ECMA 31140. PERSPECTIVES ON COMPUTATIONAL MODELING FOR ECONOMICS
PROBLEM SET 6
DUE DATE: FEBRUARY 18TH

Instructor: Sergio Salas `ssalas@uchicago.edu`

You are encouraged to work and discuss in groups, but you must submit your work individually. Answers must be legibly hand-written or typed. All assignments are due electronically on Canvas, attach code. Assignments are due at 10:00 PM. Late problem sets *will not be accepted*

1. Cake eating with Dynamic Programming and VFI. (60)

Consider again the "cake eating problem" stated as a Dynamic Programming problem

$$v(w) = \max_{0 \le c \le w} [u(c) + \beta v(w')]$$

Subject to:

$$w' = (w - c)(1 + r)$$

Assume that:

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma}$$

a) Use an educated guess for the value function, and find such function using the guess-and-verify method. (30)

Answer.

Given the particular utility function, let us guess that:

$$v(w) = B\frac{w^{1-\sigma}}{1-\sigma}$$

where B is a coefficient to be determined. The Bellman equation:

$$v(w) = \max_{c} [u(c) + \beta v(w')]$$

subject to $w' = (w - c)(1 + r)$, can be written then as:

$$v(w) = \max_{c} \left[\frac{c^{1-\sigma}}{1-\sigma} + \beta B\frac{[(w-c)(1+r)]^{1-\sigma}}{1-\sigma}\right]$$

Taking the FOC, we got:

$$c^{-\sigma} = \beta B(1 + r)^{1-\sigma}(w - c)^{-\sigma}$$

Date: February 18, 2025.

From this equation and from the budget constraint we can obtain:

$$c = \zeta w, \;\; w' = (1-\zeta)(1+r)w \tag{1}$$

where:

$$\zeta = \frac{(\beta B)^{-\frac{1}{\sigma}}(1+r)^{-\frac{1-\sigma}{\sigma}}}{1+(\beta B)^{-\frac{1}{\sigma}}(1+r)^{-\frac{1-\sigma}{\sigma}}}$$

With these policy functions, the Bellman equation is:

$$B\frac{w^{1-\sigma}}{1-\sigma} = \frac{(\zeta w)^{1-\sigma}}{1-\sigma} + \beta B\frac{[(1-\zeta)(1+r)w]^{1-\sigma}}{1-\sigma} \tag{2}$$

Therefore, if the guess is correct, it must be the case that:

$$B = \zeta^{1-\sigma} + \beta B(1+r)^{1-\sigma}(1-\zeta)^{1-\sigma}$$

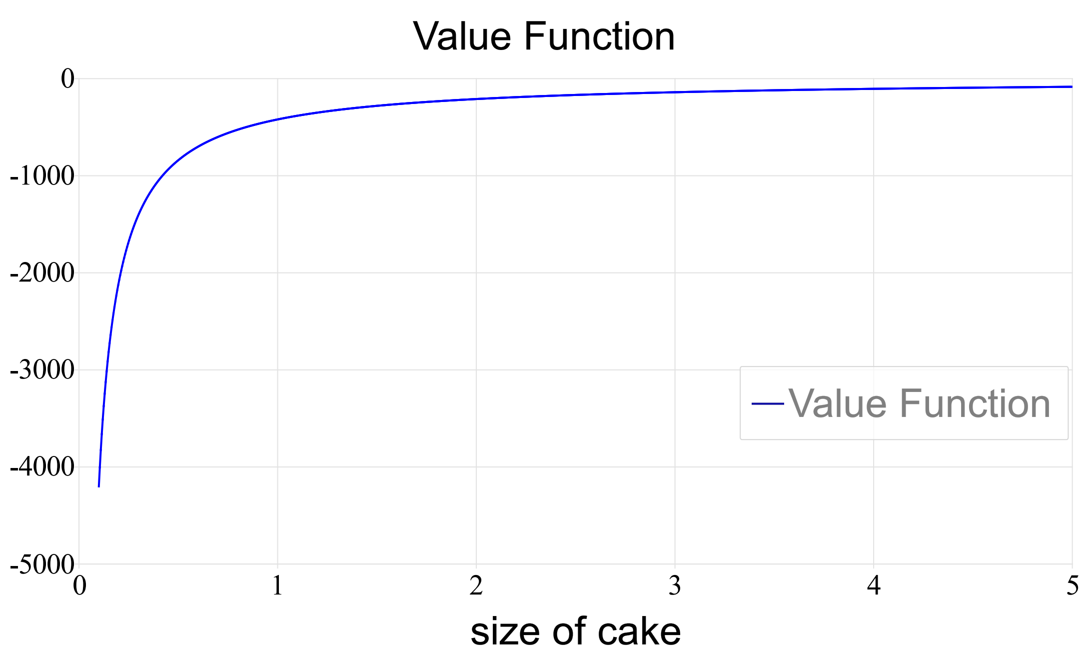
After some tedious algebra, the value of B is revealed from this equation as:

$$B = \left[1-(1+r)^{\frac{1-\sigma}{\sigma}}\beta^{\frac{1}{\sigma}}\right]^{-\sigma} \tag{3}$$

