7 Copulas and dependence

7.1 Copulas

- We now look more closely at modelling the dependence among the components of a random vector $\boldsymbol{X} \sim F$ (risk-factor changes).
- **In short:** F = dependence structure C $\circ$ marginal dfs $F_1, \dots, F_d$
- **Advantages:**
  - Most natural in a static distributional context (no time dependence; apply, for example, to residuals of an ARMA-GARCH model)
  - Copulas allow us to understand and study dependence independently of the margins (first part of Sklar's Theorem; see later)
  - Copulas allow for a bottom-up approach to multivariate model building (second part of Sklar's Theorem; see later). This is often useful for constructing tailored F, for example, when we have more information about the margins than C or for *stress testing* purposes (to challenge the existing model and see how it performs).

7.1.1 Basic properties

Definition 7.1 (Copula)

A *copula* C is a df with $\mathrm{U}(0,1)$ margins.

Characterization

$C : [0,1]^d \to [0,1]$ is a copula if and only if

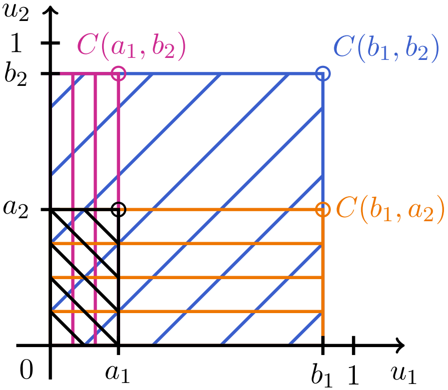
1) C is *grounded*, that is,
   $C(u_1, \dots, u_d) = 0$ if $u_j = 0$ for at least one $j \in \{1, \dots, d\}$.
2) C has standard *uniform* univariate *margins*, that is,
   $C(1, \dots, 1, u_j, 1, \dots, 1) = u_j$ for all $u_j \in [0,1]$ and $j \in \{1, \dots, d\}$.
3) C is *d-increasing*, that is, for all $\boldsymbol{a}, \boldsymbol{b} \in [0,1]^d$, $\boldsymbol{a} \le \boldsymbol{b}$,

$$\Delta_{(\boldsymbol{a},\boldsymbol{b}]} C = \sum_{\boldsymbol{i} \in \{0,1\}^d} (-1)^{\sum_{j=1}^d i_j} C(a_1^{i_1} b_1^{1-i_1}, \dots, a_d^{i_d} b_d^{1-i_d}) \ge 0.$$

   Equivalently, if existent: density $c(\boldsymbol{u}) \ge 0$ for all $\boldsymbol{u} \in (0,1)^d$.

2-increasingness explained in a picture:

$$\Delta_{(\boldsymbol{a},\boldsymbol{b}]}C = C(b_1,b_2) - C(b_1,a_2) - C(a_1,b_2) + C(a_1,a_2)$$
$$= \mathbb{P}(\boldsymbol{U} \in (\boldsymbol{a},\boldsymbol{b}]) \overset{!}{\geq} 0$$



$\Rightarrow$ $\Delta_{(\boldsymbol{a},\boldsymbol{b}]}C$ is the probability of a random vector $\boldsymbol{U} \sim C$ to be in $(\boldsymbol{a},\boldsymbol{b}]$.

Preliminaries

Lemma 7.2 (Probability transformation)
Let $X \sim F$, F continuous. Then $F(X) \sim \mathrm{U}(0,1)$.

Proof. $\mathbb{P}(F(X) \le u) = \mathbb{P}(F^{\leftarrow}(F(X)) \le F^{\leftarrow}(u)) = \mathbb{P}(X \le F^{\leftarrow}(u)) = F(F^{\leftarrow}(u)) = u$, $u \in [0,1]$; more details in the appendix. □

Note that F needs to be continuous (otherwise $F(X)$ would not reach all intervals $\subseteq [0,1]$).

Lemma 7.3 (Quantile transformation)
Let $U \sim \mathrm{U}(0,1)$ and F be any df. Then $X = F^{\leftarrow}(U) \sim F$.

Proof. $\mathbb{P}(F^{\leftarrow}(U) \le x) \underset{\text{(GI5)}}{=} \mathbb{P}(U \le F(x)) = F(x)$, $x \in \mathbb{R}$. □

Probability and quantile transformations are the key to all applications involving copulas. They allow us to go from $\mathbb{R}^d$ to $[0,1]^d$ and back.

Sklar's Theorem

Theorem 7.4 (Sklar's Theorem)

1) For any df F with margins $F_1, \dots, F_d$, there exists a copula C such that

$$F(x_1, \dots, x_d) = C(F_1(x_1), \dots, F_d(x_d)), \quad \boldsymbol{x} \in \mathbb{R}^d. \tag{27}$$

C is uniquely defined on $\prod_{j=1}^d \operatorname{ran} F_j$ and given by

$$C(u_1, \dots, u_d) = F(F_1^{\leftarrow}(u_1), \dots, F_d^{\leftarrow}(u_d)), \quad \boldsymbol{u} \in \prod_{j=1}^d \operatorname{ran} F_j,$$

where $\operatorname{ran} F_j = \{F_j(x) : x \in \mathbb{R}\}$ denotes the *range* of F_j.

2) Conversely, given any copula C and univariate dfs $F_1, \dots, F_d$, F defined by (27) is a df with margins $F_1, \dots, F_d$.

Proof.

1) Proof for continuous $F_1, \dots, F_d$ only. Let $\boldsymbol{X} \sim F$ and define $U_j = F_j(X_j)$, $j \in \{1, \dots, d\}$. By the probability transformation, $U_j \sim \mathrm{U}(0,1)$ (continuity!), $j \in \{1, \dots, d\}$, so the df C of $\boldsymbol{U}$ is a copula. Since $F_j \uparrow$ on $\operatorname{ran} X_j$, (GI3) implies that $X_j = F_j^{\leftarrow}(F_j(X_j)) = F_j^{\leftarrow}(U_j)$, $j \in \{1, \dots, d\}$. Therefore,

$$F(\boldsymbol{x}) = \mathbb{P}(X_j \le x_j \;\forall\, j) = \mathbb{P}(F_j^{\leftarrow}(U_j) \le x_j \;\forall\, j) \underset{\text{(GI5)}}{=} \mathbb{P}(U_j \le F_j(x_j) \;\forall\, j)$$

$$= C(F_1(x_1), \dots, F_d(x_d)), \quad \boldsymbol{x} \in \mathbb{R}^d.$$

Hence C is a copula and satisfies (27).

(GI4) implies that $F_j(F_j^{\leftarrow}(u_j)) = u_j$ for all $u_j \in \operatorname{ran} F_j$, so

$$C(u_1, \dots, u_d) = C(F_1(F_1^{\leftarrow}(u_1)), \dots, F_d(F_d^{\leftarrow}(u_d)))$$

$$\underset{(27)}{=} F(F_1^{\leftarrow}(u_1), \dots, F_d^{\leftarrow}(u_d)), \quad \boldsymbol{u} \in \prod_{j=1}^{d} \operatorname{ran} F_j.$$

2) For $\boldsymbol{U} \sim C$, define $\boldsymbol{X} = (F_1^{\leftarrow}(U_1), \dots, F_d^{\leftarrow}(U_d))$. Then

$$\mathbb{P}(\boldsymbol{X} \le \boldsymbol{x}) = \mathbb{P}(F_j^{\leftarrow}(U_j) \le x_j \ \forall\, j) \underset{\text{(GI5)}}{=} \mathbb{P}(U_j \le F_j(x_j) \ \forall\, j)$$
$$= C(F_1(x_1), \dots, F_d(x_d)), \quad \boldsymbol{x} \in \mathbb{R}^d.$$

Therefore, F defined by (27) is a df (that of $\boldsymbol{X}$), with margins $F_1, \dots, F_d$ (obtained by the quantile transformation). □

Example 7.5 (Bivariate Bernoulli distribution)

Let (X_1, X_2) follow a bivariate Bernoulli distribution with $\mathbb{P}(X_1 = k, X_2 = l) = 1/4$, $k, l \in \{0, 1\}$. $\Rightarrow \mathbb{P}(X_j = k) = 1/2$, $k \in \{0, 1\}$, $\operatorname{ran} F_j = \{0, 1/2, 1\}$, $j \in \{1, 2\}$. Any copula with $C(1/2, 1/2) = 1/4$ satisfies (27) (e.g. $C(u_1, u_2) = u_1 u_2$ or the copula $C(u_1, u_2) = \min\{u_1, u_2, (\delta(u_1) + \delta(u_2))/2\}$ with $\delta(u) = u^2$).

- We say that $\boldsymbol{X}$ (or F) *has copula C* if (27) holds.
- A *copula model* for $\boldsymbol{X}$ means $F(\boldsymbol{x}) = C(F_1(x_1), \dots, F_d(x_d))$ for some (parametric) copula C and (parametric) marginals $F_1, \dots, F_d$.

Invariance principle

Lemma 7.6 (Core of the invariance principle)
Let $X_j \sim F_j$, F_j continuous, $j \in \{1, \dots, d\}$. Then

$$\boldsymbol{X} \text{ has copula } C \quad \iff \quad (F_1(X_1), \dots, F_d(X_d)) \sim C.$$

Proof. See the appendix ("$\Rightarrow$" seen in the proof of Sklar's Theorem). $\square$

Theorem 7.7 (Invariance principle)
Let $\boldsymbol{X} \sim F$ with continuous margins $F_1, \dots, F_d$ and copula C. If $T_j \uparrow$ on $\operatorname{ran} X_j$ for all j, then $(T_1(X_1), \dots, T_d(X_d))$ (also) has copula C.

Proof. W.l.o.g. assume T_j to be right-continuous (T_j has at most countably many discontinuities and we thus change $T_j(X_j)$ at most on this null set). Since $T_j \uparrow$ on $\operatorname{ran} X_j$ and X_j is continuously distributed, $T_j(X_j)$ is continuously distributed and we have

$$
\begin{aligned}
F_{T_j(X_j)}(x) &= \mathbb{P}(T_j(X_j) \le x) = \mathbb{P}(T_j(X_j) < x) \underset{\text{(GI5)}}{=} \mathbb{P}(X_j < T_j^{\leftarrow}(x)) \\
&= \mathbb{P}(X_j \le T_j^{\leftarrow}(x)) = F_j(T_j^{\leftarrow}(x)), \quad x \in \mathbb{R}.
\end{aligned}
$$

This implies that $\mathbb{P}(F_{T_j(X_j)}(T_j(X_j)) \le u_j \,\forall\, j)$ equals

$$
\mathbb{P}(F_j(T_j^{\leftarrow}(T_j(X_j))) \le u_j \,\forall\, j) \underset{\text{(GI3)}}{=} \mathbb{P}(F_j(X_j) \le u_j \,\forall\, j) \overset{\text{L.7.6}}{\underset{\text{“}\Rightarrow\text{”}}{=}} C(\boldsymbol{u}).
$$

The claim follows from the if part (“$\Leftarrow$”) of Lemma 7.6. $\square$

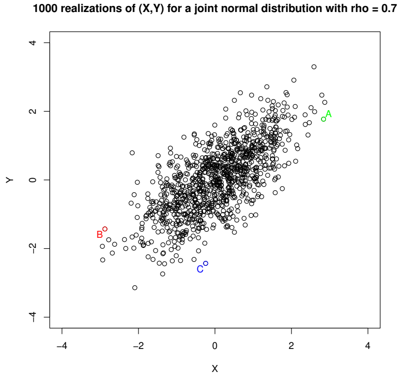
Interpretation of Sklar's Theorem (and the invariance principle)

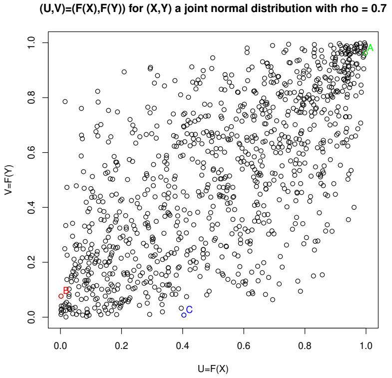
1) Part 1) of Sklar's Theorem allows one to decompose any df F into its margins and a copula. This, together with the invariance principle, allows one to study dependence independently of the margins via the margin-free $\boldsymbol{U} = (F_1(X_1), \dots, F_d(X_d))$ instead of $\boldsymbol{X} = (X_1, \dots, X_d)$ (they both have the same copula!). This is interesting for statistical applications, e.g. parameter estimation or goodness-of-fit.
2) Part 2) allows one to construct flexible multivariate distributions for particular applications (credit risk, stress testing, etc.).

Visualizing Part 1) of Sklar's Theorem

Left: Scatter plot of $n = 1000$ samples from $(X_1, X_2) \sim \mathrm{N}_2(\mathbf{0}, P)$, where $P = \begin{pmatrix} 1 & 0.7 \\ 0.7 & 1 \end{pmatrix}$. We mark three points A, B, C.

Right: Scatter plot of the corresponding Gauss copula (after applying the df Φ of $\mathrm{N}(0,1)$). Note how A, B, C change.

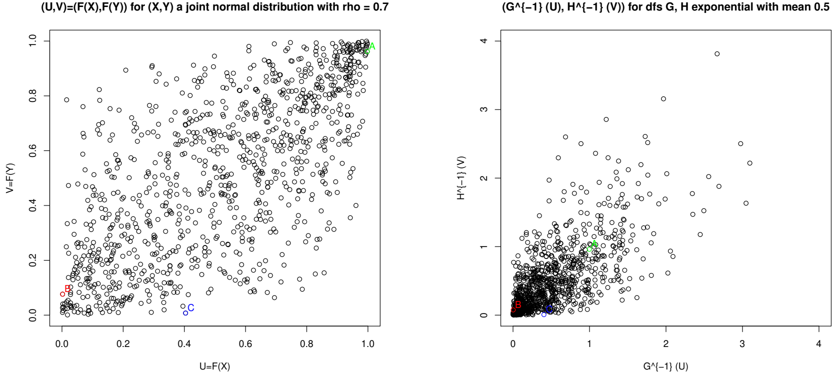




Visualizing Part 2) of Sklar's Theorem

Left: Same Gauss copula scatter plot as before. Apply marginal $\mathrm{Exp}(2)$-quantile functions ($F_j^{-1}(u) = -\log(1-u)/2$, $j \in \{1,2\}$).

