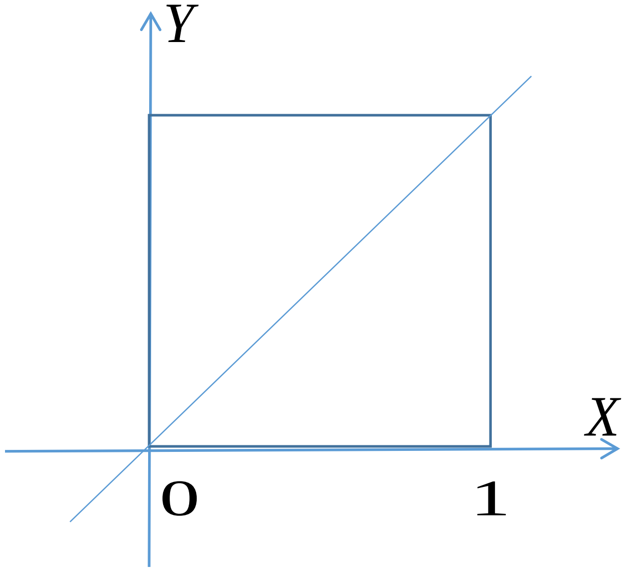
第四章 习题

1. $\left(P_{118}-1\right)$设随机变量 **X** 与 **Y** 独立，并且都在区间**(0,1)**上服从均匀分布，试求随机 变量 $Z=\left|X-Y\right|$的数学期望。



解：
$$E(Z)=E\left|X-Y\right|=\int_0^1\int_0^1\left|x-y\right|f(x,y)dxdy$$

$$=\int_0^1\int_0^y y-xdxdy+\int_0^1\int_y^1 x-ydxdy$$

$$=\int_0^1\left(xy-\frac{1}{2}x^2\right)\Bigg|_0^y dy+\int_0^1\left(\frac{1}{2}x^2-xy\right)\Bigg|_y^1 dy$$

$$=\int_0^1\frac{1}{2}y^2dy+\int_0^1\frac{1}{2}-\frac{1}{2}y^2-ydy=\frac{1}{3}$$

- **2.** $\left(P_{118}-2\right)$ 设随机变量 ***X*** 与 ***Y*** 独立，且都服从均数为**0**，方差为**1/2**的正态分布，求随机变量$\left|X-Y\right|$ 的方差。

- 解：令 $Z=X-Y$，则 $Z\sim N\left(0,1\right)$.

$$D\left|X-Y\right|=D\left|Z\right|=E\left|Z\right|^2-\left(E\left|Z\right|\right)^2=EZ^2-\left(E\left|Z\right|\right)^2=1-\left(E\left|Z\right|\right)^2$$

$$E\left|Z\right|=\int_{-\infty}^{+\infty}\left|z\right|\varphi\left(z\right)dz=\int_{-\infty}^{+\infty}\left|z\right|\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz=2\int_0^{+\infty}z\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz$$

$$=\int_0^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz^2\quad=-\frac{2}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}\Big|_0^{+\infty}=\sqrt{\frac{2}{\pi}}$$

$$D\left|X-Y\right|=1-\left(E\left|Z\right|\right)^2=1-\frac{2}{\pi}$$

- **3.**$\left(P_{118}-3\right)$ 将一枚硬币重复掷 ***n*** 次，以 ***X*** 和 ***Y*** 分别表示正面向上和反面向上的次数，试求 ***X*** 和 ***Y*** 的相关系数 ρ_{XY} .
- 解：$X \sim B\left(n, \frac{1}{2}\right)$；$Y \sim B\left(n, \frac{1}{2}\right)$；$X+Y=n$

$$Cov(X,Y)=Cov(X,n-X)=Cov(X,n)-Cov(X,X)$$

$$=-D(X)=-\frac{n}{4}$$

$$\rho_{XY}=\frac{Cov(X,Y)}{\sqrt{D(X)}\sqrt{D(Y)}}=\frac{-\frac{n}{4}}{\sqrt{\frac{n}{4}}\sqrt{\frac{n}{4}}}=-1$$

