

5 Extreme value theory

5.1 Maxima

5.2 Threshold exceedances

5.1 Maxima

Consider a series of financial losses $(X_k)_{k \in \mathbb{N}}$.

5.1.1 Generalized extreme value distribution

Convergence of sums

Let $(X_k)_{k \in \mathbb{N}}$ be iid with $\mathbb{E}(X_1^2) < \infty$ (mean μ , variance σ^2) and $S_n = \sum_{k=1}^n X_k$. As $n \rightarrow \infty$, $\bar{X}_n \xrightarrow{\text{a.s.}} \mu$ by the Strong Law of Large Numbers (SLLN), so $(\bar{X}_n - \mu)/\sigma \xrightarrow{\text{a.s.}} 0$. By the CLT,

$$\sqrt{n} \frac{\bar{X}_n - \mu}{\sigma} = \frac{S_n - n\mu}{\sqrt{n}\sigma} \xrightarrow[n \uparrow \infty]{d} N(0, 1) \text{ or } \lim_{n \rightarrow \infty} \mathbb{P}\left(\frac{S_n - d_n}{c_n} \leq x\right) = \Phi(x),$$

where the sequences $c_n = \sqrt{n}\sigma$ and $d_n = n\mu$ give normalization and where $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-z^2/2} dz$. More generally ($\sigma^2 = \infty$), the limiting distributions for appropriately normalized sums are the class of α -stable distributions ($\alpha \in (0, 2]$; $\alpha = 2$: normal distribution).

Convergence of maxima

QRM is concerned with maximal losses (worst-case losses). Let $(X_i)_{i \in \mathbb{N}} \overset{\text{ind.}}{\sim} F$ (can be relaxed to a strictly stationary time series) and F continuous. Then the *block maximum* is given by

$$M_n = \max\{X_1, \dots, X_n\}.$$

One can show that $M_n \xrightarrow[n \rightarrow \infty]{\text{a.s.}} x_F$ (similar as in the SLLN; due to monotone convergence to a constant) where

$$x_F := \sup\{x \in \mathbb{R} : F(x) < 1\} = F^{\leftarrow}(1) \leq \infty$$

denotes the *right endpoint of F* .

Question: Is there a “CLT” for block maxima?

Idea CLT: What about **linear transformations** (the simplest possible)?

Definition 5.1 (Maximum domain of attraction)

Suppose we find **normalizing sequences** of real numbers $(c_n) > 0$ and (d_n) such that $(M_n - d_n)/c_n$ **converges in distribution**, i.e.

$$\begin{aligned}\mathbb{P}((M_n - d_n)/c_n \leq x) &= \mathbb{P}(M_n \leq c_n x + d_n) \\ &= \mathbb{P}(X_i \leq c_n x + d_n, \ i = 1, \dots, n) \\ &= F^n(c_n x + d_n) \xrightarrow{n \uparrow \infty} H(x),\end{aligned}$$

for some **non-degenerate** df H (not a unit jump). Then F is in the **maximum domain of attraction of H** ($F \in \text{MDA}(H)$).

The **convergence to types theorem** (see the appendix) guarantees that H is determined up to location/scale, i.e. H specifies a **unique type** of distribution.

Question: What does H look like?

Definition 5.2 (Generalized extreme value (GEV) distribution)

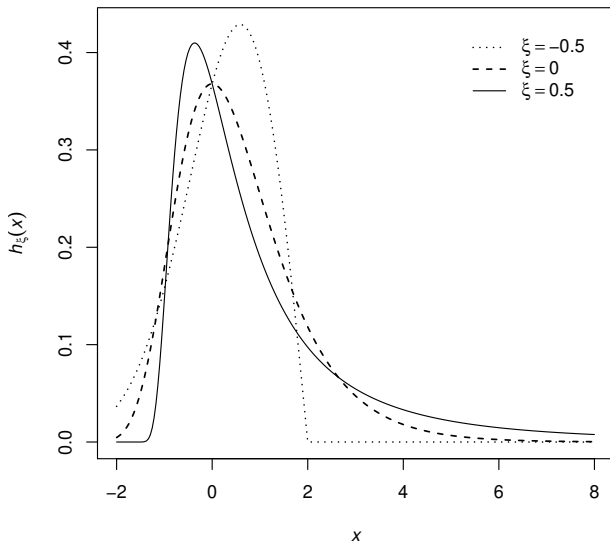
The (standard) *generalized extreme value (GEV) distribution* is given by

$$H_{\xi}(x) = \begin{cases} \exp(-(1 + \xi x)^{-1/\xi}), & \text{if } \xi \neq 0, \\ \exp(-e^{-x}), & \text{if } \xi = 0, \end{cases}$$

where $1 + \xi x > 0$ (MLE!). A three-parameter family is obtained by a location-scale transform $H_{\xi, \mu, \sigma}(x) = H_{\xi}((x - \mu)/\sigma)$, $\mu \in \mathbb{R}$, $\sigma > 0$.

- The parameterization is continuous in ξ (simplifies statistical modelling).
- The larger ξ , the heavier tailed H_{ξ} (if $\xi > 0$, $\mathbb{E}(X^k) = \infty$ iff $k \geq \frac{1}{\xi}$).
- ξ is the *shape* (determines moments, tail). Special cases:
 - 1) $\xi < 0$: the Weibull df, short-tailed, $x_{H_{\xi}} < \infty$;
 - 2) $\xi = 0$: the Gumbel df, $x_{H_0} = \infty$, decays exponentially;
 - 3) $\xi > 0$: the Fréchet df, $x_{H_{\xi}} = \infty$, heavy-tailed ($\bar{H}_{\xi}(x) \approx (\xi x)^{-1/\xi}$), most important case for practice

Density h_ξ for $\xi \in \{-0.5, 0, 0.5\}$ (dotted, dashed, solid)



Theorem 5.3 (Fisher–Tippett–Gnedenko)

If $F \in \text{MDA}(H)$ for some non-degenerate H , then H must be of GEV type, i.e. $H = H_{\xi, \mu, \sigma}$ for some $\xi \in \mathbb{R}$, $\mu \in \mathbb{R}$, $\sigma > 0$.

Proof. Non-trivial. For a sketch, see Embrechts, Klüppelberg, et al. (1997, p. 122). □

- **Interpretation:** If location-scale transformed maxima of iid random variables converge in distribution to a non-degenerate limit, the limiting distribution must be a location-scale transformed GEV distribution (that is, of GEV type).
- One can always choose normalizing sequences $(c_n) > 0$, (d_n) such that $H_{\xi, \mu, \sigma}$ appears in standard form (from a statistical point of view, $(c_n) > 0$, (d_n) can simply be estimated).
- All commonly encountered continuous distributions are in the MDA of some GEV distribution.

