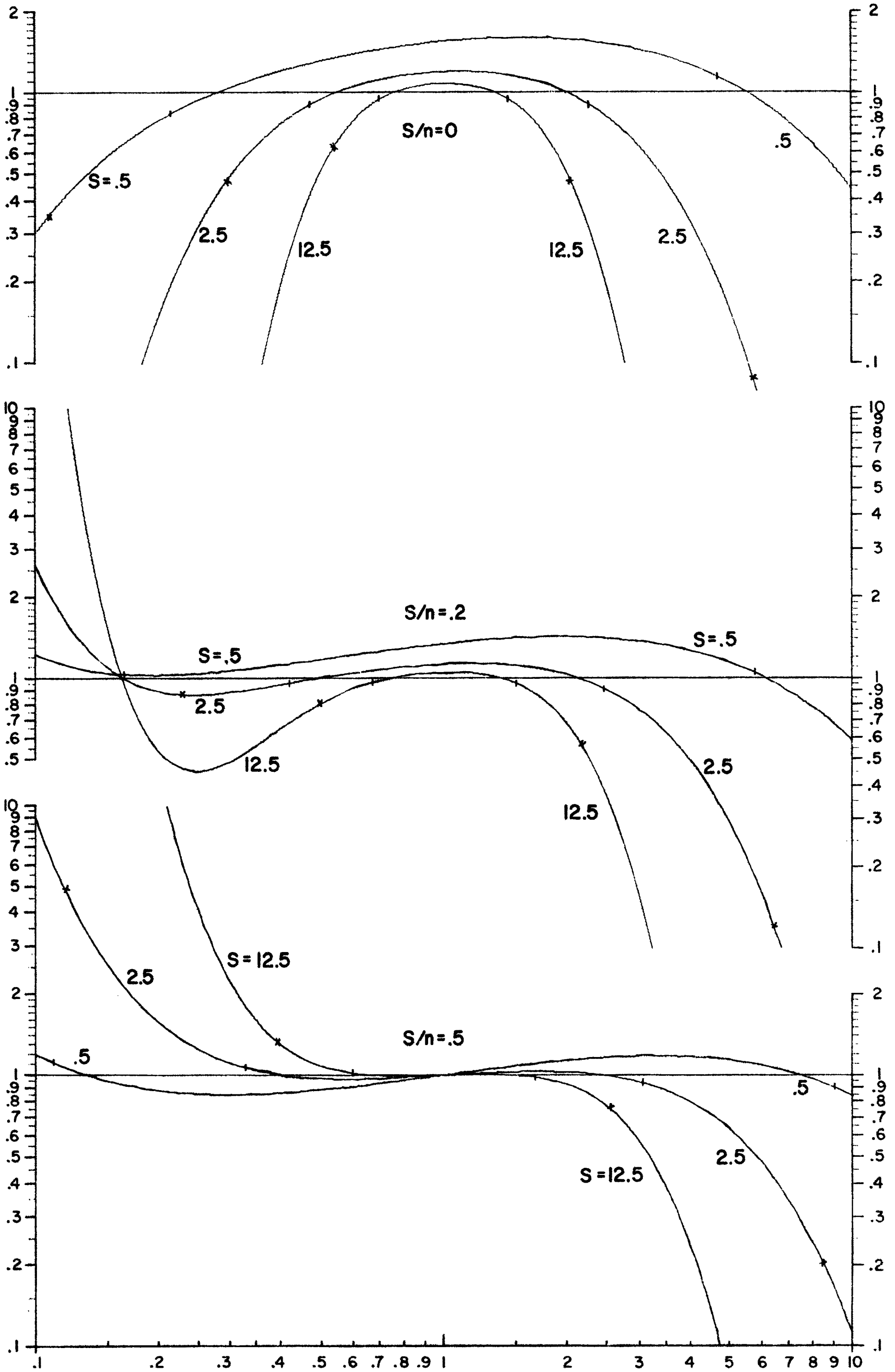

FIG. 2. (continued). Ordinary Normal approximation.

