# 14 Multivariate time series

# 14.1 Fundamentals of multivariate time series

## 14.1.1 Basic definitions

**Definition 14.1**

The mean function $\boldsymbol{\mu}(t)$ and the covariance matrix function $\Gamma(t+h,t)$ of $(\boldsymbol{X}_t)_{t\in\mathbb{Z}}$ are given by

$$\begin{aligned} \boldsymbol{\mu}(t) &= \mathbb{E}(\boldsymbol{X}_t), && t \in \mathbb{Z}, \\ \Gamma(t+h,t) &= \mathbb{E}((\boldsymbol{X}_{t+h} - \boldsymbol{\mu}(t+h))(\boldsymbol{X}_t - \boldsymbol{\mu}(t))'), && t,h \in \mathbb{Z}. \end{aligned}$$

- Analogously to the univariate case, we have $\Gamma(t,t) = \operatorname{cov}(\boldsymbol{X}_t)$. By observing that the elements $\gamma_{ij}(t+h,t)$ of $\Gamma(t+h,t)$ satisfy

$$\gamma_{ij}(t+h,t) = \operatorname{cov}(X_{t+h,i}, X_{t,j}) = \operatorname{cov}(X_{t,j}, X_{t+h,i}) = \gamma_{ji}(t,t+h),$$

  it is clear that $\Gamma(t+h,t) = \Gamma(t,t+h)'$ for all $t$, $h$.
- However, the matrix $\Gamma$ need not be symmetric, so in general $\Gamma(t+h,t) \neq \Gamma(t,t+h)$. One series can lead other series.

**Definition 14.2 (strict stationarity)**

The multivariate time series $(\boldsymbol{X}_t)_{t\in\mathbb{Z}}$ is strictly stationary if

$$(\boldsymbol{X}'_{t_1},\dots,\boldsymbol{X}'_{t_n}) \stackrel{\mathrm{d}}{=} (\boldsymbol{X}'_{t_1+k},\dots,\boldsymbol{X}'_{t_n+k}),$$

for all $t_1,\dots,t_n,k\in\mathbb{Z}$ and for all $n\in\mathbb{N}$.

**Definition 14.3 (covariance (weak, second-order) stationarity)**

The multivariate time series $(\boldsymbol{X}_t)_{t\in\mathbb{Z}}$ is covariance stationary if the first two moments exist and satisfy

$$\begin{aligned} \boldsymbol{\mu}(t) &= \boldsymbol{\mu}, && t\in\mathbb{Z}, \\ \Gamma(t+h,t) &= \Gamma(h,0), && t,h\in\mathbb{Z}. \end{aligned}$$

- For a covariance-stationary process we write $\Gamma(h) := \Gamma(h,0)$.
- Note that $\Gamma(0) = \operatorname{cov}(\boldsymbol{X}_t)$, for all $t$.

- Write $\Delta$ for the diagonal matrix whose entries are the square roots of the diagonal entries of $\Gamma(0)$ (standard deviations of component series).

**Definition 14.4 (correlation matrix function)**
The correlation matrix function $P(h)$ of a covariance-stationary multivariate time series is

$$P(h) = \Delta^{-1}\Gamma(h)\Delta^{-1}, \quad \forall h \in \mathbb{Z}. \tag{130}$$

- The diagonal entries $\rho_{ii}(h)$ of this matrix-valued function give the autocorrelation function of the $i$th component series $(X_{t,i})_{t\in\mathbb{Z}}$.
- The off-diagonal entries give so-called cross-correlations between different component series at different times.

**Definition 14.5 (multivariate white noise)**
$(\boldsymbol{X}_t)_{t\in\mathbb{Z}}$ is multivariate white noise if it is covariance stationary with correlation matrix function given by

$$P(h) = \begin{cases} P, & h = 0, \\ 0, & h \neq 0, \end{cases}$$

for some positive-definite correlation matrix $P$.

Such a process has no cross-correlation between component series, except for contemporaneous cross-correlation at lag zero.

**Definition 14.6 (multivariate strict white noise)**
$(\boldsymbol{X}_t)_{t\in\mathbb{Z}}$ is multivariate strict white noise if it is a series of iid random vectors.

A strict white noise process with mean zero and covariance matrix $\Sigma$ will be denoted $\mathrm{SWN}(\mathbf{0}, \Sigma)$.

