ECMA 31140. PERSPECTIVES ON COMPUTATIONAL MODELING FOR ECONOMICS
PROBLEM SET 8
DUE DATE: MARCH 4TH

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You are encouraged to work and discuss in groups, but you must submit your work individually. Answers must be legibly hand-written or typed. All assignments are due electronically on Canvas, attach code. Assignments are due at 12:30 PM. Late problem sets *will not be accepted*

1. An heterogeneous agent model with closed form solution. (90 points)

In class we have consider the following optimization problem:

$$\mathcal{V}(a,\theta) = \max_{c,a'} \left[\theta c + \beta \int_{\Theta} \mathcal{V}(a',\theta') dF(\theta) \right] \tag{1}$$

subject to:

$$qa' = a + y - c, \; a' \geq -b, \; c \geq 0 \tag{2}$$

Given the linearity of the utility function, postulate that:

$$\mathcal{V}(a,\theta) = A(\theta) + B(\theta)a \tag{3}$$

where $A(\theta)$ and $B(\theta)$ are unknown functions of θ.

a) Use the Guess (3) to find the policy functions $g(a,\theta)$ and $c(a,\theta)$. (20)
Hint: define z the value of θ such that:

$$z = \beta \frac{\mathbb{E}B(\theta')}{q} \tag{4}$$

Show that the policy functions adopt different forms depending on whether $\theta \leq z$ or $\theta > z$

Answer.
The Lagrangian for the problem is:

$$\mathcal{L} = \max_{a'} \left[\theta(a + y - qa') + \lambda \left(\frac{a+y}{q} - a' \right) + \mu(a' + b) + \beta \mathbb{E}A(\theta') + \beta \mathbb{E}B(\theta')a' \right], \tag{5}$$

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with λ and μ the multipliers for the non-negativity constraint on consumption and the borrowing constraint respectively. $\mathbb{E}$ is the expectation operator associated to $F(\theta)$. The first order condition:

(6) $$-q\theta - \lambda + \mu + \beta\mathbb{E}B(\theta') = 0.$$

I define z as the value of θ such that the agent with state (a, θ) would be indifferent between consuming and saving, in which case $\lambda = \mu = 0$:

(7) $$z = \beta\mathbb{E}B(\theta')/q.$$

Given linearity of utility, is natural to conjecture that agents who face $\theta > z$ would use all their resources plus maximal borrowing to consume, and agents that face $\theta \leq z$ would not consume and save all their income, this conjecture is reflected in policies (8) below.

(8) $$g(a,\theta) = \begin{cases} (a+y)/q & \text{if } \theta \leq z \\ -b & \text{if } \theta > z \end{cases} \text{ and } c(a,\theta) = \begin{cases} 0 & \text{if } \theta \leq z \\ a+y+qb & \text{if } \theta > z \end{cases},$$

b) Show that the value function then must satisfy:

(9) $$A(\theta) + B(\theta)a = \begin{cases} 0 + \beta\mathbb{E}A(\theta') + \beta\mathbb{E}B(\theta')\frac{a+y}{q} & \text{if } \theta \leq z \\ \theta(a+y+qb) + \beta\mathbb{E}A(\theta') - \beta\mathbb{E}B(\theta')b & \text{if } \theta > z. \end{cases}$$ (20)

Answer.

With policies defined in (8), using the value function (1), it is immediate by replacing elements that:

(10) $$A(\theta) + B(\theta)a = \begin{cases} 0 + \beta\mathbb{E}A(\theta') + \beta\mathbb{E}B(\theta')\frac{a+y}{q} & \text{if } \theta \leq z \\ \theta(a+y+qb) + \beta\mathbb{E}A(\theta') - \beta\mathbb{E}B(\theta')b & \text{if } \theta > z. \end{cases}$$

Note that given linearity, terms such as:

$$\int_\Theta A(\theta')dF(\theta),$$

adopt simple forms as:

$$\int_\Theta A(\theta')dF(\theta) = \mathbb{E}A(\theta).$$

c) Equate coefficients of the value function in (9) to get:

(11) $$A(\theta) = \begin{cases} \beta\mathbb{E}A(\theta') + \beta\frac{y}{q}\mathbb{E}B(\theta') & \text{if } \theta \leq z \\ \theta(y+qb) + \beta\mathbb{E}A(\theta') - \beta\mathbb{E}B(\theta')b & \text{if } \theta > z \end{cases}, \; B(\theta) = \begin{cases} \beta\frac{1}{q}\mathbb{E}B(\theta') & \text{if } \theta \leq z \\ \theta & \text{if } \theta > z \end{cases}.$$

(20)

Answer.