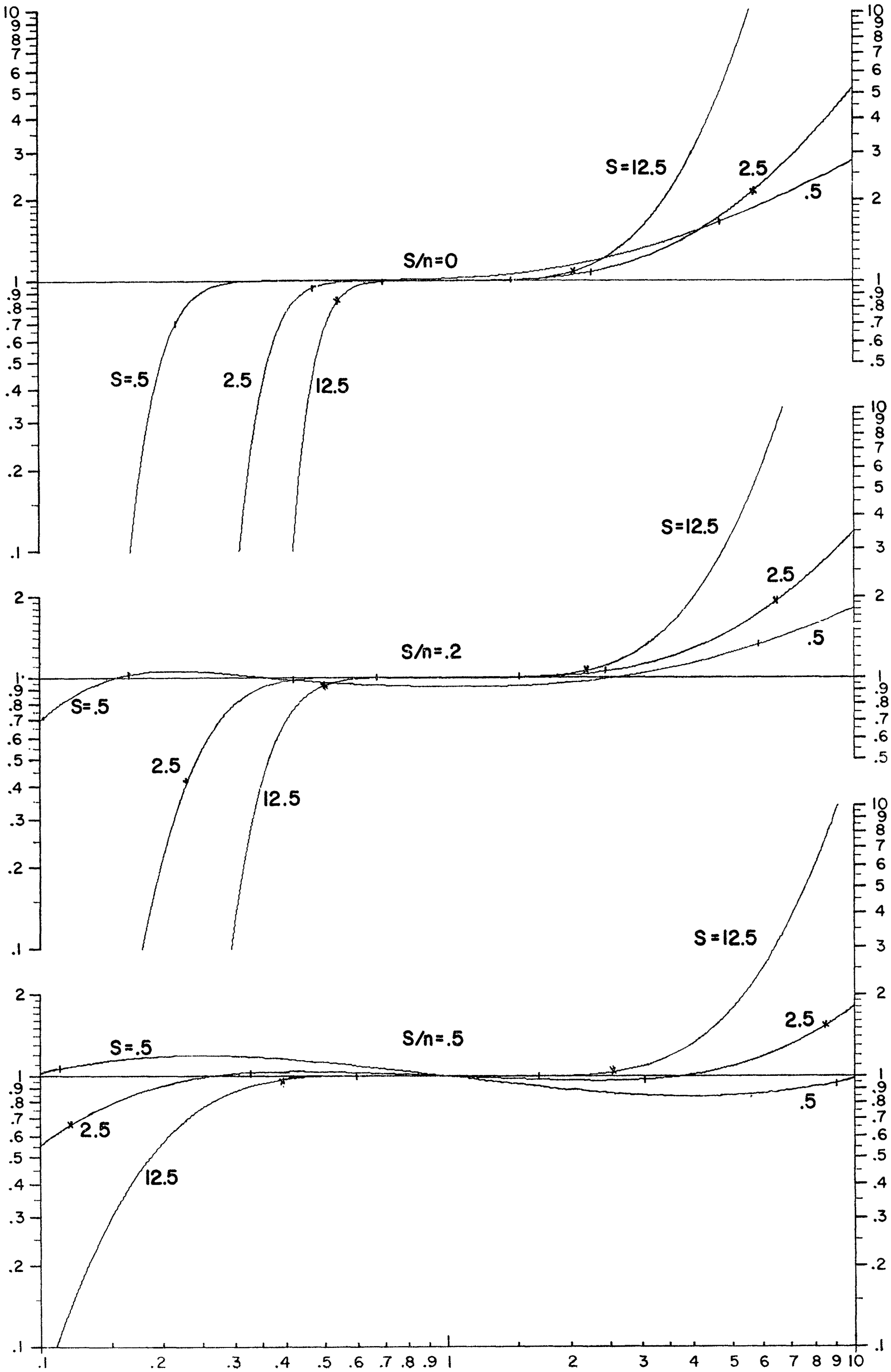

FIG. 2. (continued). Cornish-Fisher expansion with three adjustments.

be better. Of course, whichever distribution an expansion is obtained from, it can be used for all related distributions.)

Figure 2 has the same horizontal scale as Fig. 1, but the other approximations have so much larger errors than z_1 and z_2 that a new vertical scale was needed. We actually used a new variable, the ratio of true and approximate $P/(1-P)$, and a logarithmic scale, but other choices sensitive to relative error in either tail would lead to the same general conclusions. (Our choice also makes a smooth transition between tails and treats true and approximate probabilities symmetrically whatever the size of the error.)

