

2 Basics concepts in risk management

2.1 Risk management for a financial firm

2.2 Modelling value and value change

2.3 Risk measurement

2.1 Risk management for a financial firm

2.1.1 Assets, liabilities and the balance sheet

The risks of a firm (here: **bank**) can be understood from its balance sheet:

Assets Investments of the firm		Liabilities Obligations from fundraising	
Cash (and central bank balance)	£10M	Customer deposits	£80M
Securities	£50M	Bonds issued	
- bonds, stocks, derivatives		- senior bond issues	£25M
Loans and mortgages	£100M	- subordinated bond issues	£15M
- corporates		Short-term borrowing	£30M
- retail and smaller clients		Reserves (for losses on loans)	£20M
- government			
Other assets	£20M	Debt (sum of above)	£170M
- property			
- investments in companies		Equity	£30M
Short-term lending	£20M		
Total	£200M	Total	£200M

A stylized balance sheet for an **insurer** is:

Assets		Liabilities	
Investments		Reserves for policies written	£80M
- bonds	£50M	(technical provisions)	
- stocks	£5M	Bonds issued	£10M
- property	£5M		
Investments for unit-linked contracts	£30M	Debt (sum of above)	£90M
Other assets	£10M	Equity	£10M
- property			
Total	£100M	Total	£100M

- Balance sheet equation: $\text{Assets} = \text{Liabilities} = \text{Debt} + \text{Equity}$.
If equity > 0 , the company is *solvent*, otherwise *insolvent*.
- **Valuation** of the items on the balance sheet is a **non-trivial** task.
 - ▶ *Amortized cost accounting* values a position a *book value* at its inception and this is carried forward/progressively reduced over time.

- ▶ *Fair-value accounting* values assets at prices they are sold and liabilities at prices that would have to be paid in the market. This can be challenging for non-traded or illiquid assets or liabilities.

There is a tendency in the industry to move towards fair-value accounting. Market consistent valuation in Solvency II follows similar principles.

2.1.2 Risks faced by a financial firm

- Decrease in the value of the investments on the asset side of the balance sheet (e.g. losses from securities trading or credit risk).
- *Maturity mismatch* (large parts of the assets are relatively illiquid (long-term) whereas large parts of the liabilities are rather short-term obligations. This can lead to a default of a solvent bank or a bank run).
- The prime risk for an insurer is *insolvency* (risk that claims of policy holders cannot be met). On the asset side, risks are similar to those of a bank. On the liability side, the main risk is that reserves are insufficient

to cover future claim payments. Note that the **liabilities of a life insurer are of a long-term nature** and subject to multiple categories of risk (e.g. interest rate risk, inflation risk and longevity risk).

- So risk is found on **both sides** of the balance sheet and thus RM should not focus on the asset side alone.

2.1.3 Capital

- There are different notions of **capital**. One distinguishes:

- Equity capital*
 - Value of **assets** — **debt**;
 - Measures the firm's value to its shareholders;
 - Can be split into *shareholder capital* (initial capital invested in the firm) and *retained earnings* (accumulated earnings not paid to shareholders).
- Regulatory capital* — Capital required according to **regulatory rules**;

- For European insurance firms: Minimum (MCR) and solvency capital requirements (SCR);
- A regulatory framework also specifies the **capital quality**. One distinguishes **Tier 1 capital** (i.e. shareholder capital + retained earnings; **can act in full as buffer**) and **Tier 2 capital** (includes other positions on the balance sheet).

Economic capital

- Capital required to **control the probability of becoming insolvent** (typically over one year);
- **Internal assessment** of risk capital;
- Aims at a holistic view (assets and liabilities) and works with fair values of balance sheet items.

- **All of these notions** refer to items on the liability side that entail no obligations to outside creditors; they **can** thus **serve as buffer against losses**.

2.2 Modelling value and value change

2.2.1 Mapping of risks

We set up a general mathematical model for (changes in) value caused by financial risks. To this end we work on a *probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ and consider a risk or loss as a *random variable* $X : \Omega \rightarrow \mathbb{R}$ (or: L).

- Consider a *portfolio* of assets and possibly liabilities. The *value* of the portfolio at time t (*today*) is denoted by V_t (a random variable; assumed to be known at t ; its *df* is typically *not trivial to determine!*).
- We consider a given *time horizon* Δt and *assume*:
 - 1) the *portfolio composition remains fixed* over Δt ;
 - 2) there are *no intermediate payments* during Δt

\Rightarrow *Fine for small Δt* but *unlikely to hold for large Δt* .

- The *change* in value of the portfolio is given by

$$\Delta V_{t+1} = V_{t+1} - V_t$$

and we define the (random) *loss* by the *sign-adjusted* value change

$$L_{t+1} = -\Delta V_{t+1}$$

(as QRM is mainly concerned with losses).

Remark 2.1

- 1) The *distribution of L_{t+1}* is called *loss distribution*.
- 2) Practitioners often consider the *profit-and-loss (P&L) distribution* which is the distribution of $-L_{t+1} = \Delta V_{t+1}$.
- 3) For longer time intervals, $\Delta V_{t+1} = V_{t+1}/(1 + r) - V_t$ (r = *risk-free interest rate*) would be more appropriate, but we will *mostly neglect* this issue.

- V_t is typically modelled as a function f of time t and a d -dimensional random vector $\mathbf{Z} = (Z_{t,1}, \dots, Z_{t,d})$ of risk factors, that is,

$$V_t = f(t, \mathbf{Z}_t) \quad (\text{mapping of risks})$$

for some measurable $f : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}$. The choice of f and \mathbf{Z}_t is problem-specific (typically known, but possibly difficult to evaluate).

- It is often convenient to work with the risk-factor changes

$$\mathbf{X}_{t+1} = \mathbf{Z}_{t+1} - \mathbf{Z}_t.$$

We can rewrite L_{t+1} in terms of \mathbf{X}_{t+1} via

$$\begin{aligned} L_{t+1} &= -(V_{t+1} - V_t) = -(f(t+1, \mathbf{Z}_{t+1}) - f(t, \mathbf{Z}_t)) \\ &= -(f(t+1, \mathbf{Z}_t + \mathbf{X}_{t+1}) - f(t, \mathbf{Z}_t)). \end{aligned}$$

We see that the loss df is determined by the loss df of \mathbf{X}_{t+1} . We will thus also write $L_{t+1} = L(\mathbf{X}_{t+1})$, where $L(\mathbf{x}) = -(f(t+1, \mathbf{Z}_t + \mathbf{x}) - f(t, \mathbf{Z}_t))$ is known as loss operator.

