ECMA 31140: Perspectives on Computational Modeling for Economics
Problem Set 2
Due Date: January 21st, 2025
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You are encouraged to work and discuss in groups, but you must submit your work individually. Answers must be legibly hand-written or typed. All assignments are due electronically on Canvas, attach code. Assignments are due at 12:30 PM. Late problem sets will not be accepted.

Please submit one fully compiled pdf file with your mathematical solution, explanations, final scalar results, plots and comparisons all in this one pdf with clearly outlined questions. Submit your code file separately (Please note that you can submit multiple files and there is no need to submit zip files). We expect you to provide one Pdf file and one script file with all code (if you maintain multiple scripts for each question upload multiple files (again no need for zip files). Primarily focus on your pdf file as final compilation of your analysis as an economist and what you want to deliver for each model. Your code file is your supporting element and I will look at it for your logic and intuition in coding. Key focus is how you prepare your final analysis. It would not be feasible for me to run each student's code to generate the output and analyze it myself (which you should not expect while submitting your report either academically or professionally).

1 Finding Eigenvalues and Eigenvectors (15)

Consider the general matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

where $a_{i,j}$ are real values different than zero. Use software with analytical or symbolic capabilities.

(a) Find the Eigenvalues and Eigenvectors of the matrix A. Show that:

$$AV = VD$$

Solution

See details in `ps2_q1.m`.

Eigenvalues

$$\lambda_1 = \frac{a_{11}}{2} + \frac{a_{22}}{2} - \frac{\sqrt{{a_{11}}^2 - 2\,a_{11}\,a_{22} + {a_{22}}^2 + 4\,a_{12}\,a_{21}}}{2}$$

$$\lambda_2 = \frac{a_{11}}{2} + \frac{a_{22}}{2} + \frac{\sqrt{{a_{11}}^2 - 2\,a_{11}\,a_{22} + {a_{22}}^2 + 4\,a_{12}\,a_{21}}}{2}$$

Eigenvectors

$$v_1 = \begin{pmatrix} -\frac{a_{22} - a_{11} + \sqrt{{a_{11}}^2 - 2\,a_{11}\,a_{22} + {a_{22}}^2 + 4\,a_{12}\,a_{21}}}{2\,a_{21}} \\ 1 \end{pmatrix}$$

$$v_2 = \begin{pmatrix} \frac{a_{11} - a_{22} + \sqrt{{a_{11}}^2 - 2\,a_{11}\,a_{22} + {a_{22}}^2 + 4\,a_{12}\,a_{21}}}{2\,a_{21}} \\ 1 \end{pmatrix}$$

(b) Assume that $a_{21} = 0$. Compute Eigenvalues and Eigenvectors. (5 points)

Solution

Eigenvalues: $\lambda_1 = a_{11},\ \lambda_2 = a_{22}$

Eigenvectors:

$$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad v_2 = \begin{pmatrix} -\frac{a_{12}}{a_{11}-a_{22}} \\ 1 \end{pmatrix}$$

(c) Assume that $a_{12} = 0$. Compute Eigenvalues and Eigenvectors. (5 points)

Solution

Eigenvalues: $\lambda_1 = a_{11},\ \lambda_2 = a_{22}$

Eigenvectors:

$$v_1 = \begin{pmatrix} \frac{a_{11}-a_{22}}{a_{21}} \\ 1 \end{pmatrix}; \quad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

2. A Cake-Eating Problem with Interest Rate (40)

An individual wants to maximize:

$$\sum_{t=0}^{T} \beta^t u(c_t)$$

subject to:

$$w_{t+1} = (w_t - c_t)(1 + r)$$

where $r > 0$ is the real interest rate and $w_0 > 0$ is given. (You can use software with analytical or symbolic capabilities.)

(a) Iterate forward the budget constraint from period 0 onwards. (5 points)

Solution

We iterate forward from $t = 0$ up to $t = T$. For example:

$$w_1 = (w_0 - c_0)(1 + r),$$

$$w_2 = \big((w_0 - c_0)(1 + r) - c_1\big)(1 + r),$$

and so on. By induction, the closed-form expression is:

$$w_{t+1} = w_0(1 + r)^{t+1} - \sum_{\tau=0}^{t} c_\tau (1 + r)^{t-\tau+1}.$$

(b) Take now the limit as $T \to \infty$ of the problem. What is the transversality condition you obtain from the expression derived in (a)? (5 points)

Solution

Imposing the anti-Ponzi transversality condition, taking the limit as $T \to \infty$, the transversality condition is:

$$\lim_{t\to\infty} \beta^t (1+r)^t w_t = 0.$$

This condition forces the agent not to "over-save" so much that the effective present value of leftover wealth is positive in the limit. If $\beta(1+r) < 1$ this condition is usually satisfied automatically when w_t remains bounded. If $\beta(1+r) > 1$, then satisfying TVC requires that $\beta(1+r) > 1$ not explode too quickly (in fact, typically that yields corner or degenerate solutions, depending on whether borrowing is allowed, etc.).

