Consider the following problem:

$$\min_{\pi \in \mathbb{R}_+^{3\times 3}} \sum_{ij} c(x_i, y_j)\pi_{ij}$$

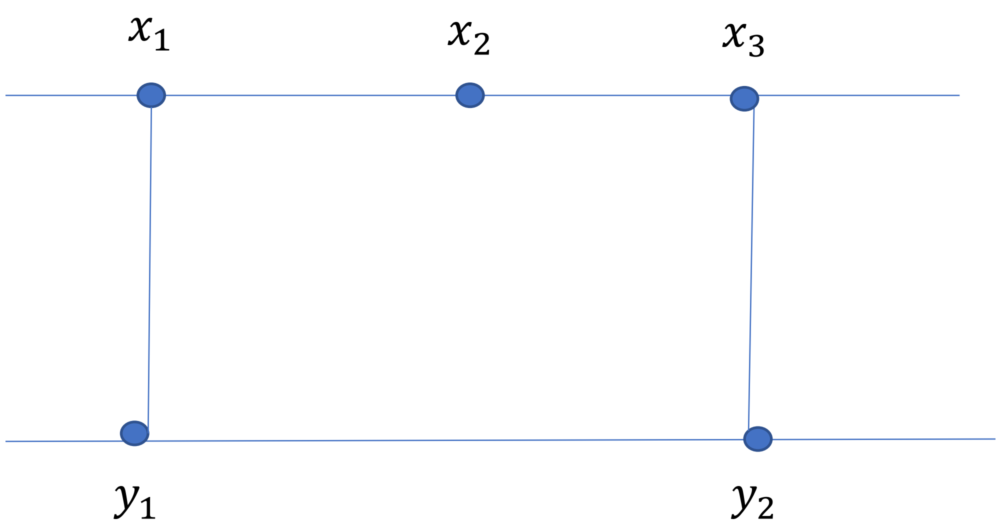
s.t. $\pi 1, \pi^T 1 \leq 1$

$\mu = \sum_{i=1}^{3} \delta_{x_i}$ $x_1 = 0, x_2 = 1, x_3 = 2$

$\nu = \sum_{i=1}^{2} \delta_{y_i}$ $y_1 = 0, y_2 = 2$

$c(x, y) = 0.1|x - y|$ and $2\lambda = 1$. Then $c < 2\lambda$ on the set $\{x_i\}_{i=1}^{3} \times \{y_j\}_{j=1}^{2}$

The optimal transportation plan is
$x_1 \to y_1, x_3 \to y_2$



Now we convert the OPT problem into OT problem, the corresponding OT problem is:

$$\min_{\pi \in \mathbb{R}_+^{4\times 4}} \sum_{ij} \tilde{c}(x, y)\pi_{ij} \text{ such that } \pi 1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix}, \pi^T 1 = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$$

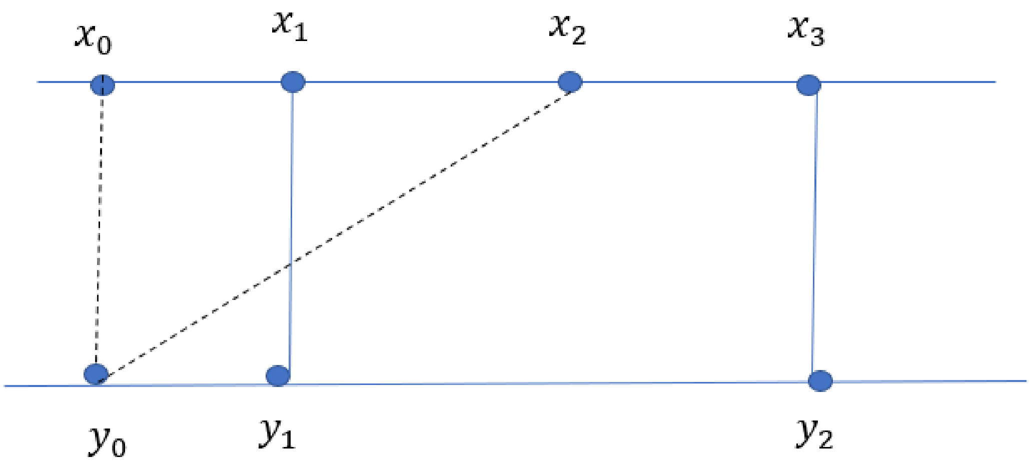
where:

$\mu = \sum_{i=1}^{3} \delta_{x_i} + 2\delta_{x_0}$, and x_0 is an isolated point and

$\nu = \sum_{i=1}^{2} \delta_{y_i} + 3\delta_{y_0}$, and y_0 is an isolated point.

$$\tilde{c}(x, y) = \begin{cases} c(x_i, y_j) - 2\lambda & if\ i, j \in [1{:}3] \times [1{:}2] \\ 0 & otherwise \end{cases}$$

The optimal plan is the following:



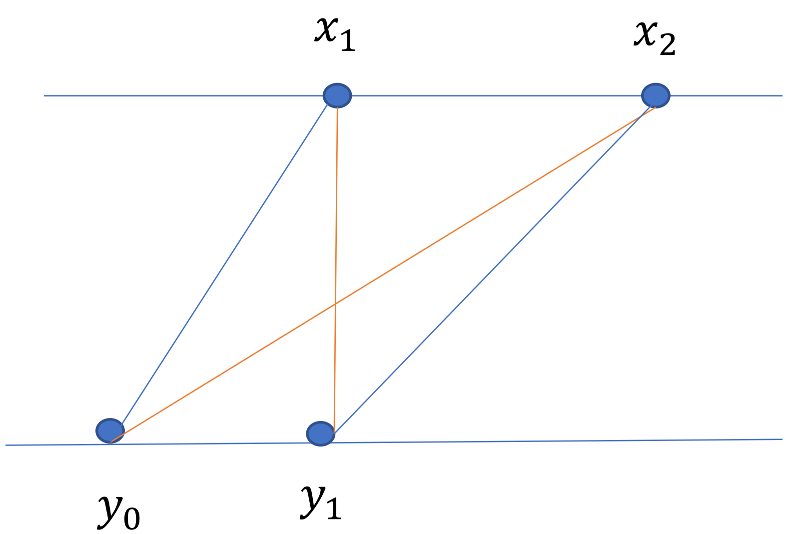
The problem is, there is cross in the optimal plan, i.e. we cannot solve the new OT problem by sorting $x_0, \ldots x_3$ and $y_0, \ldots y_3$.

Why this problem happens?

My understanding is: the reason $OT(\mu, \nu)$ in $\mathbb{R}$ can be solved by sorting is the space has an order "$\leq$" and "$\leq$" **is constant with** $c(x, y) = f(|x - y|)$.

In particular, if $y_0 \leq y_1 \leq x_2$, then $c(x_2, y_0) \geq c(x_2, y_1) + c(y_0, y_1)$. (see the graph below).

(It is a sufficient condition, but may not be necessary)



Then $cost(\, blue\ plan) = c(x_1, y_0) + c(x_2, y_1) \leq c(y_1, y_0) + c(x_2, y_1)$

$= c(x_1, y_0) + c(x_2, y_1) = cost(red\ plan).$

But, when x_0 is the isolate point, then $\tilde{c}(x, y)$ violates this condition, thus we have:

$cost(\, blue\ plan) = \tilde{c}(x_0, y_1) + \tilde{c}(x_1, y_1) = 0 + (0.1 - 2\lambda) = 0.1 - 2\lambda$

$cost(red\ plan) = \tilde{c}(x_0, y_2) + \tilde{c}(x_1, y_1) = 0 + (-2\lambda) = -2\lambda < cost(\, blue\ plan).$