

## **2 Basic concepts in risk management**

2.1 Risk management for a financial firm

2.2 Modelling value and value change

2.3 Risk measurement

## 2.1 Risk management for a financial firm

### 2.1.1 Assets, liabilities and the balance sheet

The risks of a firm (here: **bank**) can be understood from its balance sheet:

Assets Investments of the firm		Liabilities Obligations from fundraising	
Cash (and central bank balance)	£10M	Customer deposits	£80M
Securities	£50M	Bonds issued	
- bonds, stocks, derivatives		- senior bond issues	£25M
Loans and mortgages	£100M	- subordinated bond issues	£15M
- corporates		Short-term borrowing	£30M
- retail and smaller clients		Reserves (for losses on loans)	£20M
- government			
Other assets	£20M	Debt (sum of above)	£170M
- property			
- investments in companies		Equity	£30M
Short-term lending	£20M		
Total	£200M	Total	£200M

A stylized balance sheet for an **insurer** is:

Assets		Liabilities	
Investments		Reserves for policies written	£80M
- bonds	£50M	(technical provisions)	
- stocks	£5M	Bonds issued	£10M
- property	£5M		
Investments for unit-linked contracts	£30M	Debt (sum of above)	£90M
Other assets	£10M	Equity	£10M
- property			
Total	£100M	Total	£100M

- Balance sheet equation:  $\text{Assets} = \text{Liabilities} = \text{Debt} + \text{Equity}$ .  
If equity  $> 0$ , the company is *solvent*, otherwise *insolvent*.
- **Valuation** of the items on the balance sheet is a **non-trivial** task.
  - ▶ *Amortized cost accounting* values a position a *book value* at its inception and this is carried forward/progressively reduced over time.

- ▶ *Fair-value accounting* values assets at prices they are sold and liabilities at prices that would have to be paid in the market. This can be challenging for non-traded or illiquid assets or liabilities.

There is a tendency in the industry to move towards fair-value accounting. Market consistent valuation in Solvency II follows similar principles.

### 2.1.2 Risks faced by a financial firm

- Decrease in the value of the investments on the asset side of the balance sheet (e.g. losses from securities trading or credit risk).
- *Maturity mismatch* (large parts of the assets are relatively illiquid (long-term) whereas large parts of the liabilities are rather short-term obligations. This can lead to a default of a solvent bank or a bank run).
- The prime risk for an insurer is *insolvency* (risk that claims of policy holders cannot be met). On the asset side, risks are similar to those of a bank. On the liability side, the main risk is that reserves are insufficient

to cover future claim payments. Note that the **liabilities of a life insurer are of a long-term nature** and subject to multiple categories of risk (e.g. interest rate risk, inflation risk and longevity risk).

- So risk is found on **both sides** of the balance sheet and thus RM should not focus on the asset side alone.

### 2.1.3 Capital

- There are different notions of **capital**. One distinguishes:

- Equity capital*
  - Value of **assets** — **debt**;
  - Measures the firm's value to its shareholders;
  - Can be split into *shareholder capital* (initial capital invested in the firm) and *retained earnings* (accumulated earnings not paid to shareholders).
- Regulatory capital* — Capital required according to **regulatory rules**;

- For European insurance firms: Minimum (MCR) and solvency capital requirements (SCR);
- A regulatory framework also specifies the **capital quality**. One distinguishes *Tier 1 capital* (i.e. shareholder capital + retained earnings; **can act in full as buffer**) and *Tier 2 capital* (includes other positions on the balance sheet).

#### *Economic capital*

- Capital required to **control the probability of becoming insolvent** (typically over one year);
- **Internal assessment** of risk capital;
- Aims at a holistic view (assets and liabilities) and works with fair values of balance sheet items.

- **All of these notions** refer to items on the liability side that entail no obligations to outside creditors; they **can** thus **serve as buffer against losses**.

## 2.2 Modelling value and value change

### 2.2.1 Mapping of risks

We set up a general mathematical model for (changes in) value caused by financial risks. To this end we work on a *probability space*  $(\Omega, \mathcal{F}, \mathbb{P})$  and consider a risk or loss as a *random variable*  $X : \Omega \rightarrow \mathbb{R}$  (or:  $L$ ).

- Consider a *portfolio* of assets and possibly liabilities. The *value* of the portfolio at time  $t$  (*today*) is denoted by  $V_t$  (a random variable; assumed to be known at  $t$ ; its *df* is typically *not trivial to determine!*).
- We consider a given *time horizon*  $\Delta t$  and *assume*:
  - 1) the *portfolio composition remains fixed* over  $\Delta t$ ;
  - 2) there are *no intermediate payments* during  $\Delta t$

$\Rightarrow$  *Fine for small  $\Delta t$*  but *unlikely to hold for large  $\Delta t$* .

- The *change* in value of the portfolio is given by

$$\Delta V_{t+1} = V_{t+1} - V_t$$

and we define the (random) *loss* by the *sign-adjusted* value change

$$L_{t+1} = -\Delta V_{t+1}$$

(as QRM is mainly concerned with losses).

### Remark 2.1

- 1) The *distribution of  $L_{t+1}$*  is called *loss distribution*.
- 2) Practitioners often consider the *profit-and-loss (P&L) distribution* which is the distribution of  $-L_{t+1} = \Delta V_{t+1}$ .
- 3) For longer time intervals,  $\Delta V_{t+1} = V_{t+1}/(1 + r) - V_t$  ( $r =$  *risk-free interest rate*) would be more appropriate, but we will *mostly neglect* this issue.



- $V_t$  is typically modelled as a function  $f$  of time  $t$  and a  $d$ -dimensional random vector  $\mathbf{Z} = (Z_{t,1}, \dots, Z_{t,d})$  of risk factors, that is,

$$V_t = f(t, \mathbf{Z}_t) \quad (\text{mapping of risks})$$

for some measurable  $f : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}$ . The choice of  $f$  and  $\mathbf{Z}_t$  is problem-specific (typically known, but possibly difficult to evaluate).

- It is often convenient to work with the risk-factor changes

$$\mathbf{X}_{t+1} = \mathbf{Z}_{t+1} - \mathbf{Z}_t.$$

We can rewrite  $L_{t+1}$  in terms of  $\mathbf{X}_{t+1}$  via

$$\begin{aligned} L_{t+1} &= -(V_{t+1} - V_t) = -(f(t+1, \mathbf{Z}_{t+1}) - f(t, \mathbf{Z}_t)) \\ &= -(f(t+1, \mathbf{Z}_t + \mathbf{X}_{t+1}) - f(t, \mathbf{Z}_t)). \end{aligned}$$

We see that the loss df is determined by the loss df of  $\mathbf{X}_{t+1}$ . We will thus also write  $L_{t+1} = L(\mathbf{X}_{t+1})$ , where  $L(\mathbf{x}) = -(f(t+1, \mathbf{Z}_t + \mathbf{x}) - f(t, \mathbf{Z}_t))$  is known as loss operator.

