Testing Option Pricing with the Edgeworth Expansion

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Abstract

There is a well developed framework, the Black-Scholes theory, for the pricing of contracts based on the future prices of certain assets, called options. This theory assumes that the probability distribution of the returns of the underlying asset is a gaussian distribution. However, it is observed in the market that this hypothesis is flawed, leading to the introduction of a fudge factor, the so-called volatility smile. Therefore, it would be interesting to explore extensions of the Black-Scholes theory to non-gaussian distributions. In this contribution we provide an explicit formula for the price of an option when the distributions of the returns of the underlying asset is parametrized by an Edgeworth expansion, which allows for the introduction of higher independent moments of the probability distribution, namely skewness and kurtosis. We test our formula with options in the brazilian and american markets, showing that the volatility smile can be reduced. We also check whether our approach leads to more efficient hedging strategies of these instruments.

1 Introduction

There are certain contracts called options which are negotiated in the stock market worldwide. An option is a contract that gives the buyer the right, but not the obligation, to buy or sell a given asset (a particular stock, an exchage

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rate or even an index) at a future time T for a fixed price (or exchange rate or index), called the *strike* price. If the option is a buy option, its value C_T at expiration time T is given by:

$$C_T = \mathrm{Max}(S_T - K, 0) \tag{1}$$

where we denote S_t the price of the underlying asset at time t and K is the strike price. The option price problem is to find a fair price $C_0 = C(S_0, t = 0)$ for the option today.

These contracts are in principle intended to provide protection to the buyer against moves in the market that can result in large losses. In a seminal paper, Black and Scholes [1] showed that, if there is no possibility to make a profit without taking a certain amount of risk (the non-arbitrage hypothesis), then the price of contracts called options is given by the solution of a second order differential, the Black-Scholes equation. One of the basic assumptions of the model is that the returns of the underlying asset follow a geometric brownian motion:

$$\frac{dS}{S} = \mu dt + \sigma dW \tag{2}$$

where μ is the drift rate of the asset, σ is called volatility and dW is a Wiener process with probability distribution of a normal distribution with zero average and variance $\sigma^2 = dt$, that is $N(0, dt)$. This stochastic differential equation can be solved resulting in:

$$S_T = S_0 e^{(\mu - \sigma^2/2)T + \sigma\sqrt{T}W} \tag{3}$$

with $W \sim N(0, 1)$ so that, on average we have:

$$\langle S_T \rangle = \frac{1}{\sqrt{2}\pi} \int\limits_{-\infty}^{\infty} dx\ S_0 e^{(\mu - \sigma^2/2)T + \sigma\sqrt{T}x}\ e^{-x^2/2} = S_0 e^{\mu T} \tag{4}$$

Given the process [2], the price of an option is given by the solution of the Black-Scholes equation:

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial^2 S} + rS\frac{\partial C}{\partial S} - rC = 0 \tag{5}$$

with “initial” condition given by [1] and where r is the risk-free interest rate. The trained eye will recognize [5] as the backward Kolmogorov equation for the conditional probability corresponding to the process [2] with μ substituted by r.

An alternative way to find the price of an option is to use the Feynman-Kac theorem to write the solution of the Black-Scholes equation as an expectation value of the final price at expiration brought at present value by the risk-free interest rate:

$$
\begin{aligned}
C_0 &= e^{-rT} \int_{-\infty}^{\infty} dx\ P(x,t) C_T \\
&= e^{-rT} \int_{-\infty}^{\infty} dx\ P(x,t) \mathrm{Max}(S_T - K, 0) \\
&= e^{-rT} \int_{-\infty}^{\infty} dx\ P(x,t) \mathrm{Max}\left(S_0 e^{(r-\sigma^2/2)T + \sigma\sqrt{T}x} - K, 0\right)
\end{aligned}
\tag{6}
$$

where the probability distribution in this case is symply a normal $N(0,1)$ distribution. Solving the Black-Scholes differential equation [5]or the expectation value [6] yields the famous Black-Scholes pricing formula:

$$
C_0 = S_0 N(d_1) - K e^{-rT} N(d_2) \tag{7}
$$

where $N(x)$ is the commulative probability:

$$
N(x) = \int_{-\infty}^{x} dz\ \frac{e^{-z^2/2}}{\sqrt{2\pi}} \tag{8}
$$

and

$$
d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}\ , \quad d_2 = d_1 - \sigma\sqrt{T} \tag{9}
$$

