ECMA 31140. PERSPECTIVES ON COMPUTATIONAL MODELING FOR ECONOMICS
PROBLEM SET 1
DUE DATE: JANUARY 14TH

Instructor: Sergio Salas ssalas@uchicago.edu

1. The zero of an arbitrary function (30):

Consider the following function:

$$f(x) = (x - 10)e^{-x^2} + 5$$

The objective if to find the zero of the function, that is the value x^* such that $f(x^*) = 0$. Let us restrict the problem to finding solutions for positive values of x.

a) Write up a code yourself to find the zero of the function implementing the Bisection method. (15)
The numerical solution is 0.7821. (Note: we can bracket the result between 0 and 10.)

b) Write up a code yourself to find the zero of the function implementing the Newton-Raphson method. Consider different starting values, assess how reliable is this algorithm in finding the solution. Is the Bisection method better? Discuss. (15)
Newton-Raphson method relies on the first-order Taylor approximation. However, the first-order Taylor approximation is only a good approximation at the neighborhood of the real solution. And if our initial values vary to far away from it, it will return a "bad" result.

- $x_1 = 2$ The result is negative infinity.
- $x_1 = 0.5$ The result is 0.7821.
- $x_1 = 0$ The result is negative infinity.

2. Equilibrium interest rate. (50)

Assume two large groups of individuals (each of the same unit mass or measure[1]). They are identical within the group but different between groups. Let $j = a, b$ index individuals of type a and b respectively. Each live only two periods and they have endowments in each period as follows:

$$y_{i0}^j = y_0, y_{i1}^j = y_1, \text{ with } 2y_0 = y_1$$

They maximize:

$$\sum_{t=0}^{1} \beta^t u^j(c_{it}^j)$$

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[1]If you are not familiar with the concept of a measure, think that agents are distributed according to a uniform distribution over the interval $[0, 1]$. And we use the subindex i to label individuals in each group.

subject to:

$$c_{i0}^j + b_i^j = y_0,\ c_{i1}^j = y_1 + b_i^j(1+r)$$

where r is the real interest rate.

Assume:

$$u^a(c_{it}^a) = \frac{(c_{it}^a)^{1-\sigma_a}}{1-\sigma_a},\ u^b(c_{it}^b) = \frac{(c_{it}^b)^{1-\sigma_b}}{1-\sigma_b}$$

and $\beta = 0.95$.

a) Define the Competitive Equilibrium of the Economy. (10)

A competitive equilibrium of the economy are values $\{c_{i0}^a, c_{i1}^a\}$, $\{c_{i0}^b, c_{i1}^b\}$, $\{b_i^a, b_i^b\}$, and an interest rate r, such that:

- Taking as given r, $\{c_{i0}^a, c_{i1}^a\}$, $\{c_{i0}^b, c_{i1}^b\}$, $\{b_i^a, b_i^b\}$ maximize utility subject to the constraints.
- The bond market clears:

$$\int b_i^a di + \int b_i^b di = 0$$

- The goods market clears:

$$\int c_{io}^a di + \int c_{io}^b di = 2y_0,\ \int c_{i1}^a di + \int c_{i1}^b di = 2y_1$$

b) Find the equilibrium r for each combination of σ_a, σ_b where each belong to a set of evenly spaced values starting in 0.1 and ending in 2, with size 100. (For this you need to compute b^a and b^b, where these values are the aggregates across individuals, for example $b^a = \int b_i^a di$.) (20)

For individual agents in both groups, they need to maximize:

$$\sum_{t=0}^{1} \beta^t u^j(c_{it}^j)$$

And the budget constraint is:

$$y_0 = c_{i0}^j + b^j \qquad c_{i0}^j = y_1 + b^j(1+r)$$

Sorting out the equations above, we shall have:

$$c_{i0}^j + \frac{c_{i1}^j}{1+r} = y_0 + \frac{y_1}{1+r}$$

Then for agents in both groups, we can have Lagrangian:

$$L = \sum_{t=0}^{1} \beta^t u^j(c_{it}^j) + \lambda(y_0 + \frac{y_1}{1+r} - c_{i0}^j - \frac{c_{i1}^j}{1+r})$$

Now, think of an agent in group A, with utility function:

$$u_i^a = \frac{c^{1-\sigma_a}}{1-\sigma_a}$$

We shall have the first-order condition:

$$\frac{\partial L}{\partial c_{i0}} = (c_{i0}^a)^{-\sigma_a} - \lambda = 0 \qquad \frac{\partial L}{\partial c_{i1}} = \beta(c_{i1}^a)^{-\sigma_a} - \frac{\lambda}{1+r} = 0$$

