

# 7 Copulas and dependence

- 7.1 Copulas
- 7.2 Dependence concepts and measures
- 7.3 Normal mixture copulas
- 7.4 Archimedean copulas
- 7.5 A proof for subadditivity of ES
- 7.6 Fitting copulas to data

## 7.1 Copulas

- We now look more closely at modelling the dependence among the components of a random vector  $\mathbf{X} \sim F$  (risk-factor changes).
- **In short:**  $F = \text{dependence structure } C \circ \text{marginal dfs } F_1, \dots, F_d$
- **Advantages:**
  - ▶ Most natural in a static distributional context (no time dependence; apply, for example, to residuals of an ARMA-GARCH model)
  - ▶ Copulas allow us to understand and study dependence independently of the margins (first part of Sklar's Theorem; see later)
  - ▶ Copulas allow for a bottom-up approach to multivariate model building (second part of Sklar's Theorem; see later). This is often useful for constructing tailored  $F$ , for example, when we have more information about the margins than  $C$  or for stress testing purposes (to challenge the existing model and see how it performs).

## 7.1.1 Basic properties

### Definition 7.1 (Copula)

A *copula*  $C$  is a *df* with  $U(0, 1)$  margins.

### Characterization

$C : [0, 1]^d \rightarrow [0, 1]$  is a copula *if and only if*

1)  $C$  is *grounded*, that is,

$$C(u_1, \dots, u_d) = 0 \text{ if } u_j = 0 \text{ for at least one } j \in \{1, \dots, d\}.$$

2)  $C$  has standard *uniform* univariate *margins*, that is,

$$C(1, \dots, 1, u_j, 1, \dots, 1) = u_j \text{ for all } u_j \in [0, 1] \text{ and } j \in \{1, \dots, d\}.$$

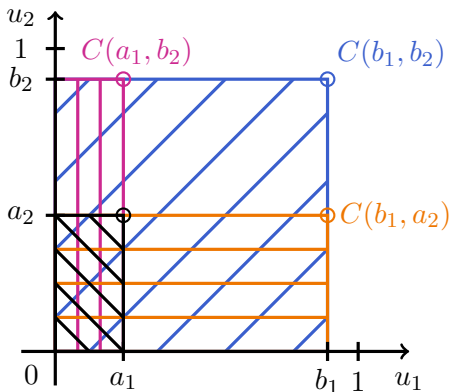
3)  $C$  is *d-increasing*, that is, for all  $\mathbf{a}, \mathbf{b} \in [0, 1]^d$ ,  $\mathbf{a} \leq \mathbf{b}$ ,

$$\Delta_{(\mathbf{a}, \mathbf{b}]} C = \sum_{\mathbf{i} \in \{0, 1\}^d} (-1)^{\sum_{j=1}^d i_j} C(a_1^{i_1} b_1^{1-i_1}, \dots, a_d^{i_d} b_d^{1-i_d}) \geq 0.$$

Equivalently, if existent: *density*  $c(\mathbf{u}) \geq 0$  for all  $\mathbf{u} \in (0, 1)^d$ .

2-increasingness explained in a picture:

$$\begin{aligned}\Delta_{(a,b]}C &= C(b_1, b_2) - C(b_1, a_2) - C(a_1, b_2) + C(a_1, a_2) \\ &= \mathbb{P}(U \in (a, b]) \geq 0\end{aligned}$$



$\Rightarrow \Delta_{(a,b]}C$  is the probability of a random vector  $U \sim C$  to be in  $(a, b]$ .

## Preliminaries

### Lemma 7.2 (Probability transformation)

Let  $X \sim F$ ,  $F$  continuous. Then  $F(X) \sim U(0, 1)$ .

*Proof.*  $\mathbb{P}(F(X) \leq u) = \mathbb{P}(F^{\leftarrow}(F(X)) \leq F^{\leftarrow}(u)) = \mathbb{P}(X \leq F^{\leftarrow}(u)) = F(F^{\leftarrow}(u)) = u, u \in [0, 1]$ ; more details in the appendix.  $\square$

Note that  $F$  needs to be continuous (otherwise  $F(X)$  would not reach all intervals  $\subseteq [0, 1]$ ).

### Lemma 7.3 (Quantile transformation)

Let  $U \sim U(0, 1)$  and  $F$  be any df. Then  $X = F^{\leftarrow}(U) \sim F$ .

*Proof.*  $\mathbb{P}(F^{\leftarrow}(U) \leq x) \stackrel{(G15)}{=} \mathbb{P}(U \leq F(x)) = F(x), x \in \mathbb{R}.$   $\square$

Probability and quantile transformations are the key to all applications involving copulas. They allow us to go from  $\mathbb{R}^d$  to  $[0, 1]^d$  and back.

# Sklar's Theorem

## Theorem 7.4 (Sklar's Theorem)

- 1) For any df  $F$  with margins  $F_1, \dots, F_d$ , there exists a copula  $C$  such that

$$F(x_1, \dots, x_d) = C(F_1(x_1), \dots, F_d(x_d)), \quad \mathbf{x} \in \mathbb{R}^d. \quad (27)$$

$C$  is uniquely defined on  $\prod_{j=1}^d \text{ran } F_j$  and given by

$$C(u_1, \dots, u_d) = F(F_1^{\leftarrow}(u_1), \dots, F_d^{\leftarrow}(u_d)), \quad \mathbf{u} \in \prod_{j=1}^d \text{ran } F_j,$$

where  $\text{ran } F_j = \{F_j(x) : x \in \mathbb{R}\}$  denotes the *range* of  $F_j$ .

- 2) Conversely, given any copula  $C$  and univariate dfs  $F_1, \dots, F_d$ ,  $F$  defined by (27) is a df with margins  $F_1, \dots, F_d$ .

*Proof.*

- 1) **Proof for continuous  $F_1, \dots, F_d$  only.** Let  $\mathbf{X} \sim F$  and define  $U_j = F_j(X_j)$ ,  $j \in \{1, \dots, d\}$ . By the probability transformation,  $U_j \sim U(0, 1)$  (continuity!),  $j \in \{1, \dots, d\}$ , so the df  $C$  of  $\mathbf{U}$  is a copula. Since  $F_j \uparrow$  on  $\text{ran } X_j$ , (G13) implies that  $X_j = F_j^{\leftarrow}(F_j(X_j)) = F_j^{\leftarrow}(U_j)$ ,  $j \in \{1, \dots, d\}$ . Therefore,

$$\begin{aligned} F(\mathbf{x}) &= \mathbb{P}(X_j \leq x_j \ \forall j) = \mathbb{P}(F_j^{\leftarrow}(U_j) \leq x_j \ \forall j) \stackrel{\text{(G15)}}{=} \mathbb{P}(U_j \leq F_j(x_j) \ \forall j) \\ &= C(F_1(x_1), \dots, F_d(x_d)), \quad \mathbf{x} \in \mathbb{R}^d. \end{aligned}$$

Hence  $C$  is a copula and satisfies (27).

(G14) implies that  $F_j(F_j^{\leftarrow}(u_j)) = u_j$  for all  $u_j \in \text{ran } F_j$ , so

$$\begin{aligned} C(u_1, \dots, u_d) &= C(F_1(F_1^{\leftarrow}(u_1)), \dots, F_d(F_d^{\leftarrow}(u_d))) \\ &\stackrel{(27)}{=} F(F_1^{\leftarrow}(u_1), \dots, F_d^{\leftarrow}(u_d)), \quad \mathbf{u} \in \prod_{j=1}^d \text{ran } F_j. \end{aligned}$$

2) For  $U \sim C$ , define  $\mathbf{X} = (F_1^{\leftarrow}(U_1), \dots, F_d^{\leftarrow}(U_d))$ . Then

$$\begin{aligned}\mathbb{P}(\mathbf{X} \leq \mathbf{x}) &= \mathbb{P}(F_j^{\leftarrow}(U_j) \leq x_j \ \forall j) \stackrel{\text{(G15)}}{=} \mathbb{P}(U_j \leq F_j(x_j) \ \forall j) \\ &= C(F_1(x_1), \dots, F_d(x_d)), \quad \mathbf{x} \in \mathbb{R}^d.\end{aligned}$$

Therefore,  $F$  defined by (27) is a df (that of  $\mathbf{X}$ ), with margins  $F_1, \dots, F_d$  (obtained by the quantile transformation).  $\square$

### Example 7.5 (Bivariate Bernoulli distribution)

Let  $(X_1, X_2)$  follow a bivariate Bernoulli distribution with  $\mathbb{P}(X_1 = k, X_2 = l) = 1/4$ ,  $k, l \in \{0, 1\}$ .  $\Rightarrow \mathbb{P}(X_j = k) = 1/2$ ,  $k \in \{0, 1\}$ ,  $\text{ran } F_j = \{0, 1/2, 1\}$ ,  $j \in \{1, 2\}$ . Any copula with  $C(1/2, 1/2) = 1/4$  satisfies (27) (e.g.  $C(u_1, u_2) = u_1 u_2$  or the copula  $C(u_1, u_2) = \min\{u_1, u_2, (\delta(u_1) + \delta(u_2))/2\}$  with  $\delta(u) = u^2$ ).

- We say that  $\mathbf{X}$  (or  $F$ ) *has copula*  $C$  if (27) holds.
- A *copula model* for  $\mathbf{X}$  means  $F(\mathbf{x}) = C(F_1(x_1), \dots, F_d(x_d))$  for some (parametric) copula  $C$  and (parametric) marginals  $F_1, \dots, F_d$ .



## Invariance principle

### Lemma 7.6 (Core of the invariance principle)

Let  $X_j \sim F_j$ ,  $F_j$  continuous,  $j \in \{1, \dots, d\}$ . Then

$$\mathbf{X} \text{ has copula } C \iff (F_1(X_1), \dots, F_d(X_d)) \sim C.$$

*Proof.* See the appendix (“ $\Rightarrow$ ” seen in the proof of Sklar’s Theorem).  $\square$

### Theorem 7.7 (Invariance principle)

Let  $\mathbf{X} \sim F$  with continuous margins  $F_1, \dots, F_d$  and copula  $C$ . If  $T_j \uparrow$  on  $\text{ran } X_j$  for all  $j$ , then  $(T_1(X_1), \dots, T_d(X_d))$  (also) has copula  $C$ .

*Proof.* W.l.o.g. assume  $T_j$  to be right-continuous ( $T_j$  has at most countably many discontinuities and we thus change  $T_j(X_j)$  at most on this null set). Since  $T_j \uparrow$  on  $\text{ran } X_j$  and  $X_j$  is continuously distributed,  $T_j(X_j)$  is continuously distributed and we have

$$\begin{aligned}
 F_{T_j(X_j)}(x) &= \mathbb{P}(T_j(X_j) \leq x) = \mathbb{P}(T_j(X_j) < x) \stackrel{(G15)}{=} \mathbb{P}(X_j < T_j^{\leftarrow}(x)) \\
 &= \mathbb{P}(X_j \leq T_j^{\leftarrow}(x)) = F_j(T_j^{\leftarrow}(x)), \quad x \in \mathbb{R}.
 \end{aligned}$$

This implies that  $\mathbb{P}(F_{T_j(X_j)}(T_j(X_j)) \leq u_j \forall j)$  equals

$$\mathbb{P}(F_j(T_j^{\leftarrow}(T_j(X_j))) \leq u_j \forall j) \stackrel{(G13)}{=} \mathbb{P}(F_j(X_j) \leq u_j \forall j) \stackrel{\text{L.7.6}}{\underset{\text{"}\Rightarrow\text{"}}{=}} C(\mathbf{u}).$$

The claim follows from the if part (" $\Leftarrow$ ") of Lemma 7.6. □

## Interpretation of Sklar's Theorem (and the invariance principle)

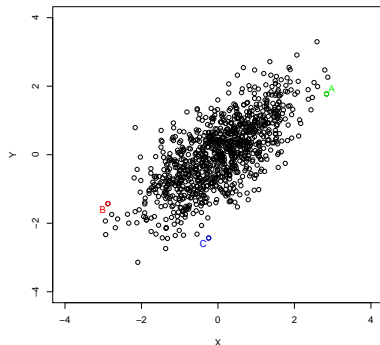
- 1) Part 1) of Sklar's Theorem allows one to **decompose any df  $F$  into its margins and a copula**. This, together with the invariance principle, allows one to **study dependence independently of the margins via the margin-free  $\mathbf{U} = (F_1(X_1), \dots, F_d(X_d))$  instead of  $\mathbf{X} = (X_1, \dots, X_d)$**  (they both have the same copula!). This is interesting for statistical applications, e.g. **parameter estimation** or **goodness-of-fit**.
- 2) Part 2) allows one to **construct flexible multivariate distributions** for particular applications (credit risk, stress testing, etc.).

## Visualizing Part 1) of Sklar's Theorem

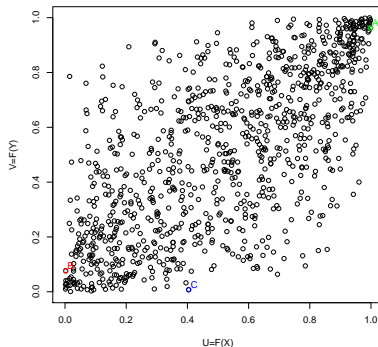
**Left:** Scatter plot of  $n = 1000$  samples from  $(X_1, X_2) \sim N_2(\mathbf{0}, P)$ , where  $P = \begin{pmatrix} 1 & 0.7 \\ 0.7 & 1 \end{pmatrix}$ . We mark three points A, B, C.

**Right:** Scatter plot of the corresponding Gauss copula (after applying the df  $\Phi$  of  $N(0, 1)$ ). Note how A, B, C change.

1000 realizations of  $(X,Y)$  for a joint normal distribution with  $\rho = 0.7$



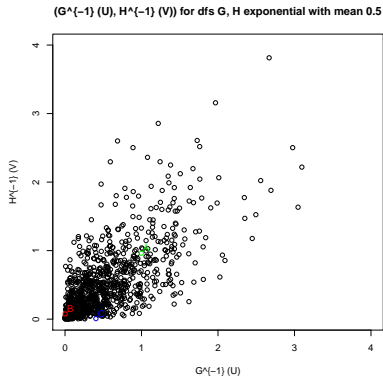
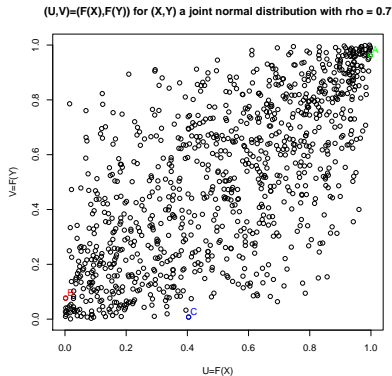
$(U,V)=(F(X),F(Y))$  for  $(X,Y)$  a joint normal distribution with  $\rho = 0.7$



## Visualizing Part 2) of Sklar's Theorem

**Left:** Same Gauss copula scatter plot as before. Apply marginal Exp(2)-quantile functions ( $F_j^{-1}(u) = -\log(1-u)/2$ ,  $j \in \{1, 2\}$ ).

