ECMA 31140. PERSPECTIVES ON COMPUTATIONAL MODELING FOR ECONOMICS
PROBLEM SET 5
DUE DATE: FEBRUARY 11TH

Instructor: Sergio Salas ssalas@uchicago.edu

You are encouraged to work and discuss in groups, but you must submit your work individually. Answers must be legibly hand-written or typed. All assignments are due electronically on Canvas, attach code. Assignments are due at 12:30 PM. Late problem sets *will not be accepted*

1. The stochastic Neoclassical Growth Model. (45 points)

Central planner uses Bellman optimality principle to solve the problem stated in class:

$$v(k,z) = \max_{c,k'} \{u(c) + \beta \mathbb{E}[v(k',z')|z]\}$$

subject to:

$$k' + c = f(k,z)$$

where $f(k,z) = zk^{\alpha}$ and $u(c) = \ln(c)$. z is a Markov process and $0 < \alpha, \beta < 1$. Guess the following value function:

$$v(k,z) = A(z) + B(z)\ln k$$

where $A(z)$ and $B(z)$ are arbitrary function of z.

a) Use the guess provided and take the FOC w.r.t. k' and c, using the Bellman equation of the problem. (10)

Answer.
Let us form the Lagrangian:

$$L = u(c) + \beta \mathbb{E}[A(z') + B(z')\ln k'|z] + \lambda(f(k,z) - c - k')$$

Taking the FOC's:

$$\frac{\partial L}{\partial c} = \frac{1}{c} - \lambda = 0$$
$$\frac{\partial L}{\partial k'} = \beta \mathbb{E}[\frac{B(z')}{k'}|z] - \lambda = 0$$
$$\frac{\partial L}{\partial \lambda} = f(k,z) - c - k' = 0$$

Date: February 11, 2025.

Moreover, by envelope theorem, we have:[1]

$$\frac{\partial V}{\partial k} = \frac{\partial L}{\partial k} = \lambda f_k$$

Then, we obtain:

$$\frac{\partial V}{\partial k} = \frac{B(z)}{k} = \frac{\alpha z k^{\alpha-1}}{c} = \lambda f_k$$

Combining terms:

$$c = \frac{\alpha z k^{\alpha}}{B(z)}$$

$$k' = \frac{B(z) - \alpha}{B(z)} z k^{\alpha}$$

b) Replace the expressions found in a) into the Bellman equation and equate coefficients on $\ln k$ to find an expression of $B(z)$ as a function of α, β and $\mathbb{E}[B(z')|z]$. (10)

Answer.

Replacing the expressions found in a) into the Bellman equation and sorting, we have

$$A(z) + B(z)\ln k = \ln(\frac{\alpha z}{B(z)}) + \beta\mathbb{E}[A(z') + B(z')\ln(z\frac{B(z)-\alpha}{B(z)})] + \alpha \ln k + \alpha\beta\mathbb{E}[B(z')]\ln k$$

Equating the coefficients on $\ln k$:

$$B(z) = \alpha + \alpha\beta\mathbb{E}[B(z')|z] \tag{1}$$

c) Assume now that z is i.i.d., use your expression derived in b) to find $\mathbb{E}B(z)$. (10)

Answer.

If z is i.i.d., it must be the case that:

$$\mathbb{E}[B(z')|z] = \mathbb{E}[B(z')] = \mathbb{E}[B(z)]$$

Then using (1), we should have:

$$B(z) = \alpha + \alpha\beta\mathbb{E}[B(z)] \tag{2}$$

Taking expectations on both sides, we have:

$$\mathbb{E}[B(z)] = \frac{\alpha}{1-\alpha\beta} \tag{3}$$

d) With the expression found in c) and your expressions found in a), state the policy functions for capital and next period consumption. (10)

Answer.

Note that the two equations (2) and (3) imply:

$$B(z) = \frac{\alpha}{1-\alpha\beta}$$

[1] We can avoid using the Envelope Theorem at this stage and just work with expressions above to find c' and k', we would obtain different but equivalent expressions.