A glance at Fig. 2 shows immediately that, for the range covered, z_2 is in fact so much more accurate than the other approximations that its errors are scarcely visible on a scale appropriate to the latter. Fig. 2 also evinces the well known superiority of the cube root approximation to the square root and ordinary Normal approximations (Wilson and Hilferty, 1931; Raff, 1956). Curves for z_1, not included here, would look much like those for z_2, although the relative error of z_1 at the ends of range of Fig. 2 reaches about $\pm 9\%$ for $S=.5$ and $\pm 5\%$ for $S=2.5$ for some values of S/n. (On the vertical scale used in Fig. 2, this is about .91 and 1.09 for $S=.5$, about .95 and 1.05 for $S=2.5$).

Looking at a fixed value of Sq/Tp other than 1, we see that as S increases (implying that the smaller tail probability approaches 0) the relative error obtained from z_2 appears to remain small, as would that from z_1, while the other approximations lose all relative accuracy, that is, the approximate tail approaches 0 infinitely faster or slower than the true tail. The difference is less dramatic when one restricts attention to a fixed range of tail probabilities, such as $.01 \leq P \leq .99$ or $.1 \leq P \leq .9$, but z_2 still appears best, z_1 would appear next best, the cube root approximation appears next, the square root next, and the ordinary Normal approximation worst. (In Fig. 2, the two ranges mentioned are indicated by crosses and bars on the curves, and any such range gets shorter as S increases.)

Asymptotic theory confirms these appearances and shows that both z_1 and z_2 have a higher order of accuracy than the other Normal approximations just mentioned. The more specific statements which follow apply even when p depends on n and also to the Poisson, gamma, and chi square distributions. They are most conveniently expressed in terms of σ^2, where $\sigma^2 = npq$ or $\sigma^2 = M$ as appropriate, and we assume that $\sigma \to \infty$. Consider first the case when z is bounded, that is, P is bounded away from 0 and 1: in this case, the errors in z_1 and z_2 and in the corresponding approximate probabilities are of order $1/\sigma^3$, at most while in general the ordinary Normal and square root approximations have errors of order $1/\sigma$ and the cube root approximation has errors of order $1/\sigma^2$. Second, even when z is unbounded so that P may approach 0, the relative error in the approximate probabilities given by z_1 and z_2 always approaches 0 as $\sigma \to \infty$ provided z is of smaller order than σ, while the ordinary Normal and square root approximations have this property only if z is of smaller order than $\sigma^{\frac{1}{3}}$ and the cube root approximation has this property only if z is of smaller order than $\sigma^{\frac{1}{2}}$. The same statement holds when "approaches 0" is replaced by "remains bounded" and "smaller order than" by "order at most." When the same approximations are applied to the t distribution via (4.2) or (4.3), the foregoing statements change as follows: first, the statements about the cube root approximation apply also the to ordinary Normal and square root approximations; second, for z bounded, the errors in z_1 and z_2 and in the corresponding approximate probabilities are of order $1/\sigma^4$ at most. Similar changes occur in other special cases, in particular, the binomial distribution with $p=\frac{1}{2}$, the F distribution with $\mu=\nu$, and the beta distribution with $a=b$. These statements are summarized in Table 2. Further details and proofs are given in Part II.

Of course simpler Normal approximations than z_1 or z_2 are adequate in certain regions. They are not quite as simple as they may seem, however, because the region needs to be checked and some other approximation used outside it. Smith (1953) exploits several approximations in giving rules for obtaining all

TABLE 2. ASYMPTOTIC BEHAVIOR OF APPROXIMATIONS

	General Case			Special Cases Including t via (4.2), Binomial ($p=\frac{1}{2}$) $F(\mu=\nu)$, Beta ($a=b$)		
Approximation:	z_1, z_2	$Z_{\frac{1}{3}}'$	$Z_{\frac{1}{2}}'$, $Z_{\frac{1}{2}}''$, z_0	z_1, z_2	$Z_{\frac{1}{3}}'$	$Z_{\frac{1}{2}}'$, $Z_{\frac{1}{2}}''$, z_0
Order of error in z or P for z bounded	$1/\sigma^3$	$1/\sigma^2$	$1/\sigma$	$1/\sigma^4$	$1/\sigma^2$	$1/\sigma^2$
Order of z guaranteeing bounded relative error in P; if z is of smaller order, relative error in P approaches 0	σ	$\sigma^{\frac{1}{2}}$	$\sigma^{\frac{1}{3}}$	σ	$\sigma^{\frac{1}{2}}$	$\sigma^{\frac{1}{2}}$

binomial probabilities within 0.01. His work is very interesting but not really comparable here, because of the nature of his accuracy requirement and because he needs a direct binomial table for $n \leq 20$, $.01 \leq p \leq .5$ and a Poisson table for $M \leq 100$. Similar comments apply to Raff (1956) who explores nine approximations and finds among other things that for $n \geq 5$, the probability of any interval is given within .005 by the cube root approximation if $np \geq .8$ and by a Poisson Gram-Charlier approximation otherwise. Of course if an error of .005 in the tail probability P occurred at $P=.01$, it would be an error of 50% relative to P.

