OPTION DELTA GAMMA AND THETA FROM GRID SCHEMES

Description

Since an American exercise option can disappear due to early exercise *before* its tenor is up, we have to price it using a numerical structure that integrates not only over underlying price, like the expectation integrals, but also over *time*. Let's call these *space-time schemes*. For American options in space-time schemes, we usually check each node versus early exercise value.

Mathematically, what we write is that the price of a European-exercise options can be given to us by Girsanov/Feyman-Kac and other fancy names by

$$V_0 = \mathbb{E}\left[-\exp\left(\int_0^T r_t dt\right) \text{Payoff}(S)\right]$$

The choices available to option holders, when American[1] exercise applies, lets those option holders make their most advantageous choice of *exercise policy* τ.

If we take the maximum advantage, or supremum (indicated in formulas by $\sup_{A\in B}()$) of such an exercise policy, we end up changing our mathematical formula for option value to indicate it must have the "best possible" policy τ_{sup}. That's the supremum over *all* such policies and can be written

$$V_0 = \sup_{\tau\in P(T)} \mathbb{E}\left[-\exp\left(\int_0^T r_t dt\right) \cdot \text{Payoff}(S)\right]$$

Checking each grid node for early exercise value versus *continuation value* of holding the option is a way of approximating the best possible policy.

1. Backwardation and Greeks

Space-time schemes backwardate from option tenor all the back to time zero on some kind of finite set of points, usually a grid or subset of a grid.

Our most basic example of this is binomial trees, though we also have examples in trinomial trees, explicit finite differences, and implicit or Crank-Nicolson finite differences[2].

When we are integrating over "both space and time" we have some built-in ways to compute greeks related to "space and time" (i.e. underlying price and time). A space-time scheme, after backwardating option values and checking early exercise policy from option tenor all the way to time zero, can give us option prices for time zero relevant not only to the current underlying price $S_0 = S^{0,0}$ but also small shifts, which I will call $S^{+1,0}$ and $S^{-1,0}$. From these, basic finite difference approximations yield estimates of delta and gamma as first and second derivative approximations.

Recapitulating some of these other course documents, we have Euler's Formula

$$F(t_0 + h) \approx F(t_0) + hF'(t_0)$$

which arises from the fact that the derivative $F'(t_0)$ is close to the finite difference

$$\frac{F(t_0 + h) - F(t_0)}{h}.$$

We want to go in the other direction. That is, given some values of F we want to generate values for the first and second derivative F' and F''. To obtain the first derivative estimate, we could in theory use the approximation we have already seen

$$F'(x) \approx \frac{F(x+h) - F(x)}{h},$$

but we obtain a much more accurate[3] estimate by using the *central difference*

$$F'(x) \approx \frac{F(x+h) - F(x-h)}{2h}. \tag{1}$$

[1]Episodic early exercise decisions are called *Bermudan* since the Bermudan islands lie sort-of between the Americas and Europe

[2]And even more, such as LSMC ("least squares Monte Carlo" or "Longstaff Schwartz Monte Carlo" depending on who you ask

[3]Second order, or $O(h^3)$ error

If we want to find a second derivative, we combine two neighboring first derivative estimates.

$$
\begin{aligned}
F''(x) &\approx \frac{F'(x) - F'(x-h)}{h} \\
&= \frac{F(x+h) - 2F(x) + F(x-h)}{h^2}.
\end{aligned}
\tag{2}
$$

It is customary, when using trees, to ignore the fact that they are *geometric* rather than *arithmetic*, and simply take

$$
h = \frac{1}{2}\left((S^{+1,0} - S^{0,0}) + (S^{0,0} - S^{-1,0})\right) \tag{3}
$$

to compensate for the fact that $(S^{+1,0} - S^{0,0})$ is a little different from $S^{0,0} - S^{-1,0}$. The error from this is $O(\Delta T^2)$ so we do not really mind it.

