Term Structures

Market Instruments And Tenors

- Recall that the essential problem in derivatives is not *pricing*, it is *calibration*
- The calibration phase includes many market instruments with price dependency on model parameters such as:
  - Risk-free short rates (or instantaneous *forward rates*) r_t
  - Credit spreads s_t, or
  - Volatilities σ_t
- They also have their own time horizons or tenors τ_i

Term Structure Interpolation

- Term structures have, in theory, independent values at every time t in the future

- In practice we cannot deal with this infinitude

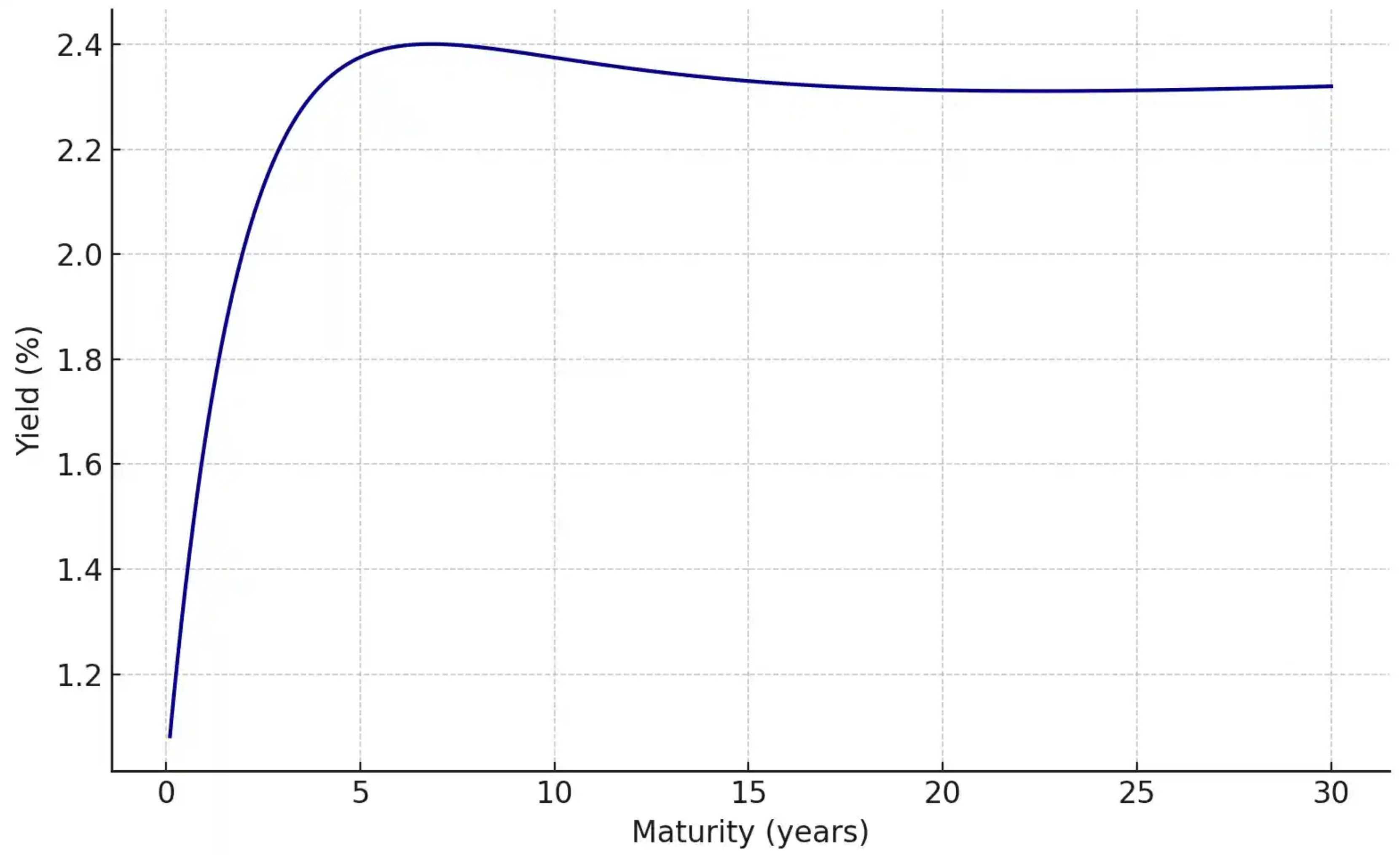
- Instead, we make assumptions about the shape. Common shapes include:

  - Nelson-Siegel-Svensson (interest rates)

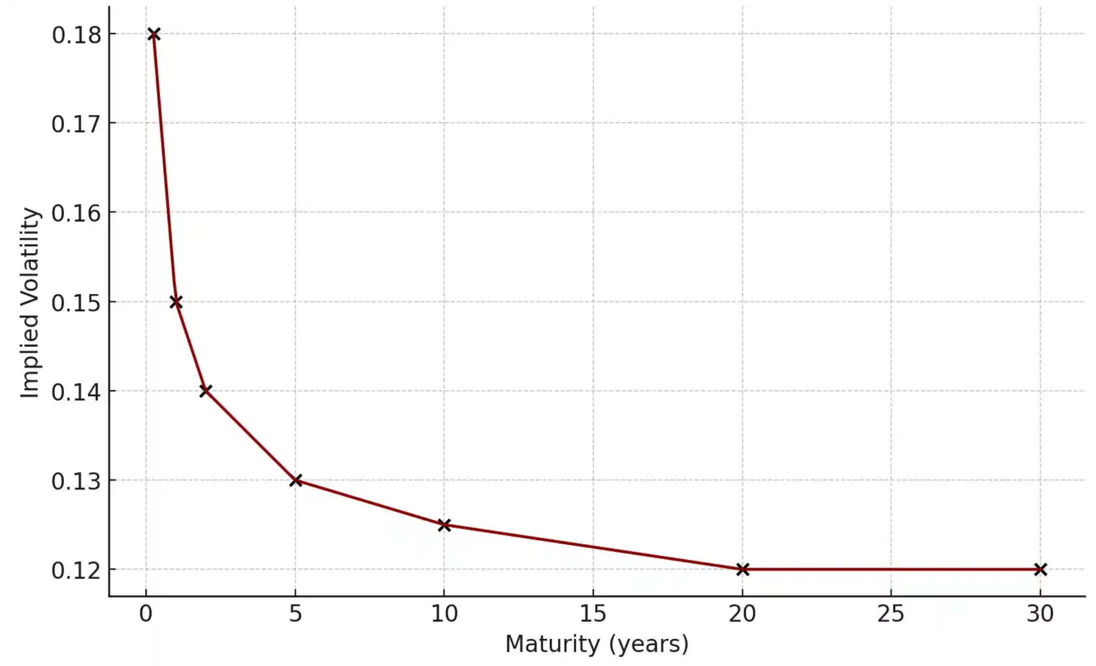
  - Piecewise constant

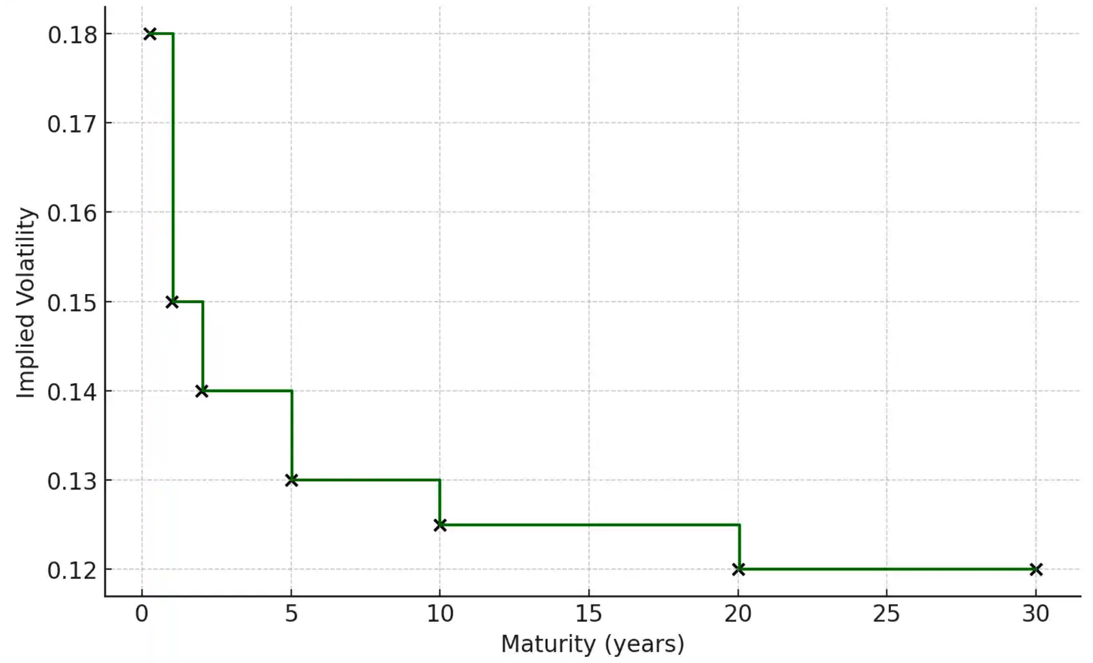
  - Piecewise linear

  - Bicubic splines (for locality)



$$y(t) = \beta_0 + \beta_1 \frac{1 - e^{-t/\tau_1}}{t/\tau_1} + \beta_2 \left(\frac{1 - e^{-t/\tau_1}}{t/\tau_1} - e^{-t/\tau_1} \right) + \beta_3 \left(\frac{1 - e^{-t/\tau_2}}{t/\tau_2} - e^{-t/\tau_2} \right)$$





Simple Functional Forms: Motivation

- Why do we prefer piecewise constant or piecewise linear term structures?
- Pricing (and calibration) have to deal with integrating the term structures
- This is reasonably easy with piecewise constant, and not too complex with piecewise linear

$$P(t) = \exp\left(-\int_0^t r(u)\,du\right) \qquad P(t) = \exp\left(-\sum_{i=1}^{k-1} r_i(T_i - T_{i-1}) - r_k(t - T_{k-1})\right)$$

Note on Credit Spreads

$$s = \frac{(1-R)\int_0^T h_t e^{-\int_0^t h_s ds} e^{-\int_0^t r_s ds}\, dt}{\int_0^T e^{-\int_0^t h_s ds} e^{-\int_0^t r_s ds}\, dt}$$

- In the modern era, bankruptcy risk is largely considered in the context of *par credit spreads*. These are (theoretically) periodic payments tuned to precisely counteract default risk
- Default comes with an entire legal process including a waterfall of obligors, from employees, accounts payable, senior and junior debt on down to contingent convertible (CoCo or "bail-in") bonds and equity
- Credit spreads are best represented in credit default swap (CDS) prices, where a recovery rate *R*, instantaneous forward rates r_t, and instantaneous hazard rates h_t are integrated to form a spread *s*
- In practice, CDS are priced at standard payment spreads, with upfront payments...

