ECMA 31140. PERSPECTIVES ON COMPUTATIONAL MODELING FOR ECONOMICS
PROBLEM SET 3
DUE DATE: JANUARY 28TH

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You are encouraged to work and discuss in groups, but you must submit your work individually. Answers must be legibly hand-written or typed. All assignments are due electronically on Canvas, attach code. Assignments are due at 12:30 PM. Late problem sets *will not be accepted*

1. The Neoclassical Growth Model I. (100)

A Central planner uses a Bellman equation to state the problem as:

$$v(k) = \max_{c,k'} [u(c) + \beta v(k')]$$

subject to:

$$k' + c = F(k,1) + (1-\delta)k$$

a) Assume $u(c) = \ln(c)$, $\delta = 1$ and $F(k,1) = k^{\alpha}$. Use a guess and verify method to find the value function and the associated policy functions. (30)

Answer.

Guess on Value function (it "inherits" utility function properties):

$$v(k) = A + B\ln k$$

Using guess on first order condition:

$$\frac{1}{k^{\alpha} - k'} = \beta \frac{B}{k'} \to k' = \frac{\beta B k^{\alpha}}{1 + \beta B}$$

Finding consumption from resource constraint:

$$c = \frac{1}{1+\beta B} k^{\alpha}$$

Note that Bellman equation is now:

$$A + B\ln k \quad = \quad \ln \frac{k^{\alpha}}{1+\beta B} + \beta \left[A + B \ln \frac{\beta B k^{\alpha}}{1+\beta B} \right]$$

Equating coefficients on $\ln k$ (A is an unimportant constant):

$$B = \alpha (1 + \beta B)$$

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And then:

$$B = \frac{\alpha}{1 - \alpha\beta}$$

Finally we can find the policy functions, for capital:

$$k' = \alpha\beta k^{\alpha}$$

And for consumption:

$$c = k^{\alpha} - \alpha\beta k^{\alpha} = (1 - \alpha\beta)k^{\alpha}$$

Note that the complicated problem has a simple solution! each period consume a fraction of income and save the rest. Note that Recursive representation is like a 2 period problem, its solution applies to ANY adjacent periods in the sequence problem:

$$\begin{aligned} k_{t+1} &= \alpha\beta k_t^{\alpha}, \ k_0 \text{ given}, \ t = 0, 1, 2, ... \\ c_t &= (1 - \alpha\beta)k_t^{\alpha} \ t = 0, 1, 2, ... \end{aligned}$$

b) Now use the following functional forms and parameters:

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma}, \ \sigma = 2 \ F(k, 1) = k^{\alpha}, \ \alpha = 0.4, \ \delta = 0.03, \ \beta = 0.95$$

Assume a range of approximation $[k_{min} = 0.1\bar{k}, k_{max} = 1.2\bar{k}]$, where $\bar{k}$ is the *steady state* stock of capital. Use the Orthogonal Collocation method to find (an approximation of) the policy function for the stock of capital $k' \equiv \hat{g}(k, \mathbf{a})$. You may want to start with the approximation under $\sigma = \delta = 1$, where you know the solution. Use at least $n = 7$ nodes for approximation. (70)

Answer.

This exercise follows closely the case developed in class. (please see the slides). With the parametrization given, the following vector of parameters have been obtained:

$$\mathbf{a} = [1.0538, -0.0885, 0.0405, -0.0196, 0.0097, -0.0049, 0.0026, -0.0012, 0.0008, -0.0005]$$

Figure 1 shows the solution along with the 45 degree line.