- If f is differentiable, its **first-order (Taylor) approximation** ($f(\mathbf{y}) \approx f(\mathbf{y}_0) + \nabla f(\mathbf{y}_0)(\mathbf{y} - \mathbf{y}_0)$ for $\mathbf{y} = (t+1, Z_{t,1} + X_{t+1,1}, \dots, Z_{t,d} + X_{t+1,d})$ and $\mathbf{y}_0 = (t, Z_{t,1}, \dots, Z_{t,d})$) is

$$f(t+1, \mathbf{Z}_t + \mathbf{X}_{t+1}) \approx f(t, \mathbf{Z}_t) + f_t(t, \mathbf{Z}_t) \cdot 1 + \sum_{j=1}^d f_{z_j}(t, \mathbf{Z}_t) \cdot X_{t+1,j}$$

We can thus approximate L_{t+1} by the **linearized loss**

$$L_{t+1}^{\Delta} = - \left(\underbrace{f_t(t, \mathbf{Z}_t)}_{=: c_t} + \sum_{j=1}^d \underbrace{f_{z_j}(t, \mathbf{Z}_t)}_{=: b_{t,j}} X_{t+1,j} \right) = -(c_t + \mathbf{b}'_t \mathbf{X}_{t+1}),$$

a linear function of $X_{t+1,1}, \dots, X_{t+1,d}$ (indices denote partial derivatives).
The **approximation is best if the risk-factor changes are small in absolute value.**

Example 2.2 (Stock portfolio)

Consider a **portfolio** \mathcal{P} of d **stocks** $S_{t,1}, \dots, S_{t,d}$ ($S_{t,j}$ = value of stock j at time t) and denote by λ_j the **number of shares** of stock j in \mathcal{P} . In finance and risk management, one typically uses **logarithmic prices as risk factors**, i.e. $Z_{t,j} = \log S_{t,j}$, $j \in \{1, \dots, d\}$. Then

$$V_t = f(t, \mathbf{Z}_t) = \sum_{j=1}^d \lambda_j S_{t,j} = \sum_{j=1}^d \lambda_j e^{Z_{t,j}}.$$

- The one-period ahead loss is then given by

$$\begin{aligned} L_{t+1} &= -(V_{t+1} - V_t) = -\sum_{j=1}^d \lambda_j (e^{Z_{t,j} + X_{t+1,j}} - e^{Z_{t,j}}) \\ &= -\sum_{j=1}^d \lambda_j e^{Z_{t,j}} (e^{X_{t+1,j}} - 1) = -\sum_{j=1}^d \underbrace{\lambda_j S_{t,j}}_{=: \tilde{w}_{t,j}} (e^{X_{t+1,j}} - 1) \quad (1) \end{aligned}$$

which is non-linear in $X_{t+1,j}$ (here: $L(\mathbf{x}) = -\sum_{j=1}^d \tilde{w}_{t,j} (e^{x_j} - 1)$).

- With $f_{z_j}(t, \mathbf{Z}_t) = \lambda_j e^{Z_{t,j}} = \lambda_j S_{t,j} = \tilde{w}_{t,j}$, the **linearized loss** is

$$\begin{aligned} L_{t+1}^{\Delta} &= -\left(f_t(t, \mathbf{Z}_t) + \sum_{j=1}^d f_{z_j}(t, \mathbf{Z}_t) X_{t+1,j}\right) = -\left(0 + \sum_{j=1}^d \tilde{w}_{t,j} X_{t+1,j}\right) \\ &= -\tilde{\mathbf{w}}_t' \mathbf{X}_{t+1}. \end{aligned}$$

- Note that $L_{t+1}^{\Delta} = -(c_t + \mathbf{b}_t' \mathbf{X}_{t+1})$ for $c_t = 0$ and $\mathbf{b}_t = \tilde{\mathbf{w}}_t$.
- If $\boldsymbol{\mu} = \mathbb{E} \mathbf{X}_{t+1}$ and $\Sigma = \text{cov} \mathbf{X}_{t+1}$ are known, then **expectation** and **variance of the (linearized) one-period ahead loss** are

$$\mathbb{E} L_{t+1}^{\Delta} = -\sum_{j=1}^d \tilde{w}_{t,j} \mathbb{E}(X_{t+1,j}) = -\tilde{\mathbf{w}}_t' \boldsymbol{\mu},$$

$$\text{var} L_{t+1}^{\Delta} = \text{var}(\tilde{\mathbf{w}}_t' \mathbf{X}_{t+1}) = \tilde{\mathbf{w}}_t' \text{cov}(\mathbf{X}_{t+1}) \tilde{\mathbf{w}}_t = \tilde{\mathbf{w}}_t' \Sigma \tilde{\mathbf{w}}_t.$$

- If \mathbf{X}_{t+1} is multivariate normal, then $L_{t+1}^{\Delta} \sim N(-\tilde{\mathbf{w}}_t' \boldsymbol{\mu}, \tilde{\mathbf{w}}_t' \Sigma \tilde{\mathbf{w}}_t)$.

Example 2.3 (European call option)

Consider a portfolio consisting of a European call option on a non-dividend-paying stock S_t with maturity T and strike (exercise price) K . The Black–Scholes formula says that today's value is

$$V_t = C^{\text{BS}}(t, S_t; r, \sigma, K, T) = S_t \Phi(d_1) - K e^{-r(T-t)} \Phi(d_2), \quad (2)$$

where

- t is the time in years;
- Φ is the df of $N(0, 1)$;
- r is the continuously compounded risk-free interest rate;
- $d_1 = \frac{\log(S_t/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$ and $d_2 = d_1 - \sigma\sqrt{T-t}$; and
- σ is the annualized volatility (standard deviation) of $\log(S_t/S_{t-1})$.

While (2) assumes r, σ to be constant, this is often not true in real markets.

Hence, besides $\log S_t$, we consider r_t, σ_t as risk factors, so

$$\mathbf{Z}_t = (\log S_t, r_t, \sigma_t) \Rightarrow \mathbf{X}_{t+1} = (\log(S_{t+1}/S_t), r_{t+1} - r_t, \sigma_{t+1} - \sigma_t).$$

This implies that the mapping f (in terms of the risk factors) is given by

$$V_t = C^{\text{BS}}(t, e^{Z_{t,1}}; Z_{t,2}, Z_{t,3}, K, T) =: f(t, \mathbf{Z}_t)$$

and the linearized one-day ahead loss (omitting the arguments of C^{BS}) is

$$\begin{aligned} L_{t+1}^{\Delta} &= -\left(f_t(t, \mathbf{Z}_t) + \sum_{j=1}^3 f_{z_j}(t, \mathbf{Z}_t) X_{t+1,j}\right) \\ &= -(C_t^{\text{BS}} \Delta t + C_{S_t}^{\text{BS}} S_t X_{t+1,1} + C_{r_t}^{\text{BS}} X_{t+1,2} + C_{\sigma_t}^{\text{BS}} X_{t+1,3}). \end{aligned}$$

If our risk management horizon is 1 d (as opposed to 1 y), we need to introduce $\Delta t := 1/250$ here. Note that the “Greeks” enter (C_t^{BS} is the *theta* of the option; $C_{S_t}^{\text{BS}}$ the *delta*; $C_{r_t}^{\text{BS}}$ the *rho*; $C_{\sigma_t}^{\text{BS}}$ the *vega*).

For portfolios of derivatives, L_{t+1}^{Δ} can be a rather poor approximation to $L_{t+1} \Rightarrow$ higher-order (Taylor) approximations such as the *delta-gamma-approximation* (second-order) can be used.