Therefore, we shall have:

$$(\frac{c_{i,0}^a}{c_{i,1}^a})^{-\sigma_a} = \beta(1+r)$$

Now we consider the budget constraint and the fact that: $y_1 = 2y_0$, we shall have

$$(\frac{y_0 - b_i^a}{2y_0 + b_i^a(1+r)})^{-\sigma_a} = \beta(1+r)$$

And solving this we shall have:

$$b_i^a = \frac{[\beta(1+r)]^{\frac{1}{\sigma_a}} y_0 - 2y_0}{1 + r + [\beta(1+r)]^{\frac{1}{\sigma_a}}}$$

Note that with each group having measure 1, we shall have:

$$b^a = \int_0^1 b_i^a di = \frac{[\beta(1+r)]^{\frac{1}{\sigma_a}} y_0 - 2y_0}{1 + r + [\beta(1+r)]^{\frac{1}{\sigma_a}}}$$

A similar derivation can be obtained for b^b.

Eventually, we will have $b^a + b^b$, which is an equation of r and we can solve $b^a + b^b = 0$ by bisection method. The result is shown in Figure 1.

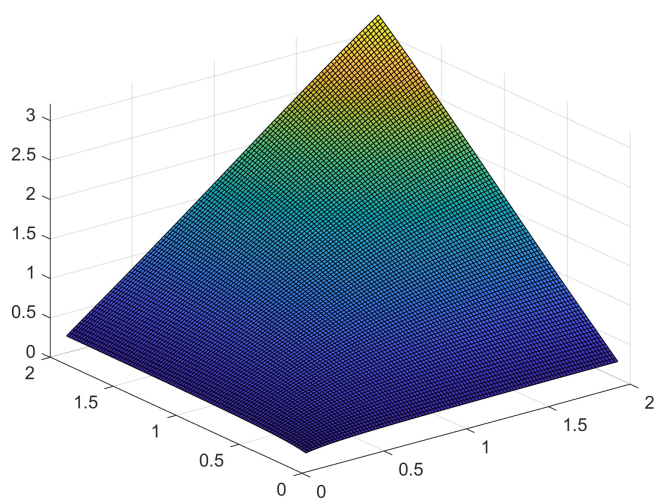
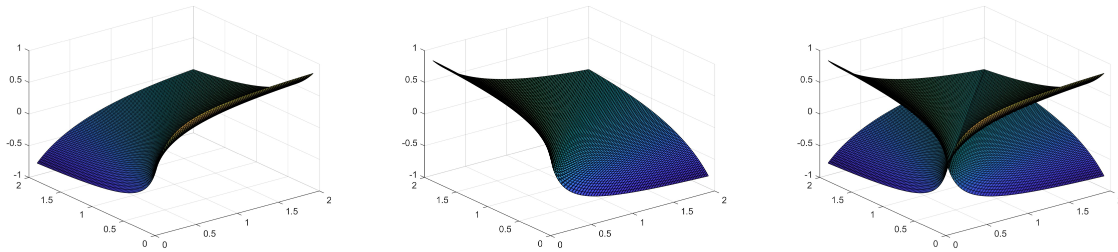


FIGURE 1. Equilibrium Interest Rate of the Model

c) Assume now that $y_0 = 1$. For the same each combination of σ_a, σ_b as before, plot bonds b^a and b^b. Explain the intuition of your results. (20)

Note that in equation (11)-(13), we already obtained b^a and b^b conditional on r and y_0. Since $y_0 = 1$ and the bond market is in equilibrium (at the equilibrium interest rate), we shall plug in the equilibrium interest rate and y_0, then we obtained:



(a) Bond A (b) Bond B (c) Bond A and B

Figure 2. Bond Restuls

3. The Neoclassical Growth Model (NGM). (30)

In class, we have seen that the NGM with finite time and inelastic labor supply delivers the following second order difference equation:

$$u'(f(k_t) - k_{t+1}) = \beta u'(f(k_{t+1}) - k_{t+2})f'(k_{t+1}),\ t = 0, 1, ..., T - 1.$$

with: $k_0 > 0,\ k_{T+1} = 0$

Let utility function be of the CRRA type:

$$u(c_t) = \frac{c_t^{1-\sigma}}{1-\sigma},\ \sigma > 0$$

Let the production function be:

$$F(k_t, n_t) = k_t^{\alpha} n_t^{1-\alpha},\ 0 < \alpha < 1$$

Recall:

$$f(k_t) = k_t^{\alpha} + (1-\delta)k_t$$

Consider $T = 100$ and assume that $k_0 = 0.1$.
We have a system of 100 equations of 100 variables:

$$\begin{aligned}
&u^{'}(f(k_0) - k_1) - \beta u^{'}(f(k_1) - k_2)f^{'}(k_1) = 0\\
&u^{'}(f(k_1) - k_2) - \beta u^{'}(f(k_2) - k_3)f^{'}(k_2) = 0\\
&u^{'}(f(k_2) - k_3) - \beta u^{'}(f(k_3) - k_4)f^{'}(k_3) = 0\\
&\cdots\\
&u^{'}(f(k_{99}) - k_{100}) - \beta u^{'}(f(k_{100}) - k_{101})f^{'}(k_{100}) = 0
\end{aligned}$$

