$$\mathrm{K}_{\mathrm{HK}_\lambda}(\mu,\nu) = \inf \int_{\Omega^2} c_\lambda(x,y)\,\mathrm{d}\pi(x,y) + \mathrm{KL}(P_{1\sharp}\pi|\mu) + \mathrm{KL}(P_{2\sharp}\pi|\nu)$$

I think there should be a term 4λ before the two KL divergence term.

Conjecture 1: Given Monge's formulation:

$$\inf_{\pi=(I\times T)_{\#}\tilde{\mu},\tilde{\mu}\ll\mu,T_{\#}\tilde{\mu}\ll\nu} \int_{\mathrm{T}} c_\lambda(x,y)d\pi(x,y) + 4\lambda KL(P_{\#1}\pi|\mu) + 4\lambda KL(P_{\#2}\pi|\nu) \quad (1)$$

where $T: S \to \{y_j\}$ for some $S \subset \{x_i\}$, and $\mu = Unif(\{x_1, \dots x_n\}), \nu = Unif(\{y_1, \dots y_n\})$,

we claim: the optimal T is a 1-1 mapping.

It is equivalent to Corollary 3.3.

Proof: Given T which is not 1-1 mapping. We reindex x_i and y_j such that:

1. $dom\ T = \{x_1, \dots x_{n_T}\}$

$range\ T = \{y_1, \dots y_{m_T}\}$.

2. $T(x_1) = \cdots T(x_k) = y_1$ where $k \geq 2$ and $T(x_{k+1}) \neq y_1$.

3. $c_\lambda(x_1, y_1) \leq c_\lambda(x_l, y_1)\ l = 2, \dots k$.

We define T' such that $T'(x_1) = y_1, T'(x_l) = T(x_l)$ for $l = k+1, \dots n_T$.
Let π and π' denote the optimal joint measure induced by T and T' respectively.

For convenience, let $\pi(x_i, T(x_i)) = \tilde{\mu}(x_i) = \frac{p_i}{n}$ denote the mass of point $(x_i, T(x_i))$ and similarly $\pi'(x_i, T(x_i)) = \frac{p_i'}{n}$. Now we analyze p_i for $i = 1,2, \dots k$.

The object function (1) becomes

$c_\lambda(x_i, y_1)\frac{p_i}{n} + 4\lambda\left(\frac{p_i}{n}\ln p_i - \frac{p_i}{n}\right) + 4\lambda\left(\frac{(p_1+\cdots p_k)}{n}\ln(p_1+\cdots p_k) - \frac{p_i}{n}\right) + C$ where C is a constant which is independent to p_i. Let L denote the function above.

Take derivative: $\frac{d}{dp_i}L = \frac{1}{n}c_\lambda(x_i, y_1) + 4\lambda\frac{1}{n}\ln p_i + 4\lambda\frac{1}{n}\ln(p_1 + \cdots p_k)$

1. There exists p_i such that $\frac{d}{dp_i}L = 0$ for each fixed $p_j, j \neq i$.
In fact, we have close form:

$$p_i = \frac{e^{-\frac{1}{4\lambda}c_\lambda(x_1,y_1)}}{\sqrt{e^{-\frac{1}{4\lambda}c_\lambda(x_1,y_1)} + \cdots e^{-\frac{1}{4\lambda}c_\lambda(x_k,y_1)}}}, (p_1 + \cdots p_k) = \sqrt{e^{-\frac{1}{4\lambda}c_\lambda(x_1,y_1)} + \cdots e^{-\frac{1}{4\lambda}c_\lambda(x_k,y_1)}}. \quad (1.1)$$

2. L is convex with respect to each p_i.

Why? $\frac{d^2}{dp_i^2}L = \frac{4\lambda}{n}\frac{1}{p_i} + \frac{4\lambda}{n}\frac{1}{p_1+\cdots.p_k} > 0.$

Then the optimal p_i satisfies $\frac{1}{n}c_\lambda(x_i, y_1) + 4\lambda\left(\frac{1}{n}\ln p_i + \frac{1}{n}\ln(p_1 + \cdots p_k)\right) = 0.$

Similarly, we have $0 = \frac{d}{dp_1'}L = \frac{1}{n}c_\lambda(x_1, y_1) + \frac{2}{n}4\lambda \ln p_1'.$

That is $p_1' = e^{-\frac{1}{8\lambda}c_\lambda(x_1,y_1)}$. (1.2)

By (1.1)(1.2), we have $-2\ln p_1' + \ln p_1 + \ln(p_1 + \cdots p_k) = 0$

Then $p_1' = \sqrt{p_1(p_1 + \cdots.p_k)} < (p_1 + \cdots p_k)$. (1.3)

For $l > k$, since $T(x_l) = T'(x_l) \neq y_1$, then $p_i(x_l) = p_i'(x_l)$.