We can also find the policy function for consumption, using the resource constraint:

$$\hat{c}(k, \mathbf{a}) = f(k) - \hat{g}(k, \mathbf{a})$$

The resulting approximation can be seen in figure 2

2. The Neoclassical Growth Model II. (110)

In this problem, you will solve for the NGM as we examined in class. A Central planner uses a Bellman equation to state the problem as:

$$v(k) = \max_{c,k'} [u(c) + \beta v(k')]$$

subject to:

$$k' + c = F(k, 1) + (1 - \delta)k$$

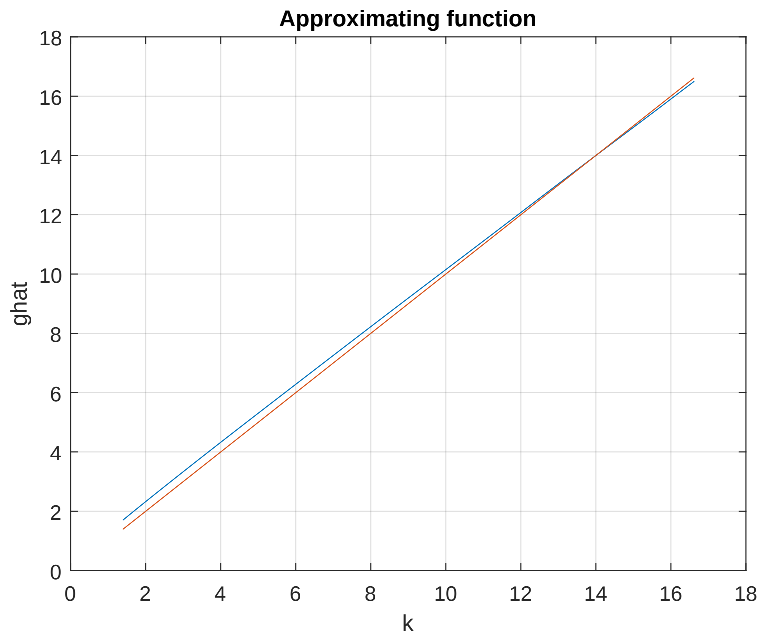

FIGURE 1. $\hat{g}(k, \mathbf{a})$ and 45°

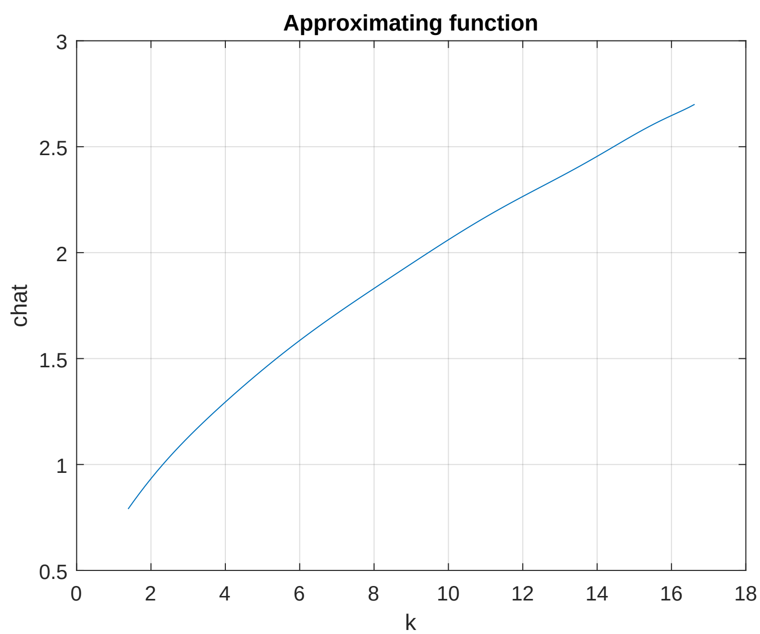

FIGURE 2. $\hat{c}(k, \mathbf{a})$

Throughout the problem, use the following functional forms and parameters:

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma}, \ \sigma = 2 \ F(k,1) = k^{\alpha}, \ \alpha = 0.36, \ \delta = 0.025, \ \beta = 0.95$$

a) Find the Euler equation. (10)

Answer.