Note on Credit Spreads, continued

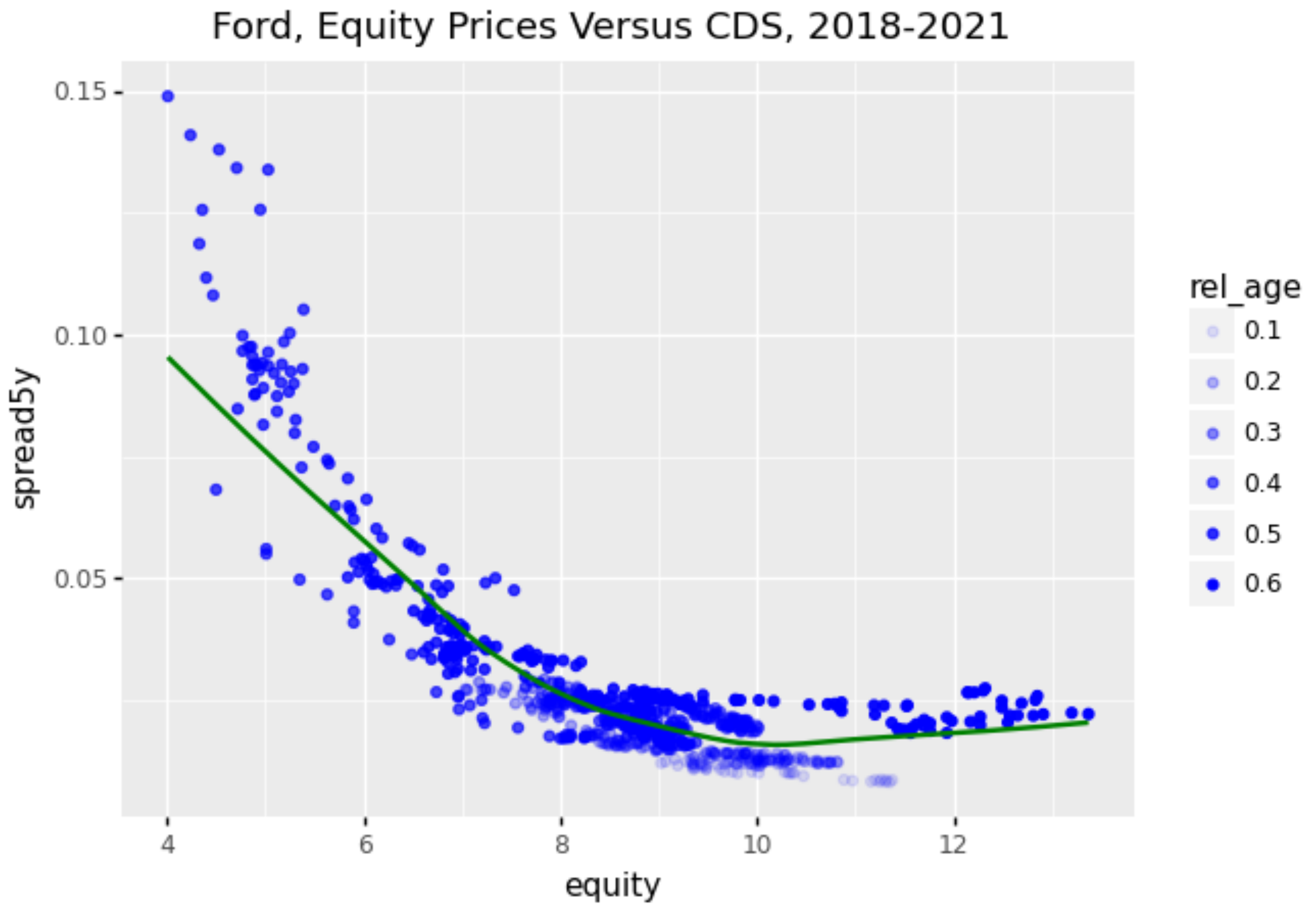
$$s = \frac{(1-R)\int_0^T h_t e^{-\int_0^t h_s ds} e^{-\int_0^t r_s ds}\, dt}{\int_0^T e^{-\int_0^t h_s ds} e^{-\int_0^t r_s ds}\, dt}$$

