

9 Market Risk

9.1 Risk factors and mapping

9.2 Market risk measurement

9.3 Backtesting

9.1 Risk factors and mapping

9.1.1 The loss operator

- The key idea in this section is that of a **loss operator** for expressing the change in value of a portfolio in terms of **risk-factor changes**.
- Let the current time be t and assume the current value V_t of an asset portfolio is known, or can be computed with appropriate valuation models.
- We are interested in value changes or losses over a relatively **short time period** $[t, t + 1]$, for example one day, two weeks or month.
- Scaling may be applied to derive capital requirements for longer periods.
- We assume there is **no change to the composition of the portfolio** over the time period.
- The future value V_{t+1} is modelled as a random variable.

- We want to determine the distribution of the loss distribution of $L_{t+1} = -(V_{t+1} - V_t)$.
- We map the value at time t using the formula

$$V_t = g(\tau_t, \mathbf{Z}_t)$$

where τ_t is time t expressed in units of **valuation time**.

The issue of time

- We will be quite precise about the modelling of time.
- The natural time unit for valuation of positions might be yearly; e.g. in Black-Scholes valuation, the volatility is expressed in annualized terms.
- On the other hand the **risk modelling time horizon** $[t, t + 1]$ is typically shorter.
- Let Δt be the length of the time horizon in **valuation time**.

- For example, suppose that valuation time is yearly. Then a monthly time horizon would be $\Delta t = 1/12$ and a trading day $\Delta t = 1/250$.
- We set $\tau_t = t(\Delta t)$ for all t so that $\tau_{t+1} - \tau_t = \Delta t$.

From the mapping to the loss operator

- The risk factor changes over the time horizon are

$$\mathbf{X}_{t+1} = \mathbf{Z}_{t+1} - \mathbf{Z}_t.$$

- Typically, historical risk factor data are available as a time series $\mathbf{X}_{t-n}, \dots, \mathbf{X}_{t-1}, \mathbf{X}_t$ and these are used to model the behaviour of \mathbf{X}_{t+1} .
- We have

$$\begin{aligned} L_{t+1} &= -(V_{t+1} - V_t) \\ &= -(g(\tau_{t+1}, \mathbf{Z}_{t+1}) - g(\tau_t, \mathbf{Z}_t)) \\ &= -(g(\tau_t + \Delta t, \mathbf{Z}_t + \mathbf{X}_{t+1}) - g(\tau_t, \mathbf{Z}_t)). \end{aligned} \quad (50)$$

- Since the risk factor values \mathbf{Z}_t are known at time t , the loss L_{t+1} is determined by the risk factor changes \mathbf{X}_{t+1} .
- Given a realization \mathbf{z}_t of \mathbf{Z}_t , the **loss operator** at time t is defined to be

$$l_{[t]}(\mathbf{x}) = -(g(\tau_t + \Delta t, \mathbf{z}_t + \mathbf{x}) - g(\tau_t, \mathbf{z}_t)), \quad (51)$$

so that

$$L_{t+1} = l_{[t]}(\mathbf{X}_{t+1}).$$

- The loss operator embodies the idea of **full revaluation**.
- From the perspective of time t the loss distribution of L_{t+1} is determined by the multivariate distribution of \mathbf{X}_{t+1} .

9.1.2 Delta and delta–gamma approximations

- If the mapping function **g is differentiable and Δt is relatively small** we can approximate g with a first-order Taylor series approximation

$$g(\tau_t + \Delta t, \mathbf{z}_t + \mathbf{x}) \approx g(\tau_t, \mathbf{z}_t) + g_\tau(\tau_t, \mathbf{z}_t)\Delta t + \sum_{i=1}^d g_{z_i}(\tau_t, \mathbf{z}_t)x_i, \quad (52)$$

where the τ -subscript and z_i -subscript denote partial derivatives with respect to (valuation) time and the risk factors respectively.

- This allows us to approximate the loss operator in (51) by the **linear loss operator** at time t given by

$$l_{[t]}^{\Delta}(\mathbf{x}) := -\left(g_{\tau}(\tau_t, \mathbf{z}_t)\Delta t + \sum_{i=1}^d g_{z_i}(\tau_t, \mathbf{z}_t)x_i\right). \quad (53)$$

- Note that, when working with a short time horizon Δt , the term $g_{\tau}(\tau_t, \mathbf{z}_t)\Delta t$ is typically small and is sometimes omitted in practice.

Example 9.1 (European call option)

- Consider portfolio consisting of one standard European call on a non-dividend paying stock S with maturity T and exercise price K .
- The Black-Scholes value of this asset at time t is $C^{BS}(t, S_t, r, \sigma)$ where

$$C^{BS}(t, S; r, \sigma) = S\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2),$$

Φ is standard normal df, r represents risk-free interest rate, σ the volatility of underlying stock, and where

$$d_1 = \frac{\log(S/K) + (r + \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}} \text{ and } d_2 = d_1 - \sigma\sqrt{T - t}.$$

- While in the BS model, it is assumed that interest rates and volatilities are constant, in reality they tend to fluctuate over time; they should be added to our set of risk factors.

- The risk factors: $\mathbf{Z}_t = (\log S_t, r_t, \sigma_t)'$.
- The risk factor changes: $\mathbf{X}_t = (\log(S_t/S_{t-1}), r_t - r_{t-1}, \sigma_t - \sigma_{t-1})'$.
- The mapping:

$$V_t = C^{BS}(\tau_t, S_t; r_t, \sigma_t) = g(\tau_t, \mathbf{Z}_t)$$

- For derivative positions it is quite common to use the linear loss operator

$$L_{t+1}^\Delta = l_{[t]}^\Delta(\mathbf{X}_{t+1}) = - \left(g_\tau(\tau_t, \mathbf{z}_t) \Delta t + \sum_{i=1}^3 g_{z_i}(\tau_t, \mathbf{z}_t) X_{t+1,i} \right),$$

where g_τ , g_{z_i} denote partial derivatives.

- Δt is the length of the time interval expressed in years since Black-Scholes parameters relate to units of one year.

- It is more common to write the linear loss operator as

$$l_{[t]}^{\Delta}(\mathbf{x}) = - \left(C_t^{BS} + C_S^{BS} S_t x_1 + C_r^{BS} x_2 + C_{\sigma}^{BS} x_3 \right),$$

in terms of the derivatives of the BS formula or [the Greeks](#).

- ▶ C_S^{BS} is known as the [delta](#) of the option.
- ▶ C_{σ}^{BS} is the [vega](#).
- ▶ C_r^{BS} is the [rho](#).
- ▶ C_t^{BS} is the [theta](#).

Note the appearance of S_t in the C_S^{BS} term. This is because the risk factor is $\ln S_t$ rather than S_t and $C_{\ln S}^{BS} = C_S^{BS} S_t$.

Quadratic loss operator

- Recall the first-order Taylor series approximation of mapping in (52).
- Let $\delta(\tau_t, \mathbf{z}_t) = (g_{z_1}(\tau_t, \mathbf{z}_t), \dots, g_{z_d}(\tau_t, \mathbf{z}_t))'$ be the first-order partial derivatives of the mapping with respect to the risk factors.

- Let $\omega(\tau_t, \mathbf{z}_t) = (g_{z_1\tau}(\tau_t, \mathbf{z}_t), \dots, g_{z_d\tau}(\tau_t, \mathbf{z}_t))'$ denote the mixed partial derivatives with respect to time and the risk factors.
- Let $\Gamma(\tau_t, \mathbf{z}_t)$ denote the matrix with (i, j) th element given by $g_{z_i z_j}(\tau_t, \mathbf{z}_t)$; this matrix contains **gamma sensitivities** to individual risk factors on the diagonal and **cross gamma sensitivities** to pairs of risk factors off the diagonal.
- The full second-order approximation of the mapping function is g is

$$g(\tau_t + \Delta t, \mathbf{z}_t + \mathbf{x}) \approx g(\tau_t, \mathbf{z}_t) + g_\tau(\tau_t, \mathbf{z}_t)\Delta t + \boldsymbol{\delta}(\tau_t, \mathbf{z}_t)'\mathbf{x} + \frac{1}{2}(g_{\tau\tau}(\tau_t, \mathbf{z}_t)(\Delta t)^2 + 2\boldsymbol{\omega}(\tau_t, \mathbf{z}_t)'\mathbf{x}\Delta t + \mathbf{x}'\Gamma(\tau_t, \mathbf{z}_t)\mathbf{x}).$$

- In practice, we would usually omit terms of order $o(\Delta_t)$ (terms that tend to zero faster than Δ_t). In standard continuous-time financial models like Black-Scholes the risk-factor changes \mathbf{x} are of order $\sqrt{\Delta_t}$.

