ECMA 31140. PERSPECTIVES ON COMPUTATIONAL MODELING FOR ECONOMICS
PROBLEM SET 7
DUE DATE: FEBRUARY 25TH

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You are encouraged to work and discuss in groups, but you must submit your work individually. Answers must be legibly hand-written or typed. All assignments are due electronically on Canvas, attach code. Assignments are due at 12:30 PM. Late problem sets *will not be accepted*

Date: February 25, 2025.

3. Endogenous Grid Method (EGM) for non-stochastic NGM. (60 points)

Consider the optimization problem for a central planner, aiming to maximize

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t),\ 0 < \beta < 1$$

subject to:

$$\begin{aligned} c_t + i_t &= F(k_t, n_t) \\ k_{t+1} &= i_t + (1-\delta)k_t \\ k_0 &> 0 \text{ given} \end{aligned}$$

The utility function and the production function are given by:

$$u(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma},\ F(k_t, 1) = k_t^{\alpha} \tag{1}$$

For this exercise use the following values for the parameters:

$$\alpha = 0.36,\ \beta = 0.98,\ \delta = 0.03, \gamma = 2$$

a) Solve the model using the EGM. Find both the value function and the policy functions for capital and consumption. (40)

Answer.
You should write the code as we have explained in class, and obtain the following results:

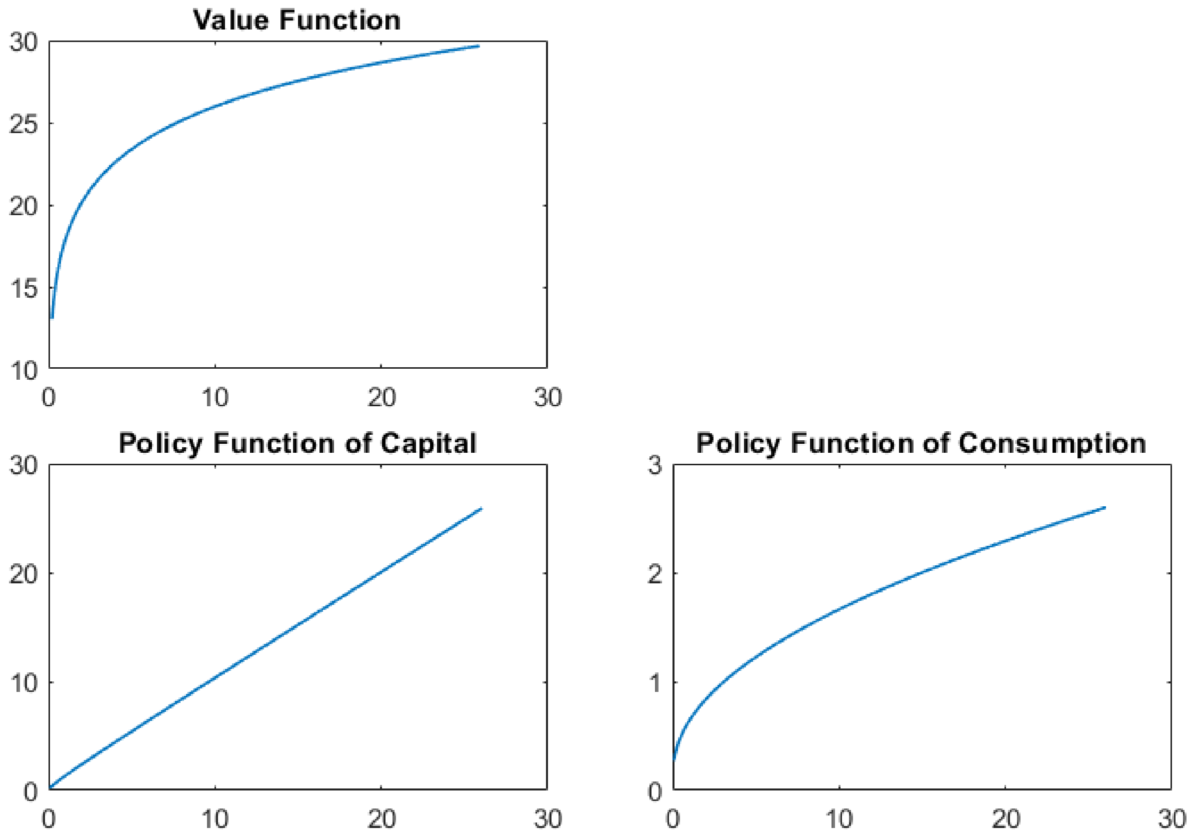

FIGURE 1. Endogenous Grid Method Result

b) Consider the case where:

