BUSN 35141 01 (Autumn 2025) Advanced Models of Security Pricing and Credit Risk

Mid Term Solution

Problem 1

Two finite difference approximations for the first partial derivative of $F(z,r)$ with respect to r, calculated at z_0, r_0 are

$$\frac{\partial F}{\partial r} \approx \frac{1}{\delta}(F(z_0, r_0+\delta) - F(z_0, r_0)) \quad \text{and} \quad \frac{\partial F}{\partial r} \approx \frac{1}{2\delta}(F(z_0, r_0+\delta) - F(z_0, r_0-\delta))$$

The second is more accurate as $\delta \to 0$. Show an example of this being the case for an F of your choice (one such possibility: $z^2 + r^2$, $(z_0, r_0) = (1,1)$). Or, if you like, present a mathematical argument for this fact.

Solution (20 points):

For $F = z^2 + r^2$, for the first approximation, we have $\frac{\partial F}{\partial r} \approx \frac{1}{\delta}[z_0^2 + (r_0+\delta)^2 - z_0^2 - r_0^2] = \frac{1}{\delta}[\delta^2 + 2r_0\delta] = \delta + 2r_0$. Evaluated at $(z_0, r_0) = (1,1)$ this gives:

$$\frac{\partial F}{\partial r} \approx \delta + 2$$

For the second approximation, we have $\frac{\partial F}{\partial r} \approx \frac{1}{2\delta}[z_0^2 + (r_0+\delta)^2 - z_0^2 - (r_0-\delta)^2] = \frac{1}{2\delta}[4r_0\delta] = 2r_0$. Evaluated at $(z_0, r_0) = (1,1)$ this gives:

$$\frac{\partial F}{\partial r} \approx 2$$

The true value of the derivative $\frac{\partial F}{\partial r} = 2r$ which evaluated at $(z_0, r_0) = (1,1)$ is 2. The error made by the first approximation is therefore $\delta + 2 - 2 = \delta$, while the second approximation, in fact, gives you the exact value of the derivative. So, this is an example where the second approximation is more accurate.

Alternatively, you can make a mathematical argument as follows. Using Taylor expansion, we have:

$$F(z_0, r_0+\delta) = F(z_0, r_0) + \frac{\partial F}{\partial r}\delta + \frac{\partial^2 F}{\partial r^2}\delta^2/2 + O(\delta^3)$$

and

$$F(z_0, r_0-\delta) = F(z_0, r_0) - \frac{\partial F}{\partial r}\delta + \frac{\partial^2 F}{\partial r^2}\delta^2/2 + O(\delta^3)$$

So, for the first approximation we have:

$$\frac{\partial F}{\partial r} \approx \frac{1}{\delta}(F(z_0, r_0+\delta) - F(z_0, r_0)) = \frac{1}{\delta}[F(z_0, r_0) + \frac{\partial F}{\partial r}\delta + \frac{\partial^2 F}{\partial r^2}\delta^2/2 + O(\delta^3) - F(z_0, r_0)]$$

$$= \frac{\partial F}{\partial r} + \frac{\partial^2 F}{\partial r^2}\delta/2 + O(\delta^2)$$

Overall, this has an error of $\delta/2 + O(\delta^2) = O(\delta)$.

For the second approximation we have:

$$\frac{\partial F}{\partial r} \approx \frac{1}{2\delta}(F(z_0, r_0+\delta) - F(z_0, r_0-\delta))$$

$$= \frac{1}{2\delta}[F(z_0, r_0) + \frac{\partial F}{\partial r}\delta + \frac{\partial^2 F}{\partial r^2}\delta^2/2 + O(\delta^3) - (F(z_0, r_0) - \frac{\partial F}{\partial r}\delta + \frac{\partial^2 F}{\partial r^2}\delta^2/2 + O(\delta^3))]$$

$$= \frac{1}{2\delta}[\frac{\partial F}{\partial r}2\delta + O(\delta^3)] = \frac{\partial F}{\partial r} + O(\delta^2)$$

Overall, this has an error of $O(\delta^2)$.