In some regions the approximations z_1 and z_2 simplify almost automatically in the course of computation. For example, z_2 can be simplified to z_1 when the term added to d_1 to get d_2 is negligible, and z_1 can be simplified to $d_1[(n+\frac{1}{6})pq]^{-\frac{1}{2}}$ when the terms involving g in (1.2b) are negligible. Similarly, $n+\frac{1}{6}$ can sometimes be simplified to n and d_1 to $S-np$. One can usually judge before making the more complicated calculation whether the simpler one will suffice. Thus z_2 will almost automatically simplify right down to the ordinary Normal approximation $z_0=(S-np)(npq)^{-\frac{1}{2}}$ in many situations where the latter is sufficiently accurate. In this sense, the simpler Normal approximations adequate only in certain regions are little simpler.

7. RECURSIVE RELATIONS

The approximations z_1 and z_2 are undefined if S or T is negative. In addition, S and T are occasionally non-negative but so small that an adequate correction cannot be read from the graphs discussed in Section 5. This can usually be ascertained before calculating the approximation. In such situations, one possibility is to use recursive relations to shift the value of S or T or both by one or more units before applying the approximation. Some standard recursive relations which can be used are given here for reference, and then a few comments are made on their use. They could be expressed in terms of S and T, but it is more convenient and suggestive to express them otherwise. They are stated for *left tails* only.

For the beta distribution (2.8), with $n=a+b-1$:

$$F_\beta(y; a, b) - F_\beta(y; a+1, b-1) = \binom{n}{a} y^a(1-y)^{b-1}, \tag{7.1}$$

$$F_\beta(y; a, b) - F_\beta(y; a+1, b) = \binom{n}{a} y^a(1-y)^b, \tag{7.2}$$

$$F_\beta(y; a, b+1) - F_\beta(y; a, b) = \binom{n}{b} y^a(1-y)^b, \tag{7.3}$$

$$F_\beta(y; a, b) - F_\beta(y; a+1, b+1) = \binom{a+b}{a} y^a(1-y)^b\left(\frac{a}{a+b} - y\right) \cdot \tag{7.4}$$

The last formula follows from the preceding two.

For the binomial distribution (2.1), corresponding to (7.1)–(7.3) are:

$$F_b(s; p, n) - F_b(s-1; p, n) = \binom{n}{s} p^s q^{n-s}, \tag{7.5}$$

$$F_b(s; p, n) - F_b(s; p, n+1) = \binom{n}{s} p^{s+1} q^{n-s}, \tag{7.6}$$

$$F_b(s+1; p, n+1) - F_b(s; p, n) = \binom{n}{s+1} p^{s+1} q^{n-s}. \tag{7.7}$$

For the Pascal distribution (2.4), the negative binomial (2.7), and the F (2.11), similar formulas can be written, but it is equivalent and not materially harder to use the beta formulas above after transforming the desired tail to a beta tail by

$$F_{Pa}(n; y, s) = F_\beta(y; s, n+1-s); \tag{7.8}$$

$$F_{nb}(f; y, s) = F_\beta(y; s, f+1); \tag{7.9}$$

$$F_F(F; \mu, \nu) = F_\beta(q; \tfrac{1}{2}\mu, \tfrac{1}{2}\nu) \tag{7.10}$$

where q is defined by Table A or (2.13).

For the Poisson, gamma, and chi square distributions, (2.21)–(2.23):

$$F_\gamma(y;r) - F_\gamma(y; r+1) = \frac{y^r e^{-y}}{\Gamma(r+1)}, \tag{7.11}$$

$$F_P(s; M) - F_P(s-1; M) = \frac{M^s e^{-M}}{s!}, \tag{7.12}$$

$$F_{\chi^2}(X^2; \nu) - F_{\chi^2}(X^2; \nu+2) = \frac{(\frac{1}{2}X^2)^{\frac{1}{2}\nu} e^{-\frac{1}{2}X^2}}{\Gamma(\frac{1}{2}\nu+1)} \cdot \tag{7.13}$$

For the t distribution (4.6), similar formulas can be written, but it is easier to use (4.3) and then use the beta formulas. Correspondingly, the beta distribution with a or $b=\frac{1}{2}$ and the F distribution with μ or $\nu=1$ can be handled by (4.4) and (4.5).