(c) From now on use the infinite version of the problem. Find the Euler equation. (5 points)

Solution

The first-order condition (Euler equation) is derived as follows:

$$u'(c_t) = \beta(1+r)u'(c_{t+1}).$$

Equivalently:

$$\frac{u'(c_t)}{u'(c_{t+1})} = \beta(1+r).$$

This condition characterizes the optimal consumption path.

(d) From now on use log utility for preferences: $u(c_t) = \ln(c_t)$. Use the Euler equation to find a second-order difference equation with two boundary conditions: One is the initial condition w_0 and the other the transversality condition you derived in (b). (10 points)

Solution

In this case, for $u(c_t) = \ln(c_t)$, we have $u'(c_t) = 1/c_t$. The Euler equation becomes:

$$\frac{1}{c_t} = \beta(1+r)\frac{1}{c_{t+1}},$$

or:

$$c_{t+1} = \beta(1+r)c_t.$$

Let $\gamma = \beta(1+r)$. Then $c_{t+1} = \gamma c_t$. From the budget constraint:

$$w_{t+1} = (1+r)(w_t - c_t).$$

Substituting for c_t, we get a second-order difference equation:

$$w_{t+2} = (1+r)(1+\gamma)w_{t+1} - \gamma(1+r)^2 w_t.$$

The two boundary conditions mentioned in the problem are 1) w_0 is given, and 2) $\lim_{t\to\infty}[\beta(1+r)]^t w_t = 0$.

(e) Solve the equation you derived in (d). Find then the solutions for w_{t+1} and c_t. How does the behavior of w_{t+1} and c_t depend on whether $\beta(1+r)$ is higher or lower than one? (10 points)

Solution

For log utility, using the transversality condition, we solve the system

$$\begin{cases} c_{t+1} = \gamma c_t, \ \gamma = \beta(1+r) \\ w_{t+1} = (r+1)(w_t - c_t) \end{cases}$$

Determining c_0,

$$w_0 = \sum_{t=0}^{\infty} \frac{c_t}{(1+r)^t} = \sum_{t=0}^{\infty} \frac{c_0 \gamma^t}{(1+r)^t} = c_0 \sum_{t=0}^{\infty} \left(\frac{\gamma}{1+r}\right)^t = c_0 \sum_{t=0}^{\infty} \beta^t = \frac{c_0}{1-\beta}$$

Hence,

$$c_0 = (1-\beta)w_0.$$

Plugging back into wealth recursion, solutions are:

$$c_t = (1-\beta)w_0[\beta(1+r)]^t, \quad w_t = w_0[\beta(1+r)]^t, \tag{1}$$

Now, we analyze cases for behavior based on whether $\beta(1+r)$ is higher or lower than one:

- $\beta(1+r) < 1$
  As $t \to \infty$, both c_t and w_t go to 0. This is the typical "impatient" case in which the consumer's effective discount factor for future consumption is high enough that they gradually decumulate wealth over time.
- $\beta(1+r) < 1$
  The formulas above show that c_t, w_t grow over time, possibly unbounded. In a textbook setting without borrowing constraints, this may violate the transversality condition (because $\lim_{t\to\infty} [\beta(1+r)]^t w_t$ fails to go to 0). In practice, either the model is deemed ill-posed or the solution is a "corner" (the agent tries to postpone consumption). But purely mathematically, from the difference equations, we see that if we do not impose TVC, the path grows forever.

(f) Now solve the model using diagonalization. That is, cast the system in the form:

$$z_{t+1} = Dz_t, \quad z_t = V^{-1}y_t$$

where $y_t = [w_t \ c_t]^T$ (T here denotes transpose). Solve the model finding the policy functions for w_{t+1} and c_t. Show that you get the exact same solution as in (e). (15 points)

Solution

Define:

$$y_t = \begin{pmatrix} w_t \\ c_t \end{pmatrix}.$$

The system

$$\begin{cases} c_{t+1} = \gamma c_t, \ \gamma = \beta(1+r) \\ w_{t+1} = (r+1)(w_t - c_t) \end{cases}$$

can be written as:

$$y_{t+1} = Ay_t,$$

where:

$$A = \begin{pmatrix} 1+r & -(1+r) \\ 0 & \gamma \end{pmatrix}.$$

Using 1b, the eigenvalues of A are:

$$\lambda_1 = a_{11} = 1+r, \quad \lambda_2 = a_{22} = \gamma.$$

and eigenvectors are

$$v_1 = \left(\begin{array}{c} 1 \\ 0 \end{array} \right), \quad v_2 = \left(\begin{array}{c} -\frac{a_{12}}{a_{11}-a_{22}} \\ 1 \end{array} \right) = \begin{pmatrix} \frac{1+r}{1+r-\gamma} \\ 1 \end{pmatrix}.$$

