ECMA 30150: Perspectives on Computational Modeling for Economics
Problem Set 3
Due Date: January 28th, 2025
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Instructions

You are encouraged to work and discuss in groups, but you must submit your work individually. Answers must be legibly hand-written or typed. All assignments are due electronically on Canvas, and you must attach your code. Assignments are due at **12:30 PM**. **Late problem sets will not be accepted.**

Problem 1: The Neoclassical Growth Model I (100 points)

A Central planner uses a Bellman equation to state the problem as:

$$v(k) = \max_{c,k'} \left[u(c) + \beta v(k')\right]$$

subject to:

$$k' + c = F(k,1) + (1-\delta)k$$

(a) (30 points)

Assume $u(c) = \ln(c)$, $\delta = 1$, and $F(k,1) = k^\alpha$. Use the guess and verify method to find the value function and the associated policy functions.

Solution

Substituting the constraint into the Bellman equation and making the guess:

$$v(k) = \max_{k'} \left[\ln(k^\alpha - k') + \beta v(k')\right] = A + B\ln(k).$$

We substitute $v(k') = A + B\ln(k')$:

$$A + B\ln(k) = \max_{k'} \left[\ln(k^\alpha - k') + \beta(A + B\ln(k'))\right]$$
$$A(1-\beta) + B\ln(k) = \max_{k'} \left[\ln(k^\alpha - k') + \beta B\ln(k')\right]$$

Taking the FOC of the RHS:

$$\frac{\partial}{\partial k'}\left[\ln(k^\alpha - k') + \beta B\ln(k')\right] = 0$$
$$k' = \frac{\beta B}{1+\beta B}k^\alpha$$

Substituting back into Bellman to solve for A, B,

$$B = \frac{\alpha}{1-\beta}; \quad A = \frac{\ln(1-\beta B)}{1-\beta}.$$

Hence, the value function is

$$v(k) = \frac{\ln\left(1 - \beta \frac{\alpha}{1-\beta}\right)}{1-\beta} + \frac{\alpha}{1-\beta}\ln(k).$$

and the policy functions are:

$$k' = \frac{\beta\alpha}{1+\beta\alpha}k^{\alpha},$$

$$c = k^{\alpha} - \frac{\beta\alpha}{1+\beta\alpha}k^{\alpha} = \frac{k^{\alpha}}{1+\beta\alpha}.$$

(b) (70 points)

Now use the following functional forms and parameters:

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma}, \quad \sigma = 2, \quad F(k,1) = k^{\alpha}, \quad \alpha = 0.4, \quad \delta = 0.03, \quad \beta = 0.95$$

Assume a range of approximation $[k_{\min} = 0.1\bar{k}, k_{\max} = 1.2\bar{k}]$, where $\bar{k}$ is the steady-state stock of capital. Use the Orthogonal Collocation method to find (an approximation of) the policy function for the stock of capital $k' \equiv \hat{g}(k,a)$. Start with the approximation under $\sigma = \delta = 1$, where the solution is known. Use at least $n = 7$ nodes for approximation.

Solution

The steady-state capital stock $\bar{k}$ is computed as:

$$\bar{k} = \left(\frac{\alpha\beta}{1-\beta(1-\delta)}\right)^{\frac{1}{1-\alpha}} = (\alpha\beta)^{\frac{1}{1-\alpha}},$$

The Chebyshev nodes z_ℓ in $[-1,1]$ are given by:

$$z_\ell = -\cos\left(\frac{2\ell-1}{2n}\pi\right), \quad \ell = 1, 2, \ldots, 7.$$

These are mapped to the interval $[k_{\min}, k_{\max}]$:

$$k_\ell = \frac{(z_\ell+1)(k_{\max}-k_{\min})}{2} + k_{\min}.$$

The Chebyshev basis functions are defined as:

$$T_i(\xi) = \cos\left(i \cdot \arccos(\xi)\right), \quad i = 0, 1, \ldots, n,$$

where:

$$\xi(k) = \frac{2(k-k_{\min})}{k_{\max}-k_{\min}} - 1.$$

The policy function $\hat{g}(k;\mathbf{a})$ is approximated as:

$$\hat{g}(k;\mathbf{a}) = \sum_{i=0}^{n} a_i T_i(\xi(k)),$$

where $\mathbf{a} = [a_0, a_1, \ldots, a_n]^\top$ are the coefficients. The Euler equation is:

$$u'(f(k) - \hat{g}(k;\mathbf{a})) = \beta u'(f(\hat{g}(k;\mathbf{a})) - \hat{g}(\hat{g}(k;\mathbf{a});\mathbf{a})) \cdot f'(\hat{g}(k;\mathbf{a})),$$

where:

$$\begin{aligned} f(k) &= k^\alpha + (1-\delta)k, \\ f'(k) &= \alpha k^{\alpha-1} + 1 - \delta, \\ u'(c) &= c^{-\sigma}. \end{aligned}$$

The residual function $R(k,\mathbf{a})$ is enforced to be zero at the $n+1$ collocation nodes:

$$R(k_\ell, \mathbf{a}) = 0, \quad \ell = 1, 2, \ldots, n+1.$$

Using a numerical solver (e.g., `fsolve`), the coefficients $\mathbf{a}$ are determined. On a fine grid $k \in [k_{\min}, k_{\max}]$, the policy function is evaluated as:

$$\hat{g}(k;\mathbf{a}) = \mathbf{T}(k) \cdot \mathbf{a},$$

where $\mathbf{T}(k)$ is the matrix of Chebyshev polynomials evaluated at the grid points. The approximated policy function is plotted alongside the 45-degree line for comparison:

$$\text{Plot: } k_{t+1} = \hat{g}(k_t;\mathbf{a}) \text{ vs. } k_{t+1} = k_t.$$

See figure (1).