2.2.2 Valuation methods

Fair value accounting

The *fair value* of an asset/liability is an *estimate of the price* which would be *received/paid* on an *active market*. One distinguishes:

- Level 1** *Mark-to-market*. *Fair value* of an investment is *determined from quoted prices* for the *same instrument*; see Example 2.2.
- Level 2** *Mark-to-model with objective inputs*. The *fair value* of an instrument is determined *using quoted prices* in active markets *for similar instruments* or by using valuation techniques/models with inputs based on observable market data; see Example 2.3.
- Level 3** *Mark-to-model with subjective inputs*. The *fair value* of an instrument is determined using valuation techniques/models for which *some inputs are not observable* in the market (e.g. determining default risk of portfolios of loans to companies for which no CDS spreads are available).

Risk-neutral valuation

- ... is **widely used** for pricing financial products, e.g. derivatives
- **value** of a financial instrument **today** = **expected discounted values of future cash flows**; the expectation is **taken w.r.t. the risk-neutral pricing measure Q** (also called *equivalent martingale measure (EMM)*; it turns discounted prices into martingales, so fair bets) as opposed to the real world/**physical measure \mathbb{P}** .
- An **risk-neutral pricing measure** is a **probability measure Q** such that the **expectation of the discounted payoff w.r.t. Q** equals V_0 (fair bet).
- **Risk-neutral valuation at t of a claim H at T** is done via the **risk-neutral pricing rule**

$$V_t^H = \mathbb{E}_{Q,t}(e^{-r(T-t)}H), \quad t < T,$$

where $\mathbb{E}_{Q,t}(\cdot)$ denotes expectation w.r.t. Q given the information up to and including time t .

- **\mathbb{P} is estimated from historical data; Q is calibrated to market prices.**

Example 2.4 (European call option continued)

- Suppose that options with strike K or maturity T are not traded, but other options on the same stock are.
- Under \mathbb{P} the stock price (S_t) is assumed to follow a geometric Brownian motion (GBM) (the so-called *Black–Scholes model*) with dynamics $dS_t = \mu S_t dt + \sigma S_t dW_t$ for constants $\mu \in \mathbb{R}$ (drift) and $\sigma > 0$ (volatility), and a standard Brownian motion (W_t) .
- Under the EMM \mathbb{Q} , $(e^{-rt}S_t)$ is a martingale and S_t follows a GBM with drift r and volatility σ .
- The European call option payoff is $H = (S_T - K)^+ = \max\{S_T - K, 0\}$ and the risk-neutral valuation formula may be shown to be

$$V_t = E_t^{\mathbb{Q}}(e^{-r(T-t)}(S_T - K)^+) = C^{\text{BS}}(t, S_t; r, \sigma, K, T), \quad t < T; \quad (3)$$

- One typically uses quoted prices $C^{\text{BS}}(t, S_t; r, \sigma, K^*, T^*)$ (for different K^*, T^*) to infer the unknown σ . Then plug this so-called *implied volatility* into (3).

2.2.3 Loss distributions

Having determined the mapping f (may involve *valuation models*, e.g. Black–Scholes, or numerical approximation), we can identify the following **key statistical tasks of QRM**:

- 1) Find a statistical **model for \mathbf{X}_{t+1}** (typically a model for forecasting \mathbf{X}_{t+1} , estimated based on historical data);
- 2) Compute/derive the **df $F_{L_{t+1}}$** (requires the df of $f(t+1, \mathbf{Z}_t + \mathbf{X}_{t+1})$);
- 3) Compute a **risk measure** (see later) **from $F_{L_{t+1}}$** .

There are **three general methods** to approach these challenges.

1) Analytical method

Idea: Choose $F_{\mathbf{X}_{t+1}}$ and f such that $F_{L_{t+1}}$ can be determined explicitly.

Prime example: *Variance-covariance method*, see RiskMetrics (1996):

Assumption 1 $\mathbf{X}_{t+1} \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$ (e.g. if (\mathbf{Z}_t) is a Brownian motion, (S_t) a geometric Brownian motion)

Assumption 2 $F_{L_{t+1}^\Delta}$ is a good approximation to $F_{L_{t+1}}$.

$$L_{t+1}^\Delta = -(c_t + \mathbf{b}_t' \mathbf{X}_{t+1}) \xRightarrow{\text{Ass. 1}} L_{t+1}^\Delta \sim \mathcal{N}(-c_t - \mathbf{b}_t' \boldsymbol{\mu}, \mathbf{b}_t' \Sigma \mathbf{b}_t)$$

Advantages:

- $F_{L_{t+1}^\Delta}$ explicit (\Rightarrow typically explicit risk measures)
- Easy to implement (unless d extremely large)

Drawbacks: Assumption 1 is unlikely to be realistic for daily (probably also weekly/monthly) data. Stylized facts about \mathbf{X}_{t+1} suggest that $F_{\mathbf{X}_{t+1}}$ is leptokurtic (thinner body, heavier tail than $\mathcal{N}(\boldsymbol{\mu}, \Sigma)$). Thus, $\mathbf{X}_{t+1} \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$ underestimates the tail of $F_{L_{t+1}}$ and thus risk measures such as VaR.

When dynamic models for \mathbf{X}_{t+1} are considered, different estimation methods are possible depending on whether we focus on conditional distributions $F_{\mathbf{X}_{t+1} | (\mathbf{X}_s)_{s \leq t}}$ or the equilibrium distribution $F_{\mathbf{X}}$ in a stationary model.

2) Historical simulation

Idea: Estimate $F_{L_{t+1}}$ by its *empirical distribution function (edf)*

$$\hat{F}_{L_{t+1},n}(x) = \frac{1}{n} \sum_{i=1}^n I_{\{L_{t-i+1} \leq x\}}, \quad x \in \mathbb{R}, \quad (4)$$

based on

$$L_k = L(\mathbf{X}_k) = -(f(t+1, \mathbf{Z}_t + \mathbf{X}_k) - f(t, \mathbf{Z}_t)), \quad (5)$$

$k \in \{t-n+1, \dots, t\}$. L_{t-n+1}, \dots, L_t show what would happen to the current portfolio if the past n risk-factor changes were to recur.

Advantages: ■ Easy to implement

■ No estimation of the distribution of \mathbf{X}_{t+1} required

Drawbacks: ■ Sufficient data for all risk-factor changes required

■ Only past losses considered (“driving a car by looking in the back mirror”)

3) Monte Carlo method

Idea: Take any model for \mathbf{X}_{t+1} , simulate \mathbf{X}_{t+1} , compute the corresponding losses as in (5) and estimate $F_{L_{t+1}}$ (typically via edf as in (4)).

Advantages: ■ Quite general (applicable to any model of \mathbf{X}_{t+1} which is easy to sample)

Drawbacks: ■ Unclear how to find an appropriate model for \mathbf{X}_{t+1} (any result is only as good as the chosen $F_{\mathbf{X}_{t+1}}$)

- Computational cost (every simulation requires to evaluate the mapping f ; expensive, e.g. if the latter contains derivatives which are priced via Monte Carlo themselves \Rightarrow Nested Monte Carlo simulations)

So-called *economic scenario generators* (i.e. economically motivated dynamic models for the evolution and interaction of risk factors) used in insurance also fall under the heading of Monte Carlo methods.

