10 Credit risk

What is credit risk?

> "*Credit risk is the risk of a loss arising from the failure of a counterparty to honour its contractual obligations. This subsumes both default risk (the risk of losses due to the default of a borrower or a trading partner) and downgrade risk (the risk of losses caused by a deterioration in the credit quality of a counterparty that translates into a downgrading in some rating system).* "

- *Obligor* = a counterparty who has a financial obligation to us; for example, a debtor who owes us money, a bond issuer who promises interest, or a counterparty in a derivatives transaction.
- *Default* = failure to fulfill that obligation, for example, failure to repay loan or pay interest/coupon on a loan/bond; generally due to lack of liquidity or insolvency; may entail bankruptcy.

A crucial risk category

- Credit risk is omnipresent in the portfolio of a typical financial institution.
- A portfolio of loans or (corporate) bonds is obviously affected by credit risk.
- Credit risk accompanies any OTC (over-the-counter) derivative transaction such as a swap, because the default of one of the parties involved may substantially affect the actual pay-off of the transaction.
- There is a specialized market for *credit derivatives*, such as credit default swaps.
- Credit risk relates to the core activities of most banks but is also highly relevant to insurance companies: Insurers are exposed to substantial credit risk in their investment portfolios and counterparty default risk in their reinsurance treaties.

Credit risk management: A range of tasks

- An enterprise needs to determine the capital it requires to absorb losses due to credit risk.
- Portfolios of credit-risky instruments should be well diversified and optimized according to risk-return considerations.
- Institutions need to manage their portfolio of traded credit derivatives, which involves pricing, hedging and managing collateral for such trades.
- Financial institutions need to control the counterparty credit risk in their trades and contracts with other institutions. This has particularly been the case since the 2007–2009 financial crisis.

10.1 Credit risky instruments

10.1.1 Loans

- May be categorized into: retail loans (to individuals and small or medium-sized companies), corporate loans (to larger companies), interbank loans and sovereign loans (to governments).
- In each of these categories there may be a number of different products. For example, retail customers may borrow money using mortgages against property, credit cards and overdrafts.
- A sum of money, known as the *principal*, is advanced to the borrower for a particular term in exchange for a series of defined *interest* payments, which may be at fixed or floating interest rates. At the end of the term the borrower is required to pay back the principal.

- A useful distinction to make is between secured and unsecured lending. If a loan is secured the borrower has pledged an asset as collateral for the loan. In a mortgage the collateral is a property. In the event of default, the lender may take possession of the asset to mitigate the loss.
- In an unsecured loan the lender has no such claim on a collateral asset.

10.1.2 Bonds

- Bonds are publicly traded securities issued by companies and governments which allow the issuer to raise funding on financial markets.
- Bonds issued by companies are corporate bonds and bonds issued by governments are known as treasuries, sovereign bonds or, particularly in the UK, gilts (gilt-edged securities).
- The security commits the bond issuer (borrower) to make a series of interest payments to the bond buyer (lender) and pay back the principal at a fixed maturity.

- The interest payments, or coupons, may be fixed at the issuance of the bond (so-called fixed-coupon bonds). Alternatively, there are also bonds where the interest payments vary with market rates (so-called floating-rate notes).
- The reference rate for the floating rates is often a LIBOR rate (London Interbank Offered Rate).
- There are also convertible bonds which allow the purchaser to convert them into shares of the issuing company at predetermined time points.

Risks faced by bondholders

- A bond holder is subject to a number of risks, particularly interest-rate risk, default risk, downgrade risk and spread risk.
- Changes in the term structure of interest rates affect the value of bonds.
- As for loans, default risk is the risk that promised coupon and principal payments are not made.

- Downgrade risk is the risk that the bond loses value because the issuer's credit rating is lowered.
- Historically government bonds issued by developed countries have been considered default-free; for obvious reasons, after the European debt crisis of 2010–2012, this notion was called into question.
- Spread risk is a form of market risk that refers to changes in credit spreads. The credit spread of a defaultable bond measures the difference in the yield of the bond and the yield of an equivalent default-free bond.
- An increase in the spread of a bond means that the market value of the bond falls, which is generally interpreted as indicating that the financial markets perceive an increased default risk for the bond.

10.1.3 Derivative contracts subject to counterparty risk

- A substantial part of all derivative transactions is carried out over the counter and there is no central clearing counterparty such as an organized exchange to guarantee the fulfilment of the contractual obligations.
- These trades are subject to the risk that a contracting party defaults during the transaction, thus affecting the cash flows that are actually received by the other party. This risk, known as counterparty credit risk, received a lot of attention during the financial crisis of 2007-2009.
- Some of the institutions heavily involved in derivative transactions experienced worsening credit quality or—in the case of Lehman Brothers—even a default event.
- Counterparty risk management is now a key issue for all financial institutions and the focus of many new regulatory developments.

Example of interest-rate swap

- Two parties A and B agree to exchange a series of interest payments on a given nominal amount of money for a given period.
- A receives payments at a fixed interest rate and makes floating payments at a rate equal to the three-month LIBOR rate.
- Suppose that A defaults at time τ_A before the maturity of the contract.
- If interest rates have risen relative to their value at inception of contract:
  - The fixed interest payments have decreased in value and the value of the contract has increased for B.
  - The default of A constitutes a loss for B.
  - The loss size depends on the term structure of interest rates at τ_A.

- If interest rates have fallen relative to their value at $t = 0$:
  - The fixed payments have increased in value so that the swap has a negative value for B.
  - B will still has to pay the value of the contract into the bankruptcy pool,
  - There is no upside for B in A's default.
- If B defaults first the situation is reversed: falling rates lead to a counterparty-risk-related loss for A.

Management of counterparty risk

- Counterparty risk has to be taken into account in pricing and valuation. This has led to the notion of credit value adjustments (CVA).
- Counterparty risk needs to be controlled using risk-mitigation techniques such as netting and collateralization.
- Under a netting agreement the value of all derivatives transactions between A and B is computed and only the aggregated value is subject to counterparty risk; since offsetting transactions cancel each other out, this has the potential to reduce counterparty risk substantially.
- Under a collateralization agreement the parties exchange collateral (cash and securities) that serves as a pledge for the receiver. The value of the collateral is adjusted dynamically to reflect changes in the value of the underlying transactions.

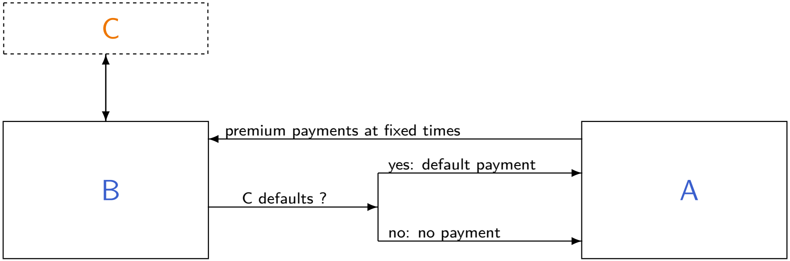
10.1.4 Credit default swaps and related credit derivatives

- Credit derivatives are securities which are primarily used for the hedging and trading of credit risk.
- The promised pay-off of a credit derivative is related to credit events affecting one or more firms.
- Major participants in the market for credit derivatives are banks, insurance companies and investment funds.
- Retail banks are typically net buyers of protection against credit events; other investors such as hedge funds and investment banks often act as both sellers and buyers of credit protection.
- Credit default swaps (CDSs) are the workhorses of the credit derivatives market and the market for CDSs written on larger corporations is fairly liquid.

Structure of CDS

Consider contract with maturity T and ignore counterparty credit risk. Three parties are involved (only two directly):

- C (reference entity); default at time $\tau_C < T$ triggers default payment.
- A (protection buyer); pays premiums to B until $\min(\tau_C, T)$.
- B (protection seller); makes default payment to A if $\tau_C < T$.



CDS: Payment flows

- If the reference entity experiences a default before the maturity date T of the contract, the protection seller makes a default payment to the protection buyer, which mimics the loss due to the default of a bond issued by the reference entity (the reference asset); this part of a CDS is called the default payment leg.
- As compensation the protection buyer makes periodic premium payments (typically quarterly or semiannually) to the protection seller (the premium payment leg); after the default of the reference entity, premium payments stop. There is no initial payment.
- The premium payments are quoted in the form of an annualized percentage x^* of the notional value of the reference asset; x^* is termed the (fair or market quoted) CDS spread.

Use of CDS

Investors enter into CDS contracts for various reasons.

- Bond investors with a large credit exposure to the reference entity may buy CDS protection to insure themselves against losses due to default of a bond (easier than reducing the original bond position as CDS contracts are more liquid).
- CDS contracts are also held for speculative reasons: so-called naked CDS positions, where the protection buyer does not own the bond are often assumed by investors who are speculating on the widening of the credit spread of the reference entity (similar to short-selling bonds issued by the reference entity.)
- Note that, in contrast to insurance, there is no requirement for the protection buyer to have insurable interest, that is, to actually own a bond issued by the reference entity.

10.1.5 PD, LGD and EAD

Exposure

- If we make a loan or buy a bond, our exposure is relatively easy to determine, since it is mainly the principal that is at stake. There is some additional uncertainty about the value of lost interest payments.
- A further source of exposure uncertainty is due to the widespread use of credit lines, essentially a ceiling up to which a corporate client can borrow money at given terms.
- For OTC derivatives the counterparty risk exposure is even more difficult to quantify, since it is a stochastic variable depending on the unknown time at which a counterparty defaults and the evolution of the value of the derivative up to that point.
- In practice the concept used is exposure at default (EAD), which recognises that exposure often depends on the exact default time.

Probability of default (PD)

- When measuring the risk of losses over a fixed time horizon, for example one year, we are particularly concerned with estimating the probability that obligors default by the time horizon, a quantity known to practitioners as probability of default or PD.
- The probability of default is related to the credit quality of an obligor and we discuss models of credit quality next.
- For instruments where the loss is dependent on the exact timing of default, for example OTC derivatives with counterparty risk, the risk of default is described by the whole distribution of possible default times and not just the probability of default by a fixed horizon.
- In simple models of default time, the probability of default may be expressed in terms of a hazard function which measures the risk of default at any instant in time.