We note that $k_0 = 0.1$ and $k_{101} = 0$. Therefore, we only need to solve the vector $\vec{K} = (k_1, k_2, \cdots, k_{100})$.

Also, noting that we have CRRA preference, we shall have:

$$(0.1^\alpha + (1-\delta)0.1 - k_1)^{-\sigma} - \beta(k_1^\alpha + (1-\delta)k_1 - k_2)^{-\sigma}(\alpha k_1^{\alpha-1} + (1-\delta)) = 0$$
$$(k_1^\alpha + (1-\delta)k_1 - k_2)^{-\sigma} - \beta(k_2^\alpha + (1-\delta)k_2 - k_3)^{-\sigma}(\alpha k_2^{\alpha-1} + (1-\delta)) = 0$$
$$(k_2^\alpha + (1-\delta)k_2 - k_3)^{-\sigma} - \beta(k_3^\alpha + (1-\delta)k_3 - k_4)^{-\sigma}(\alpha k_3^{\alpha-1} + (1-\delta)) = 0$$
$$\ldots$$
$$(k_{99}^\alpha + (1-\delta)k_{99} - k_{100})^{-\sigma} - \beta(k_{100}^\alpha + (1-\delta)k_{100})^{-\sigma}(\alpha k_{100}^{\alpha-1} + (1-\delta)) = 0$$

a) Assume that $\sigma = 2, \alpha = 0.36, \beta = 0.98$ and $\delta = 0.025$. Solve the model using a non-linear equation solver. (20)
Plug in the parameters, we have:

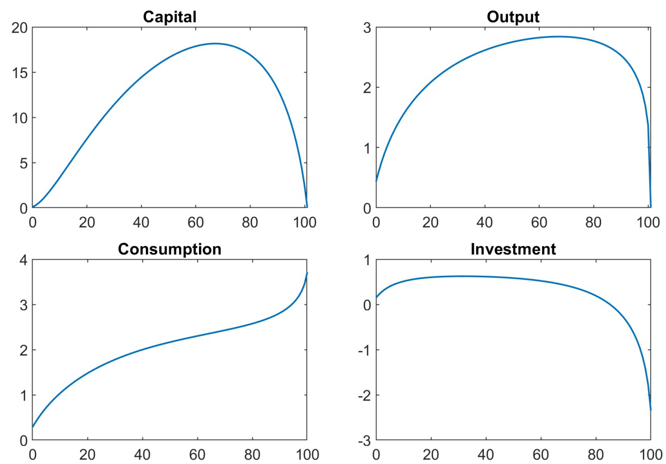


FIGURE 3. Dynamics of Neoclassical Growth Model

b) Assume now that $\sigma = 1, \alpha = 0.36, \beta = 0.98$ and $\delta = 1$. Solve the model using a non-linear equation solver. Show that the solution is the same as that given by the analytical solution for this case. (10) Plug in the parameters, we have:

It can be directly observed from the figure that analytical solution is the same as the numerical solution.

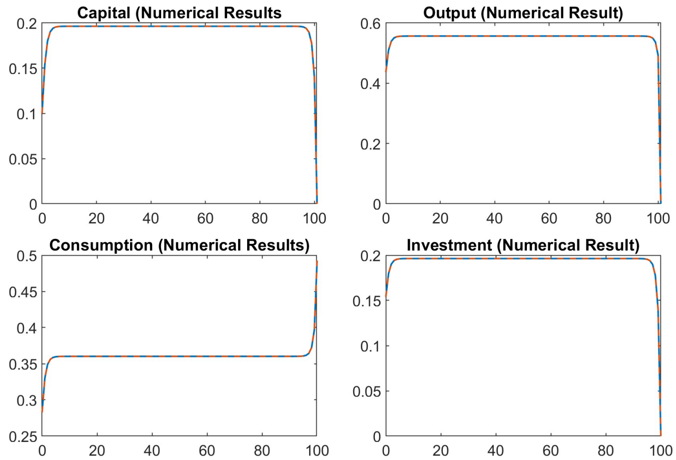


Figure 4. Dynamics of Neoclassical Growth Model