2.3 Risk measurement

- A *risk measure* for a financial position with (random) loss L is a **real number** which **measures the “riskiness of L ”**. In the Basel or Solvency context, it is often interpreted as the amount of **capital required to make a position with loss L acceptable** to an (internal/external) regulator.
- Some **reasons for using risk measures** in practice:
 - ▶ To **determine the amount of capital to hold** as a buffer against unexpected future losses on a portfolio (in order to satisfy a regulator/manager concerned with the institution's solvency).
 - ▶ As a **tool for limiting** the amount of **risk of a business unit** (e.g. by requiring that the daily 95% value-at-risk (i.e. the 95%-quantile) of a trader's position should not exceed a given bound).
 - ▶ To determine the **riskiness** (and **thus fair premium**) of an **insurance contract**.

2.3.1 Approaches to risk measurement

Existing risk measurement approaches grouped into three categories:

1) Notional-amount approach

- oldest approach; “standardized approaches” of Basel II (e.g. OpRisk)
- *risk of a portfolio* = summed notional values of the securities times their riskiness factor.
- Advantages: ► simplicity
- Drawbacks: ► No differentiation between long and short positions and no netting: For example, the risk of a long position in corporate bonds hedged by an offsetting position in credit default swaps is counted as twice the risk of the unhedged bond position.

- ▶ **No diversification** benefits: risk of a portfolio of loans to many companies = risk of a portfolio where the whole amount is lent to a single company.
- ▶ Problems for **portfolios of derivatives**: **notional** amount of the underlying **can** widely **differ from the economic value** of the derivative position.

2) Risk measures based on loss distributions

- Most modern **risk measures are characteristics** of the underlying **loss distribution** over some predetermined time horizon Δt .
- Examples: variance, **value-at-risk**, **expected shortfall** (see later)
- **Advantages:** ▶ **Makes sense on all levels** (from single portfolios to the overall position of a financial institution).
 - ▶ Loss distributions **reflect netting** and **diversification**.

- Drawbacks:
- ▶ Estimates of loss distributions are typically based on past data.
 - ▶ It is difficult to estimate loss distributions accurately (especially for large portfolios).
 - ⇒ Risk measures should be complemented by information from scenarios (forward-looking).

3) Scenario-based risk measures

- Typically considered in stress testing.
- One considers possible future risk-factor changes (*scenarios*; e.g. a 20% drop in a market index).
- *Risk of a portfolio* = maximum (weighted) loss under all scenarios.
- If $\mathcal{X} = \{x_1, \dots, x_n\}$ denote the risk-factor changes (*scenarios*) with corresponding weights $w = (w_1, \dots, w_n)$, the risk is $\psi_{\mathcal{X}, w} = \max_{1 \leq i \leq n} \{w_i L(x_i)\}$, where $L(x)$ denotes the loss the portfolio

would suffer if the hypothetical scenario x were to occur. Many risk measures are of this form; see *CME SPAN: Standard Portfolio Analysis of Risk* (2010).

- Mathematical interpretation:

- ▶ Assume $L(\mathbf{0}) = 0$ (okay if Δt small) and $w_i \in [0, 1] \forall i$.
- ▶ $w_i L(x_i) = w_i L(x_i) + (1 - w_i) L(\mathbf{0}) = \mathbb{E}_{\mathbb{P}_i}(L(\mathbf{X}))$ where $\mathbf{X} \sim \mathbb{P}_i$ and \mathbb{P}_i is such that $\mathbb{P}_i(\mathbf{X} = x_i) = w_i$ and $\mathbb{P}_i(\mathbf{X} = \mathbf{0}) = 1 - w_i$.

Therefore, $\psi_{\mathcal{X}, w} = \max\{\mathbb{E}_{\mathbb{P}}(L(\mathbf{X})) : \mathbf{X} \sim \mathbb{P} \in \{\mathbb{P}_1, \dots, \mathbb{P}_n\}\}$. Such a risk measure is known as a *generalized scenario*; they play an important role in the theory of *coherent risk measures*.

- **Advantages:**
 - ▶ Useful for portfolios with few risk factors.
 - ▶ Useful complementary information to risk measures based on loss distributions (past data).

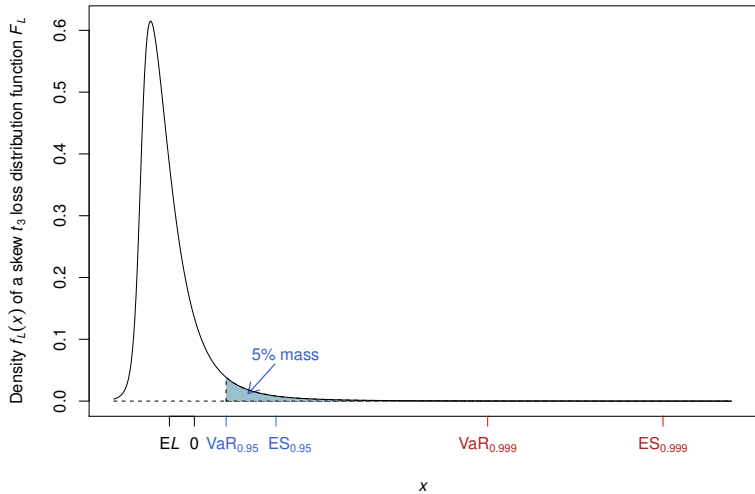
Drawbacks: ▶ Determining scenarios and weights.

2.3.2 Value-at-risk

Definition 2.5 (Value-at-risk)

For a loss $L \sim F_L$, *value-at-risk* (VaR) at confidence level $\alpha \in (0, 1)$ is defined by $\text{VaR}_\alpha = \text{VaR}_\alpha(L) = F_L^{\leftarrow}(\alpha) = \inf\{x \in \mathbb{R} : F_L(x) \geq \alpha\}$.

- VaR_α is simply the α -quantile of F_L . As such, $F_L(\text{VaR}_\alpha(L)) = F_L(F_L^{\leftarrow}(\alpha)) \geq \alpha$ and $F_L(x) < \alpha$ for all $x < \text{VaR}_\alpha(L)$.
- $\text{VaR}_\alpha(L) = \inf\{x \in \mathbb{R} : \bar{F}_L(x) \leq 1 - \alpha\}$ ($\bar{F}_L(x) = 1 - F_L(x)$), so VaR_α is the **smallest loss which is exceeded with prob. at most $1 - \alpha$** .
- Known since 1994: Weatherstone 4¹⁵ report (J.P. Morgan; RiskMetrics)
- VaR is the **most widely used risk measure** (by Basel II or Solvency II)
- $\text{VaR}_\alpha(L)$ is **not** a **what if** risk measure: It **does not provide information about the severity of losses which occur with probability $\leq 1 - \alpha$** .