The value function is:

$$v(k) = \max_{c,k'}[u(c) + \beta v(k')] \quad s.t. \quad k' + c = F(k,1) + (1-\delta)k \equiv f(k)$$

The Lagrangian is:

$$L = u(c) + \beta v(k') + \lambda\left[f(k) - c - k'\right]$$

Then, FOC w.r.t c and k':

$$\begin{aligned} u'(c) &= \lambda \\ \beta v'(k') &= \lambda \end{aligned}$$

The we have:

$$u'(c') = \beta v'(k')$$

The Envelope condition is $v'(k) = \lambda f'(k)$, then:

$$v'(k') = u'(c')f'(k')$$

Therefore, we have Euler equation:

$$u'(c) = \beta u'(c')f'(k')$$

Given $f(k) = F(k,1) + (1-\delta)k$, $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$. This is:

$$\left(\frac{c}{c'}\right)^{-\sigma} = \beta[\alpha(k')^{\alpha-1} + 1 - \delta]$$

b) Find the steady state of the model. (10)

At steady state:

$$c = c' = \bar{c} \quad k = k' = \bar{k}$$

From the Euler equation and the resource constraint:

$$\bar{k} = \left(\frac{\alpha\beta}{1-\beta(1-\delta)}\right)^{\frac{1}{1-\alpha}}, \; \bar{c} = \bar{k}^{\alpha} - \delta\bar{k}$$

Using the parameters given we have $\bar{k} = 10.991$, $\bar{c} = 2.095$

c) Find a linear approximation around the steady state of both the Euler equation and the feasibility constraint, write the system in the form:

$$y_{t+1} = Ay_t$$

(20)

Answer.

The resource constraint and Euler equation are:

$$k_{t+1} + c_t - f(k_t) = 0$$

$$u'(c_t) - \beta u'(c_{t+1})f'(k_{t+1}) = 0$$

Doing a first order Taylor approximation around steady state, we have:

$$\hat{k}_{t+1} - f'(k^*)\hat{k}_t + \hat{c}_t = 0$$

$$u''(c^*)\hat{c}_t - u''(c^*)\hat{c}_{t+1} - \beta u'(c^*)f''(k^*)\hat{k}_{t+1} = 0$$

Where we used the fact that $\beta f'(\bar{k}) = 1$.

We can write both equations as:

$$\hat{k}_{t+1} = f'(\bar{k})\hat{k}_t - \hat{c}_t$$
$$\hat{c}_{t+1} = \hat{c}_t - \beta\frac{u'(\bar{c})f''(k^*)}{u''(\bar{c})}\hat{k}_{t+1} = \hat{c}_t - \beta\frac{u'(\bar{c})f''(\bar{k})}{u''(\bar{c})}(f'(\bar{k})\hat{k}_t - \hat{c}_t)$$

And stack up $\hat{k}_t$ and $\hat{c}_t$ to get:

$$\begin{bmatrix}\hat{k}_{t+1}\\ \hat{c}_{t+1}\end{bmatrix} = \begin{bmatrix} f' & -1 \\ -\frac{u'f''}{u''}f'\beta & 1+\frac{\beta u'f''}{u''}\end{bmatrix}\begin{bmatrix}\hat{k}_t\\ \hat{c}_t\end{bmatrix} = \begin{bmatrix} \frac{1}{\beta} & -1 \\ -\frac{u'f''}{u''} & 1+\frac{\beta u'f''}{u''}\end{bmatrix}\begin{bmatrix}\hat{k}_t\\ \hat{c}_t\end{bmatrix}$$

Note that: $u'(c) = c^{-\sigma}$, $u''(c) = -\sigma c^{-\sigma-1}$, $f''(k) = \alpha(\alpha-1)k^{\alpha-2}$, and we already solve that $k^* = 10.991$ and $c^* = 2.095$. We shall have:

$$u' = 0.2278 \quad u'' = -0.2174 \quad f'' = -0.0045$$

Plug in the value, our matrix shall be:

$$\begin{bmatrix}\hat{k}_{t+1}\\ \hat{c}_{t+1}\end{bmatrix} = \begin{bmatrix} 1.0526 & -1.0000 \\ -0.0047 & 1.0045\end{bmatrix}\begin{bmatrix}\hat{k}_t\\ \hat{c}_t\end{bmatrix}$$

d) Use Matlab (or any other software) to find the eigenvectors and eigenvalues of A. Check that eigenvalue λ_1 is smaller than one and eigenvalue λ_2 is higher than one. (10)
With the help of Matlab, the eigenvalue and eigenvector matrix should be:

$$D = \begin{bmatrix} 0.9557 & 0 \\ 0 & 1.0526\end{bmatrix} \quad V = \begin{bmatrix} 0.9953 & 0.9988 \\ 0.0965 & -0.0488\end{bmatrix}$$