- If  $f$  is differentiable, its **first-order (Taylor) approximation** ( $f(\mathbf{y}) \approx f(\mathbf{y}_0) + \nabla f(\mathbf{y}_0)'(\mathbf{y} - \mathbf{y}_0)$  for  $\mathbf{y} = (t+1, Z_{t,1} + X_{t+1,1}, \dots, Z_{t,d} + X_{t+1,d})$  and  $\mathbf{y}_0 = (t, Z_{t,1}, \dots, Z_{t,d})$ ) is

$$f(t+1, \mathbf{Z}_t + \mathbf{X}_{t+1}) \approx f(t, \mathbf{Z}_t) + f_t(t, \mathbf{Z}_t) \cdot 1 + \sum_{j=1}^d f_{z_j}(t, \mathbf{Z}_t) \cdot X_{t+1,j}$$

We can thus approximate  $L_{t+1}$  by the **linearized loss**

$$L_{t+1}^{\Delta} = - \left( \underbrace{f_t(t, \mathbf{Z}_t)}_{=: c_t} + \sum_{j=1}^d \underbrace{f_{z_j}(t, \mathbf{Z}_t)}_{=: b_{t,j}} X_{t+1,j} \right) = -(c_t + \mathbf{b}'_t \mathbf{X}_{t+1}),$$

a linear function of  $X_{t+1,1}, \dots, X_{t+1,d}$  (indices denote partial derivatives).  
The **approximation is best if the risk-factor changes are small in absolute value.**

## Example 2.2 (Stock portfolio)

Consider a portfolio  $\mathcal{P}$  of  $d$  stocks  $S_{t,1}, \dots, S_{t,d}$  ( $S_{t,j}$  = value of stock  $j$  at time  $t$ ) and denote by  $\lambda_j$  the number of shares of stock  $j$  in  $\mathcal{P}$ . In finance and risk management, one typically uses logarithmic prices as risk factors, i.e.  $Z_{t,j} = \log S_{t,j}$ ,  $j \in \{1, \dots, d\}$ . Then

$$V_t = f(t, \mathbf{Z}_t) = \sum_{j=1}^d \lambda_j S_{t,j} = \sum_{j=1}^d \lambda_j e^{Z_{t,j}}.$$

- The one-period ahead loss is then given by

$$\begin{aligned} L_{t+1} &= -(V_{t+1} - V_t) = -\sum_{j=1}^d \lambda_j (e^{Z_{t,j} + X_{t+1,j}} - e^{Z_{t,j}}) \\ &= -\sum_{j=1}^d \lambda_j e^{Z_{t,j}} (e^{X_{t+1,j}} - 1) = -\sum_{j=1}^d \underbrace{\lambda_j S_{t,j}}_{=: \tilde{w}_{t,j}} (e^{X_{t+1,j}} - 1) \quad (1) \end{aligned}$$

which is non-linear in  $X_{t+1,j}$  (here:  $L(\mathbf{x}) = -\sum_{j=1}^d \tilde{w}_{t,j} (e^{x_j} - 1)$ ).

- With  $f_{z_j}(t, \mathbf{Z}_t) = \lambda_j e^{Z_{t,j}} = \lambda_j S_{t,j} = \tilde{w}_{t,j}$ , the **linearized loss** is

$$\begin{aligned} L_{t+1}^{\Delta} &= -\left(f_t(t, \mathbf{Z}_t) + \sum_{j=1}^d f_{z_j}(t, \mathbf{Z}_t) X_{t+1,j}\right) = -\left(0 + \sum_{j=1}^d \tilde{w}_{t,j} X_{t+1,j}\right) \\ &= -\tilde{\mathbf{w}}_t' \mathbf{X}_{t+1}. \end{aligned}$$

- Note that  $L_{t+1}^{\Delta} = -(c_t + \mathbf{b}_t' \mathbf{X}_{t+1})$  for  $c_t = 0$  and  $\mathbf{b}_t = \tilde{\mathbf{w}}_t$ .
- If  $\boldsymbol{\mu} = \mathbb{E} \mathbf{X}_{t+1}$  and  $\Sigma = \text{cov} \mathbf{X}_{t+1}$  are known, then **expectation** and **variance of the (linearized) one-period ahead loss** are

$$\mathbb{E} L_{t+1}^{\Delta} = -\sum_{j=1}^d \tilde{w}_{t,j} \mathbb{E}(X_{t+1,j}) = -\tilde{\mathbf{w}}_t' \boldsymbol{\mu},$$

$$\text{var} L_{t+1}^{\Delta} = \text{var}(\tilde{\mathbf{w}}_t' \mathbf{X}_{t+1}) = \tilde{\mathbf{w}}_t' \text{cov}(\mathbf{X}_{t+1}) \tilde{\mathbf{w}}_t = \tilde{\mathbf{w}}_t' \Sigma \tilde{\mathbf{w}}_t.$$

- If  $\mathbf{X}_{t+1}$  is multivariate normal, then  $L_{t+1}^{\Delta} \sim N(-\tilde{\mathbf{w}}_t' \boldsymbol{\mu}, \tilde{\mathbf{w}}_t' \Sigma \tilde{\mathbf{w}}_t)$ .

### Example 2.3 (European call option)

Consider a portfolio consisting of a European call option on a non-dividend-paying stock  $S_t$  with maturity  $T$  and strike (exercise price)  $K$ . The Black–Scholes formula says that today's value is

$$V_t = C^{\text{BS}}(t, S_t; r, \sigma, K, T) = S_t \Phi(d_1) - K e^{-r(T-t)} \Phi(d_2), \quad (2)$$

where

- $t$  is the time in years;
- $\Phi$  is the df of  $N(0, 1)$ ;
- $r$  is the continuously compounded risk-free interest rate;
- $d_1 = \frac{\log(S_t/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$  and  $d_2 = d_1 - \sigma\sqrt{T-t}$ ; and
- $\sigma$  is the annualized volatility (standard deviation) of  $\log(S_t/S_{t-1})$ .

While (2) assumes  $r, \sigma$  to be constant, this is often not true in real markets.

Hence, besides  $\log S_t$ , we consider  $r_t, \sigma_t$  as risk factors, so

$$\mathbf{Z}_t = (\log S_t, r_t, \sigma_t) \Rightarrow \mathbf{X}_{t+1} = (\log(S_{t+1}/S_t), r_{t+1} - r_t, \sigma_{t+1} - \sigma_t).$$

This implies that the mapping  $f$  (in terms of the risk factors) is given by

$$V_t = C^{\text{BS}}(t, e^{Z_{t,1}}; Z_{t,2}, Z_{t,3}, K, T) =: f(t, \mathbf{Z}_t)$$

and the linearized one-day ahead loss (omitting the arguments of  $C^{\text{BS}}$ ) is

$$\begin{aligned} L_{t+1}^{\Delta} &= -\left(f_t(t, \mathbf{Z}_t) + \sum_{j=1}^3 f_{z_j}(t, \mathbf{Z}_t) X_{t+1,j}\right) \\ &= -(C_t^{\text{BS}} \Delta t + C_{S_t}^{\text{BS}} S_t X_{t+1,1} + C_{r_t}^{\text{BS}} X_{t+1,2} + C_{\sigma_t}^{\text{BS}} X_{t+1,3}). \end{aligned}$$

If our risk management horizon is 1 d (as opposed to 1 y), we need to introduce  $\Delta t := 1/250$  here. Note that the “Greeks” enter ( $C_t^{\text{BS}}$  is the *theta* of the option;  $C_{S_t}^{\text{BS}}$  the *delta*;  $C_{r_t}^{\text{BS}}$  the *rho*;  $C_{\sigma_t}^{\text{BS}}$  the *vega*).

For portfolios of derivatives,  $L_{t+1}^{\Delta}$  can be a rather poor approximation to  $L_{t+1} \Rightarrow$  higher-order (Taylor) approximations such as the *delta-gamma-approximation* (second-order) can be used.