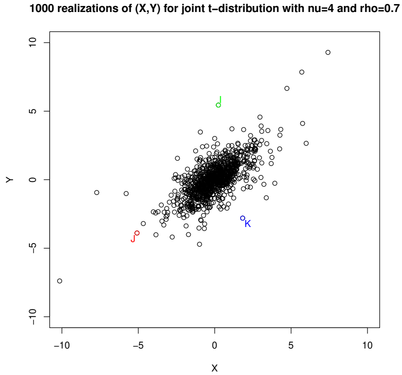
Right: The corresponding transformed random variates. Again, note the three points A, B, C.

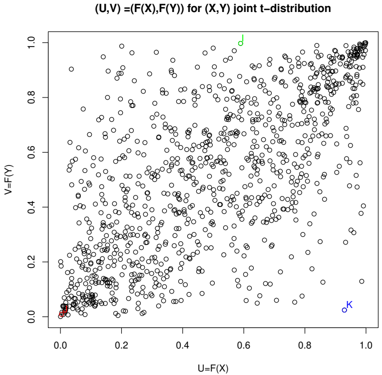


Visualizing Part 1) of Sklar's Theorem

Left: Scatter plot of $n = 1000$ samples from $(X_1, X_2) \sim t_2(4, \mathbf{0}, P)$, where $P = \begin{pmatrix} 1 & 0.7 \\ 0.7 & 1 \end{pmatrix}$. We mark three points I, J, K.

Right: Scatter plot of the corresponding t_4 copula (after applying the df t_4). Note how I, J, K change.

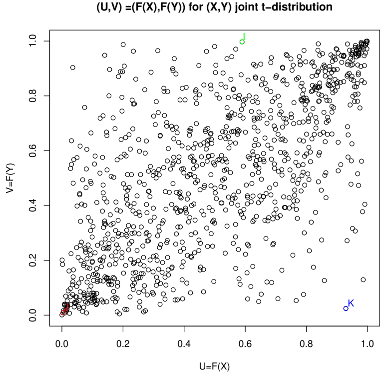


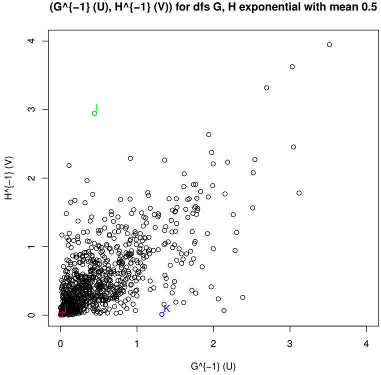


Visualizing Part 2) of Sklar's Theorem

Left: Same t_4 copula scatter plot as before. Apply marginal $\mathrm{Exp}(2)$-quantile functions ($F_j^{-1}(u) = -\log(1-u)/2$, $j \in \{1,2\}$).

Right: The corresponding transformed random variates. Again, note the three points I, J, K.





Fréchet–Hoeffding bounds

Theorem 7.8 (Fréchet–Hoeffding bounds)
Let $W(\boldsymbol{u}) = \max\{\sum_{j=1}^d u_j - d + 1, 0\}$ and $M(\boldsymbol{u}) = \min_{1\le j\le d}\{u_j\}$.

1) For any d-dimensional copula C,

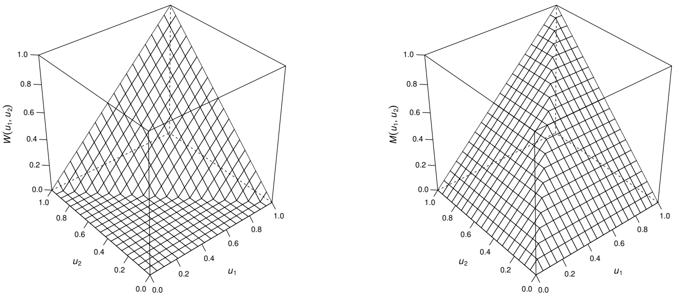
$$W(\boldsymbol{u}) \le C(\boldsymbol{u}) \le M(\boldsymbol{u}), \quad \boldsymbol{u} \in [0,1]^d.$$

2) W is a copula if and only if $d = 2$.
3) M is a copula for all $d \ge 2$.

Proof. See the appendix. □

- It is easy to verify that, for $U \sim \mathrm{U}(0,1)$,
  - $(U, \dots, U) \sim M$;
  - $(U, 1-U) \sim W$.

- Plot of W, M for $d = 2$ (compare with $(U, 1-U) \sim W$, $(U, U) \sim M$)



- The Fréchet–Hoeffding bounds correspond to perfect dependence (negative for W; positive for M); see Proposition 7.14 later.
- The Fréchet–Hoeffding bounds lead to bounds for any df F, via

$$\max\Big\{\sum_{j=1}^{d} F_j(x_j) - d + 1, 0\Big\} \le F(\boldsymbol{x}) \le \min_{1\le j\le d}\{F_j(x_j)\}.$$

We will use them later to derive bounds for the correlation coefficient.

7.1.2 Examples of copulas

- *Fundamental copulas*: important special copulas;
- *Implicit copulas*: extracted from known F via Sklar's Theorem;
- *Explicit copulas*: closed form, arising from construction principles.

Fundamental copulas

As usual, we assume the appearing margins $F_1, \dots, F_d$ to be continuous.

- $\Pi(\boldsymbol{u}) = \prod_{j=1}^{d} u_j$ is the *independence copula* since $C(F_1(x_1), \dots, F_d(x_d)) \underset{\text{Sklar}}{=} F(\boldsymbol{x}) \underset{\text{ind.}}{=} \prod_{j=1}^{d} F_j(x_j)$ if and only if $C(\boldsymbol{u}) = \Pi(\boldsymbol{u})$ (replace x_j by $F_j^{\leftarrow}(u_j)$ and apply (GI4)). Therefore, $X_1, \dots, X_d$ are independent if and only if their copula is Π; the density is thus $c(\boldsymbol{u}) = 1$, $\boldsymbol{u} \in [0,1]^d$.
- W is the *countermonotonicity copula*. It is the df of $(U, 1-U)$. It can be shown that if X_1, X_2 are perfectly negatively dependent (X_2 is a.s. a strictly decreasing function of X_1), their copula is W.

- M is the *comonotonicity copula*. It is the df of $(U, \dots, U)$. It can be shown that if $X_1, \dots, X_d$ are perfectly positively dependent ($X_2, \dots, X_d$ are a.s. strictly increasing functions of X_1), their copula is M.

Implicit copulas

Elliptical copulas are implicit copulas arising from elliptical distributions via Sklar's Theorem. The two most prominent parametric families are the Gauss copula and the t copula (stemming from normal variance mixtures).

Gauss copulas

- Consider (w.l.o.g.) $\boldsymbol{X} \sim \mathrm{N}_d(\boldsymbol{0}, P)$. The *Gauss copula* (family) is given by

$$\begin{aligned} C_P^{\mathsf{Ga}}(\boldsymbol{u}) &= \mathbb{P}(\Phi(X_1) \le u_1, \dots, \Phi(X_d) \le u_d) \\ &= \Phi_P(\Phi^{-1}(u_1), \dots, \Phi^{-1}(u_d)) \end{aligned}$$

  where Φ_P is the df of $\mathrm{N}_d(\boldsymbol{0}, P)$ and Φ the df of $\mathrm{N}(0, 1)$.

- Special cases: If $P = I_d$ then $C = \Pi$, and if $P = J_d = \mathbf{1}\mathbf{1}'$ then $C = M$. If $d = 2$ and $\rho = P_{12} = -1$ then $C = W$.
- Sklar's Theorem $\Rightarrow$ The density of $C(\boldsymbol{u}) = F(F_1^{\leftarrow}(u_1), \dots, F_d^{\leftarrow}(u_d))$ is

$$c(\boldsymbol{u}) = \frac{f(F_1^{\leftarrow}(u_1), \dots, F_d^{\leftarrow}(u_d))}{\prod_{j=1}^d f_j(F_j^{\leftarrow}(u_j))}, \quad \boldsymbol{u} \in (0,1)^d.$$

In particular, the density of C_P^{Ga} is

$$c_P^{\mathsf{Ga}}(\boldsymbol{u}) = \frac{1}{\sqrt{\det P}} \exp\Big(-\frac{1}{2}\boldsymbol{x}'(P^{-1} - I_d)\boldsymbol{x}\Big), \tag{28}$$

where $\boldsymbol{x} = (\Phi^{-1}(u_1), \dots, \Phi^{-1}(u_d))$.

t copulas

- Consider (w.l.o.g.) $\boldsymbol{X} \sim t_d(\nu, \mathbf{0}, P)$. The *$t$ copula* (family) is given by

$$\begin{aligned} C_{\nu,P}^t(\boldsymbol{u}) &= \mathbb{P}(t_\nu(X_1) \le u_1, \dots, t_\nu(X_d) \le u_d) \\ &= t_{\nu,P}(t_\nu^{-1}(u_1), \dots, t_\nu^{-1}(u_d)) \end{aligned}$$

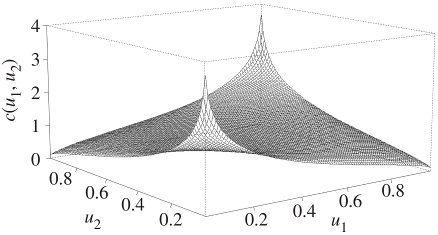
where $t_{\nu,P}$ is the df of $t_d(\nu, \mathbf{0}, P)$ and t_ν the df of the univariate t distribution with ν degrees of freedom.

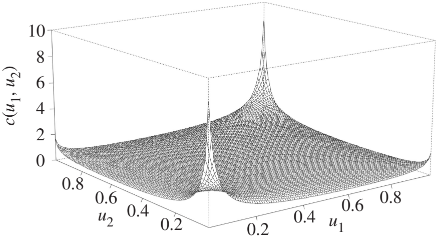
- Special cases: $P = J_d = \mathbf{1}\mathbf{1}'$ then $C = M$. However, if $P = I_d$ then $C \neq \Pi$ (unless $\nu = \infty$ in which case $C^t_{\nu,P} = C^{\mathrm{Ga}}_P$). If $d = 2$ and $\rho = P_{12} = -1$ then $C = W$.
- Sklar's Theorem $\Rightarrow$ The density of $C^t_{\nu,P}$ is

$$c^t_{\nu,P}(\boldsymbol{u}) = \frac{\Gamma((\nu+d)/2)}{\Gamma(\nu/2)\sqrt{\det P}}\left(\frac{\Gamma(\nu/2)}{\Gamma((\nu+1)/2)}\right)^d \frac{(1+\boldsymbol{x}'P^{-1}\boldsymbol{x}/\nu)^{-(\nu+d)/2}}{\prod_{j=1}^d (1+x_j^2/\nu)^{-(\nu+1)/2}},$$

for $\boldsymbol{x} = (t_\nu^{-1}(u_1), \ldots, t_\nu^{-1}(u_d))$.
- For more details, see Demarta and McNeil (2005).
- For scatter plots, see the visualization of Sklar's Theorem above. Note the difference in the tails: The smaller ν, the more mass is concentrated in the joint tails.

Perspective plots of the densities of $C^{\mathrm{Ga}}_{\rho=0.3}$ (left) and $C^{t}_{4,\,\rho=0.3}(\boldsymbol{u})$ (right).





Advantages and drawbacks of elliptical copulas:

Advantages:

- Modelling pairwise dependencies (comparably flexible)
- Density available
- Sampling simple (for Gauss, t)

Drawbacks:

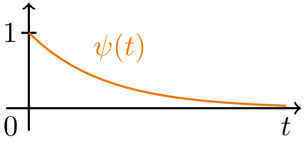
- Typically, C is not explicit
- Radially symmetric (so the same lower/upper tail behaviour)

Explicit copulas

Archimedean copulas are copulas of the form

$$C(\boldsymbol{u}) = \psi(\psi^{-1}(u_1) + \cdots + \psi^{-1}(u_d))$$

where ψ is the *(Archimedean) generator*.

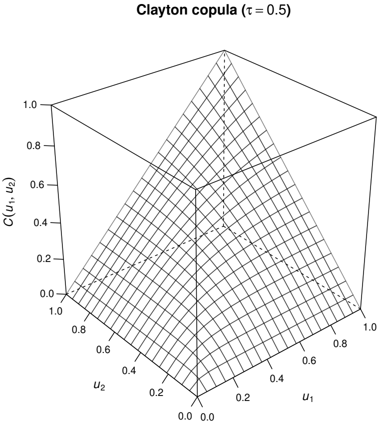


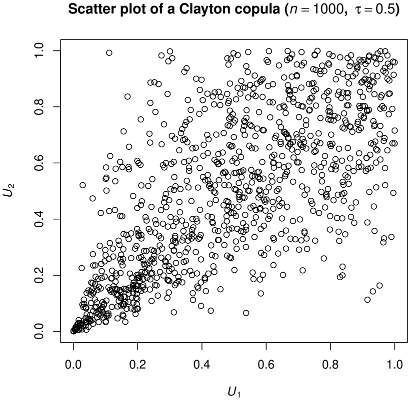
- $\psi : [0,\infty) \to [0,1]$ is $\downarrow$ on $[0, \inf\{t : \psi(t) = 0\}]$ and satisfies $\psi(0) = 1$, $\psi(\infty) = \lim_{t\to\infty} \psi(t) = 0$.
- We set $\psi^{-1}(0) = \inf\{t : \psi(t) = 0\}$.
- The set of all generators is denoted by Ψ.
- Not every generator $\psi \in \Psi$ generates indeed a proper copula (there are conditions, e.g. complete monotonicity, i.e. derivatives alternating in sign).
- If $\psi(t) > 0$, $t \in [0,\infty)$, we call ψ *strict*.

