Notes on Solitons

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1. Kink

1.1. The Weak Coupling Method in Quantum Mechanics

The weak coupling method is a semiclassical method based on static solutions of the classical equation of motion. It is essentially the perturbative expansion around some non-perturbative solutions, it is

- valid only for weak coupling, nevertheless
- it is a non-perturbative method, giving access to states which are not accessible from free field theory by a perturbative expansion in the coupling.

First let us recall some basic notions in non-relativistic quantum mechanics. For a non-relativistic particle in a one-dimensional potential well $V(x)$, given by

$$V(x) = V(x_0) + \frac{\omega^2}{2}(x - x_0)^2 + \frac{\lambda}{4}(x - x_0)^4, \tag{1.1}$$

where $\omega^2, \lambda > 0$. Then we know

1. Classically, the lowest energy trajectory for the particle happens when it sits at the minimum of the potential, i.e. $x(t) = x_0, E_0^{\text{cl.}} = V(x_0)$.
2. Such trajectory is not allowed in quantum mechanics due to uncertainty principle, a particle in the lowest energy state will have some zero point motion, resulting in a non-zero expectation value for momentum and potential,
$$E_0 = E_0^{\text{cl.}} + \Delta_0, \tag{1.2}$$
where Δ_0 is the quantum correction.
3. Also due to the zero point motion, the particle will have a non-zero spread in position around x_0,
$$\langle x \rangle = x_0, \quad \langle x^2 \rangle \neq 0.$$
4. If the energies of states under discussion are low energy, the quartic term can be neglected provided
$$\lambda \langle (x - x_0)^4 \rangle \ll \omega^2 \langle (x - x_0)^2 \rangle. \tag{1.3}$$
i.e. the spread of the particle in x is small enough.
5. In the limit of weak coupling, the potential has the form of a harmonic oscillator.

6. Then take into account the quartic term, the correction to energy can be obtained in a systematic perturbative method.
7. Obviously above procedure fails at $\omega^2 = 0$, then no matter how small the λ is, the weak coupling approximation fails. If both ω and λ are zero, the potential is flat and the wave function for energy eigenstates will spread over all space, it makes no sense to choose a special point and expand around it. In this case, the energy of the system would be
$$E_n = V + \frac{p_n^2}{2m} \tag{1.4}$$
where m is the particle mass. These can not be obtained by perturbing about classical solution, but can be obtained by semi-classical approximation.

These ideas can be generalized to n-dimensional Euclidean space and potentials having multiple wells, e.g. the famous double well potential. Supposed the local minima of potential are separated far enough, for low energy states, we can pick one of these minimum, expand the field around it, suppose the quadratic term in the potential dominates, then we will have a harmonic oscillator, whose energies and wave functions which can be used to approximate the low energy states of the full potential.

Assuming that the low energy levels can be obtained in a similar way in quantum field theory (QFT). The coordinates in quantum mechanics are replaced by field operators in QFT,
$$x, y, z, \ldots \to \phi(x_1), \phi(x_2), \phi(x_3), \ldots$$
where $x_{1,2,3}$ in $\phi(x_{1,2,3})$ are space-time positions. The lagrangian contains a potential $V[\phi(x)]$ which is a functional of the field. Its stationary points (here point means field configuration) are given by certain static field configurations, which are solution for the classical equation of motion. In weak coupling, we can expand the field about those stationary points, i.e. taking into consideration the small fluctuation about it, to find low lying energy levels.

Note that the solution for the classical equation of motion have trivial ones and non-trivial ones, the trivial ones are constants all over space and time, while the non-trivial ones are not. The non-trivial solutions can be regarded as "extended particles", and the fluctuation about which will be regarded as excitations.

1.2. ϕ^4 Kink in 1+1 Dimension

1.2.1. Vacuum Energy

The Lagrangian for ϕ^4 theory is
$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4}\phi^4, \tag{1.5}$$
where ϕ is a real scalar field. Note the sign of the mass term. Complete the square we get
$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{\lambda}{4}\left(\phi^2 - \frac{m^2}{\lambda}\right)^2 + \frac{m^4}{4\lambda}, \tag{1.6}$$

where the constant term is the Lagrangian density for the vacuum and doesn't contribute to the equation of motion, hence

$$\boxed{\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{\lambda}{4}\left(\phi^2 - \frac{m^2}{\lambda}\right)^2}. \tag{1.7}$$

The resulting Hamiltonian density reads

$$\mathcal{H} = \frac{1}{2}(\dot{\phi})^2 + \frac{1}{2}(\phi')^2 + \frac{\lambda}{4}\left(\phi^2 - \frac{m^2}{\lambda}\right)^2 - \frac{m^4}{4\lambda}, \tag{1.8}$$

where the constant term is inherited from the Lagrangian density. It is the energy density of the vacuum, which can be set to zero since it is the energy difference that we are really interested in. Hence we simply drop the constant term, then the Hamiltonian reads

$$H = \int dx \left(\frac{1}{2}(\dot{\phi})^2 + \frac{1}{2}(\phi')^2 + \frac{\lambda}{4}\left(\phi^2 - \frac{m^2}{\lambda}\right)^2\right). \tag{1.9}$$

The potential is

$$V(\phi) = \frac{1}{2}(\phi')^2 + \frac{\lambda}{4}\left(\phi^2 - \frac{m^2}{\lambda}\right)^2. \tag{1.10}$$

To obtain the potential energy we integrate the potential $V(x)$ over x,

$$V[\phi(x)] = \int_\infty^\infty dx \left[\frac{1}{2}(\phi')^2 + \frac{\lambda}{4}\left(\phi^2 - \frac{m^2}{\lambda}\right)^2\right]. \tag{1.11}$$

There is an abuse of notation here, since we used both V to denote the potential density and potential energy, but it will not cause any confusion.

The lowest energy solutions are

$$\phi(x) = \pm v, \quad v \equiv \frac{m}{\sqrt{\lambda}}. \tag{1.12}$$

To find the quantum correction to the ground state, we choose a vacuum and expand the field about it. Say we choose the positive minimum $\phi(x) = v$, writing $\phi \to \phi + v$, ($a \to b$ means substitute "a" with "b", whenever we see "a" we replace it with "b") keep only terms up to ϕ^2, we have

$$V[\phi(x)] = \int_\infty^\infty dx \phi(x) \left\{\frac{1}{2}(-\partial_x^2 + 2m^2)\right\} \phi(x), \tag{1.13}$$

note that ϕ is now the field fluctuation around v. The operator in the braces has eigenfunction e^{ikx} with eigenvalue $(k^2 + 2m^2)$. In terms of these normal modes, the potential takes the form of a set of decoupled harmonic oscillators with frequencies $\omega = \sqrt{k^2 + 2m^2}$. To see this, we can expand $\phi(x, t)$ in terms of e^{ikx}, note that we have put back the time dependence.

Perform the Fourier expansion in the spacial dimension, in a box of length L with periodic boundary condition,

$$\phi = \sum_{n\in\mathbb{Z}} c_n(t)\frac{1}{\sqrt{L}}e^{ik_n x}, \quad k_n = \frac{2\pi n}{L}. \tag{1.14}$$

Substitute it to Eq. (1.9) we have

$$H = \sum_{n\in\mathbb{Z}}\left(\frac{1}{2}\dot{c}_n^2(t) + \frac{1}{2}\omega_n^2 c_n^2\right), \quad \omega_n^2 = k_n^2 + 2m^2. \tag{1.15}$$

There it is, a set of decoupled harmonic oscillators. In order to find the quantum correction we need to quantize it, we can do this both by path integral quantization or canonical quantization, they are equivalent. To continue with canonical quantization we can first find the canonical momentum by performing a Legendre transformation, turn on the commutation relation then look for the Hamiltonian eigenvalues. The final result is the familiar quantum harmonic oscillator. The vacuum energy reads

$$E_{\text{vac}} = \frac{1}{2}\sum_n \sqrt{k_n^2 + 2m^2} = \frac{1}{2}\sum_n \sqrt{(2\pi n/L)^2 + 2m^2}, \tag{1.16}$$

which is nothing but the sum of all the zero point energies, the low energy excitation states have energy

$$E_{\{N_k\}} = \sum_k (N_k + 1/2)\sqrt{k^2 + 2m^2} = \sum_k (N_k + 1/2)\sqrt{(2\pi n/L)^2 + 2m^2}. \tag{1.17}$$

1.2.2. Kink Solution

The equation of motion resulting from Eq. (1.7) is

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \frac{\partial \mathcal{L}}{\partial \phi} \implies \ddot{\phi} - \phi'' = m^2\phi - \lambda\phi^3, \tag{1.18}$$

where the dots means differentiating with respect to (wrt) time t and the primes means differentiating wrt position x. A static solution is

$$\boxed{\phi_{\text{kink},a}(x) = \frac{m}{\sqrt{\lambda}}\tanh\left\{\frac{m(x-a)}{\sqrt{2}}\right\}} \tag{1.19}$$

where a is an arbitrary constant telling us the location of the kink. Taking the solution to Eq. (1.11), we obtain the energy of a kink:

$$\boxed{E^0_{\text{kink}} = V[\phi_{\text{kink}}] = \frac{2\sqrt{2}m^3}{3\lambda}.} \tag{1.20}$$

It is the classical rest mass of the kink, note the coupling appears in the denominator. Taking into considerations the zero point fluctuation, it will receive a quantum correction.

1.2.3. Is the Kink Solution Classically Stable?

Now the question arises: is the classical kink solution stable? By stable, we mean that the kink solution is a local minimal, i.e. small deviation from it will increase the energy. Assume the field deviates from a kink by $\delta\phi$,

$$\phi(x) = \phi_{\text{kink}} + \delta\phi, \tag{1.21}$$

the change in energy is

$$\delta E = \int_{\infty}^{\infty} dx \left(\frac{1}{2}(\delta\phi'(x))^2 + \underbrace{\delta\phi \left.\frac{\partial V}{\partial \phi}\right|_{\phi=\phi_{\text{kink}}}}_{=0,\,\text{EoM}} + \frac{(\delta\phi)^2}{2}\frac{\partial^2}{\partial\phi^2}\left[\frac{\lambda}{4}(\phi^2 - m^2/\lambda)^2\right]\Big|_{\phi_{\text{kink}}} \right) \tag{1.22}$$

where EoM is short for equation of motion, and we have neglected terms of $\mathcal{O}(\delta\phi^3)$. Integrate by part and set $\delta\phi \to 0$ at both ends, as is required by having a finite energy, we have

$$\delta E = \frac{1}{2}\int_{\infty}^{\infty} dx \left\{ \delta\phi \left[-\partial_x^2 - m^2 + 3\lambda\phi_{\text{kink}}^2 \right] \delta\phi \right\}, \tag{1.23}$$

substitute the kink solution we have

$$\delta E = \frac{1}{2}\int_{\infty}^{\infty} dx \left\{ \delta\phi \left[-\partial_x^2 - m^2 + 3m^2 \tanh^2 \frac{m(x-a)}{\sqrt{2}} \right] \delta\phi \right\}. \tag{1.24}$$

We need to find the eigenfunctions and eigenvalues for the operator in the bracket, then we can expand $\delta\phi$ in the basis of these eigenfunctions to see if the integral is semi-positive definite or not. The eigenvalue equation is

$$\left(-\partial_x^2 - m^2 + 3m^2 \tanh^2 \frac{m(x-a)}{\sqrt{2}} \right) \eta_i(x) = \omega_i^2 \eta_i(x), \tag{1.25}$$

which can be solved analytically (For more details refer to Landau and Lifshitz's textbook "the Quantum Mechanics Non-Relativistic theory").

Changing variable from $x \to z \equiv mx/\sqrt{2}$, we have

$$\left(-\frac{1}{2}\partial_z^2 + U(z) \right) \widetilde{\eta}_i(z) = \epsilon_i \widetilde{\eta}_i(z) \tag{1.26}$$

where $U(z) = -3(\tanh^2 z - 1)$ and $\epsilon_i = \omega_i^2/m^2 - 2$.

In general, for bounded solutions the eigenvalues are discrete (e.g. in 1D quantum mechanics the harmonic oscillator solutions are bounded, the corresponding eigenvalues are discrete), while for unbounded solutions the eigenvalues are continuous (e.g. the eigenfunction of free momentum operator $-i\hbar\nabla$ is $\propto e^{i\boldsymbol{p}\cdot\boldsymbol{x}}$ which is unbounded, the corresponding eigenvalues are continuous). This equation turns out to have two bound states with $\epsilon = -2, -1/2$ and unbounded states for $\epsilon > 0$ (not normalized).

$$\epsilon_0 = -2, \quad \omega_0^2 = 0, \quad \widetilde{\eta}_0(z) = \frac{1}{\cosh^2 z}, \quad \text{the zero mode}, \tag{1.27}$$

$$\epsilon_1 = -1/2, \quad \omega_1^2 = \frac{3}{2}m^2, \quad \widetilde{\eta}_1(z) = \frac{\sinh z}{\cosh^2 z}, \tag{1.28}$$

$$\epsilon = k^2/2, \quad \omega_k^2 = m^2\left(\frac{k^2}{2}+2\right), \quad \widetilde{\eta}_k(z) = e^{ikz}(3\tanh^2 z - 1 - k^2 - 3ik\tanh z). \tag{1.29}$$

Since $\omega_i^2 \geq 0 \, \forall \, i$ we can conclude that the kink solution is a local minimum, except for in the direction of the zero mode.

1.2.4. The Translation Mode

The existence of the zero mode has a simple yet important physics origin: the translation symmetry. Apparently, if we move the kink to another position it will still have the same energy,

$$V[\phi_{\text{kink},a}] = V[\phi_{\text{kink},a'}]. \tag{1.30}$$

If a' is very close to a, $a' = a + \delta a$, the difference of field configuration $\phi_{\text{kink},a'} - \phi_{\text{kink},a}$ is a fluctuation around $\phi_{\text{kink},a}$, whose resulting energy change is zero, this is the zero mode in Eq. (1.27),

$$\text{zero mode} \propto \phi_{\text{kink},a+\delta a} - \phi_{\text{kink},a} = \delta a \left.\frac{\partial \phi}{\partial a}\right|_{\phi_{\text{kink}}} = \delta a \frac{m^2}{\sqrt{2\lambda}} \cosh^{-2} \frac{m(x-a)}{\sqrt{2}}. \tag{1.31}$$

For the trivial solutions such as $\phi(x) = v$, the difference between one solution to its translated version is also trivial, hence to translate a trivial solution doesn't generate a zero mode. In general, for a non-trivial solution to the EoM, such as the kink solution, each symmetry is associated with a zero mode.

Remark 1.1.

1. *The original theory is translational invariant, meaning the actions is invariant under $\phi(x) \to \phi(x + \Delta x)$ where Δx is a constant.*

2. *The trivial vacuum is translational invariant, but the kink solution is not. $\phi_k(x) \neq \phi_k(x + \Delta x)$ since the kink solution is not constant in space. In this sense we can say the translation symmetry is broken by the kink solution. I guess we cannot call it spontaneous symmetry breaking because the kink solution is put there by hand.*

3. *For the broken symmetry, there is a zero mode. It is similar to the Goldstone mechanism, where for each spontaneous broken symmetry there is a massless boson.*

In the space of ϕ field configurations, each point represents a specific configuration for ϕ throughout the space and time. The existence of a zero mode means that at any point corresponding to a kink solution, there is a direction along which the potential energy doesn't change, while moving in other directions will increase the energy. Pictorially we can think of it as a valley of potential energy, walking at the bottom of the valley doesn't increase the potential energy.

Recall that in the weak coupling approximation, we pick a minimum point in the field space and expand the potential, obtaining a harmonic oscillator-like potential, which will provide us the higher order corrections. This doesn't work with zero modes, for in these direction the potential is flat, and there is nothing to expand about.

Zero mode will cause problems for both path integral and canonical quantization. In the case of path integral, since zero mode indicates an flat direction in the field space along which the potential is constant, integrating along this direction will naturally yield an infinite contribution. I think this is analogous to gauge theory, where the gauge invariance indicates there is a direction in the field configuration space along which the action is constant, resulting a infinite contribution in the path integral, in order to solve this problem in gauge theory we separate the path integral into two part, one over each gauge group orbit and one for the gauge group only, $\int_A = \int_{\mathcal{G}} \int_{\mathcal{A}/\mathcal{G}}$, where $\mathcal{A}, \mathcal{G}$ are the gauge field and gauge group respectively. In the case of kink in ϕ^4 we can first do the path integral without the zero mode, then integral the kink over its position to obtain the zero mode contribution. In the context of canonical quantization, for now we simply neglect the zero mode, for its contribution to kink mass is the same as giving the kink a nonzero momentum.

1.2.5. Kink Mass and Renormalization

The procedure is as following.

1. The original Lagrangian is written in terms of bare parameters, for example the mass term $\frac{1}{2}m^2\phi^2$ is actually $\frac{1}{2}m_0^2\phi_0^2$ where parameters with a subscript 0 are bare, they are not observables and are in general divergent. The renormalized perturbation theory says we should split the Lagrangian into

$$\mathcal{L}_0 = \mathcal{L}_R + \delta\mathcal{L}$$

   where we have separated the renormalized part $\mathcal{L}_R$ and counter-terms $\delta\mathcal{L}$, $\mathcal{L}_R$ only contains renormalized parameters, whose values are fixed with the help of the renormalization conditions.

2. Here is the question: the mass in the kink solution should be bare or renormalized? The answer is probably: it doesn't matter, for the kink solution itself is not an observable. To be specific, if it is

   bare parameters then we have a "bare" kink solution $\phi_{0\text{kink}} = \phi_{\text{kink}}(m_0)$, the classical energy will be divergent. This divergence can be cancelled by the divergence from mass counter term. For more details see Shifman's textbook *Advanced topics in Quantum Field Theory* where he used this method. Roughly speaking, in his book he started with the bare kink solution, then calculated the 1-loop correction, both are infinite, then putting them together results in going from m_0 to m_R with a finite energy correction.

renormalized This is what we will use in the following calculations. This is how renormalized perturbation theory works in general. The Lagrangian is originally given by bare parameters, we then separate each bare parameter into renormalized and counter parts, where the renormalized one is finite. This method is slightly more elegant (in my opinion) than dealing with the bare kink solution. In the case of kinks, we have a renormalized kink solution $\phi_{R\text{kink}} = \phi_{\text{kink}}(m_R)$ where m_R is the renormalized mass, the corresponding classical energy will be finite. The divergence comes from other terms such as $\frac{1}{2}\delta m^2 \phi^2_{R\text{kink}}$. Eventually the divergence will again be canceled by counter terms.

what we can measure is the energy difference, and that should be the same no matter whether bare or renormalized kink solution is used.