Again, simply equating coefficients in the expression above, it is clear that we get immediately:

(12) $$A(\theta) = \begin{cases} \beta\mathbb{E}A(\theta') + \beta\frac{y}{q}\mathbb{E}B(\theta') & \text{if } \theta \leq z \\ \theta(y+qb) + \beta\mathbb{E}A(\theta') - \beta\mathbb{E}B(\theta')b & \text{if } \theta > z \end{cases}, \; B(\theta) = \begin{cases} \beta\frac{1}{q}\mathbb{E}B(\theta') & \text{if } \theta \leq z \\ \theta & \text{if } \theta > z \end{cases}.$$

d) Take expectations under F in (11) and use the fact that shocks to the marginal utility of consumption are i.i.d. to find closed form expressions for both $\mathbb{E}A(\theta)$ and $\mathbb{E}B(\theta)$. (10)

Answer.
Taking expectations:

$$\text{(13a)}\quad \mathbb{E}A(\theta) = \beta\mathbb{E}A(\theta') + (y+qb)\int_z^{\bar{\theta}} \theta dF + \beta\frac{y}{q}\mathbb{E}B(\theta')F(z) - \beta b\mathbb{E}B(\theta')(1-F(z))$$

$$\text{(13b)}\quad \mathbb{E}B(\theta) = \beta\frac{1}{q}\mathbb{E}B(\theta')F(z) + \int_z^{\bar{\theta}} \theta dF$$

The i.i.d. assumption of shocks, make (13) a system of two equations in the unknowns: $\mathbb{E}A(\theta)$ and $\mathbb{E}B(\theta)$, solving:

$$\mathbb{E}A(\theta) = \frac{(y+qb)\frac{z(q-\beta\frac{bq}{y+bq})}{\beta}}{1-\beta} = \frac{(y+qb)(zq - z\beta\frac{bq}{y+qb})}{\beta(1-\beta)} = \frac{(y+qb)zq - \beta zbq}{\beta(1-\beta)}$$

$$\mathbb{E}B(\theta) = \frac{\frac{z(q-\beta\frac{bq}{y+bq}}{\beta})}{1-\frac{\beta}{q}F(z)} = \frac{zq(q-\beta\frac{qb}{y+bq})}{\beta(q-\beta F(z))} = \frac{zq(q-\beta\frac{qb}{y+qb})}{\beta(q-\beta\frac{qb}{y+qb})} = \frac{zq}{\beta}$$

e) Show that the value function is given by:

$$\text{(14)}\qquad v(a,\theta) = \begin{cases} [q^2b + (y-b\beta)q + (y+a)(1-\beta)]\,\frac{z}{1-\beta} & \text{if } \theta \le z \\ \theta(y+a+qb) + [q^2b + (y-b)q]\,\frac{z}{1-\beta} & \text{if } \theta > z \end{cases}.$$

A closed form solution! (20)

Answer.
We have:
– $\theta \le z$

$$A(\theta) = \frac{(y+qb)zq - \beta zbq}{1-\beta} + \frac{y}{q}zq = \frac{(y+qb)zq - \beta zbq + yz(1-\beta)}{1-\beta}$$
$$= \frac{z}{1-\beta}[q^2b + (y-b\beta)q + y(1-\beta)]$$
$$B(\theta) = z$$

Then, we should have:

$$A(\theta) + B(\theta)a = \frac{z}{1-\beta}[q^2b + (y-b\beta)q + y(1-\beta)] + az$$
$$= \frac{z}{1-\beta}[q^2b + (y-b\beta)q + (y+a)(1-\beta)]$$

