STOCHASTIC CALCULUS: ITÔ

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Itô Processes

An *Itô process* is a stochastic process that satisfies a stochastic differential equation of the form

$$dZ_t = a_t \, dt + b_t \, dW_t$$

Itô Processes (continued)

Here W_t is a standard Wiener process (Brownian motion), and a_t, b_t are *adapted* process, meaning their values depend only on past information, especially of W_t.

Itô Processes: Quadratic Variation

The *local quadratic variation* of the Itô process Z_t is defined by

$$d[Z, Z]_t = b_t^2 \, dt$$

As you see this basically describes the square of what we call the *volatility*.

In the particular case of geometric brownian motion

$$dZ = \mu Z dt + \sigma Z dW$$

this becomes

$$d[Z, Z]_t = \sigma^2 Z^2 \, dt$$

Itô's Formula

If Z_t is an Itô process, and if $u(x,t)$ is a smooth function of two variables, then $u(Z_t,t)$ has the SDE

$$du(Z_t,t) = \frac{\partial u}{\partial t}\,dt + \frac{\partial u}{\partial x}\,dZ_t + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\,d[Z,Z]_t$$

which we usually write

$$du(Z_t,t) = \frac{\partial u}{\partial t}\,dt + \frac{\partial u}{\partial x}\,dZ_t + \frac{1}{2}\sigma^2\frac{\partial^2 u}{\partial x^2}Z^2\,dt$$

A proof of this is in many texts, and can be done with Taylor expansions holding terms of size $\sqrt{\Delta t}$ and Δt, accounting for associated nonlinearities, and so on. It looks fairly similar to the class arguments we applied to trees.

1. A Useful Observation

Consider $u(x,t) = \exp(\sigma x)$. Applying Itô to dW_t and noting $\frac{\partial^2 u}{\partial x^2} = \sigma^2 \exp(\sigma x)$ we get

$$dZ_t = \frac{1}{2}\sigma^2 Z_t\, dt + \sigma Z_t\, dW_t$$

Note that the latter term has zero expectation, because W_t has zero expectation and so

$$d\mathbb{E}[Z_T] = \frac{1}{2}\sigma^2 T$$

Direct SDE Solutions: Setup

In special cases the SDE can be solved directly. For example, consider geometric Brownian motion with constant μ, σ and r

$$dZ_t = \mu Z_t \, dt + \sigma Z_t \, dW_t$$

The multiplications on the right side by Z itself mean this is *not* the SDE for Brownian motion.

We can take $u(x, t) = \log x$ in which case the derivatives $\frac{\partial u}{\partial x} = 1/x, \frac{\partial^2 u}{\partial x^2} = -1/x^2$ cancelling out the Z_t terms on the right hand side. So we find that

$$d \log(Z_t) = \mu \, dt + \sigma \, dW_t - (\sigma^2/2) \, dt$$

Direct SDE Solutions: Solution

We had

$$d\log(Z_t) = \mu\,dt + \sigma\,dW_t - (\sigma^2/2)\,dt$$

This is a *nonstandard* Brownian motion (since it has drift), but a Brownian motion has a known solution, and terminal distribution. We have

$$Z_t = Z_0 \exp\Big((\mu - \sigma^2/2)t + \sigma W_t\Big)$$

for a standard Brownian motion W.

So, our guess worked.

Using The Solution

When we have a known terminal distribution, which here is

$$W_T \sim \Phi(0, \sigma\sqrt{T})$$

then we can use it in Monte Carlo simulations and expectation integrals.

Of course, only with arbitrage arguments do we establish drift terms useful in mathematical finance.

Non-constant Coefficients

Consider time-dependent coefficients, as in :

$$dZ_t = \mu_t Z_t \, dt + \sigma Z_t \, dW_t$$

Our Itô formula looks similar

$$d \log(Z_t) = \mu_t \, dt + \sigma \, dW_t - (\sigma^2/2) \, dt$$

but because terminal values at any t come from integrating over everything that happened up *until* time t we have an averaging effect

Non-constant Coefficients: Integral In Solution

With our time-dependent coefficients, the solutiopn becomes :

$$Z_t = Z_0 \exp\left((\bar{\mu}_t - \sigma^2/2)t + \sigma W_t\right)$$

where

$$\bar{\mu}_t = \frac{1}{t}\int_0^t \mu_s \, ds$$