3. Put the field fluctuation (around the kink solution) into a box of length L, with periodic boundary condition (compactified), find the resulting zero point energy in the kink sector. We will find that the zero point energy in the kink sector is quadratic divergent.

4. The vacuum zero point energy is already known, see Eq. (1.16), which is also quadratic divergent. Subtract it from the kink zero point energy, the difference would be the kink mass correction. The subtraction is mode-to-mode, meaning we want the number of modes in the kink sector to be the same as that in the trivial vacuum sector. This is sometimes referred to as the lattice regularization, since having the same number of modes is the same as having the same number of degree of freedom, which is the same as having the same numbers of lattices.

5. Set $L \to \infty$. We will find that subtracting one quadratic divergence (vacuum zero point energy) from another quadratic divergence (kink zero point energy) results in a logarithmic divergence. Since the energy difference is divergent, we need to renormalize it. We will use the mass counter term in the Lagrangian to cancel the divergence, but it is by no means the only way to do it.

Remark 1.2. *There are some "questionable" regularization schemes; the quotation marks are because I don't fully understand the criteria for right or wrong. For example, we can put a hard cut-off in the momentum integral in vacuum and kink sector, and require that cut-off to be the same in both sectors, sending it to infinity independently. However that will give a different finite correction to the kink mass. The quantization of the breather solution in the sine-Gordon model appears to serve as a criterion for determining the correct scheme.*

In the mode-to-mode scheme, the modes in the kink sector include both discrete ones and continuous ones. To be more specific,

- in the kink sector, there will be two discrete modes corresponding to the two lowest energy modes, and a continuum of modes (of course with finite box size L the continuum modes are also discrete),
- in the trivial vacuum sector, there are only continuum modes (again, in the $L \to \infty$ limit).

To maintain the same number of modes, we must initially choose a finite but large box size L so that the continuum (of modes) becomes countably infinite. Then, the two lowest (eigenvalue) modes in the kink sector can be treated on equal footing with modes in the (original) continuum. Specifically, the mapping of mode numbers means

$$\sum_{k'=-N}^{N} (\text{vacuum sector}) \leftrightarrow \sum_{k=-(N-1)}^{N-1} (\text{kink sector}) + \text{ two lowest modes.}$$

Here k' labels momentum (or eigenvalue, the same) in the vacuum sector while k without prime labels the momenta in the kink sector. It means that in the kink sector, we start counting from the discrete modes. When going to infinite L, the summation becomes integral, in order to count the number of states correctly, we need to find the state density, and by state I mean the continuous eigenfunctions in Eq. (1.29).

Remark 1.3. *The kink solution is proportional to* $\tanh mx$ *and does not satisfy the periodic boundary condition, however the fluctuation around it can be put in a box with periodic boundary condition, since the fluctuation goes to zero at* $x \to \pm\infty$*.*

From now on, we will omit the subscript R on renormalized parameters; any parameter without subscript 0 will be renormalized.

In oreder to find the density of states, let's see what periodic boundary condition means for η_k. As $z \to \pm\infty$, Eq. (1.29) implies that

$$\boxed{\eta_k(z) \to \exp\{i(kz \pm \delta(k)/2)\}, \quad \delta(k) = 2\arctan\left(\frac{-3k}{2-k^2}\right)} \tag{1.32}$$

up to a multiplicative factor, which we ignored since the normalization will fix it.

Keep in mind that the arctan function is not single valued. The arctan function originates from Eq. (1.29), and whatever value we choose for arctan should be able to replicate the last term in that equation.In the function $\arctan\left(\frac{-3k}{2-k^2}\right)$, $-3k$ represents the y-coordinate and $2-k^2$ represents the x-coordinate. Therefore, each k will determine a point in the Cartesian coordinate system. $\arctan\left(\frac{-3k}{2-k^2}\right)$ is nothing but the polar angle of that point.

If we require

$$\arctan\left(\frac{-3k}{2-k^2}\right) \in [-\pi, \pi]$$

then, although it looks like δ has a pole at $k^2 = 2$, it is actually removable since

$$\arctan\left(\frac{-3k}{2-k^2}\right) \to \begin{cases} \pi/2 & k \to -\sqrt{2}_- \\ \pi/2 & k \to -\sqrt{2}_+ \\ -\pi/2 & k \to \sqrt{2}_- \\ -\pi/2 & k \to \sqrt{2}_+ \end{cases} \tag{1.33}$$

where $k \to \sqrt{2}_-$ means k approach $\sqrt{2}$ from below. This is a consequence of requiring the range to be $[-\pi, \pi]$ instead of $[-\pi/2, \pi/2]$. As one can check, the derivative of $\delta(k)$ wrt k is continuous too. The asymptotic behaviour of δ is

$$\boxed{\arctan\left(\frac{-3k}{2-k^2}\right) \to \begin{cases} -\pi + \frac{3}{|k|} & k \to \infty \\ +\pi - \frac{3}{|k|} & k \to -\infty \end{cases}}. \tag{1.34}$$

Remark 1.4. *If we instead have*

$$\arctan\left(\frac{-3k}{2-k^2}\right) \in [-2\pi, 0]$$

then the discontinuity appears not at $|k| \to \infty$ *but* $k = 0$.

Recall that $z = mx/\sqrt{2}$, the length of the box is L for x thus $mL/\sqrt{2}$ for z. The periodic condition requires that the phase must differ by $2\pi n, n \in \mathbb{Z}$ from one spacial endpoint to another, which translates to

$$\frac{mL}{\sqrt{2}} k_n + \delta(k_n) = 2\pi n, \quad n \in \mathbb{Z}. \tag{1.35}$$

As $L \to \infty$, we have

$$\frac{dn}{dk} \approx \frac{1}{2\pi}\left[\frac{mL}{\sqrt{2}} + \frac{d}{dk}\delta(k)\right] = \frac{1}{2\pi}\left[\frac{mL}{\sqrt{2}} - 6\frac{k^2+2}{(k^2+1)(k^2+4)}\right] + \mathcal{O}\left(\frac{1}{L}\right). \tag{1.36}$$

Note that k here is the momentum paired with z not x, thus it is different from the momentum in vacuum sector. For example, k is dimensionless.

From now on let's use k_n to denote the momentum in the vacuum sector which is dimensional, and q_n for the "momentum" in the kink sector which is dimensionless. Hence what we call k_n in above equation will be written as q_n from now on.

Since we already know ω as a function of q, we have

$$E_{\text{kink}} = E^0_{\text{kink}} + \frac{m}{2}\sqrt{\frac{3}{2}} + \frac{m}{2}\sum_n \sqrt{\frac{q_n^2}{2} + 2} \tag{1.37}$$

where the last two terms is the zero point energy of solution Eq. (1.28). In the meanwhile the vacuum energy is

$$E_{\text{vac}} = \frac{1}{2}\sum_{k_n} \sqrt{k_n^2 + 2m^2} \tag{1.38}$$

where $k_n = \frac{mq_n}{\sqrt{2}} + \frac{\delta}{L}, n \in \mathbb{Z}$.

Next we will cutoff the number of modes at N, subtract the vacuum energy from the kink energy, keep in mind what we discussed about matching the modes, we end up with

$$E_{\text{kink}} - E_{\text{vac}} = E^0_{\text{kink}} + \frac{m}{2}\sqrt{\frac{3}{2}} + \frac{m}{2}\sum_{n=-N+1}^{N-1}\sqrt{\frac{q_n^2}{2}+2} - \frac{m}{2}\sum_{n=-N}^{N}\sqrt{\left(\frac{q_n}{\sqrt{2}}+\frac{\delta}{mL}\right)^2+2}, \tag{1.39}$$

we need to do some work with the last two term,

$$\sum_{n=-N+1}^{N-1}\sqrt{\frac{q_n^2}{2}+2} = 2\sum_{n=0}^{N-1}\sqrt{\frac{q_n^2}{2}+2} - \sqrt{2} = 2\sum_{n=1}^{N}\sqrt{\frac{q_{n-1}^2}{2}+2} - \sqrt{2},$$

$$\sum_{n=-N}^{N}\sqrt{\left(\frac{q_n}{\sqrt{2}}+\frac{\delta}{mL}\right)^2+2} = 2\sum_{n=1}^{N}\sqrt{\left(\frac{q_n}{\sqrt{2}}+\frac{\delta}{mL}\right)^2+2} + \sqrt{2}$$

where in the second equality we have used $\delta(0) = 0$. Then

$$E_{\text{kink}} - E_{\text{vac}} = E^0_{\text{kink}} + \frac{m}{2}\sqrt{\frac{3}{2}} + m\sum_{n=1}^{N}\left[\sqrt{\frac{q_{n-1}^2}{2}+2} - \sqrt{\left(\frac{q_n}{\sqrt{2}}+\frac{\delta}{mL}\right)^2+2}\right] - \sqrt{2}m, \tag{1.40}$$

in the bracket, we can treat the second square root term as the first term with a slight different q_n, so we can write the whole bracket term as $\Delta q\, d\sqrt{\cdots}/dq$. From the periodic condition relation

$$q_n(mL/\sqrt{2}) + \delta(q_n) = 2\pi n, \quad n \in \mathbb{Z}. \tag{1.41}$$

we have

$$q_n - q_{n-1} = \frac{\sqrt{2}}{mL}\left\{2\pi - [\delta(q_n) - \delta(q_{n-1})]\right\} \tag{1.42}$$

thus

$$\frac{q_n}{\sqrt{2}} + \frac{\delta(q_n)}{mL} = \frac{q_{n-1}}{\sqrt{2}} + \frac{\delta(q_{n-1})}{mL} + \frac{2\pi}{mL}. \tag{1.43}$$

Expand in $1/L$ up to $\mathcal{O}(1/L)$, we have

$$m\sum_{n=1}^{N}\left[\sqrt{\frac{q_{n-1}^2}{2}+2} - \sqrt{\left(\frac{q_n}{\sqrt{2}}+\frac{\delta}{mL}\right)^2+2}\right]$$
$$= -\sum_{n=1}^{N}\left(\frac{\sqrt{2}\delta(q_{n-1})}{L} + \frac{2\sqrt{2}\pi}{L}\right)\frac{d}{dq}\sqrt{\frac{q^2}{2}+2}\Bigg|_{q=q_{n-1}}, \tag{1.44}$$

in the continuous limit,

$$\sum_n \to \int dq\frac{dn}{dq} \to \frac{mL}{\sqrt{2}}\left(\int\frac{dq}{2\pi} + \mathcal{O}(1/L)\right),$$

we will neglect the last term. The energy difference become

$$
\begin{aligned}
E_{\text{kink}} - E_{\text{vac}} &= \frac{2\sqrt{2}m^3}{3\lambda} + \frac{m}{2}\sqrt{\frac{3}{2}} - m\int_0^\Lambda \frac{dq}{2\pi}(\delta(q)+2\pi)\frac{d}{dq}\sqrt{\frac{q^2}{2}+2} - \sqrt{2}m \\
&= \frac{2\sqrt{2}m^3}{3\lambda} + \frac{m}{2}\sqrt{\frac{3}{2}} - \frac{m}{2\pi}\left[(\delta+2\pi)\sqrt{\frac{q^2}{2}+2}\right]\Bigg|_0^\infty \\
&\quad + \frac{m}{2\pi}\int_0^\infty dq\sqrt{\frac{q^2}{2}+2}\,\frac{d\delta(q)}{dq} - \sqrt{2}m,
\end{aligned}
\tag{1.45}
$$

where we have taken the limit $\Lambda \to \infty$. The third term in the second line contains

$$
\left[(\delta+2\pi)\sqrt{\frac{q^2}{2}+2}\right]\Bigg|_0^\infty
$$

which is a little subtle, for the value depends on the convention. We chose

$$
\boxed{\delta(q) \to -2\pi + \frac{6}{q} \quad \text{at} \quad q \to \infty,} \tag{1.46}
$$

and get

$$
\left[(\delta+2\pi)\sqrt{\frac{q^2}{2}+2}\right]\Bigg|_0^\infty = 3\sqrt{2} - 2\sqrt{2}\pi \tag{1.47}
$$

Remark 1.5. *The treatment about this surface term in Rajaraman's note is misleading, since he claims that as $q \to \infty$, $\arctan(\cdots) \sim 1/q$. But he also forgot to discard two modes so two mistakes cancel each other, as can be seen from comparing the surface term difference and the contribution from the two highest frequency modes.*

Substitute this and the value for $d\delta/dq$

$$
\frac{d\delta(q)}{dq} = -6\frac{2+q^2}{(1+q^2)(4+q^2)}
$$

to Eq. (1.45), the energy difference now reads

$$
\boxed{E_{\text{kink}} - E_{\text{vac}} = \frac{2\sqrt{2}m^3}{3\lambda} + \frac{m}{2}\sqrt{\frac{3}{2}} - \frac{3\sqrt{2}m}{2\pi} - \frac{3m}{\sqrt{2}}\int\frac{dq}{2\pi}\frac{q^2+2}{(q^2+1)\sqrt{q^2+4}}} \tag{1.48}
$$

The superficial degree of divergence is zero, meaning that the divergence is logarithmic.

The divergence can be removed by replacing terms in the Lagrangian by their normal-ordered counterpart, for example substitute $\phi^2 \to :\phi^2:$, or by adding appropriate counter terms, such as $\frac{\delta m^2}{2}\phi^2$, where δm^2 will eventually absorb the divergence, render m^2 finite. These two methods are related (where we put back the subscript R for renormalized parameters),

$$
m_R^2 :\phi_R^2: \sim m_R^2\phi_R^2 - c_1 m_R^2, \tag{1.49}
$$

$$
\lambda_R :\phi_R^4: \sim \lambda_R(\phi_R^4 - c_2\phi_R^2 - c_3) \tag{1.50}
$$

where c_1, c_2, c_3 are constants related to renormalization, and the subscript R means renormalized parameters.

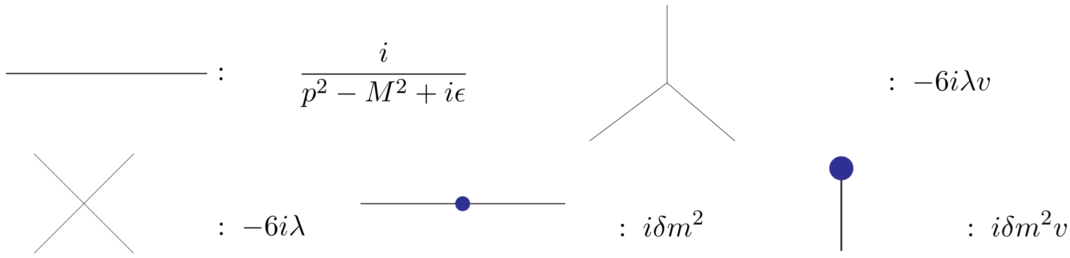

Figure 1.1: The blue dot denotes the counter terms.

1.2.6. Renormalized Perturbation Theory

We will follow steps as following,

1. Separate the original Lagrangian where all parameters are bare parameters into renormalized part and counter terms. Expand the field around the positive VEV which is $\langle \phi \rangle \equiv v$, with renormalized parameters. Keep in mind that the mass counter term δm^2 is of order λ, we will keep the leading order corrections
2. evaluate the kink zero point energy (with contribution from counter terms)
3. subtract the vacuum zero-pint energy from the kink energy, use δm^2 to cancel the divergence.

Remark 1.6. *It is tempting to replace m^2 with $m^2 + \delta m^2$ in the final result Eq.(1.48), since seems to be the only change in the Lagrangian. However that would not work, because for it to work, we need to perform this replacement everywhere in the derivation, including shifting the vacuum from v to $v + (\cdots)\delta m$, which we do not.*

Shift the field ϕ by $v \equiv m/\sqrt{\lambda}$ and denote the new field by η, namely define $\phi = \eta + v$, note that all parameters appeared are already renormalized. The Lagrangian for η with counter terms reads

$$\boxed{\mathcal{L}(x) = \frac{1}{2}\partial_\mu \eta \partial^\mu \eta - \frac{1}{2}M^2\eta^2 - \lambda v \eta^3 - \frac{\lambda}{4}\eta^4 + \frac{1}{2}\delta m^2 \eta^2 + \delta m^2 v \eta} \tag{1.51}$$

where $M^2 \equiv 2m^2$, $v^2 = m^2/\lambda$. The corresponding Feynman rules for η are shown in Fig. (1.1).

Remark 1.7. *The counter terms, although infinite, is defined perturbatively in orders of λ, since we want it to cancel the divergence order by order. δm^2 starts with order $\mathcal{O}(\lambda)$.*

The one loop diagrams are shown in Fig. (1.2). The renormalization conditions are

$$\boxed{Z_\lambda = 1, \quad Z_\eta = 1, \quad \text{No tadpole}} \tag{1.52}$$

which means $\lambda_0 \equiv \lambda$, $\phi_0 \equiv \phi \implies \eta_0 \equiv \eta$. Thus we require that in Fig. 1.2 diagram (c) get canceled by diagram (d), which is the tadpole condition.

The divergence in diagram (c) in Fig. (1.2) is

$$i\mathcal{M}_{(a)} = -\frac{6i\lambda}{2}\int \frac{d^2k}{(2\pi)^2}\frac{i}{k^2 - M^2 + i\epsilon} \to -\frac{3i\lambda}{2\pi}\ln\frac{\Lambda}{M} \tag{1.53}$$

where we have included the symmetry factor and rotated the integral into Euclidean space with a cut-off Λ,

$$\int \frac{d^2k}{(2\pi)^2}\frac{i}{k^2 - M^2 + i\epsilon} \to \int_0^\Lambda \frac{d^2k_E}{(2\pi)^2}\frac{1}{k_E^2 + M^2} = \frac{1}{2\pi}\ln\frac{\Lambda}{M} \tag{1.54}$$

where k_E is the Euclidean momentum, $k^0 \equiv ik_E^0$ and we assumed $\Lambda \gg M$.