Loss given default (LGD)

- In the event of default, it is unlikely that the entire exposure is lost.
- When a mortgage holder defaults on a residential mortgage, and there is no realistic possibility of restructuring the debt, the lender can sell the property (the collateral asset) and the proceeds from the sale will make good some of the lost principal.
- When a bond issuer goes into administration, the bond holders join the group of creditors who will be partly recompensed for their losses by the sale of the firm's assets.
- Practitioners use the term loss given default or LGD to describe the proportion of the exposure that is actually lost in the event of default, or its converse, the recovery, to describe the amount of the exposure that can be recovered through debt restructuring and asset sales.

Dependence of EAD, PD and LGD

- EAD, PD and LGD are dependent quantities. For example, in a period of financial distress, when PDs are high, asset values of firms are depressed and firms are defaulting, recoveries are likely to be correspondingly low, so that there is positive dependence between PDs and LGDs.

10.2 Measuring credit quality

Scores, ratings & measures inferred from prices

There are two philosophies for quantifying the credit quality or default probability of an obligor.

1) Credit quality can be described by a credit *rating or score* that is based on empirical data and expert judgement.
2) For obligors whose equity is traded on financial markets, prices can be used to infer the market's view of the credit quality of the obligor.

Credit ratings and scores fulfill a similar function—they allow us to order obligors by their credit risk and map that risk to an estimate of the PD.

Credit ratings tend to be expressed on an ordered categorical scale whereas credit scores are often expressed in points on a metric scale.

Rating and scoring

- The task of rating obligors is often outsourced to a rating agency such as Moody's or Standard & Poor's (S&P).
- In the S&P rating system there are seven pre-default rating categories labelled AAA, AA, A, BBB, BB, B, CCC, with AAA being the highest and CCC the lowest rating.
- Moody's uses nine pre-default rating categories labelled Aaa, Aa, A, Baa, Ba, B, Caa, Ca, C.
- A finer alpha-numeric system is also used by both agencies.
- Credit scores are traditionally used for retail customers and are based on so-called *scorecards*. Historical data is used to model default risk as a function of demographic, behavioural and financial covariates using techniques like logistic regression. The covariates are weighted and combined into a score.

10.2.1 Credit rating migration

- In the credit-migration approach each firm is assigned to a credit-rating category at any given time point.
- We assume that the current credit rating completely determines the default probability.
- The probability of moving from one credit rating to another over a given risk horizon (typically one year) is then specified.
- These probabilities, known as transition probabilities, are typically presented in the form of a matrix. They are estimated from historical data on empirical transition rates.
- The following example is taken from Ou (2013), (Exhibit 26). It gives average transition rates from one rating to another within one year. WR stands for withdrawn rating.

Initial rating	Rating at year-end (%)								WR
	Aaa	Aa	A	Baa	Ba	B	Caa	Ca–C	Default	
Aaa	87.20	8.20	0.63	0.00	0.03	0.00	0.00	0.00	0.00	3.93
Aa	0.91	84.57	8.43	0.49	0.06	0.02	0.01	0.00	0.02	5.48
A	0.06	2.48	86.07	5.47	0.57	0.11	0.03	0.00	0.06	5.13
Baa	0.039	0.17	4.11	84.84	4.05	7.55	1.63	0.02	0.17	5.65
Ba	0.01	0.05	0.35	5.52	75.75	7.22	0.58	0.07	1.06	9.39
B	0.01	0.03	0.11	0.32	4.58	73.53	5.81	0.59	3.85	11.16
Caa	0.01	0.02	0.02	0.12	0.38	8.70	61.71	3.72	13.34	12.00
Ca-C	0.00	0.00	0.00	0.00	0.40	2.03	9.38	35.46	37.93	14.80

- 1-year default probability for an A-rated company is estimated to be 0.06%, whereas for a Caa-rated company it is 13.3%.
- In practice a correction to the figures would probably be undertaken to account for rating withdrawals

- Rating agencies also publish cumulative default rates over longer time horizons.
- These provide estimates of cumulative default probabilities over several years. Alternative estimates of multi-year default probabilities can be inferred from one-year transition matrices as explained later.
- The data are taken from Ou (2013), (Exhibit 33).

Initial rating	Term						
	1	2	3	4	5	10	15
Aaa	0.00	0.01	0.01	0.04	0.11	0.50	0.93
Aa	0.02	0.07	0.14	0.26	0.38	0.92	1.75
A	0.06	0.20	0.41	0.63	0.87	2.48	4.26
Baa	0.18	0.50	0.89	1.37	1.88	4.70	8.62
Ba	1.11	3.08	5.42	7.93	10.18	19.70	29.17
B	4.05	9.60	15.22	20.13	24.61	41.94	52.22
Caa-C	16.45	27.87	36.91	44.13	50.37	69.48	79.18

TTC and PIT

- Default rates tend to vary with the state of the economy, being high during recessions and low during periods of economic expansion.
- Transition rates as estimated by rating agencies are historical averages over longer time horizons covering several *business cycles*.
- For instance the transition rates we show have been estimated from rating migration data over the period 1970–2012.
- Rating agencies focus on the average credit quality *through the business cycle* (TTC) when attributing a credit rating to a particular firm.
- Hence the default probabilities used in the rating migration approach are estimates of the average default probability, independent of the current economic environment.
- These can be contrasted with *point-in-time* (PIT) estimates of default probabilities which reflect the current macroeconomic environment.

10.2.2 Rating transitions as a Markov chain

- Let (R_t) denote a discrete-time stochastic process taking values in $S = \{0, 1, \ldots, n\}$ at times $t = 0, 1, \ldots$.
- The set S defines rating states of increasing creditworthiness with 0 representing default. (R_t) models an obligor's rating over time.
- We will assume that (R_t) is a Markov chain. This means that it has the Markov property that

$$\mathbb{P}(R_t = k \mid R_0 = r_0, R_1 = r_1, \ldots, R_{t-1} = j) = \mathbb{P}(R_t = k \mid R_{t-1} = j)$$

  for all $t \geq 1$ and all $j, r_0, r_1, r_{t-2}, k \in S$.
- Conditional probabilities of rating transitions given an obligors's rating history depend only on the previous rating $R_{t-1} = j$ at the last time point and not the more distant history.
- There is evidence that rating histories show momentum and stickiness which violates the Markov assumption (Lando and Skodeberg (2002)).

Properties of Markov chains

- The Markov chain is stationary if, for all $t \geq 1$ and rating states j, k,

$$\mathbb{P}(R_t = k \mid R_{t-1} = j) = \mathbb{P}(R_1 = k \mid R_0 = j).$$

- In this case we can define the transition matrix $\boldsymbol{P} = (p_{jk})$ with elements $p_{jk} = \mathbb{P}(R_t = k \mid R_{t-1} = j)$, for any $t \geq 1$.
- The Chapman-Kolmogorov equations say that

$$\mathbb{P}(R_t = k \mid R_{t-2} = j) = \sum_{l \in S} p_{jl} p_{lk} \,.$$

- An implication of this is that the matrix of transition probabilities over two time steps is given by $\boldsymbol{P}^2 = \boldsymbol{P} \times \boldsymbol{P}$.
- It is not clear how a matrix of transition probabilities for a fraction of a time period can be computed (one would need continuous-time chains).

Estimating default and transition probabilities

- For $t = 0, \ldots, T-1$ and $j \in S \setminus \{0\}$ let N_{tj} denote the number of companies that are rated j at time t and for which a rating is available at time $t+1$; let N_{tjk} denote the subset of those companies that are rated k at time $t+1$.
- Under the Markovian assumption the N_{tj} companies rated j can be thought of as being randomly allocated to the ratings $k \in S$ according to probabilities p_{jk} which satisfy $\sum_{k=0}^{n} p_{jk} = 1$.
- In this framework the likelihood is given by

$$L((p_{jk}); (N_{tj}), (N_{tjk})) = \prod_{t=0}^{T-1} \left(\prod_{j=1}^{n} \left(N_{tj}! \prod_{k=0}^{n} \frac{p_{jk}^{N_{tjk}}}{N_{tjk}!} \right) \right) .$$

- If this is maximized subject to the constraints that $\sum_{k=0}^{n} p_{jk} = 1$ for $j = 1, \ldots, n$ we obtain the maximum likelihood estimator

$$\hat{p}_{jk} = \frac{\sum_{t=0}^{T-1} N_{tjk}}{\sum_{t=0}^{T-1} N_{tj}} . \tag{70}$$

Continuous-time Markov transition models

- The main drawback of modelling rating transitions as a discrete-time Markov chain is that we ignore any information about intermediate transitions taking place between two times t and $t+1$.
- For this reason, better to consider transitions in continuous time. Probabilities cannot be modelled directly but are instead modelled in terms of transition rates.
- Over any small time step of duration δt we assume that the probability of a transition from rating j to k is given approximately by $\lambda_{jk}\delta t$. The probability of staying at rating j is given by $1 - \sum_{k \neq j} \lambda_{jk}\delta t$.

- If we now define a matrix Λ to have off-diagonal entries λ_{jk} and diagonal entries $-\sum_{k\neq j}\lambda_{jk}$, we can summarise these transition probabilities for a small time step in the matrix

$$I_{n+1} + \Lambda\delta t.$$

- Λ is the so-called *generator matrix*.

The generator matrix

- Let $\boldsymbol{P}(t)$ be the matrix of transition probabilities for the period $[0, t]$.
- Divide $[0, t]$into N small time steps of size $\delta t = t/N$ for N large.
- The matrix of transition probabilities can be approximated by

$$\boldsymbol{P}(t) \approx \left(I_{n+1} + \frac{\Lambda t}{N}\right)^{N}$$

- This converges, as $N \to \infty$,to the so-called matrix exponential of Λt.

$$\boldsymbol{P}(t) = \exp\left(\Lambda t\right)$$

- We can compute transition probabilities for any time horizon.
- A Markov chain with generator Λ can be constructed in the following way. An obligor remains in rating state j for an exponentially distributed amount of time with parameter $\lambda = \sum_{k \neq j} \lambda_{jk}$. When a transition takes place the probability that it is from j to state k is given by λ_{jk}/λ.