Formula (7.1) can be used either to increase a and decrease b or to decrease a and increase b. Similarly, (7.2)–(7.4) can be used in either of two directions. In choosing a formula and direction, it is desirable, if possible, to move the mean away from y and to increase the smaller of a and b. These considerations may conflict, in which case the latter is then probably more important if a or b is very small, which is the situation being addressed here. Similar remarks apply to (7.5)–(7.7).

Sometimes the foregoing considerations are irrelevant because it is possible and preferable to compute the desired probability as a sum of a small number of terms. There are two essentially different standard formulas of this kind. When b is an integer, they are obtainable by repeated application of (7.1) and (7.3) respectively and are, with $n=a+b-1$ and $p=1-q$:

$$F_\beta(q; a, b) = \binom{n}{b-1} p^{b-1} q^a + \binom{n}{b-2} p^{b-2} q^{a+1} + \cdots + q^n; \tag{7.14}$$

$$F_\beta(q; a, b) = q^a \left\{\binom{n-1}{b-1} p^{b-1} + \binom{n-2}{b-2} p^{b-2} + \cdots + 1\right\}. \tag{7.15}$$

Similarly, when a is an integer:

$$F_\beta(q; a, b) = 1 - \binom{n}{a-1} p^b q^{a-1} - \binom{n}{a-2} p^{b+1} q^{a-2} - \cdots - p^n; \tag{7.16}$$

$$F_\beta(q; a, b) = 1 - p^b \left\{\binom{n-1}{a-1} q^{a-1} + \binom{n-2}{a-2} q^{b-2} + \cdots + 1\right\} \cdot \tag{7.17}$$

If a and b are both integers, a beta tail is a binomial tail:

$$F_\beta(q; a, b) = F_b(b-1; p, n), \tag{7.18}$$

and (7.14) and (7.16) are just two expressions for this binomial tail while (7.15) and (7.17) are equivalent tails of the Pascal (or negative binomial) distribution.

The corresponding formulas for the Poisson, gamma, and chi square distributions are, for integer r,

$$\begin{aligned} F_\gamma(y; r) &= 1 - e^{-y} \left\{\frac{y^{r-1}}{(r-1)!} + \frac{y^{r-2}}{(r-2)!} + \cdots + 1\right\} \\ &= 1 - F_P(r-1; y) = F_{\chi^2}(2y; 2r). \end{aligned} \tag{7.19}$$

8. SMALL TAILS: GETTING FULL ACCURACY FROM THE NORMAL APPROXIMATION

If the graphs of the relative error (Fig. 1) indicate that the approximation has a certain relative accuracy, or will have when corrected by means of the graphs, then this accuracy holds no matter how small the tail probability under approximation. In order to realize the advantage of this accuracy fully, it is necessary to use Normal tables whose relative accuracy in the smaller tail is at least as great. Many tables give four (fixed) decimal places, which may not be enough. If sufficiently accurate tables are not readily available, the following formula may be adequate:

$$\begin{aligned}\min\{\Phi(z),\ 1-\Phi(z)\} &= \frac{e^{-\frac{1}{2}z^2}m(z)}{(2\pi)^{\frac{1}{2}}|z|} \\ &= \frac{m(z)}{|z|\operatorname{antilog}_{10}(.21714724z^2+.399090)}\end{aligned} \tag{8.1}$$

where

$$m(z) \doteq 1 - \cfrac{1}{z^2+3-\cfrac{1}{.22(z^2+3.2)}} \tag{8.2}$$

with relative error less than .0005 (and consequently error less than half a unit in the third significant place) provided $|z| \geq 1.2$. (The error increases rapidly as $|z|$ decreases below 1.2 and (8.2) is not useful there.)