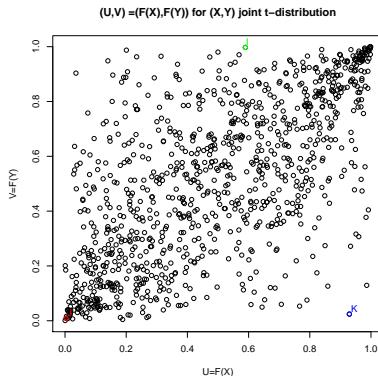
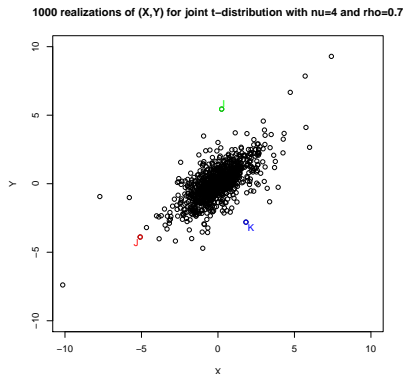
**Right:** The corresponding transformed random variates. Again, note the three points A, B, C.



## Visualizing Part 1) of Sklar's Theorem

**Left:** Scatter plot of  $n = 1000$  samples from  $(X_1, X_2) \sim t_2(4, \mathbf{0}, P)$ , where  $P = \begin{pmatrix} 1 & 0.7 \\ 0.7 & 1 \end{pmatrix}$ . We mark three points I, J, K.

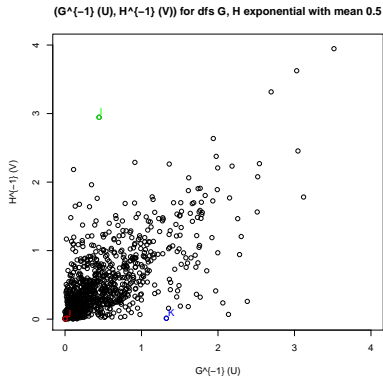
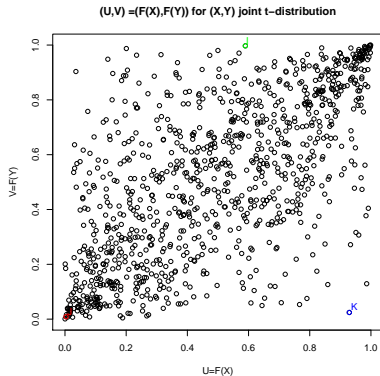
**Right:** Scatter plot of the corresponding  $t_4$  copula (after applying the df  $t_4$ ). Note how I, J, K change.



## Visualizing Part 2) of Sklar's Theorem

**Left:** Same  $t_4$  copula scatter plot as before. Apply marginal Exp(2)-quantile functions ( $F_j^{-1}(u) = -\log(1-u)/2$ ,  $j \in \{1, 2\}$ ).

**Right:** The corresponding transformed random variates. Again, note the three points I, J, K.



## Fréchet–Hoeffding bounds

### Theorem 7.8 (Fréchet–Hoeffding bounds)

Let  $W(\mathbf{u}) = \max\{\sum_{j=1}^d u_j - d + 1, 0\}$  and  $M(\mathbf{u}) = \min_{1 \leq j \leq d} \{u_j\}$ .

1) For any  $d$ -dimensional copula  $C$ ,

$$W(\mathbf{u}) \leq C(\mathbf{u}) \leq M(\mathbf{u}), \quad \mathbf{u} \in [0, 1]^d.$$

2)  $W$  is a copula if and only if  $d = 2$ .

3)  $M$  is a copula for all  $d \geq 2$ .

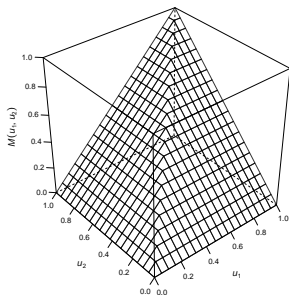
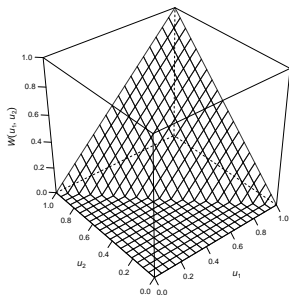
*Proof.* See the appendix. □

■ It is easy to verify that, for  $U \sim U(0, 1)$ ,

▶  $(U, \dots, U) \sim M$ ;

▶  $(U, 1 - U) \sim W$ .

- Plot of  $W, M$  for  $d = 2$  (compare with  $(U, 1 - U) \sim W$ ,  $(U, U) \sim M$ )



- The Fréchet–Hoeffding bounds correspond to perfect dependence (negative for  $W$ ; positive for  $M$ ); see Proposition 7.14 later.
- The Fréchet–Hoeffding bounds lead to bounds for any df  $F$ , via

$$\max\left\{\sum_{j=1}^d F_j(x_j) - d + 1, 0\right\} \leq F(\mathbf{x}) \leq \min_{1 \leq j \leq d} \{F_j(x_j)\}.$$

We will use them later to derive bounds for the correlation coefficient.



## 7.1.2 Examples of copulas

- *Fundamental copulas*: important special copulas;
- *Implicit copulas*: extracted from known  $F$  via Sklar's Theorem;
- *Explicit copulas*: closed form, arising from construction principles.

### Fundamental copulas

As usual, we assume the appearing margins  $F_1, \dots, F_d$  to be continuous.

- $\Pi(\mathbf{u}) = \prod_{j=1}^d u_j$  is the *independence copula* since  $C(F_1(x_1), \dots, F_d(x_d))$   
 $\stackrel{\text{Sklar}}{=} F(\mathbf{x}) \stackrel{\text{ind.}}{=} \prod_{j=1}^d F_j(x_j)$  if and only if  $C(\mathbf{u}) = \Pi(\mathbf{u})$  (replace  $x_j$  by  $F_j^{\leftarrow}(u_j)$  and apply (GI4)). Therefore,  $X_1, \dots, X_d$  are independent if and only if their copula is  $\Pi$ ; the density is thus  $c(\mathbf{u}) = 1, \mathbf{u} \in [0, 1]^d$ .
- $W$  is the *countermonotonicity copula*. It is the df of  $(U, 1 - U)$ . It can be shown that if  $X_1, X_2$  are perfectly negatively dependent ( $X_2$  is a.s. a strictly decreasing function of  $X_1$ ), their copula is  $W$ .

- $M$  is the *comonotonicity copula*. It is the df of  $(U, \dots, U)$ . It can be shown that if  $X_1, \dots, X_d$  are perfectly positively dependent ( $X_2, \dots, X_d$  are a.s. strictly increasing functions of  $X_1$ ), their copula is  $M$ .

## Implicit copulas

*Elliptical copulas* are implicit copulas arising from elliptical distributions via *Sklar's Theorem*. The two most prominent parametric families are the *Gauss copula* and the *t copula* (stemming from normal variance mixtures).

## Gauss copulas

- Consider (w.l.o.g.)  $\mathbf{X} \sim N_d(\mathbf{0}, P)$ . The *Gauss copula* (family) is given by

$$\begin{aligned} C_P^{\text{Ga}}(\mathbf{u}) &= \mathbb{P}(\Phi(X_1) \leq u_1, \dots, \Phi(X_d) \leq u_d) \\ &= \Phi_P(\Phi^{-1}(u_1), \dots, \Phi^{-1}(u_d)) \end{aligned}$$

where  $\Phi_P$  is the df of  $N_d(\mathbf{0}, P)$  and  $\Phi$  the df of  $N(0, 1)$ .

- Special cases: If  $P = I_d$  then  $C = \Pi$ , and if  $P = J_d = \mathbf{1}\mathbf{1}'$  then  $C = M$ .  
If  $d = 2$  and  $\rho = P_{12} = -1$  then  $C = W$ .
- Sklar's Theorem  $\Rightarrow$  The density of  $C(\mathbf{u}) = F(F_1^{\leftarrow}(u_1), \dots, F_d^{\leftarrow}(u_d))$  is

$$c(\mathbf{u}) = \frac{f(F_1^{\leftarrow}(u_1), \dots, F_d^{\leftarrow}(u_d))}{\prod_{j=1}^d f_j(F_j^{\leftarrow}(u_j))}, \quad \mathbf{u} \in (0, 1)^d.$$

In particular, the density of  $C_P^{\text{Ga}}$  is

$$c_P^{\text{Ga}}(\mathbf{u}) = \frac{1}{\sqrt{\det P}} \exp\left(-\frac{1}{2} \mathbf{x}'(P^{-1} - I_d) \mathbf{x}\right), \quad (28)$$

where  $\mathbf{x} = (\Phi^{-1}(u_1), \dots, \Phi^{-1}(u_d))$ .

## $t$ copulas

- Consider (w.l.o.g.)  $\mathbf{X} \sim t_d(\nu, \mathbf{0}, P)$ . The  $t$  copula (family) is given by

$$\begin{aligned} C_{\nu, P}^t(\mathbf{u}) &= \mathbb{P}(t_{\nu}(X_1) \leq u_1, \dots, t_{\nu}(X_d) \leq u_d) \\ &= t_{\nu, P}(t_{\nu}^{-1}(u_1), \dots, t_{\nu}^{-1}(u_d)) \end{aligned}$$

where  $t_{\nu,P}$  is the df of  $t_d(\nu, \mathbf{0}, P)$  and  $t_\nu$  the df of the univariate  $t$  distribution with  $\nu$  degrees of freedom.

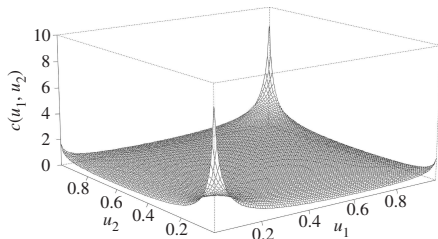
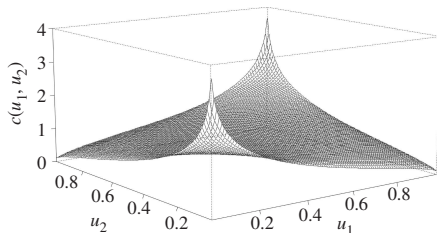
- Special cases:  $P = J_d = \mathbf{1}\mathbf{1}'$  then  $C = M$ . However, if  $P = I_d$  then  $C \neq \Pi$  (unless  $\nu = \infty$  in which case  $C_{\nu,P}^t = C_P^{\text{Ga}}$ ). If  $d = 2$  and  $\rho = P_{12} = -1$  then  $C = W$ .
- Sklar's Theorem  $\Rightarrow$  The density of  $C_{\nu,P}^t$  is

$$c_{\nu,P}^t(\mathbf{u}) = \frac{\Gamma((\nu + d)/2)}{\Gamma(\nu/2)\sqrt{\det P}} \left( \frac{\Gamma(\nu/2)}{\Gamma((\nu + 1)/2)} \right)^d \frac{(1 + \mathbf{x}'P^{-1}\mathbf{x}/\nu)^{-(\nu+d)/2}}{\prod_{j=1}^d (1 + x_j^2/\nu)^{-(\nu+1)/2}},$$

for  $\mathbf{x} = (t_\nu^{-1}(u_1), \dots, t_\nu^{-1}(u_d))$ .

- For more details, see Demarta and McNeil (2005).
- For scatter plots, see the visualization of Sklar's Theorem above. Note the difference in the tails: The smaller  $\nu$ , the more mass is concentrated in the joint tails.

Perspective plots of the densities of  $C_{\rho=0.3}^{\text{Ga}}$  (left) and  $C_{4,\rho=0.3}^t(\mathbf{u})$  (right).



Advantages and drawbacks of elliptical copulas:

### Advantages:

- Modelling pairwise dependencies (comparably flexible)
- Density available
- Sampling simple (for Gauss,  $t$ )

### Drawbacks:

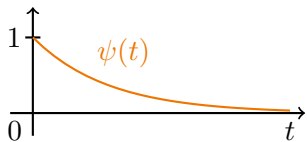
- Typically,  $C$  is not explicit
- Radially symmetric (so the same lower/upper tail behaviour)

## Explicit copulas

*Archimedean copulas* are copulas of the form

$$C(\mathbf{u}) = \psi(\psi^{-1}(u_1) + \cdots + \psi^{-1}(u_d))$$

where  $\psi$  is the *(Archimedean) generator*.

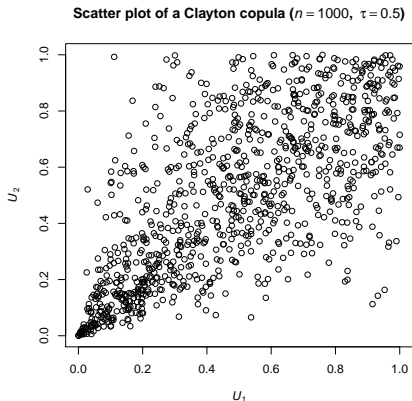
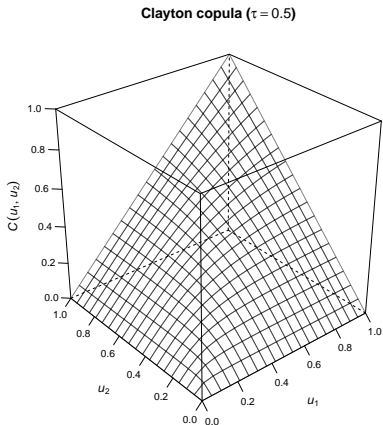


- $\psi : [0, \infty) \rightarrow [0, 1]$  is  $\downarrow$  on  $[0, \inf\{t : \psi(t) = 0\}]$  and satisfies  $\psi(0) = 1$ ,  $\psi(\infty) = \lim_{t \rightarrow \infty} \psi(t) = 0$ .
- We set  $\psi^{-1}(0) = \inf\{t : \psi(t) = 0\}$ .
- The set of all generators is denoted by  $\Psi$ .
- Not every generator  $\psi \in \Psi$  generates indeed a proper copula (there are conditions, e.g. complete monotonicity, i.e. derivatives alternating in sign).
- If  $\psi(t) > 0$ ,  $t \in [0, \infty)$ , we call  $\psi$  *strict*.