Estimating generator in continuous time

- This construction leads to natural estimators for the matrix Λ.
- Since λ_{jk} is the rate of migrating from j to k we can estimate it by

$$\hat{\lambda}_{jk} = \frac{N_{jk}(T)}{\int_0^T Y_j(t)\,\mathrm{d}t}, \tag{71}$$

  where $N_{jk}(T)$ is the total number of observed transitions from j to k in $[0, T]$ and $Y_j(t)$ is the number of obligors with rating j at time t.
- The denominator represents the total time spent in state j by all the companies in the dataset.

- Note that this is the continuous-time analogue of the maximum likelihood estimator in (70).
- It can be shown to be the maximum likelihood estimator for the transition rates in a homogenous continuous-time Markov chain.

10.3 Structural models of default

10.3.1 The Merton model

- Merton's model (1974) is the prototype of all firm value models.
- Consider firm with stochastic asset-value (V_t), financing itself by equity (i.e. by issuing shares) and debt.
- Assume that debt consists of single zero coupon bond with face or nominal value B and maturity T.
- Denote by S_t and B_t the value at time $t \leq T$ of equity and debt so that

$$V_t = S_t + B_t, \quad 0 \leq t \leq T.$$

- Assume that default occurs if the firm misses a payment to its debt holders and hence only at T.

Equity and debt as contingent claims on assets

- At T we have two possible cases:
  1) $V_T > B$. In that case the debtholders receive B; shareholders receive residual value $S_T = V_T - B$, and there is no default.
  2) $V_T \leq B$. In that case the firm cannot meet its financial obligations, and shareholders hand over control to the bondholders, who liquidate the firm; hence we have $B_T = V_T$, $S_T = 0$.
- In summary we obtain

$$
\begin{aligned}
S_T &= (V_T - B)^+ \\
B_T &= \min(V_T, B) = B - (B - V_T)^+ .
\end{aligned}
$$

- The value of equity at T equals the pay-off of a European call option on V_T with exercise price equal to B.
- The value of the debt at T equals the nominal value of debt minus the pay-off of a European put option on V_T.

- The option interpretation explains certain conflicts of interest between shareholders and bondholders.
- For example, shareholders have more interest in the firm taking on risky projects/investments since the value of an option increases with the volatility of the underlying security.
- Bondholders have a short position on the firm's assets and would like to see the volatility reduced.

The asset value process

It is assumed that asset value (V_t) follows a diffusion of the form

$$dV_t = \mu_V V_t dt + \sigma_V V_t dW_t$$

for constants $\mu_V \in$ R, $\sigma_V > 0$, and a Brownian motion $(W_t)_{t\geq 0}$, so that

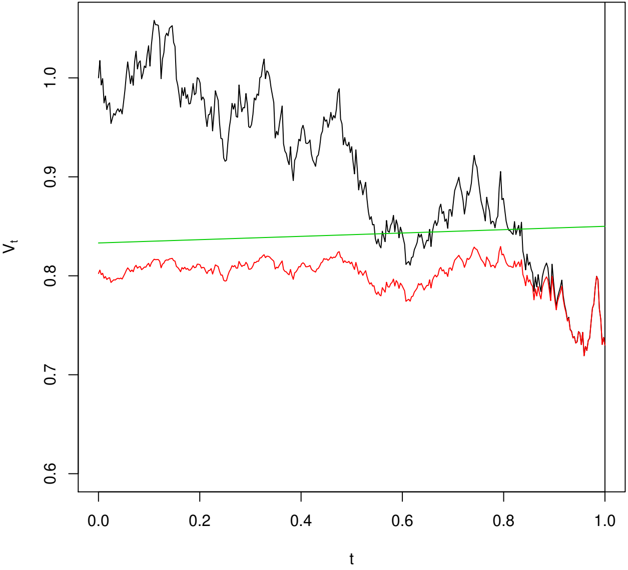
$$V_T = V_0 \exp\Big((\mu_V - \frac{1}{2}\sigma_V^2)T + \sigma_V W_T\Big)\,;$$

in particular $\ln V_T \sim \mathrm{N}(\ln V_0 + (\mu_V - \frac{1}{2}\sigma_V^2)T,\ \sigma_V^2 T)$. The default probability is thus

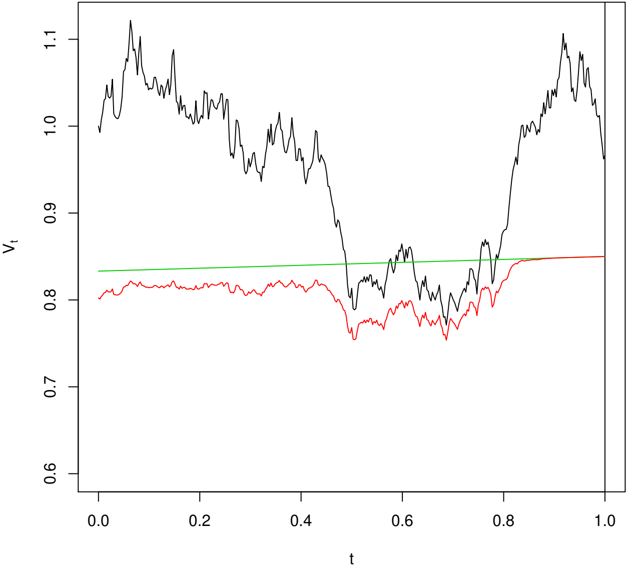
$$\mathbb{P}(V_T \leq B) = \mathbb{P}(\ln V_T \leq \ln B) = \Phi\left(\frac{\ln \frac{B}{V_0} - (\mu_V - \frac{1}{2}\sigma_V^2)T}{\sigma_V \sqrt{T}}\right); \quad (72)$$

it is increasing in B and σ_V (for $V_0 > B$) and decreasing in V_0 and μ_V.

A default path



A non-default path



10.3.2 Pricing in Merton's model

- Under some technical assumptions we can price equity and debt using the Black–Scholes formula.
- The assumptions are that:
  1) The risk-free rate is deterministic and equal to $r \geq 0$.
  2) The asset-value process (V_t) is independent of the debt level B.
  3) The asset value (V_t) can be traded on a frictionless market.
- Recall that equity is a call option on the asset value (V_t). Hence Black–Scholes formula yields

$$S_t = C^{\mathsf{BS}}(t, V_t; \sigma_V, r, T, B) := V_t \Phi(d_{t,1}) - Be^{-r(T-t)}\Phi(d_{t,2}),$$

where the arguments are given by

$$d_{t,1} = \frac{\ln \frac{V_t}{B} + (r + \frac{1}{2}\sigma_V^2)(T-t)}{\sigma_V \sqrt{T-t}}, \quad d_{t,2} = d_{t,1} - \sigma_V \sqrt{T-t}.$$

Pricing of debt

- The price at $t \le T$ of a default-free zero-coupon bond with maturity T and a face value of one equals

$$p_0(t,T) = \exp(-r(T-t)).$$

- The value of the firm's debt equals the difference between the value of default-free debt and a put option on (V_t) with strike B, i.e.

$$B_t = Bp_0(t,T) - P^{\mathsf{BS}}(t, V_t; r, \sigma_V, B, T).$$

- The Black–Scholes formula for European puts now yields

$$B_t = p_0(t,T)B\Phi(d_{t,2}) + V_t\Phi(-d_{t,1}). \tag{73}$$

- The path of (B_t) is shown on the previous plots. The value of default-free debt $Bp_0(t,T)$ is shown as a green curve.

Risk-neutral and physical default probabilities

- Under the risk-neutral measure $\mathbb{Q}$ the process (V_t) satisfies the SDE $\mathrm{d}V_t = rV_t\,\mathrm{d}t + \sigma_V V_t\,\mathrm{d}\tilde{W}_t$ for a standard $\mathbb{Q}$-Brownian motion $\tilde{W}$.
- The drift μ_V is replaced by the risk-free interest rate r.
- Hence the risk-neutral default probability is given by

$$q = \mathbb{Q}(V_T \le B) = \Phi\left(\frac{\ln B - \ln V_0 - (r - \frac{1}{2}\sigma_V^2)T}{\sigma_V\sqrt{T}}\right).$$

- Comparison with physical default probability $p = \mathbb{P}(V_T \le B)$ yields

$$q = \Phi\left(\Phi^{-1}(p) + \frac{\mu_V - r}{\sigma_V}\sqrt{T}\right). \tag{74}$$

- The correction term $(\mu_V - r)/\sigma_V$ equals the Sharpe ratio of the firm's assets (a popular measure of the risk premium earned by the firm).
- The formula is sometimes applied in practice to go from physical to risk-neutral default probabilities.

Credit spreads in Merton's model

- The credit spread measures the difference between the (continuously compounded) yield of a default-free zero coupon bond $p_0(t,T)$ and a defaultable zero coupon bond $p_1(t,T)$, i.e.

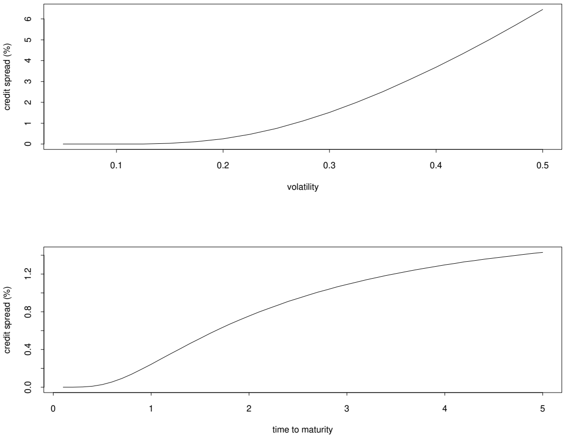
$$
\begin{aligned}
c(t,T) &= \frac{-1}{T-t}\left(\ln p_1(t,T) - \ln p_0(T-t)\right) \\
&= \frac{-1}{T-t} \ln \frac{p_1(t,T)}{p_0(t,T)} \,.
\end{aligned}
$$

- In Merton's model we have $p_1(t,T) = \frac{1}{B} B_t$ and hence

$$
c(t,T) = \frac{-1}{(T-t)} \ln \left(\Phi(d_{t,2}) + \frac{V_t}{Bp_0(t,T)}\, \Phi(-d_{t,1}) \right). \tag{75}
$$

- For a fixed time to maturity $c(t,T)$ depends only on σ_V and on the ratio $Bp_0(t,T)/V_t$ (a measure of indebtedness of the firm).
- In line with economic intuition it is increasing in both quantities.