9. EXTREME TAILS: CALCULATION BY CONTINUED FRACTIONS

If an extreme tail probability is under approximation, and if z_2 does not provide sufficient accuracy, even with the help of the graphs of Section 5, or if the graphs are inapplicable because Sq/Tp is outside the range .1 to 10, then the continued fractions given below may be useful. They correspond in a certain sense to standard series obtainable by repeated application of the formulas of Section 7, but they seem to converge much faster except when the series terminate quickly (see the end of Section 7 and Müller, 1931). In addition, they provide both upper and lower bounds. Though we are now suggesting them only for desk calculation of very small tail probabilities, where they always converge rapidly, they may well be useful also, even in moderate tails, when very high accuracy is desired, as is typical in high-speed computer work. They actually converge to the stated probabilities provided only $t \neq 0$ in (9.11), and the approximants behave as described provided also $-u_1 < v_1$ in each case. The conditions imposed in Section 9.1 are stronger yet. They were chosen so as to create convenient categories into which the problems under discussion can all be made to fall.

Successive approximants of continued fractions are easily calculated recursively, as described in texts on numerical analysis and, for instance, in the National Bureau of Standards Handbook (Abramowitz and Stegun, editors, 1964, pp. 19, 22), Wall (1948, p. 15), and Khovanskii (1956, p. 2).

9.1 *Formulas and properties*. For the binomial, Pascal, negative binomial, beta and F distributions, by transforming the problem if necessary, let $S \leq np$ and let P be a left tail. Then

$$P = \binom{n}{a} p^{b-1} q^a / (1 + u_1/(v_1 + u_2/(v_2 + u_3/(v_3 + \cdots)))) \tag{9.1}$$

where

$$a = T + \tfrac{1}{2}, \qquad b = S + \tfrac{1}{2}, \qquad \binom{n}{a} = \frac{\Gamma(n+1)}{\Gamma(a+1)\Gamma(b)}, \tag{9.2}$$

$$u_1 = -(b-1)q/p, \qquad v_k = a + k, \qquad k = 1, 2, \cdots, \tag{9.3}$$

$$u_{2j} = j(n+j)q/p, \qquad u_{2j+1} = -(a+j)(b-j-1)q/p, \quad j = 1, 2, \cdots. \tag{9.4}$$

The approximants obtained by terminating at u_k/v_k for $k=1, 2, 5, 6, 9, 10, \cdots$ decrease monotonically toward P and those for $k=0, 3, 4, 7, 8, 11, 12, \cdots$ increase monotonically toward P as long as $k \leq 2B-1$, where B is the integer part of b; for $k \geq 2B-2$, the even approximants approach P monotonically from above and the odd ones from below if B is even, vice versa if B is odd. The monotonicity is strict throughout except that, if b is an integer, the fraction terminates at $k=2b-2$.

For the Poisson, gamma, and chi square distributions, if $S \leq M$, let P be a left tail Poisson probability or a right tail gamma or chi square probability. Then

$$P = \frac{M^{b-1}}{\Gamma(b)} e^{-M} / (1 + u_1/(v_1 + u_2/(v_2 + u_3/(v_3 + \cdots)))) \tag{9.5}$$

where $b = S + \frac{1}{2}$ as before, and

$$u_{2j-1} = j - b, \qquad v_{2j-1} = M, \tag{9.6}$$

$$u_{2j} = j, \qquad v_{2j} = 1, \qquad j = 1, 2, \cdots. \tag{9.7}$$

The approximants again behave as described after (9.4).

If $S \geq M$, let P be a right tail Poisson probability or a left tail gamma or chi square probability. Then

$$P = \frac{M^b}{\Gamma(b+1)} e^{-M} / (1 + u_1/(v_1 + u_2/(v_2 + u_3/(v_3 + \cdots)))) \tag{9.8}$$

where

$$u_1 = -M, \qquad u_{2j} = jM, \tag{9.9}$$

$$u_{2j+1} = -(b+j)M, \qquad v_j = b + j, \quad j = 1, 2, \cdots. \tag{9.10}$$

The approximants obtained by terminating at u_k/v_k approach P strictly monotonically from above for $k=1, 2, 5, 6, 9, 10, \cdots$ and from below for $k=3, 4, 7, 8, 11, 12, \cdots$.