We shall have $\lambda_1 = 0.9557 < 1 < \lambda_2 = 1.0526$.

e) Find the policy functions:

$$\begin{aligned} c_t - c &= -\frac{\tilde{v}_{21}}{\tilde{v}_{22}}(k_t - k) \\ k_{t+1} - k &= \lambda_1(k_t - k) \end{aligned}$$

where $\tilde{v}_{ij}$ is the position (i,j) of matrix V^{-1}. Simulate the transition starting at 10% of steady state capital stock. (30)
With the V given, we shall have the inverse of V as:

$$V^{-1} = \begin{bmatrix} 0.3364 & 6.8904 \\ 0.6658 & -6.8864\end{bmatrix}$$

Therefore, our policy function:

$$\hat{c}_t = -\frac{v_{21}}{v_{22}}\hat{k}_t = 0.0970\hat{k}_t$$
$$\hat{k}_{t+1} = \lambda_1\hat{k}_t = 0.9557\hat{k}_t$$

Setting $k_0 = 0.1k^*$ and start simulation (T=200), the simulation result is in Figure 1.

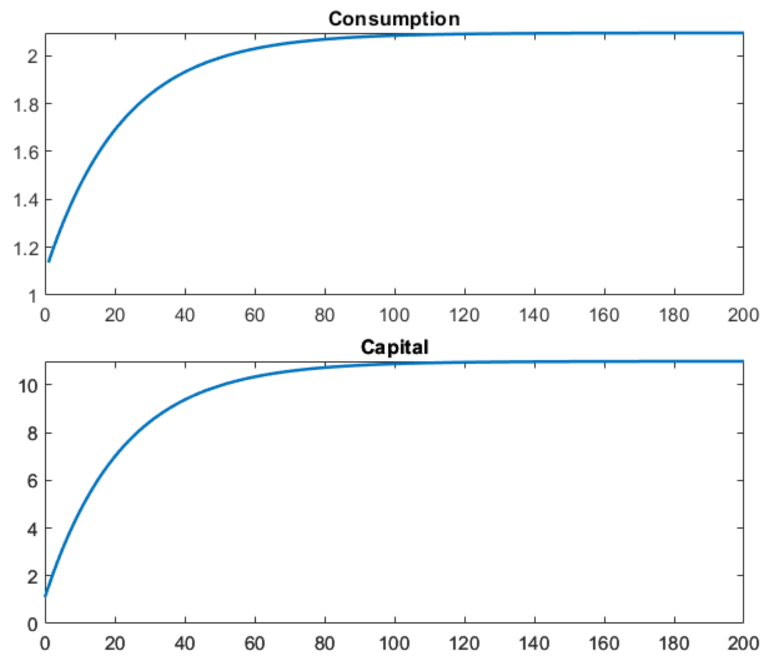

FIGURE 3. Simulation Result for Consumption and Capital

f) Use Dynare to find the time paths of both capital and consumption and capital and compare to the paths you derived in e). (15)

Answer.
With Dynare, we can simulate the transition in the Neoclassical Growth Model with perfect foresight (Dynare will solve a system of non-linear equations for 200 periods). The simulation result is in the figure 4 below. The blue line represents the result of linearization and the dashed line represents the results obtained with Dynare. There has to be a difference since Dynare's method is taking into account all non-linearities.

g) Use Dynare to find the time paths of capital and consumption under a linear approximation, compare to the paths you derived in e). (15)

Answer.
We can instruct Dynare to do a linear approximation to solve the NGM with perfect foresight. The simulation result is in the figure 5 below. The blue line represents the result of linearization through decoupling and the dashed line represents the result obtained by Dynare. The results are exactly the same.