Note that with this value found, we can also find the actual policy functions using (1).

b) Assume that $\beta = 0.95, \sigma = 2$ and $r = 0.2$. Use a uniform grid of $N = 1000$ points over an interval $[0.1, 5]$ to find $v(w)$ *numerically* using Value Function Iteration (VFI). (30)

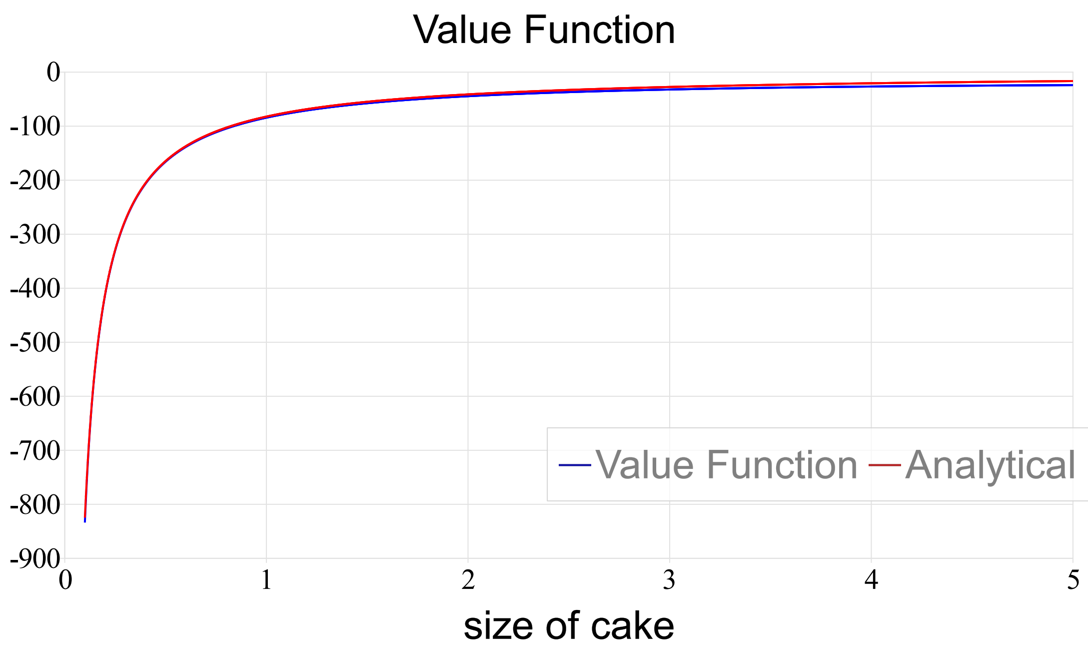
Answer.
We use VFI to get the following graphic.



c) Plot both the analytical value function you have found in a) and the numerical approximation you found in b). How good is the approximation? (20)

Answer.

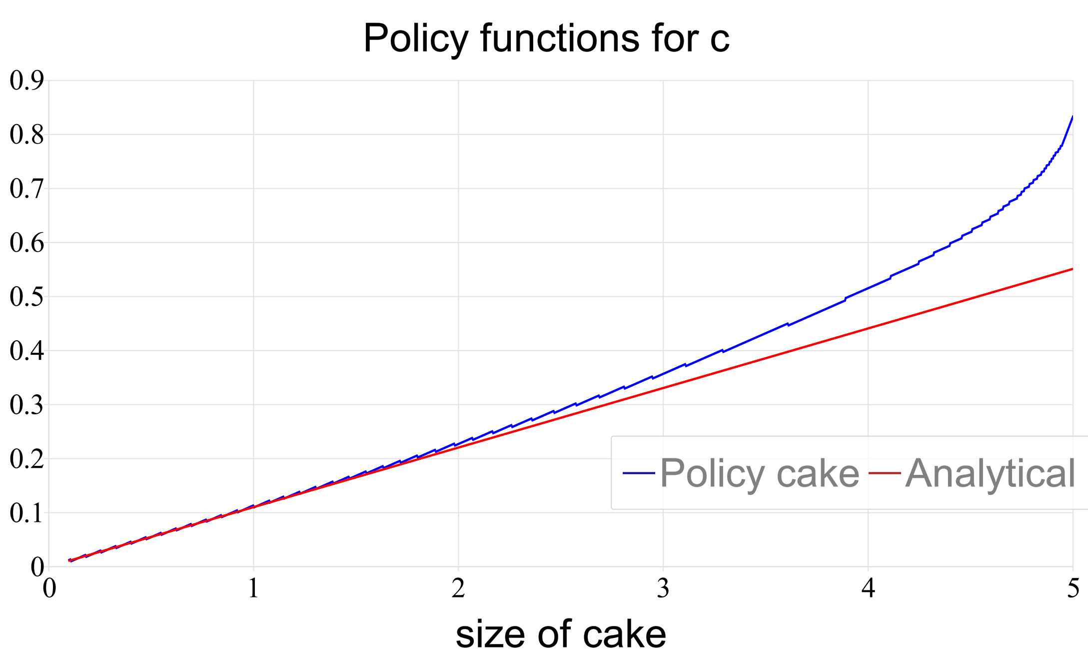
The following graphic show the value functions. The approximation appear good but this is misleading. (see rest of the incises)