The basic assumption of the Black-Scholes framework is that the returns of the assets are normally distributed. However, this is not observed in the market. There is ample evidence that the real probability distributions of the returns present heavy tails and asymmetry. The market usually corrects for this by introducing a non-constant volatility, the so-called volatility smile. Figure 1 shows schematically the relation between the volatility smile and the probability distribution of the returns. If there is a higher probability of large moves in the tails, it is compensated by an increase in the volatility implying a larger price for the option. This volatility smile, however, bears no relation to the

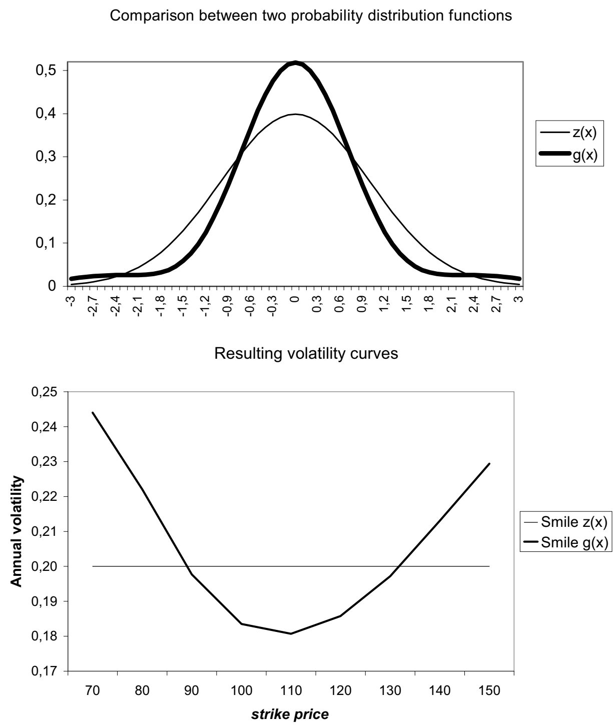

Fig. 1. The non-gaussianity of the returns probability distribution is taken into account by introducing the so-called volatility smile.

historical volatility obtained by a time series analysis and is simply a device to reproduce market data.

In this contribution, we study an extension of the Black-Scholes framework for a non-gaussian distribution. There is a large number of non-gaussian probability distributions that could be used. In particular, Lisa Borland recently obtained a solution for the option price in the context of the Tsallis distribution [2]. In a more general way, one could approximate the real probability distribution by an expansion around the normal distribution, as suggested by Jarrow and Rudd [3]. This is the so-called Edgeworth expansion. In section I we review the Edgeworth expansion, the distribution we chose to use to model the market returns. In section 2 we derive a closed form solution for the option price using this distribution in a risk-free probability measure. In section 3 we test our results with real market date and show that we can significantly reduce the volatility smile. We also perform a delta hedge test that shows no significant improvement over the traditional model. Section 5 concludes.

2 Edgeworth expansion

In this section we will briefly review the derivation of the Edgeworth expansion. More details can be found in [4]. Let's assume a series of $x_1, x_2, ..., x_n$ independent and identically distributed random variables with mean μ and finite variance σ^2. Defining the random variable

$$X_n = \frac{1}{n}\sum_{i=1}^{n} x_i \tag{10}$$

the central limit theorem states that in the $n \to \infty$, the random variable $S_n = \sqrt{n}\frac{(X_n-\mu)}{\sigma}$ approaches a normal distribution $N(0,1)$. However, we will be interested in the probability distribution before reaching this formal limit. This can be achieved by a cumulant expansion of the characteristic function of the distribution $\chi_n(t) = E\left(e^{itS_n}\right)$, which for the normal distribution results in $e^{-t^2/2}$. The characteristic function can be expanded as:

$$\chi_n(t) = \exp\left[-\frac{t^2}{2} + \frac{1}{n^{1/2}}\kappa_3(it)^3 + ... + \frac{1}{n^{(j-2)/2}}\frac{1}{j!}\kappa_j(it)^j + ...\right] \tag{11}$$

where we already used that $E(S_n) = \kappa_1 = 0$ and $\mathrm{Var}(S_n) = \kappa_2 = 1$. Performing a Taylor expansion in t of the characteristic function, collecting terms of the same order in n and performing the inverse Fourier transform in order to arrive at the probability distribution, we arrive at the Edgeworth expansion:

$$g(x) = \left(1 + \frac{\xi}{6}(x^3 - 3x) + \frac{\kappa-3}{24}(x^4 - 6x^2 + 3) + \frac{\xi^2}{72}(x^6 - 15x^4 + 45x^2 - 15)\right) z(x) \tag{12}$$

valid up to order $1/n$ and where we used the skewness $\xi = \kappa_3$ and kurtosis $\kappa = \kappa_4 + 3$, incorporating the factors of $1/n$ in these parameters. In the equation above $z(x)$ is the gaussian distribution $N(0,1)$.

