ECMA 31140. PERSPECTIVES ON COMPUTATIONAL MODELING FOR ECONOMICS
PROBLEM SET 4
DUE DATE: FEBRUARY 4TH

Instructor: Sergio Salas `ssalas@uchicago.edu`

You are encouraged to work and discuss in groups, but you must submit your work individually. Answers must be legibly hand-written or typed. All assignments are due electronically on Canvas, attach code. Assignments are due at 12:30 PM. Late problem sets *will not be accepted*

1. The stochastic Neoclassical Growth Model. (85 points)

In this problem, you will solve for the stochastic NGM as we examined in class. A Central planner uses a Bellman equation to state the problem as:

$$\mathcal{V}(k_t, z_t) = \max_{c_t, k_{t+1}} \left[u(c_t) + \beta \mathbb{E}_t \mathcal{V}(k_{t+1}, z_{t+1})\right]$$

subject to:

$$k_{t+1} + c_t = z_t F(k_t, 1) + (1-\delta)k_t \equiv f(k_t, z_t)$$

with the stochastic process:

$$\ln z_t = (1-\rho)\ln \bar{z} + \rho \ln z_{t-1} + \epsilon_t,\ \epsilon_t \sim i.i.d.(0, \eta^2)$$

Throughout the problem use the following functional forms and parameters:

$$\beta = 0.95,\ u(c_t) = \frac{c_t^{1-\sigma}}{1-\sigma},\ \sigma = 2\ F(k_t, 1) = k_t^{\alpha},\ \alpha = 0.36,\ \delta = 0.025,\ z = 1,\ \rho = 0.95$$

a) Find the relevant equations of the model, including the Euler equation. (10)

Answer.
Note that we can write the Lagrangian as:

$$L = u(c_t) + \beta \mathbb{E}_t \mathcal{V}(k_{t+1}, z_{t+1}) + \lambda_t[f(k_t, z_t) - c_t - k_{t+1}]$$

The FOCs are:

$$\begin{aligned} u'(c_t) &= \lambda_t \\ \beta \mathbb{E}_t \mathcal{V}_k(k_{t+1}, z_{t+1}) &= \lambda_t \end{aligned}$$

We conclude:

$$u'(c_t) = \beta \mathbb{E}_t \mathcal{V}_k(k_{t+1}, z_{t+1})$$

Date: February 9, 2025.

The Envelope theorem then gives us $\mathcal{V}_k(k_t, z_t) = \lambda_t f_k(k_t, z_t)$. We then get the Euler equation:

$$u'(c_t) = \beta \mathbb{E}_t f_k(k_{t+1}, z_{t+1}) u'(c_{t+1})$$

Together with the budget constraint and the dynamics of the exogenous variables, we have the equations characterizing the model:

$$\begin{aligned}
u'(c_t) &= \beta \mathbb{E}_t f_k(k_{t+1}, z_{t+1}) u'(c_{t+1}) \\
k_{t+1} + c_t &= f(k_t, z_t) \\
\ln z_t &= (1-\rho)\ln z + \rho \ln z_{t-1} + \epsilon_t
\end{aligned}$$

where $\epsilon_t \sim (0, \eta^2)$ and $f(k_t, z_t) = z_t F(k_t, 1) + (1-\delta) k_t$.

b) Find the steady state of the model, analytically. (10)

Answer.

At steady state, we have: $c_t = c_{t+1} = c$, $k_t = k_{t+1} = k$, $z_t = z_{t+1} = z = 1$. Therefore the system is:

$$\begin{aligned}
1 &= \beta f_k(k, z) \\
k + c &= f(k, z)
\end{aligned}$$

Recall $f(k, z) = zF(k, 1) + (1-\delta)k = zk^\alpha + (1-\delta)k$, then:

$$\begin{aligned}
1 &= \beta(\alpha k^{\alpha-1} + 1 - \delta) \\
k + c &= k^\alpha + (1-\delta)k
\end{aligned}$$

Solving this system:

$$k = \left[\frac{\alpha\beta}{1-(1-\delta)\beta}\right]^{\frac{1}{1-\alpha}}, \; c = k^\alpha - \delta k$$

c) Find a linear approximation around the steady state of both the Euler equation and the feasibility constraint, show that the system can be written in the form:

$$\mathbb{E}_t y_{t+1} = A y_t + B \hat{z}_t$$

where $y_t = [\hat{k}_t \; \hat{c}_t]'$. And the variables with hats denote the variables in deviations from their steady states. (20)

Answer.