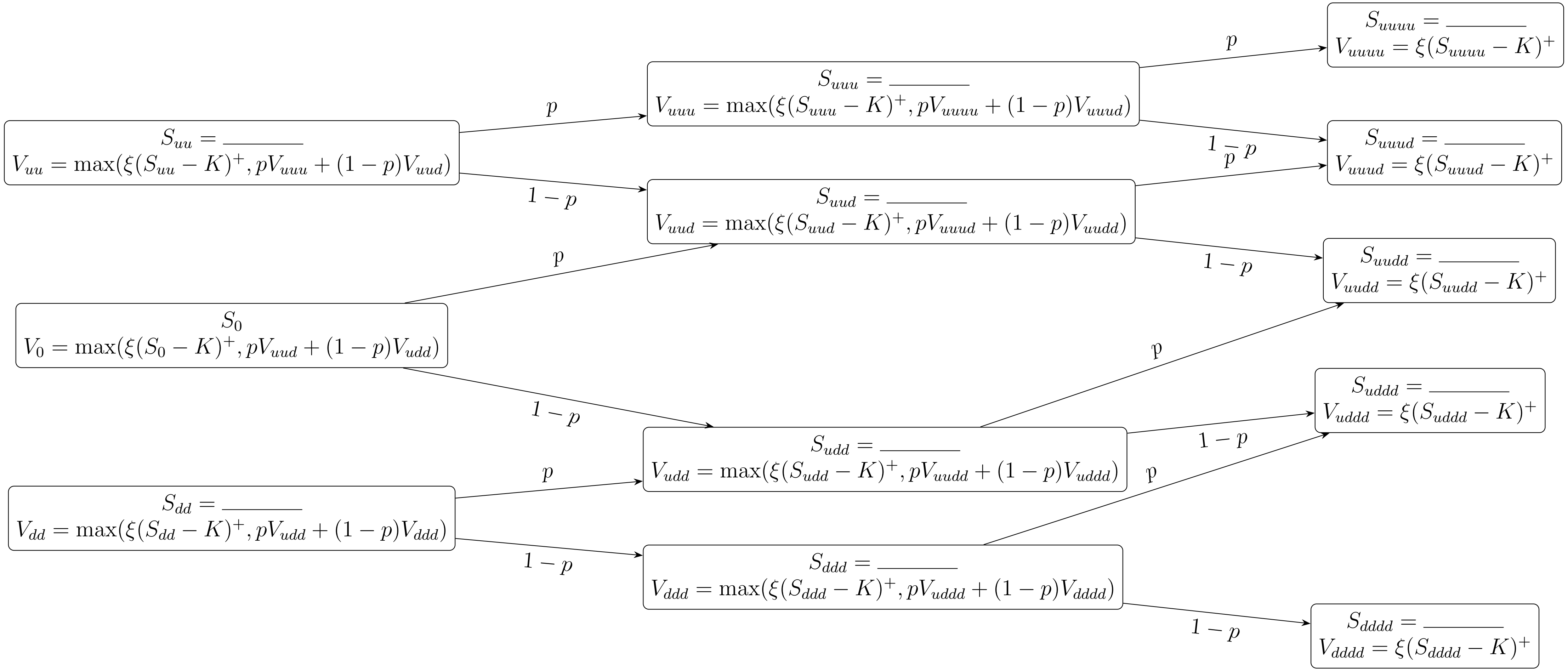
Now, if we set up our tree with three initial nodes, rather than just one, we can use this h and our formulas above to get delta and gamma.

In addition, we have prices from a timestep or two into the future (depending on the space-time scheme we happen to be using). This lets us estimate time-sensitivity (sometimes: *time-decay* or *theta-decay* or *theta-bill*) of option value. Here, we usually just use the one-sided version, taking, e.g. for a binomial tree

$$
\theta \approx \frac{1}{2\Delta T}(V_{uudd} - V_0) \tag{4}
$$

This is usually a more business-relevant estimate than a 2-sided estimate would be, since in a practical business sense we are forward-looking.

Our next page show an example of a binomial tree on which greeks can be computed.



S_0
$V_0 = \max(\xi(S_0 - K)^+, pV_{uud} + (1-p)V_{udd})$
$S_{uu} =$ ______
$V_{uu} = \max(\xi(S_{uu} - K)^+, pV_{uuu} + (1-p)V_{uud})$
$S_{dd} =$ ______
$V_{dd} = \max(\xi(S_{dd} - K)^+, pV_{udd} + (1-p)V_{ddd})$
$S_{uuu} =$ ______
$V_{uuu} = \max(\xi(S_{uuu} - K)^+, pV_{uuuu} + (1-p)V_{uuud})$
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$S_{udd} =$ ______
$V_{udd} = \max(\xi(S_{udd} - K)^+, pV_{uudd} + (1-p)V_{uddd})$
$S_{ddd} =$ ______
$V_{ddd} = \max(\xi(S_{ddd} - K)^+, pV_{uddd} + (1-p)V_{dddd})$
$S_{uuuu} =$ ______
$V_{uuuu} = \xi(S_{uuuu} - K)^+$
$S_{uuud} =$ ______
$V_{uuud} = \xi(S_{uuud} - K)^+$
$S_{uudd} =$ ______
$V_{uudd} = \xi(S_{uudd} - K)^+$
$S_{uddd} =$ ______
$V_{uddd} = \xi(S_{uddd} - K)^+$
$S_{dddd} =$ ______
$V_{dddd} = \xi(S_{dddd} - K)^+$
p
$1-p$