Example 5.4 (Exponential distribution)

For $(X_i)_{i \in \mathbb{N}} \stackrel{\text{ind.}}{\sim} \text{Exp}(\lambda)$, choosing $c_n = 1/\lambda$, $d_n = \log(n)/\lambda$, one obtains

$$\begin{aligned} F^n(c_n x + d_n) &= (1 - \exp(-\lambda((1/\lambda)x + \log(n)/\lambda)))^n \\ &= (1 - \exp(-x)/n)^n \xrightarrow{n \uparrow \infty} \exp(-e^{-x}) = H_0(x) \text{ (Gumbel)} \end{aligned}$$

Example 5.5 (Pareto distribution)

For $(X_i)_{i \in \mathbb{N}} \stackrel{\text{ind.}}{\sim} \text{Par}(\theta, \kappa)$ with $F(x) = 1 - (\frac{\kappa}{\kappa+x})^\theta$, $x \geq 0$, $\theta, \kappa > 0$, choosing $c_n = \kappa n^{1/\theta}/\theta$, $d_n = \kappa(n^{1/\theta} - 1)$, $F^n(c_n x + d_n)$ equals

$$\begin{aligned} &\left(1 - \left(\frac{\kappa}{\kappa + x(\kappa n^{1/\theta}/\theta) + (\kappa(n^{1/\theta} - 1))}\right)^\theta\right)^n \\ &= \left(1 - \left(\frac{1}{1 + x n^{1/\theta}/\theta + n^{1/\theta} - 1}\right)^\theta\right)^n = \left(1 - \left(\frac{1}{n^{1/\theta}(1 + x/\theta)}\right)^\theta\right)^n \\ &= \left(1 + \frac{-(1 + x/\theta)^{-\theta}}{n}\right)^n \xrightarrow{n \uparrow \infty} \exp(-(1 + x/\theta)^{-\theta}) = H_{1/\theta}(x) \text{ (Fréchet)} \end{aligned}$$

Therefore, $F \in \text{MDA}(H_{1/\theta})$.

5.1.2 Maximum domains of attraction

All commonly applied continuous F belong to $\text{MDA}(H_\xi)$ for some $\xi \in \mathbb{R}$ and μ, σ can be estimated. But how can we characterize/determine ξ ? All $F \in \text{MDA}(H_\xi)$, $\xi > 0$, allow for a characterization based on:

Definition 5.6 (Slowly/regularly varying functions)

- 1) A positive, Lebesgue-measurable function L on $(0, \infty)$ is *slowly varying at ∞* if $\lim_{x \rightarrow \infty} \frac{L(tx)}{L(x)} = 1$, $t > 0$. The class of all such functions is denoted by \mathcal{R}_0 ; e.g. $c, \log \in \mathcal{R}_0$.
- 2) A positive, Lebesgue-measurable function h on $(0, \infty)$ is *regularly varying at ∞ with index $\alpha \in \mathbb{R}$* if $\lim_{x \rightarrow \infty} \frac{h(tx)}{h(x)} = t^\alpha$, $t > 0$. The class of all such functions is denoted by \mathcal{R}_α ; e.g. $x^\alpha L(x) \in \mathcal{R}_\alpha$.

If $\bar{F} \in \mathcal{R}_{-\alpha}$, $\alpha > 0$, the tail of F decays like a power function (Pareto like).

The Fréchet case

Theorem 5.7 (Fréchet MDA, Gnedenko (1943))

$F \in \text{MDA}(H_\xi)$ for $\xi > 0$ if and only if $\bar{F}(x) = x^{-1/\xi}L(x)$ for some $L \in \mathcal{R}_0$. If $F \in \text{MDA}(H_\xi)$, $\xi > 0$, the normalizing sequences can be chosen as $c_n = F^{\leftarrow}(1 - 1/n)$ and $d_n = 0$, $n \in \mathbb{N}$.

Proof. Non-trivial. For a sketch, see Embrechts, Klüppelberg, et al. (1997, p. 131). \square

- **Interpretation:** Distributions in $\text{MDA}(H_\xi)$, $\xi > 0$, are those whose tails decay like power functions; $\alpha = 1/\xi$ is known as *tail index*.
- If $X \sim F \in \text{MDA}(H_\xi)$, $\xi > 0$, $X \geq 0$, then $\mathbb{E}(X^k) < \infty$ if $k < \alpha = 1/\xi$, $\mathbb{E}(X^k) = \infty$ if $k > \alpha = 1/\xi$; see Embrechts, Klüppelberg, et al. (1997, p. 568).
- **Examples in $\text{MDA}(H_\xi)$, $\xi > 0$:** Inverse gamma, Student t , log-gamma, F , Cauchy, α -stable with $0 < \alpha < 2$, Burr and Pareto

Example 5.8 (Pareto distribution)

For $F = \text{Par}(\theta, \kappa)$, $\bar{F}(x) = (\kappa/(\kappa + x))^\theta = (1 + x/\kappa)^{-\theta} = x^{-\theta}L(x)$, $x \geq 0$, $\theta, \kappa > 0$, where $L(x) = (x^{-1} + \kappa^{-1})^{-\theta} \in \mathcal{R}_0$. We (again) see that $F \in \text{MDA}(H_\xi)$, $\xi > 0$.

The Gumbel case

- The **characterization** of this class is **more complicated**; see the appendix and Embrechts, Klüppelberg, et al. (1997, p. 142).
- Essentially $\text{MDA}(H_0)$ contains dfs whose tails decay roughly exponentially (*light-tailed*), but the tails can be quite different (up to moderately heavy). All moments exist for distributions in the Gumbel class, but both $x_F < \infty$ and $x_F = \infty$ are possible.
- **Examples in $\text{MDA}(H_0)$:** Normal, log-normal, gamma (exponential, Erlang, χ^2), standard Weibull, Benktander type I and II, generalized hyperbolic (except Student t).

The Weibull case

Theorem 5.9 (Weibull MDA)

For $\xi < 0$, $F \in \text{MDA}(H_\xi)$ if and only if $x_F < \infty$ and $\bar{F}(x_F - 1/x) = x^{1/\xi} L(x)$ for some $L \in \mathcal{R}_0$; the normalizing sequences can be chosen as $c_n = x_F - F^\leftarrow(1 - 1/n)$ and $d_n = x_F$, $n \in \mathbb{N}$.

Proof. Non-trivial. For a sketch, see Embrechts, Klüppelberg, et al. (1997, p. 135). □

Examples in $\text{MDA}(H_\xi)$, $\xi < 0$: beta (uniform). All $F \in \text{MDA}(H_\xi)$, $\xi < 0$, share $x_F < \infty$.

5.1.3 Maxima of strictly stationary time series

What about maxima of strictly stationary time series?

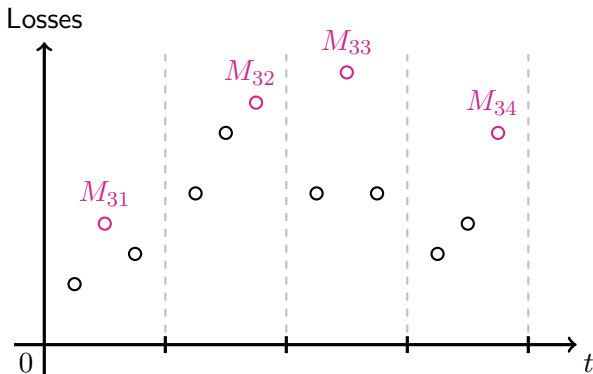
- Let $(X_k)_{k \in \mathbb{Z}}$ denote a strictly stationary time series with stationary distribution $X_k \sim F$, $k \in \mathbb{Z}$.