- This leaves us with the **quadratic loss operator**

$$l_{[t]}^{\Delta\Gamma}(\mathbf{x}) = -(g_{\tau}(\tau_t, \mathbf{z}_t)\Delta t + \boldsymbol{\delta}(\tau_t, \mathbf{z}_t)' \mathbf{x} + \frac{1}{2} \mathbf{x}' \Gamma(\tau_t, \mathbf{z}_t) \mathbf{x}) \quad (54)$$

which is more accurate than the linear loss operator (53).

Example 9.2 (European call option)

The quadratic loss operator is

$$\begin{aligned} l_{[t]}^{\Delta\Gamma}(\mathbf{x}) = & l_{[t]}^{\Delta}(\mathbf{x}) - 0.5 \left(C_{SS}^{BS} S_t^2 x_1^2 + C_{rr}^{BS} x_2^2 + C_{\sigma\sigma}^{BS} x_3^2 \right) \\ & - \left(C_{Sr}^{BS} S_t x_1 x_2 + C_{S\sigma}^{BS} S_t x_1 x_3 + C_{r\sigma}^{BS} x_2 x_3 \right). \end{aligned}$$

The names of the second-order Greeks (with the exception of gamma) are rather obscure. Here are some of them:

- C_{SS}^{BS} is known as the **gamma** of the option;
- $C_{\sigma\sigma}^{BS}$ is the **vomma**;
- $C_{S\sigma}^{BS}$ is the **vanna**.

9.1.3 Mapping bond portfolios

Basic definitions for bond pricing

- Let $p(t, T)$ denote the price at time t of a default-free zero-coupon bond paying one at time T (also called a discount factor).
- Time is measured in years.
- Many other fixed-income instruments such as coupon bonds or standard swaps can be viewed as portfolios of zero-coupon bonds.
- The mapping $T \rightarrow p(t, T)$ for different maturities is one way of describing the so-called term structure of interest rates at time t . An alternative description is based on yields.
- The term structure $T \rightarrow p(t, T)$ is known at time t .
- However the future term structure $T \rightarrow p(t + x, T)$ for $x > 0$ is not known at time t and must be modelled stochastically.

- The **continuously compounded yield** of a zero-coupon bond is

$$y(t, T) = -\frac{\ln p(t, T)}{T - t}. \quad (55)$$

- We have the relation

$$p(t, T) = \exp(-(T - t)y(t, T)).$$

- The yield is the constant, annualized rate implied by the price $p(t, T)$. Also known as spot rate.
- The mapping $T \rightarrow y(t, T)$ is referred to as the continuously compounded **yield curve** at time t .
- Yields are comparable across different times to maturity.

Detailed mapping of a bond portfolio

- Consider a portfolio of d default-free zero-coupon bonds with maturities T_i and prices $p(t, T_i)$ for $i = 1, \dots, d$. Assume $p(T_i, T_i) = 1$ for all i .

- By λ_i we denote the number of bonds with maturity T_i in the portfolio.
- The **portfolio value** at time t is given by

$$V(t) := \sum_{i=1}^d \lambda_i p(t, T_i) = \sum_{i=1}^d \lambda_i \exp(-(T_i - t)y(t, T_i)).$$

- In a detailed analysis of the change in value one takes all yields $y(t, T_i)$, $1 \leq i \leq d$, as risk factors.
- We want to put this in the general discrete-time framework of the mapping

$$V_t = g(\tau_t, \mathbf{Z}_t).$$

- We set

$$\tau_t = t(\Delta t), \quad V_t = V(\tau_t), \quad Z_{t,i} = y(\tau_t, T_i)$$

where Δt is risk management time horizon in years.

- We obtain a mapping of the form

$$V_t = V(\tau_t) = g(\tau_t, \mathbf{Z}_t) = \sum_{i=1}^d \lambda_i \exp(-(T_i - \tau_t)Z_{t,i}). \quad (56)$$

The loss operator and its approximations

- The portfolio loss is

$$\begin{aligned} L_{t+1} &= -(V_{t+1} - V_t) \\ &= -\sum_{i=1}^d \lambda_i e^{-(T_i - \tau_t)Z_{t,i}} \left(\exp(Z_{t,i}\Delta t - (T_i - \tau_{t+1})X_{t+1,i}) - 1 \right). \end{aligned}$$

- Reverting to standard bond pricing notation the loss operator is

$$l_{[t]}(\mathbf{x}) = -\sum_{i=1}^d \lambda_i p(\tau_t, T_i) \left(\exp(y(\tau_t, T_i)\Delta t - (T_i - \tau_{t+1})x_i) - 1 \right),$$

where x_i represents the change in yield of the i th bond.

- The first derivatives of the mapping function (56) are

$$g_{\tau}(\tau_t, \mathbf{z}_t) = \sum_{i=1}^d \lambda_i p(\tau_t, T_i) z_{t,i}$$

$$g_{z_i}(\tau_t, \mathbf{z}_t) = -\lambda_i (T_i - \tau_t) \exp(-(T_i - \tau_t) z_{t,i}).$$

- Inserting these in (53) and reverting to standard bond pricing notation we obtain

$$l_{[t]}^{\Delta}(\mathbf{x}) = - \sum_{i=1}^d \lambda_i p(\tau_t, T_i) \left(y(\tau_t, T_i) \Delta t - (T_i - \tau_t) x_i \right), \quad (57)$$

- For the second-order approximation we need the second derivatives with respect to yields which are

$$g_{z_i z_i}(\tau_t, \mathbf{z}_t) = \lambda_i (T_i - \tau_t)^2 \exp(-(T_i - \tau_t) z_{t,i})$$

and $g_{z_i z_j}(\tau_t, \mathbf{z}_t) = 0$ for $i \neq j$.

- The quadratic loss operator (54) is

$$l_{[t]}^{\Delta\Gamma}(\mathbf{x}) = - \sum_{i=1}^d \lambda_i p(\tau_t, T_i) \left(y(\tau_t, T_i) \Delta t - (T_i - \tau_t) x_i + \frac{1}{2} (T_i - \tau_t)^2 x_i^2 \right). \quad (58)$$

Relationship of linear operator to duration

- Consider a very simple model for the yield curve at time t in which

$$y(\tau_{t+1}, T_i) = y(\tau_t, T_i) + x$$

for all maturities T_i .

- In our mapping notation

$$Z_{t+1,i} = Z_{t,i} + X_{t+1}, \quad \forall i.$$

- In this model we assume that a **parallel shift in level** takes place along the entire yield curve.

- This is **unrealistic but frequently assumed in practice**.
- In this model the loss operator and its linear and quadratic approximations are functions of a scalar variable x , the change in level.
- Under the parallel shift model we can write

$$l_{[t]}^{\Delta}(x) = -V_t \left(A_t \Delta t - D_t x \right), \quad (59)$$

where

$$D_t := \sum_{i=1}^d \frac{\lambda_i p(\tau_t, T_i)}{v_t} (T_i - \tau_t), \quad A_t := \sum_{i=1}^d \frac{\lambda_i p(\tau_t, T_i)}{V_t} y(\tau_t, T_i).$$

- D_t is usually called the (Macaulay) **duration** of the bond portfolio.
- It is a weighted sum of the times to maturity of the different cash flows in the portfolio, the weights being proportional to the discounted values of the cash flows.