Clayton copulas are obtained for $\psi(t) = (1+t)^{-1/\theta}$, $t \in [0,\infty)$, $\theta \in (0,\infty)$. For $\theta \downarrow 0$, $C \to \Pi$; and for $\theta \uparrow \infty$, $C \to M$.

Left: Plot of a bivariate Clayton copula (Kendall's tau 0.5; see later).

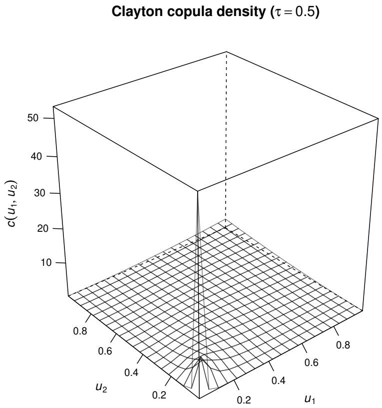
Right: Corresponding scatter plot (sample size $n = 1000$)

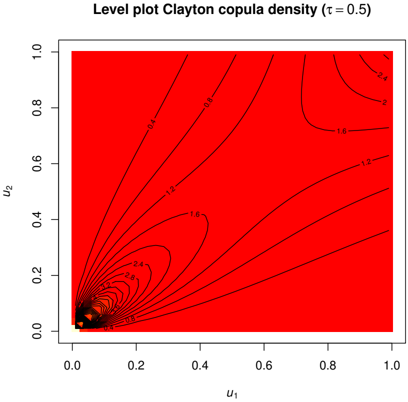




Left: Plot of the corresponding density.

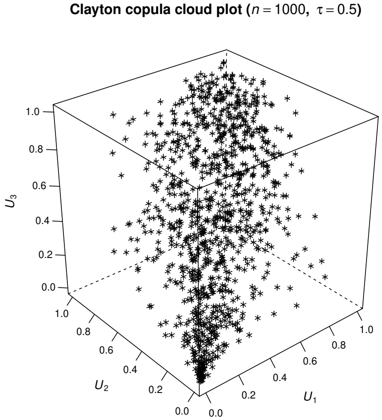
Right: Level plot of the density (with heat colors).

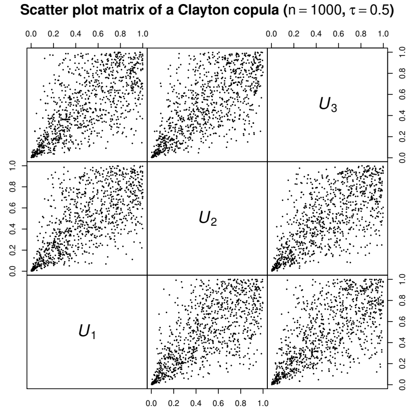




Left: Cloud plot of a trivariate Clayton copula (sample size $n = 1000$; Kendall's tau 0.5).

Right: Corresponding scatter plot matrix.

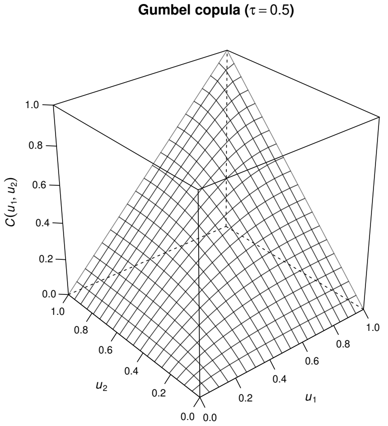


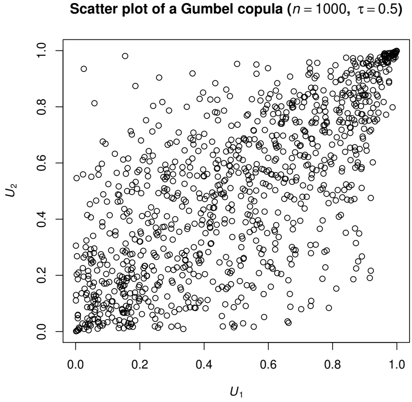


Gumbel copulas are obtained for $\psi(t) = \exp(-t^{1/\theta})$, $t \in [0, \infty)$, $\theta \in [1, \infty)$. For $\theta = 1$, $C = \Pi$; and for $\theta \to \infty$, $C \to M$.

Left: Plot of a bivariate Gumbel copula (Kendall's tau 0.5).

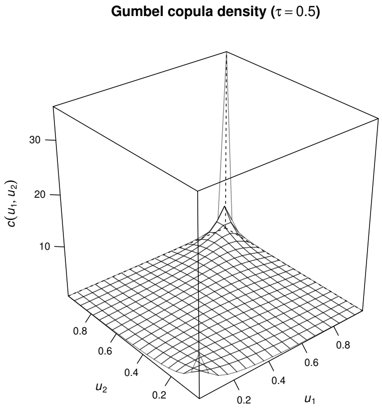
Right: Corresponding scatter plot (sample size $n = 1000$)

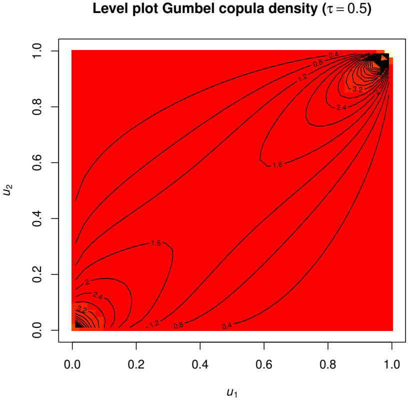




Left: Plot of the corresponding density.

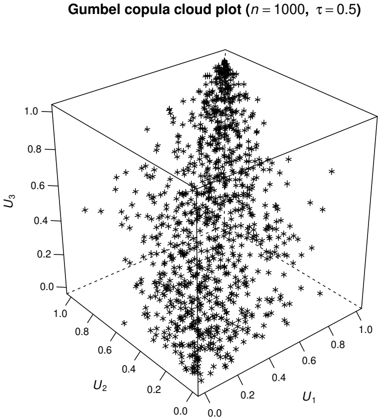
Right: Level plot of the density (with heat colors).

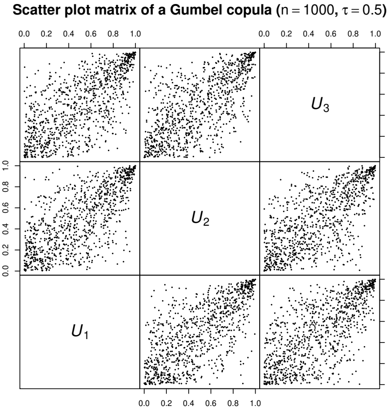




Left: Cloud plot of a trivariate Gumbel copula (sample size $n = 1000$; Kendall's tau 0.5).

Right: Corresponding scatter plot matrix.





Advantages and drawbacks of Archimedean copulas:

Advantages:

- Typically explicit (if ψ^{-1} is available)
- Useful in calculations: Properties can typically be expressed in terms of ψ
- Densities of various examples available
- Sampling often simple
- Not restricted to radial symmetry

Drawbacks:

- All margins of the same dimension are equal (symmetry or exchangeability; see later)
- Often used only with a small number of parameters (some extensions available, but still less than $d(d-1)/2$)

7.1.3 Meta distributions

- *Fréchet class*: Class of all dfs F with given marginal dfs $F_1, \dots, F_d$; *Meta-C models*: All dfs F with the same given copula C.
- **Example:** A meta-t model is a multivariate df F with t copula C and some margins $F_1, \dots, F_d$.

7.1.4 Simulation of copulas and meta distributions

Sampling implicit copulas

Due to their construction via Sklar's Theorem, implicit copulas can be sampled via Lemma 7.6.

Algorithm 7.9 (Simulation of implicit copulas)

1) Sample $\boldsymbol{X} \sim F$, where F is a df with continuous margins $F_1, \dots, F_d$.
2) Return $\boldsymbol{U} = (F_1(X_1), \dots, F_d(X_d))$ (probability transformation).

Example 7.10

- Sampling Gauss copulas C_P^{Ga}:
  1) Sample $\boldsymbol{X} \sim \mathrm{N}_d(\mathbf{0}, P)$ ($\boldsymbol{X} \stackrel{\mathrm{d}}{=} A\boldsymbol{Z}$ for $AA' = P$, $\boldsymbol{Z} \sim \mathrm{N}_d(\mathbf{0}, I_d)$).
  2) Return $\boldsymbol{U} = (\Phi(X_1), \dots, \Phi(X_d))$.
- Sampling t_ν copulas $C_{\nu,P}^t$:
  1) Sample $\boldsymbol{X} \sim t_d(\nu, \mathbf{0}, P)$ ($\boldsymbol{X} \stackrel{\mathrm{d}}{=} \sqrt{W} A\boldsymbol{Z}$ for $W = \frac{1}{V}$, $V \sim \Gamma(\frac{\nu}{2}, \frac{\nu}{2})$).
  2) Return $\boldsymbol{U} = (t_\nu(X_1), \dots, t_\nu(X_d))$.

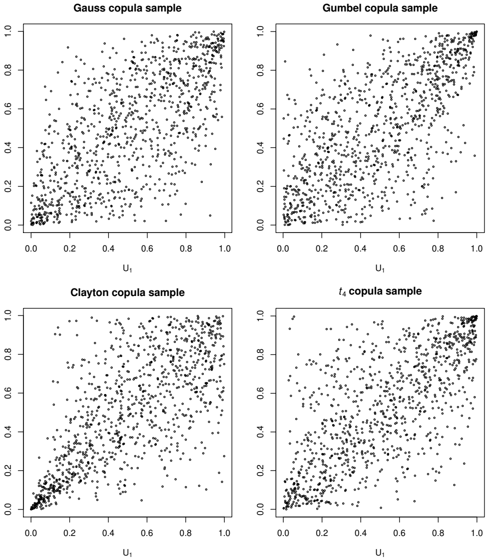
Sampling meta distributions

Meta-C distributions can be sampled via Sklar's Theorem, Part 2).

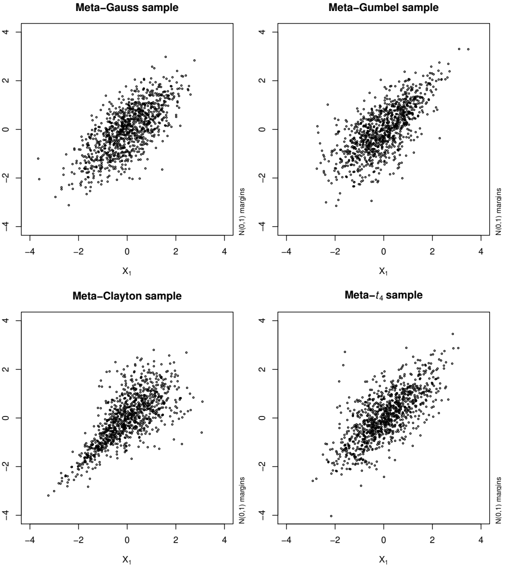
Algorithm 7.11 (Sampling meta-C models)

1) Sample $\boldsymbol{U} \sim C$.
2) Return $\boldsymbol{X} = (F_1^{\leftarrow}(U_1), \dots, F_d^{\leftarrow}(U_d))$ (quantile transformation).

2000 samples from (a): $C^{\mathsf{Ga}}_{\rho=0.7}$; (b): $C^{\mathsf{G}}_{\theta=2}$; (c): $C^{\mathsf{C}}_{\theta=2.2}$; (d): $C^{t}_{\nu=4,\,\rho=0.71}$



... transformed to $\mathrm{N}(0,1)$ margins; all have linear correlation ≈ 0.7!



A general sampling algorithm

For a general copula C (without further information), the only known sampling algorithm is the *conditional distribution method*.

Theorem 7.12 (Conditional distribution method)
If C is a d-dimensional copula and $\boldsymbol{U'} \sim \mathrm{U}(0,1)^d$ then $\boldsymbol{U} \sim C$, where

$$\begin{aligned} U_1 &= U_1', \\ U_2 &= C_{2|1}^{\leftarrow}(U_2' \,|\, U_1), \\ U_3 &= C_{3|1,2}^{\leftarrow}(U_3' \,|\, U_1, U_2), \\ &\;\;\vdots \\ U_d &= C_{d|1,\dots,d-1}^{\leftarrow}(U_d' \,|\, U_1, \dots, U_{d-1}) \end{aligned}$$

and where $C_{j|1,\dots,j-1}(u_j \,|\, u_1, \dots, u_{j-1}) = \mathbb{P}(U_j \le u_j \,|\, U_1 = u_1, \dots, U_{j-1} = u_{j-1})$, $j \in \{2, \dots, d\}$.

Theorem 7.13 (Schmitz (2003))

Let C be a d-dimensional copula which admits, for $d \geq 3$, continuous partial derivatives w.r.t. $u_1, \dots, u_{d-1}$. For a.e. $u_1, \dots, u_{j-1} \in [0,1]$,

$$C_{j|1,\dots,j-1}(u_j \,|\, u_1, \dots, u_{j-1}) = \frac{\mathrm{D}_{j-1,\dots,1}\, C^{(1,\dots,j)}(u_1, \dots, u_j)}{\mathrm{D}_{j-1,\dots,1}\, C^{(1,\dots,j-1)}(u_1, \dots, u_{j-1})},$$

where $C^{(1,\dots,j)}(u_1, \dots, u_j) = C(u_1, \dots, u_j, 1, \dots, 1)$ and $\mathrm{D}_{j-1,\dots,1}$ is the differential operator w.r.t. $u_1, \dots, u_{j-1}$.