- Let $\tilde{X}_k \stackrel{\text{ind.}}{\sim} F$, $k \in \mathbb{Z}$, and $\tilde{M}_n = \max\{\tilde{X}_1, \dots, \tilde{X}_n\}$. For many processes one can show that there exists a real number $\theta \in (0, 1]$ such that $\lim_{n \uparrow \infty} \mathbb{P}((M_n - d_n)/c_n \leq x) = H^\theta(x)$ if and only if $\lim_{n \uparrow \infty} \mathbb{P}((\tilde{M}_n - d_n)/c_n \leq x) = H(x)$ (non-degenerate); θ is known as the *extremal index*.
- If $F \in \text{MDA}(H_\xi)$ for some $\xi \Rightarrow M_n$ converges in distribution to H_ξ^θ . Since H_ξ^θ and H_ξ are of the same type, the limiting distribution of the block maxima of the dependent series is the same as in the iid case (only location/scale may change).
- For large n , $\mathbb{P}((M_n - d_n)/c_n \leq x) \approx H^\theta(x) \approx F^{n\theta}(c_n x + d_n)$, so the distribution of M_n from a time series with extremal index θ can be approximated by the distribution $\tilde{M}_{n\theta}$ of the maximum of $n\theta < n$ observations from the associated iid series. $\Rightarrow n\theta$ counts the number of roughly independent clusters in n observations (θ is often interpreted as “1/mean cluster size”).

- If $\theta = 1$, large sample maxima behave as in the iid case; if $\theta \in (0, 1)$, large sample maxima tend to cluster.
- **Examples** (see Embrechts, Klüppelberg, et al. (1997, pp. 216, pp. 415, pp. 476))
 - ▶ Strict white noise (iid rvs): $\theta = 1$;
 - ▶ ARMA processes with (ε_t) strict white noise: $\theta = 1$ (Gaussian); $\theta \in (0, 1)$ (if df of ε_t is in $\text{MDA}(H_\xi)$, $\xi > 0$);
 - ▶ GARCH processes: $\theta \in (0, 1)$.

5.1.4 The block maxima method (BMM)

The basic idea in a picture based on losses $X_1, \dots, X_{12} \stackrel{\text{ind.}}{\sim} F \in \text{MDA}(H_\xi)$:



Consider the maximal loss from each block and fit $H_{\xi, \mu, \sigma}$ to them.

Fitting the GEV distribution

- Suppose $(x_i)_{i \in \mathbb{N}}$ are realizations of $(X_i)_{i \in \mathbb{N}} \stackrel{\text{ind.}}{\sim} F \in \text{MDA}(H_\xi)$, $\xi \in \mathbb{R}$, where F is unknown. By Fisher–Tippett–Gnedenko Theorem,

$$\mathbb{P}(M_n \leq x) = \mathbb{P}((M_n - d_n)/c_n \leq (x - d_n)/c_n) \underset{n \text{ large}}{\approx} H_{\xi, \mu=d_n, \sigma=c_n}(x).$$

- For fitting $\theta = (\xi, \mu, \sigma)$, divide the realizations into m blocks of size n denoted by M_{n1}, \dots, M_{nm} (e.g. daily log-returns \Rightarrow monthly maxima)
- Assume the block size n to be sufficiently large so that (regardless of whether the underlying data are dependent or not), the block maxima can be considered independent.
- The density h_ξ of H_ξ is

$$h_\xi(x) = \begin{cases} (1 + \xi x)^{-1/\xi-1} H_\xi(x) I_{\{1+\xi x > 0\}}, & \text{if } \xi \neq 0, \\ e^{-x} H_0(x), & \text{if } \xi = 0. \end{cases}$$

The **log-likelihood** is thus

$$\ell(\boldsymbol{\theta}; M_{n1}, \dots, M_{nm}) = \sum_{i=1}^m \log\left(\frac{1}{\sigma} h_{\xi}\left(\frac{M_{ni} - \mu}{\sigma}\right)\right).$$

Maximize w.r.t. $\boldsymbol{\theta} = (\xi, \mu, \sigma)$ to get $\hat{\boldsymbol{\theta}} = (\hat{\xi}, \hat{\mu}, \hat{\sigma})$.

Remark 5.10

- 1) Sufficiently many/large blocks **require large amounts of data**.
- 2) Bias and variance must be traded off (***bias-variance tradeoff***):
 - Block size $n \uparrow \Rightarrow$ GEV approximation more accurate \Rightarrow **bias \downarrow**
 - Number of blocks $m \uparrow \Rightarrow$ more data for MLE \Rightarrow **variance \downarrow**
- 3) There is **no general best strategy for finding the optimal block size**.
- 4) **MLE regularity conditions** for consistency and asymptotic efficiency were shown by Smith (1985) for $\xi > -1/2$ (fine for practice).

Return levels and return periods

(Approximately) $M_n \sim H_{\hat{\xi}, \hat{\mu}, \hat{\sigma}}$, so $\bar{H}_{\hat{\xi}, \hat{\mu}, \hat{\sigma}}(r) = \mathbb{P}(M_n > r) = 1/k$ can be used to estimate the...

- 1) ... k n -block return level $r_{n,k}$, that is, the (smallest) r which is expected to be exceeded (at most) in one out of every k blocks of size n .
 - e.g., 10 year return level $r_{260,10} =$ (smallest) level exceeded in (at most) one out of every 10 years (where $260d \approx 1y$)
 - $r_{n,k} = H_{\hat{\xi}, \hat{\mu}, \hat{\sigma}}^{\leftarrow}(1 - 1/k) = \hat{\mu} + \frac{\hat{\sigma}}{\hat{\xi}}((- \log(1 - 1/k))^{-\hat{\xi}} - 1)$
 $\hat{\xi} \neq 0$
- 2) ... return period $k_{n,u}$ of the event $\{M_n > u\}$, that is, the smallest number of n -blocks for which we expect to see at least one n -block exceeding u .
 - $k_{n,u} = 1/\bar{H}_{\hat{\xi}, \hat{\mu}, \hat{\sigma}}(u) = 1/(1 - \exp(-(1 + \hat{\xi}(u - \hat{\mu})/\hat{\sigma})^{-1/\hat{\xi}}))$
 $\hat{\xi} \neq 0$
 - $k_{n,u}$ satisfies $r_{n,k_{n,u}} = u$

Example 5.11 (Block maxima analysis of S&P 500)

Suppose it is Friday 1987-10-16; the Friday before Black Monday (1987-10-19). The S&P 500 index fell by 9.12% this week. On that Friday alone the index is down 5.16%. We fit a GEV distribution to (bi)annual maxima of daily negative log-returns $X_t = -\log(S_t/S_{t-1})$ since 1960-01-01.