For the t distribution, let P be the smaller tail. Then

$$P=\frac{\Gamma(\frac{1}{2}\nu+\frac{1}{2})\nu^{(\nu-2)/2}}{\pi^{\frac{1}{2}}\Gamma(\frac{1}{2}\nu)(\nu+t^2)^{(\nu-1)/2}\,|\,t\,|}/(1+u_1/(v_1+u_2/(v_2+u_3/(v_3+\cdots)))) \tag{9.11}$$

where

$$u_1 = \nu/t^2, \qquad v_k = \nu + 2k, \quad k = 1, 2, \cdots, \tag{9.12}$$

$$u_j = j(\nu + j - 1)\nu/t^2, \qquad j = 2, 3, \cdots. \tag{9.13}$$

The approximants obtained by terminating at u_k/v_k approach P strictly monotonically from above for k even and from below for k odd.

9.2 *Further notes and references on continued fractions*. Both (9.1) and another continued fraction for the same tail probability, namely

$$P = \binom{n}{a} p^b q^a/(1 + u_1/(v_1 + u_2/(v_2 + u_3/(v_3 + \cdots)))) \tag{9.14}$$

where

$$u_1 = -(n+1)q, \quad v_k = a + k, \qquad k = 1, 2, \cdots, \tag{9.15}$$

$$u_{2j} = j(b-j)q, \qquad u_{2j+1} = -(a+j)(n+1+j)q, \quad j = 1, 2, \cdots, \tag{9.16}$$

are direct applications of Gauss's continued fraction for the hypergeometric function, in view of the incomplete beta function's relation to the hypergeometric function and the standard properties of the latter, especially Euler's transformation. See, for instance, Erdélyi et al (1953, in particular, Sections 2.5.4, 2.5.3 and formulas 2.1(22), 2.8(4), 2.8(8)). The odd approximants of (9.1) and (9.14) are equal, as can be shown by obtaining the odd parts of both and noting that they are algebraically identical when cast in the form

$$\binom{n}{a} p^b q^a/(1 + V_0 y + U_1 y^2/(1 + V_1 y + U_2 y^2/(1 + V_2 y + \cdots)));$$

the methods involved are standard (Wall, 1948, pp. 19–22; Khovanskii, 1956, p. 14 (2.8); our odd parts are even parts by the usual numbering). It can be shown further that the continued fraction (9.14) is slightly less good than (9.1) in two respects: for $k \leq 2B-1$, the even approximants are less accurate, though they behave the same way; and for $k \geq 2B-2$, all approximants approach P monotonically from one side, below if B is even, above if B is odd. In the binomial case, where the fractions terminate at $k=2B-2$ or $2B-1$, these relationships between them were conjectured by Bahadur (1960, p. 47), whose influence on the present work is further indicated in Part II. Other references in the statistical literature for (9.1) include Müller (1931), Aroian (1941), and, in the binomial case, Markoff (1900, pp. 135–41, also available in Uspensky, 1937, pp. 53–55). Apparently (9.14) was given first by Aroian (1941), who credits himself too little insofar as his references to Markoff and Uspensky for the binomial case really relate to (9.1) but too much insofar as the advantages he attributes to (9.14) in some cases are belied by the relationship between (9.14) and (9.1) just mentioned.

The continued fractions (9.5) and (9.8) are term-by-term limits of (9.1) and (9.14) respectively when all are written in the form $w_0/(1+w_1/(1+w_2/(1+\cdots)))$, where (e.g. Wall, 1948, p. 20) $w_1=u_1/v_1$ and $w_k=u_k/v_{k-1}v_k$ for $k\geq 2$. The fraction (9.5) is well known (e.g., Wall, 1948, p. 356, who refers to Legendre; Müller, 1931, who refers to De Morgan; NBS Handbook, p. 263, p. 941). The fraction (9.8) is a direct application of the confluent form of Gauss's continued fraction, in view of the incomplete gamma function's relation to the confluent hypergeometric function (NBS Handbook, 6.5.12). However it was apparently given first by Khovanskii (1956, p. 149), and we have not seen it in Western literature. Neither Erdélyi et al (1953), which might not be expected to, nor the NBS Handbook, which would, gives it or any equivalent or even, for that matter, the confluent form of Gauss's continued fraction. Wall (1948, p. 348) gives the latter. So does Shenton (1954), but he specializes directly to the Normal without mentioning any other distribution. (This amounts to specializing (9.8) to the chi square distribution with one degree of freedom. A similar specialization of (9.5) yields Laplace's continued fraction for the Normal distribution. We obtained (8.2) by empirical adjustment of the second approximant of the even part of the latter.)