The divergence in figure (d) in Fig. (1.2) is simply $i\delta m^2$. In order to have the desired cancellation, we need

$$\boxed{\delta m^2 = \frac{3\lambda}{2\pi}\ln\frac{\Lambda}{M}.} \tag{1.55}$$

which agrees with Rebhan, Nieuwenhuizen1997 paper. We can check that with this definition, diagram (a) and (b) in Fig. (1.2) do cancel each other. This is no miracle, since there is only one divergent loop, so one mass counter term should be able to cancel it wherever it appears.

The amplitude of diagram (e) in Fig. (1.2) is finite,

$$i\mathcal{M}_{(e)}(p) = \frac{1}{2}36\lambda^2 v^2 \int \frac{d^2k}{(2\pi)^2}\frac{1}{(k^2 - M^2)[(p-k)^2 - M^2]} \tag{1.56}$$

where the $1/2$ factor on the RHS is the symmetry factor, p is the momentum of the external line. Define $i\Sigma(p)$ as the sum of one particle irreducible diagrams at 1-loop, in our case the only such diagram surviving the regularization is diagram (e) in Fig. (1.2), thus

$$\Sigma(p^2) = \mathcal{M}_{(e)}(p^2)$$

then the pole mass M_P differers from M by

$$M_P^2 = M^2 - \Sigma(p^2 = M_P^2). \tag{1.57}$$

since $M_P = M$ at leading order, $\Sigma(M_P^2) = \Sigma(M^2)$ up to order λ, so we will use $\Sigma(M^2)$ instead of $\Sigma(M_P^2)$. We have

$$i\Sigma(p^2) = 9\lambda M^2 \int_{-\infty}^{\infty} \frac{d^2k}{(2\pi)^2}\frac{1}{(k^2 - M^2)[(p-k)^2 - M^2]} \tag{1.58}$$

where we have used $\lambda v^2 = M^2/2$. The integral is finite, in order to calculate it, we perform Feynman parametrization, which gives

$$\int_{-\infty}^{\infty} \frac{d^2k}{(2\pi)^2}\frac{1}{(k^2 - M^2)[(p-k)^2 - M^2]} = \int_0^1 dx \int \frac{d^2k}{(2\pi)^2}\frac{1}{[(k-xp)^2 - \Delta]^2} \tag{1.59}$$

where

$$\Delta \equiv M^2(1-x+x^2) > 0, \tag{1.60}$$

shift the momentum $k - xp \to k$ we have

$$\int_{-\infty}^{\infty} \frac{d^2k}{(2\pi)^2} \frac{1}{(k^2-M^2)[(p-k)^2-M^2]} = \int_0^1 dx \int \frac{d^2k}{(2\pi)^2} \frac{1}{[k^2-\Delta]^2}. \tag{1.61}$$

With the help of Wick rotation we have

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{d^2k}{(2\pi)^2} \frac{1}{(k^2-M^2)[(p-k)^2-M^2]} &= \int_0^1 dx \int \frac{d^2k}{(2\pi)^2} \frac{1}{[k^2-\Delta]^2} \\ &= \frac{i}{2\pi} \int_0^1 dx \int_0^{\infty} dk \frac{k}{(k^2+\Delta)^2} \\ &= \frac{i}{6\sqrt{3}M^2},\end{aligned} \tag{1.62}$$

take it back to the expression for Σ we have

$$\Sigma(M^2) = \frac{\sqrt{3}\lambda}{2}, \tag{1.63}$$

thus

$$\boxed{M_p^2 = M^2 - \frac{\sqrt{3}\lambda}{2}}. \tag{1.64}$$

where the subscript p stands for pole mass.

Warning: some mass-related parameters can be confusing:

m: the parameter in the Lagrangian we started with, it is not the mass of anything (has the wrong sign in the Lagrangian) but rather an interaction term for ϕ

M: The mass of η field, i.e. ϕ expanded about the positive minimum v

M_P: The pole mass of η

$M_{\textsf{kink}}$: As we will see later, it is the kink mass with counter term corrections

---

Next let us return to the kink mass. The new potential energy is

$$V'_{\text{vac}}(x) = V_{\text{vac}}(x) - \frac{m^2}{2\lambda}\delta m^2 - \frac{m}{\sqrt{\lambda}}\phi\delta m^2 - \frac{1}{2}\phi^2\delta m^2 \tag{1.65}$$

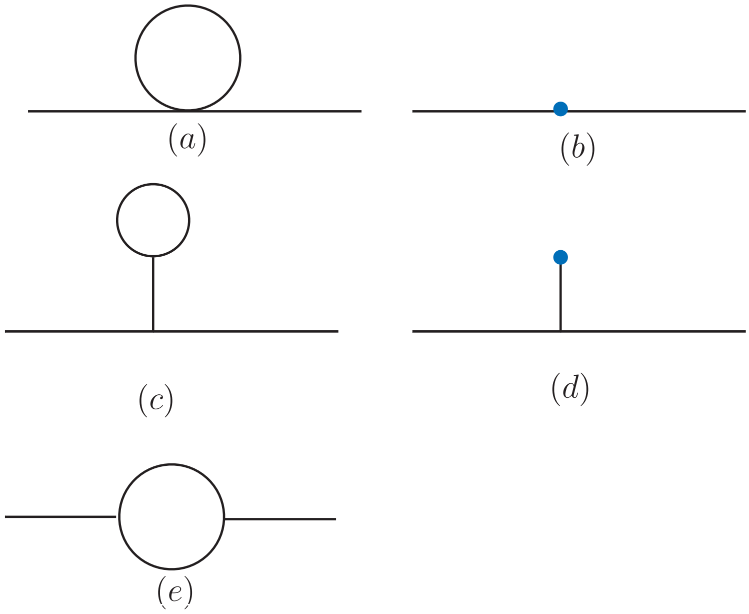

Figure 1.2: The 1-loop diagrams.

where the third term is of order $\lambda/\sqrt{\lambda} = \sqrt{\lambda}$, the last term is of order λ (δm^2 contributes a factor of λ), they can all be thrown away for now. We have

$$\boxed{E'_{\text{vac}}(x) = E_{\text{vac}}(x) - \int dx \frac{m^2}{2\lambda}\delta m^2}. \tag{1.66}$$

Since the counter term doesn't affect the vacuum, at least at the leading order, it does not affect the classical kink solution either. we need simply to consider the correction from the counter term by taking in the solution for the kink,

$$E'_{\text{kink}} = E_{\text{kink}} - \frac{1}{2}\delta m^2 \int dx \phi_k^2 \tag{1.67}$$

substitute the solution for kink, we have

$$\boxed{E'_{\text{kink}} = E_{\text{kink}} - \frac{1}{2}\delta m^2 \int dx \frac{m^2}{\lambda} \tanh^2 \frac{mx}{\sqrt{2}}}. \tag{1.68}$$

Putting everything together, the new energy difference reads

$$\begin{aligned} M_{\text{kink}} \equiv E'_{\text{kink}} - E'_{\text{vac}} = \frac{2\sqrt{2}m^3}{3\lambda} + \frac{m}{2}\sqrt{\frac{3}{2}} - \frac{3m}{\sqrt{2\pi}} \\ - \frac{3m}{\sqrt{2}} \int \frac{dq}{2\pi} \frac{q^2+2}{(q^2+1)(q^2+4)} \sqrt{q^2+4} - \frac{m^2}{2\lambda}\delta m^2 \int dx \tanh^2 \frac{mx}{\sqrt{2}} + \int dx \frac{m^2}{2\lambda}\delta m^2 \end{aligned} \tag{1.69}$$

where M_{kink} is the mass of the kink with corrections. Do the red integral we have

$$M_{\text{kink}} = \frac{2\sqrt{2}m^3}{3\lambda} + \frac{m}{2}\sqrt{\frac{3}{2}} - \frac{3m}{\sqrt{2}\pi} + \frac{\sqrt{2}m}{\lambda}\delta m^2 - \frac{3m}{\sqrt{2}}\int \frac{dq}{2\pi}\left(\frac{1}{\sqrt{q^2+4}} + \frac{1}{(q^2+1)\sqrt{q^2+4}}\right) \tag{1.70}$$

and the magenta terms are divergent separately. The δm^2 defined before is the same as defining it to be simplest form to cancel the divergence, similar to the minimal subtraction scheme. After throwing away the magenta part we end up with

$$-\frac{3m}{\sqrt{2}}\int \frac{dq}{2\pi}\frac{1}{(q^2+1)\sqrt{q^2+4}} = -\frac{m}{\sqrt{6}}$$

then all the divergence would be gone, we have a finite result.

Remark 1.8. *The counter term defined this way is the same as the counter term required to cancel the one-loop divergence, when calculating the Feynman diagrams.*

To summarize, the kink mass with order 1 correction is

$$\boxed{M_{\text{kink}} = \frac{2\sqrt{2}m^3}{3\lambda} + \frac{m}{2\sqrt{6}} - \frac{3m}{\sqrt{2}\pi}}. \tag{1.71}$$

$\frac{2\sqrt{2}m^3}{3\lambda}$ is the contribution from the classical solution, in quantize version we have replaced the parameters in the original solution by their renormalized version with certain renormalization conditions. It has a $1/\lambda$ dependence, which is typical for solitons contributions. $+\frac{m}{2\sqrt{6}} - \frac{3m}{\sqrt{2}\pi}$ comes from the quantum correction, it is due to the difference between two zero point energies, that from the kink sector and vacuum sector. It is of order 1 and proportional to the renormalized mass.

1.2.7. Modified Momentum Cutoff Regularization

Before we have mentioned that a simple momentum (energy) cutoff will yield a finite but different result, now let's take a look at it in some details. We introduced a dimensionless variable in the kink sector $q \equiv \frac{\sqrt{2}}{m}k$, where k is the momentum in the kink sector. To make it easier to compare the vacuum sector to the kink sector, we define a similar dimensionless variable in the vacuum sector $q' \equiv \frac{\sqrt{2}}{m}k$ where k is the momentum in the vacuum sector. In terms of q, q' the periodic boundary condition is

$$\frac{mL}{\sqrt{2}}q + \delta(q) = 2\pi n = \frac{mL}{\sqrt{2}}q' \tag{1.72}$$

and the resulting density of states are

$$\frac{dn}{dq} = \frac{mL}{2\sqrt{2}\pi} + \frac{1}{2\pi}\frac{d\delta(q)}{dq} \tag{1.73}$$

$$\frac{dn}{dq'} = \frac{mL}{2\sqrt{2}\pi} \tag{1.74}$$

The kink mass up to 1-loop correction is

$$M_{kink} = E^0_{kink} + \frac{m\sqrt{3}}{2\sqrt{2}} + \frac{m}{2}\sum_n \left(\sqrt{\frac{1}{2}q^2+2} - \sqrt{\frac{1}{2}q'^2+2}\right) + \frac{\sqrt{2}m}{\lambda}\delta m^2 \tag{1.75}$$

where the last term is the counter term, which can be written as an integration

$$\frac{\sqrt{2}m}{\lambda}\delta m^2 = 3\sqrt{2}m\int_0^\Lambda \frac{dk}{2\pi}\frac{1}{\sqrt{k^2+4}}. \tag{1.76}$$

In the continuous limit, the magenta sum become

$$\begin{aligned}
&\frac{m}{2}\left(\sum_n \sqrt{\frac{1}{2}q^2+2} - \sqrt{\frac{1}{2}q'^2+2}\right) \\
&= \frac{m}{2}\int_{-\Lambda}^{\Lambda} dq \left(\frac{dn}{dq} - \frac{dn}{dq'}\right)\sqrt{\frac{1}{2}q^2+2} \\
&= m\int_0^\Lambda \frac{dq}{2\pi}\frac{d\delta(q)}{dq}\sqrt{\frac{1}{2}q^2+2} \\
&= -3\sqrt{2}m\int_0^\Lambda \frac{dq}{2\pi}\frac{q^2+2}{(q^2+1)\sqrt{q^2+4}} \\
&= -3\sqrt{2}m\int_0^\Lambda \frac{dq}{2\pi}\left(\frac{1}{(q^2+1)\sqrt{q^2+4}} + \frac{1}{\sqrt{q^2+4}}\right)
\end{aligned} \tag{1.77}$$

Remark 1.9. *If we want a different boundary condition, we can do that by modifying the 2π appeared in the middle of Eq. (1.72). For example, if we want antiperiodic boundary condition, we can write $2\pi + \pi$.*

Again the divergence is canceled by the counter term, leaving us

$$\boxed{M^{\text{mc}}_{kink} = E^0_{kink} + \frac{\sqrt{3}m}{2\sqrt{2}} - \frac{m}{\sqrt{6}}} \tag{1.78}$$

where the superscript mc means momentum cutoff. This is different from the mode regularization result in Eq. (1.71).

Peter van Nieuwenhuizen et.al. have suggested a modified version of momentum cutoff regularization. The idea is that suppose the high frequency modes are insensitive to the background kink solution, for frequency higher than Λ we will no longer use the eigenfunction in Eq. (1.29), but continuous modes in the vacuum sector instead. This has influence on the density of states, namely

$$\frac{m}{\sqrt{2}}q + \theta(\Lambda - q)\delta(q) = 2\pi n = \frac{m}{\sqrt{2}}q' \tag{1.79}$$

where

$$\theta(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases} \tag{1.80}$$

thus

$$\frac{mL}{\sqrt{2}}dq + d(\theta(\Lambda - q)\delta(q)) = 2\pi dn \tag{1.81}$$

so

$$\frac{dn}{dq} = \frac{1}{2\pi}\left[\frac{mL}{\sqrt{2}} + \theta(\Lambda - q)\frac{d\delta(q)}{dq} - \delta_D(\Lambda - q)\delta(q)\right] \tag{1.82}$$

where δ_D is the usual Dirac delta function, the last term is new. Take it back to Eq. (1.75), integrate to infinity, we find

$$\boxed{\widetilde{M}^{\mathrm{mc}}_{kink} = E^0_{kink} + \frac{\sqrt{3}m}{2\sqrt{2}} - \frac{m}{\sqrt{6}} - \frac{m\Lambda}{\sqrt{2}} - \frac{3m}{\sqrt{2}\pi}} \tag{1.83}$$

The red terms are new, the second red term is exactly what we need to retrieve the mode-regularization result however the first red term is divergent.

The origin of the divergent term can be traced to the behaviour of $\delta(q)$ when $q \to \infty$. If in Eq. (1.46) the 2π disappears, then the divergent term also disappears. One way to achieve that is to exploit the arbitrary phase factor of the eigenfunction, define a new phase shift $\delta(q)$:

$$\delta(q) = \left(2\pi - 2\arctan\frac{3|q|}{2 - q^2}\right)\mathrm{sign}(q) \tag{1.84}$$

Then $\lim_{\Lambda\to\infty}\delta(\Lambda) = 6/q$, after some direct calculation we can see that the lattice-regularization result will be restored.

Remark 1.10. *Such limit for $\delta(q)$ is exactly what Rajaraman adopted in his note.*

1.3. Displacement Operator Approach

We start from the Lagrangian in terms of $\eta \equiv \phi - v$, see Eq. (1.51), where we expanded the field around the positive vacuum $\phi = v$. We have made a slight modification, namely we include a constant factor γ_0 (note that in the paper Evslin2020 they expanded around the negative vacuum).

$$\boxed{\mathcal{L}(x) = \frac{1}{2}\partial_\mu\eta\partial^\mu\eta - \frac{1}{2}M^2\eta^2 - \lambda v\eta^3 - \frac{\lambda}{4}\eta^4 + \frac{1}{2}\delta m^2\eta^2 + \delta m^2 v\eta - \gamma_0.} \tag{1.85}$$

The corresponding Hamiltonian, in terms of canonical momentum π and field operator η, reads

$$\boxed{\mathcal{H}(x) = \frac{1}{2}\pi^2 + \frac{1}{2}(\partial_x\eta)^2 + \frac{1}{2}M^2\eta^2 + \lambda v\eta^3 + \frac{\lambda}{4}\eta^4 - \frac{1}{2}\delta m^2\eta^2 - \delta m^2 v\eta + \gamma_0} \tag{1.86}$$

The field operator is denoted by η to avoid possible confusion, for when writing the note, I followed several different conventions from different papers and notations quickly got painfully misleading.

The two vacuum states at $\phi = \pm v$ are denoted by $|\pm\rangle$ respectively. The idea is as following,

1. the kink state $|K\rangle$
   - is created by some operator acting on the vacuum

   $$|K\rangle = \mathcal{O}\,|+\rangle\,, \quad a_k\,|+\rangle = 0$$

   where a_k is the usual annihilation operator, $\eta \sim \int \frac{dk}{2\pi}\frac{1}{\sqrt{2\omega_k}}(a_k e^{ipx} + a_k^\dagger e^{-ipx})$.
   - is an eigenstate of the Hamiltonian operator

   $$H\,|K\rangle = E_K\,|K\rangle$$

   - has classical kink solution as the expectation values at the leading order

   $$\langle K|\,\eta\,|K\rangle = \eta_c, \quad \eta_c = \frac{M}{\sqrt{2\lambda}}\left[\tanh\frac{M(x-a)}{2} - 1\right]$$

   where M is the mass of η field and we can set $a = 0$ without the loss of generality. η_c is the classical kink solution for η, satisfying $\eta_c(\infty) = 0$.

   In order to find the kink state, we can start from creating a state with the right expectation value first, by means of the displacement operator. It will not be the actual kink state because it will not be an eigenstate of the Hamiltonian, however we can solve that problem perturbatively.

   Remark 1.11. *The introduction of the shift operator will in general change the form of the Hamiltonian, introducing an effective mass term which is dependent on x.*

2. In the vacuum sector we expand the field ϕ using plane waves $e^{ik\cdot x}$ as basis, since the plane wave solve the free equation of motion. Then the free part of the Hamiltonian become diagonal in the form

   $$\sim \int \frac{dk}{2\pi} E_k a_k^\dagger a_k + \text{zero point energy}, \quad [a_p, a_q^\dagger] = 2\pi\delta(p-q).$$

   In the kink sector we can do the same, however the free Lagrangian is different due to the displacement operator, as a result the solutions to the equation of motion, which serve as the basis, will also be different. We can expand η in those basis so that the free Hamiltonian is diagonal.

3. Since the Lagrangian is not normal-ordered, the Hamiltonian contains divergent zero point energy, which will hopefully be canceled by the counter terms inherited from the renormalized perturbative theory, namely Lagrangian Eq. (1.51), and γ_0.