To obtain the linear approximation, consider the linear expansion of Euler equation and the steady state characterization:

$$\begin{aligned}
u' + u''(c_t - c) &= \beta \mathbb{E}_t [f_k + f_{kk}(k_{t+1} - k) + f_{kz}(z_{t+1} - z)][u' + u''(c_{t+1} - c)] \\
1 &= \beta f_k
\end{aligned}$$

This indicates:

$$u'' \hat{c}_t = \beta \mathbb{E}_t (f_k u'' \hat{c}_{t+1} + u' f_{kk} \hat{k}_{t+1} + u' f_{kz} \hat{z}_{t+1})$$

Consider the linear expansion of the resource constraint and the steady state characterization, we have:

$$k + (k_{t+1} - k) + c + (c_t - c) = f + f_k(k_t - k) + f_z(z_t - z)$$
$$k + c = f$$

This indicates:

$$\hat{k}_{t+1} + \hat{c}_t = f_k \hat{k}_t + f_z \hat{z}_t$$

And for z_{t+1} and z_t, we shall have:

$$\ln z + \frac{z_{t+1} - z}{z} = (1-\rho)\ln z + \rho(\ln z + \frac{z_t - z}{z}) + \epsilon_{t+1}$$

And this indicates:

$$\hat{z}_{t+1} = \rho \hat{z}_t + \epsilon_{t+1}$$

Plug $\hat{k}_{t+1} = \frac{1}{\beta}\hat{k}_t - \hat{c}_t + f_z \hat{z}_t$ and $\hat{z}_{t+1} = \rho\hat{z}_t + \epsilon_{t+1}$ into the Euler equation:

$$u''\hat{c}_t = \beta \mathbb{E}_t[\frac{u''}{\beta}\hat{c}_{t+1} + u' f_{kk}(\frac{1}{\beta}\hat{k}_t - \hat{c}_t + f_z\hat{z}_t) + u' f_{kz}(\rho\hat{z}_t + \epsilon_{t+1})]$$

Where the fact that $\mathbb{E}_t \epsilon_{t+1} = 0$ was used. Sorting out the equation, we have:

$$\mathbb{E}_t \hat{c}_{t+1} = (1 + \beta\frac{u' f_{kk}}{u''})\hat{c}_t - \frac{u' f_{kk}}{u''}\hat{k}_t - \beta u' \frac{f_{kk} f_z + \rho f_{kz}}{u''} z_t$$

Therefore, we can write: $\mathbb{E}y_{t+1} = Ay_t + B\hat{z}_t$:

$$\mathbb{E}y_{t+1} = \begin{bmatrix} \frac{1}{\beta} & -1 \\ -\frac{u' f_{kk}}{u''} & 1 + \beta\frac{u' f_{kk}}{u''} \end{bmatrix} y_t + \begin{bmatrix} f_z \\ -\beta u' \frac{f_{kk} f_z + \rho f_{kz}}{u''} \end{bmatrix} \hat{z}_t$$

d) Use Matlab (or other software) to find the eigenvectors and eigenvalues of A. Check that eigenvalue λ_1 is smaller than one and eigenvalue λ_2 is higher than one. (10)

Answer.