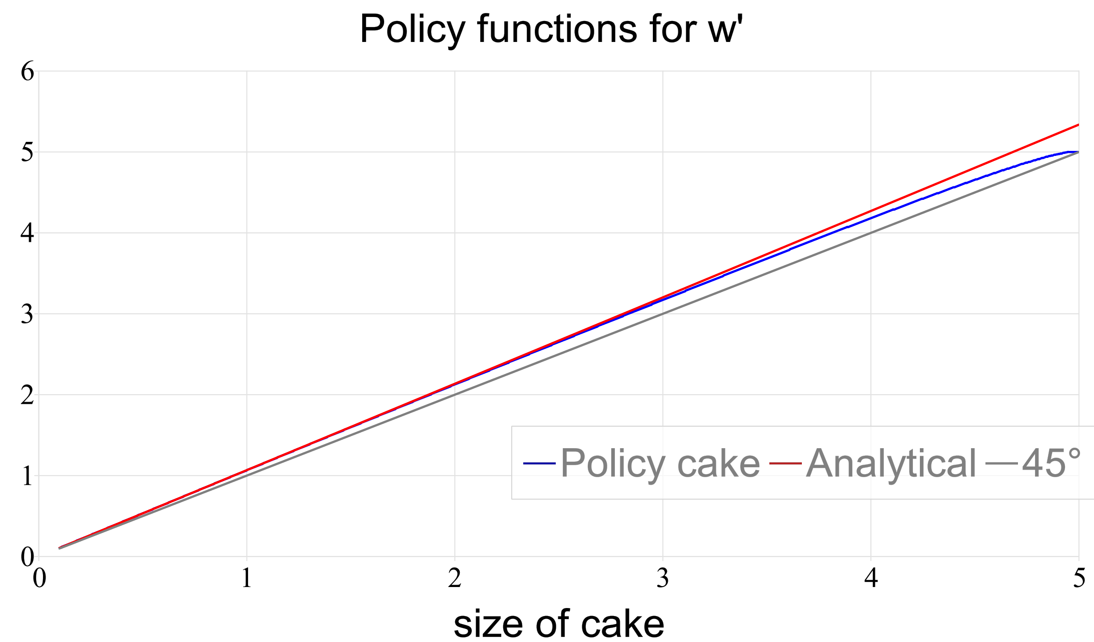


d) Plot also the analytical policy functions for w' and c and the numerical approximation. (20)

Answer.

The following figures show the policy functions, the numerical approximation and the analytical policy functions.





So we see how bad the approximation actually is.

e) Most likely you have found a weird result in c) and d). Use now the orthogonal collocation method to approximate the policy function for w'. Solve and show the optimization problem for the agent, deriving the Euler equation you must use to find the residual function for the collocation method. Plot the policy function for w'. Do you find a better approximation now? What do you think is the reason? (30)

Answer.

Using the Bellman equation, we can obtain the FOC as:

$$u'(c) = \beta(1+r)v'(w')$$

And we can use the Envelope Condition to obtain $v'(w) = u'(c)$, The Euler equation is then:

$$u'(c) = \beta(1+r)u'(c')$$

Defining the policy function for next period cake as $g(w)$, the Euler equation can be written as:

$$u'(w - g(w)/(1+r)) = \beta(1+r)u'(g(w) - g(g(w))/(1+r))$$

We can us the approximation:

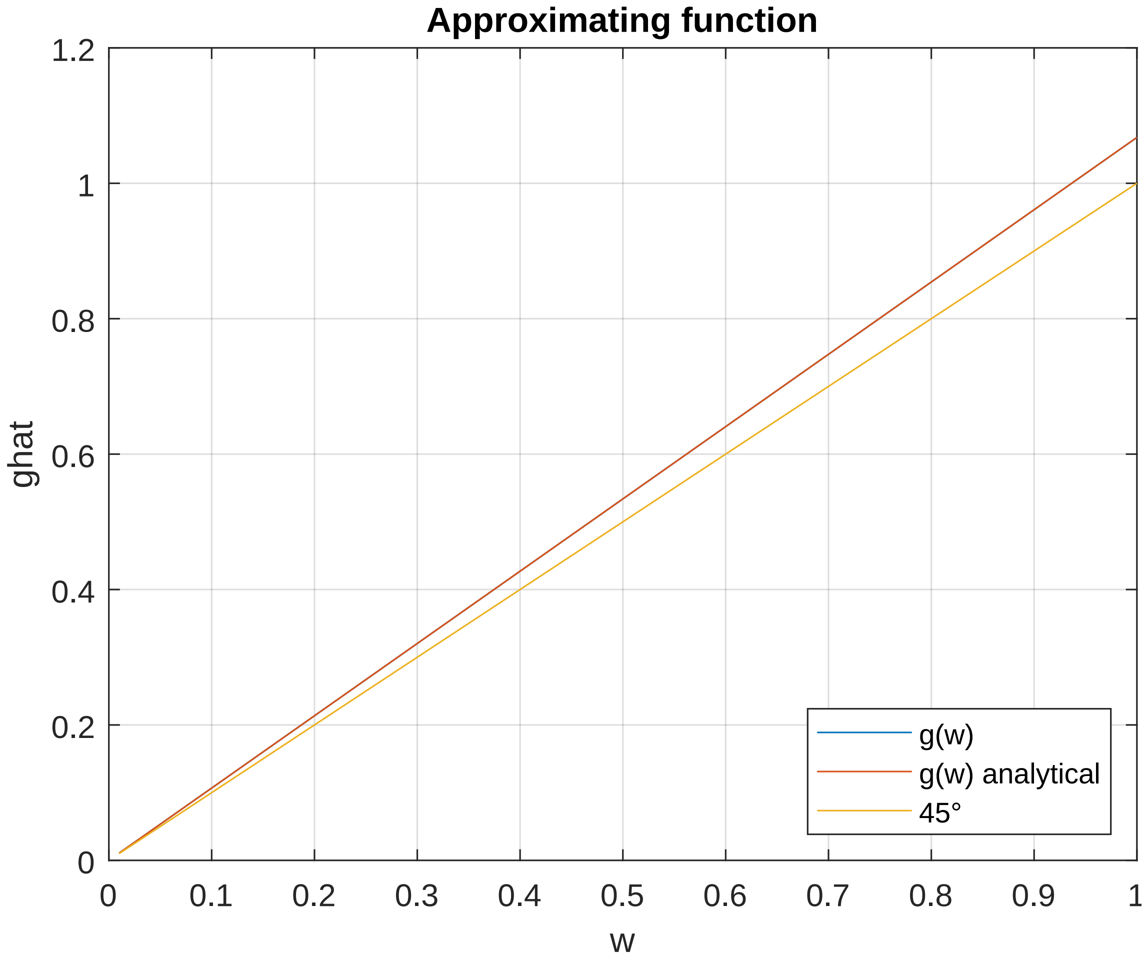
$$\hat{g}(w, \mathbf{a}) = \sum_{i=1}^{n} a_i \phi_i(w)$$

where:

$$\phi_i(w) = wT_{i-1}(w) \doteq w\cos((i-1)\arccos(w))$$

Another basis could be safely chosen, like ordinary polynomials because we know that this problem has a linear policy function. In general, we may want to use Chebychev polynomials. Under orthogonal collocation, using any basis we would use the zeros of the nth term of the Chebychev polynomial, as we have done before.

The following figure shows the resulting approximation:



As we can see the approximation is very good. Note that projection methods do not impose any restriction where the policy function should "live". In the VFI we are restricted by the grid, therefore, in this case VFI is truncating the true range of the policy function, which delivers then a bad approximation.

2. Value Function Iteration (VFI) for the stochastic NGM and simulations. (60 points)

Consider the optimization problem for a central planner, aiming to maximize:

$$\max \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t),\ \ 0 < \beta < 1$$

subject to:

$$\begin{aligned} c_t + i_t &= z_t F(k_t, n_t) \\ k_{t+1} &= i_t + (1-\delta)k_t \\ k_0 &> 0 \text{ given} \end{aligned}$$

z_t follows an AR(1) process:

(4) $$z_{t+1} = z_t^{\rho} e^{\epsilon_t}, \epsilon_t \sim \mathcal{N}(0, \sigma^2)$$

The utility function and the production function are given by:

$$u(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma},\ \ F(k_t, 1) = Ak_t^{\alpha} \tag{5}$$

For this exercise use the following values for the parameters:

$$\alpha = 0.36,\ \ \beta = 0.98,\ \ \delta = 0.03,\ \ \gamma = 2,\ \ \sigma = 0.1,\ \ \rho = 0.7$$

a) Construct a finite 5-state approximation for the Markov transition of the process $\ln z_t$, using Tauchen's method. Use 3 standard deviations to approximate the lower and upper end of the grid ($m = 3$). Find the ergodic distribution for this process. (10)

Answer.
Take the log for the $AR(1)$ process:

$$\ln z_{t+1} = \rho \ln z_t + \varepsilon_t, \varepsilon \sim \mathcal{N}(0, \sigma^2)$$

Taking unconditional expectations and variances we have:

$$\begin{aligned}
&\mathbb{E} \ln z_t = 0 \\
&\mathbb{V} \ln z_t = \frac{\sigma^2}{1-\rho^2} = \frac{0.01}{0.51}
\end{aligned}$$