Analysis 1: Annual maxima ($m = 28$; including the latest from the incomplete year 1987): $\hat{\theta} = (0.30, 0.02, 0.007) \Rightarrow$ Heavy-tailed Fréchet distribution (infinite fourth moment). The corresponding standard errors are $(0.22, 0.002, 0.001) \Rightarrow$ High uncertainty (m small) for estimating ξ .

Analysis 2: Biannual maxima ($m = 56$): $\hat{\theta} = (0.34, 0.02, 0.006)$ with standard errors $(0.15, 0.0008, 0.0005) \Rightarrow$ Even heavier tails. In what follows we work with the annual maxima.

- 1) What is the probability that next year's maximal risk-factor change exceeds all previous ones? $1 - H_{\hat{\xi}, \hat{\mu}, \hat{\sigma}}(\text{"previous maxima"})$

- 2) Was a risk-factor change as on **Black Monday** foreseeable?
- Based on data up to and including Friday 1987-10-16, the 10-year return level $r_{260,10}$ is estimated as $\hat{r}_{260,10} = 4.42\%$.
 - Index drop Black Monday: $20.47\% \Rightarrow X_{t+1} = 22.9\% \gg \hat{r}_{260,10}$.
 - One can show that 22.9% is in the 95% confidence interval of $r_{260,50}$ (estimated as $\hat{r}_{260,50} = 7.49\%$), but the 28 maxima are too few to get a reliable estimate of a once-in-50-years event.
- 3) Based on the available data, what is the (estimated) return period of a loss at least as large as on **Black Monday**?
- The estimated return period $k_{260,0.229}$ is $\hat{k}_{260,0.229} = 1876$ years.
 - One can show that the 95% confidence interval encompasses everything from 45y to essentially never! \Rightarrow Very high uncertainty!
- \Rightarrow On 1987-10-16 we did not have enough data to say anything meaningful about such an event. Quantifying such events is difficult.

5.2 Threshold exceedances

The **BMM is wasteful of data** (only the maxima of large blocks are used). It has been largely superseded in practice by methods based on **threshold exceedances** (*peaks-over-threshold (POT) approach*), where **all data above** a designated high **threshold u** are used.

5.2.1 Generalized Pareto distribution

Definition 5.12 (Generalized Pareto distribution (GPD))

The *generalized Pareto distribution (GPD)* is given by

$$G_{\xi, \beta}(x) = \begin{cases} 1 - (1 + \xi x / \beta)^{-1/\xi}, & \text{if } \xi \neq 0, \\ 1 - \exp(-x/\beta), & \text{if } \xi = 0, \end{cases}$$

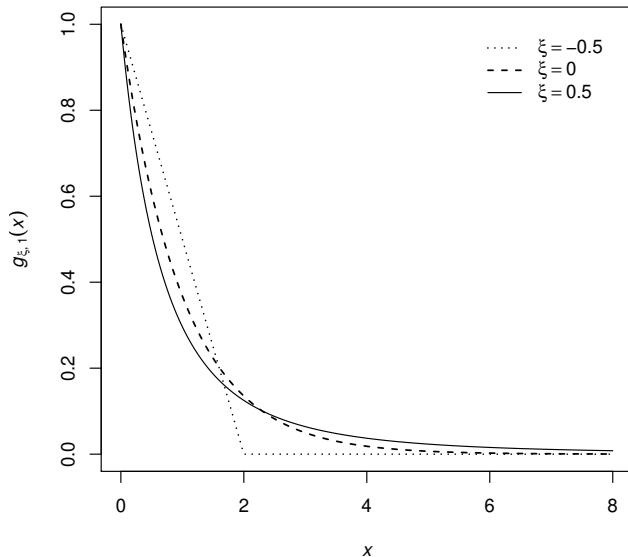
where $\beta > 0$, and the support is $x \geq 0$ when $\xi \geq 0$ and $x \in [0, -\beta/\xi]$ when $\xi < 0$.

- The parameterization is continuous in ξ .
- ξ is known as *shape*; β as *scale*. Special cases:
 - 1) $\xi > 0$: $\text{Par}(1/\xi, \beta/\xi)$
 - 2) $\xi = 0$: $\text{Exp}(1/\beta)$
 - 3) $\xi < 0$: short-tailed Pareto type II distribution
- The larger ξ , the heavier tailed $G_{\xi,\beta}$ (if $\xi > 0$, $\mathbb{E}(X^k) = \infty$ iff $k \geq \frac{1}{\xi}$; if $\xi < 1$, then $\mathbb{E}X = \beta/(1 - \xi)$).
- $G_{\xi,\beta} \in \text{MDA}(H_\xi)$, $\xi \in \mathbb{R}$, (same ξ)
- The density $g_{\xi,\beta}$ of $G_{\xi,\beta}$ is given by

$$g_{\xi,\beta}(x) = \begin{cases} \frac{1}{\beta}(1 + \xi x/\beta)^{-1/\xi-1}, & \text{if } \xi \neq 0, \\ \frac{1}{\beta} \exp(-x/\beta), & \text{if } \xi = 0, \end{cases}$$

where $x \geq 0$ when $\xi \geq 0$ and $x \in [0, -\beta/\xi)$ when $\xi < 0$ (MLE!).

Density $g_{\xi,1}$ for $\xi \in \{-0.5, 0, 0.5\}$ (dotted, dashed, solid)



Definition 5.13 (Excess distribution over u , mean excess function)

Let $X \sim F$. The *excess distribution over the threshold u* is defined by

$$F_u(x) = \mathbb{P}(X - u \leq x \mid X > u) = \frac{F(x + u) - F(u)}{1 - F(u)}, \quad x \in [0, x_F - u).$$

If $\mathbb{E}|X| < \infty$, the *mean excess function* is defined by

$$e(u) = \mathbb{E}(X - u \mid X > u) \quad (\text{i.e. the mean w.r.t. } F_u)$$

- **Interpretation:** F_u is the distribution of the excess loss $X - u$ over u , given that $X > u$. $e(u)$ is the mean of F_u as a function of u .
- One can show the useful formulas $e(u) = \int_0^{x_F - u} \bar{F}_u(x) \, dx = \frac{\int_u^{x_F} \bar{F}(x) \, dx}{\bar{F}(u)}$.
- For continuous $X \sim F$ with $\mathbb{E}|X| < \infty$, the following formula holds:

$$\text{ES}_\alpha(X) = e(\text{VaR}_\alpha(X)) + \text{VaR}_\alpha(X), \quad \alpha \in (0, 1) \quad (11)$$

Example 5.14 (F_u , $e(u)$ for $\text{Exp}(\lambda)$, $G_{\xi,\beta}$)

- 1) If F is $\text{Exp}(\lambda)$, then $F_u(x) = 1 - e^{-\lambda x}$ (so again $\text{Exp}(\lambda)$; lack-of-memory property). The mean excess function of F is thus $e(u) = 1/\lambda = \mathbb{E}X$.
- 2) If F is $G_{\xi,\beta}$, then $F_u(x) = G_{\xi,\beta+\xi u}(x)$ (so again GPD, with the same shape, only the scale grows linearly in u). The mean excess function of $F = G_{\xi,\beta}$ is thus

$$e(u) = \frac{\beta + \xi u}{1 - \xi}, \quad \text{for all } u : \beta + \xi u > 0,$$

which is linear in u (this is a characterizing property of the GPD and used (in slightly different form; see later) to determine u). Note that ξ determines the slope $\xi/(1 - \xi)$ of $e(u)$.