With the matrix given in c) and the analytical solution of steady state values, plug in $\alpha = 0.36$, $\beta = 0.95$, $\sigma = 2$, $\delta = 0.025$, $\rho = 0.95$ and $z = 1$. We shall have:

$$A = \begin{bmatrix} 1.0526 & -1 \\ -0.0047 & 1.0045 \end{bmatrix}$$

And we have two eigenvalues:

$$\lambda_1 = 0.9557 \quad \lambda_2 = 1.1015$$

And this verifies that we have one eigenvalue larger than 1 and one eigenvalue smaller than 1.

e) Find the policy functions:

$$\hat{c}_t = -\frac{\tilde{v}_{21}}{\tilde{v}_{22}}\hat{k}_t - \frac{c_2}{(\lambda_2 - \rho)\tilde{v}_{22}}\hat{z}_t$$
$$\hat{k}_{t+1} = \left(\frac{1}{\beta} + \frac{\tilde{v}_{21}}{\tilde{v}_{22}}\right)\hat{k}_t + \left(f_z + \frac{c_2}{(\lambda_2 - \rho)\tilde{v}_{22}}\right)\hat{z}_t$$

(15)

Answer.
With the eigenvalues given in d), we can also obtain the eigenvector matrix:

$$V = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = \begin{bmatrix} 0.9953 & 0.9988 \\ 0.0965 & -0.0488 \end{bmatrix}$$

And we shall have an inverse of V:

$$V = \begin{bmatrix} \tilde{v}_{11} & \tilde{v}_{12} \\ \tilde{v}_{21} & \tilde{v}_{22} \end{bmatrix} = \begin{bmatrix} 0.3365 & 6.8904 \\ 0.6658 & -6.8664 \end{bmatrix}$$

Now, we look at matrix B. We shall have $f_z = k^\alpha$ and plug in the rest of the values, we have:

$$B = \begin{bmatrix} 2.3701 \\ 0.0627 \end{bmatrix}$$

Thus, we have:

$$C = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1.2299 & 1.1473 \end{bmatrix}$$

Now, plug in these values, we shall obtain:

$$\begin{aligned} -\frac{\tilde{v}_{21}}{\tilde{v}_{22}} &= 0.0970 \\ -\frac{c_2}{(\lambda_2 - \rho)\tilde{v}_{22}} &= 1.1032 \\ \frac{1}{\beta} + \frac{\tilde{v}_{21}}{\tilde{v}_{22}} &= 0.9557 \\ f_z + \frac{c_2}{(\lambda_2 - \rho)\tilde{v}_{22}} &= 1.2670 \end{aligned}$$

Therefore, numerically, the policy function should be:

$$\begin{aligned} \hat{c}_t &= 0.0970\hat{k}_t + 1.1032\hat{z}_t \\ \hat{k}_{t+1} &= 0.9557\hat{k}_t + 1.2670\hat{z}_t \end{aligned}$$

f) Find the responses to a one time positive shock of 10% of the value of the steady state value of z_t. Use the equations derived in e) to find the time path of capital and consumption to this one time shock. (Program these–so called–, *impulse responses functions* in Matlab or other software). (10)

Answer.
Since $z = 1$ the shock has value of 0.1. Plugging in the values, in the system and tracing out the dynamics of the variables, we have the impulse response functions shown in the figure below:

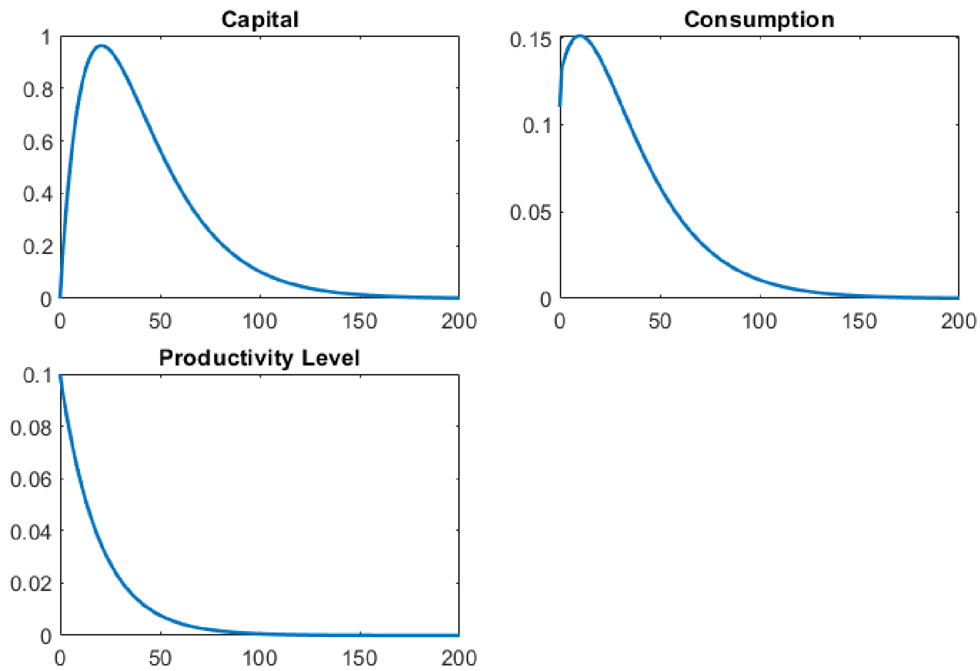

FIGURE 1. Impulse Response Function

g) Construct the matrices Ψ and Ω as explained in class. Compare with the same matrices given by Dynare. (10)

Answer.
Here, our predetermined variable (x) is k_t, non-predetermined variable (y) is c_t and our exogenous shock is z_t. Therefore, in this case, we have:

$$P = 0.9557 \quad Q = 1.2670 \quad R = 0.0970 \quad S = 1.1032 \quad N = 0.95$$

Plug in the values, we have:

$$A = \begin{bmatrix} P & QN \\ 0 & N \end{bmatrix} = \begin{bmatrix} 0.9557 & 1.2036 \\ 0 & 0.95 \end{bmatrix}$$
$$B = \begin{bmatrix} Q \\ I \end{bmatrix} = \begin{bmatrix} 1.2670 \\ 1 \end{bmatrix}$$
$$C = \begin{bmatrix} R & SN \end{bmatrix} = \begin{bmatrix} 0.0970 & 1.0480 \end{bmatrix}$$
$$D = S = 1.1032$$

Stacking everything up, we have:

$$\Psi = \begin{bmatrix} A \\ C \end{bmatrix} = \begin{bmatrix} 0.9557 & 1.2036 \\ 0 & 0.95 \\ 0.0970 & 1.0480 \end{bmatrix} \quad \Omega = \begin{bmatrix} B \\ D \end{bmatrix} = \begin{bmatrix} 1.2670 \\ 1 \\ 1.1032 \end{bmatrix}$$

And they agree with the policy and policy functions obtained with Dynare.

2. The stochastic Neoclassical Growth Model with variable labor supply. (65 points)

A Central planner uses a Bellman equation to state the problem as:

$$\mathcal{V}(k_t, z_t) = \max_{c_t, k_{t+1}, \ell_t} [u(c_t, \ell_t) + \beta \mathbb{E}_t \mathcal{V}(k_{t+1}, z_{t+1})]$$

subject to:

$$k_{t+1} + c_t = z_t F(k_t, \ell_t) + (1-\delta)k_t$$

with the stochastic process:

$$\ln z_t = (1-\rho)\ln \bar{z} + \rho \ln z_{t-1} + \epsilon_t, \ \epsilon_t \sim i.i.d.(0, \eta^2)$$

Throughout the problem use the following functional forms and parameters:

$$u(c_t, \ell_t) = \frac{[c_t^{\gamma}(1-\ell_t)^{1-\gamma}]^{1-\sigma}}{1-\sigma}, \ F(k_t, \ell_t) = k_t^{\alpha}\ell_t^{1-\alpha}, \ \alpha = 0.36, \ \delta = 0.025, \ z = 1, \ \rho = 0.95, \beta = 0.98$$

a) Find the Euler equation and the relevant FOC's. You should obtain 4 equations that describe the dynamics of the system for the variables k_t, c_t, ℓ_t and z_t. (20)