So in this case we have found the unknown! (which actually does not depend on z). Then plugging the policy functions previously found:

$$c = \frac{\alpha z k^{\alpha}}{\frac{\alpha}{1-\alpha\beta}} = (1-\alpha\beta) z k^{\alpha}$$

$$k' = \frac{\frac{\alpha^2\beta}{1-\alpha\beta}}{\frac{\alpha}{1-\alpha\beta}} z k^{\alpha} = \alpha\beta z k^{\alpha}$$

e) Discuss if you can go with the same Guess and Verify procedure under the case that z is a Markov process (not an i.i.d. random variable). (5)

Answer.
The guess and verify procedure will be stalled once we get to the equation:

$$B(z) = \alpha + \alpha\beta\mathbb{E}[B(z')|z]$$

When z is a Markov process, then we can no longer expect: $\mathbb{E}[B(z')|z] = \mathbb{E}[B(z)]$.
We can take unconditional expectation on both sides to get:

$$\mathbb{E}[B(z)] = \mathbb{E}[\alpha + \alpha\beta\mathbb{E}[B(z')|z]] = \alpha + \alpha\beta\mathbb{E}[B(z')]$$

Where we use a version of the law of iterated expectations (also called law of total expectation): $\mathbb{E}[B(z')] = \mathbb{E}[\mathbb{E}[B(z')|z]]$. The expectation should correspond to the invariant distribution of z governed by the Markov process. Therefore we can conclude that:

$$\mathbb{E}[B(z)] = \frac{\alpha}{1-\alpha\beta}$$

However there is no way to find $B(z)$ (at least analytically) and the Guess and Verify method would not work.

2. Ergodic distribution for the capital stock in the stochastic NGM

In Question 1 (NGM model under logarithmic utility, full depreciation of capital, and productivity shocks z_t being i.i.d), you found that the policy function for the stock of capital is:

$$k_{t+1} = \alpha\beta z_t k_t^{\alpha} \tag{4}$$

$k_0 > 0$ given. Assume specifically that $\ln z_t \sim \mathcal{N}(0, \sigma^2)$

(a) Derive the distribution of the log of the capital stock for any moment of time. Denote the mean of that distribution with μ_t and the variance of that distribution with σ_t^2, show the exact analytical expressions for these values. (10)

Answer.
Since we have:

$$k_{t+1} = \alpha\beta z_t k_t^{\alpha}$$

Then, taking logs:

$$\ln k_{t+1} = \ln(\alpha\beta) + \alpha \ln k_t + \ln z_t \tag{5}$$

Let $y_t = \ln k_t$, $c = \ln \alpha\beta$ and $\epsilon_t = \ln z_t$. Then the equation can be written as:

$$y_{t+1} = c + \alpha y_t + \epsilon_t$$

Because ϵ_0 is normally distributed, conditional on y_0, we have the distribution of y_1 as:

$$y_1 \sim \mathcal{N}(c + \alpha y_0, \sigma^2)$$

To derive the distribution of y_2, note that:

$$y_2 = c + \alpha y_1 + \epsilon_1 = c + c\alpha + \alpha^2 y_0 + \alpha\epsilon_0 + \epsilon_1$$

Since ε_t is i.i.d, the sum of two of these random variables will be normally distributed with the sum of mean and the sum of variances. Therefore we conclude:

$$y_2 \sim \mathcal{N}(c + c\alpha + \alpha^2 y_0, \sigma^2 + \alpha^2\sigma^2)$$

To derive the distribution of y_3 note that:

$$y_3 = c + \alpha y_2 + \epsilon_1 = c + c\alpha + c\alpha^2 + \alpha^3 y_0 + \alpha^2\epsilon_0 + \alpha\epsilon_1 + \epsilon_2$$

Therefore the distribution of y_3 is given by:

$$y_3 \sim \mathcal{N}(c + c\alpha + c\alpha^2 + \alpha^3 y_0, \sigma^2 + \alpha^2\sigma^2 + (\alpha^2)^2\sigma^2)$$

The recursion should be clear by now. For an arbitrary period t:

$$y_t \sim \mathcal{N}\left(c(1 + \alpha + \alpha^2 + ... + \alpha^{t-1}) + \alpha^t y_0, \sigma^2(1 + \alpha^2 + (\alpha^2)^2 + ... + (\alpha^2)^{t-1})\right)$$

Therefore, we can write the distribution for y_t as:

$$y_t \sim \mathcal{N}(\underbrace{c\frac{1+\alpha^t}{1-\alpha} + \alpha^t y_0}_{\mu_t}, \underbrace{\sigma^2\frac{1+\alpha^{2t}}{1-\alpha^2}}_{\sigma_t^2})$$

where $c = \ln(\alpha\beta)$.

(b) Find the ergodic distribution for the log of capital, call the mean and variance of such distribution μ_∞ and σ_∞^2, respectively. Show the exact analytical expressions for these values.