Interpreting duration

- Over short time intervals losses of value in the bond portfolio will be determined by $l_{[t]}^{\Delta}(x) \approx V_t D_t x$.
- **Increases** in level of yields lead to **losses**; **decreases** lead to **gains**.
- The duration D_t is the bond pricing **analogue of the delta of an option**.
- Any two bond portfolios with equal value and duration will be subject to similar losses when there is a small parallel shift of the yield curve.
- Duration is an important tool in traditional bond-portfolio or asset-liability management.
- An asset manager, who invests in various bonds to cover promised cash flows in the future, invests in such a way that the duration of the overall portfolio of bonds and liability cash flows is equal to zero.
- Portfolios are **immunized** against small parallel shifts in yield curve, but not changes of slope and curvature.

Relationship of quadratic operator to convexity

- It is possible to get more accurate approximations for the loss in a bond portfolio by considering second-order effects.
- The **analogue of the gamma** of an option is **convexity**. Under the parallel shift model, the quadratic loss operator (58) becomes

$$l_{[t]}^{\Delta\Gamma}(x) = -V_t \left(A_t \Delta t - D_t x + \frac{1}{2} C_t x^2 \right), \quad (60)$$

where

$$C_t := \sum_{i=1}^d \frac{\lambda_i p(\tau_t, T_i)}{V_t} (T_i - \tau_t)^2$$

is the convexity of the bond portfolio.

- The convexity is a weighted average of the squared times to maturity and is (minus) the derivative of the duration with respect to yield.

Interpreting convexity

- Consider two portfolios (1) and (2) with identical durations $D_t^{(1)} = D_t^{(2)}$ but differing convexities satisfying $C_t^{(1)} > C_t^{(2)}$.
- Ignoring terms in Δt , the difference in loss operators satisfies

$$l_{[t]}^{\Delta\Gamma,1}(x) - l_{[t]}^{\Delta\Gamma,2}(x) \approx -\frac{1}{2}V_t(C_t^{(1)} - C_t^{(2)})x^2 < 0.$$

- Since $l_{[t]}^{\Delta\Gamma,1}(x) < l_{[t]}^{\Delta\Gamma,2}(x)$ an increase in the level of yields ($x > 0$) will lead to smaller losses for portfolio (1)
- Since $-l_{[t]}^{\Delta\Gamma,1}(x) > -l_{[t]}^{\Delta\Gamma,2}(x)$ a decrease in the level of yields ($x < 0$) will lead to larger gains.
- For this reason higher convexity is considered a desirable attribute of a bond portfolio in risk management.

9.1.4 Factor models for bond portfolios

The need for factor models

- The parallel shift model is unrealistic in practice.
- For large portfolios of fixed-income instruments, such as the overall fixed-income position of a major bank, modelling changes in the yield for every cash flow maturity date becomes impractical.
- Moreover, the statistical task of estimating a distribution for X_{t+1} is difficult because the **yields are highly dependent for different times to maturity**.
- A pragmatic approach is therefore to build a factor model for yields that captures the main features of the yield curve.
- Three-factor models of the yield curve in which the factors typically represent **level**, **slope** and **curvature** are often used in practice.

The approach based on the Nelson and Siegel (1987) model

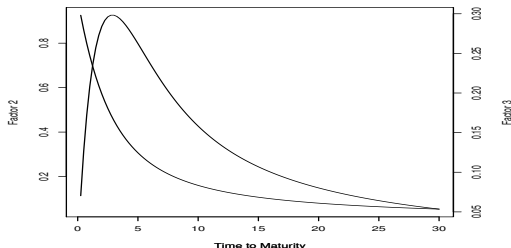
- We assume that at time t the yield curve can be modelled by

$$y(\tau_t, T) \approx Z_{t,1} + k_2(T - \tau_t, \eta_t)Z_{t,2} + k_3(T - \tau_t, \eta_t)Z_{t,3}, \quad (61)$$

where the functions k_2 and k_3 are given by

$$k_2(s, \eta) = \frac{1 - \exp(-\eta s)}{\eta s}, \quad k_3(s, \eta) = k_2(s, \eta) - \exp(-\eta s).$$

- Nelson-Siegel functions $k_2(s, \eta)$ and $k_3(s, \eta)$ for an η value of 0.623:



- η is an extra tuning parameter to improve fit.
- There are other simple factor models including the [Svensson model](#).
- Clearly $\lim_{s \rightarrow \infty} k_2(s, \eta) = \lim_{s \rightarrow \infty} k_3(s, \eta) = 0$ while $\lim_{s \rightarrow 0} k_2(s, \eta) = 1$ and $\lim_{s \rightarrow 0} k_3(s, \eta) = 0$.
- It follows that

$$\lim_{T \rightarrow \infty} y(\tau_t, T) = Z_{t,1},$$

so that the first factor is usually interpreted as a [long-term level factor](#).

- $Z_{t,2}$ is interpreted as a [slope factor](#) because the difference in short-term and long-term yields satisfies

$$\lim_{T \rightarrow \tau_t} y(\tau_t, T) - \lim_{T \rightarrow \infty} y(\tau_t, T) = Z_{t,2}.$$

- $Z_{t,3}$ has an interpretation as a [curvature factor](#).

- Using (61), the portfolio mapping (56) becomes

$$V_t = g(\tau_t, \mathbf{Z}_t) = \sum_{i=1}^d \lambda_i \exp \left(- (T_i - \tau_t) \sum_{j=1}^3 k_j (T_i - \tau_t, \eta_t) Z_{t,j} \right),$$

where $k_1(s, \eta) = 1$.

- It is then straightforward to derive the loss operator $l_{[t]}(\mathbf{x})$ or its linear version $l_{[t]}^{\Delta}(\mathbf{x})$ which are functions on \mathbb{R}^3 rather than \mathbb{R}^d .
- To use this method to evaluate the loss operator at time t we require realized values \mathbf{z}_t for the risk factors \mathbf{Z}_t . We have to overcome the fact that **the Nelson-Siegel factors \mathbf{Z}_t are not directly observed** at time t . Instead **they have to be estimated** from observable yield curve data.
- Let $\mathbf{Y}_t = (y(\tau_t, \tau_t + s_1), \dots, y(\tau_t, \tau_t + s_m))'$ denote the data vector at time t , containing the yields for m different times to maturity, s_1, \dots, s_m , where m is large.

- This is assumed to follow the factor model

$$\mathbf{Y}_t = B_t \mathbf{Z}_t + \boldsymbol{\varepsilon}_t,$$

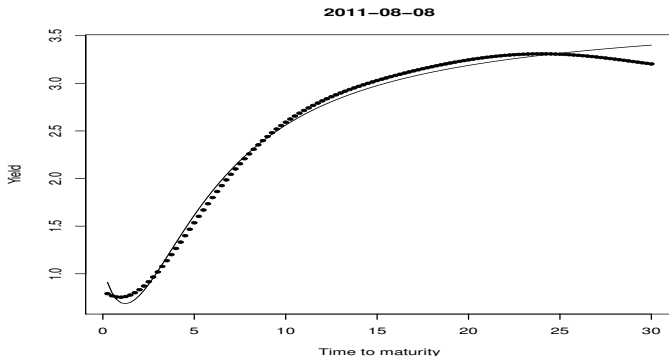
where $B_t \in \mathbb{R}^{m \times 3}$ is the matrix with i th row $(1, k_2(s_i, \eta_t), k_3(s_i, \eta_t))$ and $\boldsymbol{\varepsilon}_t \in \mathbb{R}^m$ is an error vector.

- For a given value of η_t the estimation of \mathbf{Z}_t can be carried out as a cross-sectional regression using weighted least squares. It is a **fundamental** factor model where the loading matrix B_t is known.
- To estimate η_t a more complicated optimization is carried out.

Example 9.3

- The data are daily Canadian zero-coupon bond yields for 120 different quarterly maturities ranging from 0.25 years to 30 years.
- They have been generated using pricing data for Government of Canada bonds and treasury bills.