Let $F(\pi) = \int c_\lambda(x, y)d\pi(x, y) + 4\lambda KL(P_{\#1}\pi|\mu) + 4\lambda KL(P_{\#2}\pi|\nu)$. We need to compare $F(\pi), F(\pi')$.

$F(\pi) - F(\pi')$

$.= \left\{\sum_{i=1}^k c_\lambda(x_i, y_1)\frac{p_i}{n} + 4\lambda\left(\sum_{i=1}^k \frac{p_i}{n}\ln p_i + \frac{(p_1+\cdots p_k)}{n}\ln(p_1 + \cdots p_k) - 2\sum_{i=1}^k \frac{p_i}{n} + 2\right)\right\}$

$-\left\{c_\lambda(x_1, y_1)\frac{p_1'}{n} + 4\lambda\left(\frac{p_1'}{n}\ln p_1' + \frac{p_1'}{n}\ln(p_1') - \frac{2p_1'}{n} + 2\right)\right\}$

$= 8\lambda\left(\frac{p_1'}{n} - \sum_{i=1}^k \frac{p_i}{n}\right)$ by plugging in (1.1) (1.2)

< 0. By (1.3)

Does it violate to the conjecture 1?

Particular example:

Consider the example: $x_1 = 0, x_2 = 0.1, x_3 = 0.2, y_1 = 0, y_2 = 100, y_3 = 101.\ \lambda = \frac{1}{4}$.

The optimal 1-1 mapping, dented as T' is defined as $x_1 \mapsto y_1$.

We define T by $x_1, x_2, x_3 \mapsto y_1$

Let π, π' denote the optimal joint distribution induced by T, T' respectively.

$F(\pi') = \frac{4}{3} = 2 - \frac{2}{3}$. (0 transportation cost and $4 \cdot \frac{1}{3}$ is the cost for destroying x_2, x_3 and creating y_2, y_3.)

Now we define $\pi(x_1, y_1) = \frac{1}{3}\frac{1}{\sqrt{1+\cos^2(0.1)+\cos^2(0.2)}}, \pi(x_2, y_1) = \frac{1}{3}\frac{\cos^2(0.1)}{\sqrt{1+\cos^2(0,1)+\cos^2(0.2)}}$,

$$\pi(x_3, y_1) = \frac{1}{3}\frac{\cos^2(0.2)}{\sqrt{1+\cos^2(0,1)+\cos^2(0.2)}}$$

We have $F(\pi) = \sum_{i=1}^{3} c_\lambda(x_i, y_1)\frac{p_i}{3} + \sum_{i=1}^{3}\frac{1}{3}\ln\frac{\pi(x_i,y_1)}{1/3} - \sum_{i=1}^{3}\pi(x_i, y_i) + 1$

$+\frac{1}{3}\ln\frac{\sum_{i=1}^{3}\pi(x_i,y_1)}{1/3} - \sum_{i=1}^{3}\pi(x_i, y_i) + 1$

$= 0 - \ln\cos^2(0.1)\cdot\frac{1}{3}\frac{\cos^2(0.1)}{\sqrt{1+\cos^2(0,1)+\cos^2(0.2)}} - \ln\cos^2(0.2)\cdot\frac{1}{3}\frac{\cos^2(0.2)}{\sqrt{1+\cos^2(0,1)+\cos^2(0.2)}}$

$+\frac{1}{3}\frac{1}{\sqrt{1+\cos^2(0.1)+\cos^2(0.2)}}\ln\frac{1}{\sqrt{1+\cos^2(0.1)+\cos^2(0.2)}} + \frac{1}{3}\frac{\cos^2(0.1)}{\sqrt{1+\cos^2(0,1)+\cos^2(0.2)}}\ln\frac{\cos^2(0.1)}{\sqrt{1+\cos^2(0,1)+\cos^2(0.2)}}$

$.+\frac{1}{3}\frac{\cos^2(0.2)}{\sqrt{1+\cos^2(0.1)+\cos^2(0.2)}}\ln\frac{\cos^2(0.2)}{\sqrt{1+\cos^2(0.1)+\cos^2(0.2)}}$

$+\frac{1}{3}\sqrt{1+\cos^2(0,1)+\cos^2(0.2)}\ln\sqrt{1+\cos^2(0,1)+\cos^2(0.2)}$

$-\frac{2}{3}\sqrt{1+\cos^2(0,1)+\cos^2(0.2)} + 2$

$= 2 - \frac{2}{3}\sqrt{1+\cos^2(0.1)+\cos^2(0.2)}$

$\approx 0.85 < \frac{4}{3}$.