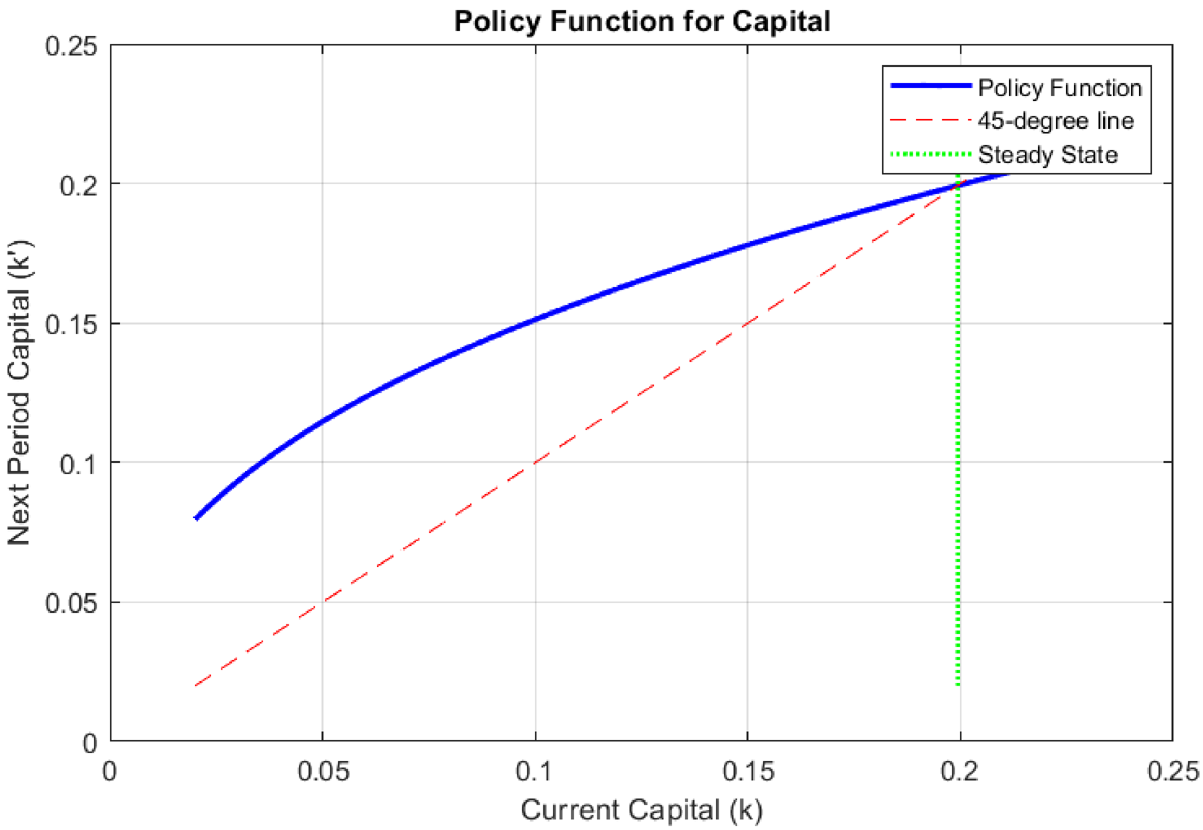


Figure 1: Orthogonal Collocation Approximation of k' for Problem 1 (b).

Problem 2: The Neoclassical Growth Model II (110 points)

A Central planner uses a Bellman equation to state the problem as:

$$v(k) = \max_{c,k'} [u(c) + \beta v(k')]$$

subject to:

$$k' + c = F(k, 1) + (1 - \delta)k$$

Use the following functional forms and parameters:

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma}, \quad \sigma = 2, \quad F(k, 1) = k^{\alpha}, \quad \alpha = 0.36, \quad \delta = 0.025, \quad \beta = 0.95$$

(a) (10 points)

Find the Euler equation.

Solution

We have:

$$\max_{\{c_t, k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u(c_t), \quad \text{subject to} \quad k_{t+1} + c_t \;=\; k_t^{\alpha} + (1-\delta)\, k_t,$$

where

$$u(c) \;=\; \frac{c^{1-\sigma}}{1-\sigma}, \quad \sigma = 2, \quad \alpha = 0.36, \quad \delta = 0.025, \quad \beta = 0.95.$$

From the planner's FOC, we have

$$u'(c_t) \;=\; \beta\, u'\big(c_{t+1}\big)\, \Big[\alpha\, k_{t+1}^{\alpha-1} + (1-\delta)\Big],$$

with

$$c_t \;=\; k_t^{\alpha} + (1-\delta)\, k_t \;-\; k_{t+1}.$$

Since $\sigma = 2$, $u'(c) = c^{-2}$. Hence, we have the Euler equation

$$c_t^{-2} \;=\; \beta\, c_{t+1}^{-2}\, \big[\alpha\, k_{t+1}^{\alpha-1} + 1 - \delta\big].$$

(b) (10 points)

Find the steady state of the model.

Solution

In steady state, $k_{t+1} = k_t = \hat{k}$ and $c_t = \bar{c}$ are constant. Then the feasibility condition implies

$$\bar{c} \;=\; (\hat{k})^{\alpha} + (1-\delta)\, \hat{k} - \hat{k}.$$

But more directly we use the Euler equation at steady state:

$$(\bar{c})^{-2} \;=\; \beta\, (\bar{c})^{-2}\, \Big[\alpha\, (\bar{k})^{\alpha-1} + 1 - \delta\Big].$$

Cancelling $(\bar{c})^{-2}$ from both sides gives

$$1 \;=\; \beta\, \big[\alpha\, (\bar{k})^{\alpha-1} + (1-\delta)\big].$$

Hence

$$\alpha\, (\bar{k})^{\,\alpha-1} + (1-\delta) \;=\; \frac{1}{\beta}.$$

One solves for $\bar{k}$:

$$\alpha\, (\bar{k})^{\,\alpha-1} \;=\; \frac{1}{\beta} - (1-\delta),$$

then $\bar{c} = \bar{k}^{\alpha} + (1-\delta)\, \bar{k} - \bar{k}$.