The Edgeworth distribution is properly normalized due to properties of the Hermite polynomials but care must be taken in the parameters since it may not be positive definite or unimodal. In Figure 2 we show a graph of the allowed region.

3 Option Pricing with the Edgeworth Expansion

The distribution of returns of assets in the real markets are known to be non-gaussian, presenting heavy tails and asymmetry. We will model these returns

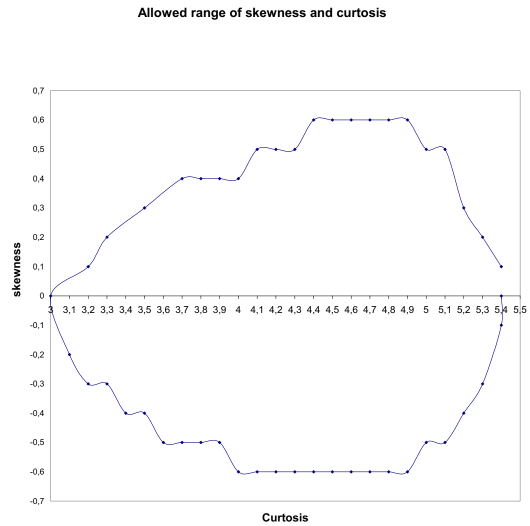

Fig. 2. Allowed range of the parameters for a positive definite and unimodal distribution.

with the Edgeworth distribution. The advantage of using the Edgeworth distribution is that it has a nice analytical form and we will see that it is possible to obtain a closed, albeit long, form for the fair price of an option on that asset. This facilitates the use of the results and the tests we will perform.

The first thing we have to do is to find the risk-free measure in order to apply equation (6). In the risk-free measure the average price of the asset should obey:

$$\langle S_T \rangle = \int_{-\infty}^{\infty} dx\; g(x)\; S_0 e^{(\mu-\sigma^2/2)T+\sigma\sqrt{T}x} = S_0 e^{rT} \tag{13}$$

This equation fixes the value of the parameter μ that must be used:

$$\mu T = rT - \log\left(1 + \frac{\kappa-3}{24}(\sigma\sqrt{T})^4 + \frac{\xi}{6}(\sigma\sqrt{T})^3 + \frac{\xi^2}{72}(\sigma\sqrt{T})^6\right) \tag{14}$$

Notice that one recovers the usual result $\mu = r$ when $\xi = 0$ and $\kappa = 3$, that is, when we have a gaussian distribution.

The option price is computed by evaluating:

$$C_0^{\text{Edge.}} = e^{-rT} \int_{-\infty}^{\infty} dx\; g(x) \text{Max}\left(S_0 e^{(\mu-\sigma^2/2)T+\sigma\sqrt{T}x} - K, 0\right) \tag{15}$$

This integral can be performed yielding a closed form solution for the price of the option in the case of the Edgeworth distribution:

$$
\begin{aligned}
C_0^{\text{Edge.}} = & C_0^{\text{BS}} + \qquad (16)\\
& \left(\frac{e^{\mu - rT - x_m^2/2 + \sigma\sqrt{T}x_m}}{72\sqrt{2\pi}} S\left((\sigma\sqrt{T})^5\xi^2 + (\sigma\sqrt{T})^4\xi^2 x_m + (\sigma\sqrt{T})^3(3(\kappa-3) + \xi^2(x_m^2-1) + \right.\right.\\
& (\sigma\sqrt{T})^2(12\xi - 3(\kappa-3)x_m + \xi^2 x_m(x_m^2-3)) + (\sigma\sqrt{T})(12\xi x_m + 3(\kappa-3)(x_m^2-1) + \\
& \left.\left.\xi^2(x_m^4 - 6x_m^2 + 3))\right)\right) + \\
& \left(\frac{e^{-rT - x_m^2/2}}{72\sqrt{2\pi}} \left(e^{\mu + \sigma\sqrt{T}x_m} S - K\right)(3(\kappa-3)x_m(x_m^2-3) + 12\xi(x_m^2-1) + \right.\\
& \left.\xi^2 x_m(x_m^4 - 10x_m^2 + 15))\right) + \\
& \left(\frac{e^{\mu - rT - \sigma^2 T/2}}{72} SN(d_1)((\sigma\sqrt{T})^4 3(\kappa-3) + (\sigma\sqrt{T})^6\xi^2 + 12(\sigma\sqrt{T})^3\xi))\right)
\end{aligned}
$$