*Clayton copulas* are obtained for  $\psi(t) = (1+t)^{-1/\theta}$ ,  $t \in [0, \infty)$ ,  $\theta \in (0, \infty)$ . For  $\theta \downarrow 0$ ,  $C \rightarrow \Pi$ ; and for  $\theta \uparrow \infty$ ,  $C \rightarrow M$ .

**Left:** Plot of a bivariate *Clayton copula* (Kendall's tau 0.5; see later).

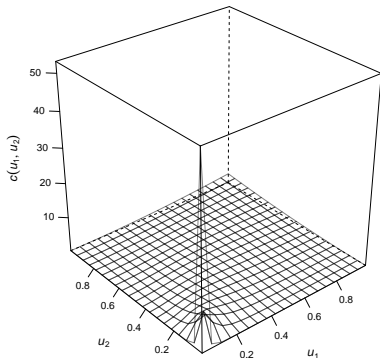
**Right:** Corresponding *scatter plot* (sample size  $n = 1000$ )



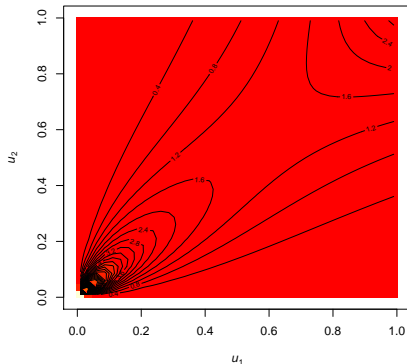
**Left:** Plot of the **corresponding density**.

**Right:** **Level plot** of the density (with heat colors).

Clayton copula density ( $\tau = 0.5$ )



Level plot Clayton copula density ( $\tau = 0.5$ )

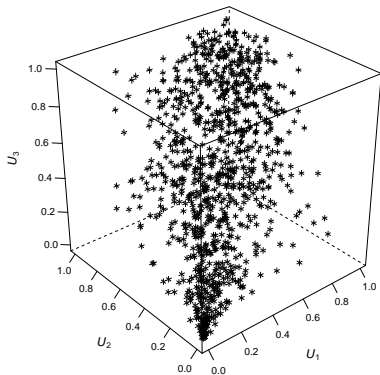




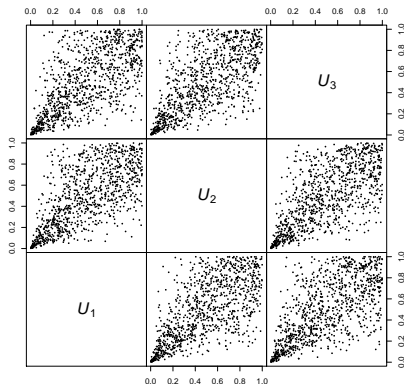
**Left:** Cloud plot of a trivariate Clayton copula (sample size  $n = 1000$ ; Kendall's tau 0.5).

**Right:** Corresponding scatter plot matrix.

Clayton copula cloud plot ( $n = 1000$ ,  $\tau = 0.5$ )



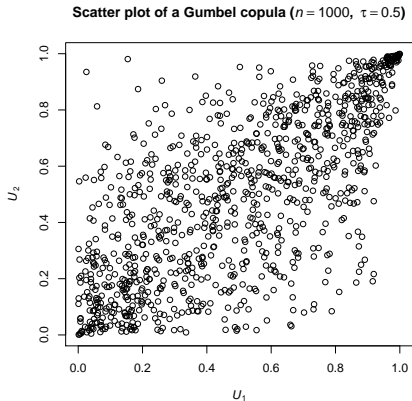
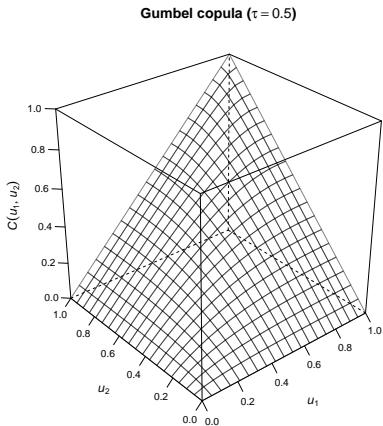
Scatter plot matrix of a Clayton copula ( $n = 1000$ ,  $\tau = 0.5$ )



*Gumbel copulas* are obtained for  $\psi(t) = \exp(-t^{1/\theta})$ ,  $t \in [0, \infty)$ ,  $\theta \in [1, \infty)$ . For  $\theta = 1$ ,  $C = \Pi$ ; and for  $\theta \rightarrow \infty$ ,  $C \rightarrow M$ .

**Left:** Plot of a bivariate *Gumbel copula* (Kendall's tau 0.5).

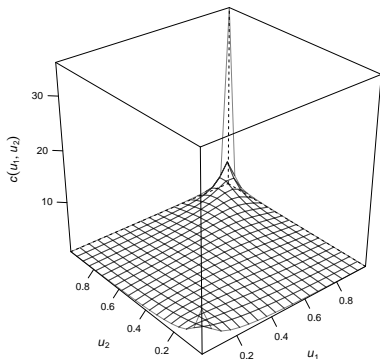
**Right:** Corresponding *scatter plot* (sample size  $n = 1000$ )



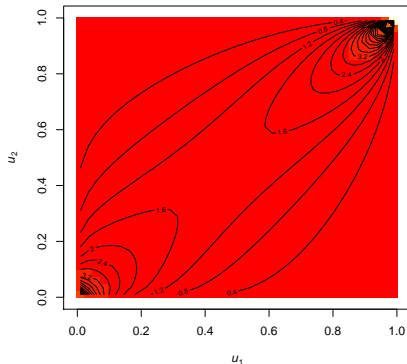
**Left:** Plot of the **corresponding density**.

**Right:** **Level plot** of the density (with heat colors).

**Gumbel copula density ( $\tau = 0.5$ )**



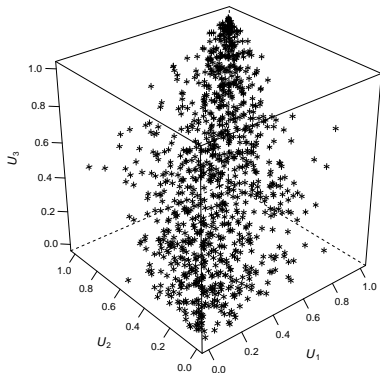
**Level plot Gumbel copula density ( $\tau = 0.5$ )**



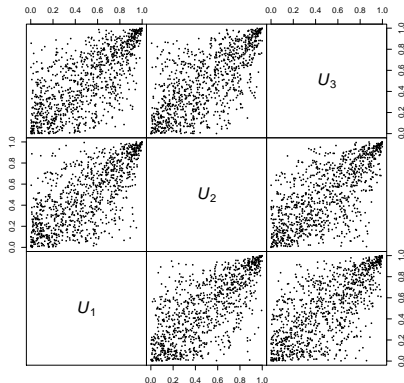
**Left:** Cloud plot of a trivariate Gumbel copula (sample size  $n = 1000$ ; Kendall's tau 0.5).

**Right:** Corresponding scatter plot matrix.

Gumbel copula cloud plot ( $n = 1000$ ,  $\tau = 0.5$ )



Scatter plot matrix of a Gumbel copula ( $n = 1000$ ,  $\tau = 0.5$ )



## Advantages and drawbacks of Archimedean copulas:

### Advantages:

- Typically **explicit** (if  $\psi^{-1}$  is available)
- Useful in calculations:  
**Properties** can typically be expressed **in terms of  $\psi$**
- **Densities** of various examples available
- **Sampling** often simple
- **Not restricted to radial symmetry**

### Drawbacks:

- All margins of the same dimension are equal (symmetry or **exchangeability**; see later)
- Often used only with a small **number of parameters** (some extensions available, but still less than  $d(d-1)/2$ )

### 7.1.3 Meta distributions

- *Fréchet class*: Class of all dfs  $F$  with given marginal dfs  $F_1, \dots, F_d$ ;  
*Meta- $C$  models*: All dfs  $F$  with the same given copula  $C$ .
- **Example**: A *meta- $t$  model* is a multivariate df  $F$  with  $t$  copula  $C$  and some margins  $F_1, \dots, F_d$ .

### 7.1.4 Simulation of copulas and meta distributions

#### Sampling implicit copulas

Due to their construction via Sklar's Theorem, implicit copulas can be sampled via Lemma 7.6.

#### Algorithm 7.9 (Simulation of implicit copulas)

- 1) Sample  $\mathbf{X} \sim F$ , where  $F$  is a df with continuous margins  $F_1, \dots, F_d$ .
- 2) Return  $\mathbf{U} = (F_1(X_1), \dots, F_d(X_d))$  (**probability transformation**).

## Example 7.10

- Sampling **Gauss copulas**  $C_P^{\text{Ga}}$ :

- 1) Sample  $\mathbf{X} \sim N_d(\mathbf{0}, P)$  ( $\mathbf{X} \stackrel{d}{=} A\mathbf{Z}$  for  $AA' = P$ ,  $\mathbf{Z} \sim N_d(\mathbf{0}, I_d)$ ).
- 2) Return  $\mathbf{U} = (\Phi(X_1), \dots, \Phi(X_d))$ .

- Sampling  **$t_\nu$  copulas**  $C_{\nu, P}^t$ :

- 1) Sample  $\mathbf{X} \sim t_d(\nu, \mathbf{0}, P)$  ( $\mathbf{X} \stackrel{d}{=} \sqrt{W}A\mathbf{Z}$  for  $W = \frac{1}{V}$ ,  $V \sim \Gamma(\frac{\nu}{2}, \frac{\nu}{2})$ ).
- 2) Return  $\mathbf{U} = (t_\nu(X_1), \dots, t_\nu(X_d))$ .

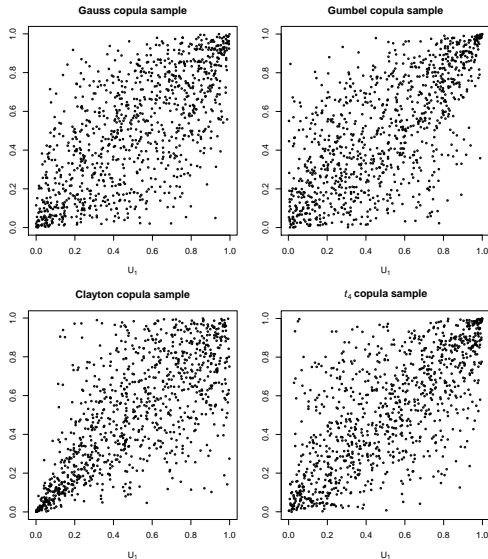
## Sampling meta distributions

Meta- $C$  distributions can be sampled via Sklar's Theorem, Part 2).

### Algorithm 7.11 (Sampling meta- $C$ models)

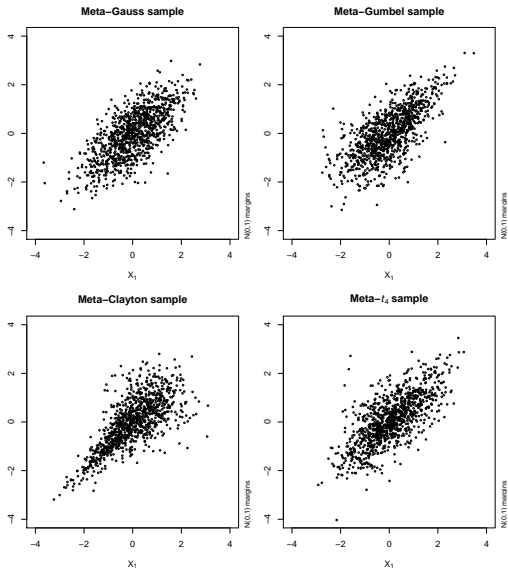
- 1) Sample  $\mathbf{U} \sim C$ .
- 2) Return  $\mathbf{X} = (F_1^{\leftarrow}(U_1), \dots, F_d^{\leftarrow}(U_d))$  (**quantile transformation**).

2000 samples from (a):  $C_{\rho=0.7}^{\text{Ga}}$ ; (b):  $C_{\theta=2}^{\text{G}}$ ; (c):  $C_{\theta=2.2}^{\text{C}}$ ; (d):  $C_{\nu=4, \rho=0.71}^t$





... transformed to  $N(0, 1)$  margins; all have linear correlation  $\approx 0.7$ !



## A general sampling algorithm

For a general copula  $C$  (without further information), the only known sampling algorithm is the conditional distribution method.

### Theorem 7.12 (Conditional distribution method)

If  $C$  is a  $d$ -dimensional copula and  $\mathbf{U}' \sim U(0, 1)^d$  then  $\mathbf{U} \sim C$ , where

$$U_1 = U'_1,$$

$$U_2 = C_{2|1}^{\leftarrow}(U'_2 | U_1),$$

$$U_3 = C_{3|1,2}^{\leftarrow}(U'_3 | U_1, U_2),$$

$$\vdots$$

$$U_d = C_{d|1,\dots,d-1}^{\leftarrow}(U'_d | U_1, \dots, U_{d-1})$$

and where  $C_{j|1,\dots,j-1}(u_j | u_1, \dots, u_{j-1}) = \mathbb{P}(U_j \leq u_j | U_1 = u_1, \dots, U_{j-1} = u_{j-1})$ ,  $j \in \{2, \dots, d\}$ .

### Theorem 7.13 (Schmitz (2003))

Let  $C$  be a  $d$ -dimensional copula which admits, for  $d \geq 3$ , continuous partial derivatives w.r.t.  $u_1, \dots, u_{d-1}$ . For a.e.  $u_1, \dots, u_{j-1} \in [0, 1]$ ,

$$C_{j|1, \dots, j-1}(u_j | u_1, \dots, u_{j-1}) = \frac{D_{j-1, \dots, 1} C^{(1, \dots, j)}(u_1, \dots, u_j)}{D_{j-1, \dots, 1} C^{(1, \dots, j-1)}(u_1, \dots, u_{j-1})},$$

where  $C^{(1, \dots, j)}(u_1, \dots, u_j) = C(u_1, \dots, u_j, 1, \dots, 1)$  and  $D_{j-1, \dots, 1}$  is the differential operator w.r.t.  $u_1, \dots, u_{j-1}$ .