(c) (20 points)

Find a linear approximation around the steady state of both the Euler equation and the feasibility constraint. Write the system in the form:

$$y_{t+1} = A y_t$$

Solution

Define small deviations:

$$\hat{k}_t \;=\; k_t - \bar{k}, \quad \hat{c}_t \;=\; c_t - \bar{c}.$$

We wish to linearize both the feasibility condition and the Euler equation to get a system of the form

$$\begin{pmatrix} \hat{k}_{t+1} \\ \hat{c}_{t+1} \end{pmatrix} \;=\; A \begin{pmatrix} \hat{k}_t \\ \hat{c}_t \end{pmatrix}.$$

$$k_{t+1} + c_t = k_t^{\alpha} + (1-\delta)\, k_t.$$

We have the following for the production and utility functions:

$$f'(k) = \alpha k^{\alpha-1} + 1 - \delta \qquad u'(c) = c^{-\sigma}$$
$$f''(k) = \alpha(\alpha-1)k^{\alpha-2} \qquad u''(c) = -\sigma c^{-(1+\delta)}$$

Plugging these in, we have the linearized system

$$\begin{pmatrix} \hat{k}_{t+1} \\ \hat{c}_{t+1} \end{pmatrix} = \begin{pmatrix} \frac{1}{\beta} & -1 \\ \frac{c^{-\sigma}\alpha(\alpha-1)k^{\alpha-2}}{\sigma c^{-(1+\delta)}} & 1 + \frac{\beta c^{-\sigma}\alpha(\alpha-1)k^{\alpha-2}}{\sigma c^{-(1+\delta)}} \end{pmatrix} \begin{pmatrix} \hat{k}_t \\ \hat{c}_t \end{pmatrix}$$

from the results in the notes.

Part (d) (10 points)

Use Matlab (or any other software) to find the eigenvectors and eigenvalues of A. Check that eigenvalue λ_1 is smaller than one and eigenvalue λ_2 is higher than one.

Solution

$$\lambda_1 = 0.9557; \quad \lambda_2 = 1.1015,$$
$$v_2 = \begin{bmatrix} 0.9953 \\ 0.0965 \end{bmatrix}; \quad v_2 = \begin{bmatrix} -0.0488 \\ 0.9988 \end{bmatrix}.$$

Part (e) (30 points)

Find the policy functions:

$$c_t - \bar{c} = -\frac{\tilde{v}_{21}}{\tilde{v}_{22}}(k_t - \bar{k}),$$
$$k_{t+1} - \bar{k} = \lambda_1(k_t - \bar{k}),$$

where $\tilde{v}_{ij}$ is the position (i, j) of the matrix V^{-1}. Simulate the transition starting at 10% of steady-state capital stock.

Solution

See figure (2).

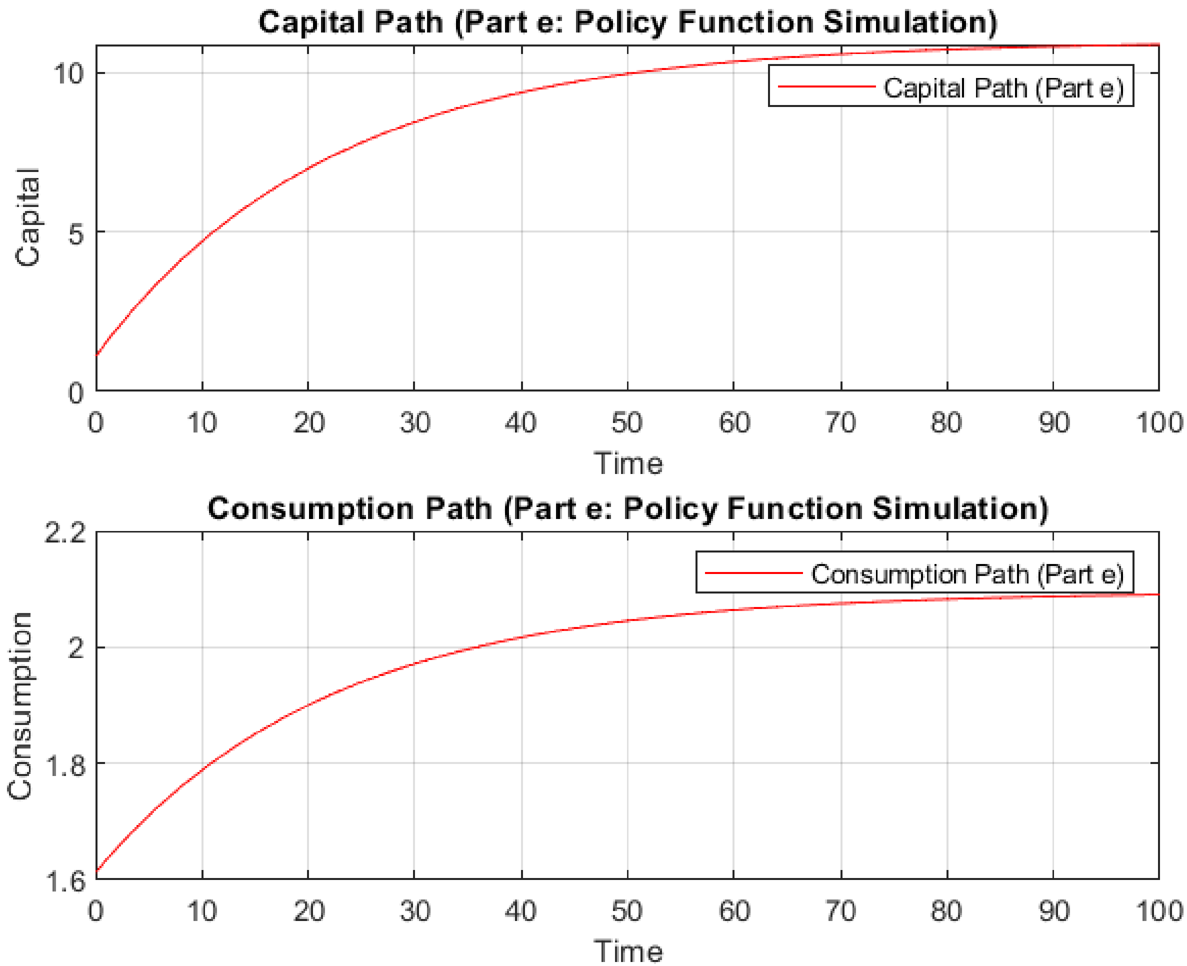


Figure 2: Simulate of transition dynamics starting at 10% of steady-state capital stock for Problem 2 (e).

