HK distance

In $\mathbb{R}^d$ space, let $c_\lambda(x,y) = -2\ln(\overline{\cos}(\frac{\|x-y\|}{2\sqrt{\lambda}}))$ and $\Omega \subset \mathbb{R}^d$ be a convex set.

Definition 0.1. *By ([1] Definition 5.5), the (square of) WFR (Wasserstein Fisher Rao) distance is defined by*

$$WFR^2(\mu,\nu) = \frac{1}{2}\inf_{\rho,\omega,\zeta}\int_{\Omega\times[0,1]}\left(|v|^2 + \lambda\left(\frac{\zeta}{\rho}\right)^2\right)\rho dxdt \tag{1}$$

where $(\rho, v\rho, \zeta) \in \mathcal{M}_+([0,1]\times\Omega)^{1+d+1}$ *satisfies continuity equation*

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho v) = \zeta.$$

Definition 0.2. *The Static Kantorovich formulation of HK is is defined as:*

$$HK_S(\mu,\nu) := \min_{\pi_1,\pi_2} 2\lambda\int\left(\frac{d\pi_1}{d\mathcal{L}} + \frac{d\pi_2}{d\mathcal{L}} - 2\sqrt{\frac{d\pi_1}{d\mathcal{L}}\frac{d\pi_2}{d\mathcal{L}}}\cdot\left(\frac{\|x-y\|}{2\sqrt{\lambda}}\right)\right)d\mathcal{L} \tag{2}$$

where $\mathcal{L}$ *is a reference such that* $\pi_1, \pi_2 \ll \mathcal{L}$ *and it does not affect the problem.*

By Theorem 5.6 in [1], $\mathrm{WFR}^2(\mu,\nu) = \mathrm{HK}_S(\mu,\nu)$.

Definition 0.3. *The dual formulation of HK distance is:*

$$HK_{duel}(\mu,\nu) := 2\lambda(\sup_{u,v}\int_\Omega(1-e^{-u})d\mu + \int_\Omega(1-e^{-v})d\nu) \tag{3}$$

where $u, v \in C_0(\Omega)$ *and* $u + v \leq c_\lambda(x,y), \forall(x,y)\in\Omega^2$.

By Corollary 5.8 in [1], we have $\mathrm{WFR}^2(\mu,\nu) = \mathrm{HK}_{duel}(\mu,\nu)$.

Definition 0.4. *The optimal entropy transport formuation of HK problem is defined as:*

$$HK_{ET}(\mu,\nu) := 2\lambda\min_{\pi\in\mathcal{M}_+(\Omega^2)}\int c_\lambda(x,y)d\pi + KL((P_1)_\#\pi \parallel \mu) + KL((P_2)_\#\pi \parallel \nu) \tag{4}$$

where $u, v \in C_0(\Omega)$ *and* $u + v \leq c_\lambda(x,y), \forall(x,y)\in\Omega^2$.

By Corollary 5.9 in [1], we have $\mathrm{WFR}^2(\mu,\nu) = \mathrm{HK}_{ET}(\mu,\nu)$

References

[1] Lenaic Chizat, Gabriel Peyré, Bernhard Schmitzer, and François-Xavier Vialard. Unbalanced optimal transport: Dynamic and kantorovich formulations. *Journal of Functional Analysis*, 274(11):3090–3123, 2018.