

3 Empirical properties of financial data

3.1 Stylized facts of financial return series

3.2 Multivariate stylized facts

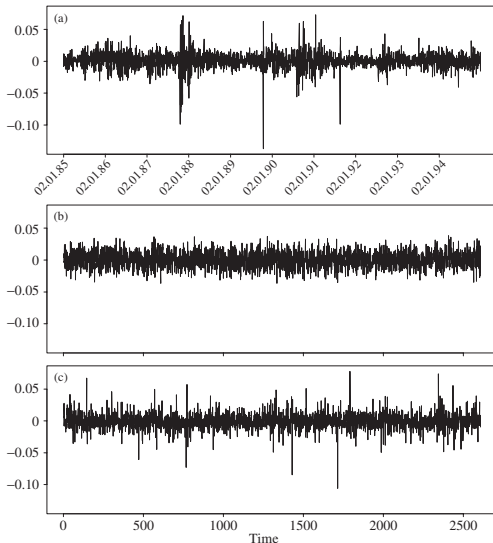
3.1 Stylized facts of financial return series

- Stylized facts are a collection of empirical observations and related inferences, which apply to many time series of risk-factor changes (e.g. log-returns on equities, indices, exchange rates, commodity prices).
- The best-known stylized facts apply to daily log-returns (also to intra-daily, weekly, monthly). Tick-by-tick (high-frequency) data have their own stylized facts (not discussed here) and annual return (low-frequency) data are more difficult to investigate (data sparseness; non-stationarity).
- Consider discrete-time risk-factor changes $X_t = Z_t - Z_{t-1}$ for a log-price or rate $Z_t = \log S_t$. In this case

$$X_t = \log(S_t/S_{t-1}) \approx S_t/S_{t-1} - 1 = (S_t - S_{t-1})/S_{t-1};$$

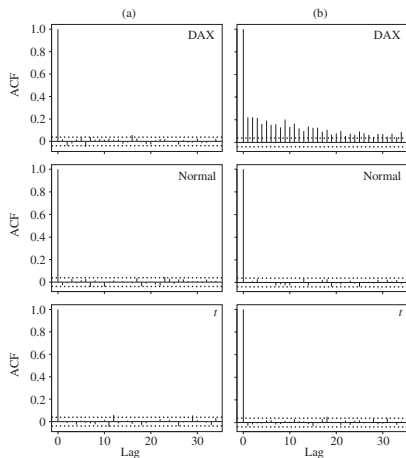
the former is often called a (log-)return, the latter a simple return.

3.1.1 Volatility Clustering



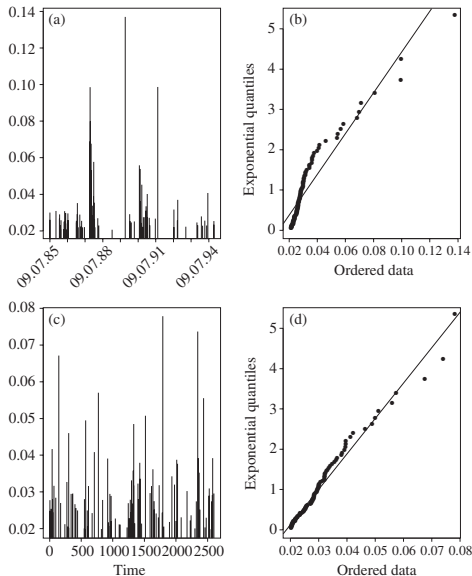
- (a) Log-returns for the DAX index from 1985-01-02 to 1994-12-30 ($n = 2608$).
- (b) Simulated iid data from a fitted normal with $\hat{\mu} = \bar{X}_n$, $\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^n (X_i - \bar{X}_n)^2$ show too few extremes.
- (c) Simulated iid data from a fitted $t_{3.8}$. Better range of values but still no volatility clustering (= tendency for extreme returns to be followed by extreme returns).