**Note:**  $C_{2|1}(u_2 | u_1) = \frac{D_1 C(u_1, u_2)}{1} = D_1 C(u_1, u_2)$  which also follows from

$$\begin{aligned} & \lim_{h \downarrow 0} \frac{C(u_1 + h, u_2) - C(u_1, u_2)}{h} \\ &= \lim_{h \downarrow 0} \frac{\mathbb{P}(U_1 \leq u_1 + h, U_2 \leq u_2) - \mathbb{P}(U_1 \leq u_1, U_2 \leq u_2)}{h} \\ &= \lim_{h \downarrow 0} \frac{\mathbb{P}(U_2 \leq u_2, u_1 < U_1 \leq u_1 + h)}{\mathbb{P}(u_1 < U_1 \leq u_1 + h)} = \lim_{h \downarrow 0} \mathbb{P}(U_2 \leq u_2 | u_1 < U_1 \leq u_1 + h). \end{aligned}$$

## 7.1.5 Further properties of copulas

### Survival copulas

- If  $U \sim C$ , then  $1 - U \sim \hat{C}$ , the *survival copula* of  $C$ .
- $\hat{C}$  can be expressed as

$$\hat{C}(u) = \sum_{J \subseteq \{1, \dots, d\}} (-1)^{|J|} C((1 - u_1)^{I_{J(1)}}, \dots, (1 - u_d)^{I_{J(d)}})$$

in terms of its corresponding copula (essentially an application of the inclusion–exclusion principle). For  $d = 2$ ,  $\hat{C}(u_1, u_2) = 1 - (1 - u_1) - (1 - u_2) + C(1 - u_1, 1 - u_2) = -1 + u_1 + u_2 + C(1 - u_1, 1 - u_2)$ . We can also verify this directly by noting that  $\hat{C}(u_1, u_2)$  equals

$$\begin{aligned} \mathbb{P}(1 - U_1 \leq u_1, 1 - U_2 \leq u_2) &= \mathbb{P}(U_1 > 1 - u_1, U_2 > 1 - u_2) \\ &= \mathbb{P}(U_1 > 1 - u_1) - \mathbb{P}(U_1 > 1 - u_1, U_2 \leq 1 - u_2) \\ &= 1 - (1 - u_1) - (\mathbb{P}(U_2 \leq 1 - u_2) - \mathbb{P}(U_1 \leq 1 - u_1, U_2 \leq 1 - u_2)) \\ &= u_1 - (1 - u_2 - C(1 - u_1, 1 - u_2)). \end{aligned}$$

- If  $C$  admits a density,  $\hat{c}(\mathbf{u}) = c(\mathbf{1} - \mathbf{u})$ .
- If  $\hat{C} = C$ ,  $C$  is called *radially symmetric*. Check that  $W$ ,  $\Pi$ , and  $M$  are radially symmetric. All elliptical copulas are radially symmetric, too.
- One can show: If  $X_j$  is symmetrically distributed about  $\mu_j$ ,  $j \in \{1, \dots, d\}$ , then  $\mathbf{X}$  is radially symmetric about  $\boldsymbol{\mu}$  if and only if  $C = \hat{C}$ .
- Sklar's Theorem can also be formulated for survival functions. In this case, the main part reads

$$\bar{F}(\mathbf{x}) = \hat{C}(\bar{F}_1(x_1), \dots, \bar{F}_d(x_d)),$$

where  $\bar{F}(\mathbf{x}) = \mathbb{P}(\mathbf{X} > \mathbf{x})$  with corresponding marginal survival functions  $\bar{F}_1, \dots, \bar{F}_d$  (with  $\bar{F}_j(x) = \mathbb{P}(X_j > x)$ ). Hence survival copulas combine marginal to joint survival functions.

## Exchangeability

- $\mathbf{X}$  is *exchangeable* if

$$(X_1, \dots, X_d) \stackrel{d}{=} (X_{\pi(1)}, \dots, X_{\pi(d)})$$

for any permutation  $(\pi(1), \dots, \pi(d))$  of  $(1, \dots, d)$ .

- A copula  $C$  is *exchangeable* if it is the df of an exchangeable  $\mathbf{U}$  with  $U(0, 1)$  margins. This holds if only if  $C(u_1, \dots, u_d) = C(u_{\pi(1)}, \dots, u_{\pi(d)})$  for all possible permutations of arguments, i.e. if  $C$  is *symmetric*.
- Exchangeable/symmetric copulas are useful for approximate modelling homogeneous portfolios.
- **Examples:**
  - ▶ Archimedean copulas
  - ▶ Elliptical copulas (such as Gauss/ $t$ ) for equicorrelated  $P$  (i.e.  $P = \rho J_d + (1 - \rho)I_d$  for  $\rho \geq -1/(d - 1)$ ); in particular,  $d = 2$

## Copula densities

- By **Sklar's Theorem**, if  $F_j$  has density  $f_j$ ,  $j \in \{1, \dots, d\}$ , and  $C$  has density  $c$ , then the density  $f$  of  $F$  satisfies

$$f(\mathbf{x}) = c(F_1(x_1), \dots, F_d(x_d)) \prod_{j=1}^d f_j(x_j). \quad (29)$$

This implies

$$c(\mathbf{u}) = \frac{f(F_1^{-1}(u_1), \dots, F_d^{-1}(u_d))}{f_1(F_1^{-1}(u_1)) \cdots f_d(F_d^{-1}(u_d))}.$$

- It follows from (29) that the **log-density** splits into

$$\log f(\mathbf{x}) = \log c(F_1(x_1), \dots, F_d(x_d)) + \sum_{j=1}^d \log f_j(x_j)$$

which **allows for a two-stage estimation** (**marginal** and **copula parameters separately**); see Section 7.6.

## 7.2 Dependence concepts and measures

*Measures of association/dependence* are scalar measures which **summarize the dependence in terms of a single number**. There are better and worse examples of such measures, which we will study in this section.

### 7.2.1 Perfect dependence

$X_1, X_2$  are *countermonotone* if  $(X_1, X_2)$  has copula  $W$ .

$X_1, \dots, X_d$  are *comonotone* if  $(X_1, \dots, X_d)$  has copula  $M$ .

#### Proposition 7.14 (Perfect dependence)

- 1)  $X_2 = T(X_1)$  a.s. with decreasing  $T(x) = F_2^{\leftarrow}(1 - F_1(x))$  (countermonotone) **if and only if**  $C(u_1, u_2) = W(u_1, u_2)$ ,  $u_1, u_2 \in [0, 1]$ .
- 2)  $X_j = T_j(X_1)$  a.s. with increasing  $T_j(x) = F_j^{\leftarrow}(F_1(x))$ ,  $j \in \{2, \dots, d\}$ , (comonotone) **if and only if**  $C(\mathbf{u}) = M(\mathbf{u})$ ,  $\mathbf{u} \in [0, 1]^d$ .

*Proof.* See the appendix. □



### Proposition 7.15 (Comonotone additivity)

Let  $\alpha \in (0, 1)$  and  $X_j \sim F_j$ ,  $j \in \{1, \dots, d\}$ , be comontone. Then  $\text{VaR}_\alpha(X_1 + \dots + X_d) = F_{X_1 + \dots + X_d}^\leftarrow(\alpha) = F_1^\leftarrow(\alpha) + \dots + F_d^\leftarrow(\alpha) = \sum_{j=1}^d \text{VaR}_\alpha(X_j)$ .

*Proof.* Assume  $F_j \uparrow$  and continuous for all  $j$  (for the general case, see the appendix). Since  $X_j \stackrel{d}{=} F_j^{-1}(U)$ ,  $j \in \{1, \dots, d\}$ , for  $U \sim U(0, 1)$ , and since  $T(u) = F_1^{-1}(u) + \dots + F_d^{-1}(u)$  is  $\uparrow$  and continuous,

$$\begin{aligned} F_{\sum_{j=1}^d X_j}(T(\alpha)) &= F_{T(U)}(T(\alpha)) = \mathbb{P}(T(U) \leq T(\alpha)) \\ &= \mathbb{P}(U \leq T^{-1}(T(\alpha))) = \alpha, \end{aligned} \tag{30}$$

hence

$$\text{VaR}_\alpha\left(\sum_{j=1}^d X_j\right) = F_{\sum_{j=1}^d X_j}^{-1}(\alpha) \stackrel{(30)}{=} T(\alpha) = \sum_{j=1}^d \text{VaR}_\alpha(X_j).$$

□

## 7.2.2 Linear correlation

For two random variables  $X_1$  and  $X_2$  with  $\mathbb{E}(X_j^2) < \infty$ ,  $j \in \{1, 2\}$ , the (*linear* or *Pearson's*) *correlation coefficient*  $\rho$  is defined by

$$\rho(X_1, X_2) = \frac{\text{cov}(X_1, X_2)}{\sqrt{\text{var } X_1} \sqrt{\text{var } X_2}} = \frac{\mathbb{E}((X_1 - \mathbb{E}X_1)(X_2 - \mathbb{E}X_2))}{\sqrt{\mathbb{E}((X_1 - \mathbb{E}X_1)^2)} \sqrt{\mathbb{E}((X_2 - \mathbb{E}X_2)^2)}}.$$

### Classical properties and drawbacks of linear correlation

Let  $X_1$  and  $X_2$  be two random variables with  $\mathbb{E}(X_j^2) < \infty$ ,  $j \in \{1, 2\}$ .

**Note that  $\rho$  depends on the marginal distributions!** In particular, **second moments have to exist** (not the case, e.g. for  $X_1, X_2 \stackrel{\text{ind.}}{\sim} F(x) = 1 - x^{-3/2}!$ )

- $|\rho| \leq 1$ . Furthermore,  $|\rho| = 1$  if and only if there are constants  $a \in \mathbb{R} \setminus \{0\}, b \in \mathbb{R}$  with  $X_2 = aX_1 + b$  a.s. with  $a \geq 0$  if and only if  $\rho = \pm 1$ . **This discards other strong functional dependence such as  $X_2 = X_1^2$ , for example.**

- If  $X_1$  and  $X_2$  are independent, then  $\rho = 0$ . However, the converse is not true in general; see Example 7.17 below.
- $\rho$  is invariant under strictly increasing linear transformations on  $\text{ran } X_1 \times \text{ran } X_2$  but not invariant under strictly increasing functions in general. To see this, consider  $(X_1, X_2) \sim N_2(\mathbf{0}, P)$ . Then  $\rho(X_1, X_2) = P_{12}$ , but (as one can show)  $\rho(F_1(X_1), F_2(X_2)) = \frac{6}{\pi} \arcsin(P_{12}/2)$ .

### Proposition 7.16 (Hoeffding's formula)

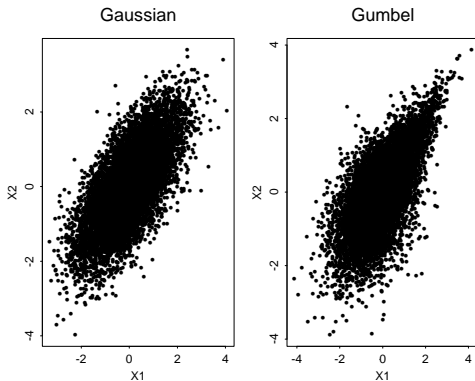
Let  $X_j \sim F_j$ ,  $j \in \{1, 2\}$ , be two random variables with  $\mathbb{E}(X_j^2) < \infty$ ,  $j \in \{1, 2\}$ , and joint distribution function  $F$ . Then

$$\text{cov}(X_1, X_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (F(x_1, x_2) - F_1(x_1)F_2(x_2)) \, dx_1 dx_2.$$

## Correlation fallacies

### Fallacy 1: $F_1$ , $F_2$ , and $\rho$ uniquely determine $F$

This is true for bivariate elliptical distributions, but wrong in general. The following samples both have  $N(0, 1)$  margins and correlation  $\rho = 0.7$ , yet come from different (copula) models:



Another example is this.

### Example 7.17 (Uncorrelated $\nRightarrow$ independent)

- Consider the two risks

$$X_1 = Z \quad (\text{Profit \& Loss Country A}),$$

$$X_2 = ZV \quad (\text{Profit \& Loss Country B}),$$

where  $V, Z$  are independent with  $Z \sim N(0, 1)$  and  $\mathbb{P}(V = -1) = \mathbb{P}(V = 1) = 1/2$ . Then  $X_2 \sim N(0, 1)$  and  $\rho(X_1, X_2) = \text{cov}(X_1, X_2) = \mathbb{E}(X_1 X_2) \underset{\text{ind.}}{=} \mathbb{E}(V)\mathbb{E}(Z^2) = 0$ , but  $X_1$  and  $X_2$  are not independent (in fact,  $V$  makes  $(X_1, X_2)$  switch between counter- and comonotonicity).

- Consider  $(X'_1, X'_2) \sim N_2(\mathbf{0}, I_2)$ . Both  $(X'_1, X'_2)$  and  $(X_1, X_2)$  have  $N(0, 1)$  margins and  $\rho = 0$ , but the copula of  $(X'_1, X'_2)$  is  $\Pi$  and the copula of  $(X_1, X_2)$  is the convex combination  $C(\mathbf{u}) = \lambda M(\mathbf{u}) + (1 - \lambda)W(\mathbf{u})$  for  $\lambda = 0.5$ .