$$\alpha = 0.36, \ \ \beta = 0.98, \ \ \delta = 1, \gamma = 1$$

Solve the model using the EGM. Find both the value function and the policy functions for capital and consumption, and compare the solutions to the analytical solutions you can find in this case. (20)

Answer
Let us derive again the closed form solution that we have in this case.

$$k_{t+1} = i_t = k_t^\alpha - c_t$$

Suppose our value function is taken as the form:

$$V(k_t) = A + B \ln k_t$$

Therefore, we shall have:

$$\frac{\beta B}{k_{t+1}} = \frac{1}{c_t}$$
$$\frac{1}{k_t} = \frac{\beta\alpha k_t^{\alpha-1}}{k_{t+1}}$$

This will suffice to show that:

$$k_{t+1} = \alpha\beta k_t^\alpha$$

Therefore, we have:

$$\alpha\beta + \frac{\alpha}{B} = 1$$

And we shall solve that:

$$B = \frac{\alpha}{1-\alpha\beta}$$

Plug $c_t = \frac{\alpha}{B} k_t^\alpha$ and $k_{t+1} = \alpha\beta k_t^\alpha$ into the Bellman equation, we shall have:

$$A + B\ln k_t = \ln\frac{\alpha}{B}k_t^\alpha + \beta A + \beta B \ln \alpha\beta k_t^\alpha$$

Then, we shall have:

$$A = \frac{\ln\frac{\alpha}{B} + \beta B\ln\alpha\beta}{1-\beta}$$
$$B = \alpha + \alpha\beta B$$

Then, we shall have:

$$A = \frac{\ln(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta}\ln\alpha\beta}{1-\beta}$$
$$B = \frac{\alpha}{1-\alpha\beta}$$

Now, you should use the same code as in incise a) to obtain a numerical approximation. The results are shown below.

