TAYLOR EXPANSIONS, BACHELIER AND BLACK-SCHOLES DRIFT

Our arbitrage argument , and the equivalence of hedging to arbitrage-consistent probabilities, tell us that the expected forward-time value in any "good" model is $S_T = e^{rT} S_0$, or in a small time increment, Δt, $S_{\Delta t} = S_0 e^{r\Delta t}$

If we consider an interval Δt, and then we cut it in half, we want the same dynamics in first and second halves. In particular, the same amount of variability in each half. That must mean each interval gets half the variance, since variances add.

Therefore we want our model variance in time Δt to be proportional to Δt. Let's consider what this looks like for symmetric 1-step trees (trees in which the up and down nodes are equidistant from the center node).

It will be useful to recall that the Taylor expansion of the exponential near $\epsilon = 0$ is

$$e^{\epsilon} = 1 + \epsilon + \frac{\epsilon^2}{2!} + \frac{\epsilon^3}{3!} + \dots$$

We will be working with Δt approaching zero, so we are interested in keeping terms of size Δt and $\sqrt{\Delta t}$ (which for small values is larger than Δt itself).

1. Bachelier Asymptotics

In the Bachelier model, we get this variance by taking the "up" node to be at

$$S^{+} = S_0(1 + \sigma_{\text{Bach}}\sqrt{\Delta t})$$

and the "down" node at

$$S^{-} = S_0(1 + \sigma_{\text{Bach}}\sqrt{\Delta t}).$$

In discussing hedging and arbitrage, we showed the probability to use is

$$p_u = \frac{S_0 e^{r\Delta t} - S^{-}}{S^{+} - S^{-}}$$

which therefore expands to

$$p_u = \frac{S_0 e^{r\Delta t} - S_0(1 - \sigma_{\text{Bach}}\sqrt{\Delta t})}{S_0(1 + \sigma_{\text{Bach}}\sqrt{\Delta t}) - S_0(1 - \sigma_{\text{Bach}}\sqrt{\Delta t})}$$

and then simplifies to

$$p_u = \frac{S_0 e^{r\Delta t} - S_0(1 - \sigma_{\text{Bach}}\sqrt{\Delta t})}{S_0(1 + \sigma_{\text{Bach}}\sqrt{\Delta t}) - S_0(1 - \sigma_{\text{Bach}}\sqrt{\Delta t})}$$

Factoring out S_0 and using Taylor expansions, we obtain

$$\frac{1 + r\Delta t + \frac{r^2 \Delta t^2}{2!} + \frac{r^3 \Delta t^3}{2!} - (1 - \sigma_{\text{Bach}}\sqrt{\Delta t})}{2\sigma_{\text{Bach}}\sqrt{\Delta t}}$$

or

$$p_u = \frac{1}{2}\left(1 + r\Delta t \cdot \frac{1}{\sigma_{\text{Bach}}\sqrt{\Delta t}} + O(\Delta t^{\frac{3}{2}})\right)$$

so our instantaneous drift is of size r

2. Black-Scholes Asymptotics

In the Black-Scholes model, we have gaussian returns, not gaussian prices. So the logarithm of prices is gaussian.

For our *returns* to have our target variance proportional to Δt, we have

$$S^+ = S_0 \exp(+\sigma_{\text{BS}}\sqrt{\Delta t})$$

and the "down" node at

$$S^- = S_0 \exp(-\sigma_{\text{BS}}\sqrt{\Delta t})$$

and again using

$$p_u = \frac{S_0 e^{r\Delta t} - S^-}{S^+ - S^-}$$

we have

$$p_u = \frac{S_0 e^{r\Delta t} - S_0 e^{-\sigma_{\text{BS}}\sqrt{\Delta t}}}{S_0 e^{+\sigma_{\text{BS}}\sqrt{\Delta t}} - S_0 S_0 e^{-\sigma_{\text{BS}}\sqrt{\Delta t}}}.$$

Again we factor out S_0 and take Taylor expansions

$$p_u = \frac{1 + r\Delta t + \frac{r^2\Delta t^2}{2!} + O(\Delta t^3) - (1 - \sigma_{\text{BS}}\sqrt{\Delta t} + \frac{1}{2}\sigma_{\text{BS}}^2\Delta t + O(\Delta t^{\frac{3}{2}}))}{(1 + \sigma_{\text{BS}}\sqrt{\Delta t} + \frac{1}{2}\sigma_{\text{BS}}^2\Delta t + O(\Delta t^{\frac{3}{2}})) - (1 - \sigma_{\text{BS}}\sqrt{\Delta t} + \frac{1}{2}\sigma_{\text{BS}}^2\Delta t + O(\Delta t^{\frac{3}{2}}))}.$$

Note here that nothing cancels the $+\ \frac{1}{2}\sigma_{\text{BS}}^2\Delta T$ term in the numerator. We simplify to

$$p_u \approx \frac{r\Delta t + \sigma_{\text{BS}}\sqrt{\Delta t} - \frac{1}{2}\sigma_{\text{BS}}^2\Delta t}{2\sigma_{\text{BS}}\sqrt{\Delta t}}.$$

and factor out a $\sigma_{\text{BS}}\sqrt{\Delta t}$ that we see in the denominator

We conclude that

$$p_u = \frac{1}{2}\left(1 + (r - \frac{1}{2}\sigma_{\text{BS}}^2)\Delta t \cdot \frac{1}{\sigma_{\text{BS}}\sqrt{\Delta t}} + O(\Delta t^{\frac{3}{2}})\right)$$

and so our drift has an extra term arising from that exponential, and is $r - \frac{1}{2}\sigma_{\text{BS}}^2$