## Fallacy 2: Given $F_1, F_2$ , any $\rho \in [-1, 1]$ is attainable

This is true for elliptically distributed  $(X_1, X_2)$  with  $\mathbb{E}(R^2) < \infty$  (as then  $\text{corr } \mathbf{X} = P$ ), but wrong in general:

- If  $F_1$  and  $F_2$  are not of the same type (no linearity),  $\rho(X_1, X_2) = 1$  is not attainable (recall that  $|\rho| = 1$  if and only if there are constants  $a \in \mathbb{R} \setminus \{0\}, b \in \mathbb{R}$  with  $X_2 = aX_1 + b$  a.s.).
- What is the attainable range then? Hoeffding's formula

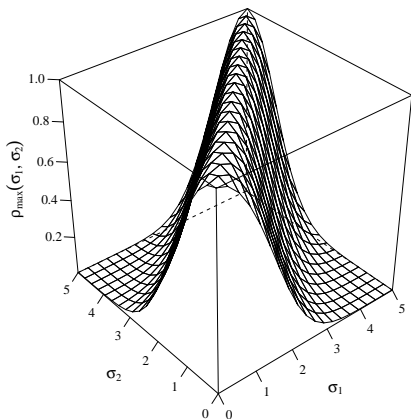
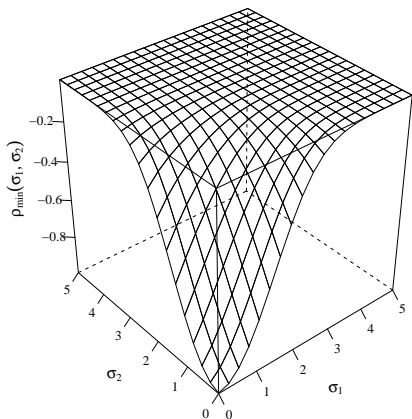
$$\text{cov}(X_1, X_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (C(F_1(x_1), F_2(x_2)) - F_1(x_1)F_2(x_2)) \, dx_1 dx_2.$$

implies bounds on attainable  $\rho$ :

$$\rho \in [\rho_{\min}, \rho_{\max}] \quad (\rho_{\min} \text{ is attained for } C = W, \rho_{\max} \text{ for } C = M).$$

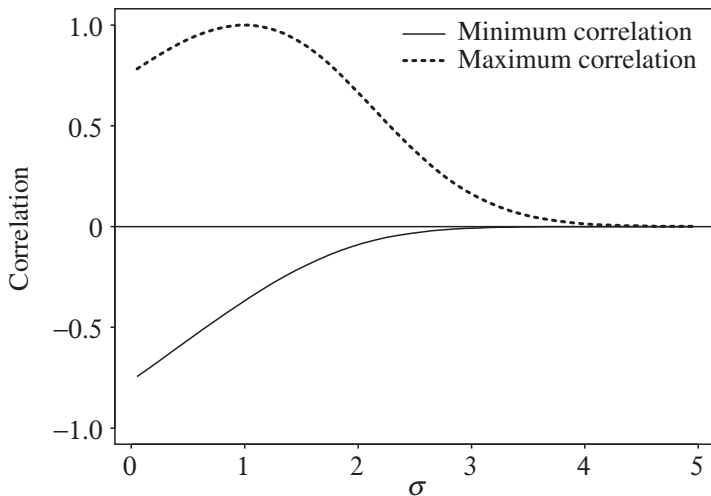
### Example 7.18 (Bounds for a model with $\text{LN}(0, \sigma_j^2)$ margins)

Let  $X_j \sim \text{LN}(0, \sigma_j^2)$ ,  $j \in \{1, 2\}$ . One can show that minimal ( $\rho_{\min}$ ; left) and maximal ( $\rho_{\max}$ ; right) correlations are given as follows.



For  $\sigma_1^2 = 1$ ,  $\sigma_2^2 = 16$  one has  $\rho \in [-0.0003, 0.0137]!$

Specifically, let  $X_1 \sim \text{LN}(0, 1)$  and  $X_2 \sim \text{LN}(0, \sigma^2)$ . Now let  $\sigma$  vary and plot  $\rho_{\min}$  and  $\rho_{\max}$  against  $\sigma$ :





### Fallacy 3: $\rho$ maximal (i.e. $C = M$ ) $\Rightarrow \text{VaR}_\alpha(X_1 + X_2)$ maximal

- This is true if  $(X_1, X_2)$  is elliptically distributed since the maximal  $\rho = 1$  implies that  $X_1, X_2$  are comonotone, so  $\text{VaR}_\alpha$  is additive (by Proposition 7.15) and additivity provides the largest possible bound in this case as  $\text{VaR}_\alpha$  is subadditive (by Proposition 6.24).
- Any superadditivity example  $\text{VaR}_\alpha(X_1 + X_2) > \text{VaR}_\alpha(X_1) + \text{VaR}_\alpha(X_2)$  serves as a counterexample as the right-hand side under comonotonicity (so maximal correlation) only equals  $\text{VaR}_\alpha(X_1 + X_2)$ ; see Section 2.3.5 and Proposition 7.15.

## 7.2.3 Rank correlation

Rank correlation coefficients are...

- ... always defined;
- ... invariant under strictly increasing transformations of the random variables (hence only depend on the underlying copula).

## Kendall's tau and Spearman's rho

### Definition 7.19 (Kendall's tau)

Let  $X_j \sim F_j$  with  $F_j$  continuous,  $j \in \{1, 2\}$ . Let  $(X'_1, X'_2)$  be an independent copy of  $(X_1, X_2)$ . *Kendall's tau* is defined by

$$\begin{aligned}\rho_\tau &= \mathbb{E}(\text{sign}((X_1 - X'_1)(X_2 - X'_2))) \\ &= \mathbb{P}((X_1 - X'_1)(X_2 - X'_2) > 0) - \mathbb{P}((X_1 - X'_1)(X_2 - X'_2) < 0),\end{aligned}$$

where  $\text{sign}(x) = I_{(0, \infty)}(x) - I_{(-\infty, 0)}(x)$  (so  $-1$  for  $x < 0$ ,  $0$  for  $x = 0$  and  $1$  for  $x > 0$ ).

By definition, Kendall's tau is *the probability of concordance* ( $\mathbb{P}((X_1 - X'_1)(X_2 - X'_2) > 0)$ ; probability of two independent points from  $F$  to have a positive slope) *minus the probability of discordance* ( $\mathbb{P}((X_1 - X'_1)(X_2 - X'_2) < 0)$ ; probability of two independent points from  $F$  to have a negative slope).

### Proposition 7.20 (Formula for Kendall's tau)

Let  $X_j \sim F_j$  with  $F_j$  continuous,  $j \in \{1, 2\}$ , and copula  $C$ . Then

$$\rho_\tau = 4 \int_0^1 \int_0^1 C(u_1, u_2) dC(u_1, u_2) - 1 = 4\mathbb{E}(C(U_1, U_2)) - 1,$$

where  $(U_1, U_2) \sim C$ .

*Proof.* See the appendix. □

An estimator of  $\rho_\tau$  is provided by the *sample version of Kendall's tau*

$$r_n^\tau = \frac{1}{\binom{n}{2}} \sum_{1 \leq i_1 < i_2 \leq n} \text{sign}((X_{i_1 1} - X_{i_2 1})(X_{i_1 2} - X_{i_2 2})). \quad (31)$$

### Definition 7.21 (Spearman's rho)

Let  $X_j \sim F_j$  with  $F_j$  continuous,  $j \in \{1, 2\}$ . *Spearman's rho* is defined by  $\rho_S = \rho(F_1(X_1), F_2(X_2))$ .

### Proposition 7.22 (Formula for Spearman's rho)

Let  $X_j \sim F_j$  with  $F_j$  continuous,  $j \in \{1, 2\}$ , and copula  $C$ . Then

$$\rho_S = 12 \int_0^1 \int_0^1 C(u_1, u_2) du_1 du_2 - 3 = 12\mathbb{E}(C(U'_1, U'_2)) - 3,$$

where  $(U'_1, U'_2) \sim \Pi$ .

*Proof.* By Hoeffding's formula, we have  $\rho_S(X_1, X_2) = \rho(F_1(X_1), F_2(X_2))$   
 $= 12 \int_0^1 \int_0^1 (C(u_1, u_2) - u_1 u_2) du_1 du_2 = 12 \int_0^1 \int_0^1 C(u_1, u_2) du_1 du_2 - 3.$

□

- An estimator  $r_n^S$  is given by the sample correlation computed from componentwise scaled ranks (the so-called *pseudo-observations*) of the data.
- For  $\kappa = \rho_\tau$  and  $\kappa = \rho_S$ , Embrechts, McNeil, et al. (2002) show that  $\kappa = \pm 1$  if and only if  $X_1, X_2$  are co-/countermonotonic. In general,  $\kappa = 0$  does not imply independence.

- **Fallacy 1** ( $F_1, F_2, \rho$  uniquely determine  $F$ ) is **not solved by** replacing  $\rho$  by **rank correlation coefficients  $\kappa$**  (it is easy to construct several copulas with the same Kendall's tau, e.g. via Archimedean copulas).
- **Fallacy 2** (For  $F_1, F_2$ , any  $\rho \in [-1, 1]$  is attainable) is **solved** when  $\rho$  is replaced by  $\rho_\tau$  or  $\rho_S$ . **Take**

$$F(x_1, x_2) = \lambda M(F_1(x_1), F_2(x_2)) + (1 - \lambda) W(F_1(x_1), F_2(x_2)).$$

This is a model with  $\rho_\tau = \rho_S = 2\lambda - 1$  (choose  $\lambda \in [0, 1]$  as desired).

- **Fallacy 3** ( $C = M$  implies  $\text{VaR}_\alpha(X_1 + X_2)$  maximal) is **also not solved by rank correlation coefficients  $\kappa = 1$** : Although  $\kappa = 1$  corresponds to  $C = M$ , this copula does not necessarily provide the largest  $\text{VaR}_\alpha(X_1 + X_2)$ ; see Fallacy 3 earlier.
- **Nevertheless**, rank correlations are useful **to summarize dependence**, to parameterize copula families to **make dependence comparable** and for copula parameter calibration or estimation.

## 7.2.4 Coefficients of tail dependence

**Goal:** Measure **extremal dependence**, i.e. dependence in the **joint tails**.

### Definition 7.23 (Tail dependence)

Let  $X_j \sim F_j$ ,  $j \in \{1, 2\}$ , be continuously distributed random variables. Provided that the limits exist, the **lower tail-dependence coefficient**  $\lambda_l$  and **upper tail-dependence coefficient**  $\lambda_u$  of  $X_1$  and  $X_2$  are defined by

$$\lambda_l = \lim_{u \downarrow 0} \mathbb{P}(X_2 \leq F_2^{\leftarrow}(u) \mid X_1 \leq F_1^{\leftarrow}(u)),$$

$$\lambda_u = \lim_{u \uparrow 1} \mathbb{P}(X_2 > F_2^{\leftarrow}(u) \mid X_1 > F_1^{\leftarrow}(u)).$$

If  $\lambda_l \in (0, 1]$  ( $\lambda_u \in (0, 1]$ ), then  $(X_1, X_2)$  is **lower (upper) tail dependent**.  
If  $\lambda_l = 0$  ( $\lambda_u = 0$ ), then  $(X_1, X_2)$  is **lower (upper) tail independent**.

As (conditional) probabilities, we clearly have  $\lambda_l, \lambda_u \in [0, 1]$ .

- Tail dependence is a copula property, since

$$\begin{aligned} \mathbb{P}(X_2 \leq F_2^{\leftarrow}(u) \mid X_1 \leq F_1^{\leftarrow}(u)) &= \frac{\mathbb{P}(X_1 \leq F_1^{\leftarrow}(u), X_2 \leq F_2^{\leftarrow}(u))}{\mathbb{P}(X_1 \leq F_1^{\leftarrow}(u))} \\ &= \frac{F(F_1^{\leftarrow}(u), F_2^{\leftarrow}(u))}{F_1(F_1^{\leftarrow}(u))} \stackrel{\text{Sklar}}{=} \frac{C(u, u)}{u}, \quad u \in (0, 1), \text{ so } \lambda_1 = \lim_{u \downarrow 0} \frac{C(u, u)}{u}. \end{aligned}$$

- If  $u \mapsto C(u, u)$  is differentiable in a neighborhood of 0 and the limit exists, then  $\lambda_1 = \lim_{u \downarrow 0} \frac{d}{du} C(u, u)$  (l'Hôpital's Rule).
- If  $C$  is totally differentiable in a neighborhood of 0 and the limit exists, then  $\lambda_1 = \lim_{u \downarrow 0} (D_1 C(u, u) + D_2 C(u, u))$  (Chain Rule). If  $C$  is exchangeable,  $\lambda_1 = 2 \lim_{u \downarrow 0} D_1 C(u, u) \stackrel{\text{Th. 7.13}}{=} 2 \lim_{u \downarrow 0} C_{2|1}(u \mid u) = 2 \lim_{u \downarrow 0} \mathbb{P}(U_2 \leq u \mid U_1 = u)$  for  $(U_1, U_2) \sim C$ . Combined with any continuous df  $F$ . (the same for both components) and  $(X_1, X_2) = (F^{\leftarrow}(U_1), F^{\leftarrow}(U_2))$ , one has

$$\lambda_1 = 2 \lim_{x \downarrow -\infty} \mathbb{P}(X_2 \leq x \mid X_1 = x). \quad (32)$$

which is useful for deriving  $\lambda_l$  for elliptical copulas.

- Similarly as above, for the upper tail-dependence coefficient,

$$\begin{aligned}\lambda_u &= \lim_{u \uparrow 1} \frac{1 - 2u + C(u, u)}{1 - u} = \lim_{u \downarrow 0} \frac{\hat{C}(u, u)}{u} \\ &= \lim_{u \uparrow 1} \frac{2(1 - u) - (1 - C(u, u))}{1 - u} = 2 - \lim_{u \uparrow 1} \frac{1 - C(u, u)}{1 - u}.\end{aligned}$$

- For all **radially symmetric copulas** (e.g. the bivariate  $C_P^{\text{Ga}}$  and  $C_{\nu, P}^t$  copulas), we have  $\lambda_l = \lambda_u =: \lambda$ .
- For **Archimedean copulas with strict  $\psi$** , a substitution and l'Hôpital's Rule show:

$$\begin{aligned}\lambda_l &= \lim_{u \downarrow 0} \frac{\psi(2\psi^{-1}(u))}{u} = \lim_{t \rightarrow \infty} \frac{\psi(2t)}{\psi(t)} = 2 \lim_{t \rightarrow \infty} \frac{\psi'(2t)}{\psi'(t)}, \\ \lambda_u &= 2 - \lim_{u \uparrow 1} \frac{1 - \psi(2\psi^{-1}(u))}{1 - u} = 2 - \lim_{t \downarrow 0} \frac{1 - \psi(2t)}{1 - \psi(t)} = 2 - 2 \lim_{t \downarrow 0} \frac{\psi'(2t)}{\psi'(t)}.\end{aligned}$$

**Clayton:**  $\lambda_l = 2^{-1/\theta}$ ,  $\lambda_u = 0$ ; **Gumbel:**  $\lambda_l = 0$ ,  $\lambda_u = 2 - 2^{1/\theta}$



## 7.3 Normal mixture copulas

... are the **copulas of multivariate normal** (mean-) **variance mixtures**  $\mathbf{X} \stackrel{d}{=} \mathbf{0} + \sqrt{W} \mathbf{A} \mathbf{Z}$ ,  $\mathbf{A} \mathbf{A}' = \mathbf{P}$ ,  $(\mathbf{X} \stackrel{d}{=} \mathbf{m}(W) + \sqrt{W} \mathbf{A} \mathbf{Z})$ ; e.g. Gauss,  $t$  copulas.