### 14.1.2 Analysis in the time domain

- Assume we have a random sample $\boldsymbol{X}_1, \dots, \boldsymbol{X}_n$ from a covariance-stationary multivariate time series model $(\boldsymbol{X}_t)_{t \in \mathbb{Z}}$.
- In the time domain we construct empirical estimators of the covariance matrix function and the correlation matrix function.
- The *sample covariance matrix function* is calculated according to

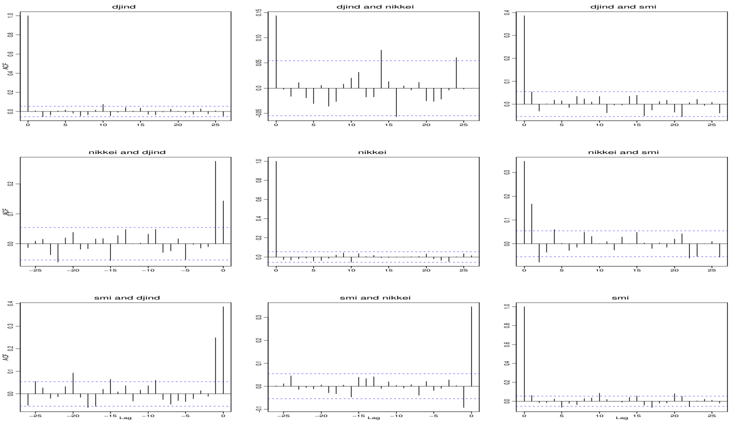
$$\hat{\Gamma}(h) = \frac{1}{n} \sum_{t=1}^{n-h} (\boldsymbol{X}_{t+h} - \bar{\boldsymbol{X}})(\boldsymbol{X}_t - \bar{\boldsymbol{X}})', \quad 0 \le h < n,$$

  where $\bar{\boldsymbol{X}} = \sum_{t=1}^{n} \boldsymbol{X}_t / n$ is the sample mean.
- Writing $\hat{\Delta}$, for the diagonal matrix of sample standard deviations (square root of the diagonal of $\hat{\Gamma}(0)$) the *sample correlation matrix function* is

$$\hat{P}(h) = \hat{\Delta}^{-1} \hat{\Gamma}(h) \hat{\Delta}^{-1}, \quad 0 \le h < n.$$

- The information contained in the elements $\hat{\rho}_{ij}(h)$ of the sample correlation matrix function is generally displayed in the *cross-correlogram*.

# Cross-correlogram of index returns




The US market leads Europe and Japan.

### 14.1.3 Multivariate ARMA processes

- ARMA models extend to higher dimensions where they are called VARMA. They provide models for the conditional mean vector.
- The VAR class is most widely used in practice.
- The first-order VAR process satisfies the set of equations

$$\boldsymbol{X}_t = \Phi \boldsymbol{X}_{t-1} + \boldsymbol{\varepsilon}_t, \quad \forall t. \tag{131}$$

  where $\Phi \in \mathsf{R}^{d \times d}$ is a matrix and $(\boldsymbol{\varepsilon}_t)$ is a white noise process.
- The process is covariance stationary if and only if all eigenvalues of the matrix $\Phi$ are less than one in absolute value.
- The covariance matrix function of this process is

$$\Gamma(h) = \Phi^h \Gamma(0), \quad h = 0, 1, 2, \ldots.$$

## 14.2 Multivariate GARCH Processes

Recall that the Cholesky factor $A$ of a positive-definite matrix $\Sigma$ is the lower-triangular matrix satisfying $AA' = \Sigma$.

**Definition 14.7**

Let $(\boldsymbol{Z}_t)_{t\in\mathbb{Z}}$ be $\mathrm{SWN}(\boldsymbol{0}, I_d)$. The process $(\boldsymbol{X}_t)_{t\in\mathbb{Z}}$ is said to be a multivariate GARCH process if it is strictly stationary and satisfies equations of the form

$$\boldsymbol{X}_t = A_t \boldsymbol{Z}_t, \quad t \in \mathbb{Z}, \tag{132}$$

where $A_t \in \mathbb{R}^{d\times d}$ is the Cholesky factor of a positive-definite matrix $\Sigma_t$ which is measurable with respect to $\mathcal{F}_{t-1} = \sigma(\{\boldsymbol{X}_s : s \leq t-1\})$, the history of the process up to time $t-1$.