Using diagonalization:

$$z_t = V^{-1}y_t, \quad z_{t+1} = Dz_t,$$

where

$$D = \begin{pmatrix} 1+r & 0 \\ 0 & \gamma \end{pmatrix}, \quad V = \begin{pmatrix} 1 & \frac{1+r}{1+r-\gamma} \\ 0 & 1 \end{pmatrix}$$

Transforming back, $y_t = VD^t z_0$, we have

$$c_t = (1-\beta)w_0\gamma^t, \quad w_t = w_0\gamma^t,$$

which is the same result as in (1).

2 The Solow Model (70)

A very well-known basic model in economics is the Solow model. The model can be described with a single (fundamental) equation:

$$\dot{k} = sAk^\alpha - (n_p + \delta)k$$

where s is the (fixed) savings rate, α is the share of capital in the Cobb-Douglas production function, A is the productivity parameter, n_p is the rate of population growth, and δ is the depreciation rate.

Equation (1) is a differential equation that has an analytical solution! The solution is given by:

$$k(t) = \left[\left(k_0^{1-\alpha} - \frac{sA}{n_p+\delta} \right) e^{-(1-\alpha)(n_p+\delta)t} + \frac{sA}{n_p+\delta} \right]^{\frac{1}{1-\alpha}}$$

for a given initial value of capital: $k(0) = k_0$.

In this question, however, we will use Projection Methods (Collocation) to solve this function on the range $[0, T]$, and we will use (2) to judge the quality of the numerical approximate solution.

Assume that you approximate the solution $k(t)$ to (1) by a function $\hat{k}$ of simple polynomials:

$$\dot{k} = k_0 + \sum_{i=1}^{n} a_i t^i$$

(a) Write down the function $R(t; a)$, where the vector $a = [a_1, a_2, \ldots, a_n]$. (20 points)

Solution

The residual function $R(t;a)$ is defined as the left-hand side minus the right-hand side of the ODE:

$$R(t;a) = \frac{d}{dt}\dot{k}(t) - \Big[s\,A\,\dot{k}(t)^{\alpha} - (n_p+\delta)\,\dot{k}(t)\Big].$$

Using the polynomial approximation:

$$\dot{k}(t) = k_0 + \sum_{i=1}^{n} a_i t^i, \quad \dot{\dot{k}}(t) = \sum_{i=1}^{n} i a_i t^{i-1},$$

the residual function becomes:

$$R(t;a) = \sum_{i=1}^{n} i a_i t^{i-1} - s\,A\left(k_0 + \sum_{i=1}^{n} a_i t^i\right)^{\alpha} + (n_p+\delta)\left(k_0 + \sum_{i=1}^{n} a_i t^i\right).$$

This is the function whose zeros we will enforce at the collocation points.

(b) Use the collocation method with $n = 6$ to find the approximate solution. For this, use the following values for the parameters:

$$A = 1,\ s = 0.3,\ \alpha = 0.36,\ \delta = 0.08,\ n_p = 0.02,\ k(0) = 0.01,\ T = 100$$

For the $n = 6$ nodes you need for the collocation method, use the zeros of the nth Chebyshev polynomial. Show the values a that you find with the method. (40 points)

Solution

We choose $n = 6$, so our polynomial approximation is:

$$\hat{k}(t) = k_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + a_6 t^6.$$

The zeros of the n-th Chebyshev polynomial $T_n(x)$ on $[-1,1]$ are given by:

$$x_k = \cos\left(\frac{(2k-1)\pi}{2n}\right), \quad k = 1,2,\ldots,n.$$

To map these to $[0,T]$ with $T = 100$, we use the transformation:

$$t_k = \frac{T}{2}(x_k+1), \quad k = 1,2,\ldots,n.$$

At each collocation point t_k, enforce the residual equation:

$$R(t_k;a) = 0, \quad k = 1,2,\ldots,n.$$

such that

$$F(a) = \begin{bmatrix} R(t_1; a) \\ R(t_2; a) \\ \vdots \\ R(t_6; a) \end{bmatrix}.$$

Explicitly, for each t_k:

$$\sum_{i=1}^{n} i a_i t_k^{i-1} - s\,A \left(k_0 + \sum_{i=1}^{n} a_i t_k^i\right)^{\alpha} + (n_p + \delta) \left(k_0 + \sum_{i=1}^{n} a_i t_k^i\right) = 0.$$

This yields a system of 6 nonlinear equations in the 6 unknowns $(a_1, \ldots, a_6)$.

