ECMA 31140. PERSPECTIVES ON COMPUTATIONAL MODELING FOR ECONOMICS
PROBLEM SET 2
DUE DATE: JANUARY 21ST

Instructor: Sergio Salas ssalas@uchicago.edu
Grader: Sania Zeb saniazeb@uchicago.edu

You are encouraged to work and discuss in groups, but you must submit your work individually. Answers must be legibly hand-written or typed. All assignments are due electronically on Canvas, attach code. Assignments are due at 12:30 PM. Late problem sets *will not be accepted*

1. Finding Eigenvalues and Eigenvectors. (15)

Consider the general matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Where $a_{i,j}$ are real values different than zero. Use software with analytical or symbolic capabilities.

a) Find the Eigenvalues and Eigenvectors of the matrix A. Show that:

$$AV = VD$$

(5)

Answers.

In Matlab, using the command eig with the symbolic tool we obtain:

$$D = \begin{bmatrix} \frac{a_{11}}{2} + \frac{a_{22}}{2} - \frac{\sqrt{{a_{11}}^2 - 2\,a_{11}\,a_{22} + {a_{22}}^2 + 4\,a_{12}\,a_{21}}}{2} & 0 \\ 0 & \frac{a_{11}}{2} + \frac{a_{22}}{2} + \frac{\sqrt{{a_{11}}^2 - 2\,a_{11}\,a_{22} + {a_{22}}^2 + 4\,a_{12}\,a_{21}}}{2} \end{bmatrix}$$

$$V = \begin{bmatrix} \frac{\frac{a_{11}}{2} + \frac{a_{22}}{2} - \frac{\sqrt{{a_{11}}^2 - 2\,a_{11}\,a_{22} + {a_{22}}^2 + 4\,a_{12}\,a_{21}}}{2}}{a_{21}} - \frac{a_{22}}{a_{21}} & \frac{\frac{a_{11}}{2} + \frac{a_{22}}{2} + \frac{\sqrt{{a_{11}}^2 - 2\,a_{11}\,a_{22} + {a_{22}}^2 + 4\,a_{12}\,a_{21}}}{2}}{a_{21}} - \frac{a_{22}}{a_{21}} \\ 1 & 1 \end{bmatrix}$$

The equality $AV = VD$ can be show to satisfy directly.

b) Assume that $a_{21} = 0$. Compute Eigenvalues and Eigenvectors. (5)

Answers.

Again, using symbolic capabilities:

$$D = \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix}, V = \begin{bmatrix} 1 & -\frac{a_{12}}{a_{11}-a_{22}} \\ 0 & 1 \end{bmatrix}$$

Date: January 21, 2025.

c) Assume that $a_{12} = 0$. Compute Eigenvalues and Eigenvectors. (5)

Answers.

Again, using symbolic capabilities:

$$D = \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix}, V = \begin{bmatrix} \frac{a_{11}-a_{22}}{a_{21}} & 0 \\ 1 & 1 \end{bmatrix}$$

2. A cake-eating problem with interest rate. (40)

An individual wants to maximize:

$$\sum_{t=0}^{T} \beta^t u(c_t)$$

subject to:

$$w_{t+1} = (w_t - c_t)(1 + r)$$

where $r > 0$ is the real interest rate. Where $w_0 > 0$ is given. (You can use software with analytical or symbolic capabilities)

a) Iterate forward the budget constraint from period 0 onwards. (5)

Answer

To ease on notation let us define $R = 1 + r$, note that if $r > 0$ (that we assume) then $R > 1$. The budget constraint can be written as: $w_t = c_t + w_{t+1}/R$. Using this equation repeatedly going forward, we have:

$$w_0 = \sum_{t=0}^{T} \frac{c_t}{R^t} + \frac{w_{T+1}}{R^{T+1}}$$

b) Take now the limit as $T \to \infty$ of the problem. What is the *transversality* condition you obtain from the expression derived in (a)? (5)

Answer

Taking the limit as $T \to \infty$ in the problem, we have:

$$\max_{c_t} \left[\lim_{T\to\infty} \sum_{t=0}^{T} \beta^t u(c_t) \right] = \max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

Subject to:

$$w_{t+1} = (w_t - c_t)R, \;\; w_0 > 0 \text{ given}$$

with the *transversality condition*:

$$\lim_{T\to\infty} \frac{w_{T+1}}{R^{T+1}} = 0$$

c) From now on use the infinite version of the problem. Find the Euler equation. (5)

Answer

A straightforward way to go is to replace consumption out of the budget constraint into the objective function:

$$\max_{\{w_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(w_t - w_{t+1}/R)$$