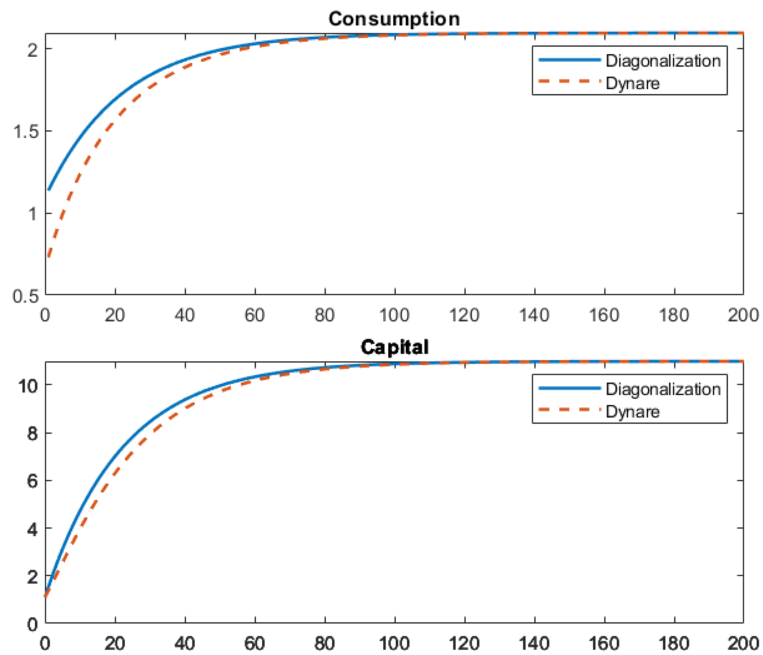

FIGURE 4. Comparison between Dynare and Diagonalization

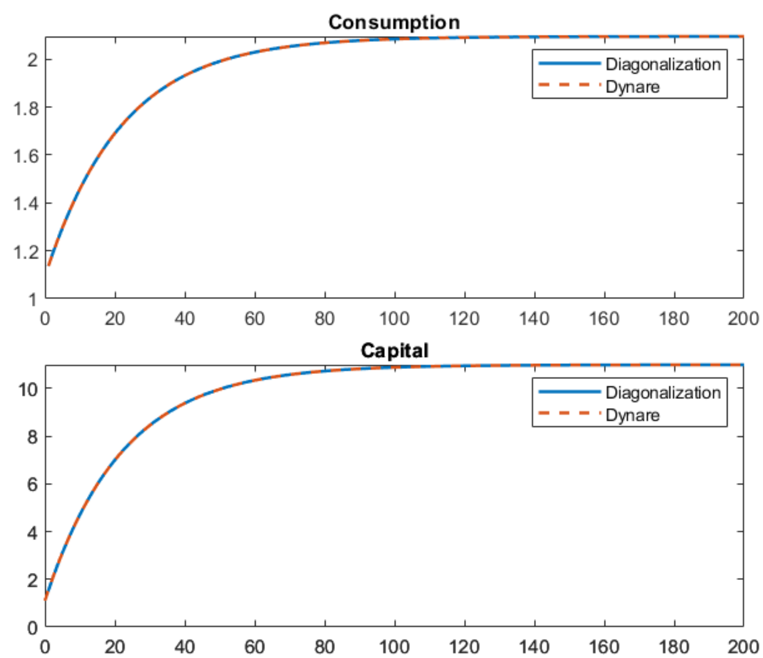

FIGURE 5. Comparison between Dynare and Diagonalization

3. The Neoclassical Growth Model with variable Labor Supply. (80)

In this problem, you will solve for the NGM with variable labor supply. A Central planner uses a Bellman equation to state the problem as:

$$v(k) = \max_{c,\ell,k'} \left[u(c,\ell) + \beta v(k') \right]$$

subject to:

$$k' + c = F(k,\ell) + (1-\delta)k$$

Throughout the problem use the following functional forms and parameters:

$$u(c,\ell) = \ln(c) + \gamma(1-\ell), \;\; F(k,\ell) = k^{\alpha}\ell^{1-\alpha}, \;\; \alpha = 0.36, \;\; \delta = 0.025, \;\; \beta = 0.95$$