Note: $C_{2|1}(u_2 \,|\, u_1) = \frac{\mathrm{D}_1 C(u_1,u_2)}{1} = \mathrm{D}_1 C(u_1, u_2)$ which also follows from

$$\begin{aligned}
&\lim_{h \downarrow 0} \frac{C(u_1 + h, u_2) - C(u_1, u_2)}{h} \\
&= \lim_{h \downarrow 0} \frac{\mathbb{P}(U_1 \leq u_1 + h, U_2 \leq u_2) - \mathbb{P}(U_1 \leq u_1, U_2 \leq u_2)}{h} \\
&= \lim_{h \downarrow 0} \frac{\mathbb{P}(U_2 \leq u_2, u_1 < U_1 \leq u_1 + h)}{\mathbb{P}(u_1 < U_1 \leq u_1 + h)} = \lim_{h \downarrow 0} \mathbb{P}(U_2 \leq u_2 | u_1 < U_1 \leq u_1 + h).
\end{aligned}$$

7.1.5 Further properties of copulas

Survival copulas

- If $\boldsymbol{U} \sim C$, then $\mathbf{1} - \boldsymbol{U} \sim \hat{C}$, the *survival copula* of C.
- $\hat{C}$ can be expressed as

$$\hat{C}(\boldsymbol{u}) = \sum_{J \subseteq \{1,\dots,d\}} (-1)^{|J|} C((1-u_1)^{I_J(1)}, \dots, (1-u_d)^{I_J(d)})$$

  in terms of its corresponding copula (essentially an application of the inclusion–exclusion principle). For $d = 2$, $\hat{C}(u_1, u_2) = 1 - (1-u_1) - (1-u_2) + C(1-u_1, 1-u_2) = -1 + u_1 + u_2 + C(1-u_1, 1-u_2)$. We can also verify this directly by noting that $\hat{C}(u_1, u_2)$ equals

$$\begin{aligned} &\mathbb{P}(1-U_1 \le u_1, 1-U_2 \le u_2) = \mathbb{P}(U_1 > 1-u_1, U_2 > 1-u_2) \\ &= \mathbb{P}(U_1 > 1-u_1) - \mathbb{P}(U_1 > 1-u_1, U_2 \le 1-u_2) \\ &= 1-(1-u_1) - (\mathbb{P}(U_2 \le 1-u_2) - \mathbb{P}(U_1 \le 1-u_1, U_2 \le 1-u_2)) \\ &= u_1 - (1-u_2 - C(1-u_1, 1-u_2)). \end{aligned}$$

- If C admits a density, $\hat{c}(\boldsymbol{u}) = c(\boldsymbol{1} - \boldsymbol{u})$.
- If $\hat{C} = C$, C is called *radially symmetric*. Check that W, Π, and M are radially symmetric. All elliptical copulas are radially symmetric, too.
- One can show: If X_j is symmetrically distributed about μ_j, $j \in \{1, \dots, d\}$, then $\boldsymbol{X}$ is radially symmetric about $\boldsymbol{\mu}$ if and only if $C = \hat{C}$.
- Sklar's Theorem can also be formulated for survival functions. In this case, the main part reads

$$\bar{F}(\boldsymbol{x}) = \hat{C}(\bar{F}_1(x_1), \dots, \bar{F}_d(x_d)),$$

where $\bar{F}(\boldsymbol{x}) = \mathbb{P}(\boldsymbol{X} > \boldsymbol{x})$ with corresponding marginal survival functions $\bar{F}_1, \dots, \bar{F}_d$ (with $\bar{F}_j(x) = \mathbb{P}(X_j > x)$). Hence survival copulas combine marginal to joint survival functions.

Exchangeability

- $\boldsymbol{X}$ is *exchangeable* if
$$(X_1, \dots, X_d) \stackrel{\text{d}}{=} (X_{\pi(1)}, \dots, X_{\pi(d)})$$
for any permutation $(\pi(1), \dots, \pi(d))$ of $(1, \dots, d)$.
- A copula C is *exchangeable* if it is the df of an exchangeable $\boldsymbol{U}$ with $\mathrm{U}(0,1)$ margins. This holds if only if $C(u_1, \dots, u_d) = C(u_{\pi(1)}, \dots, u_{\pi(d)})$ for all possible permutations of arguments, i.e. if C is symmetric.
- Exchangeable/symmetric copulas are useful for approximate modelling homogeneous portfolios.
- **Examples:**
  - Archimedean copulas
  - Elliptical copulas (such as Gauss/t) for equicorrelated P (i.e. $P = \rho J_d + (1-\rho) I_d$ for $\rho \geq -1/(d-1)$); in particular, $d = 2$

Copula densities

- By Sklar's Theorem, if F_j has density f_j, $j \in \{1, \dots d\}$, and C has density c, then the density f of F satisfies

$$f(\boldsymbol{x}) = c(F_1(x_1), \dots, F_d(x_d)) \prod_{j=1}^{d} f_j(x_j). \tag{29}$$

  This implies

$$c(\boldsymbol{u}) = \frac{f(F_1^{-1}(u_1), \dots, F_d^{-1}(u_d))}{f_1(F_1^{-1}(u_1)) \cdot \dots \cdot f_d(F_d^{-1}(u_d))}.$$

- It follows from (29) that the log-density splits into

$$\log f(\boldsymbol{x}) = \log c(F_1(x_1), \dots, F_d(x_d)) + \sum_{j=1}^{d} \log f_j(x_j)$$

  which allows for a *two-stage estimation* (marginal and copula parameters separately); see Section 7.6.

7.2 Dependence concepts and measures

Measures of association/dependence are scalar measures which summarize the dependence in terms of a single number. There are better and worse examples of such measures, which we will study in this section.

7.2.1 Perfect dependence

X_1, X_2 are *countermonotone* if (X_1, X_2) has copula W.

$X_1, \dots, X_d$ are *comonotone* if $(X_1, \dots, X_d)$ has copula M.

Proposition 7.14 (Perfect dependence)

1) $X_2 = T(X_1)$ a.s. with decreasing $T(x) = F_2^{\leftarrow}(1 - F_1(x))$ (countermonotone) if and only if $C(u_1, u_2) = W(u_1, u_2)$, $u_1, u_2 \in [0, 1]$.
2) $X_j = T_j(X_1)$ a.s. with increasing $T_j(x) = F_j^{\leftarrow}(F_1(x))$, $j \in \{2, \dots, d\}$, (comonotone) if and only if $C(\boldsymbol{u}) = M(\boldsymbol{u})$, $\boldsymbol{u} \in [0, 1]^d$.

Proof. See the appendix. □

Proposition 7.15 (Comonotone additivity)

Let $\alpha \in (0,1)$ and $X_j \sim F_j$, $j \in \{1,\dots,d\}$, be comontone. Then $\mathrm{VaR}_\alpha(X_1+\dots+X_d) = F^{\leftarrow}_{X_1+\dots+X_d}(\alpha) = F_1^{\leftarrow}(\alpha)+\dots+F_d^{\leftarrow}(\alpha) = \sum_{j=1}^d \mathrm{VaR}_\alpha(X_j)$.

Proof. Assume $F_j \uparrow$ and continuous for all j (for the general case, see the appendix). Since $X_j \overset{\mathrm{d}}{=} F_j^{-1}(U)$, $j \in \{1,\dots,d\}$, for $U \sim \mathrm{U}(0,1)$, and since $T(u) = F_1^{-1}(u)+\dots+F_d^{-1}(u)$ is $\uparrow$ and continuous,

$$\begin{aligned} F_{\sum_{j=1}^d X_j}(T(\alpha)) &= F_{T(U)}(T(\alpha)) = \mathbb{P}(T(U) \le T(\alpha)) \\ &= \mathbb{P}(U \le T^{-1}(T(\alpha))) = \alpha, \end{aligned} \tag{30}$$

hence

$$\mathrm{VaR}_\alpha\Big(\sum_{j=1}^d X_j\Big) = F^{-1}_{\sum_{j=1}^d X_j}(\alpha) \underset{(30)}{=} T(\alpha) = \sum_{j=1}^d \mathrm{VaR}_\alpha(X_j).$$

□

7.2.2 Linear correlation

For two random variables X_1 and X_2 with $\mathbb{E}(X_j^2) < \infty$, $j \in \{1, 2\}$, the (*linear* or *Pearson's*) *correlation coefficient* ρ is defined by

$$\rho(X_1, X_2) = \frac{\operatorname{cov}(X_1, X_2)}{\sqrt{\operatorname{var} X_1}\sqrt{\operatorname{var} X_2}} = \frac{\mathbb{E}((X_1 - \mathbb{E}X_1)(X_2 - \mathbb{E}X_2))}{\sqrt{\mathbb{E}((X_1 - \mathbb{E}X_1)^2)}\sqrt{\mathbb{E}((X_2 - \mathbb{E}X_2)^2)}}.$$

Classical properties and drawbacks of linear correlation

Let X_1 and X_2 be two random variables with $\mathbb{E}(X_j^2) < \infty$, $j \in \{1, 2\}$. Note that ρ depends on the marginal distributions! In particular, second moments have to exist (not the case, e.g. for $X_1, X_2 \overset{\text{ind.}}{\sim} F(x) = 1 - x^{-3/2}$!)

- $|\rho| \leq 1$. Furthermore, $|\rho| = 1$ if and only if there are constants $a \in \mathbb{R}\backslash\{0\}, b \in \mathbb{R}$ with $X_2 = aX_1 + b$ a.s. with $a \gtrless 0$ if and only if $\rho = \pm 1$. This discards other strong functional dependence such as $X_2 = X_1^2$, for example.

- If X_1 and X_2 are independent, then $\rho = 0$. However, the converse is not true in general; see Example 7.17 below.
- ρ is invariant under strictly increasing linear transformations on $\operatorname{ran} X_1 \times \operatorname{ran} X_2$ but not invariant under strictly increasing functions in general. To see this, consider $(X_1, X_2) \sim \mathrm{N}_2(\mathbf{0}, P)$. Then $\rho(X_1, X_2) = P_{12}$, but (as one can show) $\rho(F_1(X_1), F_2(X_2)) = \frac{6}{\pi} \arcsin(P_{12}/2)$.

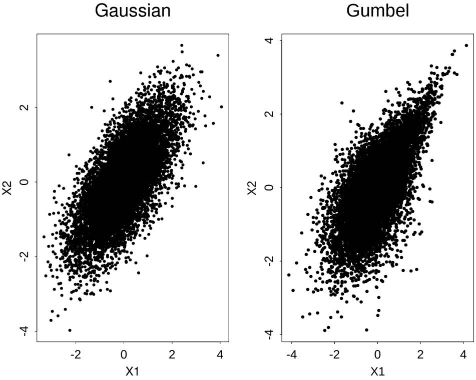
Proposition 7.16 (Hoeffding's formula)
Let $X_j \sim F_j$, $j \in \{1, 2\}$, be two random variables with $\mathbb{E}(X_j^2) < \infty$, $j \in \{1, 2\}$, and joint distribution function F. Then

$$\operatorname{cov}(X_1, X_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (F(x_1, x_2) - F_1(x_1) F_2(x_2)) \, \mathrm{d}x_1 \mathrm{d}x_2.$$

Correlation fallacies

Fallacy 1: F_1, F_2, and ρ uniquely determine F

This is true for bivariate elliptical distributions, but wrong in general. The following samples both have $\mathrm{N}(0,1)$ margins and correlation $\rho = 0.7$, yet come from different (copula) models:



Another example is this.

Example 7.17 (Uncorrelated $\not\Rightarrow$ independent)

- Consider the two risks

$$X_1 = Z \quad \text{(Profit \& Loss Country A)},$$
$$X_2 = ZV \quad \text{(Profit \& Loss Country B)},$$

  where V, Z are independent with $Z \sim \mathrm{N}(0,1)$ and $\mathbb{P}(V = -1) = \mathbb{P}(V = 1) = 1/2$. Then $X_2 \sim \mathrm{N}(0,1)$ and $\rho(X_1, X_2) = \mathrm{cov}(X_1, X_2) = \mathbb{E}(X_1 X_2) \underset{\text{ind.}}{=} \mathbb{E}(V)\mathbb{E}(Z^2) = 0$, but X_1 and X_2 are not independent (in fact, V makes (X_1, X_2) switch between counter- and comonotonicity).
- Consider $(X_1', X_2') \sim \mathrm{N}_2(\mathbf{0}, I_2)$. Both (X_1', X_2') and (X_1, X_2) have $\mathrm{N}(0,1)$ margins and $\rho = 0$, but the copula of (X_1', X_2') is Π and the copula of (X_1, X_2) is the convex combination $C(\boldsymbol{u}) = \lambda M(\boldsymbol{u}) + (1 - \lambda) W(\boldsymbol{u})$ for $\lambda = 0.5$.