Note, given 1-1 mapping T, by (1.1), we can derive the closed form of

$\inf_{\pi=(I\times T)_{\#}\tilde{\mu},\tilde{\mu}\ll\mu,T_{\#}\tilde{\mu}\ll\nu} \int_{\mathrm{T}} c_\lambda(x,y)d\pi(x,y) + 4\lambda KL(P_{\#1}\pi|\mu) + 4\lambda KL(P_{\#2}\pi|\nu)$. Suppose $T: \{x_1, \dots x_k\} \to \{y_1, \dots y_k\}$.

That is, $\{x_1, \dots x_k\} = I_{tran}$, $\{x_{k+1}, \dots x_n\} = I_{dest}$.

Then we have $F(\pi) = \sum_{i=1}^{k} c_\lambda(x_i, y_i)\frac{p_i}{n} + 2\sum_{i=1}^{k}\frac{p_i}{n}\ln p_i - 2\sum_{i=1}^{k}\frac{p_i}{n} + 1 + 1$

$= 4\lambda\left(-2\sum_{i=1}^{k}\frac{p_i}{n} + 2\right)$

$= 4\lambda\left(-\frac{2}{n}\sum_{i=1}^{k} e^{-\frac{1}{2}c_\lambda(x_i,y_i)} + 2\right)$

$= 4\lambda\left(-\frac{2}{n}\sum_{i=1}^{k} \overline{\cos}\left(\frac{\|x_i-y_i\|}{2\sqrt{\lambda}}\right) + 2\right)$

$= \sum_{i\in I_{Tran}} 2\frac{4\lambda}{n}\left(1-\overline{\cos}\left(\frac{\|x_i-y_i\|}{2\sqrt{\lambda}}\right)\right) + \sum_{i\in I_{dest}} 2\cdot\frac{4\lambda}{n}$.

Compare to the result

Proposition 3.2. *Let $\mu = \frac{1}{n}\sum_{i=1}^{n}\delta_{x_i}$ and $\nu = \frac{1}{n}\sum_{j=1}^{n}\delta_{y_j}$ where all x_i, y_j are distinct. Let T^* be the optimal transport map and define $\mathcal{I}_{\text{tran}} = \{i : T^*(x_i) \neq x_i\}$ and $\mathcal{I}_{\text{dest}} = \{i : T^*(x_i) = x_i\}$. Then,*

$$d_{\text{HK}_\lambda}(\mu,\nu)^2 = \sum_{i=1}^{n}\left(\mathbb{1}_{i\in\mathcal{I}_{\text{tran}}} D(x_i, T^*(x_i)) + 2\mathbb{1}_{i\in\mathcal{I}_{\text{dest}}}\bar{D}\right)$$

where

$$D(x,y) = \frac{4\lambda}{n}\left(1-\overline{\cos}\left(\frac{\|x-y\|}{2\sqrt{\lambda}}\right)\right)$$

$$\bar{D} = \frac{4\lambda}{n}.$$

I fee there is a typo at here. Consider $n = 1$. As $\frac{\|x-y\|}{2\sqrt{\lambda}} \to \frac{\pi}{2}$, the distance $d_{HK_\lambda}(\mu,\nu)^2$ will not be a continuous function (with respect to $\frac{\|x-y\|}{2\sqrt{\lambda}}$).

Conjecture 2: The solution of unbalanced OT problem/ Logarithmic Entropy-Transport/ Hellinger–Kantorovich OT problem:

$$\inf_{\pi} \int_{\mathrm{T}} c_{\lambda}(x,y)d\pi(x,y) + KL(P_{\#1}\pi|\mu) + KL(P_{\#2}\pi|\nu)$$

where $\mu = \sum_{i=1}^{n} \frac{1}{n} \delta_{x_i}$, $\nu = \sum_{i=1}^{n} \frac{1}{n} \delta_{y_i}$ is induced by a mapping T.

If μ has a density then, as in the Wasserstein case, the minimiser π^* of the first Kantorovich form of HK is induced by a transport map [7, Theorem 6.6]. This means there exists $T^* : \Omega \to \Omega$ and a measure $\tilde{\mu} \in \mathcal{M}_+(\Omega)$ such that $\pi^* = (\mathrm{Id} \times T^*)_\sharp \tilde{\mu}$. Unlike the Wasserstein case it is not true that $\tilde{\mu}$ is equal to μ, nor is $T^*_\sharp \tilde{\mu}$ equal to ν.

$\mu = g \cdot \tilde{\mu} + \mu^{\perp}$

Add here that this is also true if $\mu = \frac{1}{n}\sum_{i=1}^{n} \delta_{x_i}$ and $\nu = \frac{1}{n}\sum_{j=1}^{n} \delta_{y_j}$... and prove this!