As, $\delta \to 0$, the error in the second approximation goes to 0 faster than that of the first.

Problem 2

Take risk free rate r and dividend/other rates to both be zero, i.e. $r = q = 0$. We observe that our underlying has best bid price $\$99$ and offer price $\$101$. We also see that the $T = 0.25$ 3-month vanilla call option with strike $K = 80$ has a best bid $\$19.90$ and offer $\$22.90$. Give your most reasonable estimate for the price range of the $K = 80$ put.

Solution (20 points):

From put-call parity we know that $C - P = Se^{-qT} - Ke^{-rT}$ and here we have $r = q = 0$. So the equation becomes $C - P = S - K$ which can be rewritten as $P = C - S + K$.

The value of P is maximum when you use call offer price and underlying bid price i.e. $P_{max} = C_{offer} - S_{bid} + K$. Plugging in the numbers we get:

$$P_{max} = 22.90 - 99 + 80 = \$3.90$$

Similarly, the value of P is minimum when you use call bid price and underlying offer price i.e. $P_{min} = C_{bid} - S_{offer} + K$ but with a lower bound of 0. Plugging in the numbers we get:

$$P_{min} = max(19.90 - 101 + 80, 0) = max(-\$1.10, 0) = \$0$$

Problem 3

A Monte Carlo uses 800 random gaussian samples, and generates $M = 800$ terminal values for Black-Scholes model paths to price a european call option. It takes 3 seconds, and returns a price of $\$35$ with a standard error of $\$0.10$. The user can select the parameter M.

1. How much time would be needed if we wanted to demand a standard error of about $\$0.01$?
2. How long would it take to run with $M = 800,000$?

Solution (10 points):

1. The standard error for Monte Carlo is of order $O(\frac{1}{\sqrt{M}})$. The current standard error is $\$0.10$ and we want a standard error of $\$0.01$, which is a ten-fold increase. This means we need $10^2 * 800 = 80,000$ samples. Since it takes 3 seconds for $M = 800$ samples, for $M = 80,000$ we need $3 * 100 = 300$ seconds.

2. Since it takes 3 seconds for $M = 800$ samples, for $M = 800,000$ we need $3 * 1000 = 3,000$ seconds.

Problem 4

Consider CRR binomial tree, with $N = 2$ steps, pricing a $K = 100$ put where risk free rate r and dividend rates are both zero, i.e. $r = q = 0$. Tenor $T = 2/9$ and the volatility $\sigma = ln(8)$. Due to that convenient natural logarithm, the up multiplier u and down multiplier d are simple fractions a/b and b/a, where $a, b \in \{1, 2, 3, 4, 5, 6\}$. What are u and d?

Solution (20 points):

For a CRR binomial tree, the up factor u is given by $u = e^{\sigma\sqrt{\Delta t}}$, the down factor d is given by $d = e^{-\sigma\sqrt{\Delta t}}$. Here, we have $r = q = 0$ and $\sigma = ln(8)$. Since, we have $N = 2$ steps in the binomial tree and tenor $T = 2/9$, we get the time increment $\Delta t = T/2 = 1/9$. Putting it all together, the up factor:

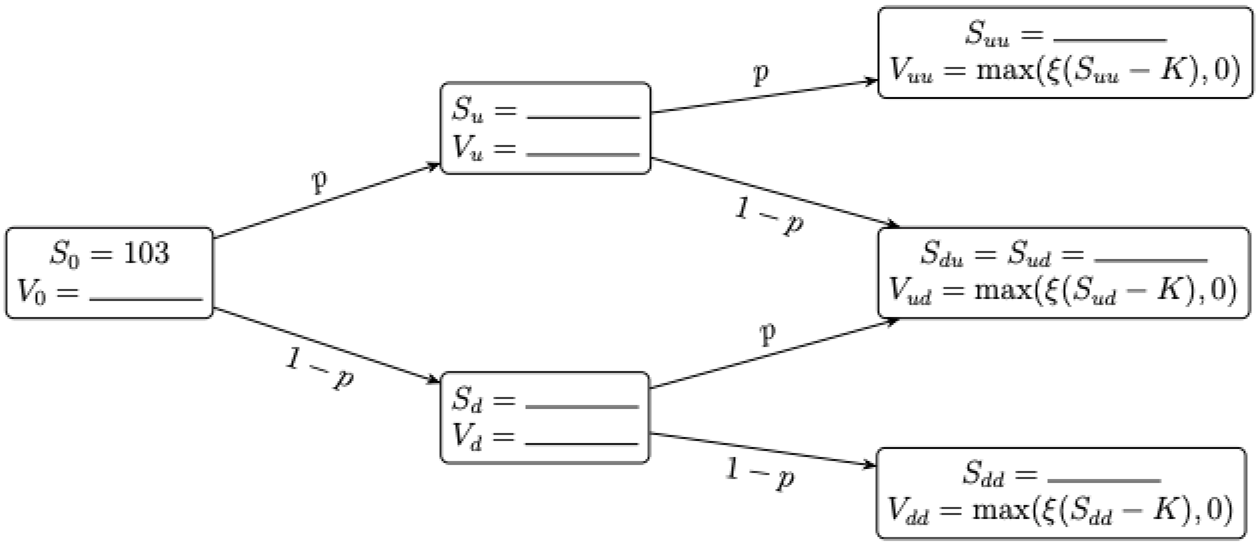
$$u = e^{ln(8)\sqrt{1/9}} = 8^{1/3} = 2$$

and the down factor:

$$d = 1/u = 1/2$$

Problem 5

Continue using the params from the previous question. Find the value of the put as computed by the tree.



Solution (30 points):

The probability of an up-move is given by $p = \frac{e^{r\Delta t} - d}{u-d}$. From the previous question, we know that $r = 0$, $\Delta t = 1/9$, $u = 2$ and $d = 1/2$. Therefore, the up probability is:

$$p = \frac{e^{r\Delta t} - d}{u - d} = \frac{1 - 1/2}{2 - 1/2} = \frac{1}{3}$$

and the down probability:

$$1 - p = 1 - \frac{1}{3} = \frac{2}{3}$$

We can then compute S_u, S_d, S_{uu}, S_{ud}, and S_{dd}, using the up and down factors as follows

$$S_u = S_0 * u = 103 \cdot 2 = 206$$

$$S_d = S_0 * d = 103 \cdot 1/2 = 103/2$$

$$S_{uu} = S_u * u = 206 \cdot 2 = 412$$

$$S_{ud} = S_u * d = 206 \cdot 1/2 = 103$$

$$S_{dd} = S_d * d = 103/2 \cdot 1/2 = 103/4$$

Since this is a put option, $\xi = -1$ and $K = 100$. The terminal values of the option at T are given by:

$$V_{uu} = max(\xi(S_{uu} - K), 0) = max(-(412 - 100), 0) = 0$$

$$V_{ud} = max(\xi(S_{ud} - K), 0) = max(-(103 - 100), 0) = 0$$

$$V_{dd} = max(\xi(S_{dd} - K), 0) = max(-(103/4 - 100), 0) = 297/4$$

and the value of the options at the prior step is given by:

$$V_u = p \cdot V_{uu} + (1-p) \cdot V_{ud} = 1/3 \cdot 0 + 2/3 \cdot 0 = 0$$

$$V_d = p \cdot V_{ud} + (1-p) \cdot V_{dd} = 1/3 \cdot 0 + 2/3 \cdot 297/4 = 99/2$$

Finally, the value at time $t = 0$ is given by:

$$V_0 = p \cdot V_u + (1-p) \cdot V_d = 1/3 \cdot 0 + 2/3 \cdot 99/2 = 33$$