- We model the yield curve on the 8th August 2011.
- The estimated value are $z_{t,1} = 3.82$, $z_{t,2} = -2.75$, $z_{t,3} = -5.22$ and $\hat{\eta}_t = 0.623$. Thus the curves $k_2(s, \eta)$ and $k_3(s, \eta)$ are as shown earlier.
- The fitted Nelson-Siegel curve and the data are shown below:



The approach based on PCA

- The key difference to the Nelson-Siegel approach is that here the dimension reduction via factor modelling is **applied at the level of the risk factor changes** \mathbf{X}_{t+1} rather than the risk factors \mathbf{Z}_t .

- We recall that PCA can be used to construct factor models of the form

$$\mathbf{X}_{t+1} = \boldsymbol{\mu} + \Gamma_1 \mathbf{F}_{t+1} + \boldsymbol{\varepsilon}, \quad (62)$$

where \mathbf{F}_{t+1} is a p -dimensional vector of principal component factors ($p < d$), $\Gamma_1 \in \mathbb{R}_{d \times p}$ contains the corresponding loading matrix, $\boldsymbol{\mu}$ is the mean vector of \mathbf{X}_{t+1} and $\boldsymbol{\varepsilon}$ is an error vector.

- Typically, the error term is neglected and $\boldsymbol{\mu} \approx \mathbf{0}$, so that we make the approximation $\mathbf{X}_{t+1} \approx \Gamma_1 \mathbf{F}_{t+1}$.
- In the case of the **linear loss operator for the bond portfolio** in (57) we

basically replace $l_{[t]}^{\Delta}(\mathbf{X}_{t+1})$ by

$$l_{[t]}^{\Delta}(\mathbf{F}_{t+1}) = - \sum_{i=1}^d \lambda_i p(\tau_t, T_i) (y(\tau_t, T_i) \Delta t - (T_i - \tau_t)(\Gamma_1 \mathbf{F}_{t+1})_i), \quad (63)$$

so that a p -dimensional function replaces a d -dimensional function.

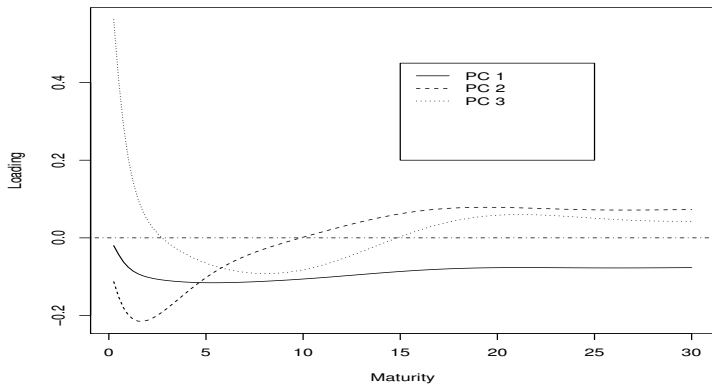
- To calibrate this function, **we require an estimate for the matrix Γ_1** . This can be obtained from historical time-series data on yield changes by estimating sample principle components.

Example 9.4

- To estimate the Γ_1 matrix of principal component loadings we require longitudinal (time-series) data rather than the cross-sectional data.
- We again analyse Canadian data. Recall that we have data vectors $\mathbf{Y}_t = (y(\tau_t, \tau_t + s_1), \dots, y(\tau_t, \tau_t + s_d))$ of yields for different maturities.

- For simplicity assume that the times-to-maturity $T_1 - \tau_t, \dots, T_d - \tau_t$ of the bonds in the portfolio correspond exactly to the times to maturity s_1, \dots, s_d available in the historical dataset.
- Assume also that the risk management horizon Δt is one day.
- We analyse the first differences of the data $\mathbf{X}_t = \mathbf{Y}_t - \mathbf{Y}_{t-1}$ using PCA under an assumption of stationarity.
- In the Canadian dataset we have 2488 days of data from 2 January 2002 to 30 December 2011.
- (Note that a small error is incurred by analysing daily returns of yields with fixed times-to-maturity rather than fixed maturity date.)
- The first principle component explains 87.0% of the variance of the data, the first two components explain 95.9% and the first three components explain 97.5%.

- We choose to work with the first three principal components, meaning that we set $p = 3$ and set the columns of Γ_1 equal to the first three principal component loading vectors.
- These vectors are shown graphically below and lend themselves to a standard interpretation.
- The first principal component has negative loadings for all maturities; the second has negative loadings up to 10 years and positive loadings thereafter; the third has positive loadings for very short maturities (less than 2.5 years) and very long maturities (greater than 15 years) but negative loadings otherwise.
- This suggests that the first principal component can be thought of as inducing a change in the level of all yields, the second induces a change of slope and the third a change in the curvature of the yield curve.



9.2 Market risk measurement

The **goal** in this section is to **estimate the distribution of**

$$L_{t+1} = l_{[t]}(\mathbf{X}_{t+1})$$

or a linear or quadratic approximation thereof, where

- \mathbf{X}_{t+1} is the vector of risk-factor changes from time t to time $t + 1$;
- $l_{[t]}$ is the known loss operator function at time t .

The problem comprises **two tasks**:

- 1) on the one hand we have the **statistical problem** of **estimating the distribution** of \mathbf{X}_{t+1} ;
- 2) on the other hand we have the **computational or numerical problem** of **evaluating the distribution** of $L_{t+1} = l_{[t]}(\mathbf{X}_{t+1})$.

9.2.1 Conditional and unconditional loss distributions

- Generally, we want to compute conditional measures of risk based on the most recent information about financial markets.
- In this case, the task is to estimate $F_{\mathbf{X}_{t+1}|\mathcal{F}_t}$, the conditional distribution of risk-factor changes, given \mathcal{F}_t , the sigma field representing the available information at time t .

- The conditional loss distribution is the distribution of the loss operator $l_{[t]}(\cdot)$ under $F_{\mathbf{X}_{t+1}|\mathcal{F}_t}$, i.e. the distribution with df

$$F_{L_{t+1}|\mathcal{F}_t}(l) = \mathbb{P}(l_{[t]}(\mathbf{X}_{t+1}) \leq l \mid \mathcal{F}_t).$$

- In the unconditional approach we assume that $(\mathbf{X}_s)_{s \leq t}$ forms a stationary time series, at least in the recent past.
- In this case we can estimate the stationary distribution $F_{\mathbf{X}}$ and then evaluate the unconditional loss distribution of $l_{[t]}(\mathbf{X})$ where $\mathbf{X} \sim F_{\mathbf{X}}$. The unconditional loss distribution is thus $F_{L_{t+1}}(l) = \mathbb{P}(l_{[t]}(\mathbf{X}) \leq l)$.

- The unconditional approach may be appropriate for longer time intervals, or for stress testing during quieter periods.
- If the risk-factor changes form an iid series, we obviously have $F_{\mathbf{X}_{t+1}|\mathcal{F}_t} = F_{\mathbf{X}}$, so that the conditional and unconditional approaches coincide.

9.2.2 Variance-covariance method

- The variance–covariance method is an analytical method in which strong assumptions of (conditional) normality and linearity are made.
- We assume that the conditional distribution of risk-factor changes $F_{\mathbf{X}_{t+1}|\mathcal{F}_t}$ is a multivariate normal distribution.
- In other words, we assume that $\mathbf{X}_{t+1} | \mathcal{F}_t \sim N_d(\boldsymbol{\mu}_{t+1}, \Sigma_{t+1})$.
- The estimation of $F_{\mathbf{X}_{t+1}|\mathcal{F}_t}$ can be carried out in a number of ways:
 - ▶ Fit multivariate ARMA-GARCH model with multivariate normal innovations; use model to derive estimates of $\boldsymbol{\mu}_{t+1}$ and Σ_{t+1} .