Fallacy 2: Given F_1, F_2, any $\rho \in [-1, 1]$ is attainable

This is true for elliptically distributed (X_1, X_2) with $\mathbb{E}(R^2) < \infty$ (as then $\mathrm{corr}\,\boldsymbol{X} = P$), but wrong in general:

- If F_1 and F_2 are not of the same type (no linearity), $\rho(X_1, X_2) = 1$ is not attainable (recall that $|\rho| = 1$ if and only if there are constants $a \in \mathbb{R}\backslash\{0\}, b \in \mathbb{R}$ with $X_2 = aX_1 + b$ a.s.).
- What is the attainable range then? Hoeffding's formula

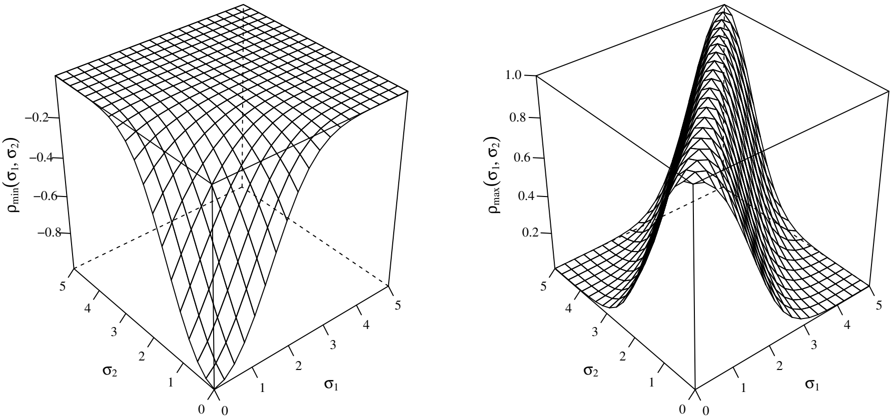
$$\mathrm{cov}(X_1, X_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(C(F_1(x_1), F_2(x_2)) - F_1(x_1)F_2(x_2))\,\mathrm{d}x_1\mathrm{d}x_2.$$

implies bounds on attainable ρ:

$$\rho \in [\rho_{\mathsf{min}},\ \rho_{\mathsf{max}}]\ (\rho_{\mathsf{min}} \text{ is attained for } C = W,\ \rho_{\mathsf{max}} \text{ for } C = M).$$

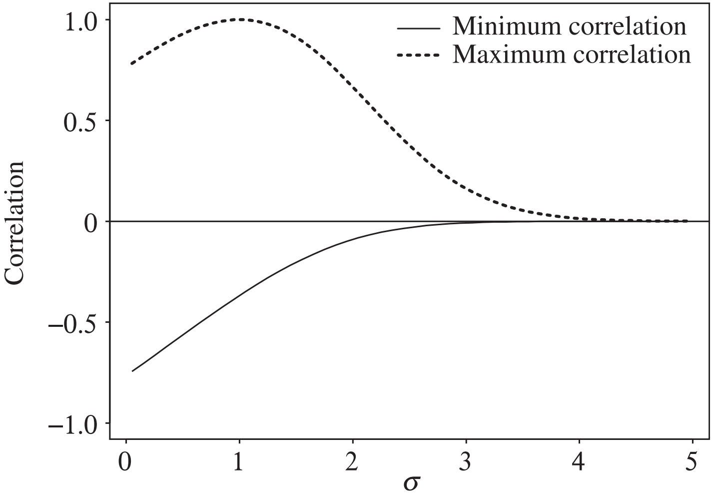
Example 7.18 (Bounds for a model with $\mathrm{LN}(0, \sigma_j^2)$ margins)

Let $X_j \sim \mathrm{LN}(0, \sigma_j^2)$, $j \in \{1, 2\}$. One can show that minimal (ρ_{min}; left) and maximal (ρ_{max}; right) correlations are given as follows.



For $\sigma_1^2 = 1$, $\sigma_2^2 = 16$ one has $\rho \in [-0.0003,\ 0.0137]$!

Specifically, let $X_1 \sim \mathrm{LN}(0,1)$ and $X_2 \sim \mathrm{LN}(0,\sigma^2)$. Now let σ vary and plot ρ_{min} and ρ_{max} against σ:



Fallacy 3: ρ maximal (i.e. $C = M$) $\Rightarrow \mathrm{VaR}_\alpha(X_1 + X_2)$ maximal

- This is true if (X_1, X_2) is elliptically distributed since the maximal $\rho = 1$ implies that X_1, X_2 are comonotone, so VaR_α is additive (by Proposition 7.15) and additivity provides the largest possible bound in this case as VaR_α is subadditive (by Proposition 6.24).
- Any superadditivity example $\mathrm{VaR}_\alpha(X_1+X_2) > \mathrm{VaR}_\alpha(X_1)+\mathrm{VaR}_\alpha(X_2)$ serves as a counterexample as the right-hand side under comonotonicity (so maximal correlation) only *equals* $\mathrm{VaR}_\alpha(X_1 + X_2)$; see Section 2.3.5 and Proposition 7.15.

7.2.3 Rank correlation

Rank correlation coefficients are...

- ... always defined;
- ... invariant under strictly increasing transformations of the random variables (hence only depend on the underlying copula).

Kendall's tau and Spearman's rho

Definition 7.19 (Kendall's tau)
Let $X_j \sim F_j$ with F_j continuous, $j \in \{1, 2\}$. Let (X_1', X_2') be an independent copy of (X_1, X_2). *Kendall's tau* is defined by

$$\begin{aligned}\rho_\tau &= \mathbb{E}(\mathrm{sign}((X_1 - X_1')(X_2 - X_2'))) \\ &= \mathbb{P}((X_1 - X_1')(X_2 - X_2') > 0) - \mathbb{P}((X_1 - X_1')(X_2 - X_2') < 0),\end{aligned}$$

where $\mathrm{sign}(x) = I_{(0,\infty)}(x) - I_{(-\infty,0)}(x)$ (so -1 for $x < 0$, 0 for $x = 0$ and 1 for $x > 0$).

By definition, Kendall's tau is the probability of *concordance* ($\mathbb{P}((X_1 - X_1')(X_2 - X_2') > 0)$; probability of two independent points from F to have a positive slope) minus the probability of *discordance* ($\mathbb{P}((X_1 - X_1')(X_2 - X_2') < 0)$; probability of two independent points from F to have a negative slope).

Proposition 7.20 (Formula for Kendall's tau)

Let $X_j \sim F_j$ with F_j continuous, $j \in \{1,2\}$, and copula C. Then

$$\rho_\tau = 4\int_0^1\int_0^1 C(u_1,u_2)\,\mathrm{d}C(u_1,u_2) - 1 = 4\,\mathbb{E}(C(U_1,U_2)) - 1,$$

where $(U_1,U_2) \sim C$.

Proof. See the appendix. □

An estimator of ρ_τ is provided by the *sample version of Kendall's tau*

$$r_n^\tau = \frac{1}{\binom{n}{2}} \sum_{1\le i_1<i_2\le n} \operatorname{sign}((X_{i_1 1} - X_{i_2 1})(X_{i_1 2} - X_{i_2 2})). \qquad (31)$$

Definition 7.21 (Spearman's rho)

Let $X_j \sim F_j$ with F_j continuous, $j \in \{1,2\}$. *Spearman's rho* is defined by $\rho_\mathsf{S} = \rho(F_1(X_1), F_2(X_2))$.

Proposition 7.22 (Formula for Spearman's rho)

Let $X_j \sim F_j$ with F_j's continuous and copula C. For $(U_1', U_2') \sim \Pi$,

$$\rho_{\mathsf{S}} = 12 \int_0^1 \int_0^1 C(u_1, u_2)\,\mathrm{d}u_1\mathrm{d}u_2 - 3 = 12\mathbb{E}(C(U_1', U_2')) - 3.$$

Proof. By Hoeffding's formula, we have $\rho_{\mathsf{S}}(X_1, X_2) = \rho(F_1(X_1), F_2(X_2))$ $= 12 \int_0^1 \int_0^1 (C(u_1, u_2) - u_1 u_2)\,\mathrm{d}u_1\mathrm{d}u_2 = 12 \int_0^1 \int_0^1 C(u_1, u_2)\,\mathrm{d}u_1\mathrm{d}u_2 - 3.$ □

- An estimator r_n^{S} is given by the sample correlation computed from compentwise scaled ranks (the so-called *pseudo-observations*) of the data.
- For $\kappa = \rho_\tau$ and $\kappa = \rho_{\mathsf{S}}$, Embrechts, McNeil, et al. (2002) show that $\kappa = \pm 1$ if and only if X_1, X_2 are co-/countermonotonic. In general, $\kappa = 0$ does not imply independence.

- Fallacy 1 (F_1, F_2, ρ uniquely determine F) is not solved by replacing ρ by rank correlation coefficients κ (it is easy to construct several copulas with the same Kendall's tau, e.g. via Archimedean copulas).
- Fallacy 2 (For F_1, F_2, any $\rho \in [-1, 1]$ is attainable) is solved when ρ is replaced by ρ_τ or ρ_S. Take

$$F(x_1, x_2) = \lambda M(F_1(x_1), F_2(x_2)) + (1-\lambda) W(F_1(x_1), F_2(x_2)).$$

  This is a model with $\rho_\tau = \rho_{\mathsf{S}} = 2\lambda - 1$ (choose $\lambda \in [0, 1]$ as desired).
- Fallacy 3 ($C = M$ implies $\mathrm{VaR}_\alpha(X_1 + X_2)$ maximal) is also not solved by rank correlation coefficients $\kappa = 1$: Although $\kappa = 1$ corresponds to $C = M$, this copula does not necessarily provide the largest $\mathrm{VaR}_\alpha(X_1 + X_2)$; see Fallacy 3 earlier.
- Nevertheless, rank correlations are useful to summarize dependence, to parameterize copula families to make dependence comparable and for copula parameter calibration or estimation.

7.2.4 Coefficients of tail dependence

Goal: Measure extremal dependence, i.e. dependence in the joint tails.

Definition 7.23 (Tail dependence)

Let $X_j \sim F_j$, $j \in \{1,2\}$, be continuously distributed random variables. Provided that the limits exist, the *lower tail-dependence coefficient λ_{l}* and *upper tail-dependence coefficient λ_{u}* of X_1 and X_2 are defined by

$$\lambda_{\mathsf{l}} = \lim_{u \downarrow 0} \mathbb{P}(X_2 \le F_2^{\leftarrow}(u) \,|\, X_1 \le F_1^{\leftarrow}(u)),$$

$$\lambda_{\mathsf{u}} = \lim_{u \uparrow 1} \mathbb{P}(X_2 > F_2^{\leftarrow}(u) \,|\, X_1 > F_1^{\leftarrow}(u)).$$

If $\lambda_{\mathsf{l}} \in (0,1]$ ($\lambda_{\mathsf{u}} \in (0,1]$), then (X_1, X_2) is *lower (upper) tail dependent*.
If $\lambda_{\mathsf{l}} = 0$ ($\lambda_{\mathsf{u}} = 0$), then (X_1, X_2) is *lower (upper) tail independent*.

As (conditional) probabilities, we clearly have $\lambda_{\mathsf{l}}, \lambda_{\mathsf{u}} \in [0,1]$.

- Tail dependence is a copula property, since
$$\mathbb{P}(X_2 \le F_2^{\leftarrow}(u) \,|\, X_1 \le F_1^{\leftarrow}(u)) = \frac{\mathbb{P}(X_1 \le F_1^{\leftarrow}(u), X_2 \le F_2^{\leftarrow}(u))}{\mathbb{P}(X_1 \le F_1^{\leftarrow}(u))}$$
$$= \frac{F(F_1^{\leftarrow}(u), F_2^{\leftarrow}(u))}{F_1(F_1^{\leftarrow}(u))} \underset{\text{(GI4)}}{\overset{\text{Sklar}}{=}} \frac{C(u,u)}{u},\ u \in (0,1),\ \text{so } \lambda_\mathsf{l} = \lim_{u \downarrow 0} \frac{C(u,u)}{u}.$$
- If $u \mapsto C(u,u)$ is differentiable in a neighborhood of 0 and the limit exists, then $\lambda_\mathsf{l} = \lim_{u\downarrow 0} \frac{d}{du} C(u,u)$ (l'Hôpital's Rule).
- If C is totally differentiable in a neighborhood of 0 and the limit exists, then $\lambda_\mathsf{l} = \lim_{u\downarrow 0}(\mathrm{D}_1\, C(u,u) + \mathrm{D}_2\, C(u,u))$ (Chain Rule). If C is exchangeable, $\lambda_\mathsf{l} = 2\lim_{u\downarrow 0} \mathrm{D}_1\, C(u,u) \underset{\text{Th. 7.13}}{=} 2\lim_{u\downarrow 0} C_{2|1}(u \,|\, u) = 2\lim_{u\downarrow 0} \mathbb{P}(U_2 \le u \,|\, U_1 = u)$ for $(U_1, U_2) \sim C$. Combined with any continuous df $F_.$ (the same for both components) and $(X_1, X_2) = (F_.^{\leftarrow}(U_1), F_.^{\leftarrow}(U_2))$, one has
$$\lambda_\mathsf{l} = 2 \lim_{x \downarrow -\infty} \mathbb{P}(X_2 \le x \,|\, X_1 = x). \tag{32}$$
which is useful for deriving λ_l for elliptical copulas.

- Similarly as above, for the upper tail-dependence coefficient,

$$\lambda_{\mathsf{u}} = \lim_{u\uparrow 1}\frac{1-2u+C(u,u)}{1-u} = \lim_{u\downarrow 0}\frac{\hat{C}(u,u)}{u}$$
$$= \lim_{u\uparrow 1}\frac{2(1-u)-(1-C(u,u))}{1-u} = 2-\lim_{u\uparrow 1}\frac{1-C(u,u)}{1-u}.$$

- For all radially symmetric copulas (e.g. the bivariate C_P^{Ga} and $C_{\nu,P}^{t}$ copulas), we have $\lambda_{\mathsf{l}} = \lambda_{\mathsf{u}} =: \lambda$.

- For Archimedean copulas with strict ψ, a substitution and l'Hôpital's Rule show:

$$\lambda_{\mathsf{l}} = \lim_{u\downarrow 0}\frac{\psi(2\psi^{-1}(u))}{u} = \lim_{t\to\infty}\frac{\psi(2t)}{\psi(t)} = 2\lim_{t\to\infty}\frac{\psi'(2t)}{\psi'(t)},$$
$$\lambda_{\mathsf{u}} = 2-\lim_{u\uparrow 1}\frac{1-\psi(2\psi^{-1}(u))}{1-u} = 2-\lim_{t\downarrow 0}\frac{1-\psi(2t)}{1-\psi(t)} = 2-2\lim_{t\downarrow 0}\frac{\psi'(2t)}{\psi'(t)}.$$

Clayton: $\lambda_{\mathsf{l}} = 2^{-1/\theta}$, $\lambda_{\mathsf{u}} = 0$; Gumbel: $\lambda_{\mathsf{l}} = 0$, $\lambda_{\mathsf{u}} = 2-2^{1/\theta}$

7.3 Normal mixture copulas

... are the copulas of multivariate normal (mean-)variance mixtures $\boldsymbol{X} \stackrel{\text{d}}{=} \boldsymbol{0} + \sqrt{W} A\boldsymbol{Z}$, $AA' = P$, $(\boldsymbol{X} \stackrel{\text{d}}{=} \boldsymbol{m}(W) + \sqrt{W} A\boldsymbol{Z})$; e.g. Gauss, t copulas.