With $\sigma_z = \sqrt{\mathbb{V} \ln z_t}$, we actually have that the support of the discretized distribution should be:

$$[-3\sigma_z, -1.5\sigma_z, 0, 1.5\sigma_z, 3\sigma_z]$$

Then, with Tauchen's method and the code provided, we have the stationary distribution as below:

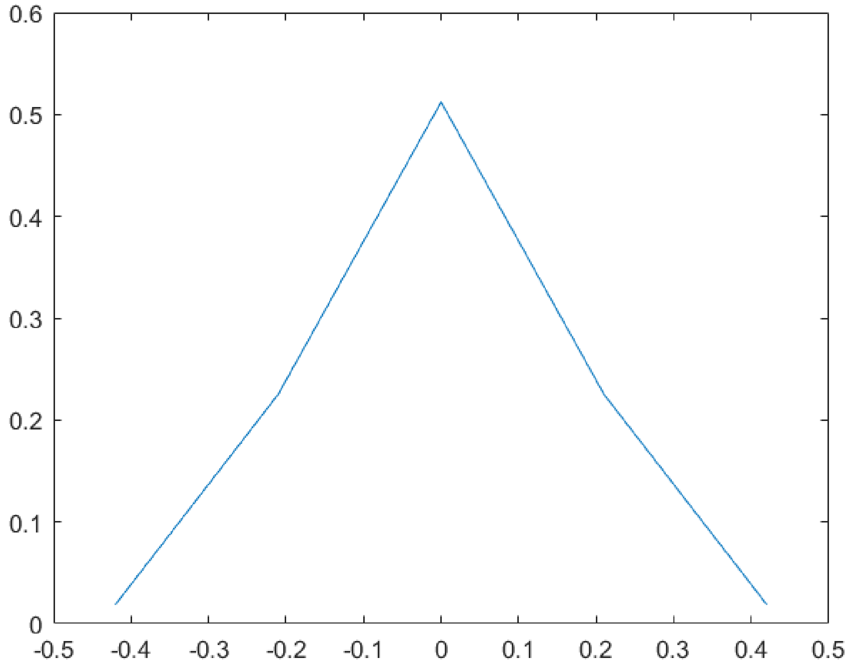
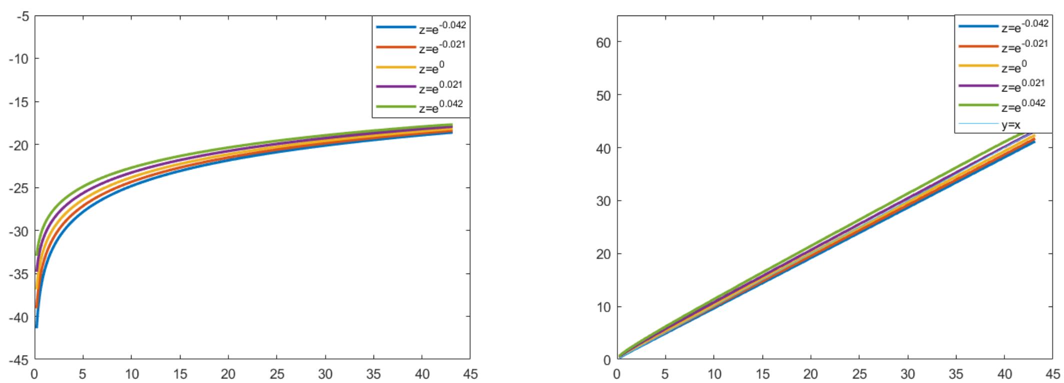


FIGURE 1. Stationary Distribution of the Process

b) With the discretized process for z_t found in question a), solve the model using VFI. Construct the grid with 1000 points with the lower bound for the capital stock of $0.01k$ and the upper bound $4k$, where k is the non-stochastic steady state of the capital stock. Find the value functions and the policy functions. (20)

Answer.
After running the value function iteration over the (exogenous) grid, we have the result of value function iteration and policy function:

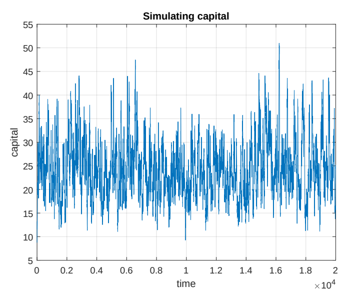


(a) Value Function (b) Policy Function

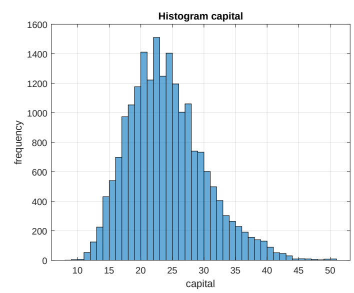
FIGURE 2. Value Function and Policy Function

c) With the policy functions for capital obtained in incise b), simulate the stock of capital. To do this depart from an arbitrary capital stock in the grid and simulate the draws of the Markov chain for the shock. You will need to use a random number generator. Simulate 10000 observations for the capital stock show the simulation and compute the histogram. This should be close to the ergodic distribution of capital! (30)

Answer.
Simulating the process I obtain:



(a) Simulation: over time



(b) Simulation: histogram

The mean of the simulation result is computed as 24.067. We can see that the distribution of capital seems not to be normal.