## 2.2.2 Valuation methods

### Fair value accounting

The *fair value* of an asset/liability is an *estimate of the price* which would be *received/paid* on an *active market*. One distinguishes:

- Level 1** *Mark-to-market*. *Fair value* of an investment is *determined from quoted prices* for the *same instrument*; see Example 2.2.
- Level 2** *Mark-to-model with objective inputs*. The *fair value* of an instrument is determined *using quoted prices* in active markets *for similar instruments* or by using valuation techniques/models with inputs based on observable market data; see Example 2.3.
- Level 3** *Mark-to-model with subjective inputs*. The *fair value* of an instrument is determined using valuation techniques/models for which *some inputs are not observable* in the market (e.g. determining default risk of portfolios of loans to companies for which no CDS spreads are available).

## Risk-neutral valuation

- ... is **widely used** for pricing financial products, e.g. derivatives
- **value** of a financial instrument **today** = **expected discounted values of future cash flows**; the expectation is **taken w.r.t. the risk-neutral pricing measure  $Q$**  (also called *equivalent martingale measure (EMM)*; it turns discounted prices into martingales, so fair bets) as opposed to the real world/**physical measure  $\mathbb{P}$** .
- An **risk-neutral pricing measure** is a **probability measure  $Q$**  such that the **expectation of the discounted payoff w.r.t.  $Q$**  equals  $V_0$  (fair bet).
- **Risk-neutral valuation at  $t$  of a claim  $H$  at  $T$**  is done via the **risk-neutral pricing rule**

$$V_t^H = \mathbb{E}_{Q,t}(e^{-r(T-t)}H), \quad t < T,$$

where  $\mathbb{E}_{Q,t}(\cdot)$  denotes expectation w.r.t.  $Q$  given the information up to and including time  $t$ .

- **$\mathbb{P}$  is estimated from historical data**;  **$Q$  is calibrated to market prices**.



## Example 2.4 (European call option continued)

- Suppose that options with strike  $K$  or maturity  $T$  are not traded, but other options on the same stock are.
- Under  $\mathbb{P}$  the stock price  $(S_t)$  is assumed to follow a geometric Brownian motion (GBM) (the so-called *Black–Scholes model*) with dynamics  $dS_t = \mu S_t dt + \sigma S_t dW_t$  for constants  $\mu \in \mathbb{R}$  (drift) and  $\sigma > 0$  (volatility), and a standard Brownian motion  $(W_t)$ .
- Under the EMM  $\mathbb{Q}$ ,  $(e^{-rt}S_t)$  is a martingale and  $S_t$  follows a GBM with drift  $r$  and volatility  $\sigma$ .
- The European call option payoff is  $H = (S_T - K)^+ = \max\{S_T - K, 0\}$  and the risk-neutral valuation formula may be shown to be

$$V_t = E_t^{\mathbb{Q}}(e^{-r(T-t)}(S_T - K)^+) = C^{\text{BS}}(t, S_t; r, \sigma, K, T), \quad t < T; \quad (3)$$

- One typically uses quoted prices  $C^{\text{BS}}(t, S_t; r, \sigma, K^*, T^*)$  (for different  $K^*, T^*$ ) to infer the unknown  $\sigma$ . Then plug this so-called *implied volatility* into (3).

## 2.2.3 Loss distributions

Having determined the mapping  $f$  (may involve *valuation models*, e.g. Black–Scholes, or numerical approximation), we can identify the following **key statistical tasks of QRM**:

- 1) Find a statistical **model for  $\mathbf{X}_{t+1}$**  (typically a model for forecasting  $\mathbf{X}_{t+1}$ , estimated based on historical data);
- 2) Compute/derive the **df  $F_{L_{t+1}}$**  (requires the df of  $f(t+1, \mathbf{Z}_t + \mathbf{X}_{t+1})$ );
- 3) Compute a **risk measure** (see later) **from  $F_{L_{t+1}}$** .

There are **three general methods** to approach these challenges.

### 1) Analytical method

**Idea:** Choose  $F_{\mathbf{X}_{t+1}}$  and  $f$  such that  $F_{L_{t+1}}$  can be determined explicitly.

Prime example: **Variance-covariance method**, see RiskMetrics (1996):

**Assumption 1**  $\mathbf{X}_{t+1} \sim N(\boldsymbol{\mu}, \Sigma)$  (e.g. if  $(\mathbf{Z}_t)$  is a Brownian motion,  $(S_t)$  a geometric Brownian motion)

**Assumption 2**  $F_{L_{t+1}^\Delta}$  is a good approximation to  $F_{L_{t+1}}$ .

$$L_{t+1}^\Delta = -(c_t + \mathbf{b}_t' \mathbf{X}_{t+1}) \xRightarrow{\text{Ass. 1}} L_{t+1}^\Delta \sim N(-c_t - \mathbf{b}_t' \boldsymbol{\mu}, \mathbf{b}_t' \Sigma \mathbf{b}_t)$$

**Advantages:**

- $F_{L_{t+1}^\Delta}$  explicit ( $\Rightarrow$  typically explicit risk measures)
- Easy to implement (unless  $d$  extremely large)

**Drawbacks:** Assumption 1 is unlikely to be realistic for daily (probably also weekly/monthly) data. Stylized facts about  $\mathbf{X}_{t+1}$  suggest that  $F_{\mathbf{X}_{t+1}}$  is leptokurtic (thinner body, heavier tail than  $N(\boldsymbol{\mu}, \Sigma)$ ). Thus,  $\mathbf{X}_{t+1} \sim N(\boldsymbol{\mu}, \Sigma)$  underestimates the tail of  $F_{L_{t+1}}$  and thus risk measures such as VaR.

When dynamic models for  $\mathbf{X}_{t+1}$  are considered, different estimation methods are possible depending on whether we focus on conditional distributions  $F_{\mathbf{X}_{t+1} | (\mathbf{X}_s)_{s \leq t}}$  or the equilibrium distribution  $F_{\mathbf{X}}$  in a stationary model.

## 2) Historical simulation

**Idea:** Estimate  $F_{L_{t+1}}$  by its *empirical distribution function (edf)*

$$\hat{F}_{L_{t+1},n}(x) = \frac{1}{n} \sum_{i=1}^n I_{\{L_{t-i+1} \leq x\}}, \quad x \in \mathbb{R}, \quad (4)$$

based on

$$L_k = L(\mathbf{X}_k) = -(f(t+1, \mathbf{Z}_t + \mathbf{X}_k) - f(t, \mathbf{Z}_t)), \quad (5)$$

$k \in \{t-n+1, \dots, t\}$ .  $L_{t-n+1}, \dots, L_t$  show what would happen to the current portfolio if the past  $n$  risk-factor changes were to recur.

**Advantages:** ■ Easy to implement

■ No estimation of the distribution of  $\mathbf{X}_{t+1}$  required

**Drawbacks:** ■ Sufficient data for all risk-factor changes required

■ Only past losses considered (“driving a car by looking in the back mirror”)

### 3) Monte Carlo method

**Idea:** Take any model for  $X_{t+1}$ , simulate  $X_{t+1}$ , compute the corresponding losses as in (5) and estimate  $F_{L_{t+1}}$  (typically via edf as in (4)).

**Advantages:** ■ Quite general (applicable to any model of  $X_{t+1}$  which is easy to sample)

**Drawbacks:** ■ Unclear how to find an appropriate model for  $X_{t+1}$  (any result is only as good as the chosen  $F_{X_{t+1}}$ )

- Computational cost (every simulation requires to evaluate the mapping  $f$ ; expensive, e.g. if the latter contains derivatives which are priced via Monte Carlo themselves  $\Rightarrow$  Nested Monte Carlo simulations)

So-called *economic scenario generators* (i.e. economically motivated dynamic models for the evolution and interaction of risk factors) used in insurance also fall under the heading of Monte Carlo methods.