Part (f) (15 points)

Use Dynare to find the time paths of both capital and consumption. Compare these paths to the ones derived in Part (e).

Solution

See figure (3).

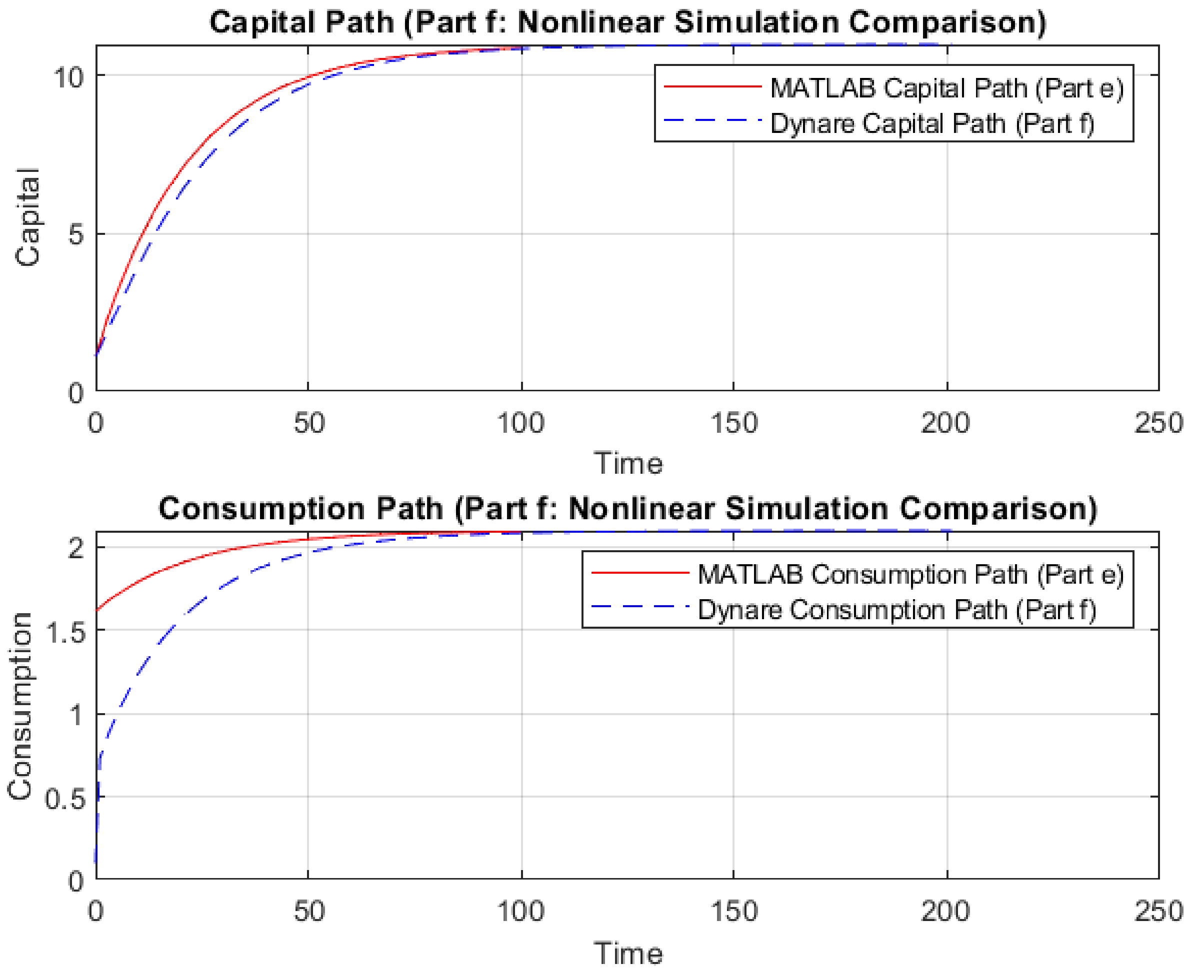


Figure 3: Dynare Time Paths of both Capital and Consumption for Problem 2 (f).

Part (g) (15 points)

Use Dynare to find the time paths of capital and consumption under a linear approximation. Compare these paths to the ones derived in Part (e).

Solution

See figure (4).