Theorem 5.15 (Pickands–Balkema–de Haan (1974/75))

There exists a positive, measurable function $\beta(u)$, such that

$$\lim_{u \uparrow x_F} \sup_{0 \leq x < x_F - u} |F_u(x) - G_{\xi, \beta(u)}(x)| = 0.$$

if and only if $F \in \text{MDA}(H_\xi)$, $\xi \in \mathbb{R}$.

Proof. Non-trivial; see, e.g. Pickands (1975) and Balkema and de Haan (1974). □

Interpretation

- The GPD is the canonical df for excess losses over high u . This leads to the peaks-over-threshold method for modeling excess losses.
- The result is also a characterization of $\text{MDA}(H_\xi)$, $\xi \in \mathbb{R}$. All $F \in \text{MDA}(H_\xi)$ form a set of df for which the excess distribution converges to the GPD $G_{\xi, \beta}$ with the same ξ as in H_ξ when u is raised.

5.2.2 Modelling excess losses

The basic idea in a picture based on losses X_1, \dots, X_{12} .



Consider all **excesses over u** and fit $G_{\xi, \beta}$ to them.

The peaks-over-threshold (POT) method

- Given losses $X_1, \dots, X_n \sim F \in \text{MDA}(H_\xi)$, $\xi \in \mathbb{R}$, let
 - 1) $N_u = |\{i \in \{1, \dots, n\} : X_i > u\}|$ denote the *number of exceedances* over the (given; see later) threshold u ;
 - 2) $\tilde{X}_1, \dots, \tilde{X}_{N_u}$ denote the *exceedances*; and
 - 3) $Y_k = \tilde{X}_k - u$, $k \in \{1, \dots, N_u\}$, the corresponding *excesses*.
- If Y_1, \dots, Y_{N_u} are *independent* and (roughly) *distributed as* $G_{\xi, \beta}$, the *log-likelihood* is given by

$$\begin{aligned}\ell(\xi, \beta; Y_1, \dots, Y_{N_u}) &= \sum_{k=1}^{N_u} \log g_{\xi, \beta}(Y_k) \\ &= -N_u \log(\beta) - (1 + 1/\xi) \sum_{k=1}^{N_u} \log(1 + \xi Y_k / \beta)\end{aligned}$$

\Rightarrow Maximize w.r.t. $\beta > 0$ and $1 + \xi Y_k / \beta > 0$ for all $k \in \{1, \dots, N_u\}$.

Excesses over higher thresholds

Once a model is fitted to F_u , we can infer a model for F_v , $v \geq u$.

Lemma 5.16

Assume, for some u , $F_u(x) = G_{\xi,\beta}(x)$ for $0 \leq x < x_F - u$. Then $F_v(x) = G_{\xi,\beta+\xi(v-u)}(x)$ for all $v \geq u$.

Proof. Recall that $F_u(x) = \mathbb{P}(X - u \leq x \mid X > u) = \frac{F(u+x)-F(u)}{F(u)}$, so $\bar{F}_u(x) = \bar{F}(u+x)/\bar{F}(u)$. For $v \geq u$, we have

$$\begin{aligned}\bar{F}_v(x) &= \frac{\bar{F}(v+x)}{\bar{F}(v)} = \frac{\bar{F}(u+(v+x-u))}{\bar{F}(u)} \frac{\bar{F}(u)}{\bar{F}(u+(v-u))} \\ &= \frac{\bar{F}_u(v+x-u)}{\bar{F}_u(v-u)} = \frac{\bar{G}_{\xi,\beta}(x+v-u)}{\bar{G}_{\xi,\beta}(v-u)} \stackrel{\text{check}}{=} \bar{G}_{\xi,\beta+\xi(v-u)}(x) \quad \square\end{aligned}$$

\Rightarrow The excess distribution over $v \geq u$ remains GPD with the same ξ (and β growing linearly in v); makes sense for a limiting distribution for $u \uparrow$.

If it exists (so if $\xi < 1$), the mean excess function over v is given by

$$e(v) = \mathbb{E}(G_{\xi, \beta + \xi(v-u)}) = \frac{\beta + \xi(v-u)}{1-\xi} = \frac{\xi}{1-\xi}v + \frac{\beta - \xi u}{1-\xi}, \quad v \in [u, x_F), \quad (12)$$

where $x_F = \infty$ if $\xi \in [0, 1)$ and $x_F = u - \beta/\xi$ if $\xi < 0$. This forms the basis for a graphical method for choosing u .

Sample mean excess plot and choice of the threshold

Definition 5.17 (Sample mean excess function, mean excess plot)

For $X_1, \dots, X_n > 0$, the sample mean excess function is defined by

$$e_n(v) = \frac{\sum_{i=1}^n (X_i - v) I_{\{X_i > v\}}}{\sum_{i=1}^n I_{\{X_i > v\}}}, \quad v < X_{(n)}.$$

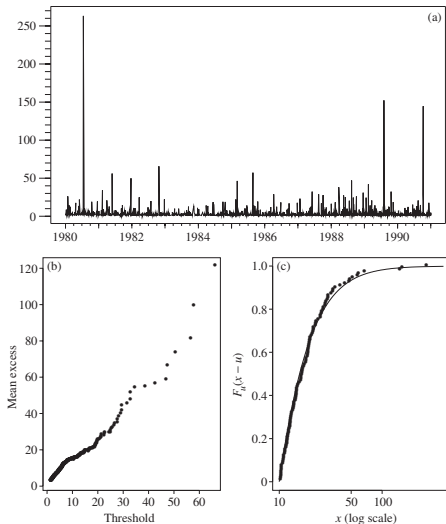
The mean excess plot is the plot of $\{(X_{(i)}, e_n(X_{(i)})) : 1 \leq i \leq n-1\}$, where $X_{(i)}$ denotes the i th order statistic.

- If the data supports the GPD model over u , $e_n(v)$ should become increasingly “linear” for higher values of $v \geq u$. An upward/zero/downward trend indicates whether $\xi > 0/\xi = 0/\xi < 0$.
- Select u as the smallest point where $e_n(v)$, $v \geq u$, becomes linear. Rule-of-thumb: One needs a couple of thousand data points and can often take u around the 0.9-quantile.
- The sample mean excess plot is rarely perfectly linear (particularly for large u where one averages over a small number of excesses).
- The choice of a good threshold u is as difficult as finding an adequate block size for the block maxima method.
- As for the block maxima method, there is a bias-variance trade-off. If u is too small, the GPD may not provide a good fit to F_u , but if u is too large, there may be too few excesses for fitting the GPD parameters adequately.
- One should always analyze the data for several u .