## 2.3 Risk measurement

- A *risk measure* for a financial position with (random) loss  $L$  is a **real number** which **measures the “riskiness of  $L$ ”**. In the Basel or Solvency context, it is often interpreted as the amount of **capital required to make a position with loss  $L$  acceptable** to an (internal/external) regulator.
- Some **reasons for using risk measures** in practice:
  - ▶ To **determine the amount of capital to hold** as a buffer against unexpected future losses on a portfolio (in order to satisfy a regulator/manager concerned with the institution's solvency).
  - ▶ As a **tool for limiting** the amount of **risk of a business unit** (e.g. by requiring that the daily 95% value-at-risk (i.e. the 95%-quantile) of a trader's position should not exceed a given bound).
  - ▶ To determine the **riskiness** (and **thus fair premium**) of an **insurance contract**.

## 2.3.1 Approaches to risk measurement

Existing risk measurement approaches grouped into three categories:

### 1) Notional-amount approach

- oldest approach; “standardized approaches” of Basel II (e.g. OpRisk)
- *risk of a portfolio* = summed notional values of the securities times their riskiness factor.
- Advantages: ► simplicity
- Drawbacks: ► No differentiation between long and short positions and no netting: For example, the risk of a long position in corporate bonds hedged by an offsetting position in credit default swaps is counted as twice the risk of the unhedged bond position.

- ▶ **No diversification** benefits: risk of a portfolio of loans to many companies = risk of a portfolio where the whole amount is lent to a single company.
- ▶ Problems for **portfolios of derivatives**: **notional** amount of the underlying **can** widely **differ from the economic value** of the derivative position.

## 2) Risk measures based on loss distributions

- Most modern **risk measures are characteristics** of the underlying **loss distribution** over some predetermined time horizon  $\Delta t$ .
- Examples: variance, **value-at-risk**, **expected shortfall** (see later)
- **Advantages:**
  - ▶ **Makes sense on all levels** (from single portfolios to the overall position of a financial institution).
  - ▶ Loss distributions **reflect netting** and **diversification**.



- Drawbacks:
- ▶ Estimates of loss distributions are typically based on past data.
  - ▶ It is difficult to estimate loss distributions accurately (especially for large portfolios).  
⇒ Risk measures should be complemented by information from scenarios (forward-looking).

### 3) Scenario-based risk measures

- Typically considered in stress testing.
- One considers possible future risk-factor changes (*scenarios*; e.g. a 20% drop in a market index).
- *Risk of a portfolio* = maximum (weighted) loss under all scenarios.
- If  $\mathcal{X} = \{x_1, \dots, x_n\}$  denote the risk-factor changes (*scenarios*) with corresponding weights  $w = (w_1, \dots, w_n)$ , the risk is  $\psi_{\mathcal{X}, w} = \max_{1 \leq i \leq n} \{w_i L(x_i)\}$ , where  $L(x)$  denotes the loss the portfolio

would suffer if the hypothetical scenario  $x$  were to occur. Many risk measures are of this form; see *CME SPAN: Standard Portfolio Analysis of Risk* (2010).

- Mathematical interpretation:

- ▶ Assume  $L(\mathbf{0}) = 0$  (okay if  $\Delta t$  small) and  $w_i \in [0, 1] \forall i$ .
- ▶  $w_i L(x_i) = w_i L(x_i) + (1 - w_i) L(\mathbf{0}) = \mathbb{E}_{\mathbb{P}_i}(L(\mathbf{X}))$  where  $\mathbf{X} \sim \mathbb{P}_i$  and  $\mathbb{P}_i$  is such that  $\mathbb{P}_i(\mathbf{X} = x_i) = w_i$  and  $\mathbb{P}_i(\mathbf{X} = \mathbf{0}) = 1 - w_i$ .

Therefore,  $\psi_{\mathcal{X}, w} = \max\{\mathbb{E}_{\mathbb{P}}(L(\mathbf{X})) : \mathbf{X} \sim \mathbb{P} \in \{\mathbb{P}_1, \dots, \mathbb{P}_n\}\}$ . Such a risk measure is known as a *generalized scenario*; they play an important role in the theory of *coherent risk measures*.

- **Advantages:**
  - ▶ Useful for portfolios with few risk factors.
  - ▶ Useful complementary information to risk measures based on loss distributions (past data).

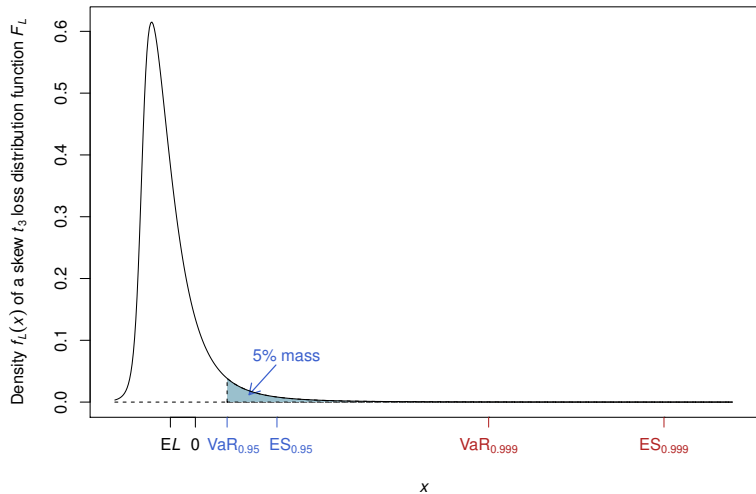
**Drawbacks:** ▶ Determining scenarios and weights.

## 2.3.2 Value-at-risk

### Definition 2.5 (Value-at-risk)

For a loss  $L \sim F_L$ , *value-at-risk* (VaR) at confidence level  $\alpha \in (0, 1)$  is defined by  $\text{VaR}_\alpha = \text{VaR}_\alpha(L) = F_L^{\leftarrow}(\alpha) = \inf\{x \in \mathbb{R} : F_L(x) \geq \alpha\}$ .

- $\text{VaR}_\alpha$  is simply the  $\alpha$ -quantile of  $F_L$ . As such,  $F_L(\text{VaR}_\alpha(L)) = F_L(F_L^{\leftarrow}(\alpha)) \geq \alpha$  and  $F_L(x) < \alpha$  for all  $x < \text{VaR}_\alpha(L)$ .
- $\text{VaR}_\alpha(L) = \inf\{x \in \mathbb{R} : \bar{F}_L(x) \leq 1 - \alpha\}$  ( $\bar{F}_L(x) = 1 - F_L(x)$ ), so  $\text{VaR}_\alpha$  is the **smallest loss which is exceeded with prob. at most  $1 - \alpha$** .
- Known since 1994: Weatherstone 4<sup>15</sup> report (J.P. Morgan; RiskMetrics)
- VaR is the **most widely used risk measure** (by Basel II or Solvency II)
- $\text{VaR}_\alpha(L)$  is **not** a **what if** risk measure: It **does not provide information about the severity of losses which occur with probability  $\leq 1 - \alpha$** .



## Interlude: Generalized inverses

$T \nearrow$  means that  $T$  is *increasing*, i.e.  $T(x) \leq T(y)$  for all  $x < y$ .  $T \uparrow$  means that  $T$  is *strictly increasing*, i.e.  $T(x) < T(y)$  for all  $x < y$ .