Interlude: Generalized inverses

$T \nearrow$ means that T is *increasing*, i.e. $T(x) \leq T(y)$ for all $x < y$. $T \uparrow$ means that T is *strictly increasing*, i.e. $T(x) < T(y)$ for all $x < y$.

Definition 2.6 (Generalized inverse)

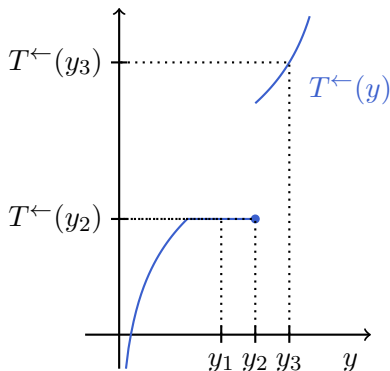
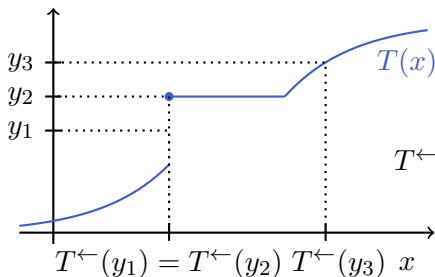
For any increasing function $T : \mathbb{R} \rightarrow \mathbb{R}$, with $T(-\infty) = \lim_{x \downarrow -\infty} T(x)$ and $T(\infty) = \lim_{x \uparrow \infty} T(x)$, the *generalized inverse* $T^{\leftarrow} : \mathbb{R} \rightarrow \bar{\mathbb{R}} = [-\infty, \infty]$ of T is defined by

$$T^{\leftarrow}(y) = \inf\{x \in \mathbb{R} : T(x) \geq y\}, \quad y \in \mathbb{R},$$

with the convention that $\inf \emptyset = \infty$. If T is a df, $T^{\leftarrow} : [0, 1] \rightarrow \bar{\mathbb{R}}$ is the *quantile function* of T .

- If T is *continuous and \uparrow* , then $T^{\leftarrow} \equiv T^{-1}$ (ordinary inverse).
- There are *rules for working with T^{\leftarrow}* (often, not always) similar to T^{-1} ; see Proposition A.17.

How to determine T^{\leftarrow} from T :



- Flat parts (jumps) of T correspond to jumps (flat parts) of T^{\leftarrow} .
- Assume T to be a df and $L \sim T$.
 - What is the probability that L falls in the region where T is flat?
 - What is $\mathbb{P}(L = T^{\leftarrow}(y_1))$?

Example 2.7 (VaR_α for $N(\mu, \sigma^2)$ and $t_\nu(\mu, \sigma^2)$)

1) Let $L \sim N(\mu, \sigma^2)$. Then

$$F_L(x) = \mathbb{P}(L \leq x) = \mathbb{P}((L - \mu)/\sigma \leq (x - \mu)/\sigma) = \Phi((x - \mu)/\sigma).$$

This implies that

$$\text{VaR}_\alpha(L) = F_L^{\leftarrow}(\alpha) = F_L^{-1}(\alpha) = \mu + \sigma\Phi^{-1}(\alpha).$$

Check: $F_L(\text{VaR}_\alpha(L)) = \Phi(((\mu + \sigma\Phi^{-1}(\alpha)) - \mu)/\sigma) = \alpha$.

2) Let $L \sim t_\nu(\mu, \sigma^2)$, so $(L - \mu)/\sigma \sim t_\nu$ and thus, as above,

$$\text{VaR}_\alpha(L) = \mu + \sigma t_\nu^{-1}(\alpha).$$

Note that $X \sim t_\nu = t_\nu(0, 1)$ has density

$$f_{t_\nu}(x) = \frac{\Gamma((\nu + 1)/2)}{\sqrt{\nu\pi}\Gamma(\nu/2)} (1 + x^2/\nu)^{-\frac{\nu+1}{2}}.$$

Furthermore, if $\nu > 1$, $\mathbb{E}X$ exists and $\mathbb{E}X = 0$; and if $\nu > 2$, $\text{var } X$ exists and $\text{var } X = \frac{\nu}{\nu-2}$.

Choices of parameters $\Delta t, \alpha$:

- Δt should reflect the time period over which the portfolio is held (unchanged) (e.g. insurance contracts: $\Delta t = 1$ y)
- Δt should be relatively small (more risk-factor change data is available).
- Typical choices:
 - ▶ For limiting traders: $\alpha = 0.95$, $\Delta t = 1$ d
 - ▶ According to Basel II:
 - Market risk: $\alpha = 0.99$, $\Delta t = 10$ d (2 trading weeks)
 - Credit risk and operational risk: $\alpha = 0.999$, $\Delta t = 1$ y
 - ▶ According to Solvency II: $\alpha = 0.995$, $\Delta t = 1$ y
- Backtesting often needs to be carried out at lower confidence levels in order to have sufficient statistical power to detect poor models.
- Be cautious with strictly interpreting $\text{VaR}_\alpha(L)$ (and other risk measure) estimates (considerable model/liquidity risk).

2.3.3 VaR in risk capital calculations

1) VaR in regulatory capital calculations for the trading book

For banks using the *internal model (IM)* approach for market risk in **Basel II** (similarly but more involved for Basel III), the daily risk capital formula is

$$RC^t = \max \left\{ VaR_{0.99}^{t,10}, \frac{k}{60} \sum_{i=1}^{60} VaR_{0.99}^{t-i+1,10} \right\} + c.$$

- $VaR_{\alpha}^{s,10}$ denotes the 10-day VaR_{α} calculated at day s ($t = \text{today}$).
- $k \in [3, 4]$ is a multiplier (or *stress factor*).
- $c =$ **stressed VaR charge** (calculated from data from a volatile market period) + **incremental risk charge (IRC;** $VaR_{0.999}$ -estimate of the annual distribution of losses due to defaults and downgrades) + **charges for specific risks.**

The averaging tends to lead to **smooth changes** in the capital charge over time **unless** $VaR_{0.99}^{t,10}$ is very **large**.

2) The Solvency Capital Requirement in Solvency II

The *Solvency Capital Requirement (SCR)* is the amount of capital that enables the insurer to meet its obligations over $\Delta t = 1$ y with $\alpha = 0.995$. Let V_t denote equity capital. The insurer wants to determine the minimum amount of extra capital x_0 to be solvent in Δt with probability $(\geq)\alpha$, so

$$\begin{aligned}x_0 &= \inf\{x \in \mathbb{R} : \mathbb{P}(V_{t+1} + x(1+r) \geq 0) \geq \alpha\} \\&= \inf\left\{x \in \mathbb{R} : \mathbb{P}\left(-\left(\frac{V_{t+1}}{1+r} - V_t\right) \leq x + V_t\right) \geq \alpha\right\} \\&= \inf\{x \in \mathbb{R} : \mathbb{P}(L_{t+1} \leq x + V_t) \geq \alpha\} \\&= \inf\{x \in \mathbb{R} : F_{L_{t+1}}(x + V_t) \geq \alpha\} \\&= \inf\{z - V_t \in \mathbb{R} : F_{L_{t+1}}(z) \geq \alpha\} = \text{VaR}_\alpha(L_{t+1}) - V_t\end{aligned}$$

and thus $\text{SCR} = V_t + x_0 = \text{VaR}_\alpha(L_{t+1})$ (available capital now + capital required to be solvent in Δt with probability $\geq \alpha$). If $x_0 < 0$, the company is already well capitalized.