Remark 1.12. *The path integral approach starts from a classical theory, with classical solution of the equation of motion, then quantizes it. The nature works in the opposite way, the quantum theory is the starting point, classical theories are just low energy or large scale approximations. The Hamiltonian approach reflect it at some degree, where the classical solution is some approximation, which can be used as a test of the quantum theory.*

1.3.1. Displacement Operator

As mentioned before, η can be expanded in plane waves, with corresponding ladder operators $a, a^\dagger$. The vacuum is annihilated by a_k for all k, hence

$$\langle +|\, \eta \,|+\rangle = 0. \tag{1.87}$$

The question is, how do we construct a state with expectation value equal to some known function $f(x)$? We want a state that satisfies

$$\langle f|\, \eta \,|f\rangle = f(x). \tag{1.88}$$

It can be done using the displacement operator, which is a lot like the translation operator $T(\Delta x^\mu)$ in quantum mechanics. The translation operator will translate the state by a distance Δx so that $T(\Delta x^\mu)\psi(x) = \psi(x^\mu - \Delta x^\mu)$, changing the expectation value of the position operator. In a similar sense the displacement operator shifts the expectation value of the field operator at each space-time point, since in quantum field theory the field operator $\phi(x^\mu)$ is a generalization of the position operator $\hat{x}$ in quantum mechanics.

The translation operator is

$$T(\Delta x^\mu) = e^{iP_\mu \Delta x^\mu}, \quad P^\mu = (H, \boldsymbol{p})$$

where H is the Hamiltonian and $\boldsymbol{p}$ the momentum operators. The spacial components are

$$T(\Delta \boldsymbol{x}) = e^{-i\boldsymbol{P}\cdot\Delta\boldsymbol{x}} \tag{1.89}$$

The displacement operator turns out to be very similar in form, we just need to make the substitution

$$p \to \pi(x), \quad \Delta x \to f(x)$$

and put an integral notation whenever needed, we end up with

$$\mathcal{D}_f \equiv \exp\left\{-i\int dx f(x)\pi(x)\right\}. \tag{1.90}$$

We can check that

$$[\mathcal{D}_f, \eta(y)] = -f(y)\mathcal{D}_f \tag{1.91}$$

provided that

$$[\eta(x), \pi(y)] = i\delta(x-y).$$

As a result if one defines $|f\rangle \equiv \mathcal{D}_f\,|+\rangle$,

$$\langle f|\, \eta(x) \,|f\rangle = f(x). \tag{1.92}$$

To obtain the state with the right expectation value (form factor), we need simply to substitute η_c for f in $\mathcal{D}_f$ where

$$\boxed{\eta_c = \frac{M}{\sqrt{2\lambda}}\left[\tanh\frac{M(x-a)}{2} - 1\right]} \tag{1.93}$$

is the classical kink solution. Next we need to further modify it so it become an eigenstate of the Hamiltonian. We do it by adding another operator $\mathcal{O}_1$,

$$|K\rangle \equiv \mathcal{D}_{\eta_c}\mathcal{O}_1 |+\rangle \equiv \mathcal{D}_{\eta_c} |\mathcal{O}_1\rangle , \quad H|K\rangle = E_K |K\rangle , \tag{1.94}$$

however to keep the generality we will keep writing η_c as a general function f.

1.3.2. Shift the Hamiltonian

We want the kink state to be an eigenstate of the Hamiltonian

$$H\mathcal{D}_f |\mathcal{O}_1\rangle = E_K \mathcal{D}_f |\mathcal{O}_1\rangle . \tag{1.95}$$

We need to push H to the right of the displacement operators, $H\mathcal{D}_f = \mathcal{D}_f H'$.

With the help of BCH formula

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2}[A, [A, B]] + \cdots \tag{1.96}$$

and

$$\pi^2 \mathcal{D}_f = \mathcal{D}_f (\mathcal{D}_f^\dagger \pi \mathcal{D}_f)(\mathcal{D}_f^\dagger \pi \mathcal{D}_f) \tag{1.97}$$

plus

$$\mathcal{D}_f^\dagger \pi \mathcal{D}_f = \pi \tag{1.98}$$

$$\mathcal{D}_f^\dagger \eta(x) \mathcal{D}_f = \eta(x) + f(x) \tag{1.99}$$

$$\mathcal{D}_f^\dagger (\partial_x \eta(x)) \mathcal{D}_f = \partial_x \eta + \partial_x f \tag{1.100}$$

we have

$$H[\pi(x), \eta(x)]\mathcal{D}_f |\mathcal{O}_1\rangle = \mathcal{D}_f H[\pi(x), \eta(x) + f(x)] |\mathcal{O}_1\rangle = \mathcal{D}_f E_K |\mathcal{O}_1\rangle \tag{1.101}$$

The last two tern tells us that the displacement operator cancels out, leaving us with a Schroedinger-like equation

$$H[\pi(x), \eta(x) + f(x)] |\mathcal{O}_1\rangle \equiv H' |\mathcal{O}_1\rangle = E_K |\mathcal{O}_1\rangle \tag{1.102}$$

Separate the Hamiltonian into free Hamiltonian, interaction terms and counter terms,

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_I + \mathcal{H}_{ct}, \tag{1.103}$$

$$\mathcal{H}_0 = \frac{1}{2}\pi^2 + \frac{1}{2}(\partial_x \eta)^2 + \frac{1}{2}M^2\eta^2, \tag{1.104}$$

$$\mathcal{H}_I = \frac{\lambda}{4}\eta^4 + \lambda v \eta^3, \tag{1.105}$$

$$\mathcal{H}_{ct} = -\frac{1}{2}\delta m^2 \eta^2 - \delta m^2 \nu \eta + \gamma_0 \tag{1.106}$$

where $M = \sqrt{2}m$. Define $H_{0,I,ct}\mathcal{D}_f \equiv \mathcal{D}_f H'_{0,I,ct}$ etc., we can write H's in the explicit form and group terms that are constant, linear in η, quadratic in η, etc. By the end of the day we have

$$\begin{aligned}\mathcal{H}' = & \frac{1}{2}M^2 f^2 + \frac{1}{2}(\partial_x f)^2 + \frac{\lambda}{4}f^4 + \lambda v f^3 - \frac{1}{2}\delta m^2 f^2 - \delta m^2 \frac{M}{\sqrt{2\lambda}} f + \gamma_0 \\ & + M^2 f\eta + (\partial_x f)(\partial_x \eta) + \lambda f^3 \eta + 3\lambda v f^2 \eta \\ & + \frac{1}{2}\pi^2 + \frac{1}{2}M^2\eta^2 + \frac{1}{2}(\partial_x \eta)^2 + \frac{3}{2}\lambda f^2 \eta^2 + 3\lambda v f \eta^2 \\ & + \lambda f \eta^3 + \lambda v \eta^3 + \frac{\lambda}{4}\eta^4 - \delta m^2 f \eta - \delta m^2 v \eta - \frac{1}{2}\delta m^2 \eta^2\end{aligned}$$

where the first line contains scalar terms, the blue terms produce the classical energy for a given field configuration f, in the last line are the terms $\sim \mathcal{O}\left(\sqrt{\lambda}\right)$ thus can be neglected when calculating the leading order kink mass correction. The second line have terms linear in η. If f satisfies the equation of motion ($f \to \eta_c$), which is (static)

$$-M^2\eta - 3\lambda v \eta^2 - \lambda \eta^3 = \partial_x^2 \eta \tag{1.107}$$

which does not contain the counter terms, then the linear terms vanish.

Remark 1.13. *We choose to keep the counter terms out of the equation of motion.*

1.3.3. Mode Expansion

From now on we will replace f with η_c given in Eq. (1.93), so we can throw the linear terms away. Without loss of generality, we can set $a = 0$ so the kink is center at $x = 0$.

From the Hamiltonian we have the Lagrangian, focusing on the quadratic terms which are

$$\mathcal{L}_2 = \frac{1}{2}\partial_\mu \eta \partial^\mu \eta - \frac{1}{2}M^2\eta^2 - 3\lambda\eta_c\left(\frac{1}{2}\eta_c + v\right)\eta^2, \tag{1.108}$$

it is the free Lagrangian.The corresponding equation of motion is

$$\left[-\frac{1}{2}\partial_t^2 + \frac{1}{2}\partial_x^2 - \frac{1}{2}M^2 + \frac{3}{4}M^2\text{sech}^2\left(\frac{Mx}{2}\right)\right]\eta(t,x) = 0 \tag{1.109}$$

write $\beta \equiv \frac{M}{2}$, the equation of motion become

$$\left(\partial_t^2 - \partial_x^2\right)\eta(t,x) = \left[-4\beta^2 + 6\beta^2\text{sech}^2(\beta x)\right]\eta(t,x). \tag{1.110}$$

Write

$$\eta(t,x) = e^{-i\omega t}f(x),$$

the equation of motion become

$$\partial_x^2 f_q(x) + 6\beta^2\text{sech}^2(\beta x) f_q(x) = -q^2 f_q(x), \quad q^2 \equiv \omega^2 - (2\beta)^2. \tag{1.111}$$

We will continue to use the convention that q denotes the momentum in the kink sector, k, p denotes the momentum in the vacuum sector. So it will be clear which term comes from which sector.

The solutions of this equation is discussed in details in the paper Evslin2020, here we only cite the results. The solutions include two discrete bounded ones and a continuum. The continuum include even and odd solutions, denoted by $\psi_q^{e,o}$ respectively, which read

$$\psi_q^e(x) = \left[\frac{q^2}{\beta^2} - 2 + 3\mathrm{sech}^2(\beta x)\right]\cos(qx) - \frac{3q}{\beta}\tanh(\beta x)\sin(qx) \tag{1.112}$$

$$\psi_q^o(x) = \left[\frac{q^2}{\beta^2} - 2 + 3\mathrm{sech}^2(\beta x)\right]\sin(qx) + \frac{3q}{\beta}\tanh(\beta x)\cos(qx) \tag{1.113}$$

with normalization conditions

$$\int dx\psi_q^i(x)\psi_{q'}^j(x) = \pi\delta^{ij}C_q^2\delta(q-q'),\, C_q = \sqrt{\left(\frac{q^2}{\beta^2}+1\right)\left(\frac{q^2}{\beta^2}+4\right)},\, i,j \in \{\text{e,o}\} \tag{1.114}$$

we can further assemble these even and odd solutions into complex solutions

$$g_q(x) \equiv \psi_q^e(x) + i\psi_q^o(x) \tag{1.115}$$

note the plus sign in the middle. The reason for this change is to easier match g_q with e^{ikx}, other wise we would have $g_q \sim e^{-iqx}$ instead of $g_q \sim e^{iqx}$.

$g_q(x)$ satisfies

$$\int g_q(x)g_p^*(x)dx = 2\pi C_q^2\delta(p-q), \quad g_q(x)^* = g_{-q}(x). \tag{1.116}$$

There are two bound states,

- The even bound state

$$\omega_{BE} = 0, \quad k_{BE} = 2i\beta,$$
$$g_{BE}(x) = \mathrm{sech}^2(\beta x),$$
$$\int dx g_{BE}^2(x) = C_{BE}^2, \quad C_{BE} = \frac{2}{\sqrt{3\beta}}.$$

  The even bound state corresponds to the zero mode.

- The odd bound state

$$\omega_{BO} = \beta\sqrt{3}, \quad k_{BO} = i\beta,$$
$$g_{BO}(x) = -i\frac{\sinh(\beta x)}{\cosh^2(\beta x)},$$
$$\int dx|g_{BO}(x)|^2 = C_{BO}^2, \quad C_{BO} = \sqrt{\frac{2}{3\beta}}$$

  where the extra factor of i in g_{BO} is to make sure the imaginary part of a general solution is always odd.

We can define new ladder operator associated with new modes, changing from (denoted as {operator|eigenfunction})

$$\{a_k|e^{ikx},\ a_k^\dagger|e^{-ikx}\}$$

to

$$\{b_q|g_q,\ |b_q^\dagger|g_q^*,\ b_{BO}|g_{BO},\ b_{BO}^\dagger|g_{BO}^*,\ \phi_0|g_{BE}\}.$$

g_{BE} is different from other modes since it is related to the zero mode.

To be specific, we have

$$\eta(x) \equiv \eta_C(x) + \eta_{BO}(x) + \eta_{BE}(x), \tag{1.117}$$

$$\eta_C(x) = \int \frac{dq}{2\pi\sqrt{2\omega_q}}(b_q^\dagger + b_{-q})\frac{g_q(x)}{C_q}, \tag{1.118}$$

$$\eta_{BO}(x) = \frac{1}{\sqrt{2\omega_{BO}}}(b_{BO}^\dagger - b_{BO})\frac{g_{BO}(x)}{C_{BO}}, \tag{1.119}$$

$$\eta_{BE}(x) = \phi_0\frac{g_{BE}(x)}{C_{BE}}. \tag{1.120}$$

The canonical momentum has mode expansion

$$\pi(x) = i\int \frac{dk}{2\pi}\sqrt{\frac{\omega_q}{2}}(b_q^\dagger - b_{-q})\frac{g_q(x)}{C_q} + i\sqrt{\frac{\omega_{BO}}{2}}(b_{BO}^\dagger + b_{BO})\frac{g_{BO}(x)}{C_{BO}} + \pi_0\frac{g_{BE}(x)}{C_{BE}} \tag{1.121}$$

From the commutation relations of η, π we can deduce the commutation relations of $b, b^\dagger$

$$\boxed{[b_q, b_{q'}^\dagger] = 2\pi\delta(q-q'), \quad [b_{BO}, b_{BO}^\dagger] = 1, \quad [\phi_0, \pi_0] = i, \quad 0 \text{ otherwise}} \tag{1.122}$$

From $[\phi_0, \pi_0] = i$ we can see that ϕ_0 doesn't create a excited state for its commutator is imaginary. It should corresponds to moving kink states.

To find the leading order correction to the kink mass, we only need the scalars, quadratic terms including the counter terms in the Hamiltonian, the rest will give a higher order correction, or have zero energy expectation value. The terms in the Hamiltonian that contribute to the mass difference are

$$\begin{aligned}\Delta\widetilde{\mathcal{H}} = &\frac{1}{2}\pi^2 - \frac{1}{2}\eta\partial_x^2\eta + \frac{1}{2}M^2\eta^2 + \frac{3}{2}\lambda\eta_c^2\eta^2 + 3\lambda v\eta_c\eta^2 \\ &- \frac{1}{2}\delta m^2\eta^2 - \frac{1}{2}\delta m^2\eta_c^2 - \delta m^2\frac{M}{\sqrt{2\lambda}}\eta_c + \gamma_0.\end{aligned} \tag{1.123}$$

It is sometimes useful to keep in mind that the dimension of $\lambda \sim M^2$. We have

$$\Delta\widetilde{\mathcal{H}} = \frac{1}{2}\pi^2 - \frac{1}{2}\eta\partial_x^2\eta + \frac{1}{2}\widetilde{M}^2(x)\eta^2 + \cdots, \quad \widetilde{M}^2(x) \equiv M^2\left(1 - \frac{3}{1+\cosh(Mx)}\right) \tag{1.124}$$

where $\widetilde{M}$ is the modified mass.

A more useful form of the Hamiltonian density is

$$\boxed{\begin{aligned}\Delta\widetilde{\mathcal{H}} = \frac{1}{2}\pi^2 - \frac{1}{2}\eta[\partial_x^2 + 6\beta^2\mathrm{sech}^2(\beta x)]\eta + \frac{1}{2}M^2\eta^2 \\ - \frac{1}{2}\delta m^2\eta_c^2 - \delta m^2\frac{M}{\sqrt{2\lambda}}\eta_c + \gamma_0\end{aligned}} \tag{1.125}$$

since it is easier to diagonalize.

In terms of ladder operators, performing the space integral, the canonical momentum in the Hamiltonian become

$$\frac{1}{2}\int dx|\pi|^2 = \frac{1}{4}\int \frac{dq}{2\pi}\omega_q(b_q^\dagger - b_{-q})(b_q - b_{-q}^\dagger) + \frac{\omega_{BO}}{4}(b_{BO}^\dagger + b_{BO})(b_{BO} + b_{BO}^\dagger) + \frac{\pi_0^2}{2}.$$

The time independent equation of motion is

$$\boxed{[\partial_x^2 + 6\beta^2\mathrm{sech}^2(\beta x)]g_q(x) = -q^2 g_q(x),} \tag{1.126}$$

where g_k could be replaced by the discrete modes $g_{BE,BO}$, then the equation would still hold. We have

$$\begin{aligned}&-\frac{1}{2}\int dx\, \eta[\partial_x^2 + 6\beta^2\mathrm{sech}^2(\beta x)]\eta \\ &\quad = \frac{1}{2}\int \frac{dq}{2\pi}\frac{q^2}{2\omega_q}(b_q + b_{-q}^\dagger)(b_q^\dagger + b_{-q}) - \frac{\beta}{4\sqrt{3}}(b_{BO} - b_{BO}^\dagger)(b_{BO}^\dagger - b_{BO}) - 2\beta^2\phi_0^2,\end{aligned}$$

and the last two terms in the hamiltonian is

$$\begin{aligned}&\frac{1}{2}\int dx\,(M^2 - \delta m^2)\eta^2 \\ &= \frac{1}{2}(M^2 - \delta m^2)\left(\int \frac{dq}{2\pi}\frac{1}{2\omega_q}(b_q + b_{-q}^\dagger)(b_q^\dagger + b_{-q}) + \frac{1}{2\sqrt{3}\beta}(b_{BO} - b_{BO}^\dagger)(b_{BO}^\dagger - b_{BO}) + \phi_0^2,\right)\end{aligned}$$

Taking above terms back to Eq. (1.125), apply the commutation relations to rewrite the operators to normal order, namely using the relation $bb^\dagger = [b, b^\dagger] + b^\dagger b$, we have the diagonal Hamiltonian,

$$\begin{aligned}\Delta\widetilde{H} = \underbrace{\frac{1}{2}\delta_D(0)\int_{-\infty}^{\infty} dq\omega_q}_{\text{from continuum}} + \underbrace{\frac{\sqrt{3}M}{4}}_{\text{from BO}} + \int dx\left(-\frac{1}{2}\delta m^2\eta_c^2 - \delta m^2\frac{M}{\sqrt{2\lambda}}\eta_c + \gamma_0\right) \\ + (\text{higher order and diagonal terms}),\end{aligned}$$

where δ_D is the Dirac delta function coming from $[b_q, b_q^\dagger]$. Note that at leading order, the terms concerning ϕ_0^2 is

$$\left(\frac{1}{2}M^2 - 2\beta^2\right)\phi_0^2 = 0$$

which is just what we need, otherwise the ϕ_0 terms will give a non-zero contribution to the vacuum energy, since ϕ_0 and π_0 don't commute with each other and $|0\rangle$ is an eigenstate of π_0, so $|0\rangle$ is not an eigenstate of ϕ_0 thus $\phi_0 |0\rangle \neq 0$.