3. The Neoclassical Growth Model with variable labor supply. (70 points)
A Central planner uses a Bellman equation to state the problem as:

$$\mathcal{V}(k) = \max_{c,k',\ell} \left[u(c,\ell) + \beta \mathcal{V}(k')\right]$$

subject to:

$$k' + c = F(k,\ell) + (1-\delta)k, \;\; \ell \in [0,1]$$

where ℓ is the amount of labor.

Throughout the problem use the following functional forms and parameters:

$$u(c,\ell) = \frac{c^{1-\sigma}}{1-\sigma} - \frac{\chi\varphi}{1+\varphi}\ell^{1+\frac{1}{\varphi}},\ F(k,\ell) = k^{\alpha}\ell^{1-\alpha},\ \alpha = 0.36,\ \delta = 0.03, \sigma = 2, \beta = 0.95, \varphi = 2$$

a) Calibrate the value for χ to a target value for $\bar{\ell}$ (steady state value) equal to 1/3. (10)

Answer.
The Bellman equation can be written as:

$$V(k) = \max_{c,\ell}[\frac{c^{1-\sigma}}{1-\sigma} - \frac{\chi\varphi}{1+\varphi}\ell^{1+\frac{1}{\varphi}} + \beta V(k^{\alpha}\ell^{1-\alpha} + (1-\delta)k - c)]$$

Take the derivative on both sides with respect to c, ℓ, we should obtain:

$$c^{-\sigma} - \beta V'(k') = 0$$
$$-\chi\ell^{\frac{1}{\varphi}} + \beta(1-\alpha)k^{\alpha}\ell^{-\alpha}V'(k') = 0$$

The Envelope condition is now:

$$V'(k) = \beta[\alpha k^{\alpha-1}\ell^{1-\alpha} + (1-\delta)]V'(k')$$

We can use these relationships to find the Euler equation and the intratemporal condition for labor supply as we have done before.

Now, at steady state, we expect $k = k' = \bar{k}$, therefore the Envelope condition states:

$$1 = \beta(\alpha\bar{k}^{\alpha-1}\bar{\ell}^{1-\alpha} + 1 - \delta)$$

Then, we have:

$$\frac{\bar{k}}{\bar{\ell}} = (\frac{\frac{1}{\beta}+\delta-1}{\alpha})^{\frac{1}{\alpha-1}}$$

From the intratemporal condition we also have:

$$(1-\alpha)\bar{k}^{\alpha}\bar{\ell}^{-\alpha}\bar{c}^{-\sigma} = \chi\bar{\ell}^{\frac{1}{\varphi}}$$

Therefore, we have:

$$\chi = \frac{(1-\alpha)(\frac{\frac{1}{\beta}+\delta-1}{\alpha})^{\frac{\alpha}{1-\alpha}}\bar{c}^{-\sigma}}{\bar{\ell}^{\frac{1}{\varphi}}}$$

while $\bar{c}$ is:

$$\bar{c} = \bar{k}^{\alpha}\bar{\ell}^{1-\alpha} - \delta\bar{k} = \bar{\ell}(\frac{\frac{1}{\beta}+\delta-1}{\alpha})^{\frac{\alpha}{\alpha-1}} - \delta\bar{\ell}(\frac{\frac{1}{\beta}+\delta-1}{\alpha})^{\frac{1}{\alpha-1}}$$

Plug in the values of the parameters, we have:

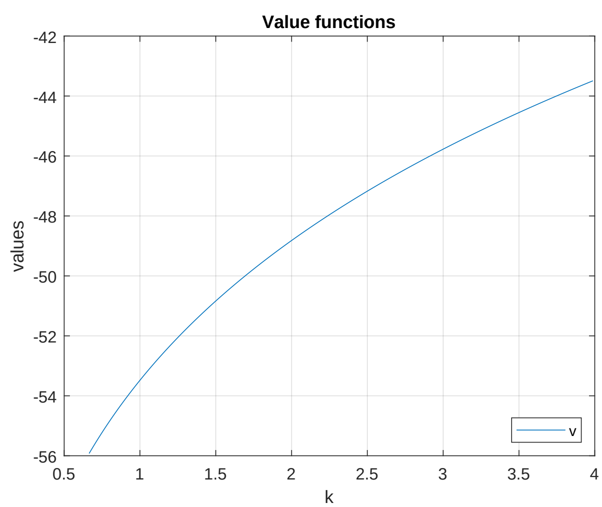
$$\bar{k} = 3.3232 \quad \bar{c} = 0.6631 \quad \bar{\ell} = \frac{1}{3} \quad \chi = 5.7692$$