2.3.4 Other risk measures based on loss distributions

1) Variance (or standard deviation)

- $\text{var}_\alpha(L)$ (or standard deviation) has a long history as a risk measure in finance (due to Markowitz).
- Drawbacks:
 - ▶ $\mathbb{E}(L^2) < \infty$ required (not justifiable for non-life insurance or operational risk)
 - ▶ no distinction between positive/negative deviations from the mean (var, or standard deviation, is only a good risk measure if F_L is roughly symmetric around $\mathbb{E}L$, but F_L is typically skewed in credit and operational risk)

2) Expected shortfall

Besides VaR, expected shortfall is the most important risk measure in practice.

Let $x_+ = \max\{x, 0\}$.

Definition 2.8 (Expected shortfall)

For a loss $L \sim F_L$ with $\mathbb{E}(L_+) < \infty$, *expected shortfall (ES)* at confidence level $\alpha \in (0, 1)$ is defined by

$$\text{ES}_\alpha = \text{ES}_\alpha(L) = \frac{1}{1 - \alpha} \int_\alpha^1 \text{VaR}_u(L) \, du. \quad (6)$$

- ES_α is the **average of VaR_u** over all $u \geq \alpha \Rightarrow \text{ES}_\alpha \geq \text{VaR}_\alpha$.
- ES_α **looks further into the tail** of F_L , it is a “what if” risk measure (VaR_α is **frequency**-based; ES_α is **severity**-based). This also becomes clear from the following result which shows that **under continuity**, expected shortfall equals *conditional tail expectation or tail value-at-risk*.

Proposition 2.9 ($\text{ES}_\alpha(L)$ under continuity)

If F_L is continuous, $\text{ES}_\alpha(L) = \mathbb{E}(L \mid L > \text{VaR}_\alpha(L))$.

Proof. If F_L is continuous, $F_L(\text{VaR}_\alpha(L)) = F_L(F_L^{\leftarrow}(\alpha)) = \alpha$ and thus, for all $x \geq \text{VaR}_\alpha(L)$,

$$\begin{aligned} F_{L|L>\text{VaR}_\alpha(L)}(x) &= \mathbb{P}(L \leq x \mid L > \text{VaR}_\alpha(L)) \\ &= \frac{\mathbb{P}(L \leq x, L > \text{VaR}_\alpha(L))}{\mathbb{P}(L > \text{VaR}_\alpha(L))} = \frac{\mathbb{P}(\text{VaR}_\alpha(L) < L \leq x)}{\mathbb{P}(L > \text{VaR}_\alpha(L))} \\ &= \frac{F_L(x) - F_L(\text{VaR}_\alpha(L))}{1 - F_L(\text{VaR}_\alpha(L))} = \frac{F_L(x) - \alpha}{1 - \alpha}. \end{aligned}$$

Since $dF_{L|L>\text{VaR}_\alpha(L)}(x) = dF_L(x)/(1 - \alpha)$,

$$\begin{aligned} \mathbb{E}(L \mid L > \text{VaR}_\alpha(L)) &= \int_{\text{VaR}_\alpha(L)}^{\infty} x dF_{L|L>\text{VaR}_\alpha(L)}(x) \\ &= \frac{1}{1 - \alpha} \int_{\text{VaR}_\alpha(L)}^{\infty} x dF_L(x) = \frac{1}{1 - \alpha} \int_{\alpha}^1 \text{VaR}_z(L) dz \\ &= \text{ES}_\alpha(L), \end{aligned}$$

where we substituted $x = \text{VaR}_z(L) = F_L^{\leftarrow}(z)$ (so $F_L(x) = z$, $dF_L(x) = dz$). □

- ES_α is more difficult to estimate and backtest than VaR_α (the variance of estimators is typically larger; larger sample size required).
- $ES_\alpha(L) < \infty$ requires $\mathbb{E}(L_+) < \infty$.
- Subadditivity and elicibility (see the appendix). One can show:
 - ▶ In contrast to VaR_α , ES_α is subadditive (more later).
 - ▶ In contrast to ES_α (see Gneiting (2011) or Kou and Peng (2014)), VaR_α exists if $\mathbb{E}|L| = \infty$ and is elicitable (i.e. minimizes some expected functional (scoring function); see Gneiting (2011). This can be used for backtesting, comparing risk measures).

Example 2.10 (A comparison of VaR and ES for stock returns)

- Consider Example 2.2 with a 1-stock portfolio and $V_t = S_t = 10\,000$. In this case, $L_{t+1}^\Delta = -S_t X_{t+1}$, where $X_{t+1} = \log(S_{t+1}/S_t)$.
- Let $\sigma = 0.2/\sqrt{250}$ (annualized volatility of 20%) and assume

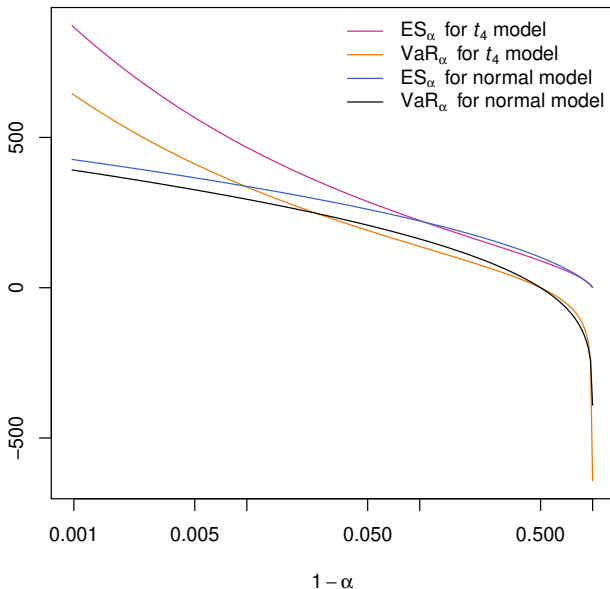
1) $X_{t+1} \sim N(0, \sigma^2) \Rightarrow L_{t+1}^\Delta \sim N(0, S_t^2 \sigma^2)$;

2) $X_{t+1} \sim t_\nu(0, \sigma^2 \frac{\nu-2}{\nu})$, $\nu > 2$ (so that $\text{var } X_{t+1} = \sigma^2$, too). Then

$$X_{t+1} = \sqrt{\sigma^2 \frac{\nu-2}{\nu}} Y \quad \text{for } Y \sim t_\nu,$$

$$\Rightarrow L_{t+1}^\Delta = -S_t \sqrt{\sigma^2 \frac{\nu-2}{\nu}} Y \sim t_\nu(0, S_t^2 \sigma^2 \frac{\nu-2}{\nu}) \quad (\text{so } \text{var}(L_{t+1}^\Delta) = S_t^2 \sigma^2, \text{ too}).$$

- Consider $\nu = 4$ and note that **only for sufficiently large α** do we have $\text{VaR}_\alpha^{t_4} \geq \text{VaR}_\alpha^{\text{normal}}$ and $\text{ES}_\alpha^{t_4} \geq \text{ES}_\alpha^{\text{normal}}$.