### Definition 2.6 (Generalized inverse)

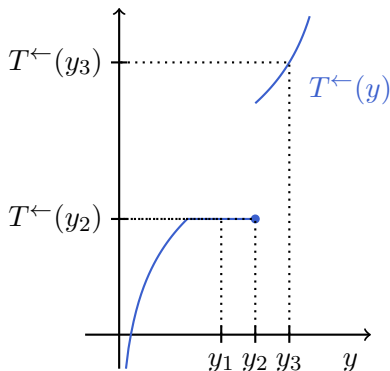
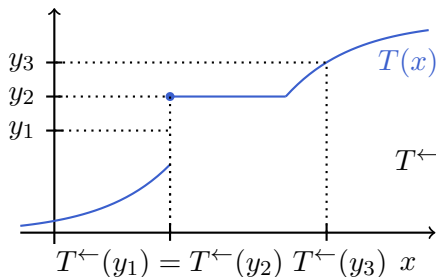
For any increasing function  $T : \mathbb{R} \rightarrow \mathbb{R}$ , with  $T(-\infty) = \lim_{x \downarrow -\infty} T(x)$  and  $T(\infty) = \lim_{x \uparrow \infty} T(x)$ , the *generalized inverse*  $T^{\leftarrow} : \mathbb{R} \rightarrow \bar{\mathbb{R}} = [-\infty, \infty]$  of  $T$  is defined by

$$T^{\leftarrow}(y) = \inf\{x \in \mathbb{R} : T(x) \geq y\}, \quad y \in \mathbb{R},$$

with the convention that  $\inf \emptyset = \infty$ . If  $T$  is a df,  $T^{\leftarrow} : [0, 1] \rightarrow \bar{\mathbb{R}}$  is the *quantile function* of  $T$ .

- If  $T$  is *continuous and  $\uparrow$* , then  $T^{\leftarrow} \equiv T^{-1}$  (ordinary inverse).
- There are *rules for working with  $T^{\leftarrow}$*  (often, not always) similar to  $T^{-1}$ ; see Proposition A.17.

How to determine  $T^{\leftarrow}$  from  $T$ :



- Flat parts (jumps) of  $T$  correspond to jumps (flat parts) of  $T^{\leftarrow}$ .
- Assume  $T$  to be a df and  $L \sim T$ .
  - What is the probability that  $L$  falls in the region where  $T$  is flat?
  - What is  $\mathbb{P}(L = T^{\leftarrow}(y_1))$ ?

### Example 2.7 ( $\text{VaR}_\alpha$ for $N(\mu, \sigma^2)$ and $t_\nu(\mu, \sigma^2)$ )

1) Let  $L \sim N(\mu, \sigma^2)$ . Then

$$F_L(x) = \mathbb{P}(L \leq x) = \mathbb{P}((L - \mu)/\sigma \leq (x - \mu)/\sigma) = \Phi((x - \mu)/\sigma).$$

This implies that

$$\text{VaR}_\alpha(L) = F_L^{\leftarrow}(\alpha) = F_L^{-1}(\alpha) = \mu + \sigma\Phi^{-1}(\alpha).$$

**Check:**  $F_L(\text{VaR}_\alpha(L)) = \Phi(((\mu + \sigma\Phi^{-1}(\alpha)) - \mu)/\sigma) = \alpha$ .

2) Let  $L \sim t_\nu(\mu, \sigma^2)$ , so  $(L - \mu)/\sigma \sim t_\nu = t_\nu(0, 1)$  and thus, as above,

$$\text{VaR}_\alpha(L) = \mu + \sigma t_\nu^{-1}(\alpha).$$

Note that  $X \sim t_\nu = t_\nu(0, 1)$  has density

$$f_{t_\nu}(x) = \frac{\Gamma((\nu + 1)/2)}{\sqrt{\nu\pi}\Gamma(\nu/2)}(1 + x^2/\nu)^{-\frac{\nu+1}{2}}.$$

Furthermore, if  $\nu > 1$ ,  $\mathbb{E}X$  exists and  $\mathbb{E}X = 0$ . If  $\nu > 2$ , then  $\text{var } X$  exists and  $\text{var } X = \frac{\nu}{\nu-2}$ ; in particular,  $Z = \sqrt{\frac{\nu-2}{\nu}}X \sim t_\nu(0, \frac{\nu-2}{\nu})$  has  $\text{var } Z = 1$ .

## Choices of parameters $\Delta t, \alpha$ :

- $\Delta t$  should reflect the time period over which the portfolio is held (unchanged) (e.g. insurance contracts:  $\Delta t = 1$  y)
- $\Delta t$  should be relatively small (more risk-factor change data is available).
- Typical choices:
  - ▶ For limiting traders:  $\alpha = 0.95$ ,  $\Delta t = 1$  d
  - ▶ According to Basel II:
    - Market risk:  $\alpha = 0.99$ ,  $\Delta t = 10$  d (2 trading weeks)
    - Credit risk and operational risk:  $\alpha = 0.999$ ,  $\Delta t = 1$  y
  - ▶ According to Solvency II:  $\alpha = 0.995$ ,  $\Delta t = 1$  y
- Backtesting often needs to be carried out at lower confidence levels in order to have sufficient statistical power to detect poor models.
- Be cautious with strictly interpreting  $\text{VaR}_\alpha(L)$  (and other risk measure) estimates (considerable model/liquidity risk).



## 2.3.3 VaR in risk capital calculations

### 1) VaR in regulatory capital calculations for the trading book

For banks using the *internal model (IM)* approach for market risk in Basel II (similarly but more involved for Basel III), the daily risk capital formula is

$$RC^t = \max \left\{ VaR_{0.99}^{t,10}, \frac{k}{60} \sum_{i=1}^{60} VaR_{0.99}^{t-i+1,10} \right\} + c.$$

- $VaR_{\alpha}^{s,10}$  denotes the 10-day  $VaR_{\alpha}$  calculated at day  $s$  ( $t = \text{today}$ ).
- $k \in [3, 4]$  is a multiplier (or *stress factor*).
- $c =$  stressed VaR charge (calculated from data from a volatile market period) + incremental risk charge (IRC;  $VaR_{0.999}$ -estimate of the annual distribution of losses due to defaults and downgrades) + charges for specific risks.

The averaging tends to lead to smooth changes in the capital charge over time unless  $VaR_{0.99}^{t,10}$  is very large.

## 2) The Solvency Capital Requirement in Solvency II

The *Solvency Capital Requirement (SCR)* is the amount of capital that enables the insurer to meet its obligations over  $\Delta t = 1$  y with  $\alpha = 0.995$ . Let  $V_t$  denote equity capital. The insurer wants to determine the minimum amount of extra capital  $x_0$  to be solvent in  $\Delta t$  with probability  $(\geq)\alpha$ , so

$$\begin{aligned}x_0 &= \inf\{x \in \mathbb{R} : \mathbb{P}(V_{t+1} + x(1+r) \geq 0) \geq \alpha\} \\&= \inf\left\{x \in \mathbb{R} : \mathbb{P}\left(-\left(\frac{V_{t+1}}{1+r} - V_t\right) \leq x + V_t\right) \geq \alpha\right\} \\&= \inf\{x \in \mathbb{R} : \mathbb{P}(L_{t+1} \leq x + V_t) \geq \alpha\} \\&= \inf\{x \in \mathbb{R} : F_{L_{t+1}}(x + V_t) \geq \alpha\} \\&= \inf\{z - V_t \in \mathbb{R} : F_{L_{t+1}}(z) \geq \alpha\} = \text{VaR}_\alpha(L_{t+1}) - V_t\end{aligned}$$

and thus  $\text{SCR} = V_t + x_0 = \text{VaR}_\alpha(L_{t+1})$  (available capital now + capital required to be solvent in  $\Delta t$  with probability  $\geq \alpha$ ). If  $x_0 < 0$ , the company is already well capitalized.