Answer.
Let's define:

$$k_{t+1} + c_t = z_t F(k_t, \ell_t) + (1-\delta)k_t = f(k_t, \ell_t, z_t)$$

Form the Lagrangian:

$$L = u(c_t, \ell_t) + \beta \mathbb{E}_t \mathcal{V}(k_{t+1}, z_{t+1}) + \lambda_t [f(k_t, \ell_t, z_t) - c_t - k_{t+1}]$$

The FOCs are:

$$\begin{aligned} u_c(c_t, \ell_t) &= \lambda_t \\ \beta \mathbb{E}_t \mathcal{V}_k(k_{t+1}, z_{t+1}) &= \lambda_t \\ u_\ell(c_t, \ell_t) + \lambda_t f_\ell(k_t, \ell_t, z_t) &= 0 \end{aligned}$$

We conclude:

$$u_c(c_t, \ell_t) = \beta \mathbb{E}_t \mathcal{V}_k(k_{t+1}, z_{t+1})$$
$$u_\ell(c_t, \ell_t) + u_c(c_t, \ell_t) f_\ell(k_t, \ell_t, z_t) = 0$$

The Envelope theorem then gives us $\mathcal{V}_k(k_t, z_t) = \lambda_t f_k(k_t, z_t)$. We then get the Euler equation and the intra-temporal condition for labor:

$$u_c(c_t, \ell_t) = \beta \mathbb{E}_t f_k(k_{t+1}, z_{t+1}) u_c(c_{t+1}, \ell_{t+1}) u_\ell(c_t, \ell_t) = -u_c(c_t, \ell_t) f_\ell(k_t, \ell_t, z_t)$$

Now, consider the functional form of utility function and production function:

$$u_c(c_t, \ell_t) = \gamma c_t^{\gamma-1}(1-\ell_t)^{1-\gamma}(c_t^{\gamma}(1-\ell_t)^{1-\gamma})^{-\sigma}$$
$$u_\ell(c_t, \ell_t) = -(1-\gamma)(1-\ell_t)^{-\gamma} c_t^{\gamma}(c_t^{\gamma}(1-\ell_t)^{1-\gamma})^{-\sigma}$$
$$f_k(k_t, \ell_t, z_t) = \alpha z_t (\frac{k_t}{\ell_t})^{\alpha-1} + 1 - \delta$$
$$f_\ell(k_t, \ell_t, z_t) = (1-\alpha) z_t (\frac{k_t}{\ell_t})^{\alpha}$$

Eventually, we shall have four equations:

$$c_t^{\gamma-1}(1-\ell_t)^{1-\gamma}(c_t^{\gamma}(1-\ell_t)^{1-\gamma})^{-\sigma} = \beta\mathbb{E}_t(\alpha z_t(\frac{k_{t+1}}{\ell_{t+1}})^{\alpha-1}+1-\delta)c_{t+1}^{\gamma-1}(1-\ell_{t+1})^{1-\gamma}(c_{t+1}^{\gamma}(1-\ell_{t+1})^{1-\gamma})^{-\sigma}$$

$$(1-\alpha)z_t(\frac{k_t}{\ell_t})^{\alpha}\gamma c_t^{\gamma-1}(1-\ell_t)^{1-\gamma}(c_t^{\gamma}(1-\ell_t)^{1-\gamma})^{-\sigma} - (1-\gamma)c_t^{\gamma}(1-\ell_t)^{-\gamma}(c_t^{\gamma}(1-\ell_t)^{1-\gamma})^{-\sigma} = 0$$

$$c_t + k_{t+1} = z_t k_t^{\alpha}\ell_t^{1-\alpha} + (1-\delta)k_t$$

$$\ln z_t = (1-\rho)\ln z + \rho\ln z_{t-1} + \varepsilon_t,\ \ \varepsilon_t \sim i.i.d.(0,\eta^2)$$

Note: the second equation can be simplified into:

$$(1-\alpha)z_t(\frac{k_t}{\ell_t})^{\alpha}\gamma(1-\ell_t) - (1-\gamma)c_t = 0$$

b) Assume $\sigma = 1$. In the steady state, and under some parametrization, the system derived in a) gives three equations for the unknowns k, c, ℓ (the steady state values). Because we don't have a value for γ, set $\ell = 1/3$ (the steady state value of labor), and use the system in a) to find the values for the unknowns k, c, γ. (15)