- ▶ Alternatively use the **exponentially weighted moving-average (EWMA)** procedure; Σ_{t+1} estimated recursively by $\hat{\Sigma}_{t+1} = \theta \mathbf{X}_t \mathbf{X}_t' + (1 - \theta) \hat{\Sigma}_t$ where θ is a small positive number (typically $\theta \approx 0.04$).
- The **second critical assumption** in the variance–covariance method is that the **linear loss operator is sufficiently accurate**. The linear loss operator is a function of the form

$$l_{[t]}^{\Delta}(\mathbf{x}) = -(c_t + \mathbf{b}_t' \mathbf{x})$$

for some constant c_t and constant vector \mathbf{b}_t , known at time t .

- We infer that, conditional on \mathcal{F}_t ,

$$L_{t+1}^{\Delta} = l_{[t]}^{\Delta}(\mathbf{X}_{t+1}) \sim \text{N}(-c_t - \mathbf{b}_t' \boldsymbol{\mu}_{t+1}, \mathbf{b}_t' \Sigma_{t+1} \mathbf{b}_t).$$

- Under normality, VaR_{α} and ES_{α} may be easily calculated:

$$\widehat{\text{VaR}}_{\alpha} = -c_t - \mathbf{b}_t' \hat{\boldsymbol{\mu}}_{t+1} + \sqrt{\mathbf{b}_t' \hat{\Sigma}_{t+1} \mathbf{b}_t} \Phi^{-1}(\alpha).$$

$$\widehat{\text{ES}}_{\alpha} = -c_t - \mathbf{b}_t' \hat{\boldsymbol{\mu}}_{t+1} + \sqrt{\mathbf{b}_t' \hat{\Sigma}_{t+1} \mathbf{b}_t} \frac{\phi(\Phi^{-1}(\alpha))}{1 - \alpha}.$$

Pros and cons, extensions

- Pros:** In contrast to the methods that follow, variance-covariance offers **analytical solution** with no simulation.
- Cons:**
- ▶ Assumption of **multivariate normality** may seriously underestimate the **tail** of the loss distribution.
 - ▶ **Linearization** may be a crude approximation.
- Extensions:** Instead of assuming normal risk factors, the method **could be** easily **adapted to** use **multivariate Student t** or multivariate hyperbolic risk-factor changes without sacrificing tractability (the method **works for all elliptical distributions** but linearization is crucial here).

9.2.3 Historical simulation

- Historical simulation is by far the most **popular method** used by banks for the trading book.
- Instead of estimating the distribution of $l_{[t]}(\mathbf{X}_{t+1})$ under an explicit parametric model for \mathbf{X}_{t+1} , the historical simulation method can be thought of as **estimating the distribution of the loss operator under the empirical distribution** of historical data $\mathbf{X}_{t-n+1}, \dots, \mathbf{X}_t$.
- **Construct** the historically simulated losses (under the current portfolio):

$$\{\tilde{L}_s = l_{[t]}(\mathbf{X}_s) : s = t - n + 1, \dots, t\}.$$

- One may apply the linear/quadratic loss operator (if that was already used; avoids revaluation).
- \tilde{L}_s shows **what would happen to the current portfolio if the risk-factor change on day s were to recur**.
- Use (\tilde{L}_s) to make inferences about the **loss distribution and risk measures**.

■ Inference about the loss distribution

- ▶ One could use **empirical quantile estimation** to estimate VaR_α .
But: What about precision (sample size; confidence intervals)?
- ▶ Or **fit a parametric** distribution to the historical losses L_{t-n+1}, \dots, L_t and calculate risk measures from this distribution.
But: Which distribution to fit (body or tail)?
- ▶ One could use **extreme value theory** to estimate the tail of the loss distribution and related risk measures based on the historical losses L_{t-n+1}, \dots, L_t .

Theoretical justification

If X_{t-n+1}, \dots, X_t are iid or, more generally, stationary, **convergence of the empirical distribution to the true distribution is ensured by a suitable version of the Law of Large Numbers** (e.g. Glivenko–Cantelli theorem).

Pros and Cons

Pros: ▶ Easy to implement.

- ▶ No statistical estimation of the distribution of \mathbf{X} necessary (the empirical df of \mathbf{X} is used implicitly).

Cons: ▶ It may be difficult to collect sufficient quantities of relevant, synchronized data for all risk factors.

- ▶ Historical data may not contain examples of extreme scenarios (“driving a car by only looking in the back mirror”).

Note: ▶ The dependence here is given by the empirical df of \mathbf{X} .

- ▶ “Historical simulation method” is a bit of a misnomer; there is no simulation in the sense of random number generation.

In its standard form HS is an unconditional method. There are a number of ways of extending historical simulation to take account of volatility dynamics (filtered HS).

9.2.4 Dynamic Historical Simulation

A univariate approach:

- Assume that $\{\tilde{L}_s = l_{[t]}(\mathbf{X}_s) : s = t - n + 1, \dots, t\}$ are realizations from a stationary process (\tilde{L}_s) of the form $\tilde{L}_s = \mu_s + \sigma_s Z_s$, where
 - μ_s and σ_s are \mathcal{F}_{s-1} -measurable;
 - (Z_s) are SWN(0, 1) innovations with distribution function F_Z .

Example: [ARMA-GARCH model](#).

- We can easily calculate that for the next loss $L_{t+1} = l_{[t]}(\mathbf{X}_{t+1})$ ahead
$$F_{L_{t+1}|\mathcal{F}_t}(l) = \mathbb{P}(\mu_{t+1} + \sigma_{t+1} Z_{t+1} \leq l \mid \mathcal{F}_t) = F_Z((l - \mu_{t+1})/\sigma_{t+1}).$$
- Writing VaR_α^t for $F_{L_{t+1}|\mathcal{F}_t}^{\leftarrow}(\alpha)$ and ES_α^t for ES, we obtain

$$\begin{aligned}\text{VaR}_\alpha^t &= \mu_{t+1} + \sigma_{t+1} \text{VaR}_\alpha(Z), \\ \text{ES}_\alpha^t &= \mu_{t+1} + \sigma_{t+1} \text{ES}_\alpha(Z),\end{aligned}$$

where Z is a random variable with distribution function F_Z .

■ Estimation

- ▶ Formal **parametric time series modelling** to estimate μ_{t+1} , σ_{t+1} , $\text{VaR}_\alpha(Z)$ and $\text{ES}_\alpha(Z)$.
- ▶ Often $\mu_{t+1} \approx 0$ and can be neglected. We can use **EWMA** to estimate $\sigma_{t-n+1}, \dots, \sigma_t, \sigma_{t+1}$ and use the **standardized residuals** $\{\hat{Z}_s = \tilde{L}_s / \hat{\sigma}_s, s = t - n + 1, \dots, t\}$ to estimate $\text{VaR}_\alpha(Z)$ and $\text{ES}_\alpha(Z)$.

A multivariate approach:

- We (implicitly) assume risk-factor change data $\mathbf{X}_{t-n+1}, \dots, \mathbf{X}_t$ are realizations from process (\mathbf{X}_s) which satisfies

$$\mathbf{X}_s = \boldsymbol{\mu}_s + \Delta_s \mathbf{Z}_s, \quad \Delta_s = \text{diag}(\sigma_{s,1}, \dots, \sigma_{s,d}),$$

where $(\boldsymbol{\mu}_s)$ is a process of vectors and (Δ_s) a process of diagonal matrices (all assumed \mathcal{F}_{s-1} -measurable) and $(\mathbf{Z}_s) \sim \text{SWN}(\mathbf{0}, P)$ for some correlation matrix P .

- The vector $\boldsymbol{\mu}_s$ contains the conditional means and the matrix Δ_s contains the volatilities of the component series at time s .
- An example of a model that fits into this framework is the [CCC-GARCH \(constant conditional correlation\)](#) process.
- The key idea of the method is to apply historical simulation to the unobserved innovations (\mathbf{Z}_s) .