Using a numeric solution from `ps2_q3.m`, we have

$$(a_1, \ldots, a_6) = [0.1153, 0.0087, -0.0004, 0.0000, 0.0000, 0.0000].$$

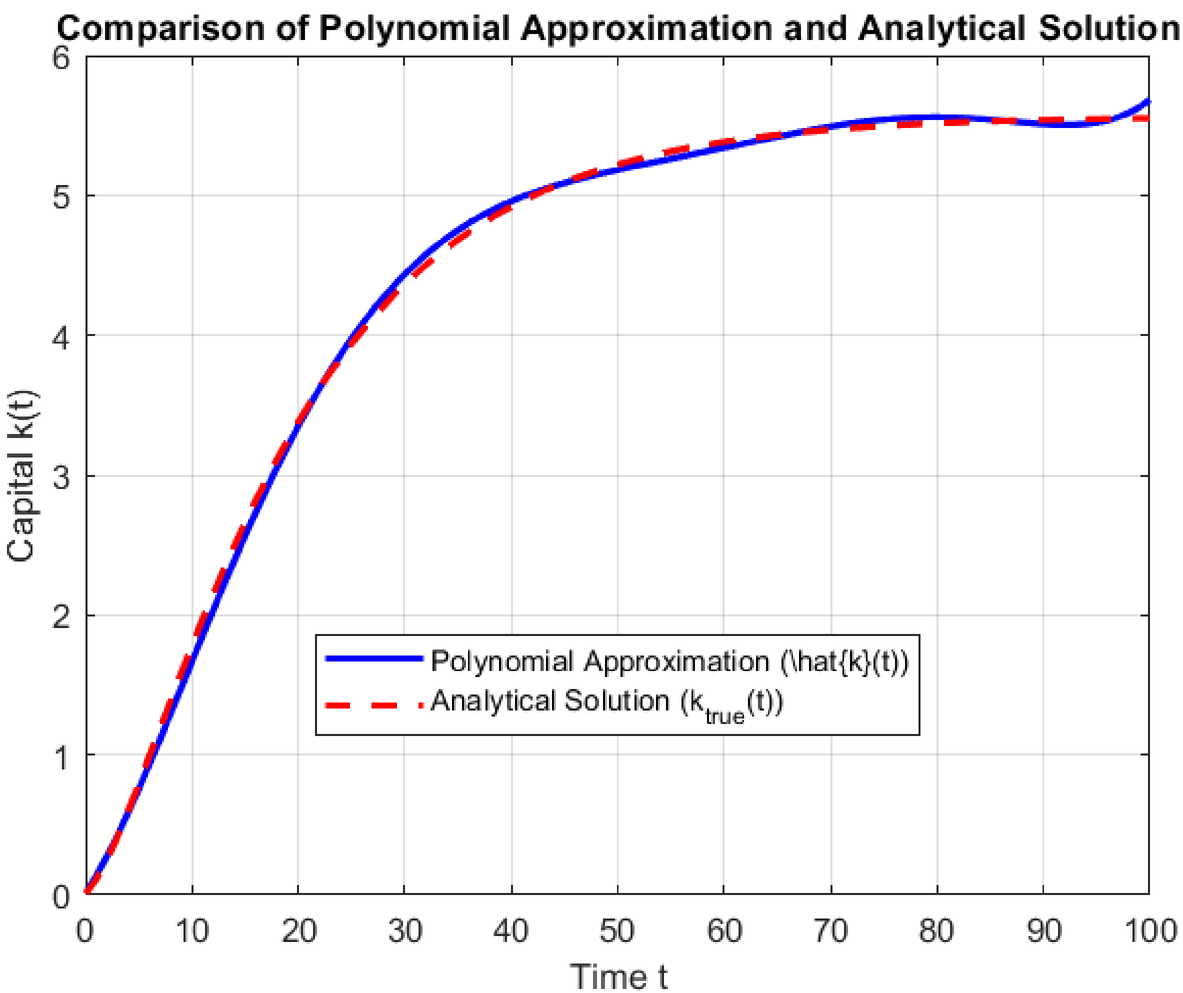
(c) Compare the approximated solution with the analytical solution. (10 points)

Solution

The analytical solution is:

$$k_{\text{true}}(t) = \left[\left(k_0^{1-\alpha} - \frac{s\,A}{n_p + \delta}\right) e^{-\left(1-\alpha\right)\left(n_p+\delta\right)t} + \frac{s\,A}{n_p + \delta}\right]^{\frac{1}{1-\alpha}}.$$

The residual norm between the solution found and the analytical solution is 0.2754. See `ps2_q3.m` for details.



Comparison of Polynomial Approximation and Analytical Solution
Capital k(t)
Time t
Polynomial Approximation (\hat{k}(t))
Analytical Solution ($k_{true}(t)$)
0
1
2
3
4
5
6
0
10
20
30
40
50
60
70
80
90
100