Illustration of credit spreads in Merton's model



Credit spread $c(t,T)$ (%) as function of σ_V (top) and time to maturity $T-t$ (bottom) for fixed debt to firm value ratio 0.6. In upper picture $T-t=2$; in lower picture $\sigma_V = 0.25$.

10.3.3 Structural models in practice: EDF and DD

- A number of industry models descend from the Merton model.
- An important example is the so-called public-firm EDF model that is maintained by Moody's Analytics.
- The methodology builds on earlier work by KMV (a private company named after its founders Kealhofer, McQuown and Vasicek) in the 1990s.
- Literature: Crosbie and Bohn (2002) and Sun et al. (2012).
- Expected Default Frequency. The EDF is an estimate of the default probability of a given firm over a one-year horizon.
- Suppose we use Merton's model for a company issuing debt with face value B maturing at time $T = 1$. The analogous quantity would be

$$\mathrm{EDF}_{Merton} = 1 - \Phi\left(\frac{\ln V_0 - \ln B + (\mu_V - \frac{1}{2}\sigma_V^2)}{\sigma_V}\right). \quad (76)$$

How Moody's adapt the Merton formula

- The decreasing function $1 - \Phi$ is replaced by an empirically estimated function.
- B is replaced by a new default threshold $\tilde{B}$ representing the structure of the firm's liabilities more closely.
- The term $(\mu_V - \frac{1}{2}\sigma_V^2)$ in the numerator is usually omitted.
- The current asset value V_0 and the asset volatility σ_V are inferred or 'backed out' from information about the firm's equity value.
- Why?
  - In contrast to the assumptions underlying Merton's model, in most cases there is no market for the assets of a firm, so that the asset value is not directly observable.
  - The market value can differ widely from the value of a company as measured by accountancy rules (the so-called book value).

Inferring asset values in Merton's model

- Recall that in Merton's model we have that

$$S_t = C^{\mathsf{BS}}(t, V_t; r, \sigma_V, B, T). \tag{77}$$

- We consider the debt structure (B and T) as well as the interest rate r to be known. Equity values (S_t) are observable.
- For fixed t, (77) is an equation with two unknowns, V_t and σ_V.
- To overcome this difficulty an iterative procedure is used.
- In step (1), an initial estimate $\sigma_V^{(0)}$ is used to infer a time series of asset values $(V_t^{(0)})$ from equity values (S_t).
- Then a new volatility estimate $\sigma_V^{(1)}$ is estimated from this time series.
- A new time series $(V_t^{(1)})$ is then constructed using (77) with $\sigma_V^{(1)}$.
- This procedure is iterated n-times, until the volatility estimates converge.

- The procedure in the public-firm EDF model is similar but a more sophisticated capital structure is assumed and the BS formula in (77) is replaced by a more complex formula.

EDF and DD

- In the public-firm EDF model a new state variable is introduced. This is the so-called distance-to-default (DD), given by
$$\mathrm{DD} := (\log V_0 - \log \tilde{B})/\sigma_V \,. \tag{78}$$
- Here $\tilde{B}$ represents the default threshold; in some versions of the model $\tilde{B}$ is modelled as the sum of the liabilities payable within one year and half of the longer term debt.
- Note that (78) is in fact an approximation of the argument of (76), since μ_V and σ_V^2 are usually small.
- It is assumed that the distance-to-default ranks firms in the sense that firms with a higher DD exhibit a higher default probability.

- The functional relationship between DD and EDF is determined empirically; using a database of historical default events, the proportion of firms with DD in a given small range that default within a year is estimated. This proportion is the empirically estimated EDF.

Variable	J&J	RadioShack	Notes
Market value of assets V_0	\$236 bn	\$1834 m	Determined from time series of equity prices.
Asset volatility σ_V	11%	24%	
Default threshold $\tilde{B}$	\$39 bn	\$1042 m	Short-term liabilities and half of long-term liabilities.
DD	16.4	2.3	Given by $(\log V_0 - \log \tilde{B})/\sigma_V$.
EDF (one year)	0.01%	3.58%	Using empirical mapping between DD and EDF.

The example is taken from Sun et al. (2012); it is concerned with the situation of Johnson and Johnson (J&J) and RadioShack as of April 2012.

10.3.4 Credit migration models revisited

- In a credit migration model, consider a firm rated j at $t = 0$ with transition probabilities p_{jk}, $0 \leq k \leq n$ for the period $[0, T]$.
- Suppose that the asset-value process (V_t) of the firm follows the Merton diffusion model so that

$$V_T = V_0 \exp((\mu_V - \tfrac{1}{2}\sigma_V^2)T + \sigma_V W_T) \tag{79}$$

  is lognormally distributed.
- We can choose thresholds $0 = \tilde{d}_0 < \tilde{d}_1 < \cdots < \tilde{d}_n < \tilde{d}_{n+1} = \infty$ such that $\mathbb{P}(\tilde{d}_k < V_T \leq \tilde{d}_{k+1}) = p_{jk}$ for $k \in \{0, \ldots, n\}$.
- Thus we have translated the transition probabilities into a series of thresholds for an assumed asset-value process.
- The threshold $\tilde{d}_1$ is the default threshold, often interpreted as the value of the firm's liabilities.

- The higher thresholds are the asset-value levels that mark the boundaries of higher rating categories.
- The firm-value model can be summarized by saying that the firm belongs to rating class k at the time horizon T if and only if $\tilde{d}_k < V_T \leq \tilde{d}_{k+1}$.
- The migration probabilities remain invariant under simultaneous strictly increasing transformations of V_T and the thresholds $\tilde{d}_j$.
- If we define

$$X_T := \frac{\ln V_T - \ln V_0 - (\mu_V - \frac{1}{2}\sigma_V^2)T}{\sigma_V \sqrt{T}}, \tag{80}$$

$$d_k := \frac{\ln \tilde{d}_k - \ln V_0 - (\mu_V - \frac{1}{2}\sigma_V^2)T}{\sigma_V \sqrt{T}}, \tag{81}$$

  then we can also say that the firm belongs to rating class k at the time horizon T if and only if $d_k < X_T \leq d_{k+1}$.
- Observe that X_T is a standardized version of the asset-value log-return $\ln V_T - \ln V_0$ and we can easily verify that $X_T = W_T/\sqrt{T} \sim N(0,1)$.

- In this case the formulas for the thresholds are easily obtained and are $d_k = \Phi^{-1}(\sum_{l=0}^{k-1} p_{jl})$ for $k = 1, \ldots, n$.

Credit migrations and public-firm EDFs compared

Advantages of EDFs.

1) The EDF reacts quickly to changes in the economic prospects of a firm, whereas agencies are often slow to adjust ratings.
2) EDFs tend to reflect the current macroeconomic environment and tend to be better predictors of default over short time horizons.

Advantages of credit migration approach.

1) The EDF approach is sensitive to over- and under-reactions in equity markets. If widely followed this might have destabilizing effects.
2) As rating agencies focus on average credit quality "through the business cycle", risk capital requirements based on rating transitions fluctuate less, helping to provide liquidity in credit markets.

10.4 Bond and CDS pricing in hazard rate models

10.4.1 Hazard rate models

- These are the simplest reduced-form credit risk models.
- A hazard rate model is a model in which the distribution of the default time of an obligor is directly specified by a hazard function without modelling the mechanism by which default occurs.
- To set up a hazard rate model we consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a random default time τ defined on this space, i.e. an $\mathcal{F}$-measurable rv taking values in $[0, \infty]$.
- We denote the df of τ by $F(t) = \mathbb{P}(\tau \leq t)$ and the tail or survival function by $\bar{F}(t) = 1 - F(t)$; we assume that $\mathbb{P}(\tau = 0) = F(0) = 0$, and that $\bar{F}(t) > 0$ for all $t < \infty$.

- The jump or default indicator process (Y_t) associated with τ is

$$Y_t = I_{\{\tau \le t\}}\,, \quad t \ge 0. \tag{82}$$

- (Y_t) is a right-continuous process which jumps from 0 to 1 at the default time τ.
- $1 - Y_t = I_{\{\tau > t\}}$ is the survival indicator of the firm at time t.

Definition 10.1 (cumulative hazard and hazard function)
The function $\Gamma(t) = -\ln(\bar{F}(t))$ is called the cumulative hazard function of the random time τ. If F is absolutely continuous with density f, the function

$$\gamma(t) = \frac{f(t)}{1 - F(t)} = \frac{f(t)}{\bar{F}(t)} = -\frac{\mathrm{d}}{\mathrm{d}t}\ln(\bar{F}(t))$$

is called the hazard function of τ.

- The hazard function $\gamma(t)$ gives the hazard rate at t, which is a measure of the instantaneous risk of default at t, given survival up to time t.

- We can represent the survival function of τ by

$$\bar{F}(t) = \exp\Big(- \int_0^t \gamma(s)\,\mathrm{d}s \Big)\,. \tag{83}$$

- We may show that

$$\lim_{h\to 0} \frac{1}{h}\mathbb{P}(\tau \le t+h \mid \tau > t) = \frac{1}{\bar{F}(t)} \lim_{h\to 0} \frac{F(t+h) - F(t)}{h} = \gamma(t).$$

Example 10.2 (Weibull distribution)

For illustrative purposes we determine the hazard function for the Weibull distribution with df $F(t) = 1 - \exp(-\lambda t^{\alpha})$ for parameters $\lambda, \alpha > 0$. Differentiation yields

$$f(t) = \lambda\alpha t^{\alpha-1}\exp(-\lambda t^{\alpha}) \text{ and } \gamma(t) = \lambda\alpha t^{\alpha-1}.$$

In particular, γ is decreasing in t if $\alpha < 1$ and increasing if $\alpha > 1$. For $\alpha = 1$ (exponential distribution) the hazard rate equals the constant λ.