Estimated *autocorrelation function (ACF)* $\rho(h) = \text{corr}(X_0, X_h)$, $h \in \mathbb{Z}$



- (a) Sample ACF of $(X_t)_{t \in \mathbb{Z}}$
 (b) and $(|X_t|)_{t \in \mathbb{Z}}$ (similarly: $(X_t^2)_{t \in \mathbb{Z}}$)
- (a): Positive ACF at lag 1 would imply a tendency for returns to be followed by returns of equal sign \Rightarrow not the case \Rightarrow predicted returns ≈ 0
- (b): Positive ACF at lag 1 would imply a tendency for large (small) returns to be followed by large (small) returns \Rightarrow the case for the DAX data \Rightarrow returns cluster (not iid).
- $(X_t)_{t \in \mathbb{Z}}$ not iid $\Rightarrow (Z_t)_{t \in \mathbb{Z}}$ not a random walk $\Rightarrow (S_t)_{t \in \mathbb{Z}}$ not a GBM.

Concerning clustering of extremes, consider the 100 largest losses of the...



- (a) ... DAX index
(c) ... simulated fitted $t_{3.8}$
- (b), (d) Q-Q plots of waiting times between these large losses (should be $\text{Exp}(\lambda)$ for iid data);
- The DAX data shows longer and shorter waiting times than the iid data, so clustering of extremes.

3.1.2 Non-normality and heavy tails

Formal statistical tests of normality

- For **general univariate df F** :
 - ▶ Kolmogorov–Smirnov (test statistic $T_n = \sup_x |\hat{F}_n(x) - F(x)|$)
 - ▶ Cramér–von Mises ($T_n = n \int_{-\infty}^{\infty} (\hat{F}_n(x) - F(x))^2 dF(x)$)
 - ▶ **Anderson–Darling** ($T_n = n \int_{-\infty}^{\infty} \frac{(\hat{F}_n(x) - F(x))^2}{F(x)(1-F(x))} dF(x)$; recommended by D’Agostino and Stephens (1986))
- For **$F = N(\mu, \sigma^2)$** :
 - ▶ Shapiro–Wilk (idea: quantify Q-Q plot in one number, biased by n)
 - ▶ D’Agostino (based on skewness and kurtosis, as Jarque–Bera)
 - ▶ **Jarque–Bera test**: Compares skewness $\beta = \frac{\mathbb{E}((X-\mu)^3)}{\sigma^3}$ and kurtosis $\kappa = \frac{\mathbb{E}((X-\mu)^4)}{\sigma^4}$ with sample versions. The test statistic is $T_n = \frac{n}{6} \left(\hat{\beta}^2 + \frac{1}{4}(\hat{\kappa} - 3)^2 \right) \underset{n \text{ large}}{\overset{H_0}{\rightsquigarrow}} \chi_2^2$.

Graphical tests

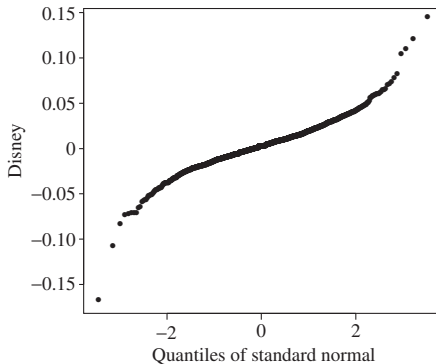
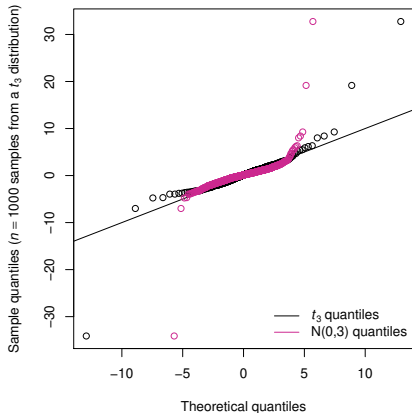
- We can also **graphically test whether $X_1, \dots, X_n \sim F$ for some df F** based on realizations of iid X_1, \dots, X_n .
- Let $X_{(1)} \leq \dots \leq X_{(n)}$ denote the corresponding **order statistics** and note that

$$\hat{F}_n(x) = \frac{1}{n} \sum_{i=1}^n I_{\{X_i \leq x\}} = \frac{1}{n} \sum_{i=1}^n I_{\{X_{(i)} \leq x\}}, \quad x \in \mathbb{R},$$

i.e. the order statistics contain all relevant information about X_1, \dots, X_n .