Since the t and beta distributions are related in two ways (see Section 4), several continued fractions for t can be obtained directly. One is (9.11), which is an application of (9.1) to $G_t(t\mid v)=\frac{1}{2}F_\beta(\nu/(t^2+\nu)\mid \frac{1}{2}\nu,\ \frac{1}{2})$ and approaches Laplace's continued fraction term by term as $\nu\to\infty$. Applying (9.1) to $F_t(t\mid\nu)-\frac{1}{2}=\frac{1}{2}F_\beta(t^2/(t^2+\nu)\mid\frac{1}{2},\ \frac{1}{2}\nu)$ gives the following continued fraction, which approaches Shenton's term by term as $\nu\to\infty$ and which should converge rapidly when t is moderate:

$$F_t(t\mid\nu)=\tfrac{1}{2}+\frac{\Gamma(\frac{1}{2}\nu+\frac{1}{2})\nu^{\frac{1}{2}\nu-1}t}{\pi^{\frac{1}{2}}\Gamma(\frac{1}{2}\nu)(\nu+t^2)^{\frac{1}{2}\nu-\frac{1}{2}}}\Big/(1+u_1/(v_1+u_2/(v_2+u_3/(v_3+\cdots)))) \qquad (9.17)$$

where

$$u_j=(-1)^j j(\nu+(-1)^j j-1)t^2/\nu, \qquad v_j=2j+1, \qquad j=1,2,\cdots. \qquad (9.18)$$

For $t^2<3\nu/(\nu-2)$, the approximants behave as described after (9.4) with B the integer part of $\frac{1}{2}\nu$.

A convenient reference for the convergence of (9.1), (9.5), (9.8) and (9.14) to the stated values is Wall (1948, pp. 339, 356, 347–8, and 339 respectively).

Under suitable conditions, monotonic convergence of certain approximants from above and the rest from below, though not necessarily to the same value, let alone to any particular value, can be shown by repeated application of the fact that $1/(1+x)$ is a decreasing function of x for $x>-1$. With the continued fractions in the form $w_0/(1+w_1/(1+w_2/(1+\cdots)))$, the easiest case occurs when the w_k are all positive, as in (9.11), and the next easiest when the w_k alternate in sign and exceed -1, as in (9.8). No new complication arises as long as $w_k>-1$ for all k and two negative w's never occur in succession; these conditions hold for each continued fraction considered here provided $-u_1<v_1$. In the present context, the first case is used by Gupta and Waknis (1965, p.

141) for $b<1$ in (9.5) and the second by Bahadur (1960, p. 46) for the binomial case of (9.14); they regard reference or proof as unncesssary. See also Shenton (1954, p. 181 and p. 178) and Khovanskii (1956, p. 7). Markoff (1900), Müller (1931), Uspensky (1937), Aroian (1941), and the NBS Handbook (p. 944) give similar results which are, however, of a weaker form and either restricted to the binomial case or not completely correct for noninteger b.

We are indebted to L. R. Shenton for several of these references.

10. DEFINITION AND PROPERTIES OF THE CORRECTION FUNCTION g

The two forms of z_1 and z_2 given in (1.2) are equal (as can be verified algebraically) if the function g in (1.2b) is defined by

$$g(x) = \frac{1 - x^2 + 2x \ln x}{(1-x)^2}, \qquad x > 0, x \neq 1, \tag{10.1}$$
$$g(0) = 1, \qquad g(1) = 0.$$

This, then, is the analytical definition of g. For calculation, of course, Table 1 should be used rather than this definition.

It can be verified from the definition that

$$g(x) = -g(x^{-1}), \qquad x > 0. \tag{10.2}$$

This must be used when $x>1$ to evaluate $g(x)$ from Table 1.

The function g is continuous, strictly decreasing, and strictly convex. For $x \leq 1$, this follows from the expansion

$$g(x) = \sum_{j=1}^{\infty} \frac{2(1-x)^j}{(j+1)(j+2)}, \qquad 0 \leq x \leq 2, \tag{10.3}$$

which can be obtained from (10.1). For $x \geq 1$, it then follows from (10.2) and

$$g''(x) = 4x^{-3} \sum_{j=1}^{\infty} \frac{j(1-x^{-1})^{j-1}}{(j+2)(j+3)}, \qquad \tfrac{1}{2} \leq x, \tag{10.4}$$

which can be obtained by substituting x^{-1} for x in (10.3), using (10.2), and differentiating twice.

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