**Conditional moments:**

- $\mathbb{E}(\boldsymbol{X}_t \mid \mathcal{F}_{t-1}) = \boldsymbol{0}$

- $\mathrm{cov}(\boldsymbol{X}_t \mid \mathcal{F}_{t-1}) = A_t A_t' = \Sigma_t$ is the conditional covariance matrix.
- Could add a non-zero conditional mean term $\boldsymbol{\mu}_t$ so that $\mathbf{X}_t = \boldsymbol{\mu}_t + A_t \mathbf{Z}_t$ where, for example,
  - $\boldsymbol{\mu}_t = \boldsymbol{\mu}$ for a constant conditional mean;
  - or $\boldsymbol{\mu}_t$ could follow a VARMA specification, such as $\boldsymbol{\mu}_t = \Phi \boldsymbol{X}_{t-1}$.
- We can write $\Sigma_t = \Delta_t P_t \Delta_t$, where $\Delta_t$ is the diagonal *volatility matrix* and $P_t$ is the *conditional correlation matrix*.
- The art of building multivariate GARCH models is to specify the dependence of $\Sigma_t$ (or of $\Delta_t$ and $P_t$) on the past in such a way that $\Sigma_t$ always remains symmetric and positive definite.
- The innovations are generally taken to be from either a multivariate Gaussian distribution ($\boldsymbol{Z}_t \sim N_d(\mathbf{0}, I_d)$) or an appropriately scaled spherical multivariate $t$ distribution ($\boldsymbol{Z}_t \sim t_d(\nu, \mathbf{0}, (\nu-2)I_d/\nu)$). Any distribution with mean zero and covariance matrix $I_d$ is permissible.

### 14.2.1 Models for conditional correlation

**Definition 14.8**

The process $(\boldsymbol{X}_t)_{t \in \mathbb{Z}}$ is a CCC-GARCH process if it is a multivariate GARCH process with conditional covariance matrix of the form $\Sigma_t = \Delta_t P_c \Delta_t$, where

- $P_c$ is a constant, positive-definite correlation matrix; and
- $\Delta_t$ is a diagonal volatility matrix with elements $\sigma_{t,k}$ satisfying

$$\sigma_{t,k}^2 = \alpha_{k0} + \sum_{i=1}^{p_k} \alpha_{ki} X_{t-i,k}^2 + \sum_{j=1}^{q_k} \beta_{kj} \sigma_{t-j,k}^2, \quad k = 1, \dots, d, \tag{133}$$

  where $\alpha_{k0} > 0$, $\alpha_{ki} \geq 0$, $i = 1, \dots, p_k$, $\beta_{kj} \geq 0$, $j = 1, \dots, q_k$.

- Alternatives to ordinary GARCH$(p_k, q_k)$ model may of course be used.
- In the CCC GARCH model the process $\boldsymbol{Y}_t = \Delta_t^{-1} \boldsymbol{X}_t$ (known as the de-volatilized process) satisfies $(\boldsymbol{Y}_t)_{t \in \mathbb{Z}} \sim \mathrm{SWN}(\boldsymbol{0}, P_c)$.

- Estimation can be accomplished in two stages:
  1) Fit univariate GARCH models to each component series;
  2) Form residuals $\hat{\boldsymbol{Y}}_t = \hat{\Delta}_t^{-1} \boldsymbol{X}_t$, for $t = 1, \ldots, n$ and estimate $P_c$ (either by using the standard correlation estimator or by fitting an appropriate distribution).
- Alternatively all parameters can be maximized in one step.
- The CCC model is often a useful starting point from which to proceed to more complex models.
- In some empirical settings it gives an adequate performance, but it is generally considered that the constancy of conditional correlation in this model is an unrealistic feature and that the impact of news on financial markets requires models that allow a dynamic evolution of conditional correlation as well as a dynamic evolution of volatilities.

**Definition 14.9**

The process $(\boldsymbol{X}_t)_{t\in\mathbb{Z}}$ is a DCC-GARCH process if it is a multivariate GARCH process where the volatilities comprising $\Delta_t$ follow univariate GARCH specifications as in (133) and the conditional correlation matrices $P_t$ satisfy, for $t \in \mathbb{Z}$, the equations

$$P_t = \wp\left(\left(1-\sum_{i=1}^{p}\alpha_i-\sum_{j=1}^{q}\beta_j\right)P_c+\sum_{i=1}^{p}\alpha_i\boldsymbol{Y}_{t-i}\boldsymbol{Y}_{t-i}'+\sum_{j=1}^{q}\beta_j P_{t-j}\right), \quad (134)$$

where

- $P_c$ is a positive-definite correlation matrix,
- $\wp$ is the operator that extracts correlation matrices from covariance matrices,
- $\boldsymbol{Y}_t = \Delta_t^{-1}\boldsymbol{X}_t$ denotes the devolatized process,
- and the coefficients satisfy $\alpha_i \geq 0$, $\beta_j \geq 0$ and $\sum_{i=1}^{p}\alpha_i + \sum_{j=1}^{q}\beta_j < 1$.