Taking the FOC:

$$-\beta^t \frac{u'(c_t)}{R} + \beta^{t+1} u'(c_{t+1}) = 0$$

Then we get:

$$u'(c_t) = \beta R u'(c_{t+1})$$

d) From now on use log utility for preferences: $u(c_t) = \ln(c_t)$. Use the Euler equation to find a second order difference equation with two boundary conditions: One is the initial condition w_0 and the other the *transversality* condition you derived in b). (10)

Answer

I will work with the more general utility function:

$$u(c_t) = \frac{c_t^{1-\sigma}}{1-\sigma}$$

Note that when $\sigma = 1$, then this utility is $u(c_t) = \ln(c_t)$. The Euler equation is then:

$$c_t = (\beta R)^{-\frac{1}{\sigma}} c_{t+1}$$

Replacing consumption out of the budget constraint:

$$w_t - \frac{w_{t+1}}{R} = (\beta R)^{-\frac{1}{\sigma}} \left(w_{t+1} - \frac{w_{t+2}}{R} \right)$$

This equation can be written as:

$$R w_t - [1 + (\beta R)^{-\frac{1}{\sigma}}] w_{t+1} + (\beta R)^{-\frac{1}{\sigma}} w_{t+2} = 0$$

with boundary conditions:

$$w_0 > 0 \text{ given}, \ \lim_{t\to\infty} \frac{w_{t+1}}{R^{t+1}} = 0$$

e) Solve the equation you derived in (d). Find then the solutions for w_{t+1} and c_t. How does the behavior of w_{t+1} and c_t depend on whether $\beta(1+r)$ is higher or lower than one? (10)

Answer

The characteristic equation is:

$$(\beta R)^{-\frac{1}{\sigma}} \lambda^2 - [1 + (\beta R)^{-\frac{1}{\sigma}}] \lambda + R = 0$$

This equation can be solved analytically to get:

$$\lambda_1 = R, \ \lambda_2 = (\beta R)^{\frac{1}{\sigma}}$$

Note that $\lambda_1 > 1$ but $\lambda_2 \gtreqless 1$

The general solution for the equation is:

$$w_t = A \lambda_1^t + B \lambda_2^t$$

The initial value for w_0 gives:

$$w_0 = A + B$$

Also, the general solution can be used to obtain:

$$\frac{w_{t+1}}{R^{t+1}} = \frac{AR^{t+1}}{R^{t+1}} + B\frac{(\beta R)^{\frac{t+1}{\sigma}}}{R^{t+1}} = A + B\left(\beta R^{1-\sigma}\right)^{\frac{t+1}{\sigma}}$$

Then if the *transversality condition* is to be satisfied it must be the case that $A = 0$ and:

$$\beta R^{1-\sigma} < 1$$

Note that when $\sigma = 1$ this is satisfied. Since $w_0 = B$, the solution is:

$$w_t = w_0(\beta R)^{\frac{t}{\sigma}} \rightarrow w_{t+1} = w_0(\beta R)^{\frac{t+1}{\sigma}}$$

Replacing these values in the budget constraint:

$$c_t = \frac{R - (\beta R)^{\frac{1}{\sigma}}}{R}(\beta R)^{\frac{t}{\sigma}} w_0$$

f) Now solve the model using diagonalization. That is, cast the system in the form:

$$z_{t+1} = Dz_t,\ z_t = V^{-1}y_t$$

where: $y_t = [w_t\ c_t]^T$ (T here denotes transpose). And solve the model finding the policy functions for w_{t+1} and c_t. Show that you get the exact same solution as in e). (15)

Answer

The budget constraint and the Euler equation can be written as:

$$\begin{aligned} w_{t+1} &= Rw_t - Rc_t \\ c_{t+1} &= (\beta R)^{\frac{1}{\sigma}} c_t \end{aligned}$$

This can be written as:

$$\underbrace{\begin{bmatrix} w_{t+1} \\ c_{t+1} \end{bmatrix}}_{y_{t+1}} = \underbrace{\begin{bmatrix} R & -R \\ 0 & (\beta R)^{\frac{1}{\sigma}} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} w_t \\ c_t \end{bmatrix}}_{y_t}$$

We can find analytically the eigenvalues and eigenvectors of the matrix A:

$$\lambda_1 = (\beta R)^{\frac{1}{\sigma}},\ \lambda_2 = R,\ V = \begin{bmatrix} \frac{R}{R-(\beta R)^{\frac{1}{\sigma}}} & 1 \\ 1 & 0 \end{bmatrix}$$

These matrices will satisfy:

$$V^{-1}AV = D, \text{ where } D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

And we know that: $z_{t+1} = Dz_t$. Hence:

$$\begin{aligned} z_{1t+1} = \lambda_1 z_{1t} \rightarrow z_{1t} = \lambda_1^t z_{10} \\ z_{2t+1} = \lambda_2 z_{2t} \rightarrow z_{2t} = \lambda_2^t z_{20} \end{aligned}$$