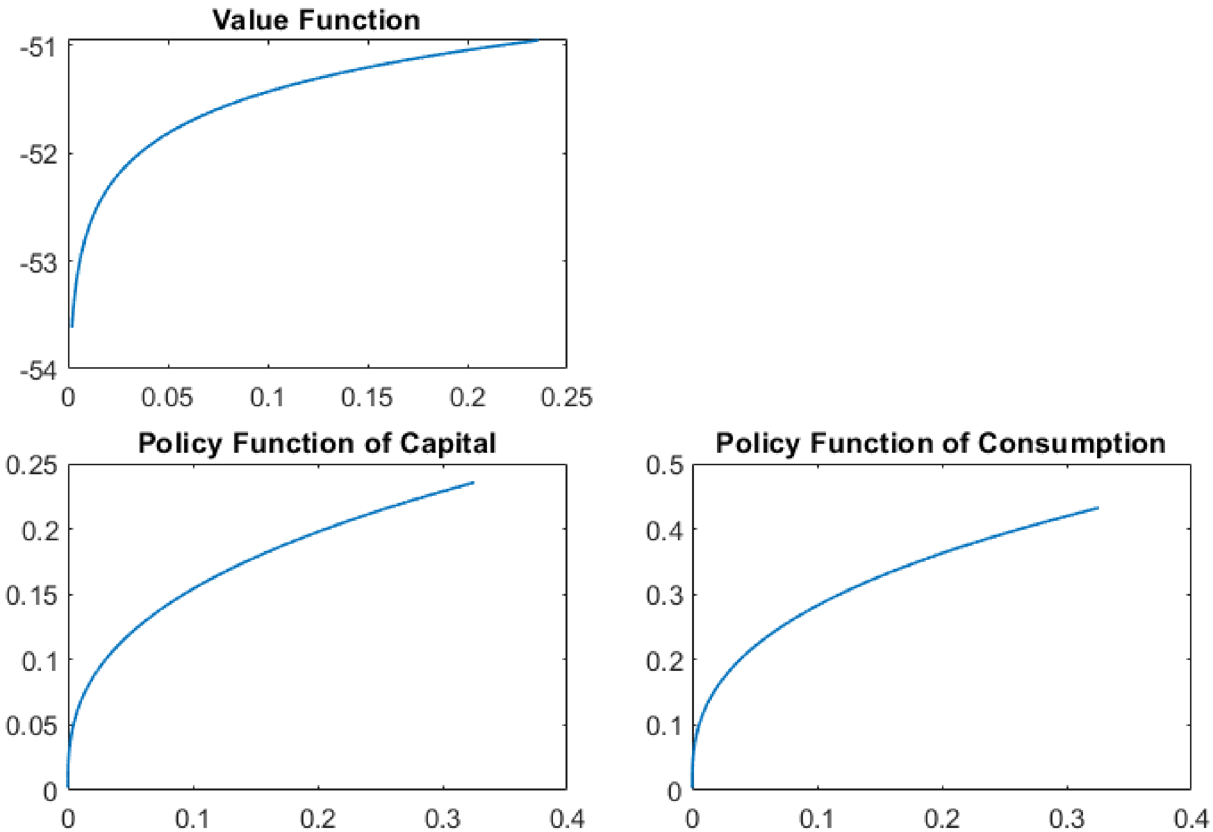

FIGURE 2. Endogenous Grid Method Result

Note that we also have the analytical solution by the procedure stated in the previous page, we can compare the result:

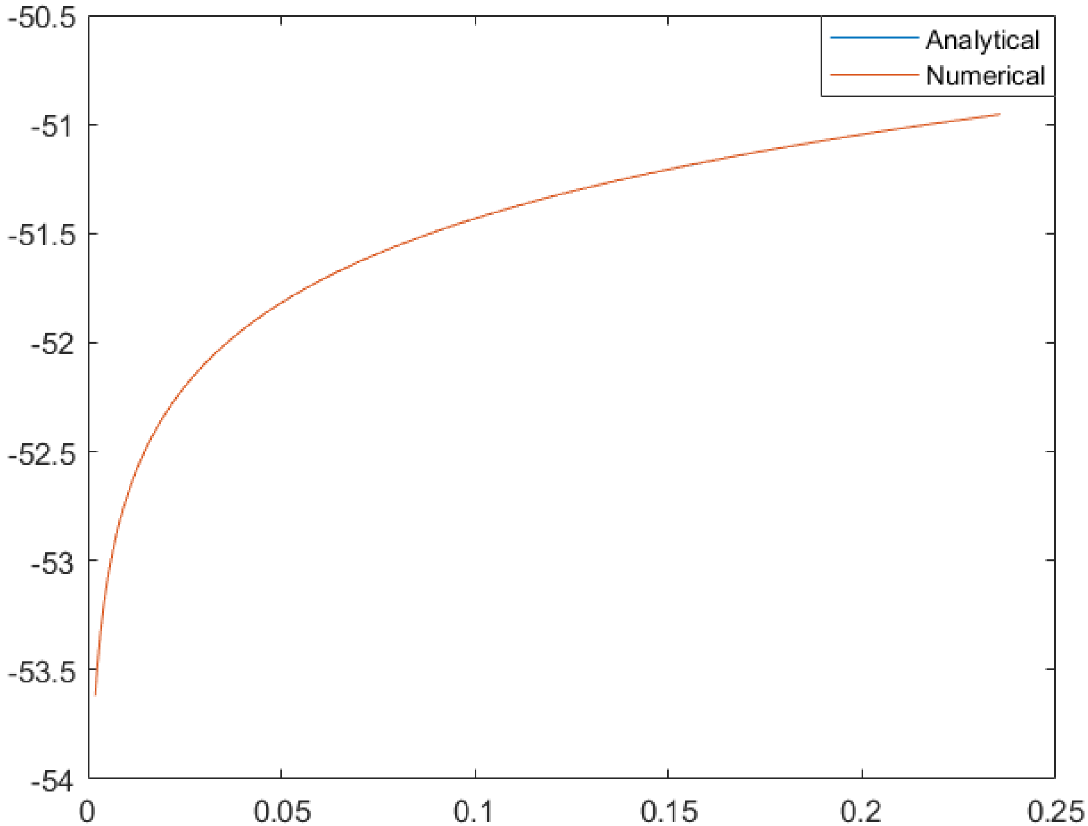

FIGURE 3. Comparison of Value Function

And we can observe from the figure that the analytical solution and numerical solution agree with each other.