Answer.
Taking the limit as $t \to \infty$ in the expressions above, we immediately get:

$$\mu_\infty = \frac{\ln(\alpha\beta)}{1-\alpha}, \;\; \sigma_\infty^2 = \frac{\sigma^2}{1-\alpha^2}$$

Note that these are the same values we would obtain if we take the unconditional mean and unconditional variance in the process (5):

$$\mathbb{E}\ln k_{t+1} = \ln(\alpha\beta) + \alpha \ln \mathbb{E}\ln k_t$$

Setting $\mu_\infty \equiv \mathbb{E}\ln k_t$ we obtain:

$$\mathbb{E}\ln k_t = \frac{\ln(\alpha\beta)}{1-\alpha}$$

Similarly for the unconditional variance:

$$\mathbb{V}\ln k_{t+1} = \alpha^2\mathbb{V}\ln k_t + \sigma^2$$

Setting $\sigma_\infty^2 \equiv \mathbb{V} \ln k_t$ we obtain:

$$\mathbb{V} \ln k_t = \frac{\sigma^2}{1-\alpha^2}$$

(c) What is the relationship between β and the mean of the ergodic distribution of capital? Can you give an intuitive explanation? (10)

Answer.
With:

$$\mu_\infty = \frac{\ln \alpha\beta}{1-\alpha}$$

It is clear that there is a positive relationship.

The intuitive explanation for this is that: β is the discount factor, indicating how patient an agent is. Higher β indicates the agent is more patient and value the utility from the future more. And the agent is more willing to save for the consumption in the future. As the result, the agent has higher saving rate and the mean of the invariant distribution of capital is also higher.

3. Ergodic distribution for the capital stock in the stochastic NGM, computational approach

Take again the solution for the stochastic NGM under the special case described in Question 2. The policy function is given by equation (4)

(a) Construct a finite state approximation for the Markov transition of the process $\ln k_t$. That is, find the vector for the log of capital stocks (call it if you want $\mathbf{k} = [\ln k_1, ..., \ln k_N]'$) and the transition matrix $\Pi = (p_{ij})$, using Tauchen's method. Use 3 standard deviations to approximate the lower and upper end of the grid ($m = 3$). Use $N = 20$ states of approximation. Use the following values for the parameters: $\alpha = 0.36$, $\beta = 0.90$ and $\sigma = 0.01$. (20)

Answer.
Using the code provided we can find the vector for capital:

$$\begin{bmatrix} 0.193 \\ 0.194 \\ 0.194 \\ 0.195 \\ 0.196 \\ 0.196 \\ 0.197 \\ 0.198 \\ 0.198 \\ 0.199 \\ 0.200 \\ 0.200 \\ 0.201 \\ 0.202 \\ 0.203 \\ 0.203 \\ 0.204 \\ 0.205 \\ 0.205 \\ 0.206 \end{bmatrix}$$

While the transition matrix (p_{ij}) is given by:

0.029	0.031	0.052	0.078	0.105	0.126	0.134	0.128	0.109	0.083	0.056	0.034	0.018	0.009	0.004	0.001	0.001	0.000	0.0
0.022	0.025	0.044	0.069	0.096	0.119	0.133	0.132	0.117	0.093	0.065	0.041	0.023	0.012	0.005	0.002	0.001	0.000	0.0
0.016	0.020	0.036	0.059	0.086	0.112	0.130	0.134	0.124	0.102	0.075	0.049	0.029	0.015	0.007	0.003	0.001	0.000	0.0
0.012	0.016	0.030	0.050	0.076	0.103	0.125	0.134	0.129	0.111	0.085	0.058	0.036	0.019	0.009	0.004	0.002	0.001	0.0
0.009	0.012	0.024	0.042	0.067	0.094	0.118	0.132	0.133	0.119	0.095	0.068	0.043	0.024	0.012	0.006	0.002	0.001	0.0
0.006	0.009	0.019	0.035	0.057	0.084	0.110	0.129	0.134	0.125	0.104	0.077	0.051	0.030	0.016	0.008	0.003	0.001	0.0
0.004	0.007	0.015	0.028	0.048	0.074	0.101	0.123	0.134	0.130	0.113	0.087	0.060	0.037	0.020	0.010	0.004	0.002	0.0
0.003	0.005	0.011	0.023	0.040	0.064	0.092	0.116	0.132	0.133	0.120	0.097	0.070	0.045	0.026	0.013	0.006	0.002	0.0
0.002	0.004	0.009	0.018	0.033	0.055	0.082	0.108	0.128	0.134	0.126	0.106	0.080	0.053	0.032	0.017	0.008	0.003	0.0
0.001	0.003	0.006	0.014	0.027	0.047	0.072	0.099	0.122	0.134	0.131	0.115	0.089	0.062	0.039	0.022	0.011	0.005	0.0
0.001	0.002	0.005	0.011	0.022	0.039	0.062	0.089	0.115	0.131	0.134	0.122	0.099	0.072	0.047	0.027	0.014	0.006	0.0
0.001	0.001	0.003	0.008	0.017	0.032	0.053	0.080	0.106	0.126	0.134	0.128	0.108	0.082	0.055	0.033	0.018	0.009	0.0
0.000	0.001	0.002	0.006	0.013	0.026	0.045	0.070	0.097	0.120	0.133	0.132	0.116	0.092	0.064	0.040	0.023	0.011	0.0
0.000	0.001	0.002	0.004	0.010	0.020	0.037	0.060	0.087	0.113	0.130	0.134	0.123	0.101	0.074	0.048	0.028	0.015	0.0
0.000	0.000	0.001	0.003	0.008	0.016	0.030	0.051	0.077	0.104	0.125	0.134	0.129	0.110	0.084	0.057	0.035	0.019	0.0
0.000	0.000	0.001	0.002	0.006	0.012	0.024	0.043	0.068	0.095	0.119	0.133	0.132	0.118	0.094	0.067	0.042	0.024	0.0
0.000	0.000	0.001	0.002	0.004	0.009	0.019	0.036	0.058	0.085	0.111	0.129	0.134	0.125	0.103	0.076	0.050	0.030	0.0
0.000	0.000	0.000	0.001	0.003	0.007	0.015	0.029	0.049	0.075	0.102	0.124	0.134	0.130	0.112	0.086	0.059	0.036	0.0
0.000	0.000	0.000	0.001	0.002	0.005	0.012	0.023	0.041	0.065	0.093	0.117	0.132	0.133	0.119	0.096	0.069	0.044	0.0
0.000	0.000	0.000	0.001	0.001	0.004	0.009	0.018	0.034	0.056	0.083	0.109	0.128	0.134	0.126	0.105	0.078	0.052	0.0