Note now that $z_t = V^{-1}y_t$ implies $Vz_t = y_t$. Then this expression is:

$$\begin{aligned} \lambda_1^t z_{10} v_{11} + \lambda_2^t z_{20} v_{12} = w_t \\ \lambda_1^t z_{10} v_{21} + \lambda_2^t z_{20} v_{22} = c_t \end{aligned}$$

Which is the same as:

$$\begin{aligned} \lambda_1^t z_{10} v_{11} + \lambda_2^t z_{20} &= w_t \\ \lambda_1^t z_{10} &= c_t \end{aligned}$$

Taking the first of these equations:

$$\begin{aligned} \frac{w_{t+1}}{R^{t+1}} &= \frac{(\beta R)^{\frac{t+1}{\sigma}}}{R^{t+1}} z_{10} v_{11} + \frac{R^{t+1}}{R^{t+1}} z_{20} \\ \lambda_1^t z_{10} &= c_t \end{aligned}$$

So, under the same considerations as before, if the *transversality condition* is to be satisfied, we got to have:

$$z_{20} = 0$$

Therefore "solutions" are:

$$\begin{aligned} w_t &= (\beta R)^{\frac{t}{\sigma}} z_{10} v_{11} \\ c_t &= (\beta R)^{\frac{t}{\sigma}} z_{10} \end{aligned}$$

What is z_{10}?... Note that: $w_0 = z_{10} v_{11}$, then solutions are:

$$w_t = w_0 (\beta R)^{\frac{t}{\sigma}} \rightarrow w_{t+1} = w_0 (\beta R)^{\frac{t+1}{\sigma}}$$

And consumption is:

$$c_t = \frac{w_0}{v_{11}} (\beta R)^{\frac{t}{\sigma}} \rightarrow c_t = \frac{R - (\beta R)^{\frac{1}{\sigma}}}{R} (\beta R)^{\frac{t}{\sigma}} w_0$$

Which is the exact same solution as before.

An alternative route

Note that because $z_{20} = 0$, we have that $z_{2t} = 0$ for all t. Then from $z_{t+1} = D z_t$ we have:

$$z_{1t+1} = \lambda_1 z_{1t} \tag{1a}$$

$$z_{2t+1} = \lambda_2 z_{2t} \tag{1b}$$

Note that:

$$V^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -\frac{R}{R - (\beta R)^{\frac{1}{\sigma}}} \end{bmatrix}$$

Then (1) delivers:

$$\begin{aligned} c_{t+1} &= (\beta R)^{\frac{1}{\sigma}} c_t \\ w_{t+1} - \frac{R}{R - (\beta R)^{\frac{1}{\sigma}}} c_{t+1} &= R \left(w_t - \frac{R}{R - (\beta R)^{\frac{1}{\sigma}}} c_t \right) \end{aligned}$$

The first equation just replicates the Euler equation, the second is satisfied since $z_{2t} = 0$ (due to transversality condition), this means:

$$w_t = \frac{R}{R - (\beta R)^{\frac{1}{\sigma}}} c_t \rightarrow c_t = \frac{R - (\beta R)^{\frac{1}{\sigma}}}{R} w_t$$

Which is the policy function for consumption and using the budget constraint we get:

$$w_{t+1} = Rw_t - \frac{R - (\beta R)^{\frac{1}{\sigma}}}{R} w_t \to w_{t+1} = (\beta R)^{\frac{1}{\sigma}} w_t$$

Which is the policy function for next period cake.

3. The Solow Model. (70)

A very well-known basic model in economics is the Solow model. The model can be described with a single (fundamental) equation:

$$\dot{k} = sAk^{\alpha} - (n_p + \delta)k \tag{2}$$

Where s is the (fixed) savings rate. α is the share of capital in the Cobb-Douglas production function, A is the productivity parameter, n_p is the rate of population growth, and δ is the depreciation rate.

Equation (2) is a differential equation which has an analytical solution! Solution is given by:

$$k(t) = \left[\left(k_0^{1-\alpha} - \frac{sA}{n_p + \delta} \right) e^{-(1-\alpha)(n_p+\delta)t} + \frac{sA}{n_p + \delta} \right]^{\frac{1}{1-\alpha}} \tag{3}$$

For a given initial value of capital: $k(0) = k_0$.

In this question however, we will use Projection Methods (Collocation) to solve this function on the range $[0, T]$, and we will use (3) to judge the quality of the numerical approximate solution.

Assume that you approximate the solution $k(t)$ to (2) by a function $\hat{k}$ of simple polynomials:

$$\hat{k} = k_0 + \sum_{i=1}^{n} a_i t^i \tag{4}$$

a) Write down the function $R(t; \mathbf{a})$, where the vector $\mathbf{a} = [a_1, a_2, ..., a_n]$. (20)

Answer.