- The first step is to compute estimates $\{\hat{\boldsymbol{\mu}}_s : s = t - n + 1, \dots, t\}$ and $\{\hat{\Delta}_s : s = t - n + 1, \dots, t\}$.
- This can be achieved by fitting univariate time series models of ARMA-GARCH type to each of the component series in turn; alternatively we can use the univariate EWMA approach for each series.
- In the second step we construct residuals

$$\{\hat{\mathbf{Z}}_s = \hat{\Delta}_s^{-1}(\mathbf{X}_s - \hat{\boldsymbol{\mu}}_s) : s = t - n + 1, \dots, t\}$$

and treat these as “observations” of the unobserved innovations.

- We then construct the dataset

$$\{\tilde{L}_s = l_{[t]}(\hat{\boldsymbol{\mu}}_{t+1} + \hat{\Delta}_{t+1}\hat{\mathbf{Z}}_s) : s = t - n + 1, \dots, t\} \quad (64)$$

and treat these as observations of $L_{t+1} = l_{[t]}(\mathbf{X}_{t+1})$.

- To estimate VaR (or expected shortfall) we can apply simple empirical estimators directly to these data.

9.2.5 Monte Carlo

- Estimate the distribution of $L = \ell_{[t]}(\mathbf{X}_{t+1})$ under some explicit parametric model for \mathbf{X}_{t+1} .
- In contrast to the variance-covariance approach we do not necessarily make the problem analytically tractable by linearizing the loss and making an assumption of normality for the risk factors.
- Instead, make inference about L using simulated risk factor data.

The method

- 1) Based on the historical risk-factor data $\mathbf{X}_{t-n+1}, \dots, \mathbf{X}_t$, estimate a suitable statistical model for the risk-factor changes.
- 2) Simulate N new risk-factor changes $\mathbf{X}_{t+1}^{(1)}, \dots, \mathbf{X}_{t+1}^{(N)}$ from this model.
- 3) Construct the simulated losses $L_k = \ell_{[t]}(\mathbf{X}_{t+1}^{(k)})$, $k \in \{1, \dots, N\}$.

- 4) **Make inference** about the loss distribution F_L and risk measures **using** $L_k, k \in \{1, \dots, N\}$ (similar possibilities as for the historical simulation method: non-parametric/parametric/EVT).

Pros and Cons

- Pros:** ▶ General. **Any distribution** for \mathbf{X}_{t+1} can be taken.
- Cons:** ▶ **Can be time consuming** if loss operator is difficult to evaluate (depends on size and complexity of the portfolio).
- ▶ Note that MC approach does not address the **problem of determining the distribution of \mathbf{X}_{t+1}** .

9.2.6 Estimating risk measures

Aim: In both the historical simulation and Monte Carlo methods we estimate risk measures using simulated loss data. Let us suppose that we have data L_1, \dots, L_n from an underlying loss distribution F_L and the aim is to estimate $\text{VaR}_\alpha = q_\alpha(F_L) = F_L^{\leftarrow}(\alpha)$ or $\text{ES}_\alpha = (1 - \alpha)^{-1} \int_\alpha^1 q_\theta(F_L) d\theta$. In the book we consider two possibilities:

- **L-estimators.** These are **linear** combinations of sample order statistics. Easiest to use notation for lower order statistics $L_{(1)} \leq \dots \leq L_{(n)}$.
- **GPD-based estimators.** These are semi-parametric estimators based on GPD approximations described in EVT chapter.

L-estimators:

VaR: $\text{VaR}_\alpha(L) = \inf\{x \in \mathbb{R} : F_L(x) \geq \alpha\}$. Replacing F_L by \hat{F}_L we obtain an L-estimator.

$$\begin{aligned}
\widehat{\text{VaR}}_\alpha(L) &= \inf\{x \in \mathbb{R} : \hat{F}_L(x) \geq \alpha\} \\
&= \inf\left\{x \in \mathbb{R} : \sum_{i=1}^n I_{\{L_i \leq x\}} \geq \lceil n\alpha \rceil\right\} \\
&= \inf\left\{x \in \mathbb{R} : \sum_{i=1}^n I_{\{L_{(i)} \leq x\}} \geq \lceil n\alpha \rceil\right\} = L_{(\lceil n\alpha \rceil)}.
\end{aligned}$$

In practice, most software uses an average of two order statistics.

ES: Assume F_L is continuous so that

$$\text{ES}_\alpha(L) = \frac{\mathbb{E}(LI_{\{L > F_L^{\leftarrow}(\alpha)\}})}{1 - \alpha} = \frac{\mathbb{E}(LI_{\{L > F_L^{\leftarrow}(\alpha)\}})}{\mathbb{E}(I_{\{L > F_L^{\leftarrow}(\alpha)\}})}.$$

Replacing F_L by \hat{F}_L leads to the canonical estimator

$$\widehat{\text{ES}}_\alpha(L) = \frac{\sum_{i=1}^n L_i I_{\{L_i > \widehat{\text{VaR}}_\alpha(L)\}}}{\sum_{i=1}^n I_{\{L_i > \widehat{\text{VaR}}_\alpha(L)\}}}.$$

GPD-based estimators:

We set a high threshold $u = L_{(n-k)}$ at an order statistic and fit a GPD distribution to the k excess losses over u to obtain maximum likelihood estimates $\hat{\xi}$ and $\hat{\beta}$.

For $k/n > 1 - \alpha$ we can form the risk measure estimates:

$$\widehat{\text{VaR}}_{\alpha} = u + \frac{\hat{\beta}}{\hat{\xi}} \left(\left(\frac{1 - \alpha}{k/n} \right)^{-\hat{\xi}} - 1 \right)$$
$$\widehat{\text{ES}}_{\alpha} = \frac{\widehat{\text{VaR}}_{\alpha}}{1 - \hat{\xi}} + \frac{\hat{\beta} - \hat{\xi}u}{1 - \hat{\xi}}.$$

9.2.7 Losses over several periods and scaling

- **Goal:** Go from single-period risk measure (e.g. one day/one week VaR/ES) to multi-period risk measure using simple formula.
- **Idea:** The loss between today and h periods ahead is

$$\begin{aligned}L_{t+h}^{(h)} &= -(V_{t+h} - V_t) = -(g(\tau_{t+h}, \mathbf{Z}_{t+h}) - g(\tau_t, \mathbf{Z}_t)) \\&= -(g(\tau_{t+h}, \mathbf{Z}_t + \mathbf{X}_{t+1} + \cdots + \mathbf{X}_{t+h}) - g(\tau_t, \mathbf{Z}_t)) \\&= L\left(\sum_{i=1}^h \mathbf{X}_{t+i}\right).\end{aligned}$$

- **Question:** How do risk measures scale with h ?
- There is no general answer.
- If $\mathbf{X}_{t+i} \stackrel{\text{ind.}}{\sim} \mathbf{N}(\boldsymbol{\mu}, \Sigma)$, then $\mathbf{Y} = \sum_{i=1}^h \mathbf{X}_{t+i} \sim \mathbf{N}(h\boldsymbol{\mu}, h\Sigma)$. Then

$$L_{t+h}^{(h)\Delta} = -g_{\tau}(\tau_t, \mathbf{Z}_t) - \sum_{j=1}^d g_{z_j}(\tau_t, \mathbf{Z}_t) \left(\sum_{i=1}^h X_{t+i,j} \right) = -(c_t + \mathbf{b}'_t \mathbf{Y}).$$

- We infer that $L_{t+h}^{(h)\Delta} \sim N(-c_t - h\mathbf{b}'_t\boldsymbol{\mu}, h\mathbf{b}'_t\Sigma\mathbf{b}_t)$.
- If we assume $c_t \approx 0$, $\boldsymbol{\mu} \approx \mathbf{0}$ (typical for daily data) we obtain square-root-of-time scaling formulas for VaR and ES.
- $\text{VaR}_\alpha(L_{t+h}^{(h)\Delta}) = 0 + \sqrt{h\mathbf{b}'_t\Sigma\mathbf{b}_t}\Phi^{-1}(\alpha) = \sqrt{h}\text{VaR}_\alpha(L_{t+1}^\Delta)$.
- $\text{ES}_\alpha(L_{t+h}^{(h)\Delta}) = 0 + \sqrt{h\mathbf{b}'_t\Sigma\mathbf{b}_t}\frac{\varphi(\Phi^{-1}(\alpha))}{1-\alpha} = \sqrt{h}\text{ES}_\alpha(L_{t+1}^\Delta)$.
- Note the many underlying assumptions:
 - ▶ \mathbf{X}_{t+i} independent;
 - ▶ \mathbf{X}_{t+i} multivariate normal;
 - ▶ The linearized loss provides a sufficiently good approximation to the true loss distribution.
- Note also that we have only considered the scaling of unconditional risk measures.