⇒ The t_4 model is not always “riskier” than the normal model.

Example 2.11 (Example 2.7 continued; ES_α for $N(\mu, \sigma^2)$ and $t_\nu(\mu, \sigma^2)$)

1) Let $\tilde{L} \sim N(0, 1)$. Then $\text{VaR}_\alpha(\tilde{L}) = 0 + 1 \cdot \Phi^{-1}(\alpha)$ and thus

$$\text{ES}_\alpha(\tilde{L}) = \frac{1}{1 - \alpha} \int_\alpha^1 \Phi^{-1}(u) \, du \stackrel{x = \Phi^{-1}(u)}{=} \frac{1}{1 - \alpha} \int_{\Phi^{-1}(\alpha)}^\infty x \varphi(x) \, dx,$$

where $\varphi(x) = \Phi'(x) = \exp(-x^2/2)/\sqrt{2\pi}$. Since $x\varphi(x) = -\varphi'(x)$,

$$\text{ES}_\alpha(\tilde{L}) = \frac{-[\varphi(x)]_{\Phi^{-1}(\alpha)}^\infty}{1 - \alpha} = \frac{-(0 - \varphi(\Phi^{-1}(\alpha)))}{1 - \alpha} = \frac{\varphi(\Phi^{-1}(\alpha))}{1 - \alpha}.$$

By linearity (or see soon), $L \sim N(\mu, \sigma^2)$ has expected shortfall

$$\text{ES}_\alpha(L) = \mu + \sigma \text{ES}_\alpha(\tilde{L}) = \mu + \sigma \frac{\varphi(\Phi^{-1}(\alpha))}{1 - \alpha}.$$

2) Let $L \sim t_\nu(\mu, \sigma^2)$, $\nu > 1$. Similarly as above, one obtains that

$$\text{ES}_\alpha(L) = \mu + \sigma \frac{1}{1 - \alpha} \frac{\nu}{\nu - 1} f_{t_\nu}(t_\nu^{-1}(\alpha)^2)(1 + t_\nu^{-1}(\alpha)^2/\nu),$$

where f_{t_ν} denotes the density of t_ν ; see Example 2.7.

- By l'Hôpital's Rule (case "0/0"), one can show that

$$1 \leq \lim_{\alpha \uparrow 1} \frac{\text{ES}_\alpha(L)}{\text{VaR}_\alpha(L)} = \frac{\nu}{\nu - 1}.$$

- In finance, often $\nu \in (3, 5)$. With $\nu = 3$, $\text{ES}_\alpha(L)$ is 50% larger than $\text{VaR}_\alpha(L)$ (in the limit for large α).
- For $\nu \uparrow \infty$, $\lim_{\alpha \uparrow 1} \frac{\text{ES}_\alpha(L)}{\text{VaR}_\alpha(L)} \downarrow 1$.
- For $\nu \downarrow 1$, $\lim_{\alpha \uparrow 1} \frac{\text{ES}_\alpha(L)}{\text{VaR}_\alpha(L)} \uparrow \infty$.

Conclusion:

For losses with *heavy tails* (power-like), the *difference between VaR and ES can be huge* (for large α as required by Basel II).

2.3.5 Coherent and convex risk measures

- Artzner et al. (1999) (coherent risk measures) and Föllmer and Schied (2002) (convex risk measures) propose **axioms of a good risk measure**.
- Assume that **risk measures** ϱ are defined on a **linear space of random variables** \mathcal{M} (including constants; we can thus add rvs, multiply them with constants etc.), so $\varrho : \mathcal{M} \rightarrow \mathbb{R}$.
- There are **two possible interpretations** of elements of \mathcal{M} :
 - 1) **Elements of \mathcal{M} are net asset values** V_{t+1} : $\tilde{\varrho}(V_{t+1})$ denotes the **capital to be added to a position** with future value V_{t+1} to make it acceptable to a regulator.
 - 2) **Elements of \mathcal{M} are losses** $L_{t+1} = -(V_{t+1} - V_t)$: $\varrho(L_{t+1})$ denotes the **total amount of capital** necessary to back a position with loss L .
- 1) and 2) are **related via** $\varrho(L_{t+1}) = V_t + \tilde{\varrho}(V_{t+1})$ (total capital = available capital + additional capital). **We focus on 2) and drop “ $t + 1$ ”.**

Axioms of coherence

Axiom 1 (**monotonicity**) $L_1, L_2 \in \mathcal{M}$, $L_1 \leq L_2$ (a.s., i.e. almost surely)
 $\Rightarrow \varrho(L_1) \leq \varrho(L_2)$

Interpr.: Positions which lead to a higher loss in every state of the world require more risk capital.

Criticism: None

Axiom 2 (**translation invar.**) $\varrho(L + l) = \varrho(L) + l$ for all $L \in \mathcal{M}$, $l \in \mathbb{R}$

Interpr.:

- By shifting a position with loss L , we alter the capital requirements accordingly.
- If $\varrho(L) > 0$, and $l = -\varrho(L)$, then $\varrho(L - \varrho(L)) = \varrho(L + l) = \varrho(L) + l = 0$ so that adding $\varrho(L)$ to a position with loss L makes it acceptable.

Criticism: Most people believe this to be reasonable.

Axiom 3 (subadditivity) $\varrho(L_1 + L_2) \leq \varrho(L_1) + \varrho(L_2)$ for all $L_1, L_2 \in \mathcal{M}$

- Interpr.:
- Reflects the idea of diversification. Using a non-subadditive (that is, a *superadditive*) ϱ encourages institutions to legally break up into subsidiaries to reduce regulatory capital requirements.
 - Subadditivity makes decentralization possible: Assume $L = L_1 + L_2$ and that we want to bound $\varrho(L)$ by M . Choose M_j such that $\varrho(L_j) \leq M_j$, $j \in \{1, 2\}$, and $M_1 + M_2 \leq M$. Then $\varrho(L) \leq_{\text{subadd.}} \varrho(L_1) + \varrho(L_2) \leq M_1 + M_2 \leq M$.

Criticism: VaR is ruled out under certain scenarios (see later). VaR is monotone, translation invariant, and positive homogeneous, but in general not subadditive.

Axiom 4 (**positive homogeneity**) $\varrho(\lambda L) = \lambda \varrho(L)$ for all $L \in \mathcal{M}$, $\lambda > 0$

Interpr.: (or motivation): For $L_1 = \dots = L_n = L$, subadditivity implies $\varrho(nL) \leq n\varrho(L)$, but there is no diversification, so equality should hold.