## 2.3.4 Other risk measures based on loss distributions

### 1) Variance (or standard deviation)

- $\text{var}_\alpha(L)$  (or standard deviation) has a long history as a risk measure in finance (due to Markowitz).
- Drawbacks:
  - ▶  $\mathbb{E}(L^2) < \infty$  required (not justifiable for non-life insurance or operational risk)
  - ▶ no distinction between positive/negative deviations from the mean (var, or standard deviation, is only a good risk measure if  $F_L$  is roughly symmetric around  $\mathbb{E}L$ , but  $F_L$  is typically skewed in credit and operational risk)

### 2) Expected shortfall

Besides VaR, expected shortfall is the most important risk measure in practice.

### Definition 2.8 (Expected shortfall)

For a loss  $L \sim F_L$  with  $\mathbb{E}(L_+) = \mathbb{E}(\max\{L, 0\}) < \infty$ , *expected shortfall (ES)* at confidence level  $\alpha \in (0, 1)$  is defined by

$$\text{ES}_\alpha = \text{ES}_\alpha(L) = \frac{1}{1 - \alpha} \int_\alpha^1 \text{VaR}_u(L) \, du. \quad (6)$$

- $\text{ES}_\alpha$  is the **average of  $\text{VaR}_u$**  over all  $u \geq \alpha \Rightarrow \text{ES}_\alpha \geq \text{VaR}_\alpha$ .
- $\text{ES}_\alpha$  **looks further into the tail** of  $F_L$ , it is a “what if” risk measure ( $\text{VaR}_\alpha$  is **frequency**-based;  $\text{ES}_\alpha$  is **severity**-based). This also becomes clear from the following result which shows that **under continuity**, expected shortfall equals *conditional tail expectation or tail value-at-risk*.

### Proposition 2.9 ( $\text{ES}_\alpha(L)$ under continuity)

If  $F_L$  is continuous,  $\text{ES}_\alpha(L) = \mathbb{E}(L \mid L > \text{VaR}_\alpha(L))$ .

*Proof.* If  $F_L$  is continuous,  $F_L(\text{VaR}_\alpha(L)) = F_L(F_L^{\leftarrow}(\alpha)) = \alpha$  and thus, for all  $x \geq \text{VaR}_\alpha(L)$ ,

$$\begin{aligned} F_{L|L>\text{VaR}_\alpha(L)}(x) &= \mathbb{P}(L \leq x \mid L > \text{VaR}_\alpha(L)) \\ &= \frac{\mathbb{P}(L \leq x, L > \text{VaR}_\alpha(L))}{\mathbb{P}(L > \text{VaR}_\alpha(L))} = \frac{\mathbb{P}(\text{VaR}_\alpha(L) < L \leq x)}{\mathbb{P}(L > \text{VaR}_\alpha(L))} \\ &= \frac{F_L(x) - F_L(\text{VaR}_\alpha(L))}{1 - F_L(\text{VaR}_\alpha(L))} = \frac{F_L(x) - \alpha}{1 - \alpha}. \end{aligned}$$

Since  $dF_{L|L>\text{VaR}_\alpha(L)}(x) = dF_L(x)/(1 - \alpha)$ ,

$$\begin{aligned} \mathbb{E}(L \mid L > \text{VaR}_\alpha(L)) &= \int_{\text{VaR}_\alpha(L)}^{\infty} x dF_{L|L>\text{VaR}_\alpha(L)}(x) \\ &= \frac{1}{1 - \alpha} \int_{\text{VaR}_\alpha(L)}^{\infty} x dF_L(x) = \frac{1}{1 - \alpha} \int_{\alpha}^1 \text{VaR}_z(L) dz \\ &= \text{ES}_\alpha(L), \end{aligned}$$

where we substituted  $x = \text{VaR}_z(L) = F_L^{\leftarrow}(z)$  (so  $F_L(x) = z$ ,  $dF_L(x) = dz$ ). □

- $ES_\alpha$  is more difficult to estimate and backtest than  $VaR_\alpha$  (the variance of estimators is typically larger; larger sample size required).
- $ES_\alpha(L) < \infty$  requires  $\mathbb{E}(L_+) < \infty$ .
- Subadditivity and elicibility (see the appendix). One can show:
  - ▶ In contrast to  $VaR_\alpha$ ,  $ES_\alpha$  is subadditive (more later).
  - ▶ In contrast to  $ES_\alpha$  (see Gneiting (2011) or Kou and Peng (2014)),  $VaR_\alpha$  exists if  $\mathbb{E}|L| = \infty$  and is elicitable (i.e. minimizes some expected functional (scoring function); see Gneiting (2011). This can be used for backtesting, comparing risk measures).

### Example 2.10 (A comparison of VaR and ES for stock returns)

- Consider Example 2.2 with a 1-stock portfolio and  $V_t = S_t = 10\,000$ . In this case,  $L_{t+1}^\Delta = -S_t X_{t+1}$ , where  $X_{t+1} = \log(S_{t+1}/S_t)$ .
- Let  $\sigma = 0.2/\sqrt{250}$  (annualized volatility of 20%) and assume

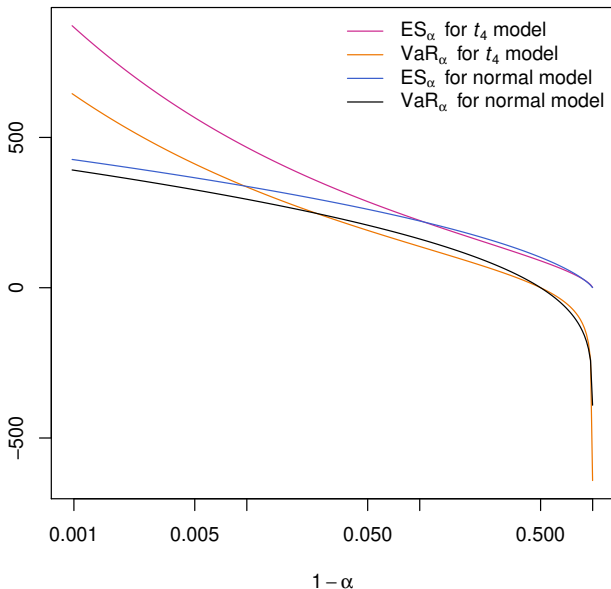
1)  $X_{t+1} \sim N(0, \sigma^2) \Rightarrow L_{t+1}^\Delta \sim N(0, S_t^2 \sigma^2)$ ;

2)  $X_{t+1} \sim t_\nu(0, \sigma^2 \frac{\nu-2}{\nu})$ ,  $\nu > 2$  (so that  $\text{var } X_{t+1} = \sigma^2$ , too). Then

$$X_{t+1} = \sqrt{\sigma^2 \frac{\nu-2}{\nu}} Y \quad \text{for } Y \sim t_\nu,$$

$$\Rightarrow L_{t+1}^\Delta = -S_t \sqrt{\sigma^2 \frac{\nu-2}{\nu}} Y \sim t_\nu(0, S_t^2 \sigma^2 \frac{\nu-2}{\nu}) \quad (\text{so } \text{var}(L_{t+1}^\Delta) = S_t^2 \sigma^2, \text{ too}).$$

- Consider  $\nu = 4$  and note that **only for sufficiently large  $\alpha$**  do we have  $\text{VaR}_\alpha^{t_4} \geq \text{VaR}_\alpha^{\text{normal}}$  and  $\text{ES}_\alpha^{t_4} \geq \text{ES}_\alpha^{\text{normal}}$ .