- **4.** $\left(P_{118}-5\right)$ **对于事件 *A* 和 *B*，满足** $0<P(A)<1,\ 0<P(B)<1,$ **定义**
- $$X=\begin{cases}0 & A\text{不发生}\\ 1 & A\text{发生}\end{cases}\qquad Y=\begin{cases}0 & B\text{不发生}\\ 1 & B\text{发生}\end{cases}\qquad \text{则 } X\sim\begin{pmatrix}0 & 1\\ 1-P(A) & P(A)\end{pmatrix}$$
- **证明随机变量 *X* 与 *Y* 独立的充分必要条件是*X* 与 *Y* 不相关。**
- **证明：** $X\sim B\{1,P(A)\},\quad Y\sim B\{1,P(B)\},\quad XY\sim B\{1,P(AB)\}$
- $$EX=P(X=1)=P(A),\qquad EY==P(Y=1)=P(B)$$

$$EXY=P(X=1,\ Y=1)=P(AB)$$

- **X 与 Y 独立，必有 X 与 Y 不相关。　反之，若X 与 Y 不相关：**
- 则 $Cov(X,Y) = EXY - EXEY = P(AB) - P(A)P(B) = 0$
- $P(AB) = P(A)P(B)$　**A与B独立**

$$P(\bar{A}B) = P(\bar{A})P(B);\quad P(A\bar{B}) = P(A)P(\bar{B});\quad P(\overline{AB}) = P(\bar{A})P(\bar{B})$$

$X\|Y$	**0**	**1**	
0	$P(\overline{AB})$	$P(\bar{A}B)$	$P(\bar{A})$
1	$P(A\bar{B})$	$P(AB)$	$P(A)$
	$P(\bar{B})$	$P(B)$	

$X\|Y$	**0**	**1**	
0	$P(\bar{A})P(\bar{B})$	$P(\bar{A})P(B)$	$P(\bar{A})$
1	$P(A)P(\bar{B})$	$P(A)P(B)$	$P(A)$
	$P(\bar{B})$	$P(B)$	

- **5.** $\left(P_{118}-6\right)$ 设 X 为只取非负整数值的离散型随机变量，
- 试证明：$EX=\sum_{n=1}^{+\infty}P\left(X\geq n\right)$
- 证明：$E\left(X\right)=\sum_{n=0}^{+\infty}nP\left(X=n\right)$ $\left(\text{设}\ P\left(X=n\right)=p_n, n=0,1,2\cdots\right)$

$$=0\times P\left(X=0\right)+1\times P\left(X=1\right)+\cdots+nP\left(X=n\right)+\cdots$$

$$=P\left(X=1\right)\quad+\left\{P\left(X=2\right)+P\left(X=2\right)\right\}$$

$$+\left\{P\left(X=n\right)+P\left(X=n\right)+\cdots+P\left(X=n\right)\right\}+\cdots$$

$$=\left\{P\left(X=1\right)+P\left(X=2\right)+\cdots\right\}+\left\{P\left(X=2\right)+P\left(X=3\right)+\cdots\right\}$$

$$+\left\{P\left(X=n\right)+P\left(X=n+1\right)+\cdots\right\}+\cdots\quad=\sum_{n=1}^{+\infty}P\left(X\geq n\right)$$

- **6.** $\left(P_{118}-7\right)$对于两个随机变量 X 与 Y，若EX^2, EY^2都存在，证明

$$\left(EXY\right)^2 \le EX^2EY^2$$

- **证明：对任意常数 c，考虑**

$$E\left(cX+Y\right)^2 = c^2EX^2+2cEXY+EY^2 \ge 0$$

- 有 $\left(2EXY\right)^2-4\times EX^2EY^2 \le 0$

- 则 $\left(EXY\right)^2 \le EX^2EY^2$

- **7.** $\left(P_{118}-8\right)$ 设随机变量$X_1, X_2, \cdots X_n$ 相互独立,且 $EX_i=\mu,\ DX_i=\sigma^2,$
- $i=1,2,\cdots n.$ 令 $\overline{X}=\frac{1}{n}\sum_{i=1}^{n}X_i,\quad S^2=\frac{1}{n\text{-}1}\sum_{i=1}^{n}\left(X-\overline{X}\right)^2$

及 $Y_i=X_i-\overline{X},\ i=1,2,\cdots n.\quad (1)$ 求 $E\overline{X}, D\overline{X}$。

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解:(1) $E\overline{X}=E\left(\frac{1}{n}\sum_{i=1}^{n}X_i\right)=\frac{1}{n}\sum_{i=1}^{n}EX_i=\frac{1}{n}\times n\mu=\mu.$

$$D\overline{X}=D\left(\frac{1}{n}\sum_{i=1}^{n}X_i\right)=\frac{1}{n^2}\sum_{i=1}^{n}D\left(X_i\right)=\frac{1}{n^2}\times n\sigma^2=\frac{\sigma^2}{n}$$

$$E\overline{X}=EX_i=\mu;\qquad D\overline{X}=\frac{\sigma^2}{n}$$

- (2)证明$S^2=\dfrac{1}{n-1}\left(\sum_{i=1}^{n}X_i^2-n\left(\overline{X}\right)^2\right)$，并求$ES^2$