Introducing filtrations

- Filtrations model information available to investors over time.
- A filtration $(\mathcal{F}_t)$ on $(\Omega, \mathcal{F})$ is an increasing family $\{\mathcal{F}_t : t \geq 0\}$ of sub-σ-algebras of $\mathcal{F} : \mathcal{F}_t \subset \mathcal{F}_s \subset \mathcal{F}$ for $0 \leq t \leq s < \infty$.
- $\mathcal{F}_t$ represents the state of knowledge of an observer at time t. $A \in \mathcal{F}_t$ means that at time t we can determine if A has occurred.
- In this section we assume that only observable quantity is the default indicator (Y_t) associated with τ. The appropriate filtration is $(\mathcal{H}_t)$ with

$$\mathcal{H}_t = \sigma(\{Y_u : u \leq t\}), \tag{84}$$

  the default history up to and including time t.
- τ is a $(\mathcal{H}_t)$-stopping time, since $\{\tau \leq t\} = \{Y_t = 1\} \in \mathcal{H}_t$ for all $t \geq 0$.
- In order to study bond and CDS pricing in hazard rate models we need to compute conditional expectations with respect to the σ-algebra $\mathcal{H}_t$.

A useful result

Lemma 10.3

Let τ be a default time with jump indicator process $Y_t = I_{\{\tau \leq t\}}$ and natural filtration $(\mathcal{H}_t)$. Then, for any integrable rv X and any $t \geq 0$, we have

$$\mathbb{E}(I_{\{\tau>t\}} X \mid \mathcal{H}_t) = I_{\{\tau>t\}} \frac{\mathbb{E}(I_{\{\tau>t\}} X)}{\mathbb{P}(\tau > t)}. \tag{85}$$

This result can be used to determine conditional survival probabilities. For $t < T$, applying (85) with $X := I_{\{\tau>T\}}$ we get

$$\mathbb{P}(\tau > T \mid \mathcal{H}_t) = I_{\{\tau>t\}} \exp\Big(-\int_t^T \gamma(s)\,\mathrm{d}s\Big), \quad t < T\,. \tag{86}$$

Martingale property of jump indicator process

Proposition 10.4

The process (M_t) defined as

$$M_t = Y_t - \int_0^t I_{\{\tau > u\}} \gamma(u) \, \mathrm{d}u, \quad t \geq 0$$

is an $(\mathcal{H}_t)$-martingale, that is $\mathbb{E}(M_s \mid \mathcal{H}_t) = M_t$ for all $0 \leq t \leq s < \infty$.

10.4.2 Risk-neutral pricing revisited

- According to the first fundamental theorem of asset pricing, a model for security prices is arbitrage free if and (essentially) only if it admits at least one equivalent martingale measure $\mathbb{Q}$.
- When building a model for pricing derivatives it is a natural shortcut to model the objects of interest—such as interest rates and default times—directly, under a martingale measure $\mathbb{Q}$.

Martingale modelling

- So-called martingale modelling is particularly convenient if the value H of the underlying assets at some maturity date T is exogenously given, as in the case of zero-coupon bonds.
- The underlying asset at time $t < T$ can be computed as the conditional expectation under $\mathbb{Q}$ of the discounted value at maturity via the risk-neutral pricing rule
$$V_t = \mathbb{E}^{\mathbb{Q}}\left(e^{-\int_t^T r_s \mathrm{d}s} H \mid \mathcal{F}_t\right). \tag{87}$$
- Model parameters are determined using the requirement that at time $t = 0$ the model price should coincide with the market price of the security; this is known as calibration to market data.

Pros and cons of Martingale modelling

- Martingale modelling ensures that the resulting model is arbitrage free, which is important for pricing many different securities simultaneously.
- The approach is frequently adopted in default-free term structure models and in reduced-form models for credit-risky securities.
- Martingale modelling is however problematic if the underlying market is incomplete (meaning that not all risk cannot be hedged away). In practice martingale modelling is best applied in situations where many liquidly traded derivatives are available.

10.4.3 Bond pricing

- It suffices to consider zero-coupon bonds.
- We use martingale modelling and work directly under some martingale measure $\mathbb{Q}$.

- We assume that under $\mathbb{Q}$ the default time τ is a random time with deterministic risk-neutral hazard function $\gamma^{\mathbb{Q}}(t)$.
- The information available to investors at time t is given by the sigma algebra $\mathcal{H}_t = \sigma(\{Y_u : u \leq t\})$.
- We take interest rates and recovery rates to be deterministic.
- The percentage loss given default is denoted by $\delta \in (0, 1)$.
- The continuously compounded interest rate is denoted by $r(t) \geq 0$.
- The price of the default-free zero-coupon bond with maturity $T \geq t$ is $p_0(t, T) = \exp(-\int_t^T r(s)\,\mathrm{d}s)$.

Analysing the payments

- The payments of a defaultable zero-coupon bond can be represented as a combination of a survival claim that pays one unit at the maturity date T and a recovery payment in case of default.
- The survival claim has pay-off $I_{\{\tau>T\}}$.
- Recall from (86) that

$$\mathbb{Q}(\tau > T \mid \mathcal{H}_t) = I_{\{\tau>t\}} \exp\Big(-\int_t^T \gamma^{\mathbb{Q}}(s)\,\mathrm{d}s\Big)$$

  and define $R(t) = r(t) + \gamma^{\mathbb{Q}}(t)$.
- Then the price of a survival claim at time t equals

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}(p_0(t,T) I_{\{\tau>T\}} \mid \mathcal{H}_t) &= \exp\Big(-\int_t^T r(s)\,\mathrm{d}s\Big)\mathbb{Q}(\tau > T \mid \mathcal{H}_t) \\ &= I_{\{\tau>t\}} \exp\Big(-\int_t^T R(s)\,\mathrm{d}s\Big). \end{aligned} \tag{88}$$

- Note that for $\tau > t$, this can be viewed as the price of a default-free zero-coupon bond with adjusted interest rate $R(t) > r(t)$.
- A similar relationship between defaultable and default-free bond prices can be established in many reduced-form credit risk models.

Recovery models

1) Recovery of Treasury (RT).
   - The RT model was proposed by Jarrow and Turnbull (1995).
   - If default occurs at some point in time $\tau \leq T$, the owner of the defaulted bond receives $(1 - \delta_\tau)$ units of the default-free zero-coupon bond $p_0(\cdot\,, T)$ at time τ, where $\delta_\tau \in [0, 1]$ models the percentage loss given default.
   - At maturity T the holder of the defaultable bond therefore receives the payment $I_{\{\tau > T\}} + (1 - \delta_\tau) I_{\{\tau \leq T\}}$.

2) Recovery of Face Value (RF).

- Under RF, if default occurs at $\tau \leq T$, the holder of the bond receives a recovery payment of size $(1-\delta_\tau)$ immediately at the default time τ.
- Note that even with deterministic loss given default and deterministic interest rates, the value at maturity of the recovery payment is random as it depends on the exact timing of default.

RF is slightly more realistic; RT is slightly easier to analyse.

Pricing recovery payment under RT

- The value of the recovery payment at the maturity date T is

$$(1-\delta)I_{\{\tau \leq T\}} = (1-\delta) - (1-\delta)I_{\{\tau > T\}}.$$

- Using (88), the value of the recovery payment at time $t < T$ is hence

$$(1-\delta)p_0(t,T) - (1-\delta)I_{\{\tau > t\}} \exp\Big(-\int_t^T R(s)\,\mathrm{d}s\Big).$$

- Hence the value of the bond is

$$p_1(t,T) = (1-\delta)p_0(t,T) + \delta I_{\{\tau>t\}} \exp\Big(-\int_t^T R(s)\,\mathrm{d}s\Big).$$

Pricing recovery payment under RF

- Under the RF-hypothesis the recovery payment takes the form $(1-\delta)I_{\{\tau\le T\}}$ where the payment occurs directly at time τ.
- A payments of this form is a payment-at-default claim.
- The value of the recovery payment at time $t \le T$ equals

$$\mathbb{E}^{\mathbb{Q}}\Big((1-\delta)I_{\{t<\tau\le T\}} \exp\Big(-\int_t^\tau r(s)\,\mathrm{d}s\Big) \,\Big|\, \mathcal{H}_t\Big).$$

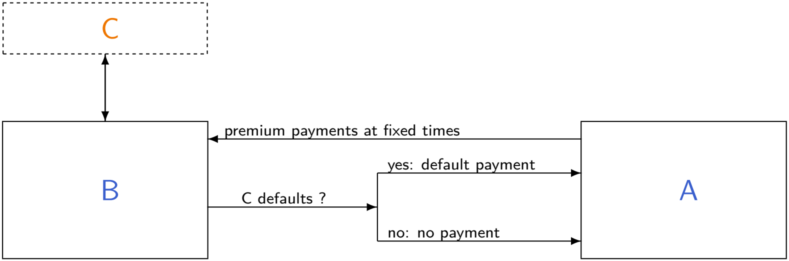
- Using (85) we may show that

$$\mathbb{E}^{\mathbb{Q}}\Big((1-\delta)I_{\{t<\tau\le T\}} \exp\Big(-\int_t^\tau r(s)\,\mathrm{d}s\Big) \,\Big|\, \mathcal{H}_t\Big)$$
$$= (1-\delta)I_{\{\tau>t\}} \int_t^T \gamma^{\mathbb{Q}}(s) \exp\Big(-\int_t^s R(u)\,\mathrm{d}u\Big)\,\mathrm{d}s.$$

10.4.4 CDS pricing

Recap: Structure of CDS

- C (reference entity); default at time $\tau_C < T$ triggers default payment.
- A (protection buyer); pays premiums to B until $\min(\tau_C, T)$.
- B (protection seller); makes default payment to A if $\tau_C < T$.



Payment flows

For simplicity write $\tau = \tau_C$ and consider the following contract:

- Premium payments.
  - These are due at times $0 < t_1 < \cdots < t_N$ measured in years.
  - If $\tau > t_k$, A pays a premium of size $x^*(t_k - t_{k-1})$ at t_k, where x^* denotes the fair swap spread.
  - After τ premium payments stop.
  - No initial payment.
- Default payment.
  - If $\tau < t_N = T$, B makes a default payment δ at τ.
  - Sometimes B receives an accrued premium payment of size $x^*(\tau - t_k)$ for $\tau \in (t_k, t_{k-1})$. We ignore this feature for simplicity.