- If all the $\alpha_i$ and $\beta_j$ coefficients (134) are zero, model reduces to CCC.
- In a covariance-stationary univariate GARCH model with unconditional variance $\sigma^2$, the volatility equation can be written

$$\sigma_t^2 = \Big(1 - \sum_{i=1}^{p} \alpha_i - \sum_{j=1}^{q} \beta_j\Big)\sigma^2 + \sum_{i=1}^{p} \alpha_i X_{t-i} + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2.$$

- Thus, in DCC, the correlation matrix $P_c$ in (134) can be thought of as representing the long-run correlation structure.
- The usual estimation method for the DCC model is as follows.
  1) Fit univariate GARCH-type models to the component series to estimate the volatility matrix $\Delta_t$. Form an estimated realization of the devolatized process by taking $\hat{\boldsymbol{Y}}_t = \hat{\Delta}_t^{-1}\boldsymbol{X}_t$.
  2) Estimate $P_c$ by estimating correlation matrix of the devolatized data.
  3) Estimate the remaining parameters $\alpha_i$ and $\beta_j$ in equation (134) by fitting the implied dynamic model to the devolatized data ($\hat{\boldsymbol{Y}}_t$).

**Consider a first-order model ($p = q = 1$):**

- Given $\mathcal{F}_{t-1}$ (comprising $\boldsymbol{Y}_{t-k}, P_{t-k}, k = 1, 2, \ldots$) and using an estimate of $P_c$ (known as variance targeting), we have

$$
\begin{aligned}
\boldsymbol{Y}_t &= B_t \boldsymbol{Z}_t, \quad \text{where} \\
B_t B_t' &= P_t, \quad \text{(Cholesky decomposition)} \\
P_t &= \wp(Q_t), \quad \text{(correlation from covariance)} \\
Q_t &= (1 - \alpha_1 - \beta_1) P_c + \alpha_1 \boldsymbol{Y}_{t-1} \boldsymbol{Y}_{t-1}' + \beta_1 P_{t-1}
\end{aligned}
$$

- Usually estimated by conditional maximum likelihood.
- The likelihood is built up recursively from starting values (for example $P_0 = \boldsymbol{Y}_0 \boldsymbol{Y}_0' = P_c$).
- There are two parameters to estimate for dynamics, in addition to parameters of innovation distribution (if non-Gaussian).

**Relationship to dynamic copula models:**

- In terms of copulas, using a Gaussian innovation distribution means estimating a 2-parameter model where

$$\boldsymbol{Y}_t \mid \mathcal{F}_{t-1} \sim C^{\mathsf{Ga}}_{P_t}(\Phi, \ldots, \Phi).$$

- Using a Student innovation distribution means estimating a 3-parameter model where

$$\boldsymbol{Y}_t \mid \mathcal{F}_{t-1} \sim C^{\mathsf{t}}_{\nu, P_t}(F_\nu, \ldots, F_\nu)$$

  where $F_\nu$ is a scaled Student $t$ distribution.

**Copula-MGARCH models**

- Note that models of the form

$$\boldsymbol{Y}_t \mid \mathcal{F}_{t-1} \sim C^{\mathsf{t}}_{\nu, P_t^*}(F_{\nu_1}, \ldots, F_{\nu_d})$$

  with $P_t^*$ updating as in (134) have also been considered.
  This has $d + 3$ parameters.

- Previous model doesn't quite fit into the DCC class as we have defined it because $\operatorname{cov}(\boldsymbol{Y}_t \mid \mathcal{F}_{t-1}) = P_t \neq P_t^*$.
- It is not the parameters of the conditional correlation matrix but rather the parameters of the copula that update according to (134).
- However it fits into a bigger class of copula-MGARCH models where

$$\boldsymbol{X}_t = \boldsymbol{\mu}_t + \Delta_t \boldsymbol{Y}_t, \quad \boldsymbol{Y}_t \mid \mathcal{F}_{t-1} \sim C_t(F_1, \dots, F_d)$$

  and

  - the volatility components of $\Delta_t$ follow GARCH schemes;
  - the conditional mean terms $\boldsymbol{\mu}_t$ follow VARMA schemes;
  - the conditional copula $C_t$ evolves as function of information in $\mathcal{F}_{t-1}$;
  - $F_1, \dots, F_d$ are zero-mean, unit-variance distributions.
- See A. J. Patton (2006), A. Patton (2012), and Fan and A. Patton (2014)

### 14.2.2 Dimension reduction in MGARCH

- While the multi-stage estimation procedure for DCC makes it possible to estimate in quite high dimensions, it is usual to first apply dimension reduction through factor modelling and then fit MGARCH models to the most important factors.
- Can easily fit MGARCH models to factors derived from macroeconomic and fundamental factor models. The factors are typically correlated.
- The use of so-called PC-GARCH (principal components GARCH) is quite popular and avoids need for multivariate models.
  - Here we assume that the principal components of $X_t$ follow a CCC model with $P = I_d$.
  - To estimate such a model, we estimate the principal components of the data and fit univariate GARCH models to each principal components series.