Example 5.18 (Danish fire loss data)

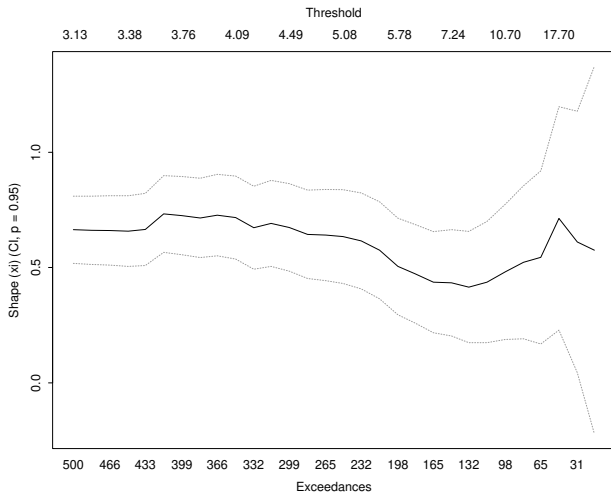
- 2156 fire insurance losses over 1M Danish kroner from 1980-01-03 to 1990-12-31; combined loss for a building and its contents, in some cases also a loss of business earnings. The losses are inflation adjusted to reflect values as of 1985.
- The sample mean excess function shows a “kink” below 10; “straightening out” above 10 \Rightarrow Our choice is $u = 10$ (so 10M Danish kroner).
- MLE $(\hat{\xi}, \hat{\beta}) = (0.50, 7.0)$ (with standard errors $(0.14, 1.1)$)
 \Rightarrow very heavy-tailed, infinite-variance model
- We can then estimate the expected loss given exceedance of 10M kroner or any higher threshold (via $e(v)$ in (12) based on $\hat{\xi}, \hat{\beta}$ and the chosen u), even beyond the data.
 \Rightarrow EVT allows us to estimate “in the data” and then “scale up”.

(a): Losses ($> 1M$; in M); (b): $e_n(u)$ (\uparrow); (c) $\hat{F}_{u,n}(x - u)$, $G_{\hat{\xi}, \hat{\beta}}(x - u)$



\Rightarrow Choose the threshold $u = 10$.

Sensitivity of the estimated shape parameter $\hat{\xi}$ to changes in u :



\Rightarrow The higher u , the wider the confidence intervals (also support $u = 10$).

Example 5.19 (AT&T weekly loss data)

- Let (X_t) denote weekly log-returns and consider the percentage one-week loss as a fraction of S_t , given by

$$100L_{t+1}/S_t \stackrel{(1)}{=} 100(-S_t(\exp(X_{t+1}) - 1))/S_t = 100(1 - \exp(X_{t+1})).$$

- We have 521 such losses (period 1991–2000).
- The estimated GPD parameters are $\hat{\xi} = 0.22$ and $\hat{\beta} = 2.1$ (MLEs) with standard errors 0.13 and 0.34, respectively. The fitted model is thus close to having an infinite fourth moment.
- Note that we ignored here that monthly data over 1993–2000 is not consistent with the iid assumption (absolute values of log-returns reject the hypothesis of serial uncorrelatedness via the Ljung–Box test).

(a): % losses (1991–2000); (b): $e_n(u)$; (c): $\hat{F}_{u,n}(x - u)$, $G_{\hat{\xi}, \hat{\beta}}(x - u)$.



⇒ Choose the threshold $u = 2.75\%$ (102 exceedances)

5.2.3 Modelling tails and measures of tail risk

- How can the fitted GPD model be used to estimate the tail of the loss distribution F and associated risk measures?
- Assume $F_u(x) = G_{\xi, \beta}(x)$ for $0 \leq x < x_F - u$, $\xi \neq 0$ and some u .
- We obtain the following GPD-based formula for tail probabilities:

$$\begin{aligned}\bar{F}(x) &= \mathbb{P}(X > x) = \mathbb{P}(X > u) \mathbb{P}(X > x \mid X > u) \\ &= \bar{F}(u) \mathbb{P}(X - u > x - u \mid X > u) = \bar{F}(u) \bar{F}_u(x - u) \\ &= \bar{F}(u) \left(1 + \xi \frac{x - u}{\beta}\right)^{-1/\xi}, \quad x \geq u.\end{aligned}\tag{13}$$

- Assuming we know $\bar{F}(u)$, inverting this formula for $\alpha \geq F(u)$ leads to

$$\text{VaR}_\alpha = F^{\leftarrow}(\alpha) = u + \frac{\beta}{\xi} \left(\left(\frac{1 - \alpha}{\bar{F}(u)} \right)^{-\xi} - 1 \right),\tag{14}$$

$$\text{ES}_\alpha = \frac{\text{VaR}_\alpha}{1 - \xi} + \frac{\beta - \xi u}{1 - \xi}, \quad \xi < 1.\tag{15}$$

The formula for ES_α can also be obtained from $e(\cdot)$ via (11) and (12).

- $\bar{F}(x)$, VaR_α and ES_α are all of the form $g(\xi, \beta, \bar{F}(u))$. If we have sufficient samples above u , we obtain semi-parametric plug-in estimators via $g(\hat{\xi}, \hat{\beta}, N_u/n)$. We hope to gain over empirical estimators by using a kind of extrapolation based on the GPD for more extreme tail probabilities and risk measures.
- In this spirit, Smith (1987) proposed the *tail estimator*

$$\hat{\bar{F}}(x) = \frac{N_u}{n} \left(1 + \hat{\xi} \frac{x - u}{\hat{\beta}} \right)^{-1/\hat{\xi}}, \quad x \geq u \quad (\text{see (13)});$$

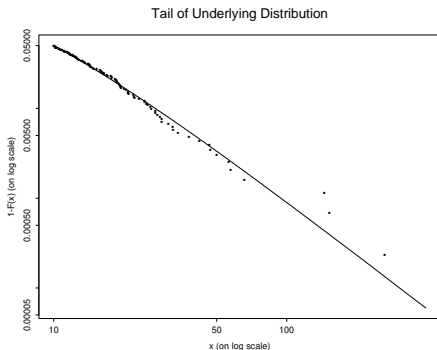
also known as the *Smith estimator* (note that it is only valid for $x \geq u$). It faces a **bias-variance tradeoff**: If u is increased, the bias of parametrically estimating $\bar{F}_u(x - u)$ decreases, but the variance of it and the nonparametrically estimated $\bar{F}(u)$ increases.

- Similarly, semi-parametric GPD-based $\widehat{\text{VaR}}_\alpha$, $\widehat{\text{ES}}_\alpha$ for $\alpha \geq 1 - N_u/n$ can be obtained from (14), (15).

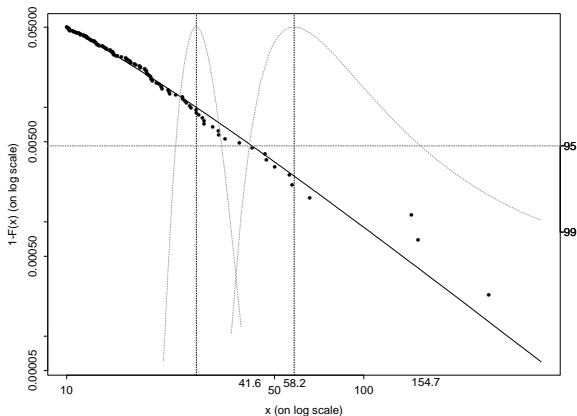
- Confidence intervals for $\bar{F}(x)$, $x \geq u$, VaR_α , ES_α can be obtained likelihood-based (neglecting the uncertainty in N_u/n): Reparametrize the GPD model in terms of $\phi = g(\xi, \beta, N_u/n)$ and construct a confidence interval for ϕ based on the likelihood ratio test.