Note that ℓ it the amount of working time. Time itself is normalized to unity and $\gamma > 0$ is a parameter that measures the disutility of work. Note that a value for γ is not given, you will have to calibrate it.

a) Find the Euler equation and the relevant FOC's using a Lagrangian. You should obtain 3 equations that form a dynamic system for the variables k_t, c_t and ℓ_t. (20)

Answer.
Let $f(k,\ell) = F(k,\ell) + (1-\delta)k$. Then the Lagrangian of the problem is:

$$L = u(c,\ell) + \beta v(k') + \lambda[f(k,\ell) - c - k']$$

The relevant FOC are:

$$\begin{aligned} u_c(c,\ell) &= \lambda \\ u_\ell(c,\ell) &= -\lambda f_\ell(k,\ell) \\ \beta v'(k') &= \lambda \end{aligned}$$

The Envelope condition is $v'(k) = \lambda f_k(k,\ell)$, which is the same as $v'(k') = \lambda' f_k(k',\ell')$. The system above then can be written as:

$$\begin{aligned} u_c(c,\ell) &= u_c(c',\ell')f_k(k',\ell') \\ u_\ell(c,\ell) &= -u_c(c,\ell)f_\ell(k,\ell) \end{aligned}$$

These are the Euler equation and the intra-temporal labor supply condition.
With the functional forms provided these are::

$$\frac{c'}{c} = \beta[\alpha(\frac{k'}{\ell'})^{\alpha-1} + 1 - \delta]$$

$$\gamma = \frac{1}{c}(1-\alpha)(\frac{k}{\ell})^{\alpha}$$

To complete the system we add the resource constraint:

$$c + k' = k^{\alpha}\ell^{1-\alpha} + (1-\delta)k$$

b) In the steady state, and under some parametrization, the system derived in a) gives three equations for the unknowns $\bar{k}, \bar{c}, \bar{\ell}$ (the steady state values). Because we don't have a value for γ, set $\bar{\ell} = 1/3$ (the steady state value of labor), and use the system in a) to find the values for the unknowns k, c, γ. (20)

Answer.
In steady state:

$$1 = \beta f'(\bar{k}) \to \beta(\alpha(\frac{\bar{k}}{\bar{\ell}})^{\alpha-1} + 1 - \delta) = 1$$

Note that $\beta = 0.95$, $\alpha = 0.36$, $\delta = 0.025$ and $\bar{\ell} = \frac{1}{3}$, plug in these data. We shall have:

$$\frac{\bar{k}}{\bar{\ell}} = (\frac{\frac{1}{\beta} + \delta - 1}{\alpha})^{\frac{1}{\alpha-1}} \rightarrow \bar{k} = 3.6637$$

With $\bar{k}$ given we can use the resource constraint to get:

$$\bar{c} = \bar{k}^{\alpha}\bar{\ell}^{1-\alpha} - \delta\bar{k} \rightarrow \bar{c} = 0.6985$$

And the intratemporal labor condition can be used to get γ:

$$\gamma = \frac{1}{c}(1-\alpha)(\frac{\bar{k}}{\bar{\ell}})^{\alpha} = 2.1718$$

c) Use Dynare to find the time paths of capital, consumption and labor. Simulate the transition starting at 10% of steady state capital stock. (20)

Answers.