7.3.1 Tail dependence

Coefficients of tail dependence

Let (X_1, X_2) be distributed according to a normal variance mixture and assume (w.l.o.g.) that $\boldsymbol{\mu} = (0,0)$ and $AA' = P = \left(\begin{smallmatrix} 1 & \rho \\ \rho & 1 \end{smallmatrix}\right)$. In this case, $F_1 = F_2$ and C is symmetric and radially symmetric. We thus obtain that

$$\lambda \underset{\text{symm.}}{\overset{\text{radial}}{=}} \lambda_{\text{l}} \underset{(32)}{\overset{\text{symm.}}{=}} 2 \lim_{x \downarrow -\infty} \mathbb{P}(X_2 \le x \mid X_1 = x).$$

Example 7.24 (λ for the Gauss and t copula)

- Considering the bivariate $\mathrm{N}(\boldsymbol{0}, P)$ density, one can show (via $f_{X_2|X_1}(x_2 \mid x_1) = \frac{f_{X_1,X_2}(x_1,x_2)}{f_{X_1}(x_1)}$) that $(X_2 \mid X_1 = x) \sim \mathrm{N}(\rho x, 1-\rho^2)$. This implies that

$$\lambda = 2\lim_{x\downarrow-\infty}\mathbb{P}(X_2 \le x \,|\, X_1 = x) = 2\lim_{x\downarrow-\infty}\Phi\Big(\frac{x(1-\rho)}{\sqrt{1-\rho^2}}\Big) = I_{\{\rho=1\}}.$$

- For $C^t_{\nu,P}$, one can show that $(X_2 \,|\, X_1 = x) \sim t_{\nu+1}(\rho x, \frac{(1-\rho^2)(\nu+x^2)}{\nu+1})$ and thus $\mathbb{P}(X_2 \le x \,|\, X_1 = x) = t_{\nu+1}\Big(\frac{x-\rho x}{\sqrt{\frac{(1-\rho^2)(\nu+x^2)}{\nu+1}}}\Big)$. Hence

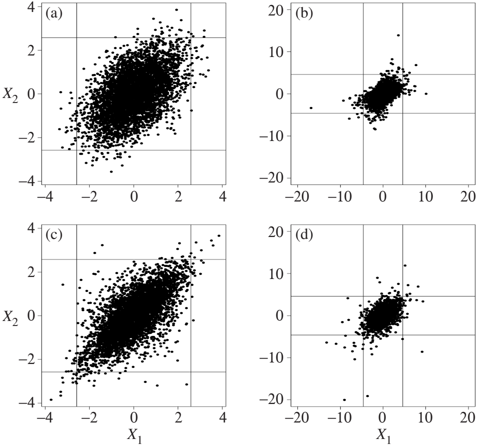
$$\lambda = 2t_{\nu+1}\Big(-\sqrt{\frac{(\nu+1)(1-\rho)}{1+\rho}}\Big)$$ (tail dependence; $\lambda\uparrow$ in $\rho\uparrow$ and $\nu\downarrow$).

- λ values for various ν, ρ:

ν	$\rho = -0.5$	$\rho = 0$	$\rho = 0.5$	$\rho = 0.9$	$\rho = 1$
∞	0	0	0	0	1
10	0.00	0.01	0.08	0.46	1
4	0.01	0.08	0.25	0.63	1
2	0.06	0.18	0.39	0.72	1

One can show that if W has a power tail, $\lambda > 0$, otherwise $\lambda = 0$.

Joint quantile exceedance probabilities



5000 samples from

(a) $\mathrm{N}_2(\mathbf{0}, P = \left(\begin{smallmatrix} 1 & \rho \\ \rho & 1 \end{smallmatrix}\right))$, $\rho = 0.5$;

(b) C_ρ^{Ga} with t_4 margins (same dependence as in (a));

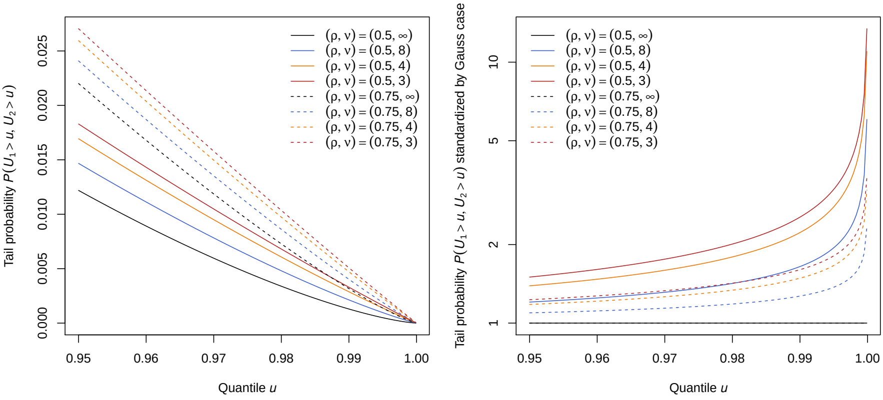
(c) $C_{4,\rho}^t$ with $\mathrm{N}(0,1)$ margins;

(d) $t_2(4, \mathbf{0}, P)$ (same dependence as in (c)).

Lines denote the true marginal 0.005- and 0.995-quantiles.

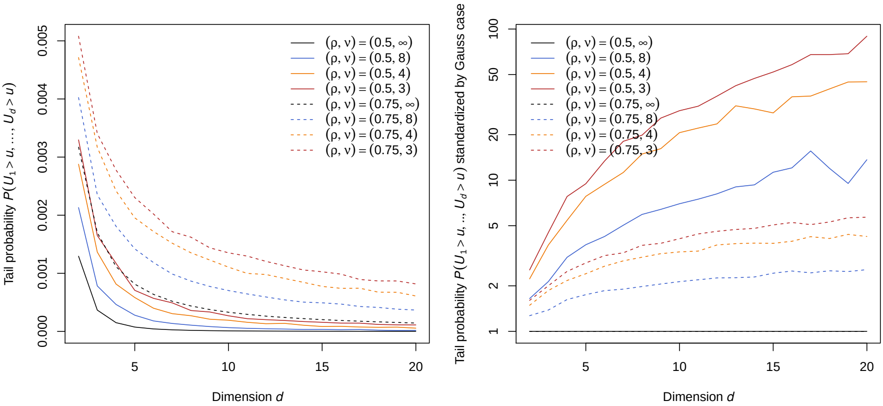
Note the different number of points in the bivariate tails (all models have the same Kendall's tau!)

Joint tail probabilities $\mathbb{P}(U_1 > u, U_2 > u)$ for $d = 2$



- **Left:** The higher ρ or the smaller ν, the larger $\mathbb{P}(U_1 > u, U_2 > u)$.
- **Right:** $u \mapsto \dfrac{\mathbb{P}(U_1 > u, U_2 > u)}{\mathbb{P}(V_1 > u, V_2 > u)} \underset{\text{symm.}}{\overset{\text{radial}}{=}} \dfrac{C^t_{\nu,\rho}(u,u)}{C^{\text{Ga}}_\rho(u,u)}$

Joint tail probabilities $\mathbb{P}(U_1 > u, \ldots, U_d > u)$ for $u = 0.99$



- Homogeneous P (off-diagonal entry ρ). Note the MC randomness.
- **Left:** Clear; less mass in corners in higher dimensions.
- **Right:** $d \mapsto \dfrac{\mathbb{P}(U_1 > u, \ldots, U_d > u)}{\mathbb{P}(V_1 > u, \ldots, V_d > u)} \overset{\text{radial}}{\underset{\text{symm.}}{=}} \dfrac{C^t_{\nu,\rho}(u, \ldots, u)}{C^{\text{Ga}}_{\rho}(u, \ldots, u)}$ for $u = 0.99$.

Example 7.25 (Interpretation of joint tail probabilities)

- Consider 5 daily negative log-returns $\boldsymbol{X} = (X_1, \dots, X_5)$. Assume they follow an elliptical distribution and have pairwise correlations $\rho = 0.5$. However, we are unsure about the best joint model.

- If $\boldsymbol{X}$ are multivariate normal (and thus $C^{\mathrm{Ga}}_{\rho=0.5}$), the probability that on any day all 5 negative returns lie above their $u = 0.99$ quantiles is

$$\mathbb{P}(X_1 > F_1^{\leftarrow}(u), \dots, X_5 > F_5^{\leftarrow}(u)) = \mathbb{P}(U_1 > u, \dots, U_5 > u)$$
$$\underset{\text{MC error}}{\approx} 7.48 \times 10^{-5}.$$

  In the long run such an event will happen once every $1/7.48 \times 10^{-5} \approx 13\,369$ trading days on average ($\approx$ once every 51.4 years; assuming 260 trading days in a year).

- If $\boldsymbol{X}$ is multivariate t_3 (and thus $C^t_{\nu=3,\rho=0.5}$), however, such an event will happen approximately 10 times more often, i.e. $\approx$ once every 5.14 years. This gets worse the larger d!

7.3.2 Rank correlations

Proposition 7.26 (Spearman's rho for normal variance mixtures)
Let $\boldsymbol{X} \sim M_2(\mathbf{0}, P, \hat{F}_W)$ with $\mathbb{P}(\boldsymbol{X} = \mathbf{0}) = 0$, $\rho = P_{12}$. Then

$$\rho_{\mathsf{S}} = \frac{6}{\pi}\mathbb{E}\Bigg(\arcsin \frac{W\rho}{\sqrt{(W+\tilde{W})(W+\bar{W})}}\Bigg),$$

for $W, \tilde{W}, \bar{W} \overset{\text{ind.}}{\sim} F_W$ with Laplace–Stieltjes transform $\hat{F}_W$. For Gauss copulas, $\rho_{\mathsf{S}} = \frac{6}{\pi}\arcsin(\frac{\rho}{2})$.

Proof. See the appendix. □

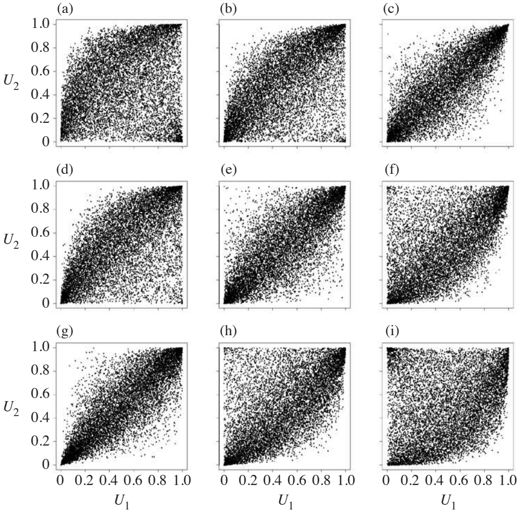
Proposition 7.27 (Kendall's tau for elliptical distributions)
Let $\boldsymbol{X} \sim \mathrm{E}_2(\mathbf{0}, P, \psi)$ with $\mathbb{P}(\boldsymbol{X} = \mathbf{0}) = 0$, $\rho = P_{12}$. Then $\rho_\tau = \frac{2}{\pi}\arcsin\rho$.

Proof. See the appendix. □

7.3.3 Skewed normal mixture copulas

- *Skewed normal mixture copulas* are the copulas of normal mixture distributions which are not elliptical, e.g. the *skewed t copula* $C^t_{\nu,P,\gamma}$ is the copula of a generalized hyperbolic distribution; see MFE (2015, Sections 6.2.3 and 7.3.3) for more details.
- It can be sampled as other implicit copulas; see Algorithm 7.9 (the evaluation of the margins requires numerical integration of a skewed t density).
- The main advantage of such a copula over $C^t_{\nu,P}$ is its radial asymmetry (e.g. for modelling $\lambda_{\mathsf{l}} \neq \lambda_{\mathsf{u}}$)

10 000 samples from $C^t_{\nu=5,\ \rho=0.8,\ \gamma=0.8(I_{\{i<2\}}-I_{\{i>2\}},I_{\{j>2\}}-I_{\{j<2\}})}$:



(a) $\gamma = (\ \ 0.8, -0.8)$

(b) $\gamma = (\ \ 0.8, \ \ 0)$

(c) $\gamma = (\ \ 0.8, \ \ 0.8)$

(d) $\gamma = (\ \ 0, -0.8)$

(e) $\gamma = (\ \ 0, \ \ 0)$

(f) $\gamma = (\ \ 0, \ \ 0.8)$

(g) $\gamma = (-0.8, -0.8)$

(h) $\gamma = (-0.8, \ \ 0)$

(i) $\gamma = (-0.8, \ \ 0.8)$

7.3.4 Grouped normal mixture copulas

- *Grouped normal mixture copulas* are copulas which attach together a set of normal mixture copulas.
- Let $\boldsymbol{Y} \sim \mathrm{N}_d(\boldsymbol{0}, P)$ (so $\boldsymbol{Y} \stackrel{\mathrm{d}}{=} A\boldsymbol{Z}$ as before). The *grouped t copula* is the copula of

  $$\boldsymbol{X} = (\sqrt{W_1}Y_1, \dots, \sqrt{W_1}Y_{s_1}, \dots, \sqrt{W_S}Y_{s_1+\dots+s_{S-1}+1}, \dots, \sqrt{W_S}Y_d)$$

  for $(W_1, \dots, W_S) \sim M(\mathrm{IG}(\frac{\nu_1}{2}, \frac{\nu_1}{2}), \dots, \mathrm{IG}(\frac{\nu_S}{2}, \frac{\nu_S}{2}))$; see Demarta and McNeil (2005) for details.
- Clearly, the marginals are t distributed, hence

  $$\boldsymbol{U} = (t_{\nu_1}(X_1), \dots, t_{\nu_1}(X_{s_1}), \dots, t_{\nu_S}(X_{s_1+\dots+s_{S-1}+1}), \dots, t_{\nu_S}(X_d))$$

  follows a grouped t copula. This is straightforward to simulate.
- It can be fitted with pairwise inversion of Kendall's tau.
- If $S = d$, grouped t copulas are also known as *generalized t copulas*; see Luo and Shevchenko (2010).

7.4 Archimedean copulas

Recall that an (Archimedean) generator ψ is a function $\psi : [0,\infty) \to [0,1]$ which is $\downarrow$ on $[0, \inf\{t : \psi(t) = 0\}]$ and satisfies $\psi(0) = 1$, $\psi(\infty) = \lim_{t\to\infty} \psi(t) = 0$; the set of all generators is denoted by Ψ.