Answer.
Evaluate the steady state, we shall have:

$$1 = \beta(\alpha z(\frac{k}{\ell})^{\alpha-1} + 1 - \delta)$$

$$(1-\alpha)z(\frac{k}{\ell})^{\alpha}\gamma(1-\ell) - (1-\gamma)c = 0$$

$$c + k = zk^{\alpha}\ell^{1-\alpha} + (1-\delta)k$$

Therefore, we shall have:

$$\ell = \frac{1}{3}$$

$$k = \ell(\frac{\frac{1}{\beta}+\delta-1}{\alpha z})^{\frac{1}{\alpha-1}} = \frac{1}{3}(\frac{\frac{1}{\beta}+\delta-1}{\alpha})^{\frac{1}{\alpha-1}}$$

$$c = \frac{1}{\ell}(\frac{k}{\ell})^{\alpha} - \delta k = 3(\frac{\frac{1}{\beta}+\delta-1}{\alpha})^{\frac{\alpha}{\alpha-1}} - \delta\frac{1}{3}(\frac{\frac{1}{\beta}+\delta-1}{\alpha})^{\frac{1}{\alpha-1}}$$

Think of the substitution relationship between c and ℓ, we have:

$$(1-\alpha)z(\frac{k}{\ell})^{\alpha} = \frac{(1-\gamma)c}{\gamma(1-\ell)}$$

This will induce:

$$\frac{1}{\gamma} - 1 = (1-\alpha)z(\frac{k}{\ell})^{\alpha}\frac{1-\ell}{c}$$

Eventually, we shall have the analytical form of γ as:

$$\gamma = \frac{1}{(1-\alpha)z(\frac{k}{\ell})^{\alpha}\frac{1-\ell}{c}+1}$$

Plug in the values of parameters into the analytical form, we shall have:

$$\ell = \frac{1}{3} \quad k = 8.4690 \quad c = 0.8565 \quad \gamma = 0.3851$$

c) Assume $\sigma = 1$. Use Dynare to find the impulse response function for a shock of a one time positive shock of 10% of the value of the steady state value of z_t. Show the responses of consumption, capital and labor. (15)

Answer.
With Dynare, we shall have the impulse response function for a shock of a one time positive shock of 10% as below:

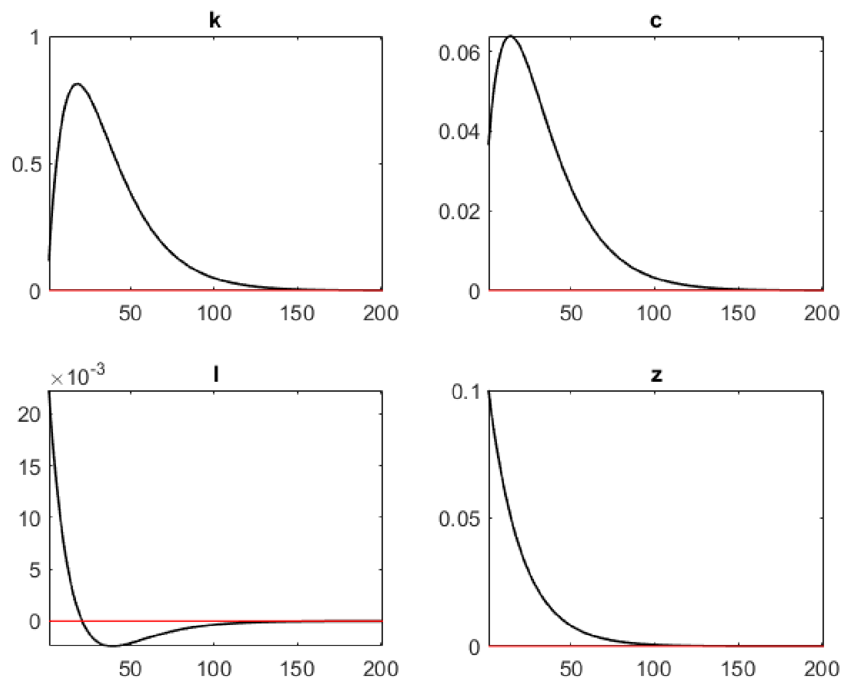

FIGURE 2. Impulse Response Function

d) Find the matrices Ψ and Ω in Dynare and write down the system:

$$\underset{(m+n+k)\times 1}{Y_t} = \underset{(m+n+k)\times(m+k)}{\Psi} \underset{(m+k)\times 1}{s_{t-1}} + \underset{(m+n+k)\times k}{\Omega} \underset{k\times 1}{\epsilon_t}$$

explicitly. (15)

Answer.

With the following order:

$$Y_t = \begin{bmatrix} k_t \\ z_t \\ c_t \\ \ell_t \end{bmatrix} \quad s_t = \begin{bmatrix} k_{t-1} \\ z_{t-1} \end{bmatrix}$$

The corresponding matrices in Dynare are:

$$\Psi = \begin{bmatrix} 0.9418 & 1.0767 \\ 0 & 0.95 \\ 0.0583 & 0.3596 \\ -0.0099 & 0.2055 \end{bmatrix} \quad \Omega = \begin{bmatrix} 1.1313 \\ 1 \\ 0.3785 \\ 0.2163 \end{bmatrix}$$

3. Parametrizing with the method of moments. (65 points)

Consider the exact same model as Question 2. You will use the method of moments to find the value of the parameters such that the theoretical moments from the model are as close as possible to the data counterparts.

Let Θ be the 6×1 vector of parameters:

$$\Theta = [\alpha, \beta, \delta, \sigma, \rho, \eta]$$

Let the data moments be collected in the 8×1 vector:

$$M = [sd_c, sd_i, sd_\ell, sd_w, corr_{c,y}, corr_{i,y}, corr_{\ell,y}, corr_{w,y}] = [0.69, 1.35, 0.90, 1.14, 0.85, 0.60, 0.07, 0.76]$$

Where sd_j for $j = c, i, \ell, w$ is the standard deviation of variable j relative to the standard deviation of GDP. $corr_{j,y}$ for $j = c, i, \ell, w$ is the contemporaneous correlation of variable j with GDP.

Let the model moments (counterpart of M) be:

$$\mathcal{M} = [sd_c^*, sd_i^*, sd_\ell^*, sd_w^*, corr_{c,y}^*, corr_{i,y}^*, corr_{\ell,y}^*, corr_{w,y}^*]$$

a) Use Matlab and Dynare to find:

$$\Theta^* = \arg\min_{\Theta} \left[(\mathcal{M} - M)^T W (\mathcal{M} - M) \right]$$

where T denotes transpose. Assume for simplicity $W = I$. Consider the following bounds for the parameters:

$$\begin{aligned} \alpha &\in [0.2, 0.9] \\ \beta &\in [0.7, 0.999] \\ \delta &\in [0.01, 0.05] \\ \sigma &\in [0.5, 6] \\ \rho &\in [0.5, 0.99] \\ \eta &\in [0.001, 0.1] \end{aligned}$$

(50)

Answer.

Searching starting with some initial parameter values, the following results are found:

$$\Theta = [0.2835,\ 0.9022,\ 0.0100,\ 0.5000,\ 0.5000,\ 0.0581]$$

Other values might be obtained if you started with a different initial guess. The value of the objective function that I found is 1.41314.

b) Report the model moments obtained, which are close to the actual moments? (15)

Answer.

The following table compares the moments:

$$\begin{bmatrix} \mathcal{M} & M \\ \hline \\ 0.527 & 0.690 \\ 0.776 & 1.350 \\ 0.365 & 0.900 \\ 1.350 & 1.140 \\ 0.641 & 0.850 \\ 0.853 & 0.600 \\ 0.845 & 0.070 \\ 0.898 & 0.760 \end{bmatrix}$$

Evidently, the moments grossly mismatched are sd_ℓ and $corr_{y,\ell}$. They both involve labor supply.