The related theorem is the following:

Theorem 6.6 (Uniqueness) *Let $\mu_i \in \mathcal{M}(X)$ and $\boldsymbol{\gamma} \in \mathrm{Opt}_{\mathsf{LET}}(\mu_1, \mu_2)$.*

(i) *The marginals $\gamma_i = \pi^i_\sharp \boldsymbol{\gamma}$ are uniquely determined.*

(ii) *If $X = \mathbb{R}$ with the usual distance then $\boldsymbol{\gamma}$ is the unique element of $\mathrm{Opt}_{\mathsf{LET}}(\mu_1, \mu_2)$.*

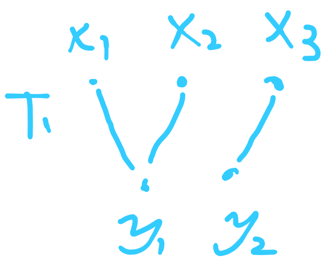
(iii) *If $X = \mathbb{R}^d$ with the usual distance, $\mu_1 \ll \mathscr{L}^d$ is absolutely continuous, and $A_i \subset \mathbb{R}^d$ and $\sigma_i : A_i \to (0, \infty)$ are as in Theorem 6.3 b), then σ_1 is approximately differentiable at μ_1-a.e. point of A_1 and $\boldsymbol{\gamma}$ is the unique element of $\mathrm{Opt}_{\mathsf{LET}}(\mu_1, \mu_2)$. The transport plan $\boldsymbol{\gamma}$ is concentrated on the graph of a function $\boldsymbol{t} : \mathbb{R}^d \to \mathbb{R}^d$ satisfying*

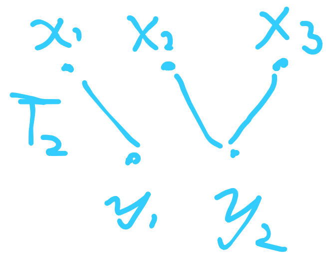
$$\boldsymbol{t}(x_1) = x_1 - \frac{\arctan(|\boldsymbol{\xi}(x_1)|)}{|\boldsymbol{\xi}(x_1)|}\boldsymbol{\xi}(x_1),$$
$$\boldsymbol{\xi}(x_1) = -\frac{1}{2}\tilde{\mathrm{D}} \log \sigma_1(x_1) \tag{6.29}$$

Consider the following example:

$x_1 = -2, x_2 = 0, x_3 = 2$

$y_1 = -1, y_2 = 1, y_3 = 100$





We first consider the following two mappings:

$T_1: \{x_1, x_2\} \mapsto y_1, x_3 \mapsto y_2$

$T_2: x_1 \mapsto y_1, \{x_2, x_3\} \mapsto y_2$. We aim to show T_1, T_2 are not optimal mappings.

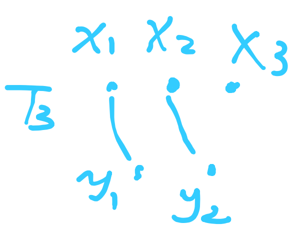
We have: $F(T_1) = F(T_2)$.

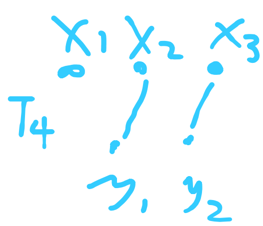
Let π_1, π_2 denoted the optimal joint distribution induced by T_1, T_2 respectively. We have:

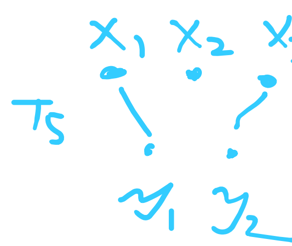
$P_{2,\#}\pi_1 \neq P_{2,\#}\pi_1$, indeed, $P_{2,\#}\pi_1(y_1) > P_{1,\#}\pi_1(x_1)$ while $P_{2,\#}\pi_2(y_1) = P_{1,\#}\pi_2(x_1)$.

Then by Theorem 6.6 (i), both T_1, T_2 can not be optimal mappings.

Similarly, we consider the following mappings T_3, T_4, T_5.







$T_3: x_1 \mapsto y_1, x_2 \mapsto y_2$

$.T_4: x_2 \mapsto y_1, x_3 \mapsto y_2$

$T_5: x_1 \mapsto y_1, x_3 \mapsto y_2$

We have $F(T_3) = F(T_4) = F(T_5)$. But $P_{1,\#}\pi(T_3), P_{1,\#}\pi(T_4), P_{1,\#}\pi(T_5)$ are pair-wisely different.

Then T_3, T_4, T_5 can not be the optimal mappings.

We do not have other choices of (monotonic increasing) mapping T.

So, does it violate the conjecture 2?