Valuing the premium leg

- The premium leg consists of a set of survival claims.
- Introduce a function of x given by

$$
\begin{aligned}
&V_t^{\text{prem}}(x;\gamma^{\mathbb{Q}}) \\
&= \mathbb{E}^{\mathbb{Q}}\Bigg(\sum_{k\colon\, t_k>t} \exp\Big(-\int_t^{t_k} r(u)\,\mathrm{d}u\Big) x(t_k - t_{k-1}) I_{\{\tau>t_k\}} \mid \mathcal{H}_t\Bigg) \\
&= x \sum_{k\colon\, t_k>t} p_0(t,t_k)(t_k - t_{k-1})\mathbb{Q}(\tau > t_k \mid \mathcal{H}_t),
\end{aligned}
$$

  which is easily computed using $\mathbb{Q}(\tau > t_k \mid \mathcal{H}_t) = \exp(-\int_t^{t_k} \gamma^{\mathbb{Q}}(s)\,\mathrm{d}s)$.

- We obtain

$$
V_t^{\text{prem}}(x;\gamma^{\mathbb{Q}}) = I_{\{\tau>t\}}\, x \sum_{k\colon\, t_k>t} (t_k - t_{k-1}) \exp\Big(-\int_t^{t_k} R(u)\,\mathrm{d}u\Big).
$$

Valuing the default leg

- The default payment leg is a typical payment-at-default claim.
- We obtain

$$\begin{aligned}
&V_t^{\mathsf{def}}(\gamma^{\mathbb{Q}}) \\
&= \mathbb{E}^{\mathbb{Q}}\Big(\delta I_{\{t<\tau\le t_N\}} \exp\Big(-\int_t^{\tau} r(s)\,\mathrm{d}s\Big) \,\Big|\, \mathcal{H}_t\Big) \\
&= I_{\{\tau>t\}}\,\delta \int_t^{t_N} \gamma^{\mathbb{Q}}(s) \exp\Big(-\int_t^{s} R(u)\,\mathrm{d}u\Big)\,\mathrm{d}s.
\end{aligned}$$

The fair CDS spread

- The fair CDS spread x_t^* quoted for the contract at time t is chosen such that the value of the contract is equal to zero.

- The equation $V_t^{\text{prem}}(x_t^*; \gamma^{\mathbb{Q}}) = V_t^{\text{def}}(\gamma^{\mathbb{Q}})$ yields

$$x_t^* = I_{\{\tau > t\}} \frac{\delta \int_t^{t_N} \gamma^{\mathbb{Q}}(s) \exp\Big(- \int_t^s R(u)\,\mathrm{d}u \Big)\,\mathrm{d}s}{\sum_{k:\, t_k > t} (t_k - t_{k-1}) \exp\Big(- \int_t^{t_k} R(s)\,\mathrm{d}s \Big)} \,. \tag{89}$$

Model calibration

- We have to calibrate our model to the available market information. Hence we have to determine the implied risk-neutral hazard function $\gamma^{\mathbb{Q}}(t)$, which ensures that the fair CDS spreads implied by the model equal the spreads quoted in the market.
- Suppose that the market information at time $t = 0$ consists of the fair spread x^* of one CDS with maturity t_N.
- In that case $\gamma^{\mathbb{Q}}(s)$ is taken constant: for all $s \geq 0$, $\gamma^{\mathbb{Q}}(s) = \bar{\gamma}^{\mathbb{Q}}$ for some $\bar{\gamma}^{\mathbb{Q}} > 0$.

- $\bar{\gamma}^{\mathbb{Q}}$ has to solve the equation

$$x^* \sum_{k=1}^{N} p_0(0, t_k)(t_k - t_{k-1}) e^{-\bar{\gamma}^{\mathbb{Q}} t_k} = \delta \bar{\gamma}^{\mathbb{Q}} \int_0^{t_N} p_0(0, t) e^{-\bar{\gamma}^{\mathbb{Q}} t} \, \mathrm{d}t.$$

- There is a unique solution.
- If we observe spreads for several CDSs on the same reference entity but with different maturities, a constant function is not sufficient. Instead one typically uses piecewise constant or linear hazard functions.
- An exception occurs in the special case where: (1) the spread curve is flat (i.e. all CDSs on the reference entity have the same spread x^*, independent of the maturity); (2) the risk-free interest rate is constant; (3) the time points t_k are equally spaced ($t_k - t_{k-1} = \Delta t$ for all k).
- In that case the implied risk-neutral hazard rate $\bar{\gamma}^{\mathbb{Q}}$ is the solution of

$$x^* \Delta t \, p_0(0, \Delta t) e^{-\bar{\gamma}^{\mathbb{Q}} \Delta t} = \delta \bar{\gamma}^{\mathbb{Q}} \int_0^{\Delta t} e^{-rt} e^{-\bar{\gamma}^{\mathbb{Q}} t} \, \mathrm{d}t. \tag{90}$$

- For Δt relatively small (quarterly or semi-annual spread payments) a good approximation to the solution of (90) is given by $\bar{\gamma}^{\mathbb{Q}} \approx x^*/\delta$.
- This approximation is frequently used in practice and implies that the one-year default probability satisfies $\mathbb{Q}(\tau \leq 1) = 1 - e^{-\bar{\gamma}^{\mathbb{Q}}} \approx \bar{\gamma}^{\mathbb{Q}} \approx x^*/\delta$.

Shortcomings of simple hazard rate models

- In the models of this section the only risk factor affecting a defaultable bond or CDS is default risk.
- In these models credit spreads evolve deterministically prior to default, which is unrealistic.
- The models are not sophisticated enough to price options on defaultable bonds or CDSs.
- To obtain more realistic models we can replace the deterministic hazard functions by stochastic hazard processes.

- This means that default times are modelled as so-called doubly-stochastic random times.
- We might also consider adding stochastic interest rate models; and more complex assumptions on recoveries in the event of default.

10.4.5 P versus Q: Empirical results

- There are some empirical studies of the relationship between physical and risk-neutral default probabilities.
- Risk-neutral default probabilities are generally estimated from CDS spreads. These can be compared, for example, with EDFs.
- Berndt et al. (2008) compare five-year CDSs against five-year EDFs for a large pool of firms. The five-year EDF is an annualized estimate of the physical five-year default probability.

- Let $x^*_{t,i}$ and $\mathrm{EDF}_{t,i}$ denote the CDS spread and five-year EDF of firm i at date t. Their (most basic) model took the form

$$x^*_{t,i} = \alpha + \beta \, \mathrm{EDF}_{t,i} + \varepsilon_{t,i} \, ,$$

  with estimates $\alpha = 33bp$ and $\beta = 1.6$; the R^2 was 0.73.

- More crudely $x^*_{t,i} / \mathrm{EDF}_{t,i} \approx 1.6$.
- Using $q_{t,i} = x^*_{t,i}/\delta$ as a proxy for the risk-neutral default probability yields

$$\frac{q_{t,i}}{p_{t,i}} \approx \frac{x^*_{t,i}}{\delta \, \mathrm{EDF}_{t,i}} \approx 1.6 \, \delta^{-1} \, .$$

10.5 Pricing with stochastic hazard rates

Why stochastic hazard rates?

- In hazard rate models the only risk factor is default risk $\Rightarrow$ Credit spreads evolve deterministically prior to default, which is clearly unrealistic.
- Moreover, it is not possible to price options on bonds or CDSs or to do risk management for bond portfolios in such models.
- Hence it is of interest to consider models where hazard rate is a stochastic process $(\gamma_t)_{t\geq 0}$; typically hazard rate is driven by a second stochastic process Ψ, that is $\gamma_t = \gamma(\Psi_t)$.
- Simplest such model class are doubly-stochastic random times.

10.5.1 Doubly stochastic random times

Setup. We work on $(\Omega, \mathcal{F}, \mathbb{P})$ with background filtration $(\mathcal{F}_t)$ containing information about all other economic events except the default event.

Consider a random time τ, that is a measurable rv with values in $(0, \infty)$.

- $Y_t = I_{\{\tau \leq t\}}$ is the associated default indicator and $(\mathcal{H}_t) = \sigma\{Y_s, s \leq t\}$ is the default history up to t.
- Define new filtration $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$, $t \geq 0$, i.e. $\mathcal{G}_t$ contains background info $\mathcal{F}_t$ and default history up to t (this is the information available to investors).

Definition 10.5

τ is called *doubly stochastic* if there is a positive $(\mathcal{F}_t)$-adapted process (γ_t) (the hazard rate process) such that for all $t \geq 0$

$$\mathbb{P}(\tau > t \,|\, \mathcal{F}^{\infty}) = \exp\Big(-\int_0^t \gamma_s \,\mathrm{d}s\Big). \tag{91}$$

Comments.

- Here $\mathcal{F}_\infty = \sigma(\bigcup_{t\geq 0} \mathcal{F}_t)$. Conditioning on $\mathcal{F}_\infty$ thus means that we know the past and future economic environment and in particular the entire trajectory $(\gamma_s(\omega))_{s\geq 0}$ of the hazard rate.
- Relation (91) implies that, given the economic environment $\mathcal{F}^{\infty}$, τ is a random time with deterministic hazard function $s \mapsto \gamma_s(\omega)$.
- In the literature doubly stochastic random times are also known as conditional Poisson or Cox random times.

Sampling doubly stochastic random times

A simple algorithm is based on the following result

Lemma 10.6

Let E be a standard exponentially distributed rv independent of $\mathcal{F}^\infty$, that is $\mathbb{P}(\tau > t \mid \mathcal{F}^\infty) = e^{-t}$. Let (γ_t) be a positive $\mathcal{F}_t$-adapted process with $\int_0^t \gamma_s \,\mathrm{d}s) < \infty$ for all t. Define τ by

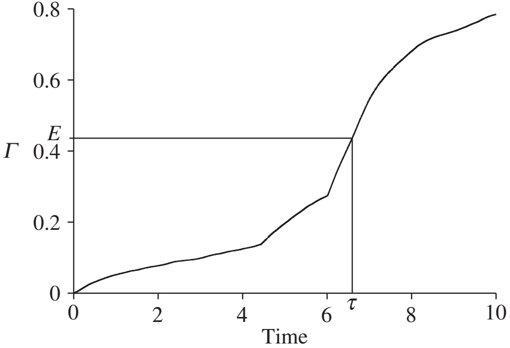
$$\tau := \inf\Big\{t \geq 0 : \int_0^t \gamma_s \,\mathrm{d}s \geq E\Big\}. \tag{92}$$

Then τ is doubly stochastic with hazard-rate process (γ_t).