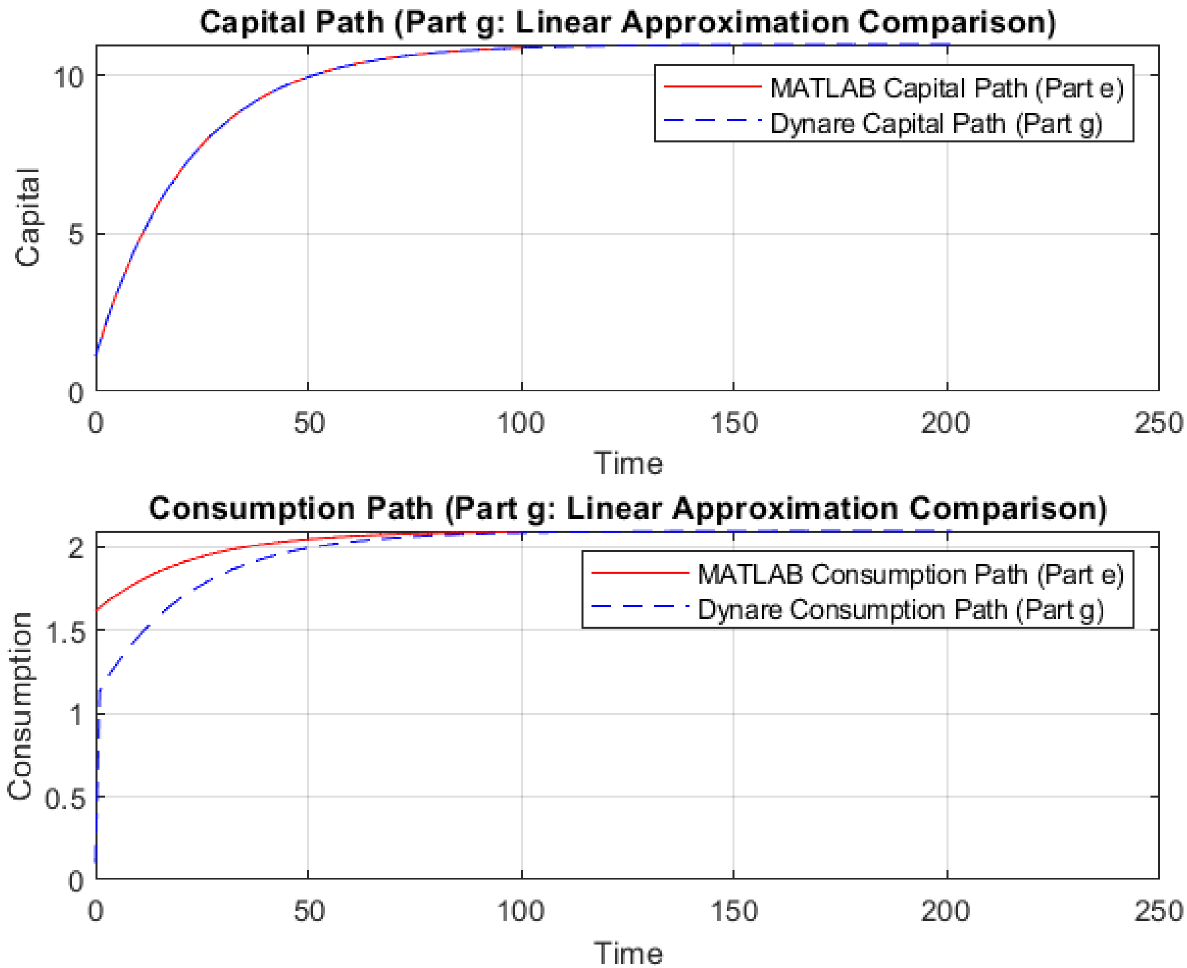


Figure 4: Dynare Time Paths of both Capital and Consumption (Linear Approximation) for Problem 2 (g).

Problem 3: The Neoclassical Growth Model with Variable Labor Supply (80 points)

A Central planner uses a Bellman equation to state the problem as:

$$v(k) = \max_{c,\ell,k'} [u(c,\ell) + \beta v(k')]$$

subject to:

$$k' + c = F(k,\ell) + (1-\delta)k$$

Use the following functional forms and parameters:

$$u(c,\ell) = \ln(c) + \gamma(1-\ell), \quad F(k,\ell) = k^{\alpha}\ell^{1-\alpha}, \quad \alpha = 0.36, \quad \delta = 0.025, \quad \beta = 0.95$$

Here, ℓ is the amount of working time, normalized to unity. $\gamma > 0$ measures the disutility of work. Note that γ is not given and must be calibrated.

(a) (20 points)

Find the Euler equation and the relevant first-order conditions (FOCs) using a Lagrangian. Derive three equations that form a dynamic system for k_t, c_t, ℓ_t.

Solution

The Lagrange is given by

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left[\ln(c_t) + \gamma(1 - l_t) + \lambda_t \left(k_t^{\alpha} l_t^{1-\alpha} + (1-\delta)k_t - k_{t+1} - c_t\right)\right]$$

with Lagrange multipliers λ_t. FOCs are given by

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial c_t} &= \beta^t \left[\frac{1}{c_t} - \lambda_t\right] = 0 && \Longrightarrow \lambda_t = \frac{1}{c_t} \\
\frac{\partial \mathcal{L}}{\partial l_t} &= \beta^t \left[-\gamma + \lambda_t(1-\alpha)k_t^{\alpha} l_t^{-\alpha}\right] = 0 && \Longrightarrow \lambda_t = \frac{\gamma l_t^{\alpha}}{(1-\alpha)k_t^{\alpha}} \\
& && \Longrightarrow \frac{1}{c_t} = \frac{\gamma l_t^{\alpha}}{(1-\alpha)k_t^{\alpha}} \\
& && \Longrightarrow c_t = \frac{(1-\alpha)k_t^{\alpha}}{\gamma l_t^{\alpha}} \\
\frac{\partial \mathcal{L}}{\partial k_{t+1}} &= \beta^t(-\lambda_t) + \beta^{t+1}\left[\frac{1}{c_{t+1}}(\alpha k_{t+1}^{\alpha-1} l_{t+1}^{1-\alpha} + 1 - \delta)\right] && \Longrightarrow c_t = \frac{c_{t+1}}{\beta(\alpha k_{t+1}^{\alpha-1} l_{t+1}^{1-\alpha} + 1 - \delta)}
\end{aligned}$$

Hence we have our Euler equation

$$c_t = \frac{c_{t+1}}{\beta(\alpha k_{t+1}^{\alpha-1} l_{t+1}^{1-\alpha} + 1 - \delta)} \tag{1}$$

subject to the labor-supply condition

$$c_t = \frac{(1-\alpha)k_t^{\alpha}}{\gamma l_t^{\alpha}} \tag{2}$$

and the resource constraint

$$c_t + k_{t+1} = k_t^{\alpha} \ell_t^{1-\alpha} + (1-\delta)k_t. \tag{3}$$

(b) (20 points)

In the steady state, under some parameterization, the system derived in Part (a) gives three equations for the unknowns $\bar{k}, \bar{c}, \bar{\ell}$ (steady-state values). Since γ is not provided, set $\bar{\ell} = \frac{1}{3}$ and use the system to solve for $\bar{k}, \bar{c}, \gamma$.