- Possible graphical tests (see also the appendix):
 - ▶ **P-P plot:** For $p_i = \frac{i-1/2}{n} \approx \frac{i}{n}$, plot $\{(p_i, F(X_{(i)})) : i = 1, \dots, n\}$.
If $F \approx \hat{F}_n$, $F(X_{(i)}) \approx p_i$, so the points lie on a **line with slope 1**.
 - ▶ **Q-Q plot:** Plot $\{(F^{\leftarrow}(p_i), X_{(i)}) : i = 1, \dots, n\}$ (tail differences better visible).

Interpreting Q-Q plots (S-shape hints at a leptokurtic distribution, i.e., narrower center, heavier tails than $N(\mu, \sigma^2)$ (kurtosis $\kappa = 3$)):



Daily returns typically have kurtosis $\kappa > 3$.

3.1.3 Longer-interval return series

- By going from daily to weekly, monthly, quarterly and yearly data, these effects become less pronounced (returns look more iid, less heavy-tailed).
- Since the h -period log-return at $t \in \{h, 2h, \dots, \lfloor \frac{n}{h} \rfloor h\}$ is

$$X_t^{(h)} = \log\left(\frac{S_t}{S_{t-h}}\right) = \log\left(\frac{S_t}{S_{t-1}} \frac{S_{t-1}}{S_{t-2}} \dots \frac{S_{t-h+1}}{S_{t-h}}\right) = \sum_{k=0}^{h-1} X_{t-k},$$

a Central Limit Theorem (CLT) effect takes place (less heavy-tailed, less evidence of serial correlation).

- Problem: the larger h , the less data are available.
- One could consider overlapping returns

$$\left\{ X_t^{(h)} : t \in \left\{ h, h+k, \dots, h + \left\lfloor \frac{n-h}{k} \right\rfloor k \right\} \right\}, \quad 1 \leq k < h.$$

⇒ more data but serially dependent now (typically not a good idea).

To summarize, we can infer the following stylized facts about univariate financial return series:

- (U1) Return series are not iid although they show little serial correlation;
- (U2) Series of absolute or squared returns show profound serial correlation;
- (U3) Conditional expected returns are close to zero;
- (U4) Volatility (conditional standard deviation) appears to vary over time;
- (U5) Extreme returns appear in clusters;
- (U6) Return series are leptokurtic or heavy-tailed (power-like tail).

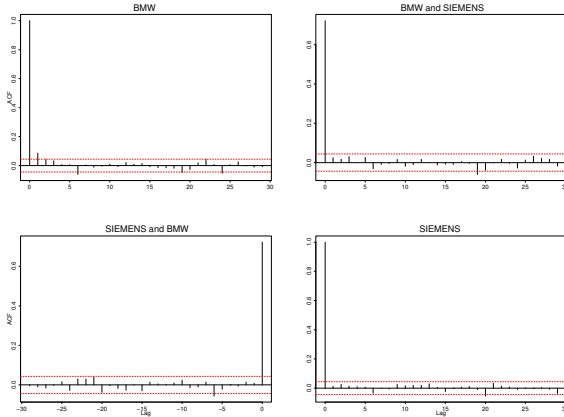
3.2 Multivariate stylized facts

Consider **multivariate** (componentwise) log-return data X_1, \dots, X_n .

3.2.1 Correlation between series

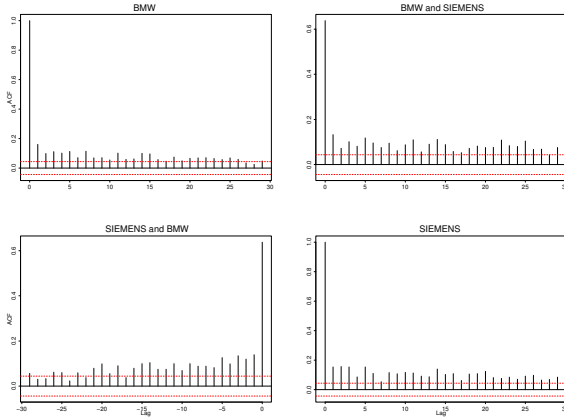
- By (U1), the **returns of stock A at t and $t+h$** show little (auto)correlation. The **returns of stock A at t and stock B at $t+h$, $h > 0$** , also show little **cross-correlation**. However, **Stock A and stock B on day t may be correlated** due to factors that affect the whole market (*contemporaneous dependence*).
- **These correlations of returns at t vary over time** (difficult to detect whether changes are continual or constant within regimes; fit different models for changing correlation, then make a formal comparison).
- **Periods of high/low volatility** are typically **common to more than one stock**, so returns of large magnitude in A at t may be followed by returns of large magnitude in A and B at $t+h$.