Algorithm. (threshold simulation)

1) Generate $E \sim \mathrm{Exp}(1)$.
2) Generate a trajectory $(\gamma_s)_{s=0}^\infty$ of hazard rate process.
3) Return $\tau := \inf\{t \geq 0 : \int_0^t \gamma_s \,\mathrm{d}s \geq E\}$

Graphical illustration.



A graphical illustration of threshold simulation; $E \approx 0.44$, $\tau \approx 6.59$.

Intensities

Definition 10.7

Consider a filtration $(\mathcal{G}_t)$ and a random time τ with $(\mathcal{G}_t)$-adapted jump indicator process (Y_t). A non-negative $(\mathcal{G}_t)$-adapted process (λ_t) is called $(\mathcal{G}_t)$-(default) intensity of the random time τ if $M_t := Y_t - \int_0^{t\wedge\tau} \lambda_s \,\mathrm{d}s$ is a $(\mathcal{G}_t)$-martingale.

The next result extends Proposition 10.4 to doubly stochastic τ.

Proposition 10.8

Let τ be a doubly stochastic random time with $(\mathcal{F}_t)$-conditional hazard rate process (γ_t). Then $M_t := Y_t - \int_0^{t\wedge\tau} \gamma_s \,\mathrm{d}s$ is a $(\mathcal{G}_t)$-martingale, that is the hazard rate γ_t is the $(\mathcal{G}_t)$ default intensity.

Conditional expectations

Conditional expectations wrt $\mathcal{G}_t$ are crucial for pricing formulas.

Proposition 10.9 (Dellacherie formula)

Let τ be an arbitrary random time (not necessarily doubly stochastic) such that $\mathbb{P}(\tau > t \mid \mathcal{F}_t) > 0$ for all $t \geq 0$. Then we have for every integrable rv X that

$$\mathbb{E}(I_{\{\tau>t\}} X \mid \mathcal{G}_t) = I_{\{\tau>t\}} \frac{\mathbb{E}(I_{\{\tau>t\}} X \mid \mathcal{F}_t)}{\mathbb{P}(\tau > t \mid \mathcal{F}_t)}.$$

Corollary 10.10

Let $T > t$ and assume that τ is doubly-stochastic with hazard rate process (γ_t). If $\tilde{X}$ is integrable and $\mathcal{F}_T$-measurable, we have $\mathbb{E}(I_{\{\tau>T\}} \tilde{X} \mid \mathcal{G}_t) = I_{\{\tau>t\}} \mathbb{E}(e^{-\int_t^T \gamma_s \mathrm{d}s} \tilde{X} \mid \mathcal{F}_t)$.

Application: 1-year default probabilities

γ_t gives good approximation to the one-year default probability: Let $T = t+1$ and $\tilde{X} = 1$ to obtain

$$\mathbb{P}(\tau > t+1 \mid \mathcal{G}_t) = I_{\{\tau > t\}} \mathbb{E}\left(\exp\left(-\int_t^{t+1} \gamma_s \, \mathrm{d}s \right) \,\middle|\, \mathcal{F}_t \right). \tag{93}$$

For $\tau > t$ and a fairly stable hazard rate over the time interval $[t, t+1]$ the right-hand side of (93) is $\approx \exp(-\gamma_t)$ and for γ_t small,

$$\mathbb{P}(\tau \leq t+1 \mid \mathcal{G}_t) \approx 1 - \exp(-\gamma_t) \approx \gamma_t. \tag{94}$$

10.5.2 Pricing formulas

Setup.

- Consider arbitrage-free security market model on $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{Q})$ where $\mathbb{Q}$ is equivalent martingale measure. Prices of default-free securities $(\mathcal{F}_t)$-adapted; $B_t = \exp(\int_0^t r_s \, \mathrm{d}s)$ models default-free savings account.
- Let τ be the default time of some company. As before we set $\mathcal{H}_t = \sigma(\{Y_s : s \leq t\})$ and $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$; this is the information available to investors at time t.
- We use martingale-modelling. Hence price at t of an $\mathcal{G}_T$-measurable contingent claim H is given by

$$H_t = \mathbb{E}^{\mathbb{Q}}\Big(\exp\Big(-\int_t^T r_s \, \mathrm{d}s\Big) H \,\Big|\, \mathcal{G}_t\Big). \tag{95}$$

- Under $\mathbb{Q}$, τ is a doubly stochastic random time with background filtration $(\mathcal{F}_t)$ and hazard rate process (γ_t).

Key building blocks

The pricing of bonds and CDSs can be reduced to the pricing of the following building blocks:

- A survival claim, i.e. a promised $\mathcal{F}_T$-measurable payment X which is made at time T if there is no default; the actual payment of the survival claim equals $XI_{\{\tau>T\}}$.
- A payment-at-default claim of the form $Z_\tau I_{\{\tau\leq T\}}$, where $Z=(Z_t)_{t\geq 0}$ is an $(\mathcal{F}_t)$ adapted stochastic process and where Z_τ is short for $Z_{\tau(\omega)}(\omega)$. Note that the payment is made directly at τ, provided that $\tau \leq T$ where T is the maturity date of the claim.

Example. Defaultable bond is a combination of a survival claim and a payment at default claim (the recovery payment).

Pricing the building blocks

Next result shows that pricing of building blocks can be reduced to pricing problem for default-free claims with adjusted interest rate.

Theorem 10.11

Define adjusted interest rate $R_t = r_t + \gamma_t$. Under the above assumptions (in particular for τ doubly stochastic) it holds that

$$\mathbb{E}^{\mathbb{Q}}\Big(\exp\Big(-\int_t^T r_s\,\mathrm{d}s\Big) I_{\{\tau>T\}} X \,\Big|\, \mathcal{G}_t\Big) = I_{\{\tau>t\}} \mathbb{E}^{\mathbb{Q}}\Big(\exp\Big(-\int_t^T R_s\,\mathrm{d}s\Big) X \,\Big|\, \mathcal{F}_t\Big),$$

$$\begin{aligned} &\mathbb{E}^{\mathbb{Q}}\Big(I_{\{t<\tau\leq T\}} \exp\Big(-\int_t^\tau r_s\,\mathrm{d}s\Big) Z_\tau \,\Big|\, \mathcal{G}_t\Big) \\ &= I_{\{\tau>t\}} \mathbb{E}^{\mathbb{Q}}\Big(\int_t^T Z_s \gamma_s \exp\Big(-\int_t^s R_u\,\mathrm{d}u\Big)\,\mathrm{d}s \,\Big|\, \mathcal{F}_t\Big). \end{aligned}$$

10.5.3 Applications

Corporate bonds and RF. The price at t of a defaultable zero-coupon bond with maturity $T \geq t$ is $p_1(t, T)$; price of corresponding default-free bond is $p_0(t, T)$.

Recall the following recovery models

1) Recovery of Treasury (RT). Under RT, if default occurs at $\tau \leq T$, the bond holder receives $(1 - \delta_\tau)$ units of $p_0(\cdot\,, T)$ at time τ, where $\delta \in [0, 1]$ models the percentage loss given default (LGD). Under RT the holder of the defaultable bond therefore receives the payment

$$I_{\{\tau > T\}} + (1 - \delta) I_{\{\tau \leq T\}} = 1 - \delta + \delta I_{\{\tau > T\}}.$$

2) Recovery of Face Value (RF). Under RF, if default occurs at $\tau \leq T$, the bondholder receives $(1 - \delta_\tau)$ immediately at τ. $\Rightarrow$ Value of the recovery payment depends on the timing of default.

3) Recovery of market value (RM). Duffie and Singleton (1999) Under RM recovery payment equals $(1-\delta_\tau)V_\tau I_{\{\tau \le T\}}$, where $(\delta_t) \in (0,1)$ gives the percentage LGDI and where the $(\mathcal{F}_t)$-adapted process (V_t) gives the pre-default value of the claim. This is a recursive definition (but explicit solution exists)

Application to corporate bonds.

1) Under RT the bond price in $t < T$ is

$$p_1(t,T) = (1-\delta)p_0(t,T) + I_{\{\tau > t\}}\delta\mathbb{E}^{\mathbb{Q}}\Big(\exp\Big(-\int_t^T R_s\,\mathrm{d}s\Big)\,\Big|\,\mathcal{F}_t\Big)$$

2) Under RF the bond is the sum of a survival claim and a payment at

default claim. One has

$$p_1(t,T) = I_{\{\tau>t\}}\Big(\mathbb{E}^{\mathbb{Q}}\Big(\exp\Big(-\int_t^T R_s\,\mathrm{d}s\Big) \mid \mathcal{F}_t\Big) \\ + (1-\delta)\mathbb{E}^{\mathbb{Q}}\Big(\int_t^T \gamma_s \exp\Big(-\int_t^s R_u\,\mathrm{d}u\Big)\,\mathrm{d}s \mid \mathcal{F}_t\Big)\Big)$$

3) Under RM assumption one has

Proposition 10.12

Suppose that, under $\mathbb{Q}$, τ is doubly stochastic with hazard rate process (γ_t). Then under RM the pre-default value (V_t) of a corporate bond is uniquely determined and given by

$$V_t = \mathbb{E}^{\mathbb{Q}}\Big(\exp\Big(-\int_t^T (r_s+\delta_s\gamma_s)\,\mathrm{d}s\Big)\,\Big|\,\mathcal{F}_t\Big), \quad 0 \le t \le T. \qquad (96)$$

Special cases: $\delta = 1 \Rightarrow$ standard survival claim; $\delta = 0 \Rightarrow$ default-free.

Credit spreads

With doubly stochastic default times hazard rate process (γ_t) and credit spread $c(t,T) = -\frac{1}{T-t}(\ln p_1(t,T) - \ln p_0(t,T))$ of defaultable bonds are closely related. Analytic results for the instantaneous credit spread

$$c(t,t) = \lim_{T \to t} c(t,T) = -\frac{\partial}{\partial T}\Big|_{T=t}(\ln p_1(t,T) - \ln p_0(t,T)). \qquad (97)$$

Proposition 10.13

In all recovery models $c(t,t) = \delta \gamma_t^{\mathbb{Q}}$.

- Instantaneous credit spreads are product of LGD and instantaneous default probability.
- In hazard-rate models short term spreads strictly positive.
- For $T > t$ spreads in the three models differ.