### 7.3.1 Tail dependence

#### Coefficients of tail dependence

Let  $(X_1, X_2)$  be distributed according to a normal variance mixture and assume (w.l.o.g.) that  $\boldsymbol{\mu} = (0, 0)$  and  $\mathbf{A} \mathbf{A}' = \mathbf{P} = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$ . In this case,  $F_1 = F_2$  and  $C$  is symmetric and radially symmetric. We thus obtain that

$$\lambda \stackrel{\text{radial}}{=} \lambda_l \stackrel{\text{symm.}}{=} \underset{(32)}{2} \lim_{x \downarrow -\infty} \mathbb{P}(X_2 \leq x \mid X_1 = x).$$

#### Example 7.24 ( $\lambda$ for the Gauss and $t$ copula)

- Considering the bivariate  $N(\mathbf{0}, \mathbf{P})$  density, one can show (via  $f_{X_2|X_1}(x_2 \mid x_1) = \frac{f_{X_1, X_2}(x_1, x_2)}{f_{X_1}(x_1)}$ ) that  $(X_2 \mid X_1 = x) \sim N(\rho x, 1 - \rho^2)$ . This implies that

$$\lambda = 2 \lim_{x \downarrow -\infty} \mathbb{P}(X_2 \leq x \mid X_1 = x) = 2 \lim_{x \downarrow -\infty} \Phi\left(\frac{x(1-\rho)}{\sqrt{1-\rho^2}}\right) = I_{\{\rho=1\}}.$$

- For  $C_{\nu, \rho}^t$ , one can show that  $(X_2 \mid X_1 = x) \sim t_{\nu+1}(\rho x, \frac{(1-\rho^2)(\nu+x^2)}{\nu+1})$  and thus  $\mathbb{P}(X_2 \leq x \mid X_1 = x) = t_{\nu+1}\left(\frac{x-\rho x}{\sqrt{\frac{(1-\rho^2)(\nu+x^2)}{\nu+1}}}\right)$ . Hence

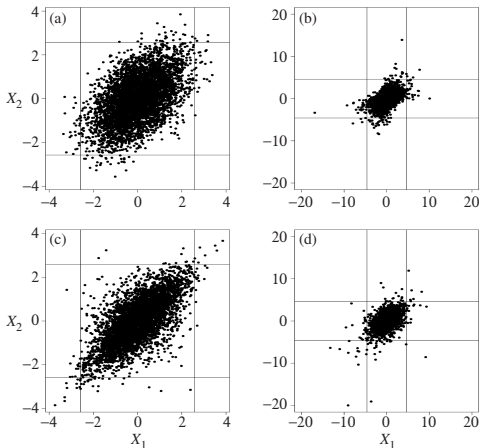
$$\lambda = 2t_{\nu+1}\left(-\sqrt{\frac{(\nu+1)(1-\rho)}{1+\rho}}\right) \quad (\text{tail dependence; } \lambda \uparrow \text{ in } \rho \uparrow \text{ and } \nu \downarrow).$$

- $\lambda$  values for various  $\nu, \rho$ :

$\nu$	$\rho = -0.5$	$\rho = 0$	$\rho = 0.5$	$\rho = 0.9$	$\rho = 1$
$\infty$	0	0	0	0	1
10	0.00	0.01	0.08	0.46	1
4	0.01	0.08	0.25	0.63	1
2	0.06	0.18	0.39	0.72	1

One can show that if  $W$  has a power tail,  $\lambda > 0$ , otherwise  $\lambda = 0$ .

## Joint quantile exceedance probabilities



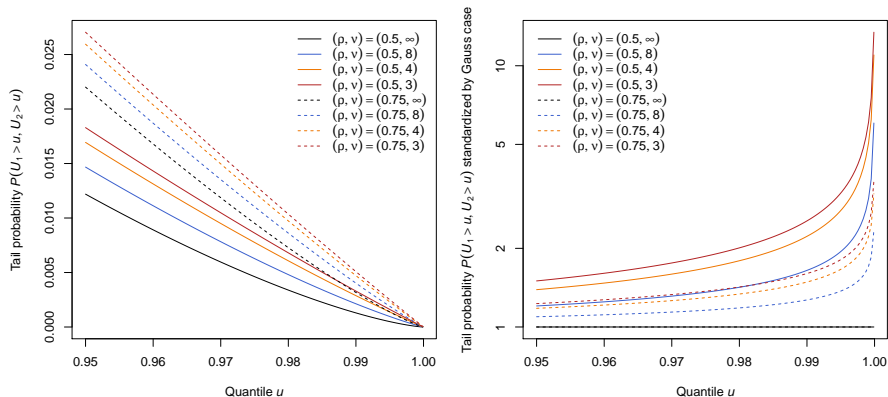
5000 samples from

- (a)  $N_2(\mathbf{0}, P = (\frac{1}{\rho} \begin{pmatrix} \rho & 1 \\ 1 & \rho \end{pmatrix}))$ ,  $\rho = 0.5$ ;
- (b)  $C_{\rho}^{\text{Ga}}$  with  $t_4$  margins (same dependence as in (a));
- (c)  $C_{4,\rho}^t$  with  $N(0, 1)$  margins;
- (d)  $t_2(4, \mathbf{0}, P)$  (same dependence as in (c)).

Lines denote the true marginal 0.005- and 0.995-quantiles.

Note the different number of points in the bivariate tails (all models have the same Kendall's tau!)

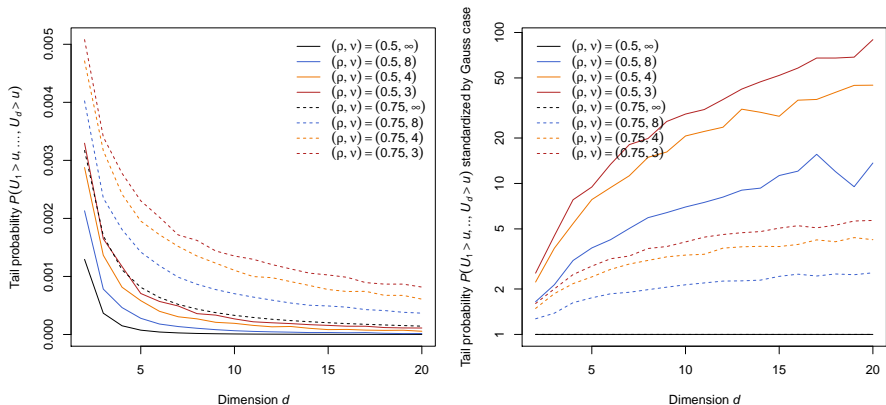
## Joint tail probabilities $\mathbb{P}(U_1 > u, U_2 > u)$ for $d = 2$



■ **Left:** The higher  $\rho$  or the smaller  $\nu$ , the larger  $\mathbb{P}(U_1 > u, U_2 > u)$ .

■ **Right:**  $u \mapsto \frac{\mathbb{P}(U_1 > u, U_2 > u)}{\mathbb{P}(V_1 > u, V_2 > u)} \stackrel{\text{radial}}{=} \frac{C_{\nu, \rho}^t(u, u)}{\stackrel{\text{symm.}}{C_{\rho}^{\text{Ga}}(u, u)}}$

## Joint tail probabilities $\mathbb{P}(U_1 > u, \dots, U_d > u)$ for $u = 0.99$



- Homogeneous  $P$  (off-diagonal entry  $\rho$ ). Note the MC randomness.
- **Left:** Clear; less mass in corners in higher dimensions.
- **Right:**  $d \mapsto \frac{\mathbb{P}(U_1 > u, \dots, U_d > u)}{\mathbb{P}(V_1 > u, \dots, V_d > u)} \stackrel{\text{radial}}{\underset{\text{symm.}}{=}} \frac{C_{\nu, \rho}^t(u, \dots, u)}{C_{\rho}^{\text{Ga}}(u, \dots, u)}$  for  $u = 0.99$ .

### Example 7.25 (Interpretation of joint tail probabilities)

- Consider 5 daily negative log-returns  $\mathbf{X} = (X_1, \dots, X_5)$ . Assume they follow an elliptical distribution and have pairwise correlations  $\rho = 0.5$ . However, we are unsure about the best joint model.
- If  $\mathbf{X}$  are multivariate normal (and thus  $C_{\rho=0.5}^{\text{Ga}}$ ), the probability that on any day all 5 negative returns lie above their  $u = 0.99$  quantiles is

$$\mathbb{P}(X_1 > F_1^{\leftarrow}(u), \dots, X_5 > F_5^{\leftarrow}(u)) = \mathbb{P}(U_1 > u, \dots, U_5 > u) \\ \approx \underset{\text{MC error}}{7.48 \times 10^{-5}}.$$

In the long run such an event will happen once every  $1/7.48 \times 10^{-5} \approx 13\,369$  trading days on average ( $\approx$  once every 51.4 years; assuming 260 trading days in a year).

- If  $\mathbf{X}$  is multivariate  $t_3$  (and thus  $C_{\nu=3, \rho=0.5}^t$ ), however, such an event will happen approximately 10 times more often, i.e.  $\approx$  once every 5.14 years. This gets worse the larger  $d$ !

## 7.3.2 Rank correlations

### Proposition 7.26 (Spearman's rho for normal variance mixtures)

Let  $\mathbf{X} \sim M_2(\mathbf{0}, P, \hat{F}_W)$  with  $\mathbb{P}(\mathbf{X} = \mathbf{0}) = 0$ ,  $\rho = P_{12}$ . Then

$$\rho_S = \frac{6}{\pi} \mathbb{E} \left( \arcsin \frac{W\rho}{\sqrt{(W + \tilde{W})(W + \bar{W})}} \right),$$

for  $W, \tilde{W}, \bar{W} \stackrel{\text{ind.}}{\sim} F_W$  with Laplace–Stieltjes transform  $\hat{F}_W$ . For Gauss copulas,  $\rho_S = \frac{6}{\pi} \arcsin(\frac{\rho}{2})$ .

*Proof.* See the appendix. □

### Proposition 7.27 (Kendall's tau for elliptical distributions)

Let  $\mathbf{X} \sim E_2(\mathbf{0}, P, \psi)$  with  $\mathbb{P}(\mathbf{X} = \mathbf{0}) = 0$ ,  $\rho = P_{12}$ . Then  $\rho_\tau = \frac{2}{\pi} \arcsin \rho$ .

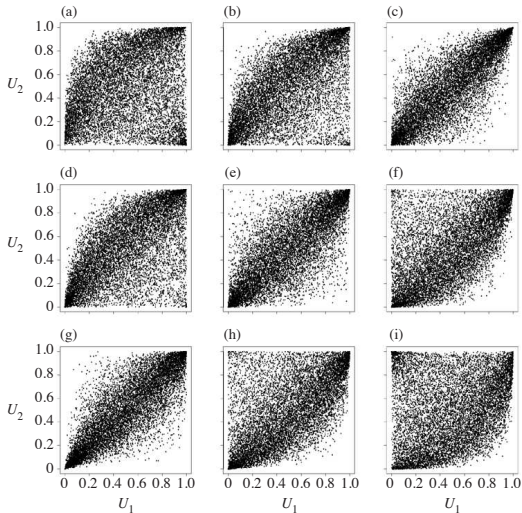
*Proof.* See the appendix. □

### 7.3.3 Skewed normal mixture copulas

- *Skewed normal mixture copulas* are the copulas of normal mixture distributions which are not elliptical, e.g. the *skewed  $t$  copula*  $C_{\nu, P, \gamma}^t$  is the copula of a generalized hyperbolic distribution; see McNeil, Frey, and Embrechts (2015, Sections 6.2.3 and 7.3.3) for more details.
- It can be sampled as other implicit copulas; see Algorithm 7.9 (the evaluation of the margins requires numerical integration of a skewed  $t$  density).
- The main advantage of such a copula over  $C_{\nu, P}^t$  is its radial asymmetry (e.g. for modelling  $\lambda_l \neq \lambda_u$ )



10 000 samples from  $C_{\nu=5, \rho=0.8, \gamma=0.8(I_{\{i<2\}}-I_{\{i>2\}}, I_{\{j>2\}}-I_{\{j<2\}})}$ :



(a)  $\gamma = (0.8, -0.8)$

(b)  $\gamma = (0.8, 0)$

(c)  $\gamma = (0.8, 0.8)$

(d)  $\gamma = (0, -0.8)$

(e)  $\gamma = (0, 0)$

(f)  $\gamma = (0, 0.8)$

(g)  $\gamma = (-0.8, -0.8)$

(h)  $\gamma = (-0.8, 0)$

(i)  $\gamma = (-0.8, 0.8)$

### 7.3.4 Grouped normal mixture copulas

- *Grouped normal mixture copulas* are copulas which attach together a set of normal mixture copulas.
- Let  $\mathbf{Y} \sim N_d(\mathbf{0}, P)$  (so  $\mathbf{Y} \stackrel{d}{=} A\mathbf{Z}$  as before). The *grouped  $t$  copula* is the copula of

$$\mathbf{X} = (\sqrt{W_1}Y_1, \dots, \sqrt{W_1}Y_{s_1}, \dots, \sqrt{W_S}Y_{s_1+\dots+s_{S-1}+1}, \dots, \sqrt{W_S}Y_d)$$

for  $(W_1, \dots, W_S) \sim M(\text{IG}(\frac{\nu_1}{2}, \frac{\nu_1}{2}), \dots, \text{IG}(\frac{\nu_S}{2}, \frac{\nu_S}{2}))$ ; see Demarta and McNeil (2005) for details.

- Clearly, the marginals are  $t$  distributed, hence

$$\mathbf{U} = (t_{\nu_1}(X_1), \dots, t_{\nu_1}(X_{s_1}), \dots, t_{\nu_S}(X_{s_1+\dots+s_{S-1}+1}), \dots, t_{\nu_S}(X_d))$$

follows a *grouped  $t$  copula*. This is straightforward to simulate.