The purpose of γ_0 in the hamiltonian is to cancel the zero point energy of the trivial vacuum, thus

$$\boxed{\int dx\, \gamma_0 = -\frac{1}{2}\delta_D(0)\int_{-\infty}^{\infty} dk\omega_k} \tag{1.127}$$

which is the negative zero point energy of the vacuum.

1.3.4. Phase shift and Mode-Matching Regularization

γ_0 enables us to subtract two different zero point energies. However, to do that, it's easier to put the system in a box of length L, adopt the periodic boundary condition (of course it's not the only possible boundary condition that can be used here), and use mode-matching regularization.

In the kink sector, the eigenfunctions with eigenvalue q can be rewritten as

$$\boxed{g_q(x) = \left(\frac{q^2}{\beta^2} - 2 + \frac{3}{\cosh^2(\beta x)} + i\frac{3q}{\beta}\tanh(\beta x)\right) e^{iqx}} \tag{1.128}$$

Rewrite terms in the parenthesis as $\rho(q, x)\exp\{i\delta(q, x)/2\}$, we can neglect ρ since it is part of the normalization factor.

We require

$$\rho(q,x)e^{i\delta(q,x)/2} \sim e^{i\delta(q,x)/2} = \left(\frac{q^2}{\beta^2} - 2 + \frac{3}{\cosh^2(\beta x)}\right) + i\frac{3q}{\beta}\tanh(\beta x) \equiv z, \tag{1.129}$$

the phase $\delta(q, x)/2$ is given by the real and imaginary part of z, its range covers 2π. Draw z in the complex plane, since $\tan(\delta/2) = \text{imaginary/real}$, $\delta/2$ can be calculated by taking the inverse of tangent. However we must be careful here, for the principal value of tangent function is from $-\pi/2$ to $\pi/2$, and satisfies the identity $\tan\theta = \tan(\theta + \pi)$, while the phase shift $\delta/2 = \arg z$ takes value in $(0, 2\pi)$, plus $e^{\delta/2}$ and $e^{\delta/2+\pi}$ will give two different complex numbers. So instead of simply writing $\delta/2 = \arctan(\cdots)$, we really should write $\arg z$. But if we take into account the fact that the orthonormal eigenfunctions g_q are defined up to a phase shift, especially $-g_q$ works as good as g_q, then $e^{\delta/2+\pi}$ indeed gives the eigenfunction with the same momentum q, so we can identify $e^{\delta/2+\pi}$ with $e^{\delta/2}$, thus we can use arctan function to calculate the phase shift.

$$\delta(q,x) = 2\arctan\left(\frac{\frac{3q}{\beta}\tanh(\beta x)}{\frac{q^2}{\beta^2} - 2 + \frac{3}{\cosh^2(\beta x)}}\right).$$

There is still a free choice to add to arctan any multiple of π, so we should really write

$$\delta(q,x) = 2\arctan\left(\frac{\frac{3q}{\beta}\tanh(\beta x)}{\frac{q^2}{\beta^2} - 2 + \frac{3}{\cosh^2(\beta x)}}\right) + 2\pi n, \quad n \in \mathbb{Z},$$

and we will take whatever value of n that eliminates the unnecessary divergence.

At $x \to \pm\infty$ we have

$$g_q|_{x\to\pm\infty} \equiv e^{i(qx\pm\delta(q)/2)}, \quad \delta(q) \equiv 2\arctan\frac{3q\beta}{q^2-2\beta^2}. \tag{1.130}$$

The most important reason for putting the system in a box is so that the modes are discrete and we can match modes from different sectors one-by-one. Adopting the periodic boundary condition means

$$q_nL + \delta(q_n) = 2\pi n = k_nL, \quad n \in \mathbb{N}, \tag{1.131}$$

more useful is the differential form,

$$\boxed{dq + \frac{1}{L}\frac{d\delta}{dq}dq = \frac{2\pi}{L}dn = dk} \tag{1.132}$$

of course dn can only be one if $n \in \mathbb{N}$ but we can imagine as if n is continuous and treat dn as a small quantity. It will not cause any problems if we set $L \to \infty$ eventually.

$$\boxed{\frac{d\delta}{dq} = -\frac{6M(M^2+2q^2)}{(M^2+q^2)(M^2+4q^2)}, \quad \left.\frac{d\delta}{dq}\right|_{q=0} = -\frac{6}{M}} \tag{1.133}$$

Note that after we put the system into a box of length L, the Dirac delta function $\delta_D(0)$ become $\frac{L}{2\pi}$, since

$$\delta_D(k) = \frac{1}{2\pi}\int dx e^{ikx} \implies \delta_D(0) = \frac{1}{2\pi}\int dx \to \delta_D(0) = \frac{1}{2\pi}\int_{-L/2}^{L/2} dx = \frac{L}{2\pi}. \tag{1.134}$$

First let's calculate

$$\frac{1}{2}\int_{-\infty}^{\infty} dq\omega_q\delta_D(0) + \gamma_0, \quad \gamma_0 = -\frac{1}{2}\int_{-\infty}^{\infty} dk\omega_k\delta_D(0). \tag{1.135}$$

We have

$$\sum_n = \sum_n \Delta n \sim \int dn = \int \frac{dn}{dq}dq = \int \frac{dq}{2\pi}\left(L + \frac{d\delta(q)}{dq}\right) \tag{1.136}$$

where the last term can be neglected in comparison with L, thus we can simply use the replacement

$$\int \frac{dq}{2\pi} \to \frac{1}{L}\sum_n \tag{1.137}$$

Also, in the kink sector, there are two modes not included in the continuum, namely g_{BO}, g_{BE}, so we should take two less modes in the continuum. If the sum in the vacuum sector is

$$\sum_{n=-N}^{N} (\text{some function of } k_n),$$

then in the kink sector it should be

$$\sum_{n=-N+1}^{N-1} (\text{some function of } q_n).$$

We have

$$\begin{aligned}
\frac{1}{2}\int_{-\infty}^{\infty} dq\omega_q\delta_D(0) + \gamma_0 &= \frac{1}{2}\int_{-\infty}^{\infty} dq\omega_q\delta_D(0) - \frac{1}{2}\int_{-\infty}^{\infty} dk\omega_k\delta_D(0) \\
&= \frac{L}{2}\left[\frac{1}{L}\sum_{-N+1}^{N-1}\sqrt{q_n^2+M^2} - \frac{1}{L}\sum_{-N}^{N}\sqrt{k_n^2+M^2}\right] \\
&= \frac{1}{2}\left(2\sum_{n=0}^{N-1}\sqrt{q_n^2+M^2} - M - 2\sum_{n=0}^{N-1}\sqrt{k_{n+1}^2+M^2} - M\right) \\
&= \sum_{n=0}^{N-1}\left(\sqrt{q_n^2+M^2} - \sqrt{k_{n+1}^2+M^2}\right) - M
\end{aligned}$$

From Eq. (1.131) we have

$$k_{n+1} = q_{n+1} + \frac{1}{L}\delta(q_{n+1}) \tag{1.138}$$

and

$$(q_{n+1}-q_n)L + [\delta(q_{n+1}) - \delta(q_n)] = 2\pi \implies (q_{n+1}-q_n) = \frac{2\pi}{L} + \mathcal{O}\left(1/L^2\right) \tag{1.139}$$

since

$$\delta(q_{n+1}) - \delta(q_n) = \frac{d\delta}{dq}\Delta q, \quad \Delta q = (\cdots)\frac{1}{L} + (\cdots)\frac{1}{L^2}.$$

We can throw $\mathcal{O}\left(1/L^2\right)$ away, as a result we have

$$k_{n+1} = q_{n+1} + \frac{1}{L}\delta(q_{n+1}) = q_n + \frac{1}{L}(2\pi + \delta(q_n)) \tag{1.140}$$

where by writing $\delta(q_{n+1})$ to $\delta(q_n)$ we have introduced an higher order error which can be neglected. We have

$$\begin{aligned}
\frac{1}{2}\int_{-\infty}^{\infty} dq\omega_q\delta_D(0) + \gamma_0 &= \frac{1}{2}\int_{-\infty}^{\infty} dq\omega_q\delta_D(0) - \frac{1}{2}\int_{-\infty}^{\infty} dk\omega_k\delta_D(0) \\
&= \sum_{n=0}^{N-1}\left(\sqrt{q_n^2+M^2} - \sqrt{q_n^2+M^2} - \frac{1}{L}(2\pi+\delta)\frac{d}{dq}\sqrt{q_n^2+M^2}\right) - M \\
&= -\frac{1}{L}\sum_{n=0}^{N-1}(2\pi+\delta)\frac{d}{dq}\sqrt{q_n^2+M^2} \\
&\to -\int_0^{\Lambda}\frac{dq}{2\pi}(2\pi+\delta)\frac{d}{dq}\sqrt{q^2+M^2} \\
&= -\frac{1}{2\pi}\left.(2\pi+\delta)\sqrt{q^2+M^2}\right|_0^{\Lambda} + \frac{1}{2\pi}\int_0^{\Lambda} dq\sqrt{q^2+M^2}\frac{d\delta}{dq},
\end{aligned} \tag{1.141}$$

from Eq. (1.130) we have

$$\delta(q)|_{q\to\infty} = \frac{6\beta}{q} + 2\pi n, \quad \delta(q)|_{q=\infty} = 2\pi + 2\pi n \tag{1.142}$$

we need to set $n = -1$, which leads to

$$\left[(\delta + 2\pi)\sqrt{q^2 + M^2}\right]\Big|_0^\infty = 3\sqrt{2} - 2\sqrt{2}\pi \tag{1.143}$$

other choices of n will give rise to a divergent term $\sim \Lambda$.

On the other hand, in the last line of Eq.(1.141), the last terms can be separated into two sub-terms, one is finite and the other divergent,

$$\begin{aligned}\frac{1}{2\pi}\int_0^\Lambda dq\sqrt{q^2+M^2}\frac{d\delta}{dq} &= \frac{1}{2\pi}\int_0^\Lambda dq\frac{-6M(2q^2+M^2)}{(4q^2+M^2)\sqrt{q^2+M^2}} \\ &= -\frac{3M}{2\pi}\int_0^\Lambda dq\left(\frac{1}{\sqrt{q^2+M^2}} + \frac{M^2}{(4q^2+M^2)\sqrt{q^2+M^2}}\right).\end{aligned} \tag{1.144}$$

Putting things together, the kink mass correction reads

$$\begin{aligned}\Delta\widetilde{H} = \frac{1}{2\pi}(2\pi - 3)M - M - \frac{3M}{2\pi}\int_0^\Lambda dq\left(\frac{1}{\sqrt{q^2+M^2}} + \frac{M^2}{(4q^2+M^2)\sqrt{q^2+M^2}}\right) \\ + \delta m^2\frac{M}{\lambda} + \frac{\sqrt{3}}{4}M,\end{aligned} \tag{1.145}$$

again, we require the two blue terms to cancel each other. This definition of the counter term δm^2 is in agreement with the definition in renormalized perturbation approach.

Now there is no divergence, we can evaluate the remaining integral, after some simplification we get

$$\boxed{\Delta\widetilde{H} = -\frac{3M}{2\pi} + \frac{M}{4\sqrt{3}}.} \tag{1.146}$$

Recall that $M \equiv \sqrt{2}m$, Eq. (1.146) is in agreement with Eq. (1.71).

1.3.5. Similarity Transformation

Redefine the continuous eigenfunctions in the kink sector as

$$g_q(x) = \psi_q^e(x) - i\psi_q^o(x),$$

recall that we used plus sign instead of minus sign in an earlier definition in order to simplify the mode-number regularization. We now switch back to minus sign so it is in agreement with Jarah's convention.

The Hamiltonian that responsible for the quantum mass correction is

$$\boxed{\begin{aligned}\Delta\widetilde{\mathcal{H}} = \frac{1}{2}\pi^2 - \frac{1}{2}\eta[\partial_x^2 + 6\beta^2\mathrm{sech}^2(\beta x)]\eta + \frac{1}{2}M^2\eta^2 \\ - \frac{1}{2}\delta m^2\eta_c^2 - \delta m^2\frac{M}{\sqrt{2\lambda}}\eta_c + \gamma_0\end{aligned}} \tag{1.147}$$

Note that in the second line, η_c is not an operator but the classical solution to the equation of motion.

We need to find the expressions for $a, a^\dagger$ in terms of $b, b^\dagger$. Since

$$\eta(x) = \int \frac{dp}{2\pi}\frac{1}{\sqrt{2\omega_p}}(a_p^\dagger + a_{-p})e^{-ipx}, \tag{1.148}$$

$$\pi(x) = i\int \frac{dp}{2\pi}\frac{\sqrt{\omega_p}}{\sqrt{2}}(a_p^\dagger - a_{-p})e^{-ipx} \tag{1.149}$$

we can inverse it to get

$$a_p^\dagger = \int dx\left[\sqrt{\frac{\omega_p}{2}}\eta(x) - \frac{i}{\sqrt{2\omega_p}}\pi(x)\right]e^{ipx}, \tag{1.150}$$

$$a_{-p} = \int dx\left[\sqrt{\frac{\omega_p}{2}}\eta(x) + \frac{i}{\sqrt{2\omega_p}}\pi(x)\right]e^{ipx}. \tag{1.151}$$

They can be projected to kink sector,

$$a_p^\dagger = a_{C,p}^\dagger + a_{BO,p}^\dagger + a_{BE,p}^\dagger, \quad a_p = a_{C,p} + a_{BO,p} + a_{BE,p}. \tag{1.152}$$

Using Eq. (1.117) to Eq. (1.121) we have

$$\begin{aligned}
a_{C,p}^\dagger &= \int \frac{dq}{2\pi}\frac{\tilde{g}_q(p)}{2C_q}\left(\frac{\omega_p+\omega_q}{\sqrt{\omega_p\omega_q}}b_q^\dagger + \frac{\omega_p-\omega_q}{\sqrt{\omega_p\omega_q}}b_{-q}\right), \\
a_{C,-p} &= \int \frac{dq}{2\pi}\frac{\tilde{g}_q(p)}{2C_q}\left(\frac{\omega_p-\omega_q}{\sqrt{\omega_p\omega_q}}b_q^\dagger + \frac{\omega_p+\omega_q}{\sqrt{\omega_p\omega_q}}b_{-q}\right), \\
a_{BO,p}^\dagger &= \frac{\tilde{g}_{BO}(p)}{2C_{BO}}\left(\frac{\omega_p+\omega_{BO}}{\sqrt{\omega_p\omega_{BO}}}b_{BO}^\dagger - \frac{\omega_p-\omega_{BO}}{\sqrt{\omega_p\omega_{BO}}}b_{BO}\right), \\
a_{BO,-p} &= \frac{\tilde{g}_{BO}(p)}{2C_{BO}}\left(\frac{\omega_p-\omega_{BO}}{\sqrt{\omega_p\omega_{BO}}}b_{BO}^\dagger - \frac{\omega_p+\omega_{BO}}{\sqrt{\omega_p\omega_{BO}}}b_{BO}\right), \\
a_{BE,p}^\dagger &= \frac{\tilde{g}_{BE}(p)}{C_{BE}}\left[\sqrt{\frac{\omega_p}{2}}\phi_0 - \frac{i}{\sqrt{2\omega_p}}\pi_0\right], \\
a_{BE,-p} &= \frac{\tilde{g}_{BE}(p)}{C_{BE}}\left[\sqrt{\frac{\omega_p}{2}}\phi_0 + \frac{i}{\sqrt{2\omega_p}}\pi_0\right].
\end{aligned} \tag{1.153}$$

where $\tilde{g}$ is the inverse Fourier transformation, inverse since it is e^{iqx} in the integrand instead of e^{-iqx},

$$\tilde{g}(q) = \int dxg(x)e^{iqx}. \tag{1.154}$$

To simplify the notations, write

$$\begin{aligned}
a^\dagger_{C,p} &= \int \frac{dq}{2\pi} \left(A_{p,q}\, b^\dagger_q + \bar{A}_{p,q}\, b_{-q} \right), \\
a_{C,-p} &= \int \frac{dq}{2\pi} \left(\bar{A}_{p,q} b^\dagger_q + A_{p,q} b_{-q} \right), \\
a^\dagger_{BO,p} &= A_{p,BO} b^\dagger_{BO} - \bar{A}_{p,BO} b_{BO}, \\
a_{BO,-p} &= \bar{A}_{p,BO} b^\dagger_{BO} - A_{p,BO} b_{BO}, \\
a^\dagger_{BE,p} &= A_{p,BE} \phi_0 - \bar{A}_{p,BE} \pi_0, \\
a_{BE,-p} &= A_{p,BE} \phi_0 + \bar{A}_{p,BE} \pi_0.
\end{aligned} \tag{1.155}$$

where

$$\begin{aligned}
A_{p,q} &= \frac{\tilde{g}_q(p)}{2C_q} \frac{\omega_p + \omega_q}{\sqrt{\omega_p \omega_q}}, \quad \bar{A}_{p,q} = \frac{\tilde{g}_q(p)}{2C_q} \frac{\omega_p - \omega_q}{\sqrt{\omega_p \omega_q}}, \\
A_{p,BO} &= \frac{\tilde{g}_{BO}(p)}{2C_{BO}} \frac{\omega_p + \omega_{BO}}{\sqrt{\omega_p \omega_{BO}}}, \quad \bar{A}_{p,BO} = \frac{\tilde{g}_{BO}(p)}{2C_{BO}} \frac{\omega_p - \omega_{BO}}{\sqrt{\omega_p \omega_{BO}}} \\
A_{p,BE} &= \frac{\tilde{g}_{BE}(p)}{C_{BE}} \sqrt{\frac{\omega_p}{2}}, \quad \bar{A}_{p,BE} = \frac{\tilde{g}_{BE}(p)}{C_{BE}} \frac{i}{\sqrt{2\omega_p}}.
\end{aligned}$$

Next, we need to substitute them in the Hamiltonian Eq. (1.147).