Criticism: If $\lambda > 1$ is large, liquidity risk plays a role and one should rather have $\varrho(\lambda L) > \lambda \varrho(L)$ (also to penalize risk concentration), but this contradicts subadditivity. This has led to *convex risk measures*, i.e. monotone, translation invariant ϱ satisfying $\varrho(\lambda L_1 + (1-\lambda)L_2) \leq \lambda \varrho(L_1) + (1-\lambda)\varrho(L_2)$ for all $L_1, L_2 \in \mathcal{M}$, $0 \leq \lambda \leq 1$.

Definition 2.12 (Coherent risk measure)

A risk measure ϱ which satisfies Axioms 1–4 is called *coherent*.

Coherent risk measures are convex. The converse is not true in general (but for positive homogeneous risk measures ϱ).

Example 2.13 (Coherence of generalized scenario risk measures)

Let $L(x)$ denote the hypothetical loss under scenario x (risk-factor change).

The generalized scenario risk measure

$$\psi_{\mathcal{X},w}(L) = \max\{\mathbb{E}_{\mathbb{P}}(L(\mathbf{X})) : \mathbf{X} \sim \mathbb{P} \in \{\mathbb{P}_1, \dots, \mathbb{P}_n\}\}$$

is coherent. Monotonicity, translation invariance, positive homogeneity are clear (by monotonicity and linearity of $\mathbb{E}_{\mathbb{P}}(\cdot)$). For subadditivity, note that

$$\begin{aligned}\psi_{\mathcal{X},w}(L_1 + L_2) &= \max\{\underbrace{\mathbb{E}_{\mathbb{P}}(L_1(\mathbf{X}) + L_2(\mathbf{X}))}_{=\mathbb{E}_{\mathbb{P}}(L_1(\mathbf{X})) + \mathbb{E}_{\mathbb{P}}(L_2(\mathbf{X}))} : \mathbf{X} \sim \mathbb{P} \in \{\mathbb{P}_1, \dots, \mathbb{P}_n\}\} \\ &\leq \psi_{\mathcal{X},w}(L_1) + \psi_{\mathcal{X},w}(L_2).\end{aligned}$$

Remark 2.14

One can show that all coherent risk measures can be represented as generalized scenarios via $\varrho(L) = \sup\{\mathbb{E}_{\mathbb{P}}(L) : \mathbb{P} \in \mathcal{P}\}$ for a suitable set \mathcal{P} of probability measures.

Theorem 2.15 (Coherence of ES)

ES is a coherent risk measure.

Proof. Monotonicity, translation invariance and positive homogeneity follow from VaR. Subadditivity is more involved but can be shown. \square

Superadditivity scenarios for VaR

Under the following scenarios, VaR_α is typically superadditive:

- 1) L_1, L_2 have skewed distributions;
- 2) Independent, light-tailed L_1, L_2 and small α ;
- 3) L_1, L_2 have special dependence;
- 4) L_1, L_2 have heavy tailed distributions.

Let's have a look at examples for 1), 2) and 4); for 3), see later.

Example 2.16 (Skewed loss distributions)

Consider two independent losses of the form

$$L_j = \begin{cases} -5, & \text{with prob. } 1 - p = 0.991, \\ 100, & \text{with prob. } p = 0.009, \end{cases} \quad j \in \{1, 2\}.$$

Set $\alpha = 0.99$. Then $\text{VaR}_\alpha(L_j) = -5$, $j \in \{1, 2\}$. The loss $L_1 + L_2$ is

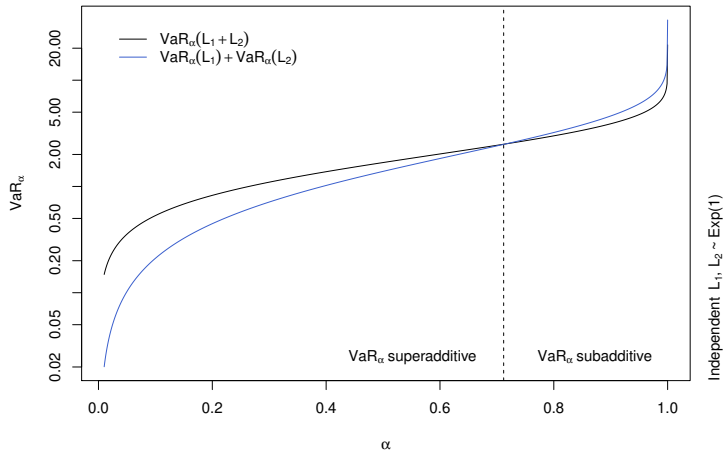
$$L_1 + L_2 = \begin{cases} -10, & \text{with prob. } (1 - p)^2 = 0.982081, \\ 95, & \text{with prob. } 2p(1 - p) = 0.017838, \\ 200, & \text{with prob. } p^2 = 0.000081. \end{cases}$$

Therefore, $\text{VaR}_\alpha(L_1 + L_2) = 95 > -10 = \text{VaR}_\alpha(L_1) + \text{VaR}_\alpha(L_2)$.

- For d such losses, one can show that VaR_α is superadditive if and only if $(1 - p)^d < \alpha \leq 1 - p$.
- From the money lender's (investor) view, the losses could be **two independently defaultable zero-coupon bonds** (maturity $T = 1$ y, face value 100, interest 5%, default probability $p = 0.009$, no recovery).

Example 2.17 (Independent, light-tailed L_1, L_2 and small α)

If $L_1, L_2 \stackrel{\text{ind.}}{\sim} \text{Exp}(1)$, VaR_α is superadditive $\iff \alpha < 0.71$.



One can show that (independently of λ) VaR_α is superadditive if and only if $(1 - \alpha)(1 - 2 \log(1 - \alpha)) > 1$.

Example 2.18 (Heavy tailed loss distributions)

Let $L_1, L_2 \stackrel{\text{ind.}}{\sim} F(x) = 1 - x^{-1/2}$, $x \in [1, \infty)$. One can show (via density convolution formula; tedious!) that $F_{L_1+L_2}(x) = 1 - 2\sqrt{x-1}/x$, $x \geq 2$. By solving a quadratic equation one obtains that VaR_α is superadditive for all $\alpha \in (0, 1)$.

Remark 2.19 (Special case of comonotone risks; elliptical risks)

- $L_1 \stackrel{\text{a.s.}}{=} L_2$ (special case of “comonotonicity”) does not lead to the largest $\text{VaR}_\alpha(L_1 + L_2)$ since $\text{VaR}_\alpha(L_1 + L_2) = \text{VaR}_\alpha(2L_1) = 2 \text{VaR}_\alpha(L_1) = \text{VaR}_\alpha(L_1) + \text{VaR}_\alpha(L_2)$, so “only” equality (whereas all above scenarios produced “>”). All previous examples thus gave a larger VaR under independence than comonotonicity!
- ES_α is subadditive (see Theorem 2.15) and comonotone additive (same idea as for VaR_α) and thus largest under comonotonicity.
- VaR_α is subadditive (so coherent) for all elliptical distributions (strictly including the multivariate normal and t) when $\alpha \geq 1/2$; see later.