(b) Once you have Π, find the ergodic distribution of capital π that would satisfy:

$$\pi'\Pi = \pi', \ \sum_{i=1}^{n} \pi_i = 1$$

Plot the ergodic distribution of capital. (20)

Answer.
Using any methods discussed in class you should obtain:

$$\begin{bmatrix} 0.002 \\ 0.004 \\ 0.008 \\ 0.016 \\ 0.028 \\ 0.046 \\ 0.068 \\ 0.092 \\ 0.112 \\ 0.124 \\ 0.124 \\ 0.112 \\ 0.092 \\ 0.068 \\ 0.046 \\ 0.028 \\ 0.016 \\ 0.008 \\ 0.004 \\ 0.002 \end{bmatrix}$$

And the ergodic distribution shall be in the figure below:

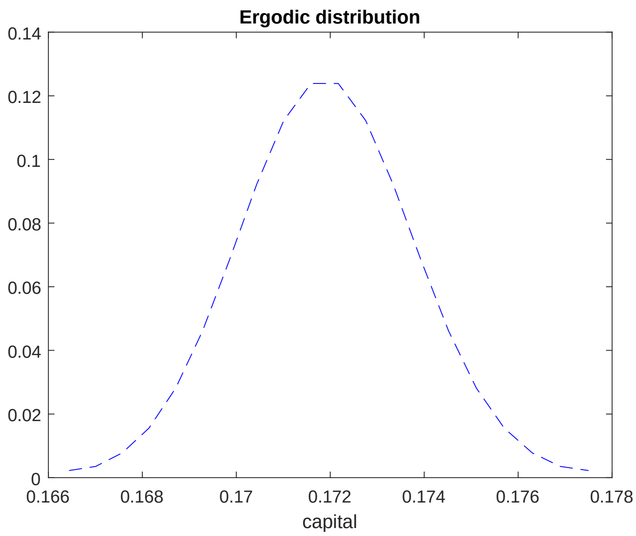

FIGURE 1. Ergodic Distribution for k_t

(c) Compute the expected value for the stock of capital in this economy using π. Find the expected value for $\beta = 0.95$, $\beta = 0.97$ and $\beta = 0.99$. Is there a monotonic relationship? How do these expected values compare with μ_∞ computed in question 2? (10)

Answer.

To obtain the expected value, we shall take the sum of the grid value times π, which can be written as:

$$\mathbb{E}k = \sum_{i=1}^{n} \text{grid}_i * \pi_i$$

Also note that in question 2, we have obtained the analytical solution of expectation as:

$$\mu_t = \frac{\log(\alpha\beta)}{1-\alpha}$$

Then, under different β, the results are shown in Table 1:

TABLE 1. Comparison of Result

β	Analytical	With Tauchen's
0.95	-1.67647	-1.67647
0.97	-1.64392	-1.64392
0.99	-1.61203	-1.61203