Solution

In steady state, $k_t = \bar{k}, c_t = \bar{c}, l_t = \bar{l}$. Hence, from (1):

$$\begin{aligned}
\bar{c} &= \frac{\bar{c}}{\beta(\alpha\bar{k}^{\alpha-1}\bar{l}^{1-\alpha} + 1 - \delta)} \\
\alpha\bar{k}^{\alpha-1}\bar{l}^{1-\alpha} + 1 - \delta &= \frac{1}{\beta} \\
\bar{k} &= \left(\frac{\frac{1}{\beta} - 1 + \delta}{\alpha\bar{l}^{1-\alpha}}\right)^{\frac{1}{1-\alpha}} \qquad (4)
\end{aligned}$$

which is fully calculable since $\bar{\ell} = \frac{1}{3}$. From (2):

$$\bar{c} = \frac{(1-\alpha)\bar{k}^{\alpha}}{\gamma\bar{l}^{\alpha}} \qquad (5)$$

From (3):

$$\begin{aligned}
\bar{c} + \bar{k} &= \bar{k}^{\alpha}\bar{l}^{1-\alpha} + (1-\delta)\bar{k} \\
\bar{c} &= \bar{k}^{\alpha}\bar{l}^{1-\alpha} - \delta\bar{k} \qquad (6)
\end{aligned}$$

which is fully calculable. Since we have (4) and (6), we have

$$\gamma = \frac{(1-\alpha)\bar{k}^{\alpha}}{\bar{c}\bar{l}^{\alpha}} \qquad (7)$$

from (5). Therefore, k_t, c_t, ℓ_t are given by (4),(6), and (7) respectively.

Part (c) (20 points)

Use Dynare to find the time paths of capital, consumption, and labor. Simulate the transition starting at 10% of steady-state capital stock.

Solution

See figure (5).