In this case, the operator that we want to be zero is:

$$\mathcal{N}(k(t)) = \dot{k} - sAk(t)^{\alpha} + (n_p + \delta)k(t)$$

With the approximation $\hat{k}(t)$, then the residual function is:

$$R(t, \mathbf{a}) = \sum_{i=1}^{n} i a_i t^{i-1} - sA \left(k_0 + \sum_{i=1}^{n} a_i t^i \right)^{\alpha} + (n_p + \delta) \left(k_0 + \sum_{i=1}^{n} a_i t^i \right) \tag{5}$$

b) Use the collocation method with $n = 6$ to find the approximate solution. For this, use the following values for the parameters:

$$A = 1, s = 0.3, \alpha = 0.36, \delta = 0.08, n_p = 0.02, k(0) = 0.01, T = 100$$

For the $n = 6$ nodes you need for the collocation method, use the zeros of the nth Chebychev polynomial. Show the values $\mathbf{a}$ that you find with the method. (40)

Answer.

We generate the 6 nodes of approximation using the formula (the zeros of the 6th degree Chebychev polynomial):

$$z_\ell \equiv -\cos\left(\frac{(2\ell-1)\pi}{2n}\right),\ \ell = 1, 2, ..., 6$$

In order to use them in the range required ($[0, T]$), we use the transformation:

$$t_\ell \equiv t(z_\ell) = \frac{(z_\ell + 1)(T - 0)}{2} + 0,\ z_\ell \in [-1, 1]$$

Then we can use (5) to evaluate in these points: $R(t_\ell, a) = 0,\ \ell = 1, 2, ..., 6$. Which leads to a system of 6 equations in the unknowns $[a_1, a_2, ..., a_6]$. Using any software (or own coding) we can find:

$$a^* = [0.144, 0.005406, -0.00030584, 5.472e-06, -4.329e-08, 1.286e-10]$$

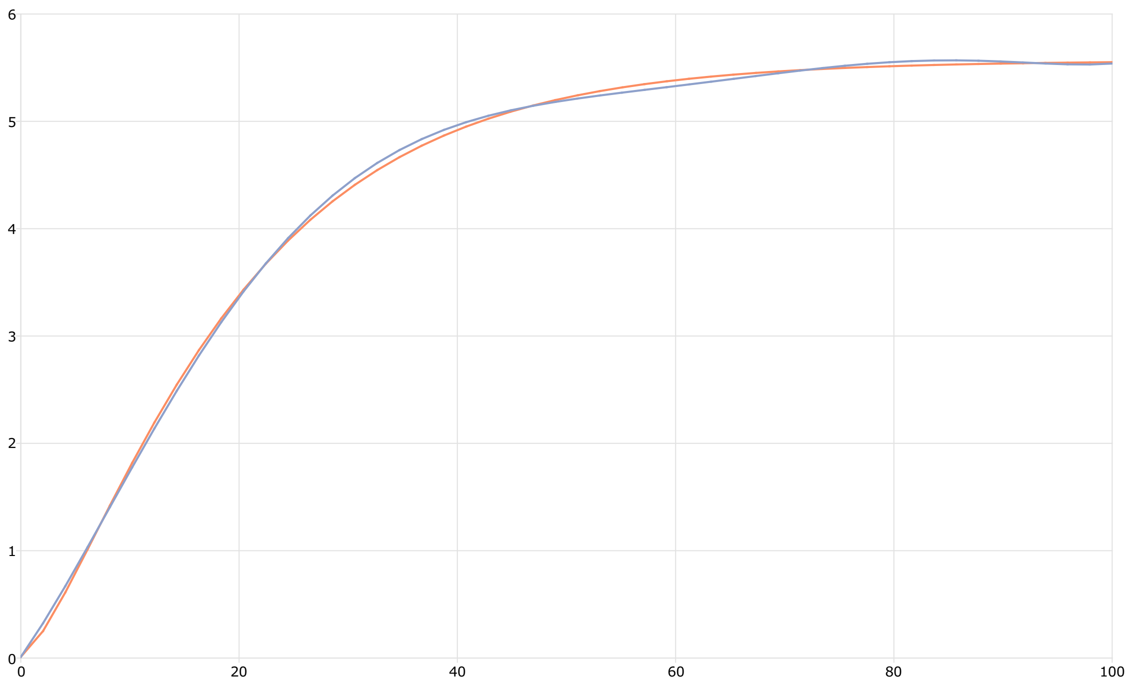

Figure 1. Solution to Solow

Results can be see in figure 1. The orange path is the solution using the approximation.

c) Compare the approximated solution with the analytical solution. (10)

Answer.

The comparison can be seen in figure 1. The blue path is the analytical solution in (3).