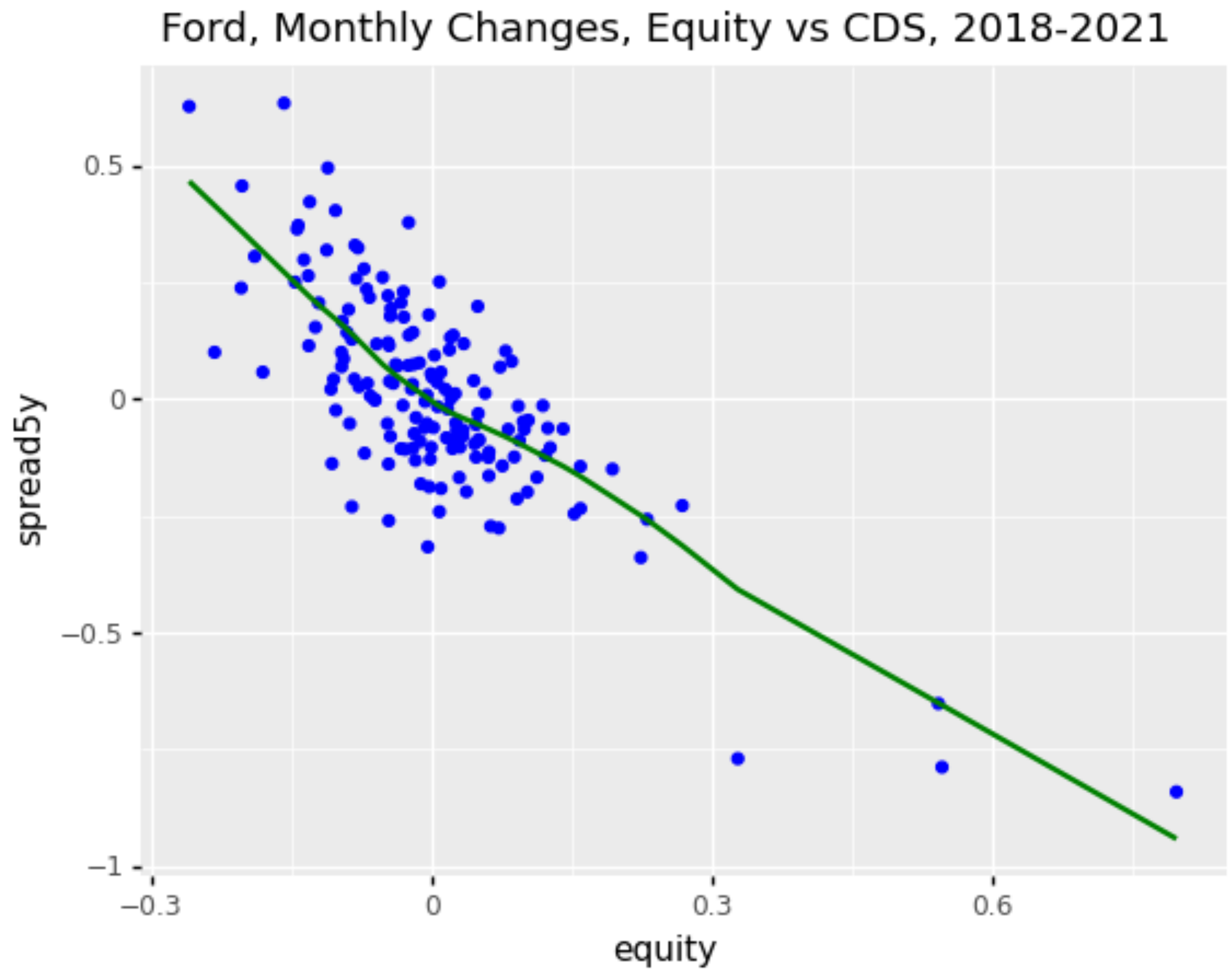
- A swap has two sides, with periodic payments over time increments δ_i
- A CDS has the *protection* leg, paying out if default happens, and the *premium* leg buying the insurance
- We discretize time into t_i, and choose *c* as one of two standard coupons, 1.00% or 5.00%
- Take $Q_i = Q(t_i)$ as the probability of survival until time t_i and $P_i = P(t_i)$ as the risk-free discount factor to that time
- Upfront value: $U = (s-c) \cdot \text{risk} = (1-R)\sum_{i=1}^{N}\left(Q_{i-1} - Q_i\right) P_i - c\sum_{i=1}^{N} \delta_i Q_i P_i$

Note on Credit Spreads, continued

- In addition to standardized coupons and upfront payments, CDS markets have instituted aggregates
- CDX indexes incorporate protection on a weighted sum of dozens to hundreds of individual names
    - The CDX indexes are generally far more liquid than individual name CDS
    - Nominally coverage is over many categories, in practice liquidity is further concentrated in IG and HY with March and September new series issues taking the lion's share of liquidity
- To first approximation, defaults are independent and index value is a weighted linear sum of individual protection values
- When defaults are cointegrated, this is more complex. We will discuss later in the context of collateralized debt obligations (CDO)