Estimated correlations between/within series:



Based on 2000 values from period 1985-01-23 to 1994-09-22. **Little autocorrelation**, **little crosscorrelation** (at different lags), **contemporaneous correlation**.

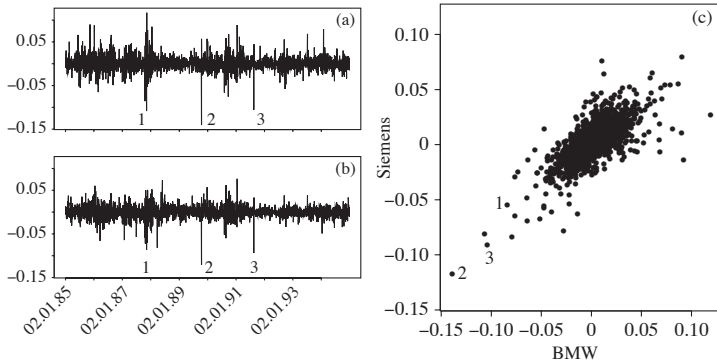
Estimated correlations between/within series of absolute values:



Autocorrelation of absolute returns (indication of volatility clustering).
Common to more than one stock.

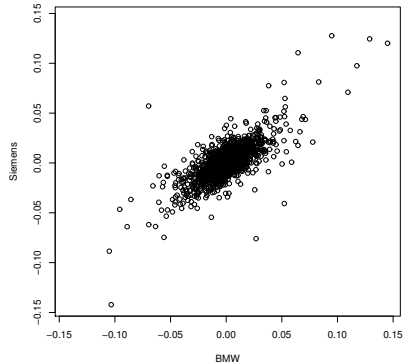
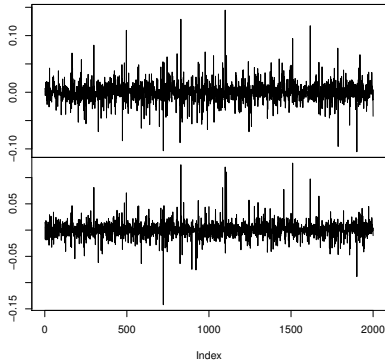
3.2.2 Tail dependence (for quantifying joint extremes)

(BMW, Siemens) log-returns from 1985-01-23 to 1994-09-22 ($n = 2000$)



In **volatile/extreme**(ly bad) **periods**, **dependence seems stronger** (1: 1987-10-19 **Black Monday** (DJ drop by 22%); 2: 1989-10-16 **Monday demonstrations** (Wende); 3: 1991-08-19 **coup against** soviet president **M. Gorbachev**).

Simulated log-returns from a fitted bivariate t distribution ($n = 2000$; $\rho = 0.72$, $\nu = 2.8$ both fitted to (BMW, Siemens))



- The multivariate t distribution can replicate joint large gains/losses but in a symmetric way.
- The multivariate normal distribution cannot replicate such behaviour, known as tail dependence; see Chapter 7.

To summarize, we can infer the following **stylized facts** about **multivariate** financial return series:

- (M1) **Multivariate return series** show **little** evidence of **cross-correlation**, **except** for **contemporaneous returns** (i.e. at the same t);
- (M2) Multivariate series of **absolute returns** show **profound cross-correlation**;
- (M3) **Correlations** between contemporaneous returns **vary over time** (difficult to infer from empirical correlations due to estimation error in small samples);
- (M4) **Extreme returns** in one series **often coincide with extreme returns** in several **other series** (e.g. tail dependence).