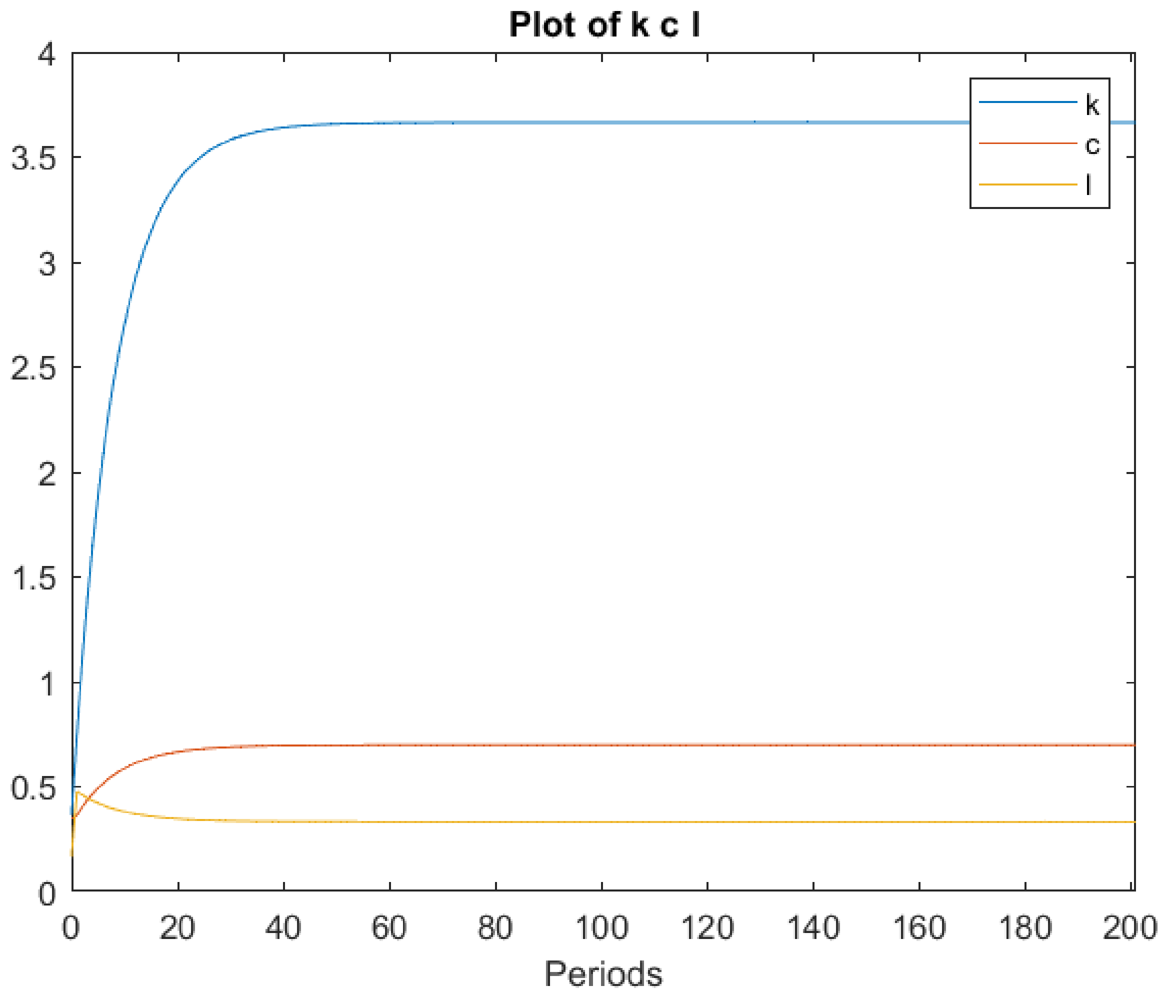


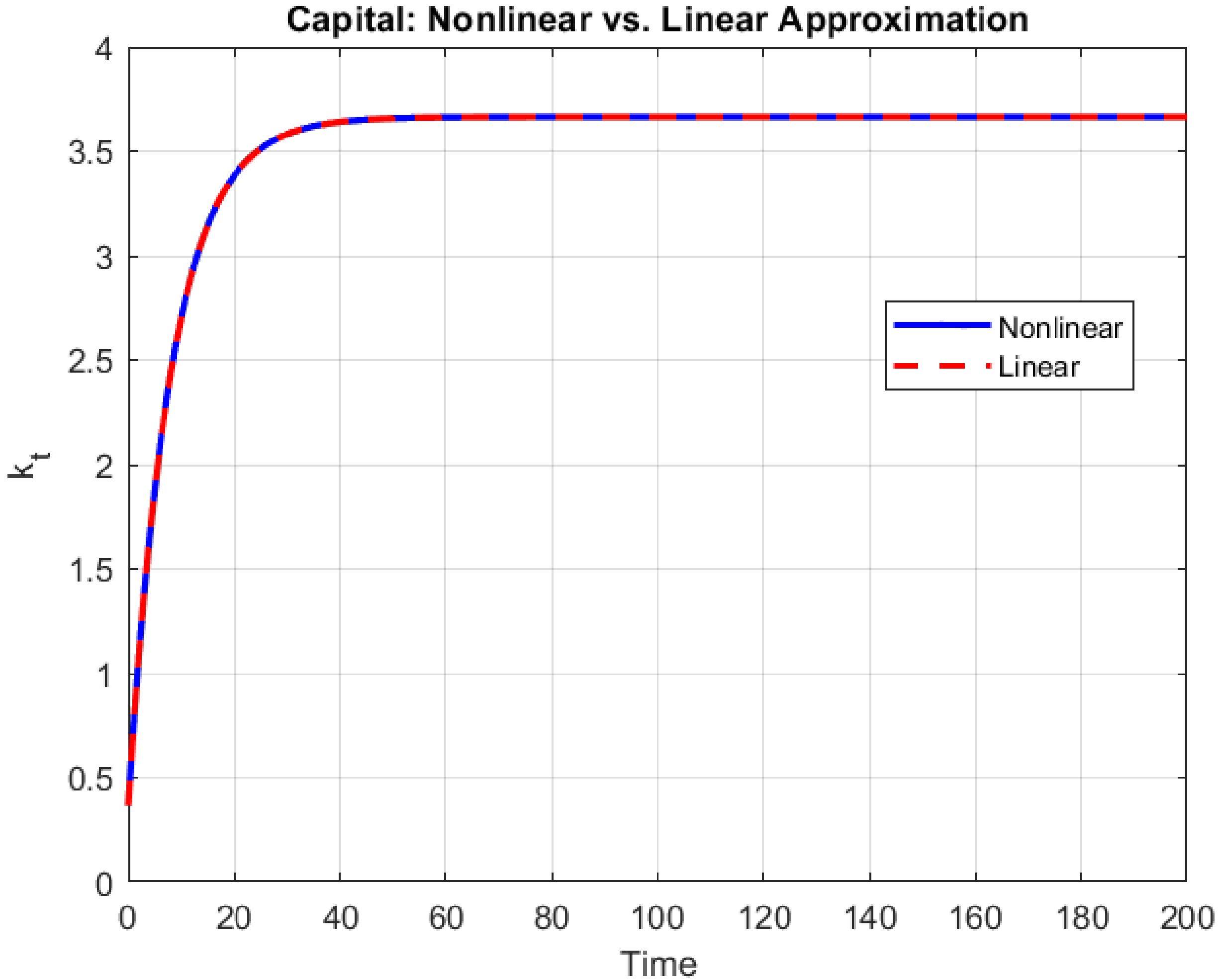
Figure 5: Simulate the time paths of capital, consumption, and labor starting at 10% of steady-state capital stock for Problem 3 (c).

Part (d) (20 points)

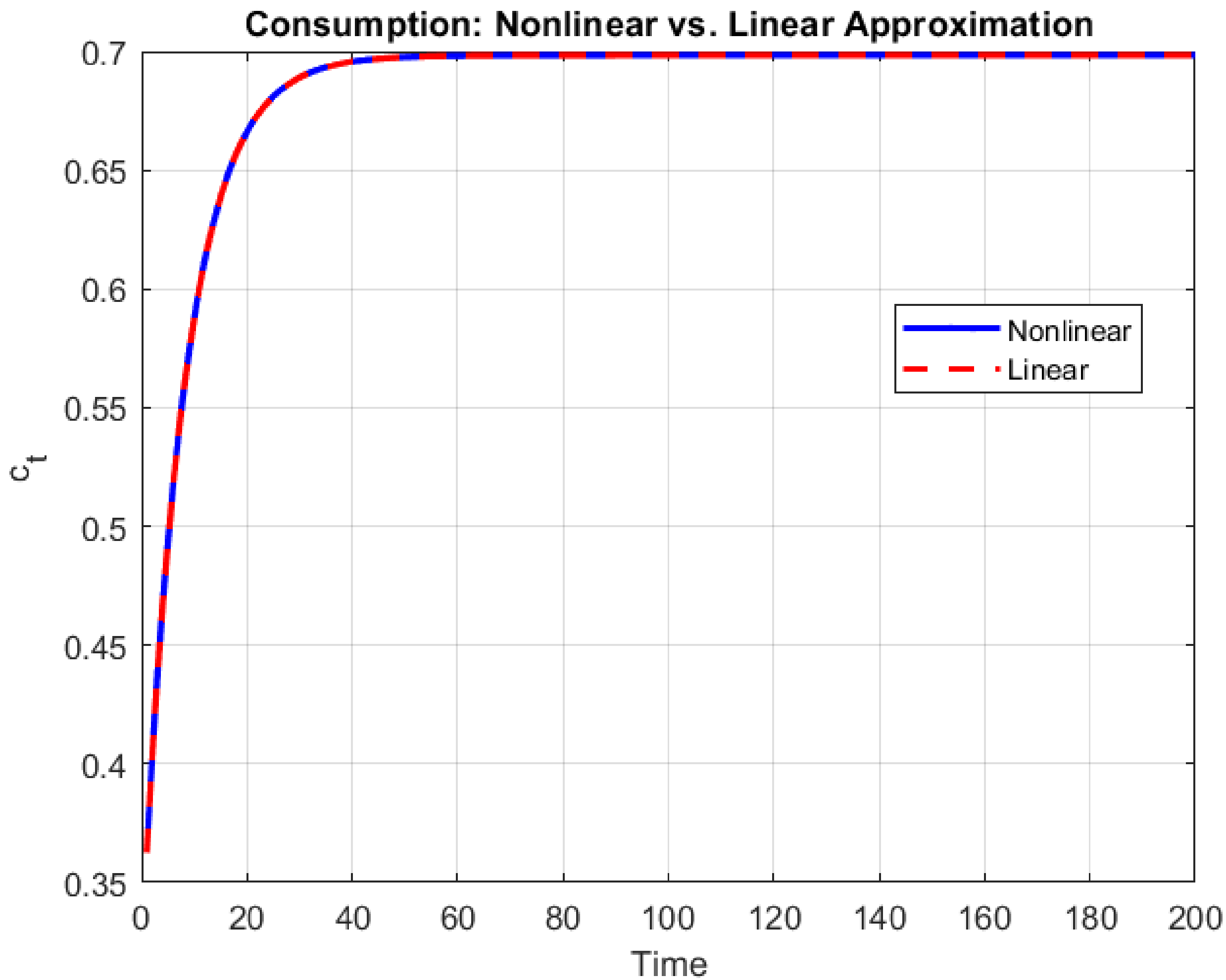
Use Dynare to find the time paths of capital, consumption, and labor under a linear approximation. Compare these paths to the ones derived in Part (c). Simulate the transition starting at 10% of steady-state capital stock.