where $x_m = \frac{\log(K/S_0) - (\mu - \sigma^2/2)T}{\sigma\sqrt{T}}$ is the minimum value that the random variable can have for a non-zero integrand in equation (15). Notice that the usual Black-Scholes result is recovered when $\xi = 0$ and $\kappa = 3$. In the next section we will use equation (17) in order to check if the volatility smile can be reduced.

4 Edgeworth smile

In this section we compare the Edgeworth model with the usual Black-Scholes type of model for the call options of the future dollar/real exchange rate negotiated at the brazilian BM&F and also call options of the S&P500 index negotiated at the Chicago Mercantile Exchange. The dollar/real option price are the daily opening prices in the market.

We want to compare the volatitily smiles obtained from the Black-Scholes and the Edgeworth model. The volatility smile of the Black-Scholes model is calculated in the standard way, by numerically finding the volatility in the model that corresponds to the market price for a given a strike price. For the Edgeworth model, we first have to estimate the skewness and kurtosis parameter. This is achieved by minimizing the total quadratic error with respect to the model parameters (σ, ξ, κ):

$$
\mathrm{M}in \sum \frac{(P_{Market} - P_{Edge})^2}{P_{Edge}} \qquad (17)
$$

We make sure that the parameters are within the region of validity of the

Edgeworth expansion. Once ξ and κ are fixed by this procedure, we again numerically evaluate the volatility for different strikes in a given day.

Some representative results are shown in figures 3. We can see that the smile is significantly reduced when using the Edgeworth distribution.

In addition, we have also studied the performance of the Edgeworth model compared to the traditional Black-Scholes models in the delta hedge of portfolio, that is, if there is a real competitive advantage in using an alternative model in terms of making money. We concluded that the Edgeworth model does not present a quantitative gain in the delta hedge test as compared to the Black-Scholes model.

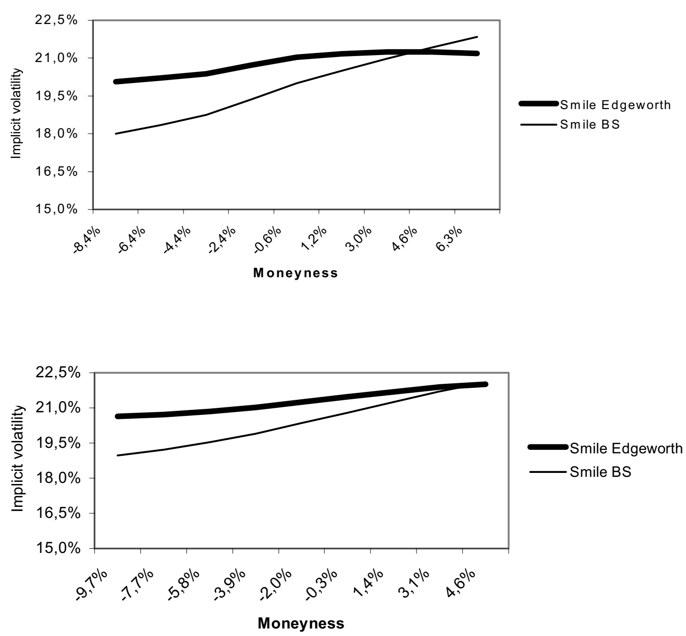

Fig. 3. Volatility smile for the Black-Scholes and Edgeworth distributions. These are smiles obtained from future Real/Dollar exchange rates with two different expiration dates.

5 Conclusion

We have studied a non-gaussian extension of the traditional Black-Scholes model, using the Edgeworth expansion to parametrize the probability distributions of the asset returns. We derived a closed form expression for the price of an european option on this asset. We verified with market data that this extension reduces significantly the volatility smile, implying that the model parameters are more robust than in the traditional approach. We also performed a delta hedge test of a given portfolio, where the extension did not show a quantitative improvement over the Black-Scholes model. This reduces somewhat the interest of large corporations in a practical use of this extension at this point.

It would be very important to find a microscopic process that would lead to the Edgeworth distribution. This would provide a solid basis for performing Monte Carlo simulations for option pricing in this framework.

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