The free part of the Hamiltonian is

$$\int dx \left(\frac{1}{2}\pi^2 + \frac{1}{2}(\partial_x \eta)^2 + \frac{1}{2}M^2 \right), \tag{1.156}$$

which in terms of the ladder operators $a, a^\dagger$ reads

$$\int \frac{dp}{2\pi} \omega_p \left(a^\dagger_p a_p + \frac{1}{2}[a_p, a^\dagger_p] \right) = \int \frac{dp}{2\pi} \omega_p (a^\dagger_p a_p) + \frac{1}{2}\delta_D(0) \int_{-\infty}^{\infty} dk\omega_k \tag{1.157}$$

where δ_D is the Dirac delta function. The second term on the right hand side is canceled by the zero point energy counter term γ_0, by definition. Throw away these two terms, we can define the free Hamiltonian H_0 and the counter Hamiltonian H_δ as

$$\begin{aligned}
H_0 &= \int \frac{dp}{2\pi} \omega_p (a^\dagger_p a_p), \\
H_\delta &= -\frac{1}{2}\delta m^2 \eta_c^2 - \delta m^2 \frac{M}{\sqrt{2\lambda}} \eta_c.
\end{aligned} \tag{1.158}$$

There is also the Pöschl-Teller potential in the Hamiltonian,

$$\boxed{T_2 = \int dx \left(-3\beta^2 \mathrm{sech}^2(\beta x)\eta^2\right),} \tag{1.159}$$

So the Hamiltonian that gives the mass correction is

$$\boxed{\Delta\widetilde{\mathcal{H}} = H_0 + T_2 + H_\delta.} \tag{1.160}$$

Since we have separated the plane wave ladder operators $a, a^\dagger$ into three parts in terms of kink sector eigenfunctions, i.e. $a = a_{C,p} + a_{BO,p} + a_{BE,p}$, we can do the same to the Hamiltonian, namely

$$H_0 = H_{C,0} + H_{BO,0} + H_{BE,0}, \quad T_2 = T_{C,2} + T_{BO,2} + T_{BE,2}.$$

Continuum State and Counter Terms Contribution

The free Hamiltonian contains[1]

$$H_{C,0} = \frac{1}{4}\int \frac{dk}{2\pi}\frac{dp}{2\pi}\frac{(\omega_p - \omega_k)^2}{C_k^2\omega_k}\tilde{g}_k^2(p) + \cdots, \tag{1.161}$$

where we have neglected the terms with zero vacuum expectation value, for instance terms of form $(\cdots)b^\dagger b$, $(\cdots)b^\dagger b^\dagger$, $(\cdots)bb$. This integral can be done numerically. We have [2]

$$\tilde{g}_k(p) = (k^2/\beta^2 - 1)2\pi\delta(p-k) + \frac{3\pi p}{\beta^2}\mathrm{csch}\left(\frac{\pi(p-k)}{2\beta}\right) \tag{1.162}$$

where the quantum correction comes from the second term, thus

$$\begin{aligned} H_{C,0} &= \frac{1}{4}\int \frac{dk}{2\pi}\frac{dp}{2\pi}\frac{(\omega_p - \omega_k)^2}{C_k^2\omega_k}\tilde{g}_k^2(p) \\ &= \frac{9\pi^2}{4}\beta\int \frac{dk}{2\pi}\frac{dp}{2\pi}\frac{p^2\left(\sqrt{p^2+4} - \sqrt{k^2+4}\right)^2}{(k^2+1)(k^2+4)^{3/2}}\mathrm{csch}^2\left(\frac{\pi}{2}(p-k)\right), \end{aligned} \tag{1.163}$$

the function $\mathrm{csch}^2\left(\frac{\pi}{2}(p-k)\right)$ has a simple pole at $p = k$, while $(\sqrt{p^2+4} - \sqrt{k^2+4})$ has a root at $p = k$, they cancel each other. To see the integrand better we can define

$$\alpha = \frac{1}{2}(p-k), \quad \beta = \frac{1}{2}(p+k),$$

then

$$H_{C,0} = \frac{9\pi^2}{4}\beta\int \frac{dk}{2\pi}\frac{dp}{2\pi}\frac{(\alpha+\beta)^2\left(\sqrt{(\alpha+\beta)^2+4} - \sqrt{(\alpha-\beta)^2+4}\right)^2}{((\alpha-\beta)^2+4)^{3/2}}\mathrm{csch}^2\left(\frac{\pi\beta}{2}\right), \tag{1.164}$$

[1] arXiv:1908.06710, Jarah Evslin, Eq.(4.16)
[2] arXiv:1908.06710, Jarah Evslin, Eq.(3.23)

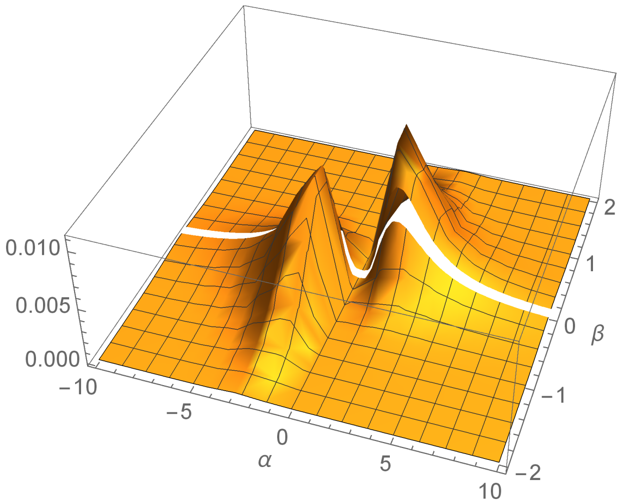

Figure 1.3: The integrand of Eq. (1.164) has a double-peak shape in the direction of α near the center.

the integrand (without $(2\pi)^2$) is shown in Fig. (1.3). The numerical result is

$$\boxed{H_{C,0} \approx 0.02386\beta.} \tag{1.165}$$

Keep the terms with nonzero vev only, the $T_{C,2}$ contribution is

$$T_{C,2} = -3\beta^2 \int dx\, \mathrm{sech}^2(\beta x) \int \frac{dpdp'}{(2\pi)^2} \frac{e^{-i(p+p')x}}{2\sqrt{\omega_p \omega_{p'}}} \int \frac{dq}{2\pi} \times \mathrm{I}. \tag{1.166}$$

where

$$\mathrm{I} = \bar{A}_{p,q} A_{p',-q} + \bar{A}_{p,q} \bar{A}_{p',-q} + A_{p,q} A_{p',-q} + A_{p,q} \bar{A}_{p',-q}.$$

From now on we will neglect the terms with zero vev. Substitute the expressions for $A, \bar{A}$, we have

$$\mathrm{I} = \frac{\sqrt{\omega_p \omega_{p'}}}{C_q^2 \omega_q} \tilde{g}_{-q}(p') \tilde{g}_q(p). \tag{1.167}$$

Take it back to $T_{C,2}$, group terms containing p, p' separately, since

$$\int \frac{dp}{2\pi} e^{-ipx} \tilde{g}_q(p) = g_q(x), \quad \int \frac{dp'}{2\pi} e^{-ip'x} \tilde{g}_{-q}(p) = g_{-q}(x) \tag{1.168}$$

we have

$$T_{C,2} = -\frac{3}{2}\beta^2 \int \frac{dq}{2\pi} \frac{1}{C_q^2 \omega_q} \int dx\, \mathrm{sech}^2(\beta x) g_q(x) g_{-q}(x). \tag{1.169}$$

Now, there are two ways to continue, one is to write $g_q(x)$ to its inverse Fourier transformation, then use the equation of motion, which implies

$$
\begin{aligned}
6\beta^2 \int dx \int \frac{dk}{2\pi} e^{-ikx} \mathrm{sech}^2(\beta x) g_{-q}(x) &= \int dx \frac{dk}{2\pi} e^{-ikx}(-q^2 - \partial_x^2) g_{-q}(x) \\
&= \int dx \frac{dk}{2\pi}(-q^2 + k^2) e^{-ikx} g_{-q}(x) \\
&= \int dx \frac{dk}{2\pi}(\omega_k^2 - \omega_q^2) e^{-ikx} g_{-q}(x) \\
&= \int \frac{dk}{2\pi}(\omega_k^2 - \omega_q^2) \tilde{g}_{-q}(-k) \\
&= \int \frac{dk}{2\pi}(\omega_k^2 - \omega_q^2) \tilde{g}_q(k),
\end{aligned}
$$

hence

$$
T_{C,2} = -\frac{1}{4} \int \frac{dqdk}{(2\pi)^2} \frac{(\omega_k^2 - \omega_q^2)}{C_q^2 \omega_q} \tilde{g}_q^2(k), \tag{1.170}
$$

where we have neglected terms with zero vev. But this integral is divergent and difficult to compare with the counter terms.

Remark 1.14. *If the normal-ordered Hamiltonian were used, $T_{C,2}$ would contain an extra term which makes it finite.*

So here we instead try a different approach, namely perform the integral over x in Eq. (1.169) first. We have

$$
\begin{aligned}
\int dx\, \mathrm{sech}^2(\beta x) |g_q(x)|^2 &= \frac{1}{\beta} \int dx\, \mathrm{sech}^2 x |g_q(\beta x)|^2 \\
&= \frac{1}{\beta} \int dx\, \mathrm{sech}^2 x \left| \frac{q^2}{\beta^2} - 2 + \frac{3}{\cosh^2(x)} - i\frac{3q}{\beta} \tanh(x) \right|^2 \\
&= \frac{1}{\beta} \left(\frac{8}{5} + \frac{6q^2}{\beta^2} + \frac{2q^4}{\beta^4} \right),
\end{aligned} \tag{1.171}
$$

Where we have used Eq. (1.128). Take it back to $T_{C,2}$,

$$
\begin{aligned}
T_{C,2} &= -3\beta \int \frac{dq}{2\pi} \frac{1}{C_q^2 \omega_q} \left(\frac{4}{5} + \frac{3q^2}{\beta^2} + \frac{q^4}{\beta^4} \right) \\
&= -3\beta \int \frac{dq}{2\pi} \frac{1}{(q^2+1)(q^2+4)^{3/2}} \left(\frac{4}{5} + 3q^2 + q^4 \right),
\end{aligned} \tag{1.172}
$$

where in the second line we have rescaled q by a factor of β.

In this form it is clear that divergence arises from q^4 term in the last line. But to see exactly how it get eliminated, we need to first see the counter term in the Hamiltonian. A direct calculation tells us

$$
\boxed{H_\delta = -\delta m^2 \int dx \left(\frac{1}{2}\eta_c^2 + \frac{M}{\sqrt{2\lambda}} \eta_c \right) = 6\beta \int_0^\Lambda \frac{dq}{2\pi} \frac{1}{\sqrt{q^2+4}},} \tag{1.173}
$$

where we have used the expression for δm^2, which is fixed by the renormalization conditions, I only quote the result here,

$$\delta m^2 = 3\lambda \int_0^\Lambda \frac{dq}{2\pi} \frac{1}{\sqrt{q^2+4}}. \tag{1.174}$$

We need to isolate from $T_{C,2}$ a divergent part equal to H_δ. Since

$$q^4 + 3q^2 = (q^2+4)(q^2-1) + 4$$

we have

$$\begin{aligned} T_{C,2} &= -3\beta \int \frac{dq}{2\pi} \frac{1}{(q^2+1)(q^2+4)^{3/2}} \left((q^2+4)(q^2-1) + \frac{24}{5} \right) \\ &= -\frac{72}{5}\beta \int \frac{dq}{2\pi} \frac{1}{(q^2+1)(q^2+4)^{3/2}} + 9\beta \int \frac{dq}{2\pi} \frac{1}{(q^2+1)\sqrt{q^2+4}} + \mathrm{II} \end{aligned}$$

where

$$\begin{aligned} \mathrm{II} &= -3\beta \int_{-\Lambda}^{\Lambda} \frac{dq}{2\pi} \frac{q^2+2}{(q^2+1)\sqrt{q^2+4}} \\ &= -3\beta \int_{-\Lambda}^{\Lambda} \frac{dq}{2\pi} \left(\frac{1}{\sqrt{q^2+4}} + \frac{1}{(q^2+1)\sqrt{q^2+4}} \right) \\ &= -6\beta \int_0^\Lambda \frac{dq}{2\pi} \frac{1}{\sqrt{q^2+4}} - 3\beta \int_{-\Lambda}^{\Lambda} \frac{dq}{2\pi} \frac{1}{(q^2+1)\sqrt{q^2+4}}. \end{aligned} \tag{1.175}$$

Compare to H_δ in Eq. (1.173), we find that when added together, these divergences cancel each other, just like we have expected.

We can now calculate various integrals, either numerically or analytically,

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{dq}{2\pi} \frac{1}{(q^2+1)\sqrt{q^2+4}} &= \frac{1}{3\sqrt{3}}, \\ \int_{-\infty}^{\infty} \frac{dq}{2\pi} \frac{1}{(q^2+1)(q^2+4)^{3/2}} &= \frac{4\sqrt{3}\pi - 9}{108\pi} \end{aligned}$$

We have

$$\boxed{T_{C,2} + H_\delta = \frac{2}{5\sqrt{3}} + \frac{6}{5\pi} \approx 0.613\beta} \tag{1.176}$$

Putting this together with $H_{C,0}$, define Q_C be the total quantum correction from the continuum modes and counter terms, then

$$\boxed{Q_C = H_{C,0} + T_{C,2} + H_\delta \approx \beta(0.047 + \frac{2}{5\sqrt{3}} + \frac{6}{5\pi}) \approx 0.6321\beta} \tag{1.177}$$

Odd Bound State Contribution

The contribution from H_0 is

$$H_{BO,0} = \frac{1}{4C_{BO}^2\omega_{BO}} \int \frac{dp}{2\pi}(\omega_p - \omega_{BO})^2 \tilde{g}_{BO}^2(p) + \cdots \tag{1.178}$$

where

$$\begin{aligned}
\int \frac{dp}{2\pi}(\omega_p - \omega_{BO})^2 \tilde{g}_{BO}^2(p) &= \beta^3 \int \frac{dp}{2\pi}(\sqrt{p^2+4} - \sqrt{3})^2 \tilde{g}_{BO}^2(p\beta) &(1.179)\\
&= \beta^3 \int \frac{dp}{2\pi}\left(\sqrt{p^2+4} - \sqrt{3}\right)^2 \left(\frac{\pi p}{\beta}\text{sech}\left(\frac{\pi p}{2}\right)\right)^2 &(1.180)\\
&\approx 0.277\beta, &(1.181)
\end{aligned}$$

where we have used[3]

$$\tilde{g}_{BO}(p) = \int dx g_{BO}(x) e^{ipx} = \frac{\pi p}{\beta^2}\text{sech}\left(\frac{\pi p}{2\beta}\right).$$

Recall that $\omega_{BO} = \sqrt{3}\beta$, $C_{BO}^2 = 2/(3\beta)$, we have

$$H_{BO,0} = 0.060\beta. \tag{1.182}$$

The contribution from T_2 is

$$T_{BO,2} = -3\beta^2 \int dx \frac{dpdp'}{(2\pi)^2}\text{sech}^2(\beta x)\frac{e^{-i(p+p')x}}{2\sqrt{\omega_p\omega_{p'}}}\text{III} \tag{1.183}$$

where

$$\begin{aligned}
\text{III} &= (a_{BO,p}^\dagger + a_{BO,-p})(a_{BO,p'}^\dagger + a_{BO,-p'})\\
&= \left(A_{p,BO}b_{BO}^\dagger - \bar{A}_{p,BO}b_{BO} + \bar{A}_{p,BO}b_{BO}^\dagger - A_{p,BO}b_{BO}\right) \times (p \to p')\\
&= \left(-\bar{A}_{p,BO}A_{p',BO} - \bar{A}_{p,BO}\bar{A}_{p',BO} - \bar{A}_{p,BO}\bar{A}_{p',BO} - A_{p,BO}\bar{A}_{p',BO}\right) b_{BO}b_{BO}^\dagger + \cdots\\
&= -\frac{\sqrt{\omega_p\omega_{p'}}}{C_{BO}^2\omega_{BO}}\tilde{g}_{BO}(p)\tilde{g}_{BO}(p') &(1.184)
\end{aligned}$$

taking it back to $T_{BO,2}$, we have

$$T_{BO,2} = \frac{3\sqrt{3}\beta^2}{4}\int dx\,\text{sech}^2(\beta x) g_{BO}^2(x) = -\frac{\sqrt{3}}{5}\beta. \tag{1.185}$$

Putting them together, the mass correction from odd bound state Q_{BO} is

$$\boxed{Q_{BO} = H_{BO,0} + T_{BO,2} \approx -0.0694\beta} \tag{1.186}$$

[3] arXiv:1908.06710, Jarah Evslin, Eq. (3.36)

Even Bound State Contribution

The part originated from the free Hamiltonian:

$$H_{BE,0} = -\frac{1}{2}\int \frac{dp}{2\pi}\frac{\tilde{g}_{BE}^2(p)}{C_{BE}^2}\omega_p, \tag{1.187}$$

where [4]

$$\tilde{g}_{BE}(p) = \frac{\pi p}{\beta^2}\text{csch}\left(\frac{\pi p}{2\beta}\right),\ C_{BE}^2 = \frac{4}{3\beta}. \tag{1.188}$$

The integral can be done numerically,[5] we have

$$H_{BE,0} = -1.088\beta. \tag{1.189}$$

The contribution from T_2 is

$$T_{BE,2} = -3\beta^2 \int dx\text{sech}^2(\beta x)\int \frac{dpdp'}{(2\pi)^2}\frac{e^{-i(p+p')x}}{2\sqrt{\omega_p\omega_{p'}}}\times \text{IV}, \tag{1.190}$$

where

$$\begin{aligned}\text{IV} &= \left(A_{p,BE}\phi_0 - \bar{A}_{p,BE}\pi_0 + A_{p,BE}\phi_0 + \bar{A}_{p,BE}\pi_0\right)\times(p\to p')\\ &= \frac{2}{C_{BE}^2}\sqrt{\omega_p\omega_{p'}}\tilde{g}_{BE}(p)\tilde{g}_{BE}(p')\phi_0^2.\end{aligned}$$

The vev of ϕ_0^2 is zero, so we can throw it away.

The quantum correction to the kink mass from the even bound state is

$$\boxed{Q_{BO} = H_{BE,0} \approx -1.088\beta} \tag{1.191}$$

Total Contribution

The total contribution is

$$Q = Q_C + Q_{BE} + Q_{BO} \approx -0.5253\beta \tag{1.192}$$

while it should be -0.666β, so where did we do wrong?