CDS contracts

Here premium payments constitute a sequence of survival claims; default payment is a payment-at-default claim. This gives the following formula for the fair CDS spread x^* at t:

$$x^* = \frac{\delta \mathbb{E}^{\mathbb{Q}}\big(\int_t^T \gamma_s e^{-\int_t^s R_u \mathrm{d}u} \mathrm{d}s \,|\, \mathcal{F}_t\big)}{\sum_{T_k > t} (t_k - t_{k-1}) \mathbb{E}^{\mathbb{Q}}\big(\exp\big(-\int_t^{t_k} R_s \mathrm{d}s \,|\, \mathcal{F}_t\big)\big)}. \tag{98}$$

- For $T \to t$ we get that x^* converges to $\delta\gamma_t$.
- The formula is a generalization of (89).

10.6 Affine models

In most models with doubly stochastic default time used in practice it is assumed that (r_t) and (γ_t) are functions of some Markov process (Ψ_t) on $D \subset \mathbb{R}^p$.

- Natural background filtration is $(\mathcal{F}_t) = \sigma(\{\Psi_s : s \leq t\})$.
- $R_t := r_t + \gamma_t$ is of the form $R_t = R(\Psi_t)$ for some $R : D \subseteq \mathbb{R}^p \to \mathbb{R}_+$.

To evaluate general pricing formulas we hence have to compute conditional expectations of the form

$$\mathbb{E}\Big(e^{-\int_t^T R(\Psi_s)\,\mathrm{d}s} g(\Psi_T) + \int_t^T h(\Psi_s) e^{-\int_t^s R(\Psi_u)\,\mathrm{d}u}\,\mathrm{d}s \mid \mathcal{F}_t\Big) \tag{99}$$

for generic $g, h : D \to \mathsf{R}_+$. Since (Ψ_t) is Markov, (99) is a function $f(t, \Psi_t)$ of time and of Ψ_t. f can sometimes be computed by solving a PDE; this is the well-known Feynman-Kac formula.

Theorem 10.14 (Feynman–Kac)
Consider generic $R, g\colon D \to \mathsf{R}_+$. Suppose that $f : [0,T] \times D \to \mathsf{R}$ is bounded, continuous and solves the terminal-value problem

$$f_t + \mu(\psi) f_\psi + \frac{1}{2}\sigma^2(\psi) f_{\psi\psi} = R(\psi) f,\ (t,\psi) \in [0,T) \times D, \qquad (100)$$

with $f(T,\psi) = g(\psi),\ \psi \in D$. Suppose that (Ψ_t) is the unique solution of the SDE

$$d\Psi_t = \mu(\Psi_t)\,\mathrm{d}t + \sigma(\Psi_t)\,\mathrm{d}W_t, \quad \Psi_0 = \psi \in D, \qquad (101)$$

with state space $D \subseteq \mathsf{R}$, (W_t) a standard, Brownian motion and μ and σ continuous functions from D to R resp. R_+. Then

$$E\left(e^{-\int_t^T R(\Psi_s)\mathrm{d}s} g(\Psi_T) \mid \mathcal{F}_t\right) = f(t,\Psi_t)\,. \qquad (102)$$

Comments

The Feynman Kac formula can be used in two ways:

- We can use probabilistic techniques or Monte-Carlo simulation to compute the (conditional) expectation (102) in order to solve numerically the PDE (100).
- We can solve the PDE (100) perhaps numerically, in order to compute the expectation (102).

For an extension to the d-dimensional case (and weaker regularity conditions on f) we refer to the literature such as Karatzas and Shreve (1988)

Affine term structure

Consider a model where r and γ are functions of a diffusion Ψ. Define a function f by

$$f(t, \Psi_t) = \mathbb{E}\Big(e^{-\int_t^T R(\Psi_s)\mathrm{d}s} e^{u\Psi_T} \;\Big|\; \mathcal{F}_t\Big)$$

where u, D are such $u\psi \leq 0$ for all $\psi \in D$. Note that for $u = 0$ we have a bond price with zero recovery.

Definition 10.15

The model has an *affine* (defaultable) term structure if

$$f(t, \psi) = \exp(\alpha(t, T) + \beta(t, T)\psi) \tag{103}$$

for deterministic functions $\alpha(\cdot, T)$ and $\beta(\cdot, T)$.

The following assumption guarantees an affine term structure.

Assumption 10.16 (affine term structure)
R, μ and σ^2 are *affine functions* of ψ, i.e. there are constants ρ^0, ρ^1, k^0, k^1, h^0 and h^1 such that

$$R(\psi) = \rho^0 + \rho^1\psi,\ \mu(\psi) = k^0 + k^1\psi,\ \sigma^2(\psi) = h^0 + h^1\psi.$$

Moreover, for all $\psi \in D$ we have $h^0 + h^1\psi \geq 0$ and $\rho_0 + \rho_1\psi \geq 0$.

An ODE system for α and β

The educated guess $f(t,\psi) = \exp(\alpha(t,T) + \beta(t,T)\psi)$ gives

$$f_t = (\alpha' + \beta'\psi)f, \quad f_\psi = \beta f \text{ and } f_{\psi\psi} = \beta^2 f.$$

Substituting this in the PDE $f_t + \mu(\psi) f_\psi + \frac{1}{2}\sigma^2(\psi) f_{\psi\psi}) = R(\psi) f$, using the special form of μ, σ^2 and rearranging terms gives the following ODE system

$$\beta'(t,T) = \rho^1 - k^1 \beta(t,T) - \tfrac{1}{2} h^1 \beta^2(t,T), \quad \beta(T,T) = u, \tag{104}$$

$$\alpha'(t,T) = \rho^0 - k^0 \beta(t,T) - \tfrac{1}{2} h^0 \beta^2(t,T), \quad \alpha(T,T) = 0. \tag{105}$$

Comments.

- The ODE (104) for $\beta(\cdot\,, T)$ is a so-called Ricatti equation.
- The ODE (105) for $\alpha(\cdot\,, T)$ can be solved by (numerical) integration once β has been determined.

Summary. Suppose that the affine-term-structure assumption holds, that the ODE system (104), (105) has a unique solution (α, β) on $[0, T]$ and that there is some C such that $\beta(t, T)\psi \leq C$ for all $t \in [0, T]$, $\psi \in D$. Then the model has an affine term structure.

10.6.1 The CIR square-root diffusion

The CIR or square-root diffusion model due to Cox et al. (1985) is a popular affine model.

CIR dynamics.

$$d\Psi_t = \kappa(\bar{\theta} - \Psi_t)\,dt + \sigma\sqrt{\Psi_t}\,dW_t, \quad \Psi_0 = \psi > 0, \tag{106}$$

for parameters $\kappa, \bar{\theta}, \sigma > 0$ and state space $D = [0, \infty)$.

Properties.

- (106) is an affine model; the parameters are given by $k^0 = \kappa\bar{\theta}$, $k^1 = -\kappa$, $h^0 = 0$ and $h^1 = \sigma^2$.
- The SDE (106) admits a global solution (non-trivial)
- (106) implies that (Ψ_t) is *mean reverting*.
- Mean reversion sufficiently strong $\Rightarrow$ trajectories never reach zero: Let $\tau_0(\Psi) := \inf\{t \geq 0 : \Psi_t = 0\}$. For $\kappa\bar{\theta} \geq \frac{1}{2}\sigma^2$, $\mathbb{P}(\tau_0(\Psi) < \infty) = 0$; for $\kappa\bar{\theta} < \frac{1}{2}\sigma^2$, $\mathbb{P}(\tau_0(\Psi) < \infty) = 1$.

CIR term structure

Theorem 10.17

Suppose that the factor Ψ follows the CIR model and that adjusted interest rate is an affine function of the state, $R(\psi) = \rho^0 + \rho^1\psi$. Then it holds that

$$\mathbb{E}\Big(\exp\Big(-\int_t^T (\rho^0 + \rho^1\Psi_s)\,\mathrm{d}s\Big)\,\Big|\,\Psi_t\Big) = \exp(\alpha(T-t) + \beta(T-t)\Psi_t),$$

where

$$\beta(\tau) = \frac{-2\rho^1(e^{\gamma\tau}-1)}{\gamma-\kappa+e^{\gamma\tau}(\gamma+\kappa)},$$
$$\alpha(\tau) = -\rho^0\tau + 2\frac{\kappa\bar{\theta}}{\sigma^2}\ln\left(\frac{2\gamma e^{\tau(\gamma+\kappa)/2}}{\gamma-\kappa+e^{\gamma\tau}(\gamma+\kappa)}\right),$$

and $\tau := T-t$, $\gamma := \sqrt{\kappa^2+2\sigma^2\rho^1}$

10.6.2 Extensions

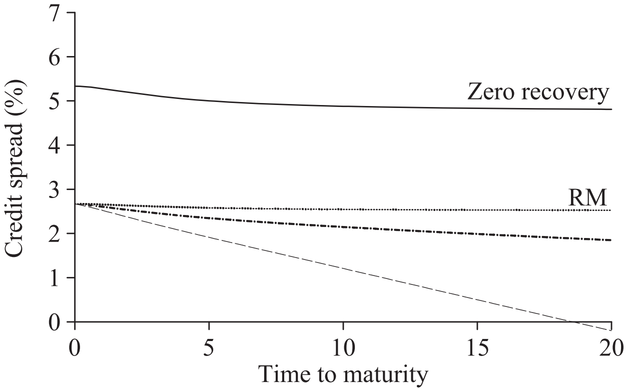
- It is possible to extend the above to CIR models with jumps,

$$d\Psi_t = \kappa(\bar{\theta} - \Psi_t)\,dt + \sigma\sqrt{\Psi_t}\,dW_t + dZ_t$$

  where $Z_t = \sum_{T_n \le t} Z_n$ is a compound Poisson process with jump intensity $\lambda_t = \lambda_0 + \lambda_1 \Psi_t$, λ_0 and $\lambda_1 > 0$ and the Z_n are iid positive rvs, for instance exponentially distributed.

- The computation of payment-at-default claims is also possible with "affine model technology".

Numerical example



Spreads of defaultable zero-coupon bonds in an affine model for various recovery assumptions. It holds $\Psi_0 \approx 0.0533$, $r = 6\%$ and $\delta = 0.5$. Note that under the RF recovery model (dashed line) the spread becomes negative for large times to maturity; this is not true under other recovery assumptions.