Relating Credit Spread And Equity



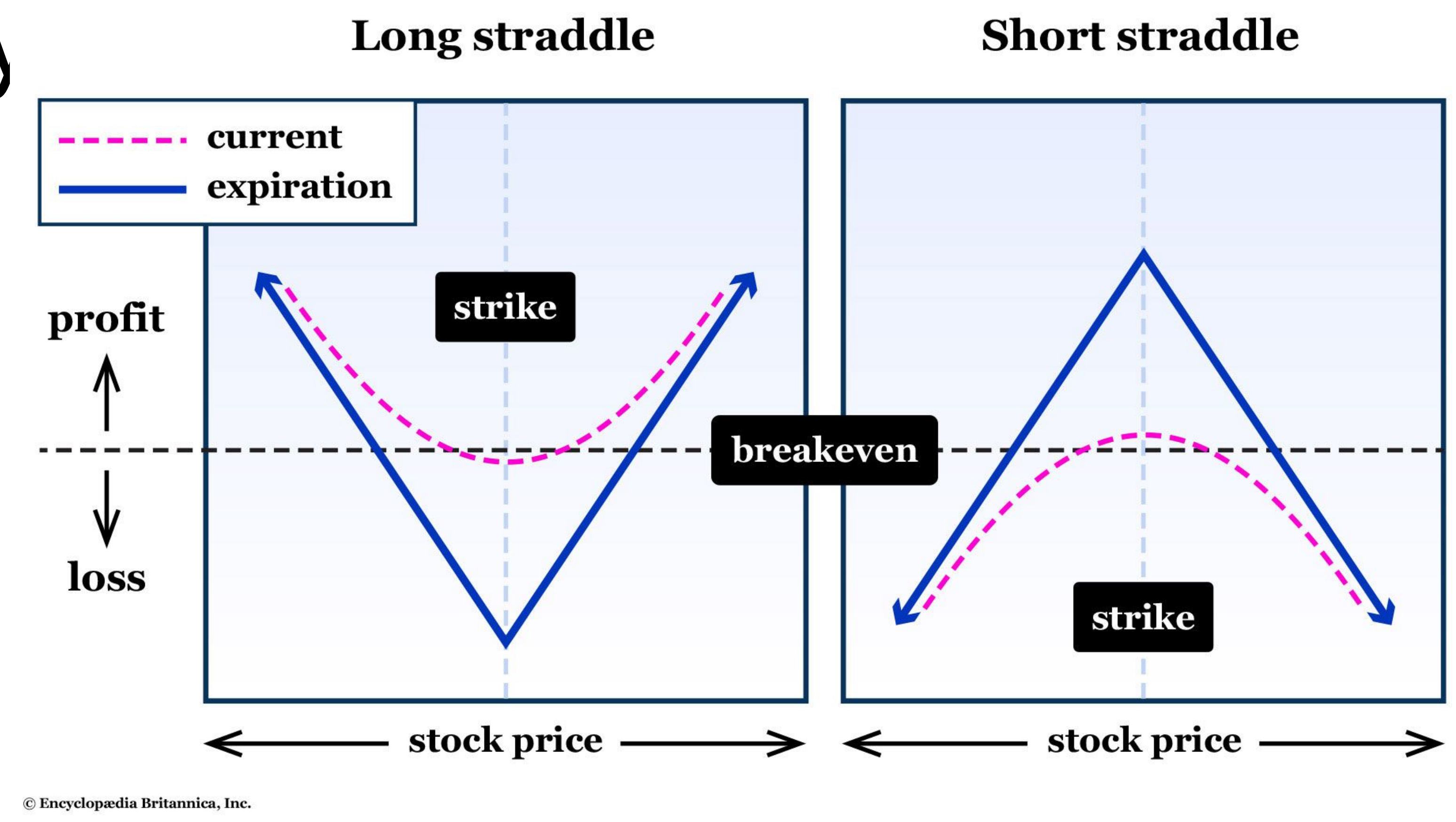


Term Structures And Arbitrage

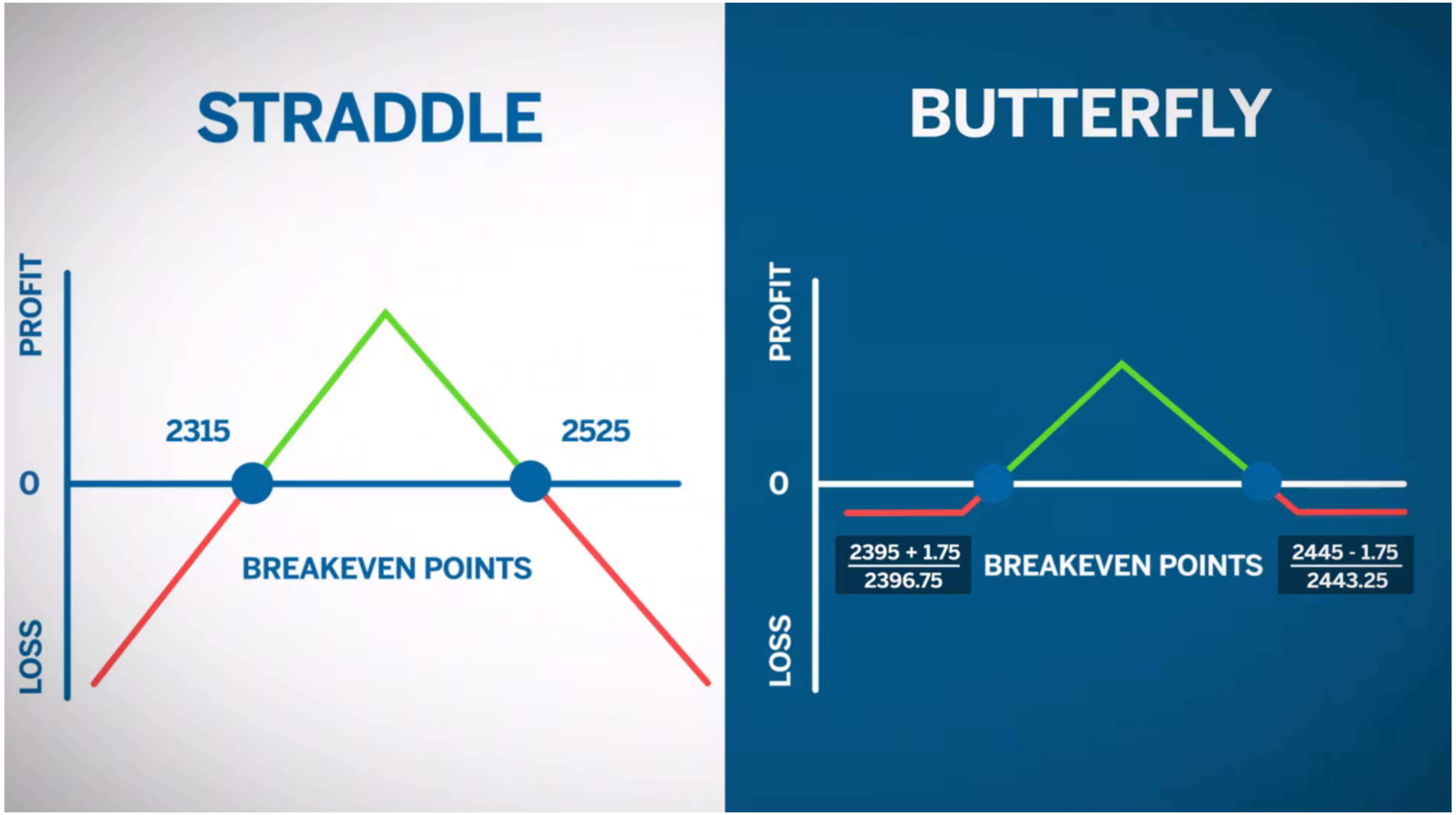
- We like to assume no arbitrage in our modeling
- Sometimes, market prices we use admit, or appear to admit, arbitrage
- Especially when we ignore bid-offer spreads or other friction
- Examples:
  - No negative interest rates: $\exp\left(-\int_{t=0}^{T} r_t dt\right) \leq \exp\left(-\int_{t=0}^{T+\Delta T} r_t dt\right)$
  - No negative volatilities: $\exp\left(-\int_{t=0}^{T} \sigma_t^2 dt\right) \leq \exp\left(-\int_{t=0}^{T+\Delta T} \sigma_t^2 dt\right)$
  - These may seem obvious, but actual prices may fail to agree!

Term Structures And Arbitrage (continued)

- Also, volatility skew: implied risk-free probability should be positive
    - Skew fits arising from stochastic models will naturally preclude arbitrage
    - Some more empirical fits work on the terminal distribution, calendar arbitrage may be possible
    - Other empirical fits risk arbitrage without constraints checks
- If our model allows arbitrage, in theory a counterparty can take free money from us. We might offer free straddles
- In practice, bid-offer spreads often cancel the possibility



Volatility Plays



CME Group

Term Structure Levels

- For a particular skew tenor, we characterize the level by at-the-money-forward level

- We can also judge the overall level of a term structure, out to some maturity, by taking its time-weighted average

$$r^T_{\text{Avg}} = \frac{1}{T}\int_{t=0}^{T} r_t dt$$

$$\sigma^T_{\text{Avg}} = \sqrt{\frac{1}{T}\int_{t=0}^{T} \sigma_t^2 dt}$$

Term Structure Tilts

- Characterizing tilts requires some arbitrary choices of points from which to infer the slope
- Interest rates: 2 years and 10 years
- Volatility term structures: 1 month and 6 months
- Volatility skew: 25 and 75 delta