Example 5.20 (Danish fire loss data (continued))

The semi-parametric Smith/tail estimator $\hat{\bar{F}}(x)$, $x \geq u$ is given by:

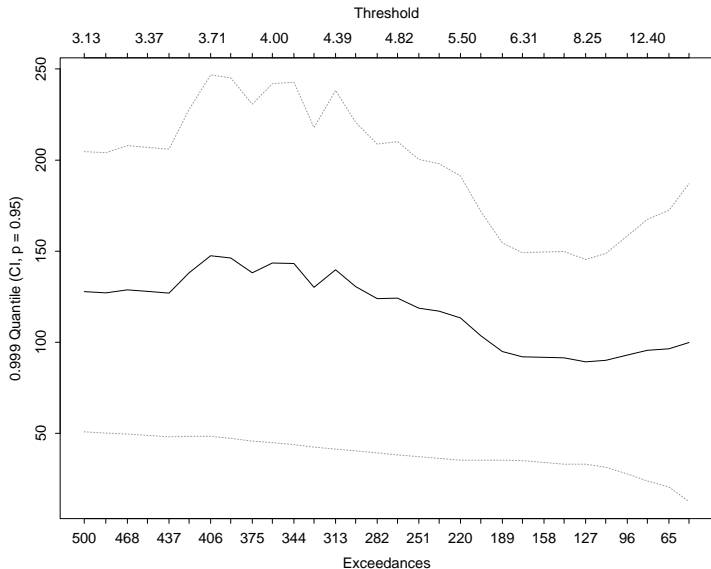


Here are $\hat{\bar{F}}(x)$, $x \geq u$, $\widehat{\text{VaR}}_{0.99}$, $\widehat{\text{ES}}_{0.99}$ including confidence intervals.

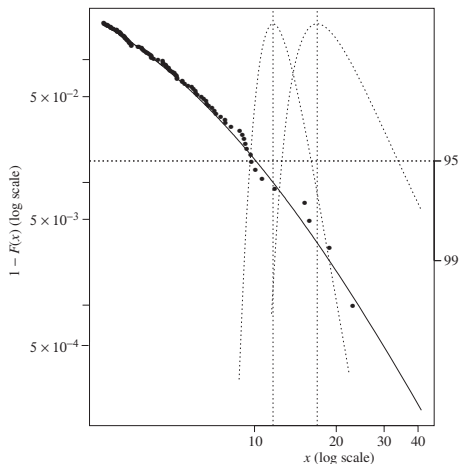


Log-log scale often helpful: If $\bar{F}(x) = x^{-\alpha}L(x)$, $\log \bar{F}(x) = -\alpha \log(x) + \log L(x)$ which is approximately linear in $\log x$.

One can check the **sensitivity of $\hat{F}(0.999)$** (or $\widehat{\text{VaR}}_\alpha$, $\widehat{\text{ES}}_\alpha$) **w.r.t. u** .



Example 5.21 (AT&T weekly loss data (continued))



- Fitted GPD model as in Example 5.19.
- Plot of $\hat{F}(x)$.
- Vertical lines: $\widehat{\text{VaR}}_{0.99}$, $\widehat{\text{ES}}_{0.99}$



- Sensitivity w.r.t. u
- **Top:** $\hat{\xi}$ for different u or N_u , including a 95% CI based on standard error
- **Bottom:** Corresponding $\widehat{\text{VaR}}_{0.99}$ (solid line), $\widehat{\text{ES}}_{0.99}$ (dotted line)

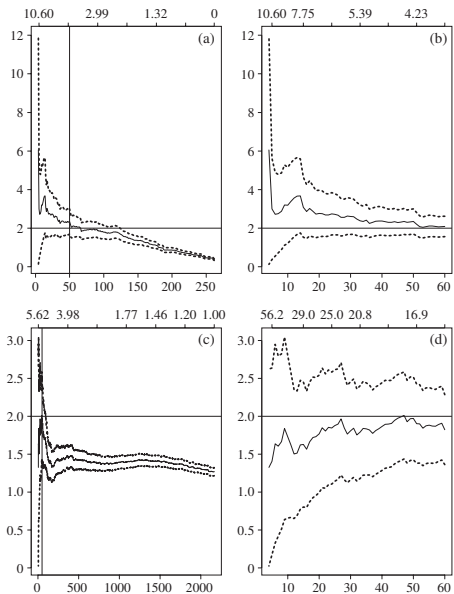
5.2.4 The Hill estimator

- Assume $F \in \text{MDA}(H_\xi)$, $\xi > 0$, so that $\bar{F}(x) = x^{-\alpha}L(x)$, $\alpha > 0$.
- The standard form of the *Hill estimator of the tail index α* is

$$\hat{\alpha}_{k,n}^{(H)} = \left(\frac{1}{k} \sum_{i=1}^k \log X_{i,n} - \log X_{k,n} \right)^{-1}, \quad 2 \leq k \leq n, \quad k \text{ sufficiently small.}$$

Idea: This can be derived by noting that the mean excess function $e(\log u)$ of $\log X$ at $\log u$ is roughly $1/\alpha$ for large u (by Karamata's Theorem), then using $e_n(\log X_{k,n})$ as an estimator for $e(\log u)$ and solving for α ; see the appendix. Note: $X_{1,n} \geq \dots \geq X_{n,n}$.

- Choosing k : Find a small k where the *Hill plot* $\{(k, \hat{\alpha}_{k,n}^{(H)}) : 2 \leq k \leq n\}$ stabilizes (typically, $k = \lceil \beta n \rceil$, $\beta \in [0.01, 0.05]$).
- Interpreting Hill plots can be difficult. If F does not have a regularly varying tail (or if it has serial dependence), Hill plots can be very misleading.