证明：$S^2=\dfrac{1}{n-1}\sum_{i=1}^{n}\left(X_i-\overline{X}\right)^2=\dfrac{n}{n-1}\left\{\dfrac{1}{n}\sum_{i=1}^{n}\left(X_i-\overline{X}\right)^2\right\}=\dfrac{n}{n-1}B_2$

令 $B_2=\dfrac{1}{n}\sum_{i=1}^{n}\left(X_i-\overline{X}_n\right)^2=\dfrac{1}{n}\sum_{i=1}^{n}\left(X_i^2-2X_i\overline{X}_n+\overline{X}_n^2\right)$

$$=\frac{1}{n}\sum_{i=1}^{n}X_i^2-2\overline{X}_n\frac{1}{n}\sum_{i=1}^{n}X_i+\overline{X}_n^2=\frac{1}{n}\sum_{i=1}^{n}X_i^2-\overline{X}_n^2$$

$$S^2=\frac{n}{n-1}B_2=\frac{n}{n-1}\left(\frac{1}{n}\sum_{i=1}^{n}X_i^2-\overline{X}_n^2\right)=\frac{1}{n-1}\left(\sum_{i=1}^{n}X_i^2-n\overline{X}_n^2\right)$$

- $\because EB_2=\frac{1}{n}\sum_{i=1}^{n}EX_i^2-E\left(\overline{X}_n\right)^2=\frac{1}{n}\sum_{i=1}^{n}\left(\sigma^2+\mu^2\right)-\left(\frac{\sigma^2}{n}+\mu^2\right)$

$$=\sigma^2+\mu^2-\left(\frac{\sigma^2}{n}+\mu^2\right)=\frac{n-1}{n}\sigma^2$$

$$EX_i^2=DX_i+E\overline{X}=\sigma^2+\mu^2$$

$$E\left(\overline{X}_n\right)^2=D\overline{X}_n+\left(E\overline{X}_n\right)^2=\frac{\sigma^2}{n}+\mu^2$$

即：$EB_2=\frac{n-1}{n}\sigma^2$

而 $S^2=\frac{1}{n-1}\sum_{i=1}^{n}\left(X_i-\overline{X}\right)^2=\frac{n}{n-1}B_2$, 则 $ES^2=E\left(\frac{n}{n-1}B_2\right)=\sigma^2$

- **(3) 解：** $$Cov\left(X_i, \overline{X}\right) = Cov\left(X_i, \frac{1}{n}\sum_{j=1}^{n} X_j\right) \qquad i = 1,2,\cdots n$$

$$= Cov\left(X_i, \frac{1}{n} X_i\right) + Cov\left(X_i, \frac{1}{n}\sum_{\substack{j=1 \\ j\neq i}}^{n} X_j\right) = \frac{1}{n} Cov\left(X_i, X_i\right)$$

$$= \frac{1}{n} DX_i = \frac{\sigma^2}{n}$$

- **(4) 解：** $$D\left(Y_i\right) = D\left(X_i - \overline{X}\right) = D\left(\left(1 - \frac{1}{n}\right)X_i - \frac{1}{n}\sum_{\substack{j=1 \\ j\neq i}}^{n} X_j\right)$$

$$= \left(1 - \frac{1}{n}\right)^2 DX_i + \frac{n-1}{n^2} DX_j = \frac{n-1}{n}\sigma^2$$

- **(5) 解：** $Cov(Y_i, Y_j) = Cov\left(X_i - \overline{X}_n, X_j - \overline{X}_n\right) \quad i \neq j \qquad Y_i = X_i - \overline{X}_n$

$$= Cov(X_i, X_j) + Cov\left(\overline{X}_n, \overline{X}_n\right) - Cov\left(X_i, \overline{X}_n\right) - Cov\left(X_j, \overline{X}_n\right)$$

$$= 0 + D\left(\overline{X}_n\right) - \frac{\sigma^2}{n} - \frac{\sigma^2}{n} = -\frac{\sigma^2}{n}$$

- **(6) 解：** $D(Y_i - Y_j) = D\left\{\left(X_i - \overline{X}_n\right) - \left(X_j - \overline{X}_n\right)\right\} \qquad i \neq j$

$$= D(X_i - X_j)$$

$$= D(X_i) + D(X_j) = 2\sigma^2$$