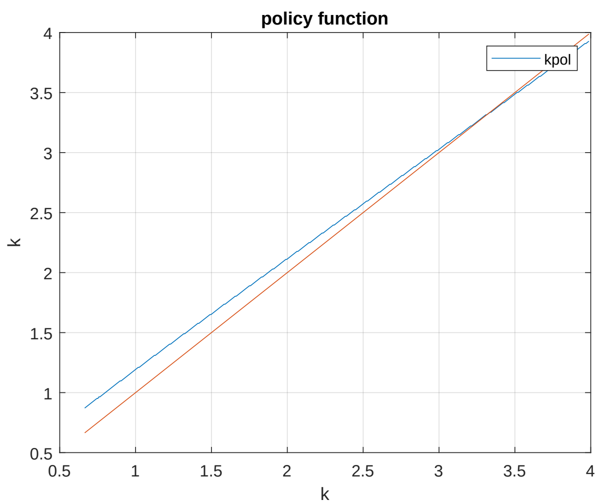
b) Compute the policy functions for capital, consumption and labor supply using VFI. Present graphics for the three functions. For this consider a grid of 500 points for the capital stock staring at $0.2k$ and ending at $1.2k$, where k is the steady state value of capital.[1] (30)

[1]You need to discretize time between $[0,1]$, use also 500 points.

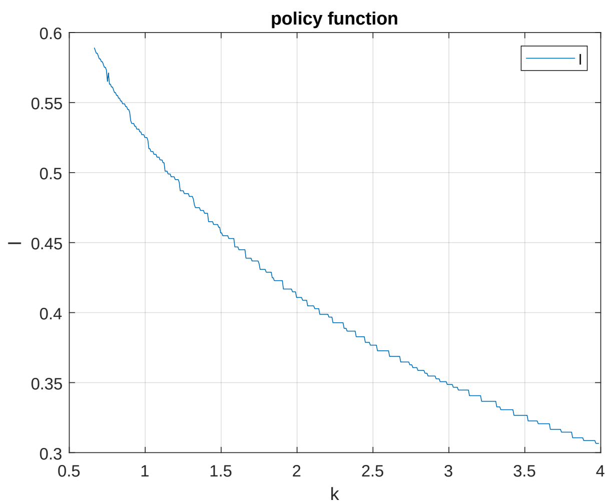
Answer.

Using VFI, we obtain the following value function and the corresponding policy functions:





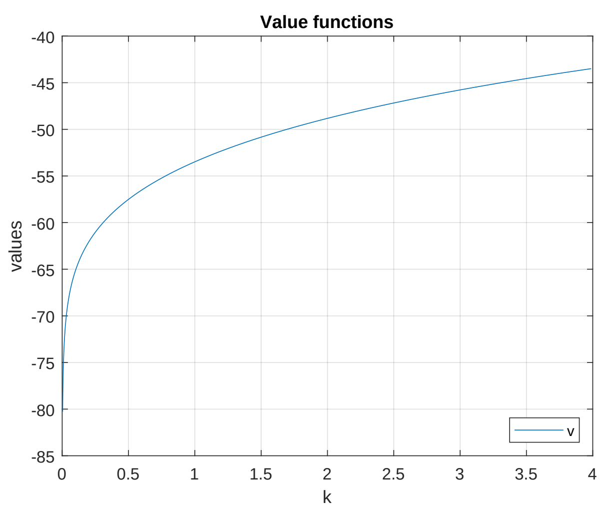
policy function
kpol
k
k

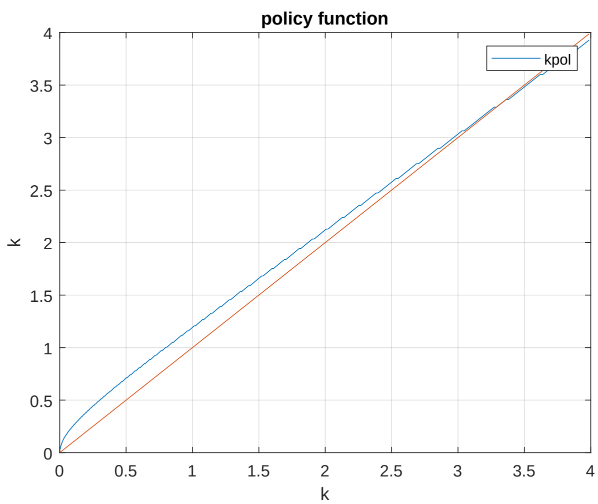


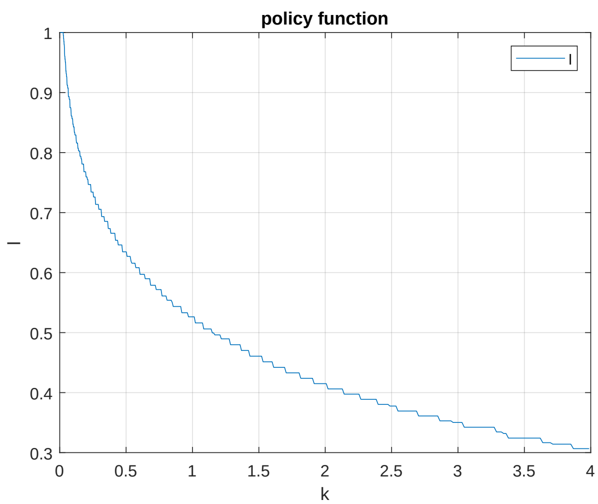
c) Take the first point of approximation to be equal to $0.001k$. Use a non-uniform grid to find the policy functions for capital, consumption and labor supply using VFI. Present graphics for the three functions. Comment on the shape of the policy functions, especially the policy function for labor supply. Can you give an intuitive explanation of its shape? What happen when the stock capital is near the lower bound? (20)

Answer.