7.4.1 Bivariate Archimedean copulas

Theorem 7.28 (Bivariate Archimedean copulas)
For $\psi \in \Psi$, $C(u_1, u_2) = \psi(\psi^{-1}(u_1) + \psi^{-1}(u_2))$ is a copula if and only if ψ is convex.

- For a strict and twice-continuously differentiable ψ, one can show that

$$\rho_\tau = 1 - 4\int_0^\infty t(\psi'(t))^2 \,\mathrm{d}t = 1 + 4\int_0^1 \frac{\psi^{-1}(t)}{(\psi^{-1}(t))'} \,\mathrm{d}t.$$

- If ψ is strict, $\lambda_\mathrm{l} = 2\lim_{t\to\infty} \frac{\psi'(2t)}{\psi'(t)}$ and $\lambda_\mathrm{u} = 2 - 2\lim_{t\downarrow 0} \frac{\psi'(2t)}{\psi'(t)}$ (as seen before).

- The most widely used one-parameter Archimedean copulas are:

Family	ρ_τ	λ_l	λ_u
A	$1-2(\theta+(1-\theta)^2\log(1-\theta))/(3\theta^2)$	0	0
C	$\theta/(\theta+2)$	$2^{-1/\theta}$	0
F	$1+4(D_1(\theta)-1)/\theta$	0	0
G	$(\theta-1)/\theta$	0	$2-2^{1/\theta}$
J	$1-4\sum_{k=1}^{\infty}1/(k(\theta k+2)(\theta(k-1)+2))$	0	$2-2^{1/\theta}$

Family	θ	$\psi(t)$	$V\sim F=\mathcal{LS}^{-1}(\psi)$
A	$[0,1)$	$(1-\theta)/(\exp(t)-\theta)$	$\mathrm{Geo}(1-\theta)$
C	$(0,\infty)$	$(1+t)^{-1/\theta}$	$\Gamma(1/\theta,1)$
F	$(0,\infty)$	$-\log\big(1-(1-e^{-\theta})\exp(-t)\big)/\theta$	$\mathrm{Log}(1-e^{-\theta})$
G	$[1,\infty)$	$\exp(-t^{1/\theta})$	$\mathrm{S}(1/\theta,1,\cos^\theta(\pi/(2\theta)),I_{\{\theta=1\}};1)$
J	$[1,\infty)$	$1-(1-\exp(-t))^{1/\theta}$	$\mathrm{Sibuya}(1/\theta)$

7.4.2 Multivariate Archimedean copulas

ψ is *completely monotone (c.m.)* if $(-1)^k\psi^{(k)}(t) \geq 0$ for all $t \in (0,\infty)$ and all $k \in \mathbb{N}_0$. The set of all c.m. generators is denoted by Ψ_∞.

Theorem 7.29 (Kimberling (1974))

If $\psi \in \Psi$, $C(\boldsymbol{u}) = \psi\Big(\sum_{j=1}^{d} \psi^{-1}(u_j)\Big)$ is a copula $\forall\, d$ if and only if $\psi \in \Psi_\infty$.

Bernstein's Theorem characterizes all $\psi \in \Psi_\infty$.

Theorem 7.30 (Bernstein (1928))

$\psi(0) = 1$, ψ c.m. if and only if $\psi(t) = \mathbb{E}(\exp(-tV))$ for $V \sim G$ with $V \geq 0$ and $G(0) = 0$.

We thus use the notation $\psi = \hat{G}$ and call all Archimedean copulas with $\psi \in \Psi_\infty$ *LT-Archimedean copulas*.

Proposition 7.31 (Stochastic representation, related properties)
Let $\psi \in \Psi_\infty$ with $V \sim G$ such that $\hat{G} = \psi$ and let $E_1, \dots, E_d \overset{\text{ind.}}{\sim} \mathrm{Exp}(1)$ be independent of V. Then

1) The survival copula of $\boldsymbol{X} = (\frac{E_1}{V}, \dots, \frac{E_d}{V})$ is Archimedean (with ψ).
2) $\boldsymbol{U} = (\psi(X_1), \dots, \psi(X_d)) \sim C$ and the U_j's are conditionally independent given V with $\mathbb{P}(U_j \le u \,|\, V = v) = \exp(-v\psi^{-1}(u))$.

Proof.

1) The joint survival function of $\boldsymbol{X}$ is given by

$$\begin{aligned}
\bar{F}(\boldsymbol{x}) &= \mathbb{P}(X_j > x_j \;\forall\, j) = \int_0^\infty \mathbb{P}(E_j/V > x_j \;\forall\, j \,|\, V = v)\, \mathrm{d}G(v) \\
&= \int_0^\infty \mathbb{P}(E_j > v x_j \;\forall\, j)\, \mathrm{d}G(v) = \int_0^\infty \prod_{j=1}^d \exp(-v x_j)\, \mathrm{d}G(v) \\
&= \int_0^\infty \exp\Big(-v \sum_{j=1}^d x_j\Big)\, \mathrm{d}G(v) = \psi\Big(\sum_{j=1}^d x_j\Big).
\end{aligned}$$

The jth marginal survival function is thus (set $x_k = 0\ \forall\, k \neq j$) $\bar{F}_j(x_j) = \mathbb{P}(X_j > x_j) = \psi(x_j)$ ($\downarrow$ and continuous) and therefore $\hat{C}(\boldsymbol{u}) = \bar{F}(\bar{F}_1^{\leftarrow}(u_1), \dots, \bar{F}_d^{\leftarrow}(u_d)) = \psi(\sum_{j=1}^d \psi^{-1}(u_j))$.

2) $\mathbb{P}(\boldsymbol{U} \leq \boldsymbol{u}) = \mathbb{P}(X_j > \psi^{-1}(u_j)\ \forall\, j) \underset{1)}{=} \psi(\sum_{j=1}^d \psi^{-1}(u_j))$. Conditional independence is clear by construction and $\mathbb{P}(U_j \leq u \,|\, V = v) = \mathbb{P}(X_j > \psi^{-1}(u) \,|\, V = v) = \mathbb{P}(E_j > v\psi^{-1}(u)) = \exp(-v\psi^{-1}(u))$. $\square$

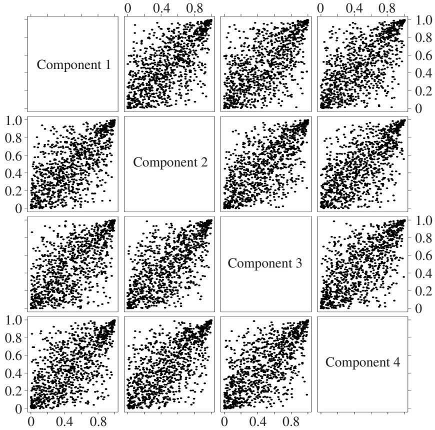
Algorithm 7.32 (Marshall and Olkin (1988))

1) Sample $V \sim G$ (df corresponding to ψ).

2) Sample $E_1, \dots, E_d \overset{\text{ind.}}{\sim} \mathrm{Exp}(1)$ independently of V.

3) Return $\boldsymbol{U} = (\psi(E_1/V), \dots, \psi(E_d/V))$ (conditional independence).

1000 samples of a 4-dim. Gumbel copula ($\rho_\tau = 0.5$; $\lambda_\mathrm{u} \approx 0.5858$)



- Various non-exchangeable extensions to Archimedean copulas exist.
- For fixed d, c.m. can be relaxed to d-monotonicity; see McNeil and Nešlehová (2009).

7.4.3 Copulas and credit risk

Felix Salmon: "Recipe for Disaster: The Formula That Killed Wall Street"

$$\Pr[T_A<1, T_B<1] = \phi_2(\phi^{-1}(F_A(1)), \phi^{-1}(F_B(1)), \gamma)$$

Here's what killed your 401(k) *David X. Li's Gaussian copula function as first published in 2000. Investors exploited it as a quick—and fatally flawed—way to assess risk. A shorter version appears on this month's cover of* Wired.

Probability
Specifically, this is a joint default probability—the likelihood that any two members of the pool (A and B) will both default. It's what investors are looking for, and the rest of the formula provides the answer.

Survival times
The amount of time between now and when A and B can be expected to default. Li took the idea from a concept in actuarial science that charts what happens to someone's life expectancy when their spouse dies.

Equality
A dangerously precise concept, since it leaves no room for error. Clean equations help both quants and their managers forget that the real world contains a surprising amount of uncertainty, fuzziness, and precariousness.

Copula
This couples (hence the Latinate term copula) the individual probabilities associated with A and B to come up with a single number. Errors here massively increase the risk of the whole equation blowing up.

Distribution functions
The probabilities of how long A and B are likely to survive. Since these are not certainties, they can be dangerous: Small miscalculations may leave you facing much more risk than the formula indicates.

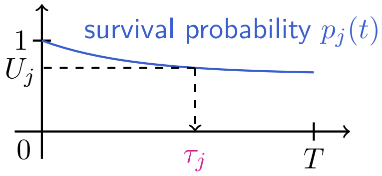
Gamma
The all-powerful correlation parameter, which reduces correlation to a single constant—something that should be highly improbable, if not impossible. This is the magic number that made Li's copula function irresistible.

How intensity-/copula-based default models work

Intensity-based default model:

$$p_j(t) = \exp\Big(-\int_0^t \lambda_j(s)\,ds\Big)$$

$$\tau_j = \inf\{t \ge 0 : p_j(t) \le U_j\}$$



Note: $\lambda_{\mathrm{u}} = 0$ (as for the Gauss copula!)

$\Rightarrow$ (Almost) no joint defaults! (p_j typically very flat)

Copulas for the triggers $\boldsymbol{U}$:

1) Li (2000): Gauss (Sibuya (1960): $\lambda_{\mathrm{u}} = 0$)

2) Schönbucher and Schubert (2001): Archimedean ($\lambda_{\mathrm{u}} > 0$)

3) Hofert and Scherer (2011): nested Archimedean ($\lambda_{\mathrm{u}} > 0$, hierarchies)

Typical application: CDO pricing models based on iTraxx data.

7.5 A proof for subadditivity of ES

Proposition 7.33 (Subadditivity of ES)
$\mathrm{ES}_\alpha(L) = \dfrac{\sup_{\{\tilde{Y}\sim \mathrm{B}(1,1-\alpha)\}} \mathbb{E}(L\tilde{Y})}{1-\alpha}$, which is subadditive; the supremum is taken over all copulas between $L \sim F_L$ and $\tilde{Y} \sim \mathrm{B}(1, 1-\alpha)$.

Proof.

- Let $L = F_L^{\leftarrow}(U)$ and $Y = I_{\{U>\alpha\}} \sim \mathrm{B}(1, 1-\alpha)$ for $U \sim \mathrm{U}(0, 1)$.
- Then $\mathrm{ES}_\alpha(L) = \frac{1}{1-\alpha}\int_\alpha^1 F_L^{\leftarrow}(u)\,\mathrm{d}u = \frac{1}{1-\alpha}\int_0^1 F_L^{\leftarrow}(u) I_{\{u>\alpha\}} \cdot 1\,\mathrm{d}u = \frac{1}{1-\alpha}\mathbb{E}(F_L^{\leftarrow}(U)I_{\{U>\alpha\}}) = \frac{1}{1-\alpha}\mathbb{E}(LY)$.
- L and Y are comontone. For any other $(L, \tilde{Y})$ with $\tilde{Y} \sim \mathrm{B}(1, 1-\alpha)$,

  $$\mathbb{E}(L\tilde{Y}) = \mathrm{cov}(L, \tilde{Y}) + \mathbb{E}(L)\mathbb{E}(\tilde{Y}) \underset{\text{Hoeffding}}{\leq} \mathrm{cov}(L, Y) + \mathbb{E}(L)\mathbb{E}(Y) = \mathbb{E}(LY)$$

  and thus $\mathrm{ES}_\alpha(L) = \frac{1}{1-\alpha} \sup_{\{\tilde{Y}\sim \mathrm{B}(1,1-\alpha)\}} \mathbb{E}(L\tilde{Y})$. □

7.6 Fitting copulas to data

- Let $\boldsymbol{X}, \boldsymbol{X}_1, \dots, \boldsymbol{X}_n \overset{\text{ind.}}{\sim} F$ with cont. margins $F_1, \dots, F_d$ and copula C.
- We assume that we have data $\boldsymbol{x}_1, \dots, \boldsymbol{x}_n$, interpreted as realizations of $\boldsymbol{X}_1, \dots, \boldsymbol{X}_n$; in what follows we work with the latter.
- Assume
  - $F_j = F_j(\cdot; \boldsymbol{\theta}_{0,j})$ for some $\boldsymbol{\theta}_{0,j} \in \Theta_j$, $j \in \{1, \dots, d\}$; ($F_j(\cdot; \boldsymbol{\theta}_j)$ is assumed to be continuous $\forall\, \boldsymbol{\theta}_j \in \Theta_j$, $j \in \{1, \dots, d\}$)
  - $C = C(\cdot; \boldsymbol{\theta}_{0,C})$ for some $\boldsymbol{\theta}_{0,C} \in \Theta_C$.

  Thus F has the true but unknown parameter vector $\boldsymbol{\theta}_0 = (\boldsymbol{\theta}'_{0,C}, \boldsymbol{\theta}'_{0,1}, \dots, \boldsymbol{\theta}'_{0,d})'$ to be estimated.
- Here, we focus particularly on $\boldsymbol{\theta}_{0,C}$. Whenever necessary, we assume that the margins $F_1, \dots, F_d$ and the copula C are absolutely continuous with corresponding densities $f_1, \dots, f_d$ and c, respectively.
- We assume the chosen copula to be appropriate (w.r.t. symmetry etc.).