– $\theta > z$

$$A(\theta) = \theta(y+qb) + \frac{(y+qb)zq - \beta zbq}{1-\beta} - zqb = \theta(y+qb) + \frac{(y+qb)zq - zbq}{1-\beta}$$
$$B(\theta) = \theta$$

Then, we should have:

$$
\begin{aligned}
A(\theta) + B(\theta)a &= \theta(y + qb) + \frac{(y + qb)zq - zbq}{1 - \beta} + \theta a \\
&= \theta(y + a + qb) + \frac{z}{1 - \beta}[q^2 b + yq - qb] \\
&= \theta(y + a + qb) + \frac{z}{1 - \beta}[q^2 b + (y - b)q]
\end{aligned}
$$

2. Computing policies in the Huggett Economy. (70 points)

There is a measure one of agents. Each agent wants to maximize:

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t), \;\; 0 < \beta < 1$$

and each period an agent can have a high or a low endowment e_1 or e_0 with $e_1 > e_0$, with transition matrix:

$$\Pr(e_{t+1}|e_t) \equiv \begin{bmatrix} Pr(e_1|e_1) & Pr(e_0|e_1) \\ Pr(e_1|e_0) & Pr(e_0|e_0) \end{bmatrix}$$

Agents have the following budget constraint:

$$c_t + qa_{t+1} = a_t + e_t, \;\; a_{t+1} \geq \bar{a}$$

where depending on the realization of the shock e_t will be either e_1 or e_0.

Assume that $q = 1.0124$, $\beta = 0.99322$, $\bar{a} = -2$, $e_1 = 1$ and $e_0 = 0.1$. Use also the following values for the probabilities: $Pr(e_1|e_1) = 0.925, Pr(e_0|e_0) = 0.5$. The utility function is:

$$u(c_t) = \frac{1}{1 - \sigma} c_t^{1-\sigma}$$

where $\sigma = 1.5$.

a) State the Bellman equation for an individual with current value of the asset a and facing labor endowment e. (20)

Answer.

The Bellman equation is:

$$\mathcal{V}(a, e_i) = \max_{c,a'} [u(c) + \beta \sum_{j=0}^{1} \mathcal{V}(a', e_j) Pr(e_j|e_i)],$$

subject to:

$$c + qa' = a + e_i.$$

For $i = 1, 0$

b) Use value function iteration to obtain value and policy functions for next period asset and consumption. Use a 1000 points in the grid for assets with a maximum value of 4. (50)

Answer.

The policy functions for next period asset can be seen in the figure below. Green line is 45°, blue curve is $g(a, e_1)$ and orange curve is $g(a, e_0)$. Note that when the asset is low and facing a low shock, the borrowing constraint might be binding.

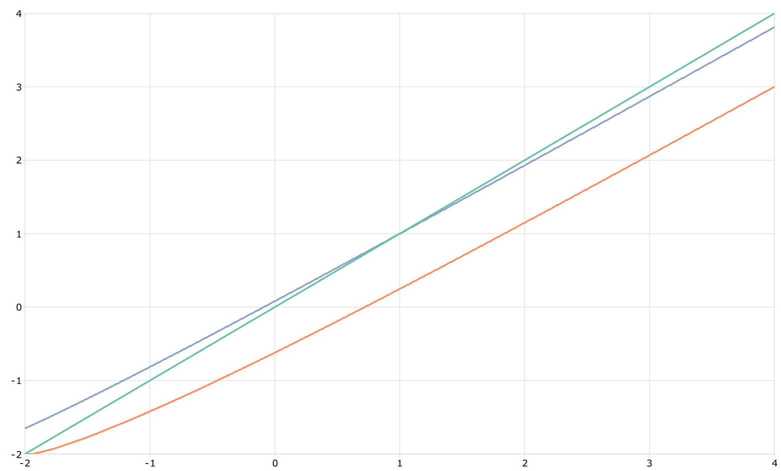

FIGURE 1. Policy functions assets

The value functions can be seen in the following figure. Blue curve is $\mathcal{V}(a, e_1)$ and orange curve is $\mathcal{V}(a, e_0)$.

