

STOCHASTIC CALCULUS: FINANCIAL PRINCIPLES

One-period Markets

Consider a one period market with K initial asset values $\vec{S}_0 = \{S_0^i\}$ and terminal asset values $\vec{S}_1 = \{S_1^i\}$

Given portfolio weights $\vec{\theta} = \{\theta_i\}$ the initial value of the portfolio is $V_0^\theta := \vec{\theta} \cdot \vec{S}_0$ and the terminal value is $V_1^\theta := \vec{\theta} \cdot \vec{S}_1 = \sum_{k=1}^K \theta_i S_1^k$

We enumerate all the N possible market scenarios as $\vec{\omega} = \{\omega_i\}$. In our case this consists of a set of terminal asset values for $\vec{S}_1 = \vec{S}_1(\omega)$.

Arbitrage

We say this market has *arbitrage* if there is some $\vec{\theta}^A$ such that $V_0^{\theta^A} \leq 0$ and $V_1^{\theta^A} > 0$ for *every* market scenario ω_i .

Replication

A *replicating portfolio* or *hedging portfolio* for some asset B is a choice of weights θ such that for *any* market scenario ω_ν ,

$$S_0^B = \sum_{k=1}^N \theta_k S_1^k(\omega_\nu)$$

We say a position in B is *hedged*.

Equilibrium Measure

An *equilibrium measure* is a some probability distribution $\vec{\pi}$ over all the possible market scenarios $\vec{\omega}$ such that, for *every* one of the scenarios,

$$S_0^i = e^{-rT} \sum_{i=1}^N \pi(\omega_i) S_1^i(\omega_i)$$

Colloquially, this is equivalent to every risky asset being hedgeable, all at once. We saw the 2-asset version in binomial trees.

Completeness

A *complete market* is a market that has a *unique* equilibrium measure.

Fundamental Theorem Of Arbitrage Pricing

The *Fundamental Theorem Of Asset Pricing* states that there exists an equilibrium measure if, and only if, no arbitrage exists.

Completeness Theorem

Every security S^i has a replicating portfolio if, and only if, the market is (a) complete and (b) has an equilibrium measure.

Consequences

If we can construct replicating portfolios, then we can look for an equilibrium measure and compute prices with it. The uniqueness of that measure ensures there will be only one way to do this.

How To Use The Consequences

Searching for an equilibrium measure sounds difficult.
Can we make it easier?

Then answer is yes, and the *Girsanov theorem* will tell us a convenient way.