9.3 Backtesting

- Backtesting is the practice of evaluating risk measurement procedures by comparing *ex ante estimates/forecasts of risk measures* with *ex post realized losses and gains*.
- It allows us to evaluate whether a model and estimation procedure produce *credible risk measure estimates*.

9.3.1 Violation-based tests for VaR

- Let VaR_α^t denote the α -quantile of the conditional loss distribution $F_{L_{t+1}|\mathcal{F}_t}$ and consider the event indicator variable $I_{t+1} = I_{\{L_{t+1} > \text{VaR}_\alpha^t\}}$.
- The event $\{L_{t+1} > \text{VaR}_\alpha^t\}$ is a VaR *violation* or *exception*.
- Assuming a continuous loss distribution, we have, by definition of the quantile,

$$\mathbb{E}(I_{t+1} \mid \mathcal{F}_t) = \mathbb{P}(L_{t+1} > \text{VaR}_\alpha^t \mid \mathcal{F}_t) = 1 - \alpha, \quad (65)$$

- I_{t+1} is a **Bernoulli variable** with event probability $(1 - \alpha)$.
- Moreover, the sequence of VaR exception indicators (I_t) is an **iid sequence**.
- The sum of exception indicators is **binomially distributed**:

$$M = \sum_{t=1}^m I_t \sim B(m, 1 - \alpha).$$

- Assume exceptions occur at times $1 \leq T_1 < \dots < T_M \leq m$ and set $T_0 = 0$. The spacings $S_j = T_j - T_{j-1}$ will be independent **geometrically distributed** rvs with mean $1/(1 - \alpha)$, so that

$$\mathbb{P}(S_j = k) = \alpha^{k-1}(1 - \alpha), \quad k \in \mathbb{N}.$$

- Both of these properties are **testable in empirical data**.
- For small event probability $1 - \alpha$, the Bernoulli Trials Process may be well approximated by a **Poisson process**.

- Also for small $1 - \alpha$ the geometric distribution may be approximated by an **exponential distribution**.
- Suppose we **estimate** VaR_α^t at time point t by $\widehat{\text{VaR}}_\alpha^t$.
- In a backtest we consider empirical indicator variables

$$\hat{I}_{t+1} = I_{\{L_{t+1} > \widehat{\text{VaR}}_\alpha^t\}}.$$

- The sequence $(\hat{I}_t)_{1 \leq t \leq m}$ **should behave** like a realization from a Bernoulli trials process with event probability $(1 - \alpha)$.
- To test binomial behaviour for number of violations we compute a **score test statistic**

$$Z_m = \frac{(\sum_{t=1}^m \hat{I}_t) - m(1 - \alpha)}{\sqrt{m\alpha(1 - \alpha)}}$$

and reject Bernoulli hypothesis at 5% level if $Z_m > \Phi^{-1}(0.95)$.

- Exponential spacings can be tested numerically or with a Q-Q plot.

9.3.2 Violation-based tests of expected shortfall

- Let ES_{α}^t denote the one-period expected shortfall and \widehat{ES}_{α}^t its estimate.
- Assume (L_t) follows a model of the form $L_t = \sigma_t Z_t$, where σ_t is a function of \mathcal{F}_{t-1} and the (Z_t) are $\text{SWN}(0, 1)$ innovations.
- Then we can define a process (K_t) by

$$K_{t+1} = \frac{(L_{t+1} - ES_{\alpha}^t)}{ES_{\alpha}^t} I_{\{L_{t+1} > \text{VaR}_{\alpha}^t\}} = \frac{Z_{t+1} - ES_{\alpha}(Z)}{ES_{\alpha}(Z)} I_{\{Z_{t+1} > q_{\alpha}(Z)\}},$$

and note that it is a **zero-mean iid sequence**.

- This suggests we form **violation residuals** of the form

$$\widehat{K}_{t+1} = \frac{(L_{t+1} - \widehat{ES}_{\alpha}^t)}{\widehat{ES}_{\alpha}^t} \widehat{I}_{t+1}. \quad (66)$$

- We test for mean-zero behaviour using a bootstrap test on the non-zero violation residuals (McNeil and Frey (2000)).

9.3.3 Elicitability and comparison of risk measure estimates

- The elicibility concept has been introduced into the backtesting literature by Gneiting (2011); see also important papers by Bellini and Bignozzi (2013) and Ziegel (2014).
- A key concept is that of a **scoring function** $S(y, l)$ which measures the discrepancy between a forecast y and a realized loss l .
- Forecasts are made by applying real-valued statistical functionals T (such as mean, median or other quantile) to the distribution of the loss F_L to obtain the forecast $y = T(F_L)$.
- Suppose that for some class of loss distribution functions a real-valued statistical functional T satisfies

$$T(F_L) = \arg \min_{y \in \mathbb{R}} \int_{\mathbb{R}} S(y, l) \, dF_L(l) = \arg \min_{y \in \mathbb{R}} \mathbb{E}(S(y, L)) \quad (67)$$

for a scoring function S and any loss distribution F_L in that class.

- Suppose moreover that $T(F_L)$ is the unique minimizing value.
- The scoring function S is said to be **strictly consistent** for T .
- The functional $T(F_L)$ is said to be **elicitable**.
- Note that (67) implies that

$$\begin{aligned} \left. \frac{d}{dy} \mathbb{E}(S(y, L)) \right|_{y=T(F_L)} &= \left. \int_{\mathbb{R}} \frac{d}{dy} S(y, l) dF_L(l) \right|_{y=T(F_L)} \\ &= \mathbb{E}(h(T(F_L), L)) = 0 \end{aligned}$$

where h is the derivative of the scoring function.

- The **VaR risk measure** corresponds to $T(F_L) = F_L^{\leftarrow}(\alpha)$. For any $0 < \alpha < 1$ this functional is **elicitable** for strictly increasing distribution functions. The scoring function

$$S_{\alpha}^q(y, l) = |1_{\{l \leq y\}} - \alpha| |l - y| \quad (68)$$

is strictly consistent for T .

- The α -expectile of L is defined to be the risk measure that minimizes $\mathbb{E}(S_{\alpha}^e(y, L))$ where the scoring function is

$$S_{\alpha}^e(y, l) = |1_{\{l \leq y\}} - \alpha|(l - y)^2. \quad (69)$$

This risk measure is **elicitable by definition**.

- Bellini and Bignozzi (2013) and Ziegel (2014) show that a risk measure is coherent and elicitable if and only if it is the α -expectile risk measure for $\alpha \geq 0.5$; see also Weber (2006). **Expected shortfall is not elicitable.**
- VaR_{α}^t minimizes

$$\mathbb{E}\left(S_{\alpha}^q(\text{VaR}_{\alpha}^t, L_{t+1}) \mid \mathcal{F}_t\right)$$

for the scoring function in (68). We refer to $S_{\alpha}^q(\text{VaR}_{\alpha}^t, L_{t+1})$ as a (theoretical) **VaR score**.

- Assume VaR_{α}^t is replaced by an estimate at each time point and consider the VaR scores $\{S_{\alpha}^q(\widehat{\text{VaR}}_{\alpha}^t, L_{t+1}) : t = 1, \dots, m\}$
- These can be used to address questions of **relative model performance**.

- The statistic

$$Q_0 = \frac{1}{m} \sum_{t=1}^m S_{\alpha}^q(\widehat{\text{VaR}}_{\alpha}^t, L_{t+1})$$

can be used as a measure of relative model performance.