With the equations specified in a) and the parameter calibrated in b), we can start the simulation at the point: $k_0 = 0.1k^*$, we have the simulation path shown below:

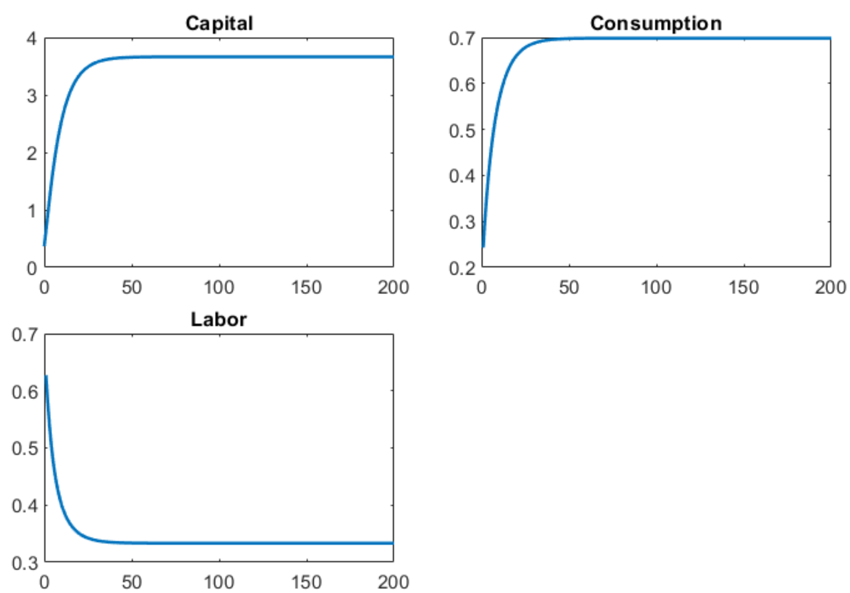

FIGURE 6. Simulation Result with Dynare: $k_0 = 0.1\bar{k}$

With $k_0 < \bar{k}$, we shall have $c_0 < \bar{c}$ and $\ell_0 > \bar{l}$ then slowly converge to steady state.

d) Use Dynare to find the time paths of capital, consumption and labor under a linear approximation, compare to the paths you derived in c). Simulate the transition starting at 10% of steady state capital stock. (20)

Answer.

We can instruct Dynare to do linearization with the command perfect_foresight_solver(linear_approximation).

So we obtain the paths in the following figure. In the figures the label "linearization" corresponds to the linearization done in Dynare as well.

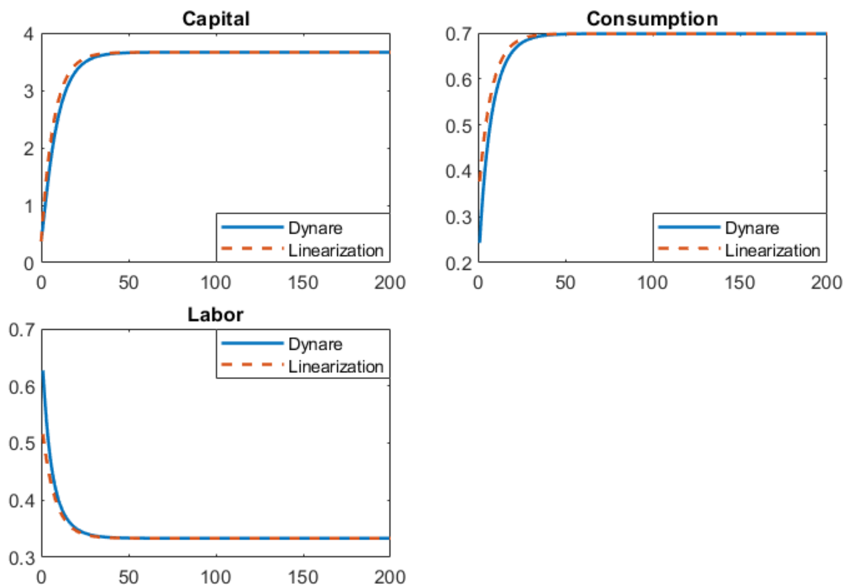

FIGURE 7. Comparison between Dynare (non-linear) and Dynare (linear approximation)

We can observe that the linearization path is different from the Dynare result: with linearization given a faster convergence to the steady state for all variables.