⇒ The  $t_4$  model is not always “riskier” than the normal model.



**Example 2.11 (Example 2.7 continued;  $\text{ES}_\alpha$  for  $N(\mu, \sigma^2)$  and  $t_\nu(\mu, \sigma^2)$ )**

1) Let  $\tilde{L} \sim N(0, 1)$ . Then  $\text{VaR}_\alpha(\tilde{L}) = 0 + 1 \cdot \Phi^{-1}(\alpha)$  and thus

$$\text{ES}_\alpha(\tilde{L}) = \frac{1}{1-\alpha} \int_\alpha^1 \Phi^{-1}(u) \, du \stackrel{x=\Phi^{-1}(u)}{=} \frac{1}{1-\alpha} \int_{\Phi^{-1}(\alpha)}^\infty x \varphi(x) \, dx,$$

where  $\varphi(x) = \Phi'(x) = \exp(-x^2/2)/\sqrt{2\pi}$ . Since  $x\varphi(x) = -\varphi'(x)$ ,

$$\text{ES}_\alpha(\tilde{L}) = \frac{-[\varphi(x)]_{\Phi^{-1}(\alpha)}^\infty}{1-\alpha} = \frac{-(0 - \varphi(\Phi^{-1}(\alpha)))}{1-\alpha} = \frac{\varphi(\Phi^{-1}(\alpha))}{1-\alpha}.$$

By linearity (or see soon),  $L \sim N(\mu, \sigma^2)$  has expected shortfall

$$\text{ES}_\alpha(L) = \mu + \sigma \text{ES}_\alpha(\tilde{L}) = \mu + \sigma \frac{\varphi(\Phi^{-1}(\alpha))}{1-\alpha}.$$

2) Let  $L \sim t_\nu(\mu, \sigma^2)$ ,  $\nu > 1$ . Similarly as above, one obtains that

$$\text{ES}_\alpha(L) = \mu + \sigma \frac{1}{1-\alpha} \frac{\nu}{\nu-1} f_{t_\nu}(t_\nu^{-1}(\alpha)^2)(1 + t_\nu^{-1}(\alpha)^2/\nu),$$

where  $f_{t_\nu}$  denotes the density of  $t_\nu$ ; see Example 2.7.

- By l'Hôpital's Rule (case “0/0”), one can show that

$$1 \leq \lim_{\alpha \uparrow 1} \frac{\text{ES}_\alpha(L)}{\text{VaR}_\alpha(L)} = \frac{\nu}{\nu - 1}.$$

- In finance, often  $\nu \in (3, 5)$ . With  $\nu = 3$ ,  $\text{ES}_\alpha(L)$  is 50% larger than  $\text{VaR}_\alpha(L)$  (in the limit for large  $\alpha$ ).
- For  $\nu \uparrow \infty$ ,  $\lim_{\alpha \uparrow 1} \frac{\text{ES}_\alpha(L)}{\text{VaR}_\alpha(L)} \downarrow 1$ .
- For  $\nu \downarrow 1$ ,  $\lim_{\alpha \uparrow 1} \frac{\text{ES}_\alpha(L)}{\text{VaR}_\alpha(L)} \uparrow \infty$ .

### Conclusion:

For losses with *heavy tails* (power-like), the difference between VaR and ES can be huge (for large  $\alpha$  as required by Basel II).

## 2.3.5 Coherent and convex risk measures

- Artzner et al. (1999) (coherent risk measures) and Föllmer and Schied (2002) (convex risk measures) propose **axioms of a good risk measure**.
- Assume that **risk measures**  $\varrho$  are defined on a **linear space of random variables**  $\mathcal{M}$  (including constants; we can thus add rvs, multiply them with constants etc.), so  $\varrho : \mathcal{M} \rightarrow \mathbb{R}$ .
- There are **two possible interpretations** of elements of  $\mathcal{M}$ :
  - 1) **Elements of  $\mathcal{M}$  are net asset values**  $V_{t+1}$ :  $\tilde{\varrho}(V_{t+1})$  denotes the **capital to be added to a position** with future value  $V_{t+1}$  to make it acceptable to a regulator.
  - 2) **Elements of  $\mathcal{M}$  are losses**  $L_{t+1} = -(V_{t+1} - V_t)$ :  $\varrho(L_{t+1})$  denotes the **total amount of capital** necessary to back a position with loss  $L$ .
- 1) and 2) are **related via**  $\varrho(L_{t+1}) = V_t + \tilde{\varrho}(V_{t+1})$  (total capital = available capital + additional capital). **We focus on 2) and drop “ $t + 1$ ”.**

## Axioms of coherence

**Axiom 1** (**monotonicity**)  $L_1, L_2 \in \mathcal{M}$ ,  $L_1 \leq L_2$  (a.s., i.e. almost surely)  
 $\Rightarrow \varrho(L_1) \leq \varrho(L_2)$

**Interpr.:** Positions which lead to a higher loss in every state of the world require more risk capital.

**Criticism:** None

**Axiom 2** (**translation invar.**)  $\varrho(L + l) = \varrho(L) + l$  for all  $L \in \mathcal{M}$ ,  $l \in \mathbb{R}$

**Interpr.:**

- By shifting a position with loss  $L$ , we alter the capital requirements accordingly.
- If  $\varrho(L) > 0$ , and  $l = -\varrho(L)$ , then  $\varrho(L - \varrho(L)) = \varrho(L + l) = \varrho(L) + l = 0$  so that adding  $\varrho(L)$  to a position with loss  $L$  makes it acceptable.

**Criticism:** Most people believe this to be reasonable.

**Axiom 3** (subadditivity)  $\varrho(L_1 + L_2) \leq \varrho(L_1) + \varrho(L_2)$  for all  $L_1, L_2 \in \mathcal{M}$

- Interpr.:
- Reflects the idea of diversification. Using a non-subadditive (that is, a superadditive)  $\varrho$  encourages institutions to legally break up into subsidiaries to reduce regulatory capital requirements.
  - Subadditivity makes decentralization possible: Assume  $L = L_1 + L_2$  and that we want to bound  $\varrho(L)$  by  $M$ . Choose  $M_j$  such that  $\varrho(L_j) \leq M_j$ ,  $j \in \{1, 2\}$ , and  $M_1 + M_2 \leq M$ . Then  $\varrho(L) \leq_{\text{subadd.}} \varrho(L_1) + \varrho(L_2) \leq M_1 + M_2 \leq M$ .

Criticism: VaR is ruled out under certain scenarios (see later). VaR is monotone, translation invariant, and positive homogeneous, but in general not subadditive.

**Axiom 4** (**positive homogeneity**)  $\varrho(\lambda L) = \lambda \varrho(L)$  for all  $L \in \mathcal{M}$ ,  $\lambda > 0$

**Interpr.:** (or motivation): For  $L_1 = \dots = L_n = L$ , subadditivity implies  $\varrho(nL) \leq n\varrho(L)$ , but there is no diversification, so equality should hold.