7.6.1 Method-of-moments using rank correlation

- For $d = 2$ and one-parameter copulas, Genest and Rivest (1993) suggested estimating $\theta_{0,C}$ by solving $\rho_\tau(\theta_C) = r_n^\tau$ w.r.t. θ_C, i.e.

  $$\hat{\theta}_{n,C}^{\mathsf{IKTE}} = \rho_\tau^{-1}(r_n^\tau), \quad \text{(inversion of Kendall's tau estimator (IKTE))}$$

  where $\rho_\tau(\cdot)$ denotes Kendall's tau as a function of θ and r_n^τ is the sample version of Kendall's tau (computed via (31)from $\boldsymbol{X}_1, \dots, \boldsymbol{X}_n$ or pseudo-observations $\boldsymbol{U}_1, \dots, \boldsymbol{U}_n$; see later).
- The standardized dispersion matrix P for elliptical copulas can be estimated via *pairwise inversion of Kendall's tau*. If $r_{n,j_1j_2}^\tau$ denotes the sample version of Kendall's tau for data pair (j_1, j_2), then

  $$\hat{P}_{n,j_1j_2}^{\mathsf{IKTE}} = \sin\left(\frac{\pi}{2} r_{n,j_1j_2}^\tau\right).$$

  A proper correlation matrix P can be constructed as in Higham (2002).

- One can also use Spearman's rho. For Gauss copulas,
$$\rho \approx \frac{6}{\pi} \arcsin \frac{\rho}{2} \underset{\text{Prop.7.26}}{=} \rho_{\mathsf{S}}.$$
The approximation error is comparably small, so that the matrix of pairwise sample versions of Spearman's rho is an estimator for P.
- For t copulas, $\hat{P}_n^{\mathsf{IKTE}}$ can be used to estimate P and then ν can be estimated via its MLE based on $\hat{P}_n^{\mathsf{IKTE}}$; see Mashal and Zeevi (2002).

7.6.2 Forming a pseudo-sample from the copula

- $\boldsymbol{X}_1, \dots, \boldsymbol{X}_n$ typically does not have $\mathrm{U}(0,1)$ margins. For applying the "copula approach" we thus need *pseudo-observations* from C.
- In general, we take $\hat{\boldsymbol{U}}_i = (\hat{U}_{i1}, \dots, \hat{U}_{id}) = (\hat{F}_1(X_{i1}), \dots, \hat{F}_d(X_{id}))$, $i \in \{1, \dots, n\}$, where $\hat{F}_j$ denotes an estimator of F_j; see Lemma 7.6. Note that $\hat{\boldsymbol{U}}_1, \dots, \hat{\boldsymbol{U}}_n$ are typically neither independent (even if $\boldsymbol{X}_1, \dots, \boldsymbol{X}_n$ are) nor perfectly $\mathrm{U}(0,1)^d$ distributed.

- Possible choices for $\hat{F}_j$:
  - Parametric estimators (typically if n is small). One often still uses (33) below for estimating $\boldsymbol{\theta}_{0,C}$ (to keep the error due to misspecification of the margins small).
  - Semi-parametric estimators (for example EVT-based: Bodies are modelled empirically, tails semiparametrically via the GPD-based tail estimator of Smith (1987)).
  - Non-parametric estimators with scaled empirical dfs, so

$$\hat{U}_{ij} = \frac{n}{n+1}\hat{F}_{n,j}(X_{ij}) = \frac{R_{ij}}{n+1}, \tag{33}$$

    where R_{ij} denotes the *rank* of X_{ij} among all $X_{1j}, \dots, X_{nj}$. The scaling is to avoid density evaluation on the boundary of $[0,1]^d$.

  If n is sufficiently large, one typically uses (33).

7.6.3 Maximum likelihood estimation

The (classical) maximum likelihood estimator

- If it exists, the density of $F(\boldsymbol{x}) = C(F_1(x_1), \dots, F_d(x_d))$ is

$$f(\boldsymbol{x};\boldsymbol{\theta}_0) = c(F_1(x_1;\boldsymbol{\theta}_{0,1}), \dots, F_d(x_d;\boldsymbol{\theta}_{0,d});\boldsymbol{\theta}_{0,C})\prod_{j=1}^d f_j(x_j;\boldsymbol{\theta}_{0,j}).$$

- The log-likelihood based on $\boldsymbol{X}_1, \dots, \boldsymbol{X}_n$ is thus

$$\begin{aligned}\ell(\boldsymbol{\theta};\boldsymbol{X}_1,\dots,\boldsymbol{X}_n) &= \sum_{i=1}^n \ell(\boldsymbol{\theta};\boldsymbol{X}_i)\\ &= \sum_{i=1}^n \ell_C(\boldsymbol{\theta}_C;F_1(X_{i1};\boldsymbol{\theta}_1),\dots,F_d(X_{id};\boldsymbol{\theta}_d)) + \sum_{i=1}^n\sum_{j=1}^d \ell_j(\boldsymbol{\theta}_j;X_{ij}),\end{aligned}$$

  where

$$\begin{aligned}\ell_C(\boldsymbol{\theta}_C;u_1,\dots,u_d) &= \log c(u_1,\dots,u_d;\boldsymbol{\theta}_C)\\ \ell_j(\boldsymbol{\theta}_j;x) &= \log f_j(x;\boldsymbol{\theta}_j), \quad j\in\{1,\dots,d\}.\end{aligned}$$

- The *maximum likelihood estimator (MLE)* of $\boldsymbol{\theta}_0$ is

$$\hat{\boldsymbol{\theta}}_n^{\mathsf{MLE}} = \operatorname*{argsup}_{\boldsymbol{\theta}\in\Theta} \ell(\boldsymbol{\theta}; \boldsymbol{X}_1, \dots, \boldsymbol{X}_n).$$

This optimization is typically done by numerical means. Note that this can be quite demanding, especially in high dimensions.

The inference functions for margins estimator

- Joe and Xu (1996) suggested the two-step estimation approach:

Step 1: For $j \in \{1, \dots, d\}$, estimate $\boldsymbol{\theta}_{0,j}$ by its MLE $\hat{\boldsymbol{\theta}}_{n,j}^{\mathsf{MLE}}$.

Step 2: Estimate $\boldsymbol{\theta}_{0,C}$ by

$$\hat{\boldsymbol{\theta}}_{n,C}^{\mathsf{IFME}} = \operatorname*{argsup}_{\boldsymbol{\theta}_C\in\Theta_C} \ell(\boldsymbol{\theta}_C, \hat{\boldsymbol{\theta}}_{n,1}^{\mathsf{MLE}}, \dots, \hat{\boldsymbol{\theta}}_{n,d}^{\mathsf{MLE}}; \boldsymbol{X}_1, \dots, \boldsymbol{X}_n).$$

The *inference functions for margins estimator (IFME)* of $\boldsymbol{\theta}_0$ is thus

$$\hat{\boldsymbol{\theta}}_n^{\mathsf{IFME}} = (\hat{\boldsymbol{\theta}}_{n,C}^{\mathsf{IFME}}, \hat{\boldsymbol{\theta}}_{n,1}^{\mathsf{MLE}}, \dots, \hat{\boldsymbol{\theta}}_{n,d}^{\mathsf{MLE}})$$

- This is typically much easier to compute than $\hat{\boldsymbol{\theta}}_n^{\mathsf{MLE}}$ while providing good results; see Joe and Xu (1996) or Kim et al. (2007).
- $\hat{\boldsymbol{\theta}}_n^{\mathsf{IFME}}$ can also be used as initial value for computing $\hat{\boldsymbol{\theta}}_n^{\mathsf{MLE}}$.
- In terms of likelihood equations, $\hat{\boldsymbol{\theta}}_n^{\mathsf{IFME}}$ compares to $\hat{\boldsymbol{\theta}}_n^{\mathsf{MLE}}$ as follows:

$$\hat{\boldsymbol{\theta}}_n^{\mathsf{MLE}} \text{ solves } \left(\frac{\partial}{\partial \boldsymbol{\theta}_C}\ell, \frac{\partial}{\partial \boldsymbol{\theta}_1}\ell, \dots, \frac{\partial}{\partial \boldsymbol{\theta}_d}\ell\right) = \mathbf{0},$$
$$\hat{\boldsymbol{\theta}}_n^{\mathsf{IFME}} \text{ solves } \left(\frac{\partial}{\partial \boldsymbol{\theta}_C}\ell, \frac{\partial}{\partial \boldsymbol{\theta}_1}\ell_1, \dots, \frac{\partial}{\partial \boldsymbol{\theta}_d}\ell_d\right) = \mathbf{0},$$

where

$$\ell = \ell(\boldsymbol{\theta}; \boldsymbol{X}_1, \dots, \boldsymbol{X}_n),$$
$$\ell_j = \ell_j(\boldsymbol{\theta}_j; X_{1j}, \dots, X_{nj}) = \sum_{i=1}^n \ell_j(\boldsymbol{\theta}_j; X_{ij}) = \sum_{i=1}^n \log f_j(X_{ij}; \boldsymbol{\theta}_j).$$

Example 7.34 (A computationally convincing example)

Suppose $X_j \sim \mathrm{N}(\mu_j, \sigma_j^2)$, $j \in \{1, \dots, d\}$, for $d = 100$, and C has (just) one parameter.

1) MLE requires to solve a 201-dimensional optimization problem.
2) IFME only requires 100 optimizations in two dimensions and 1 one-dimensional optimization.

- If the marginals are estimated parametrically one often still uses the pseudo-observations built from the marginal empirical dfs to estimate $\boldsymbol{\theta}_{0,C}$ (see MPLE below) in order to avoid misspecifiation of the margins.
- In this case (and under more complicated marginal models), one can execute the 101 optimizations in parallel, independently of each other.

The maximum pseudo-likelihood estimator

- The *maximum pseudo-likelihood estimator (MPLE)*, introduced by Genest, Ghoudi, et al. (1995), works similarly to $\hat{\boldsymbol{\theta}}_n^{\mathsf{IFME}}$, but estimates the margins non-parametrically:

  Step 1: Compute rank-based pseudo-observations $\hat{\boldsymbol{U}}_1, \dots, \hat{\boldsymbol{U}}_n$.

  Step 2: Estimate $\boldsymbol{\theta}_{0,C}$ by

$$\hat{\boldsymbol{\theta}}_{n,C}^{\mathsf{MPLE}} = \underset{\boldsymbol{\theta}_C \in \Theta_C}{\operatorname{argsup}} \sum_{i=1}^{n} \ell_C(\boldsymbol{\theta}_C; \hat{U}_{i1}, \dots, \hat{U}_{id}) = \underset{\boldsymbol{\theta}_C \in \Theta_C}{\operatorname{argsup}} \sum_{i=1}^{n} \log c(\hat{\boldsymbol{U}}_i; \boldsymbol{\theta}_C).$$

- Genest and Werker (2002) show that $\hat{\boldsymbol{\theta}}_{n,C}^{\mathsf{MPLE}}$ is not asymptotically efficient in general.
- Kim et al. (2007) compare $\hat{\boldsymbol{\theta}}_n^{\mathsf{MLE}}$, $\hat{\boldsymbol{\theta}}_n^{\mathsf{IFME}}$, and $\hat{\boldsymbol{\theta}}_{n,C}^{\mathsf{MPLE}}$ in a simulation study ($d = 2$ only!) and argue in favor of $\hat{\boldsymbol{\theta}}_{n,C}^{\mathsf{MPLE}}$ overall, especially w.r.t. robustness against misspecification of the margins; but see Embrechts and Hofert (2013b) for $d \gg 2$.

Example 7.35 (Fitting the Gauss copula)

- Use pairwise inversion of Spearman's rho or Kendall's tau.
- Or the MPLE via the (copula-related) log-likelihood

$$\ell_C(P; \hat{\boldsymbol{U}}_1, \dots, \hat{\boldsymbol{U}}_n) = \sum_{i=1}^{n} \ell_C(P; \hat{\boldsymbol{U}}_i) \underset{\text{Eq. (28)}}{=} \sum_{i=1}^{n} \log c_P^{\mathsf{Ga}}(\hat{\boldsymbol{U}}_i).$$

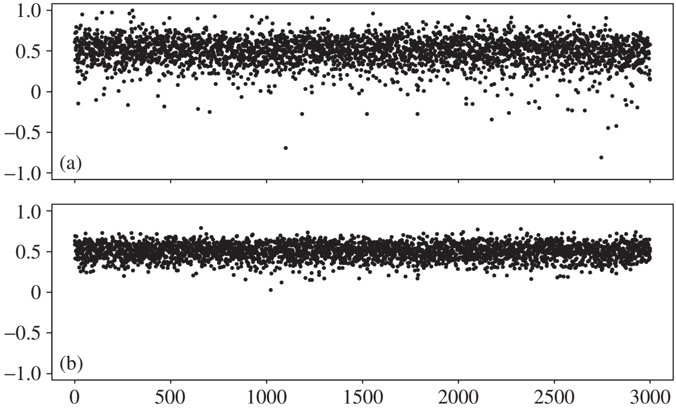
  For maximization over all correlation matrices P, we can use the Cholesky factor A as reparameterization and maximize over all lower triangular matrices A with 1s on the diagonal; still this is $\mathcal{O}(d^2)$.

Example 7.36 (Fitting the t copula)

- For small d, maximize the likelihood over all correlation matrices (as for the Gauss copula case) and the d.o.f. ν.
- For moderate/larger d, use Mashal and Zeevi (2002):
  1) Estimate P via pairwise inversion of Kendall's tau (see above).
  2) Plug $\hat{P}$ into the likelihood and maximize it w.r.t. ν to obtain $\hat{\nu}_n$.

Example 7.37 (Correlation estimation for heavy-tailed data)

Consider $n = 3000$ realizations of independent samples of size 90 from $t_2\left(3, \mathbf{0}, \left(\begin{smallmatrix} 1 & 0.5 \\ 0.5 & 1 \end{smallmatrix}\right)\right)$ ($\Rightarrow$ linear correlation $\rho = 0.5$). Shall we estimate ρ via the sample correlation (estimates are shown in (a)) or via inversion of Kendall's tau (shown in (b))? The variance of the latter is smaller!



Estimation is only one side of the coin. The other is *goodness-of-fit* (i.e. to find out whether our estimated model indeed represents the given data well) and model selection (i.e. to decide which model is best among all adequate fitted models). Goodness-of-fit can be (computationally) challenging, particularly for large d. There are also graphical approaches not further discussed here.