- Hill plots showing estimates of $\alpha = 1/\xi$ for (a), (b) the AT&T data and (c),(d) the Danish fire loss data (rhs = zoomed-in version of the lhs).
- (a),(b) suggest estimates of $\alpha \in [2, 4]$ ($\xi \in [1/4, 1/2]$; larger than the estimated $\hat{\xi} = 0.22$, see Example 5.19); (c),(d) suggest estimates of $\alpha \in [1.5, 2]$ ($\xi \in [1/2, 2/3]$ (infinite variance!); close to the estimated $\hat{\xi} = 0.50$, see Example 5.18)

Hill-based tail and risk measure estimates

- Assume $\bar{F}(x) = cx^{-\alpha}$, $x \geq u > 0$ (replacing L by a constant). Estimate α by $\hat{\alpha}_{k,n}^{(H)}$ and u by $X_{k,n}$ (for k sufficiently small).
- Note that $c = u^\alpha \bar{F}(u)$ so $\hat{c} = X_{k,n}^{\hat{\alpha}_{k,n}^{(H)}} \hat{\bar{F}}_n(X_{k,n}) \approx X_{k,n}^{\hat{\alpha}_{k,n}^{(H)}} \frac{k}{n}$. We thus obtain the semi-parametric *Hill tail estimator*

$$\hat{\bar{F}}(x) = \frac{k}{n} \left(\frac{x}{X_{k,n}} \right)^{-\hat{\alpha}_{k,n}^{(H)}}, \quad x \geq X_{k,n}.$$

- From this result we obtain the semi-parametric *Hill VaR estimator*

$$\widehat{\text{VaR}}_\gamma(X) = \left(\frac{n}{k}(1 - \gamma) \right)^{-\frac{1}{\hat{\alpha}_{k,n}^{(H)}}} X_{k,n}, \quad \gamma \geq F(u) \approx 1 - \frac{k}{n},$$

and, for $\hat{\alpha}_{k,n}^{(H)} > 1$, $\gamma \geq F(u) \approx 1 - \frac{k}{n}$, the semi-param. *Hill ES estimator*

$$\widehat{\text{ES}}_\gamma(X) = \frac{\left(\frac{n}{k} \right)^{-\frac{1}{\hat{\alpha}_{k,n}^{(H)}}} X_{k,n}}{1 - \gamma} \int_\gamma^1 (1 - z)^{-\frac{1}{\hat{\alpha}_{k,n}^{(H)}}} dz = \frac{\hat{\alpha}_{k,n}^{(H)}}{\hat{\alpha}_{k,n}^{(H)} - 1} \widehat{\text{VaR}}_\gamma(X).$$

5.2.5 Simulation study of EVT quantile estimators

We compare estimators for ξ (Study 1) and $\text{VaR}_{0.99}$ (Study 2) based on

$$\begin{aligned}\text{MSE}(\hat{\theta}) &= \mathbb{E}((\hat{\theta} - \theta)^2) = \mathbb{E}((\hat{\theta} - \mathbb{E}[\hat{\theta}] + \mathbb{E}[\hat{\theta}] - \theta)^2) \\ &= \mathbb{E}((\hat{\theta} - \mathbb{E}[\hat{\theta}])^2) + \mathbb{E}(2(\hat{\theta} - \mathbb{E}[\hat{\theta}])(\mathbb{E}[\hat{\theta}] - \theta)) + \mathbb{E}((\mathbb{E}[\hat{\theta}] - \theta)^2) \\ &= (\mathbb{E}[\hat{\theta}] - \theta)^2 + \text{var}(\hat{\theta}) = \text{bias}(\hat{\theta})^2 + \text{var}(\hat{\theta})\end{aligned}$$

with a Monte Carlo study (based on 1000 samples from a t_4 distribution with corresponding true $\xi = 1/4$) since analytical evaluation of bias and variance is not possible.

Study 1: Estimating ξ

We estimate ξ with a fitted GPD (via MLE; $k \in \{30, 40, \dots, 400\}$) and with the Hill estimator ($\hat{\xi} = 1/\hat{\alpha}_{k,n}^{(H)}$; $k \in \{2, 3, \dots, 200\}$). Note that the t_4 distribution has a well-behaved regularly varying tail.

(a): $\widehat{\text{MSE}}(\hat{\xi})$; (b): $\widehat{\text{bias}}(\hat{\xi})$; (c): $\widehat{\text{var}}(\hat{\xi})$ (solid: GPD; dotted: Hill)



- The Hill estimator outperforms the GPD estimator (optimal k around 20–30) according to the variance for small k (number of order statistics)
- The biases are closer: the Hill (GPD) estimator tends to overestimate (underestimate) ξ .
- For the GPD method, the optimal u is around 100–150 exceedances.

Study 2: Estimating $\text{VaR}_{0.99}$

Estimate $\text{VaR}_{0.99}$ based on a fitted GPD, with the Hill VaR estimator and with the empirical quantile estimator. Here the situation changes.

(a): $\widehat{\text{MSE}}(\widehat{\text{VaR}}_{0.99})$; (b): $\widehat{\text{bias}}(\widehat{\text{VaR}}_{0.99})$; (c): $\widehat{\text{var}}(\widehat{\text{VaR}}_{0.99})$ (solid: GPD; dotted: Hill; dashed: empirical quantile estimator)



- The empirical $\text{VaR}_{0.99}$ estimator has a negative bias.
- The Hill $\text{VaR}_{0.99}$ estimator has a negative bias for small k but a rapidly growing positive bias for larger k .
- The GPD $\text{VaR}_{0.99}$ estimator has a positive bias which grows much more slowly.
- The GPD $\text{VaR}_{0.99}$ estimator attains lowest MSE for a value of k around 100, and the MSE is very robust to the choice of k (because of the slow growth of the bias) \Rightarrow Choice of u less critical
- The Hill $\text{VaR}_{0.99}$ estimator performs well for $20 \leq k \leq 75$ (we only use k values that lead to a quantile estimate beyond the effective threshold $X_{k,n}$) but then deteriorates rapidly.
- Both EVT methods outperform the empirical quantile estimator.

5.2.6 Conditional EVT for financial time series

- The GPD method is an unconditional approach for estimating \bar{F} and associated risk measures. A conditional (time-dependent) risk-measurement approach may be more appropriate.
- We now consider a simple adaptation of the GPD method to obtain conditional risk-measure estimates in a GARCH context.
- Assume X_{t-n+1}, \dots, X_t are negative log-returns generated by a strictly stationary time series process (X_t) of the form

$$X_t = \mu_t + \sigma_t Z_t,$$

where μ_t and σ_t are \mathcal{F}_{t-1} -measurable and $Z_t \stackrel{\text{ind.}}{\sim} F_Z$; e.g. ARMA model with GARCH errors. Furthermore, let $Z \sim F_Z$.

- VaR_α^t and ES_α^t based on $F_{X_{t+1}|\mathcal{F}_t}$ are given by

$$\text{VaR}_\alpha^t(X_{t+1}) = \mu_{t+1} + \sigma_{t+1} \text{VaR}_\alpha(Z),$$

$$\text{ES}_\alpha^t(X_{t+1}) = \mu_{t+1} + \sigma_{t+1} \text{ES}_\alpha(Z).$$

- To obtain estimates $\widehat{\text{VaR}}_{\alpha}^t(X_{t+1})$ and $\widehat{\text{ES}}_{\alpha}^t(X_{t+1})$, proceed as follows:
 - 1) Fit an ARMA-GARCH model (via exponential smoothing or QMLE based on normal innovations (since we do not assume a particular innovation distribution)). \Rightarrow Estimates of μ_{t+1} and σ_{t+1} .
 - 2) Fit a GPD to the excesses corresponding to F_Z (treat the residuals from the GARCH fitting procedure as iid from F_Z) \Rightarrow GPD-based estimates of $\text{VaR}_{\alpha}(Z)$ (see (14)) and $\text{ES}_{\alpha}(Z)$ (see (15)).