[4] arXiv:1908.06710, Jarah Evslin, Eq. (3.29)
[5] arXiv:1908.06710, Jarah Evslin, Eq. (4.45), Eq. (4.52), note the difference of a factor of 2.

2. Monopoles

2.1. Basics and Conventions

The gauge Lie group G has an underlying Lie algebra $\mathfrak{g}$, whose generator satisfies

$$[T^a, T^b] = if_c^{ab}T^c \tag{2.1}$$

where the indices can be lowered or raised by the Cartan-Killing metric $g^{ab} = \mathrm{tr}\left\{T^a T^b\right\}$, but for our discussion $g^{ab} = \delta^{ab}$ so it doesn't matter where we put the indices.

We require the generators in the fundamental representations to satisfy normalization condition

$$\mathrm{tr}\left\{T^a T^b\right\} = \frac{1}{2}\delta^{ab}. \tag{2.2}$$

Gauge field is a $\mathfrak{g}$-valued field,

$$A_\mu = A_\mu^a T^a,\ T^a \in \mathfrak{g}. \tag{2.3}$$

The $\mathfrak{g}$-valued field strength is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu], \tag{2.4}$$

our convention for covariant derivative acting on a field in the fundamental representation is

$$\mathcal{D}_\mu \psi \equiv \partial_\mu \psi - iA_\mu \psi, \tag{2.5}$$

while acting on a field in the adjoint representation is

$$\mathcal{D}_\mu \phi \equiv \partial_\mu \phi - i[A_\mu, \phi], \tag{2.6}$$

and

$$[\mathcal{D}_\mu, \mathcal{D}_\nu]\psi = -iF_{\mu\nu}\psi, \tag{2.7}$$

$$[\mathcal{D}_\mu, \mathcal{D}_\nu]\phi = -i[F_{\mu\nu}, \phi], \tag{2.8}$$

where again ψ, ϕ are fields in the fundamental and adjoint representation respectively.

The gauge transformation for various fields are

gauge field: $A_\mu \to \Omega(A_\mu + i\partial_\mu)\Omega^\dagger$, $\Omega = e^{i\omega^i T^i} \in SU(N)$

fundamental scalar field: $\phi \to \Omega\phi$

fundamental spinor field: $\psi \to \Omega\psi$

adjoint scalar field: $\phi \to \Omega\phi\Omega^\dagger$

It is sometimes useful to know the infinitesimal form of gauge transformation. In such case

$$\Omega = 1 + i\omega^a(x)T^a \tag{2.9}$$

and the infinitesimal change of the gauge field is

$$\delta A = \partial_\mu \omega - i[A_\mu, \omega] = \mathcal{D}_\mu \omega \tag{2.10}$$

where $\omega = \omega^a T^a$ is a matrix, or a $\mathfrak{g}$-valued term. And the infinitesimal change of the field strength is

$$\delta F_{\mu\nu} = i[\omega, F_{\mu\nu}]. \tag{2.11}$$

Another useful identity is

$$\boxed{\delta F_{\mu\nu} = D_\mu \delta A_\nu - \mu \leftrightarrow \nu}. \tag{2.12}$$

The action for pure Yang-Mills field is

$$S_{YM} = -\frac{1}{2g^2}\int d^4x \, \mathrm{tr}\left\{F_{\mu\nu}F^{\mu\nu}\right\}, \tag{2.13}$$

where g^2 is the Yang-Mills coupling constant, which is not a constant at all. The fact that it appears in the denominator instead of numerator, as in Peskin&Shroeder, is due to a rescaling of the Yang-Mills field A, we can reproduce the Yang-Mills action used in Peskin&Shroeder by writing $A \to A/g$, $F \to \partial A - \partial A - ig[A, A]$.

The advantage of normalization in Eq. (2.13) is that g is factored out from the Lagrangian, and sits where $\hbar$ sits, which implies that in the weak coupling limit, the paths that contribute most to the path integral are the solution to the classical equation of motion, we have the classical limit. If $g \to \infty$, then all path contribute almost equally, we would be living in an entirely quantum world.

Introduce the Hodge star operator on the field strength,

$$\star F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}, \tag{2.14}$$

Then there is a Bianchi identity

$$D_\mu \star F_{\mu\nu} = 0. \tag{2.15}$$

2.2. SU(2) Magnetic Monopoles

Magnetic Monopole solutions in Yang-Mills theory appears more naturally than that in U(1) theory. The adjoint Higgs fields have non-zero vacuum expectation values (VEV, $\langle\bullet\rangle$), breaking the gauge symmetry. Those gauge transformations that leave $\langle\phi\rangle$ invariant are symmetries not spontaneously broken by the Higgs bosons. Loosely speaking there are two kinds of symmetries,

- the symmetry of the Lagrangian, and

- the symmetry of the vacuum.

Sometimes a symmetry is preserved in the Lagrangian but not in the vacuum, meaning that performing the symmetry transformation will change a chosen vacuum, e.g. the double well potential in quantum mechanics, the $x \leftrightarrow -x$ symmetry doesn't change the Lagrangian, but it makes us jump from one vacuum (valley in the potential) to the other, hence it is not a symmetry of the vacuum. In this case we say it is spontaneously broken.

Since the spontaneously broken symmetry is still a symmetry of the Hamiltonian, starting from a vacuum and perform the operation, we will always get a new vacuum, meaning the vacua is degenerate. The vacua form a manifold, which is invariant under the broken symmetry.

Why SU(2) monopole? because it is

- the simplest monopole in Yang-Mills theory, and
- in the BPS limit, SU(2) monopole solution is the building block for monopole solutions in larger groups.

In SU(2) Yang-Mills theory, the gauge group being $SU(2)$, the generator is $T^a = \frac{1}{2}\sigma^a$, where σs are the Pauli matrices, with normalization

$$\left\langle T^a \middle| T^b \right\rangle \equiv \operatorname{tr}\left\{T^a T^b\right\} = \frac{1}{2}\delta^{ab} \tag{2.16}$$

Besides the gauge field we add the triplet Higgs scalar in the adjoint representation,

$$\phi = \frac{\sigma^a}{2}\phi^a \tag{2.17}$$

with convention $|\phi|^2 \equiv \phi^a\phi^a = 2\operatorname{tr}\left\{(\phi^a T^a)^2\right\}$. If ϕ without absolute value notation is squared, we are simply squaring the matrix field itself, while if $|\phi|$ is squared, we perfome the module square.

The lagrangian is

$$\mathcal{L} = \int d^4x \operatorname{tr}\left\{-\frac{1}{2e^2}F_{\mu\nu}F^{\mu\nu} + \frac{1}{e^2}|D_\mu\phi|^2 - V(\phi)\right\}, \tag{2.18}$$

where

$$V(\phi) = -\mu^2 \operatorname{tr}\left\{\phi^2\right\} + \lambda(\operatorname{tr}\left\{\phi^2\right\})^2 = -\frac{\mu^2}{2}\phi^a\phi^a + \frac{\lambda}{4}(\phi^a\phi^a)^2 \tag{2.19}$$

generates a S^2 of vacua. To see this we can complete the square

$$V = \frac{\lambda}{4}\left(\phi^a\phi^a - v^2\right)^2 - \frac{\mu^4}{4\lambda}, \quad v^2 = \frac{\mu^2}{\lambda}. \tag{2.20}$$

Fix the vacuum to be

$$\langle\phi^1\rangle = \langle\phi^2\rangle = 0, \langle\phi^3\rangle = v, \langle A\rangle = 0 \tag{2.21}$$

i.e. the VEV of $\boldsymbol{\phi}$ field points to the z-direction. The gauge symmetry SU(2) is broken to $U(1)$, since only $e^{i\theta^3 T^3}$ preserves the vacuum.

It is often useful to separate the field strength into "magnetic" and "electric" part,

$$B_i = -\frac{1}{2}\epsilon_{ijk}F_{jk}, \tag{2.22}$$

$$E_i = F_{0i}. \tag{2.23}$$

Note that in Yang-Mills theory, B and E are not gauge invariant as they are in U(1) theory. It means that they are no longer physical observables. In Yang-Mills theory, the physical observables are gauge invariant objects, such as the trace of $F^2_{\mu\nu}$ or the Wilson loops.

Since we have chosen the vacuum to be $\langle\phi\rangle = vT^3$, the quanta of the theory are

- two massive gauge bosons, $W^\pm_\mu = \frac{1}{\sqrt{2}}(A^1_\mu \pm iA^2_\mu)$. The mass comes solely from the covariant derivative $|D_\mu\phi|^2$, after shifting ϕ^3 to $\phi^3 + v$, the coupling between A and ϕ^3 is
$$\frac{1}{2}\frac{(\phi^3)^2}{e^2}A^a_\mu A^{a\mu} \equiv \frac{1}{2}m^2_W A^2, \quad a = 1, 2$$
- A^3 will remain massless, it is the "photon" decoupled from ϕ^3
- ϕ^3 particle is electrically neutral.

For the total energy to be finite, we must have at spacial infinity,

$$|\phi| \to v, \quad |D_\mu\phi|^2 \to 0, \quad A \to i\Omega\partial_\mu\Omega^{-1} \tag{2.24}$$

$|D_\mu\phi|^2$ must drop to zero faster than $r^{-3/2}$ to make the energy integral finite.

The topology of the space boundary is S^2 i.e. $\partial\mathbb{R}^3 = S^2$, the vacua of the Higgs field is also S^2, the Higgs field at $r \to \infty$ is a map $S^2 \to S^2$, which is classified by homotopy group $\pi_2(S^2) = \mathbb{Z}$, where $\mathbb{Z}$ is the addition group of integers. Hence the a finite energy Higgs configuration can be labeled by an integer n, which counts if we go through the spacial boundary, how many times the ϕ vacua winds around. Given a ϕ field configuration, the winding number is given by

$$\boxed{n = \frac{1}{8\pi}\epsilon_{ijk}\epsilon_{abc}\int d^2S_i\,\hat{\phi}^a\partial_j\hat{\phi}^b\partial_k\hat{\phi}^c} \tag{2.25}$$

where $\hat{\phi}^i \equiv \phi^i/|\phi|$ is the unit field vector. The trivial vacuum where $\phi = $ const everywhere clearly has $n = 0$.

Let us look at what happens at $n = 1$. $\langle\phi\rangle$ would depend on the direction. The unbroken U(1) of SU(2) is that commutes with $\langle\phi\rangle$, hence is also position-dependent, which can be written in a gauge invariant form

$$a_\mu = \frac{2}{v}\mathrm{tr}\,\{\phi A_\mu\}, \tag{2.26}$$

for example, if $\phi = vT^3$, $a_\mu = A^3_\mu$.

Remark 2.1. *Regard $A_\mu = A_\mu^a T^a$ as a $\mathfrak{su}(2)$-valued field, the unbroken U(1) component is a vector in the Lie-algebra whose direction, after choosing a ϕ vacuum, is parallel to ϕ. For example, if ϕ vacuum is chosen to be vT^3, then the unbroken U(1) group is proportional to T^3. In the case of a monopole solution, the U(1) direction is position-dependent.*

Let's look at the covariant derivative. In order to make sure $|D_\mu\phi| = |\partial_\mu\phi - i[A_\mu, \phi]| \to 0$, we need to find a corresponding A_μ that can cancel the $\partial_\mu\phi$, for $\phi = v\hat{\phi}$. Note that

$$\phi^2 = \phi^a T^a \phi^b T^b = \frac{1}{2}\phi^a\phi^b\{T^a, T^b\} = \frac{v^2}{4} \tag{2.27}$$

where we have used the relation that

$$\sigma^i\sigma^j = \delta^{ij} + i\epsilon^{ijk}\sigma^k. \tag{2.28}$$

We have

$$\partial_\mu\phi^2 = 0 = \{\partial_\mu\phi, \phi\} \tag{2.29}$$

thus

$$[\partial_\mu\phi, \phi] = -2\phi\partial_\mu\phi \tag{2.30}$$

and

$$[[\partial_\mu\phi, \phi], \phi] = v^2\partial_\mu\phi. \tag{2.31}$$

Letting

$$\boxed{A_\mu \to -\frac{i}{v^2}[\partial_\mu\phi, \phi] + \frac{a_\mu}{v}\phi,} \tag{2.32}$$

then

$$-i[A_\mu, \phi] = -\partial_\mu\phi, \tag{2.33}$$

a_μ above is the same as that in Eq.(2.26), namely $a_\mu\phi$ is the unbroken U(1) field.

Knowing the asymptotic form of the gauge field, we can work out the asymptotic form of $F_{\mu\nu}$, however we are mostly interested in the U(1) part, that is the field strength in the same direction of ϕ,

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu] \\ &= \frac{\phi}{v}(\partial_\mu a_\nu - \partial_\nu a_\mu) + \frac{i}{v^2}[\partial_\mu\phi, \partial_\nu\phi] + \frac{a_\mu}{v}\partial_\nu\phi - \frac{a_\nu}{v}\partial_\mu\phi \end{aligned} \tag{2.34}$$

The last two terms are orthogonal to ϕ thus we can discard them. Writing

$$F_{\mu\nu} = \cdots + f_{\mu\nu}\frac{\phi}{v} \tag{2.35}$$

where

$$f_{\mu\nu} = \partial_\mu a_\nu - \partial_\mu a_\nu + \frac{2i}{v^3}\mathrm{tr}\,\{\phi[\partial_\mu\phi, \partial_\nu\phi]\} \tag{2.36}$$

is the U(1) field strength. The corresponding magnetic field is

$$B_i = -\frac{1}{2}\epsilon_{ijk}f_{jk} = \cdots + \frac{1}{2v^3}\epsilon_{ijk}\epsilon_{abc}\phi^a\partial_j\phi^b\partial_k\phi^c \tag{2.37}$$

The last term is a surprise since it comes from the non-abelian part, whose contribution to the magnetic charge is proportional to the topological charge

$$m = \int d^2S_i\, B_i = \frac{1}{2}\int d^2S_i\, \epsilon_{ijk}\epsilon_{abc}\hat{\phi}^a\partial_j\hat{\phi}^b\partial_k\hat{\phi}^c = 4\pi n \tag{2.38}$$

where n is the winding number. The $\partial_\mu a_\nu - \partial_\mu a_\nu$ term doesn't contribution to the magnetic charge since its contribution to the magnetic field is a curl term, whose divergence vanishes. Such solution with a quantized U(1) magnetic charge goes by the name 't Hooft-Polyakov monopole.

2.2.1. Monopole Solutions

Above we have given a general analyses, to find the explicit monopole solution, we introduce the hedgehog ansatz for winding number $n = 1$, a natural guess for the solution for ϕ:

$$\phi^i = v\hat{x}^i h(r), \quad h(r) \to \begin{cases} 1 & r \to \infty \\ 0 & r \to 0 \end{cases}. \tag{2.39}$$

We can find corresponding A_μ at spacial infinity by solving

$$D_i\phi = 0 \implies \partial_i\hat{x}^a + \epsilon_{abd}A^b_\mu\hat{x}^c = 0 \tag{2.40}$$

by multiplying both sides by $x_j\epsilon_{adj}$. The final results is

$$A^a_i(x) = a(r)\epsilon_{aij}x_j/er^2, \quad a(r) \to \begin{cases} 1 & r \to \infty \\ 0 & r \to 0 \end{cases}. \tag{2.41}$$

The explicit form of $h(r)$ and $a(r)$ can be worked out by solving the equation of motion, numerically if not analytically.

Question: is this form of A a pure gauge, meaning $A_\mu = i\Omega\partial_\mu\Omega^{-1}$ for some Ω? It looks to me that it differs from pure gauge by an extra term.

2.3. Generalization to SU(N)

The SU(N) gauge symmetry can be maximally broken to $U(1)^{N-1}$ abelian components, they are the sub Lie algebra of $\mathfrak{su}(N)$. Here we use the Gothic font to denote the Lie algebra. Consequently,

- we have N-1 different kinds of electric charges, field may have various linear combination of these charges.
- There will be monopole solutions with respect to $U(1)^{N-1}$

There are already some pretty nice references on SU(3) monopole[1].

[1]DOI:https://doi.org/10.1103/PhysRevD.14.2016

3. Instanton

What is an instanton? Roughly speaking, instanton are topologically nontrivial solutions to Euclidean equation of motion. By Wick rotating t to $-i\tau$, we get an imaginary time, it is a mathematical trick but has physical consequences. I still don't quite understand why this trick is legal. Anyway, an imaginary times makes the regions inaccessible in the classical case accessible. Combined with WKB (Wentzel, Kramers and Brillouin) method, we can use the Euclidean space path integral to calculate a lot of things.

Wick rotation in space-time is essential $t \to -i\tau$. The Partition function for a scalar theory become

$$Z = \int \mathcal{D}\phi e^{iS} \to Z = \int \mathcal{D}\phi e^{-S_E} \tag{3.1}$$

where S_E is the Euclidian action and we have used

$$d^d x \to -i d_E^d x$$
$$\left(\frac{d\phi}{dt}\right)^2 - \left(\frac{d\phi}{dx^i}\right)^2 \to -\left(\frac{d\phi}{d\tau}\right)^2 - \left(\frac{d\phi}{dx^i}\right)^2$$

On the one hand, the path integral also gives the amplitude from some initial field configuration to some final field. On the other hand, such amplitudes can also be given by a time evolvement operator in Schrödinger picture, which is e^{-itH}. Equating them together, we find a connection between path integral and Hamiltonian eigenvalues

$$\langle f| e^{-iHT} |i\rangle = \sum_n \langle f|n\rangle e^{-iE_nT} \langle n|i\rangle = \int_i^f \mathcal{D}\phi e^{iS} \tag{3.2}$$

where T is time not temperature, $|i\rangle, |f\rangle$ are initial and final states respectively, $|n\rangle$ is the n-th eigenstate of the Hamiltonian, E_n is the nth hamiltonian eigenvalue. In order to get rid of the annoying factors $\langle\cdot|\cdot\rangle$ we can as the initial and final states to be the same and sum over it, namely we take the trace of e^{iHT}

$$\operatorname{tr}\left\{e^{-iHT}\right\} = \sum_{n,i} \langle n|i\rangle \langle i|n\rangle e^{-iE_nT} = \sum_n e^{-iE_nT} = \int_{\text{PBC}} \mathcal{D}\phi e^{iS} \tag{3.3}$$

where PBC means periodic boundary condition. Turning to the Euclidean space we have

$$\int_{\text{PBC}} \mathcal{D}\phi e^{-S_E} = \sum_n e^{-\beta E_n} \tag{3.4}$$

where β is the expansion in Euclidean time. Now this looks just like the partition function in statistical mechanics

$$Z = \operatorname{tr}\left\{e^{-\beta H}\right\} \tag{3.5}$$

where β is the inverse of the temperature. So the inverse of time in Euclidean field theory is connected with the temperature.