- It can be fitted with pairwise inversion of Kendall's tau.
- If  $S = d$ , grouped  $t$  copulas are also known as *generalized  $t$  copulas*; see Luo and Shevchenko (2010).

## 7.4 Archimedean copulas

Recall that an (Archimedean) generator  $\psi$  is a function  $\psi : [0, \infty) \rightarrow [0, 1]$  which is  $\downarrow$  on  $[0, \inf\{t : \psi(t) = 0\}]$  and satisfies  $\psi(0) = 1$ ,  $\psi(\infty) = \lim_{t \rightarrow \infty} \psi(t) = 0$ ; the set of all generators is denoted by  $\Psi$ .

### 7.4.1 Bivariate Archimedean copulas

#### Theorem 7.28 (Bivariate Archimedean copulas)

For  $\psi \in \Psi$ ,  $C(u_1, u_2) = \psi(\psi^{-1}(u_1) + \psi^{-1}(u_2))$  is a copula if and only if  $\psi$  is convex.

- For a strict and twice-continuously differentiable  $\psi$ , one can show that

$$\rho_{\tau} = 1 - 4 \int_0^{\infty} t(\psi'(t))^2 dt = 1 + 4 \int_0^1 \frac{\psi^{-1}(t)}{(\psi^{-1}(t))'} dt.$$

- If  $\psi$  is strict,  $\lambda_l = 2 \lim_{t \rightarrow \infty} \frac{\psi'(2t)}{\psi'(t)}$  and  $\lambda_u = 2 - 2 \lim_{t \downarrow 0} \frac{\psi'(2t)}{\psi'(t)}$  (as seen before).

- The most widely used one-parameter Archimedean copulas are:

Family	$\rho_\tau$	$\lambda_l$	$\lambda_u$
A	$1 - 2(\theta + (1 - \theta)^2 \log(1 - \theta)) / (3\theta^2)$	0	0
C	$\theta / (\theta + 2)$	$2^{-1/\theta}$	0
F	$1 + 4(D_1(\theta) - 1) / \theta$	0	0
G	$(\theta - 1) / \theta$	0	$2 - 2^{1/\theta}$
J	$1 - 4 \sum_{k=1}^{\infty} 1 / (k(\theta k + 2)(\theta(k - 1) + 2))$	0	$2 - 2^{1/\theta}$

Family	$\theta$	$\psi(t)$	$V \sim F = \mathcal{LS}^{-1}(\psi)$
A	$[0, 1)$	$(1 - \theta) / (\exp(t) - \theta)$	$\text{Geo}(1 - \theta)$
C	$(0, \infty)$	$(1 + t)^{-1/\theta}$	$\Gamma(1/\theta, 1)$
F	$(0, \infty) - \log(1 - (1 - e^{-\theta}) \exp(-t)) / \theta$		$\text{Log}(1 - e^{-\theta})$
G	$[1, \infty)$	$\exp(-t^{1/\theta})$	$S(1/\theta, 1, \cos^\theta(\pi/(2\theta)), I_{\{\theta=1\}}; 1)$
J	$[1, \infty)$	$1 - (1 - \exp(-t))^{1/\theta}$	$\text{Sibuya}(1/\theta)$

## 7.4.2 Multivariate Archimedean copulas

$\psi$  is *completely monotone (c.m.)* if  $(-1)^k \psi^{(k)}(t) \geq 0$  for all  $t \in (0, \infty)$  and all  $k \in \mathbb{N}_0$ . The set of all c.m. generators is denoted by  $\Psi_\infty$ .

### Theorem 7.29 (Kimberling (1974))

If  $\psi \in \Psi$ ,  $C(\mathbf{u}) = \psi\left(\sum_{j=1}^d \psi^{-1}(u_j)\right)$  is a copula  $\forall d$  if and only if  $\psi \in \Psi_\infty$ .

Bernstein's Theorem characterizes all  $\psi \in \Psi_\infty$ .

### Theorem 7.30 (Bernstein (1928))

$\psi(0) = 1$ ,  $\psi$  c.m. if and only if  $\psi(t) = \mathbb{E}(\exp(-tV))$  for  $V \sim G$  with  $V \geq 0$  and  $G(0) = 0$ .

We thus use the notation  $\psi = \hat{G}$  and call all Archimedean copulas with  $\psi \in \Psi_\infty$  *LT-Archimedean copulas*.

### Proposition 7.31 (Stochastic representation, related properties)

Let  $\psi \in \Psi_\infty$  with  $V \sim G$  such that  $\hat{G} = \psi$  and let  $E_1, \dots, E_d \stackrel{\text{ind.}}{\sim} \text{Exp}(1)$  be independent of  $V$ . Then

- 1) The survival copula of  $\mathbf{X} = (\frac{E_1}{V}, \dots, \frac{E_d}{V})$  is Archimedean (with  $\psi$ ).
- 2)  $\mathbf{U} = (\psi(X_1), \dots, \psi(X_d)) \sim C$  and the  $U_j$ 's are conditionally independent given  $V$  with  $\mathbb{P}(U_j \leq u \mid V = v) = \exp(-v\psi^{-1}(u))$ .

*Proof.*

- 1) The joint survival function of  $\mathbf{X}$  is given by

$$\begin{aligned}\bar{F}(\mathbf{x}) &= \mathbb{P}(X_j > x_j \ \forall j) = \int_0^\infty \mathbb{P}(E_j/V > x_j \ \forall j \mid V = v) \, dG(v) \\ &= \int_0^\infty \mathbb{P}(E_j > vx_j \ \forall j) \, dG(v) = \int_0^\infty \prod_{j=1}^d \exp(-vx_j) \, dG(v) \\ &= \int_0^\infty \exp\left(-v \sum_{j=1}^d x_j\right) \, dG(v) = \psi\left(\sum_{j=1}^d x_j\right).\end{aligned}$$

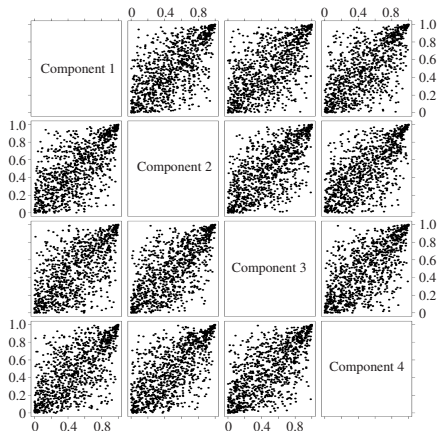
The  $j$ th marginal survival function is thus (set  $x_k = 0 \ \forall k \neq j$ )  
 $\bar{F}_j(x_j) = \mathbb{P}(X_j > x_j) = \psi(x_j)$  ( $\downarrow$  and continuous) and therefore  
 $\hat{C}(\mathbf{u}) = \bar{F}(\bar{F}_1^{\leftarrow}(u_1), \dots, \bar{F}_d^{\leftarrow}(u_d)) = \psi(\sum_{j=1}^d \psi^{-1}(u_j))$ .

- 2)  $\mathbb{P}(\mathbf{U} \leq \mathbf{u}) = \mathbb{P}(X_j > \psi^{-1}(u_j) \ \forall j) \stackrel{1)}{=} \psi(\sum_{j=1}^d \psi^{-1}(u_j))$ . Conditional independence is clear by construction and  $\mathbb{P}(U_j \leq u \mid V = v) = \mathbb{P}(X_j > \psi^{-1}(u) \mid V = v) = \mathbb{P}(E_j > v\psi^{-1}(u)) = \exp(-v\psi^{-1}(u))$ .  $\square$

### Algorithm 7.32 (Marshall and Olkin (1988))

- 1) Sample  $V \sim G$  (df corresponding to  $\psi$ ).
- 2) Sample  $E_1, \dots, E_d \stackrel{\text{ind.}}{\sim} \text{Exp}(1)$  independently of  $V$ .
- 3) Return  $\mathbf{U} = (\psi(E_1/V), \dots, \psi(E_d/V))$  (conditional independence).

1000 samples of a 4-dim. Gumbel copula ( $\rho_\tau = 0.5$ ;  $\lambda_u \approx 0.5858$ )

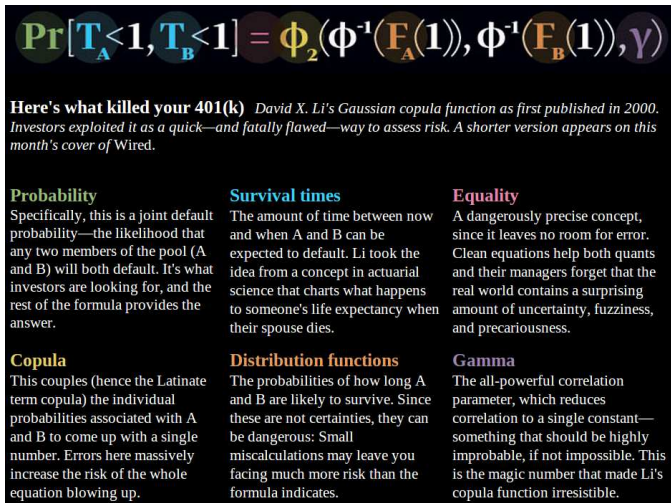


- Various non-exchangeable extensions to Archimedean copulas exist.
- For fixed  $d$ , c.m. can be relaxed to  $d$ -monotonicity; see McNeil and Nešlehová (2009).



## 7.4.3 Copulas and credit risk

Felix Salmon: “Recipe for Disaster: The Formula That Killed Wall Street”



**Pr** $[T_A < 1, T_B < 1] = \Phi_2(\Phi^{-1}(F_A(1)), \Phi^{-1}(F_B(1)), \gamma)$

**Here's what killed your 401(k)** *David X. Li's Gaussian copula function as first published in 2000. Investors exploited it as a quick—and fatally flawed—way to assess risk. A shorter version appears on this month's cover of Wired.*

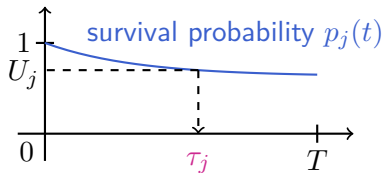
<p><b>Probability</b></p> <p>Specifically, this is a joint default probability—the likelihood that any two members of the pool (A and B) will both default. It's what investors are looking for, and the rest of the formula provides the answer.</p>	<p><b>Survival times</b></p> <p>The amount of time between now and when A and B can be expected to default. Li took the idea from a concept in actuarial science that charts what happens to someone's life expectancy when their spouse dies.</p>	<p><b>Equality</b></p> <p>A dangerously precise concept, since it leaves no room for error. Clean equations help both quants and their managers forget that the real world contains a surprising amount of uncertainty, fuzziness, and precariousness.</p>
<p><b>Copula</b></p> <p>This couples (hence the Latinate term copula) the individual probabilities associated with A and B to come up with a single number. Errors here massively increase the risk of the whole equation blowing up.</p>	<p><b>Distribution functions</b></p> <p>The probabilities of how long A and B are likely to survive. Since these are not certainties, they can be dangerous: Small miscalculations may leave you facing much more risk than the formula indicates.</p>	<p><b>Gamma</b></p> <p>The all-powerful correlation parameter, which reduces correlation to a single constant—something that should be highly improbable, if not impossible. This is the magic number that made Li's copula function irresistible.</p>

## How intensity-/copula-based default models work

Intensity-based **default model**:

$$p_j(t) = \exp\left(-\int_0^t \lambda_j(s) ds\right)$$

$$\tau_j = \inf\{t \geq 0 : p_j(t) \leq U_j\}$$



**Note:**  $\lambda_U = 0$  (as for the Gauss copula!)

$\Rightarrow$  (Almost) **no joint defaults!** ( $p_j$  typically **very flat**)

Copulas for the **triggers**  $U$ :

- 1) D. X. Li (2000): **Gauss** (Sibuya (1960):  $\lambda_U = 0$ )
- 2) Schönbucher and Schubert (2001): **Archimedean** ( $\lambda_U > 0$ )
- 3) Hofert and Scherer (2011): **nested Archimedean** ( $\lambda_U > 0$ , **hierarchies**)

**Typical application:** **CDO pricing models** based on iTraxx data.

## 7.5 A proof for subadditivity of ES

### Proposition 7.33 (Subadditivity of ES)

$$\text{ES}_\alpha(L) = \frac{\sup_{\{\tilde{Y} \sim B(1, 1-\alpha)\}} \mathbb{E}(L\tilde{Y})}{1 - \alpha},$$
 which is subadditive; the supremum is taken over all copulas between  $L \sim F_L$  and  $\tilde{Y} \sim B(1, 1 - \alpha)$ .

*Proof.*

- Let  $L = F_L^\leftarrow(U)$  and  $Y = I_{\{U > \alpha\}} \sim B(1, 1 - \alpha)$  for  $U \sim U(0, 1)$ .
  - Then  $\text{ES}_\alpha(L) = \frac{1}{1-\alpha} \int_\alpha^1 F_L^\leftarrow(u) \, du = \frac{1}{1-\alpha} \int_0^1 F_L^\leftarrow(u) I_{\{u > \alpha\}} \cdot 1 \, du = \frac{1}{1-\alpha} \mathbb{E}(F_L^\leftarrow(U) I_{\{U > \alpha\}}) = \frac{1}{1-\alpha} \mathbb{E}(LY)$ .
  - $L$  and  $Y$  are comontone. For any other  $(L, \tilde{Y})$  with  $\tilde{Y} \sim B(1, 1 - \alpha)$ ,  

$$\mathbb{E}(L\tilde{Y}) = \text{cov}(L, \tilde{Y}) + \mathbb{E}(L)\mathbb{E}(\tilde{Y}) \underset{\text{Hoeffding}}{\leq} \text{cov}(L, Y) + \mathbb{E}(L)\mathbb{E}(Y) = \mathbb{E}(LY)$$
- and thus  $\text{ES}_\alpha(L) = \frac{1}{1-\alpha} \sup_{\{\tilde{Y} \sim B(1, 1-\alpha)\}} \mathbb{E}(L\tilde{Y})$ . □

## 7.6 Fitting copulas to data

- Let  $\mathbf{X}, \mathbf{X}_1, \dots, \mathbf{X}_n \stackrel{\text{ind.}}{\sim} F$  with cont. margins  $F_1, \dots, F_d$  and copula  $C$ .
- We assume that we have data  $\mathbf{x}_1, \dots, \mathbf{x}_n$ , interpreted as realizations of  $\mathbf{X}_1, \dots, \mathbf{X}_n$ ; in what follows we work with the latter.
- Assume
  - ▶  $F_j = F_j(\cdot; \theta_{0,j})$  for some  $\theta_{0,j} \in \Theta_j$ ,  $j \in \{1, \dots, d\}$ ;  
( $F_j(\cdot; \theta_j)$  is assumed to be continuous  $\forall \theta_j \in \Theta_j$ ,  $j \in \{1, \dots, d\}$ )
  - ▶  $C = C(\cdot; \theta_{0,C})$  for some  $\theta_{0,C} \in \Theta_C$ .