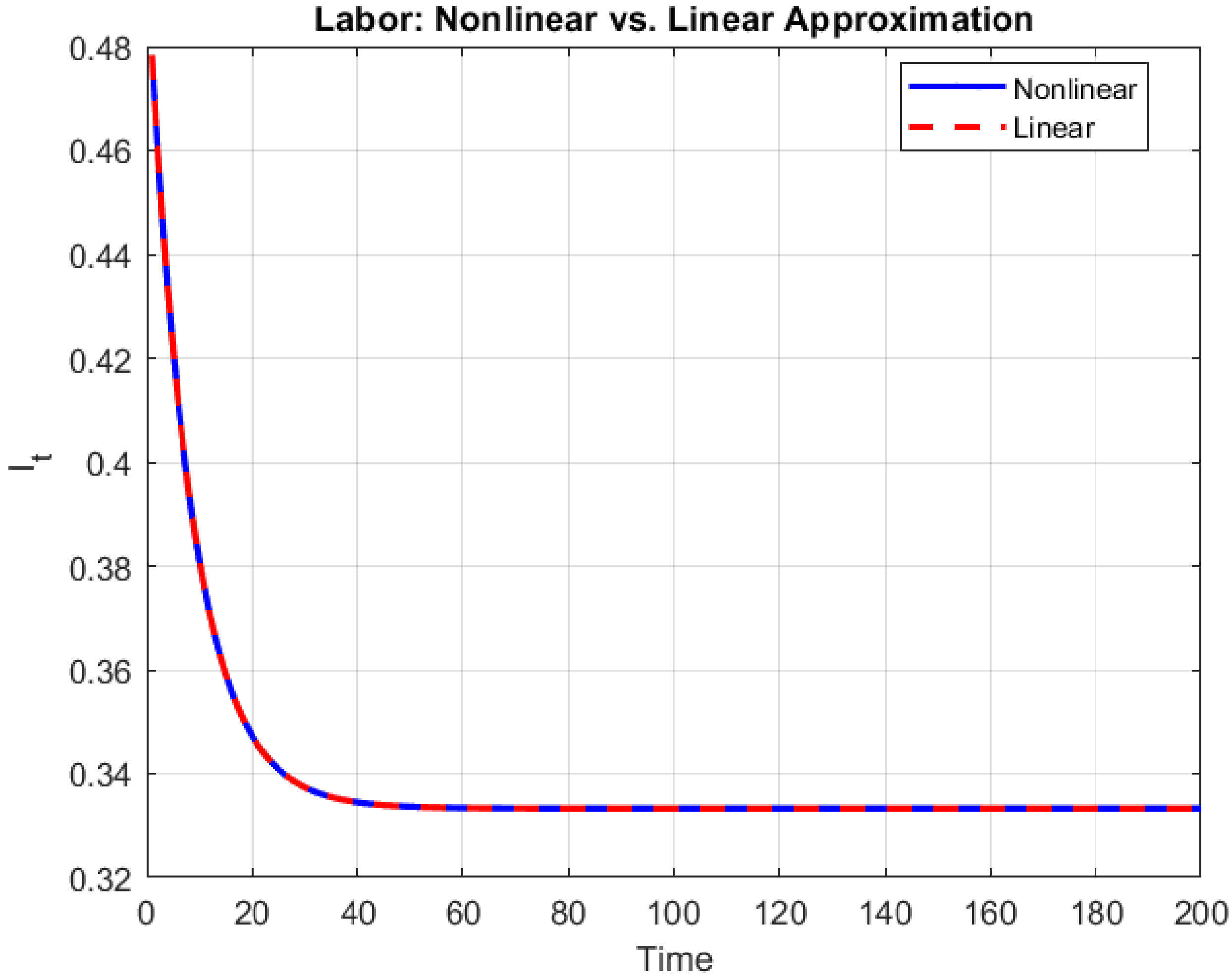
Solution

See the figures below.



Capital: Nonlinear vs. Linear Approximation
Nonlinear
Linear
k_t
Time
0
0.5
1
1.5
2
2.5
3
3.5
4
0
20
40
60
80
100
120
140
160
180
200


Consumption: Nonlinear vs. Linear Approximation
0.7
0.65
0.6
0.55
0.5
0.45
0.4
0.35
c_t
0
20
40
60
80
100
120
140
160
180
200
Time
Nonlinear
Linear


Labor: Nonlinear vs. Linear Approximation
Nonlinear
Linear
l_t
0.48
0.46
0.44
0.42
0.4
0.38
0.36
0.34
0.32
0
20
40
60
80
100
120
140
160
180
200
Time