2. The stochastic Neoclassical Growth Model with variable labor supply. (120 points)

Consider again the non-stochastic NGM where the Central planner uses a Bellman equation to state the problem as:

$$\mathcal{V}(k) = \max_{c,k',\ell'} [u(c,\ell) + \beta \mathcal{V}(k')]$$

subject to:

$$k' + c = F(k,\ell) + (1-\delta)k$$

Throughout the problem use the following functional forms:

$$u(c,\ell) = \frac{[c^\gamma(1-\ell)^{1-\gamma}]^{1-\sigma}}{1-\sigma}, \quad F(k,\ell) = k^\alpha \ell^{1-\alpha}$$

a) Assume for this incise that $\sigma = \delta = 1$. Use a Guess and Verify method for the value function to find the policy function for consumption, next period capital and labor. (30)

Answer.

We again use the guess $\mathcal{V} = A + B\ln k$. The the Bellman equation can be written:

$$\mathcal{V}(k) = \gamma \ln(k^\alpha \ell^{1-\alpha}) + (1-\gamma)\ln(1-\ell) + \beta A + \beta B \ln k'$$

The FOC w.r.t. k' and ℓ are given by:

$$\frac{\gamma}{c} = \beta \frac{B}{k'}$$

$$\frac{\gamma}{c}(1-\alpha)k^\alpha \ell^{-\alpha} = \frac{1-\gamma}{1-\ell}$$

The resource constraint and the FOC w.r.t. k' can be solved for consumption and next period capital as:

$$c = \frac{\gamma}{\gamma+\beta B} k^\alpha \ell^{1-\alpha}, \quad k' = \frac{\beta B}{\gamma} c$$

We can use then the FOC w.r.t. ℓ to get:

$$\ell = \frac{(\gamma+\beta B)(1-\alpha)}{\gamma(1-\gamma) + (\gamma+\beta B)(1-\alpha)}$$

Note that in this case labor is constant, independent of the stock of capital. Under $\sigma = 1 = \delta$ income and substitution effects cancel exactly so labor becomes constant.

To find B, we equate coefficient in the Bellman equation (usual procedure, details omitted here) and we obtain:

$$B = \frac{\gamma\alpha}{1-\alpha\beta}$$

Replacing this value in the policy functions for capital and consumption we get:

$$k' = \alpha\beta k^\alpha \ell^{1-\alpha}, \quad c = (1-\alpha\beta)k^\alpha \ell^{1-\alpha}$$

The solutions for the constant value of ℓ can be found replacing B in the expression above.

b) Consider the policy functions for capital and labor:

$$k' = g(k), \quad \ell = h(k)$$

Write down explicitly the operator $\mathcal{N}(g,h) = 0$. (30)

Answer.

Let me use the notation $f(k,\ell) = k^\alpha \ell^{1-\alpha} + (1-\delta)k$. Finding the optimality conditions of the problem we have:

$$u_c(c,\ell) = \beta f_k(k',\ell') u_c(c',\ell') \tag{2}$$

$$-u_\ell(c,\ell) = u_c(c,\ell) f_\ell(k,\ell) \tag{3}$$

Along with the resource constraint $k' + c = f(k, \ell)$.

Consider not the policy functions $k' = g(k), \ell = h(k)$ and the following relationships:

$$
\begin{aligned}
c &= f(k, h(k)) - g(k) \\
\ell' &= h(g(k)) \\
c' &= f(g(k), h(g(k))) - g(g(k))
\end{aligned}
$$