Thus  $F$  has the true but unknown parameter vector  $\theta_0 = (\theta'_{0,C}, \theta'_{0,1}, \dots, \theta'_{0,d})'$  to be estimated.

- Here, we focus particularly on  $\theta_{0,C}$ . Whenever necessary, we assume that the margins  $F_1, \dots, F_d$  and the copula  $C$  are absolutely continuous with corresponding densities  $f_1, \dots, f_d$  and  $c$ , respectively.
- We assume the chosen copula to be appropriate (w.r.t. symmetry etc.).

## 7.6.1 Method-of-moments using rank correlation

- For  $d = 2$  and one-parameter copulas, Genest and Rivest (1993) suggested estimating  $\theta_{0,C}$  by solving  $\rho_\tau(\theta_C) = r_n^\tau$  w.r.t.  $\theta_C$ , i.e.

$$\hat{\theta}_{n,C}^{\text{IKTE}} = \rho_\tau^{-1}(r_n^\tau), \quad (\text{inversion of Kendall's tau estimator (IKTE)})$$

where  $\rho_\tau(\cdot)$  denotes Kendall's tau as a function of  $\theta$  and  $r_n^\tau$  is the sample version of Kendall's tau (computed via (31) from  $\mathbf{X}_1, \dots, \mathbf{X}_n$  or pseudo-observations  $\mathbf{U}_1, \dots, \mathbf{U}_n$ ; see later).

- The standardized dispersion matrix  $P$  for elliptical copulas can be estimated via *pairwise inversion of Kendall's tau*. If  $r_{n,j_1j_2}^\tau$  denotes the sample version of Kendall's tau for data pair  $(j_1, j_2)$ , then

$$\hat{P}_{n,j_1j_2}^{\text{IKTE}} = \sin\left(\frac{\pi}{2} r_{n,j_1j_2}^\tau\right).$$

A proper correlation matrix  $P$  can be constructed as in Higham (2002).

- One can also use **Spearman's rho**. For **Gauss copulas**,

$$\rho \approx \frac{6}{\pi} \arcsin \frac{\rho}{2} \stackrel{\text{Prop. 7.26}}{=} \rho_S.$$

The approximation error is comparably small, so that **the matrix of pairwise sample versions of Spearman's rho is an estimator for  $P$** .

- For  **$t$  copulas**,  $\hat{P}_n^{\text{IKTE}}$  can be used to estimate  $P$  and then  $\nu$  can be estimated via its MLE based on  $\hat{P}_n^{\text{IKTE}}$ ; see Mashal and Zeevi (2002).

## 7.6.2 Forming a pseudo-sample from the copula

- $\mathbf{X}_1, \dots, \mathbf{X}_n$  typically does not have  $U(0, 1)$  margins. **For applying the "copula approach" we thus need pseudo-observations from  $C$** .
- In general, we take  $\hat{\mathbf{U}}_i = (\hat{U}_{i1}, \dots, \hat{U}_{id}) = (\hat{F}_1(X_{i1}), \dots, \hat{F}_d(X_{id}))$ ,  $i \in \{1, \dots, n\}$ , where  $\hat{F}_j$  denotes an estimator of  $F_j$ ; see Lemma 7.6. Note that  **$\hat{\mathbf{U}}_1, \dots, \hat{\mathbf{U}}_n$  are typically neither independent** (even if  $\mathbf{X}_1, \dots, \mathbf{X}_n$  are) **nor perfectly  $U(0, 1)^d$  distributed**.

- Possible choices for  $\hat{F}_j$ :
  - ▶ Parametric estimators (typically if  $n$  is small). One often still uses (33) below for estimating  $\theta_{0,C}$  (to keep the error due to misspecification of the margins small).
  - ▶ Semi-parametric estimators (for example EVT-based: Bodies are modelled empirically, tails semiparametrically via the GPD-based tail estimator of Smith (1987)).
  - ▶ Non-parametric estimators with scaled empirical dfs, so

$$\hat{U}_{ij} = \frac{n}{n+1} \hat{F}_{n,j}(X_{ij}) = \frac{R_{ij}}{n+1}, \quad (33)$$

where  $R_{ij}$  denotes the rank of  $X_{ij}$  among all  $X_{1j}, \dots, X_{nj}$ . The scaling is to avoid density evaluation on the boundary of  $[0, 1]^d$ .

If  $n$  is sufficiently large, one typically uses (33).

## 7.6.3 Maximum likelihood estimation

### The (classical) maximum likelihood estimator

- If it exists, the density of  $F(\mathbf{x}) = C(F_1(x_1), \dots, F_d(x_d))$  is

$$f(\mathbf{x}; \boldsymbol{\theta}_0) = c(F_1(x_1; \boldsymbol{\theta}_{0,1}), \dots, F_d(x_d; \boldsymbol{\theta}_{0,d}); \boldsymbol{\theta}_{0,C}) \prod_{j=1}^d f_j(x_j; \boldsymbol{\theta}_{0,j}).$$

- The log-likelihood based on  $\mathbf{X}_1, \dots, \mathbf{X}_n$  is thus

$$\begin{aligned} \ell(\boldsymbol{\theta}; \mathbf{X}_1, \dots, \mathbf{X}_n) &= \sum_{i=1}^n \ell(\boldsymbol{\theta}; \mathbf{X}_i) \\ &= \sum_{i=1}^n \ell_C(\boldsymbol{\theta}_C; F_1(X_{i1}; \boldsymbol{\theta}_1), \dots, F_d(X_{id}; \boldsymbol{\theta}_d)) + \sum_{i=1}^n \sum_{j=1}^d \ell_j(\boldsymbol{\theta}_j; X_{ij}), \end{aligned}$$

where

$$\ell_C(\boldsymbol{\theta}_C; u_1, \dots, u_d) = \log c(u_1, \dots, u_d; \boldsymbol{\theta}_C)$$

$$\ell_j(\boldsymbol{\theta}_j; x) = \log f_j(x; \boldsymbol{\theta}_j), \quad j \in \{1, \dots, d\}.$$



- The *maximum likelihood estimator (MLE)* of  $\theta_0$  is

$$\hat{\theta}_n^{\text{MLE}} = \underset{\theta \in \Theta}{\operatorname{argsup}} \ell(\theta; \mathbf{X}_1, \dots, \mathbf{X}_n).$$

This optimization is typically done by numerical means. Note that this can be quite demanding, especially in high dimensions.

## The inference functions for margins estimator

- Joe and Xu (1996) suggested the *two-step estimation approach*:

**Step 1:** For  $j \in \{1, \dots, d\}$ , estimate  $\theta_{0,j}$  by its MLE  $\hat{\theta}_{n,j}^{\text{MLE}}$ .

**Step 2:** Estimate  $\theta_{0,C}$  by

$$\hat{\theta}_{n,C}^{\text{IFME}} = \underset{\theta_C \in \Theta_C}{\operatorname{argsup}} \ell(\theta_C, \hat{\theta}_{n,1}^{\text{MLE}}, \dots, \hat{\theta}_{n,d}^{\text{MLE}}; \mathbf{X}_1, \dots, \mathbf{X}_n).$$

The *inference functions for margins estimator (IFME)* of  $\theta_0$  is thus

$$\hat{\theta}_n^{\text{IFME}} = (\hat{\theta}_{n,C}^{\text{IFME}}, \hat{\theta}_{n,1}^{\text{MLE}}, \dots, \hat{\theta}_{n,d}^{\text{MLE}})$$

- This is typically much easier to compute than  $\hat{\theta}_n^{\text{MLE}}$  while providing good results; see Joe and Xu (1996) or Kim et al. (2007).
- $\hat{\theta}_n^{\text{IFME}}$  can also be used as initial value for computing  $\hat{\theta}_n^{\text{MLE}}$ .
- In terms of likelihood equations,  $\hat{\theta}_n^{\text{IFME}}$  compares to  $\hat{\theta}_n^{\text{MLE}}$  as follows:

$$\hat{\theta}_n^{\text{MLE}} \text{ solves } \left( \frac{\partial}{\partial \theta_C} \ell, \frac{\partial}{\partial \theta_1} \ell, \dots, \frac{\partial}{\partial \theta_d} \ell \right) = \mathbf{0},$$

$$\hat{\theta}_n^{\text{IFME}} \text{ solves } \left( \frac{\partial}{\partial \theta_C} \ell, \frac{\partial}{\partial \theta_1} \ell_1, \dots, \frac{\partial}{\partial \theta_d} \ell_d \right) = \mathbf{0},$$

where

$$\ell = \ell(\boldsymbol{\theta}; \mathbf{X}_1, \dots, \mathbf{X}_n),$$

$$\ell_j = \ell_j(\boldsymbol{\theta}_j; X_{1j}, \dots, X_{nj}) = \sum_{i=1}^n \ell_j(\boldsymbol{\theta}_j; X_{ij}) = \sum_{i=1}^n \log f_j(X_{ij}; \boldsymbol{\theta}_j).$$

### Example 7.34 (A computationally convincing example)

Suppose  $X_j \sim N(\mu_j, \sigma_j^2)$ ,  $j \in \{1, \dots, d\}$ , for  $d = 100$ , and  $C$  has (just) one parameter.

- 1) MLE requires to solve a 201-dimensional optimization problem.
  - 2) IFME only requires 100 optimizations in two dimensions and 1 one-dimensional optimization.
- If the marginals are estimated parametrically one often still uses the pseudo-observations built from the marginal empirical dfs to estimate  $\theta_{0,C}$  (see MPLE below) in order to avoid misspecification of the margins.
  - In this case (and under more complicated marginal models), one can execute the 101 optimizations in parallel, independently of each other.

## The maximum pseudo-likelihood estimator

- The *maximum pseudo-likelihood estimator (MPLE)*, introduced by Genest, Ghoudi, et al. (1995), works similarly to  $\hat{\theta}_n^{\text{IFME}}$ , but estimates the margins non-parametrically:

**Step 1:** Compute rank-based pseudo-observations  $\hat{U}_1, \dots, \hat{U}_n$ .

**Step 2:** Estimate  $\theta_{0,C}$  by

$$\hat{\theta}_{n,C}^{\text{MPLE}} = \underset{\theta_C \in \Theta_C}{\operatorname{argsup}} \sum_{i=1}^n \ell_C(\theta_C; \hat{U}_{i1}, \dots, \hat{U}_{id}) = \underset{\theta_C \in \Theta_C}{\operatorname{argsup}} \sum_{i=1}^n \log c(\hat{U}_i; \theta_C).$$

- Genest and Werker (2002) show that  $\hat{\theta}_{n,C}^{\text{MPLE}}$  is not asymptotically efficient in general.
- Kim et al. (2007) compare  $\hat{\theta}_n^{\text{MLE}}$ ,  $\hat{\theta}_n^{\text{IFME}}$ , and  $\hat{\theta}_{n,C}^{\text{MPLE}}$  in a simulation study ( $d = 2$  only!) and argue in favor of  $\hat{\theta}_{n,C}^{\text{MPLE}}$  overall, especially w.r.t. robustness against misspecification of the margins; but see Embrechts and Hofert (2013b) for  $d \gg 2$ .

### Example 7.35 (Fitting the Gauss copula)

- Use pairwise inversion of Spearman's rho or Kendall's tau.
- Or the MPLE via the (copula-related) log-likelihood

$$\ell_C(P; \hat{U}_1, \dots, \hat{U}_n) = \sum_{i=1}^n \ell_C(P; \hat{U}_i) \stackrel{\text{Eq. (28)}}{=} \sum_{i=1}^n \log c_P^{\text{Ga}}(\hat{U}_i).$$

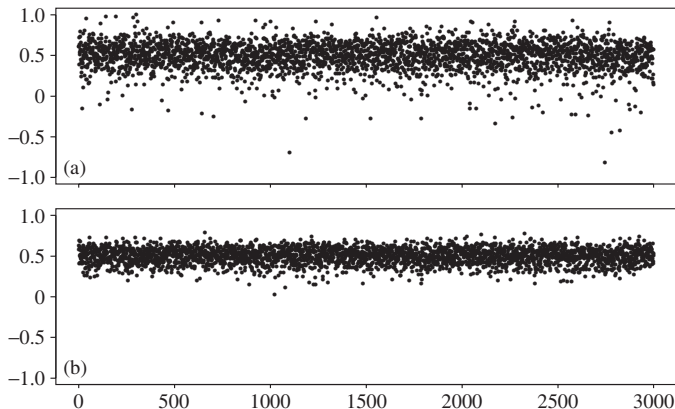
For maximization over all correlation matrices  $P$ , we can use the Cholesky factor  $A$  as reparameterization and maximize over all lower triangular matrices  $A$  with 1s on the diagonal; still this is  $\mathcal{O}(d^2)$ .

### Example 7.36 (Fitting the $t$ copula)

- For small  $d$ , maximize the likelihood over all correlation matrices (as for the Gauss copula case) and the d.o.f.  $\nu$ .
- For moderate/larger  $d$ , use Mashal and Zeevi (2002):
  - 1) Estimate  $P$  via pairwise inversion of Kendall's tau (see above).
  - 2) Plug  $\hat{P}$  into the likelihood and maximize it w.r.t.  $\nu$  to obtain  $\hat{\nu}_n$ .

### Example 7.37 (Correlation estimation for heavy-tailed data)

Consider  $n = 3000$  realizations of independent samples of size 90 from  $t_2(3, \mathbf{0}, \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix})$  ( $\Rightarrow$  linear correlation  $\rho = 0.5$ ). Shall we estimate  $\rho$  via the sample correlation (estimates are shown in (a)) or via inversion of Kendall's tau (shown in (b))? The variance of the latter is smaller!



Estimation is only one side of the coin. The other is *goodness-of-fit* (i.e. to find out whether our estimated model indeed represents the given data well) and *model selection* (i.e. to decide which model is best among all adequate fitted models). Goodness-of-fit can be (computationally) *challenging*, particularly for large  $d$ . There are also *graphical approaches* not further discussed here.