- If two models A and B deliver VaR estimates $\{\widehat{\text{VaR}}_{\alpha}^{tA}, t = 1, \dots, m\}$ and $\{\widehat{\text{VaR}}_{\alpha}^{tB}, t = 1, \dots, m\}$ with corresponding average scores Q_0^A and Q_0^B , then we expect the better model to give estimates closer to the true VaR numbers and thus a value of Q_0 that is lower.
- Of course, the power to discriminate between good models and inferior models will depend on the length of the backtest.

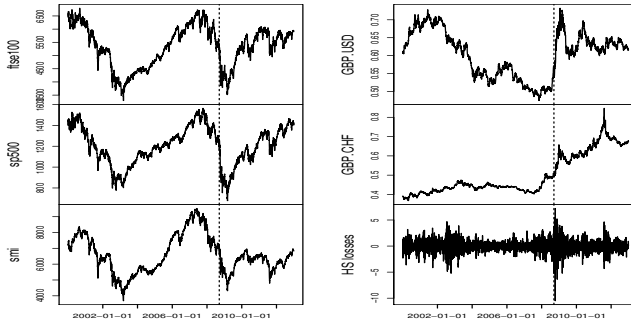
9.3.4 Empirical comparison of methods using backtesting concepts

- We apply various VaR estimation methods to the portfolio of a hypothetical investor in international equity indexes.
- The investor is assumed to have domestic currency sterling (GBP) and to invest in the Financial Times 100 Shares Index (FTSE 100), the Standard & Poor's 500 (S&P 500) and the Swiss Market Index (SMI).
- The portfolio is **influenced by five risk factors**.
- On any day t we standardize the total portfolio value V_t in sterling to be one and assume portfolio weights are 30%, 40% and 30%, respectively.
- The loss operator and linear loss operator are:

$$l_{[t]}(\mathbf{x}) = 1 - (0.3e^{x_1} + 0.4e^{x_2+x_4} + 0.3e^{x_3+x_5})$$

$$l_{[t]}^{\Delta}(\mathbf{x}) = -(0.3x_1 + 0.4(x_2 + x_4) + 0.3(x_3 + x_5))$$

- x_1 , x_2 and x_3 represent log-returns on the three indexes and x_4 and x_5 are log-returns on the GBP/USD and GBP/CHF exchange rates.



- The final picture shows the corresponding historical simulation data. The vertical dashed line is Lehman Brothers bankruptcy.

Estimation methods:

VC. The [variance–covariance method](#) assuming multivariate Gaussian risk-factor changes and using the multivariate EWMA method to estimate the conditional covariance matrix of risk-factor changes.

HS. The standard [unconditional historical simulation](#) method.

HS-GARCH. The univariate [dynamic approach to historical simulation](#) in which a GARCH(1, 1) model with a constant conditional mean term and Gaussian innovations is fitted to the historically simulated losses to estimate the volatility of the next day's loss.

HS-GARCH- t . A similar method to HS-GARCH but Student t innovations are assumed in the GARCH model.

HS-MGARCH. The [multivariate dynamic approach to historical simulation](#) in which GARCH(1, 1) models with constant conditional mean terms are fitted to each time series of risk-factor changes to estimate volatilities.

| Year | 2005 | 2006 | 2007 | 2008 | 2009 | 2010 | 2011 | 2012 | All |
|--------------|------|------|------|------|------|------|------|------|------|
| Trading days | 258 | 257 | 258 | 259 | 258 | 259 | 258 | 258 | 2065 |

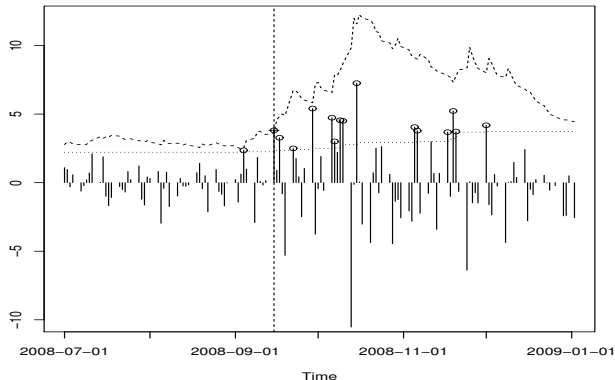
Results for 95% VaR

| | | | | | | | | | |
|----------------------------|----|----|----|----|----|----|----|----|-----|
| Expected no. of violations | 13 | 13 | 13 | 13 | 13 | 13 | 13 | 13 | 103 |
| VC | 8 | 16 | 17 | 19 | 13 | 15 | 14 | 14 | 116 |
| HS | 0 | 6 | 28 | 49 | 19 | 6 | 10 | 1 | 119 |
| HS-GARCH | 9 | 13 | 22 | 22 | 13 | 14 | 9 | 15 | 117 |
| HS-GARCH- t | 9 | 14 | 23 | 22 | 14 | 15 | 10 | 15 | 122 |
| HS-MGARCH | 5 | 14 | 21 | 19 | 12 | 9 | 11 | 12 | 103 |

Results for 99% VaR

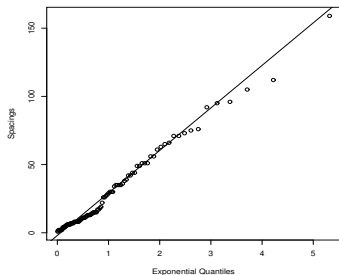
| | | | | | | | | | |
|----------------------------|-----|-----|-----|-----|-----|-----|-----|-----|----|
| Expected no. of violations | 2.6 | 2.6 | 2.6 | 2.6 | 2.6 | 2.6 | 2.6 | 2.6 | 21 |
| VC | 2 | 8 | 8 | 8 | 2 | 4 | 5 | 6 | 43 |
| HS | 0 | 0 | 10 | 22 | 2 | 0 | 2 | 0 | 36 |
| HS-GARCH | 2 | 8 | 8 | 10 | 5 | 4 | 3 | 3 | 43 |
| HS-GARCH- t | 2 | 8 | 6 | 8 | 1 | 4 | 2 | 1 | 32 |
| HS-MGARCH | 0 | 4 | 4 | 5 | 0 | 1 | 2 | 1 | 17 |

- The HS method **does not react to changing volatility:**



- Dotted line is HS; dashed line is HS-MGARCH; vertical line is Lehmann.
- Circle is VaR violation for HS; cross is VaR violation for HS-MGARCH.

- Q-Q plot of **spacings between exceptions** (for HS-MGARCH):



- Violation residual test for ES (n : number of VaR violations):

| | ES _{0.95} | n | ES _{0.99} | n |
|---------------|--------------------|-----|--------------------|-----|
| VC | 0.00 | 116 | 0.05 | 43 |
| HS | 0.02 | 119 | 0.25 | 36 |
| HS-GARCH | 0.00 | 117 | 0.05 | 43 |
| HS-GARCH- t | 0.12 | 122 | 0.68 | 32 |
| HS-MGARCH | 0.99 | 103 | 0.55 | 17 |

9.3.5 Backtesting the predictive distribution

- As well as backtesting VaR and expected shortfall we can also devise tests that assess the overall quality of the estimated conditional loss distribution, or its tail.
- If L_{t+1} is a random variable with (continuous) distribution function $F_{L_{t+1}|\mathcal{F}_t}$, then $U_{t+1} = F_{L_{t+1}|\mathcal{F}_t}(L_{t+1})$ is uniform (probability transform).
- In actual applications we estimate $F_{L_{t+1}|\mathcal{F}_t}$ from data up to time t and we backtest our estimates by forming $\hat{U}_{t+1} = \hat{F}_{L_{t+1}|\mathcal{F}_t}(L_{t+1})$ on day $t + 1$.
- Suppose we estimate the predictive distribution on days $t = 0, \dots, n - 1$ and form backtesting data $\hat{U}_1, \dots, \hat{U}_n$; we expect these to behave like a sample of iid uniform data.
- The distributional assumption can be assessed by standard goodness-of-fit tests like the Kolmogorov–Smirnov test.