**Criticism:** If  $\lambda > 1$  is large, liquidity risk plays a role and one should rather have  $\varrho(\lambda L) > \lambda \varrho(L)$  (also to penalize risk concentration), but this contradicts subadditivity. This has led to *convex risk measures*, i.e. monotone, translation invariant  $\varrho$  satisfying  $\varrho(\lambda L_1 + (1-\lambda)L_2) \leq \lambda \varrho(L_1) + (1-\lambda)\varrho(L_2)$  for all  $L_1, L_2 \in \mathcal{M}$ ,  $0 \leq \lambda \leq 1$ .

### Definition 2.12 (Coherent risk measure)

A risk measure  $\varrho$  which satisfies Axioms 1–4 is called *coherent*.

Coherent risk measures are convex. The converse is not true in general (but for positive homogeneous risk measures  $\varrho$ ).

### Example 2.13 (Coherence of generalized scenario risk measures)

Let  $L(x)$  denote the hypothetical loss under scenario  $x$  (risk-factor change).

The generalized scenario risk measure

$$\psi_{\mathcal{X},w}(L) = \max\{\mathbb{E}_{\mathbb{P}}(L(\mathbf{X})) : \mathbf{X} \sim \mathbb{P} \in \{\mathbb{P}_1, \dots, \mathbb{P}_n\}\}$$

is coherent. Monotonicity, translation invariance, positive homogeneity are clear (by monotonicity and linearity of  $\mathbb{E}_{\mathbb{P}}(\cdot)$ ). For subadditivity, note that

$$\begin{aligned}\psi_{\mathcal{X},w}(L_1 + L_2) &= \max\{\underbrace{\mathbb{E}_{\mathbb{P}}(L_1(\mathbf{X}) + L_2(\mathbf{X}))}_{=\mathbb{E}_{\mathbb{P}}(L_1(\mathbf{X})) + \mathbb{E}_{\mathbb{P}}(L_2(\mathbf{X}))} : \mathbf{X} \sim \mathbb{P} \in \{\mathbb{P}_1, \dots, \mathbb{P}_n\}\} \\ &\leq \psi_{\mathcal{X},w}(L_1) + \psi_{\mathcal{X},w}(L_2).\end{aligned}$$

### Remark 2.14

One can show that all coherent risk measures can be represented as generalized scenarios via  $\varrho(L) = \sup\{\mathbb{E}_{\mathbb{P}}(L) : \mathbb{P} \in \mathcal{P}\}$  for a suitable set  $\mathcal{P}$  of probability measures.

## Theorem 2.15 (Coherence of ES)

ES is a coherent risk measure.

*Proof.* Monotonicity, translation invariance and positive homogeneity follow from VaR. Subadditivity is more involved but can be shown.  $\square$

## Superadditivity scenarios for VaR

Under the following scenarios,  $\text{VaR}_\alpha$  is typically superadditive:

- 1)  $L_1, L_2$  have skewed distributions;
- 2) Independent, light-tailed  $L_1, L_2$  and small  $\alpha$ ;
- 3)  $L_1, L_2$  have special dependence;
- 4)  $L_1, L_2$  have heavy tailed distributions.

Let's have a look at examples for 1), 2) and 4); for 3), see later.



### Example 2.16 (Skewed loss distributions)

Consider two independent losses of the form

$$L_j = \begin{cases} -5, & \text{with prob. } 1 - p = 0.991, \\ 100, & \text{with prob. } p = 0.009, \end{cases} \quad j \in \{1, 2\}.$$

Set  $\alpha = 0.99$ . Then  $\text{VaR}_\alpha(L_j) = -5$ ,  $j \in \{1, 2\}$ . The loss  $L_1 + L_2$  is

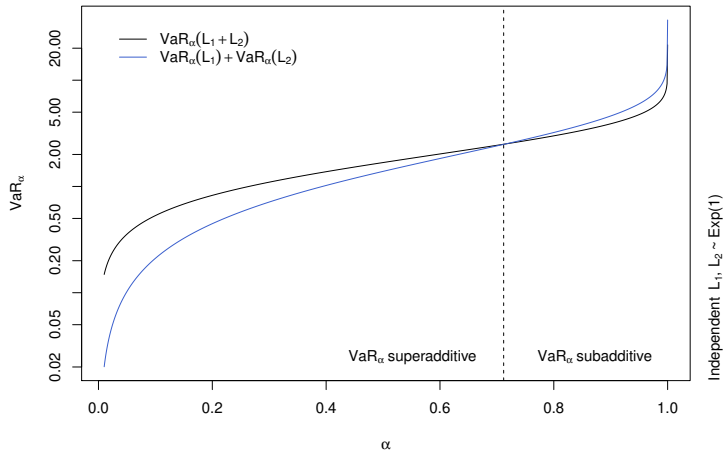
$$L_1 + L_2 = \begin{cases} -10, & \text{with prob. } (1 - p)^2 = 0.982081, \\ 95, & \text{with prob. } 2p(1 - p) = 0.017838, \\ 200, & \text{with prob. } p^2 = 0.000081. \end{cases}$$

Therefore,  $\text{VaR}_\alpha(L_1 + L_2) = 95 > -10 = \text{VaR}_\alpha(L_1) + \text{VaR}_\alpha(L_2)$ .

- For  $d$  such losses, one can show that  $\text{VaR}_\alpha$  is superadditive if and only if  $(1 - p)^d < \alpha \leq 1 - p$ .
- From the money lender's (investor) view, the losses could be **two independently defaultable zero-coupon bonds** (maturity  $T = 1$  y, face value 100, interest 5%, default probability  $p = 0.009$ , no recovery).

### Example 2.17 (Independent, light-tailed $L_1, L_2$ and small $\alpha$ )

If  $L_1, L_2 \stackrel{\text{ind.}}{\sim} \text{Exp}(1)$ ,  $\text{VaR}_\alpha$  is superadditive  $\iff \alpha < 0.71$ .



One can show that (independently of  $\lambda$ )  $\text{VaR}_\alpha$  is superadditive if and only if  $(1 - \alpha)(1 - 2 \log(1 - \alpha)) > 1$ .

### Example 2.18 (Heavy tailed loss distributions)

Let  $L_1, L_2 \stackrel{\text{ind.}}{\sim} F(x) = 1 - x^{-1/2}$ ,  $x \in [1, \infty)$ . One can show (via density convolution formula; tedious!) that  $F_{L_1+L_2}(x) = 1 - 2\sqrt{x-1}/x$ ,  $x \geq 2$ . By solving a quadratic equation one obtains that  $\text{VaR}_\alpha$  is superadditive for all  $\alpha \in (0, 1)$ .

### Remark 2.19 (Special case of comonotone risks; elliptical risks)

- $L_1 \stackrel{\text{a.s.}}{=} L_2$  (special case of “comonotonicity”) does not lead to the largest  $\text{VaR}_\alpha(L_1 + L_2)$  since  $\text{VaR}_\alpha(L_1 + L_2) = \text{VaR}_\alpha(2L_1) = 2 \text{VaR}_\alpha(L_1) = \text{VaR}_\alpha(L_1) + \text{VaR}_\alpha(L_2)$ , so “only” equality (whereas all above scenarios produced “ $>$ ”). All previous examples thus gave a larger VaR under independence than comonotonicity!
- $\text{ES}_\alpha$  is subadditive (see Theorem 2.15) and comonotone additive (same idea as for  $\text{VaR}_\alpha$ ) and thus largest under comonotonicity.
- $\text{VaR}_\alpha$  is subadditive (so coherent) for all elliptical distributions (strictly including the multivariate normal and  $t$ ) when  $\alpha \geq 1/2$ ; see later.