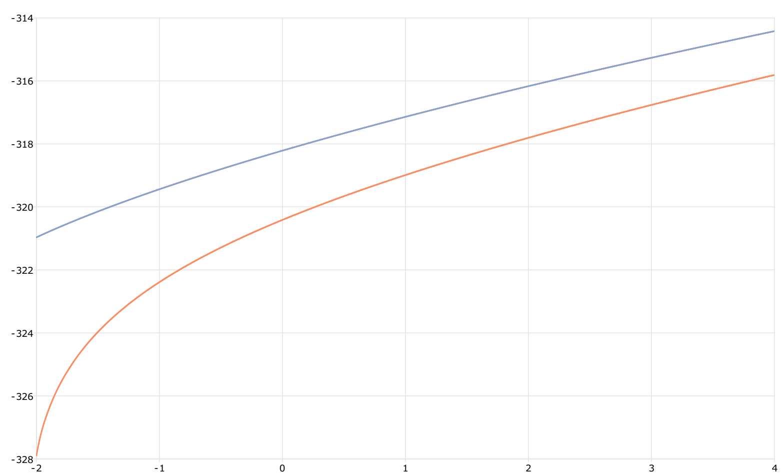

FIGURE 2. Value functions

c) Use EGM to obtain the policy functions for next period asset and consumption. Use again 1000 points on the grid for assets with a maximum value of 4. (50)

Answer.

Implementing the EGM we find policy functions for asset accumulation as very similar to the one with VFI, but the time to convergence is much smaller. Below is the figure for the policy functions.

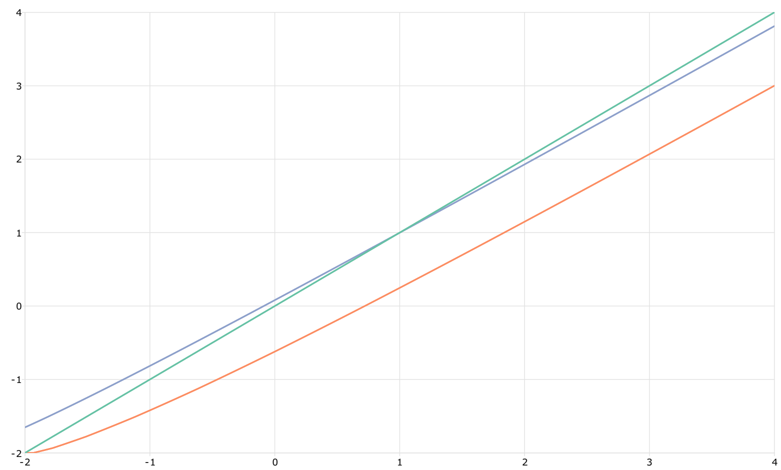

FIGURE 3. Policy functions assets, EGM

Here I present the policy functions for consumption.

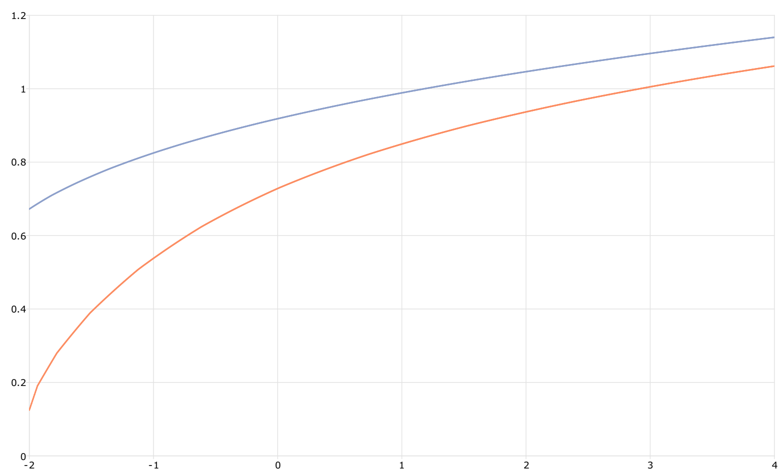

FIGURE 4. Policy functions consumption, EGM