Now, we take the first point equal to $0.001\bar{k}$. And to implement the non-uniform grid, we use a transformation such that the grid is finer for low values of the capital stock.




policy function
kpol
k
k
0
0.5
1
1.5
2
2.5
3
3.5
4



The approximation is better near the lower bound. The policy function for labor indicates that labor tends to 1. That is because capital is so low the marginal product of capital is very large. The marginal product of labor is large too and agents work a lot there.

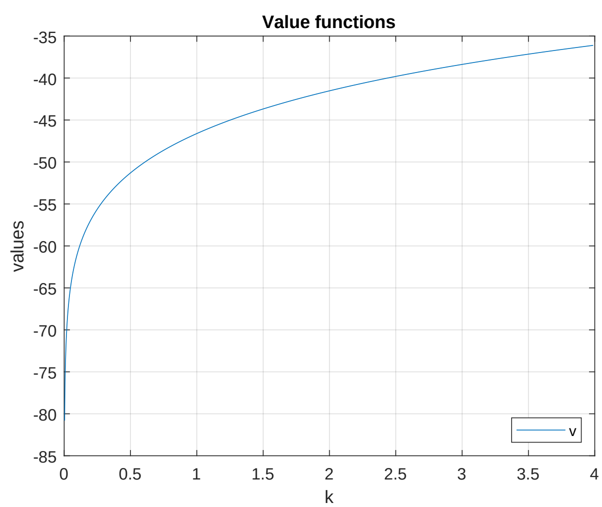
d) Repeat the exercise in c) assuming $\varphi = 1/2$. Explain the intuition. (10)

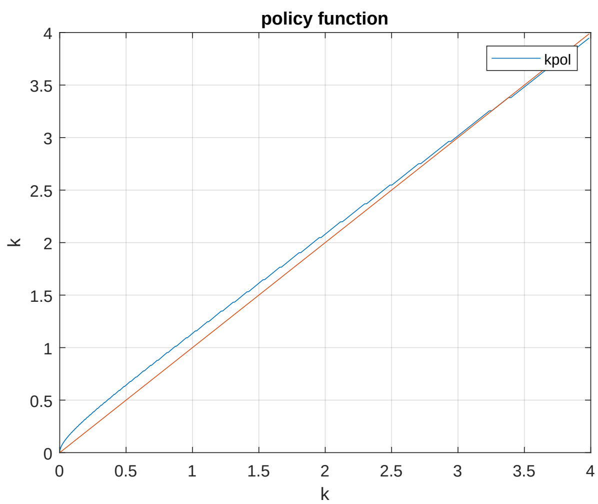
Answer.

When $\varphi = \frac{1}{2}$, we note that there should be a change in χ. And we should obtain:

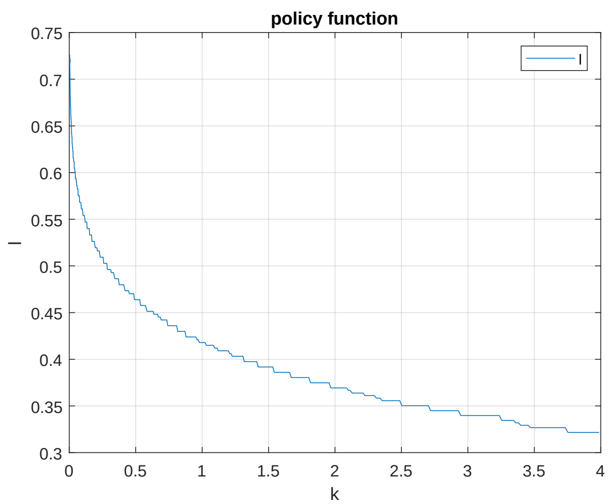
$$k_{ss} = 3.3232 \quad c_{ss} = 0.6631 \quad l_{ss} = \frac{1}{3} \quad \chi = 29.97$$

After we run the value function iteration, we should obtain the value function and corresponding policy function:





policy function
kpol
k
k
0
0.5
1
1.5
2
2.5
3
3.5
4



From the figure, comparing with the case of $\varphi = 2$, the biggest change is the policy function for labor. The mechanism driving the increasing of χ is the decrease in φ, the Frisch elasticity of labor supply. When φ decreases, the supply of labor is less elastic. We can see that when the policy function for labor shifts downward.