Using this formulation the operator $\mathcal{N}$ is composed of two functionals:

$$
\begin{aligned}
\mathcal{N}_1(g(k), h(k)) &= u_c(f(k, h(k)) - g(k), h(k)) - \beta f_k(g(k), h(g(k)))u_c(f(g(k), h(g(k))) - g(g(k)), h(g(k))) \\
\mathcal{N}_2(g(k), h(k)) &= -u_\ell(f(k, h(k)) - g(k), \ell) - u_c(f(k, h(k)) - g(k), h(k))f_\ell(k, h(k))
\end{aligned}
$$

c) From now on, consider the following parametrization:

$$\alpha = 0.36,\ \delta = 0.03,\ \beta = 0.98,\ \sigma = 2$$

Because we don't have a value for γ, set $\bar{\ell} = 1/3$ (the steady state value of labor), use the optimality conditions of the problem to find the values in steady state of capital and consumption $\bar{k}, \bar{c}$ and the calibrated value for γ. (10)

Answer.

The optimality conditions in steady state deliver:

$$
\begin{aligned}
\bar{\ell} &= \frac{1}{3} \\
\bar{k} &= \frac{1}{3}\left(\frac{\frac{1}{\beta} + \delta - 1}{\alpha}\right)^{\frac{1}{\alpha-1}} \\
\bar{c} &= 3\left(\frac{\frac{1}{\beta} + \delta - 1}{\alpha}\right)^{\frac{\alpha}{\alpha-1}} - \delta\frac{1}{3}\left(\frac{\frac{1}{\beta} + \delta - 1}{\alpha}\right)^{\frac{1}{\alpha-1}}
\end{aligned}
$$

With the closed form solution for γ as:

$$\gamma = \frac{1}{(1-\alpha)(\frac{\bar{k}}{\bar{\ell}})^{\alpha}\frac{1-\bar{\ell}}{\bar{c}} + 1}$$

d) Use an approximation for the policy functions for capital k' and labor ℓ as follows:

$$k' \equiv \hat{g}(k; \mathbf{a}) = \sum_{i=1}^{n} a_i \phi_i(k),\ \ \ell \equiv \hat{h}(k; \mathbf{b}) = \sum_{i=1}^{n} b_i \psi_i(k)$$

where:

$$\phi_i(k) = kT_{i-1}(k) \doteq k\cos((i-1)\arccos(k)),\ \ \psi_i(k) = T_{i-1}(k) \doteq \cos((i-1)\arccos(k))$$

Use orthogonal collocation to find approximations to the policy functions for capital labor and consumption for a range $[0.3\bar{k}, 1.2\bar{k}]$. Use $n = 8$. (Hint: if it helps, you can start with a good guess for the case $\sigma = 1 = \delta$.) (50)

Answer.

Coding the orthogonal collocation method as we have explained in class, you should obtain:

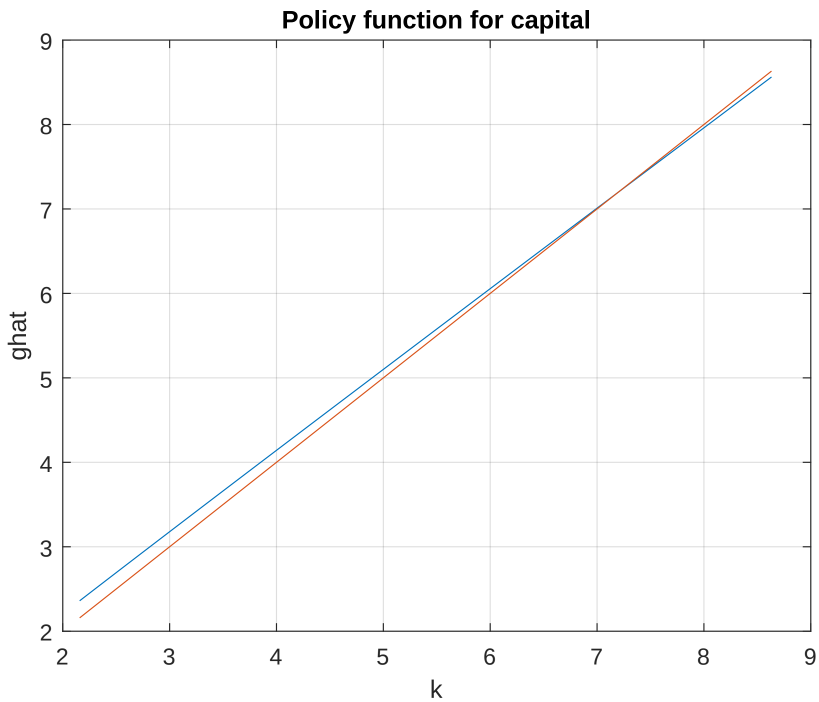

FIGURE 4. $\hat{g}$, organge line is 45°

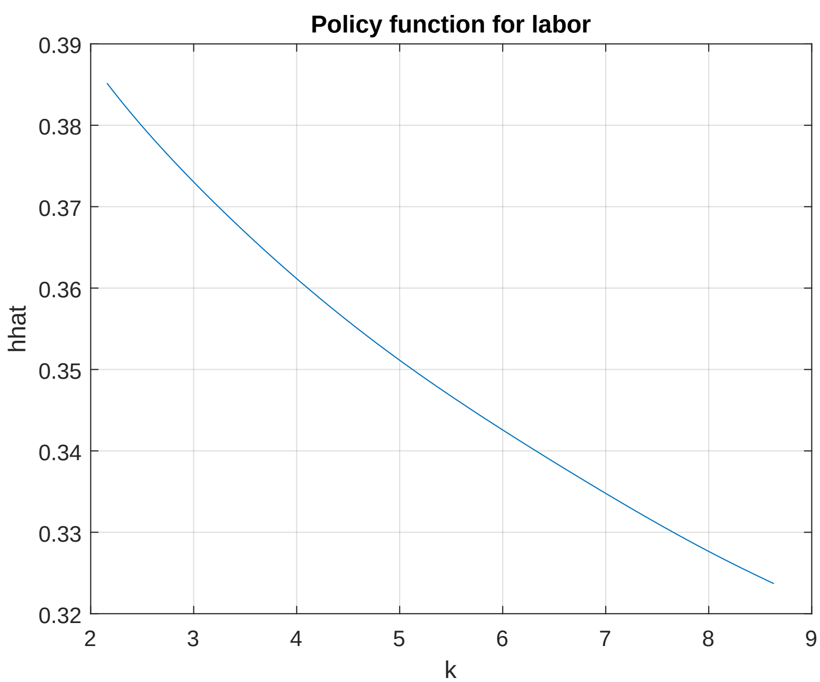

FIGURE 5. $\hat{h}$

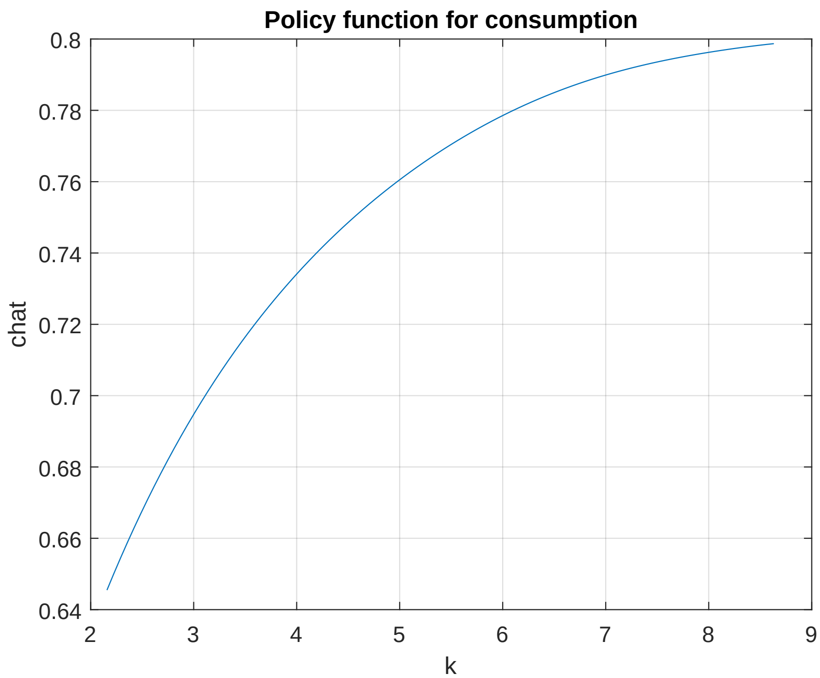

FIGURE 6. $\hat{c}$