3.1. Instanton in Quantum Mechanics

Instanton effect is connected with a very important quantum phenomenon: quantum tunneling effect. In a double well potential, there are two degenerate vacua, by vacua I mean the minimum of potential. The ground state wave function, however, will permeate between two vacua locations.

There is a useful theorem called the node theorem which can help us to analysis the solution to 1D Schrödinger equation, which says *any eigenfunction $\Psi_n(x)$ corresponding to the nth eigenvalue of the one-dimensional Schrödinger equation, ordered in increasing magnitude, has exactly n zeros.*

It can be understood as regarding the Schrödinger equation as a equation of motion for a particle, treat $\Psi(x)$ as $q(t)$ where q is some coordinate and t the time, and analyse the possible motion of the particle based on the potential.

The ground energy $\sim e^{-\frac{A}{g}}$ where g is some coupling constant in the Lagrangian, this is characteristic instanton effect contribution. Note that $y(g) = e^{-A/g}$ is a transcendental function in g, meaning it can not be Taylor expanded around $g = 0$, which is a essential singularity of $y(g)$. Another way to look at it is to note that $y^{(n)}(g = 0)$ is zero for any order. This is to say that instanton effects can not be reproduced by perturbation to any order.

By introducing the set of improper states $|x\rangle, |p\rangle$ and the unity operator, we can integral the paths. The improper states are not vectors in the Hilbert space of states due to the infinity norm, they are actually vector-valued distributions, i.e., linear maps from the space of the L^2 functions to actual vectors in the Hilbert space,

$$|x\rangle : \quad f(x) \to |f\rangle \sim \int dx\, |x\rangle\, . \tag{3.6}$$

3.2. Decay of false vacuum

3.3. Large Orders in Perturbation Theory

3.4. Quantum Electrodynamics in 1+1 dimensions

3.5. The Polyakov Proof of Confinement

3.6. Monopole Pair Production

3.7. Quantum Chromodynamics

3.8. Instantons, Supersymmetry and Morse Theory

Appendix A.

Conventions and Formula

The master formulae in QFT:

$$\int \mathcal{D}\phi^* \mathcal{D}\phi \exp\left\{i \int d^4x(\phi^* M\phi + JM)\right\} = \frac{\mathcal{N}}{\det M} \exp(iJM^{-1}J) \tag{A.1}$$

where $\mathcal{N}$ is some normalization factor.

For fermions,

$$\int \mathcal{D}\psi \mathcal{D}\overline{\psi} \exp\left\{i \int d^4x(\overline{\psi} M\psi)\right\} = \mathcal{N} \det M. \tag{A.2}$$

Some conventions

$$\begin{aligned}
g_{\mu\nu} &= \mathrm{diag}\{1, -1, -1, -1\}, \\
\epsilon_{0123} &= 1, \\
f(x) &= \int \frac{d^n k}{(2\pi)^n} e^{-ip\cdot x} \widetilde{f}(p), \\
\widetilde{f}(p) &= \int dx e^{ip\cdot x} f(x), \\
\delta^{(n)}(k) &= \frac{1}{(2\pi)^n} \int d^n x e^{ikx}.
\end{aligned}$$

The gauge Lie group G has an underlying Lie algebra $\mathfrak{g}$, whose generator satisfies

$$[T^a, T^b] = if^{abc}T^c \tag{A.3}$$

We require the generators in the fundamental representations to satisfy normalization condition

$$\mathrm{tr}\left\{T^a T^b\right\} = \frac{1}{2}\delta^{ab}. \tag{A.4}$$

Gauge field is a $\mathfrak{g}$-valued field,

$$A_\mu = A^a_\mu T^a,\ T^a \in \mathfrak{g}. \tag{A.5}$$

The $\mathfrak{g}$-valued field strength is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu], \tag{A.6}$$

our convention for covariant derivative acting on a field in the fundamental representation is

$$\mathcal{D}_\mu \psi \equiv \partial_\mu \psi - i A_\mu \psi, \tag{A.7}$$

while acting on a field in the adjoint representation is

$$\mathcal{D}_\mu \phi \equiv \partial_\mu \phi - i[A_\mu, \phi], \tag{A.8}$$

and

$$[\mathcal{D}_\mu, \mathcal{D}_\nu]\psi = -iF_{\mu\nu}\psi, \tag{A.9}$$
$$[\mathcal{D}_\mu, \mathcal{D}_\nu]\phi = -i[F_{\mu\nu}, \phi], \tag{A.10}$$

where again ψ, ϕ are fields in the fundamental and adjoint representation respectively.

The gauge transformation for various fields are

gauge field: $A_\mu \to \Omega(A_\mu + i\partial_\mu)\Omega^\dagger$, $\Omega = e^{i\omega^i T^i} \in SU(N)$

fundamental scalar field: $\phi \to \Omega\phi$

fundamental spinor field: $\psi \to \Omega\psi$

adjoint scalar field: $\phi \to \Omega\phi\Omega^\dagger$

The non-abelian, gauge dependent magnetic field (chromo-magnetic field) is

$$B_i \equiv -\frac{1}{2}\epsilon_{ijk}F_{jk} \tag{A.11}$$

and the non-abelian electric field

$$E_i \equiv F_{0i} \tag{A.12}$$

The Weyl(chiral) basis of gamma matrices is

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \quad \bar{\sigma}^\mu \equiv (\mathbb{1}, \sigma^i). \tag{A.13}$$

Going to the Euclidean spacetime:

$$t \to -i\tau \tag{A.14}$$

Appendix B.

Differential form and Hodge theory

The conventions follow Miko Nakahara.

The Levi-Civita tensor is defined to have all lower indices,

$$\epsilon_{\mu_1 \dots \mu_m} = \begin{cases} +1 & \text{even permutation} \\ -1 & \text{odd permutation} \\ 0 & \text{otherwise} \end{cases} \tag{B.1}$$

The Hodge $\star$ is a linear map defined by

$$\star\left(dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_r}\right) = \frac{\sqrt{|g|}}{(m-r)!} \epsilon^{\mu_1 \mu_2 \dots \mu_r}{}_{\nu_{r+1} \dots \nu_m} dx^{\nu_{r+1}} \wedge \cdots \wedge dx^{\nu_m}. \tag{B.2}$$

For example, $\star 1 = \sqrt{|g|} dx^1 \dots dx^m$ is the volume element. From now on we will neglect the wedge symbol.

If we take the non-coordinate basis $\{\hat{\theta}^\alpha\}$, then things are usually simpler since the non-coordinate symbols are orthonormal to eachother, so $g^{\mu\nu} \to \delta^{\mu\nu}$.

Hodge star is its own inverse operator up to a minus sign:

$$\star \star \omega = (-1)^{r(m-r)} \omega, \omega \in \Omega^r(M) \tag{B.3}$$

if (M, g) is Riemannian,

$$\star \star \omega = (-1)^{r(m-r)+s} \omega, \omega \in \Omega^r(M) \tag{B.4}$$

if (M, g) is Lorentzian, s is the signature of the metric.

Given two $r-$forms ω, η, the exterior product $\omega \wedge \star \eta$ is an $m-$form, thus its integral on M is well-defined. The inner product is defined to be

$$\langle \omega | \eta \rangle \equiv \int \omega \wedge \star \eta \tag{B.5}$$

$$= \frac{1}{r!} \int_M \omega_{\mu_1 \dots \mu_r} \eta^{\mu_1 \dots \mu_r} \sqrt{|g|} dx^1 \dots dx^m. \tag{B.6}$$

We can also define the local inner product of forms,

$$\langle \omega | \eta \rangle \equiv \int \langle \omega | \eta \rangle_{\text{local}} \, d\text{Vol} \tag{B.7}$$

then

$$\omega \wedge \star\eta = \langle\omega|\eta\rangle_{\text{local}}\, d\text{Vol}.$$

We will neglect the subscript "local" and which is which can be told from the context.

The adjoint exterior derivative $d^\dagger : \Omega^r(M) \to \Omega^{r-1}(M)$ is defined to be

$$d^\dagger = (-1)^{m(r+1)+1} \star d\star \tag{B.8}$$

if (M, g) is Riemannian and

$$d^\dagger = (-1)^{m(r+1)} \star d\star \tag{B.9}$$

if Lorentzian, where $m = \dim(M)$. We have $(d^\dagger)^2$ just like for d.

Let (M, g) be a compact orientable manifold without a boundary, then

$$\langle d\beta|\alpha\rangle = \left\langle \beta \middle| d^\dagger\alpha \right\rangle. \tag{B.10}$$

Instead of showing it by brutal calculation, it can be proved from $d(\beta \wedge \star\alpha) = 0$ and so is its integral.

The Laplacian $\Delta : \Omega^r(M) \to \Omega^r(M)$ is defined by

$$\Delta = (d + d^\dagger)^2 = dd^\dagger + d^\dagger d. \tag{B.11}$$

For instance, when acting on a function $f \in \mathcal{F}(M)$ we have

$$\Delta f = -\frac{1}{\sqrt{|g|}}\partial_\nu \left[\sqrt{|g|}g^{\nu\mu}\partial_\mu f\right], \tag{B.12}$$

note that this definition differs from others by a minus sign. There is another easier way to derive the Laplacian from the Lagragian thanks to A.Zee. Since the action is invariant under the change of coordinates,

$$S = \int d^d x \,\sqrt{|g|}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \tag{B.13}$$

we can perform integral by part and rearrange things, discard the surface term

$$S = -\int d^d x\,\partial_\nu[\sqrt{|g|}g^{\mu\nu}\partial_\mu\phi\phi] \tag{B.14}$$

$$= -\int d^d x\,\phi\sqrt{|g|}\frac{1}{\sqrt{|g|}}\partial_\nu[\sqrt{|g|}g^{\mu\nu}\partial_\mu\phi] \tag{B.15}$$

The (half) Maxwell equation read

$$d^\dagger F = -j,\; j = j_\mu dx^\mu, \tag{B.16}$$

note that j^μ rather than j_μ is our familiar current, $j^\mu = (\rho, \boldsymbol{j})$.

Appendix C.

Center Symmetry

The confinement in QCD is notoriously difficult, however we can make it easier to study in QCD's cousin—pure Yang-Mills theory. In a pure Yang-Mills theory we do not have dynamic quarks, only static quarks as infinitely heavy test particles. In this case we will see that there exists an order parameter for the confinement-deconfinement phase transition, namely the vacuum expectation value (VEV) of Polyakov loop.

The logic is roughly as following,

1. turn to Euclidian space-time,
2. introduce the center transformation, it is a symmetry of pure Yang-Mills action. The center transformation requires that the time dimension be compactified with some peculiar boundary conditions, i.e. periodic boundary condition up to group center.
3. See how the Polyakov loop changes as we change the boundary condition. The Polyakov loop will break the center symmetry.
4. Show that the Polyakov loop is the order parameter of confinement phase transition, as well as a bridge connecting the center symmetry and confinement.

The center Z of a group G is the set of elements in the group which commutes with all the elements of the group,

$$Z(G) = \{z \in G \,|\, gzg^{-1} = z, \, \forall g \in G\}. \tag{C.1}$$

Z itself is a group and $Z \lhd G$ (meaning Z is a normal subgroup of G).

For SU(N), the center group is Z_N. To be specific

$$Z(SU(N)) = \left\{ e^{i2\pi n/N} \mathbb{1} \,|\, n = 1, \cdots, N \right\},$$

for example the center of SU(2) is $\{\mathbb{1}, -\mathbb{1}\}$.

The center transformation in a SU(N) gauge theory looks like a special case of the usual gauge transformation,

$$A_\mu \to A'_\mu \equiv \Omega(A_\mu + i\partial_\mu)\Omega^{-1} \tag{C.2}$$

but with a different boundary condition,

$$\Omega(t + \beta) = C\Omega(t), \, C \in Z(SU(N))$$

where β is the length of Euclidean time-dimension. Recall that the gauge transformation should satisfy periodic boundary condition.

In a pure Yang-Mills theory, i.e. with gauge bosons only, the center symmetry respects the periodic boundary condition of the gauge field, since if A satisfies periodic boundary condition then so does A'. Hence the center transformation is a symmetry of the action.

The key difference between usual gauge transformation and the center transformation:

- The Polyakov loop is gauge invariant, but not invariant under center transformation

The Wilson line is defined as a path-ordered production along some path in space-time from an initial point to a final point:

$$W_{fi} = P \exp \left\{ i \int_{x_i}^{x_f} A_\mu dx^\mu \right\} \tag{C.3}$$

where $x_i = (\boldsymbol{x}_i, t_i)$ is the initial point, P means path-ordered production.

The Wilson line is a SU(N) matrix in color space, under gauge transformation the Wilson line transforms as

$$W_{fi} \to \Omega_f W_{fi} \Omega_i^\dagger. \tag{C.4}$$

Compactifying the line to form a closed loop, we will have a Wilson loop. Since the Euclidean time dimension has the topology of S^1, we can wind the Wilson loop around the time-S^1 for integer times. If we wind only once and keep the spacial coordinates fixed, we will have a Wilson line from t to $t+\beta$, the trace of it is called Polyakov loop,

$$\Phi(\boldsymbol{x}) = \frac{1}{N} \mathrm{tr} \left\{ T \exp \left(i \int_0^\beta d\tau A_0(\boldsymbol{x}, \tau) \right) \right\}. \tag{C.5}$$

One can verify that under

gauge transformation: $\Phi(\boldsymbol{x})$ is invariant

center transformation: $\Phi(\boldsymbol{x}) \to z\Phi(\boldsymbol{x})$, $z \in Z(SU(N))$. More generally, the vev of the trace of Polyakov loop transforms as

$$\langle \mathrm{tr}\,\{P\} \rangle \to z^{k_R} \langle \mathrm{tr}\,\{P\} \rangle \tag{C.6}$$

where k_R is called the N-ality which depends on the representation of the gauge group.

On the one hand, if the center symmetry is preserved, we must have zero Polyakov loop, for if the Polyakov loop is non-zero, it will break the center symmetry explicitly. On the other hand, the Polyakov loop has a physical meaning: it is related to the free energy of adding an external static color charge F, which lives in the fundamental representation, for example quarks with infinite mass.

$$\boxed{|\Phi(\boldsymbol{x})| = \exp\{-F\beta\}} \tag{C.7}$$

Remark C.1.

- *In QED we can add a test charge by adding the corresponding source term in the Lagrangian. In Yang-Mills theory we need to do this in a gauge-invariant way.The Wilson line in the time direction is the cheapest way to do it.*
- *The absolute value is needed because the Polyakov loop is complex.*

In a confining phase, the free energy with an added colour charge should be infinite, resulting a zero Polyakov loop. If it is not confined then the Polyakov loop will be non-zero. In this sense, for pure Yang-Mills theory the Polyakov loop is the order parameter for confinement phase transition.

Putting everything together, we can see the connection between the center symmetry and confinement, namely

$$\boxed{\text{preserved center symmetry} \implies \text{zero Polyakov loop vev} \implies \text{confinement.}}$$

Appendix D.

Meissner Effect

People had long suspected that superconduction has something to do with Bose-Einstein condensation, the questions is that electrons are fermions and Bose-Einstein condensation is for Bosons, so is there any way that electrons can actually condensate? A naive conjecture would be that electrons can first pair up, these pairs can then condensate. This conjecture turns out to be essentially correct, these pairs are called Cooper pairs.

In the meanwhile, Landau and Ginzburg circumvented the problem by suggesting an effective field model to account for the superconductivity, namely the Landau-Ginzburg model.

Recall from the statistical physics that, for a macroscopic system, the free energy tends to be minimized, which is a result from the fact that entropy tends to be maximized. Landau-Ginzburg assumes that for a system in temperature T and external static magnetic field $\boldsymbol{B}$, the free energy of the system has the form

$$\boxed{\mathcal{F} = \frac{1}{4}F_{ij}^2 + |D_i\phi|^2 + a(T)|\phi|^2 + b(T)|\phi|^4} \tag{D.1}$$

where a, b are coefficients dependent on the temperature. Clearly the free energy satisfies the $U(1)$ gauge symmetry. The degree of freedom is the Cooper pairs, thus having electric charge 2, resulting in the covariant derivative

$$D_i = \partial_i - i2eA_i. \tag{D.2}$$

There is the critical temperature T_c below which the superconductivity happens. Supposed near T_c we have

$$a \sim a_0(T - T_c), \quad b \sim b_0, \tag{D.3}$$

where a_0, b_0 are both positive constant. In this way, the temperature dependence have been introduced, which allows us to see what happens at different temperature.

We are mostly interested in the role played by magnetic field in the free energy. The first term in Eq. (D.1) is the ordinary electromagnetic term, yielding the familiar magnetic contribution

$$\frac{1}{4}F_{ij}^2 = \frac{1}{2}\boldsymbol{B}^2 \tag{D.4}$$

to the free energy. Since we assume $\boldsymbol{B}$ is a constant in space-time, this term, when integrated out, has a contribution proportional to the volume of the system $\sim \boldsymbol{B}^2 V$. What happens with the rest of the terms?

- Above the critical temperature, namely $T > T_c$, we can treat $\mathcal{F}$ just like the Hamiltonian. Then the potential terms $a\phi^2 + b\phi^4$ would have the shape of a single well. Nothing interesting happens.
- Below the critical temperature, namely $T < T_c$, the sign for a is flipped, now we have a familiar Mexico-hat shaped potential.

Remark D.1. *The phase at $T > T_c$ is different from the phase at $T < T_c$, at $T = T_c$ we have $a = 0$, it corresponds to a second-order phase transition critical point.*

When $T < T_c$, the Mexico-hat potential will cause the spontaneous symmetry breaking, the symmetry being the $U(1)$ gauge symmetry. Recall what happens when the gauge symmetry is spontaneously broken in a gauge theory? The Higgs mechanism! Meaning we will have a mass term for the original-mass-free gauge boson

$$\mathcal{F} = \cdots + (2ev)^2 A_i^2 + \cdots \tag{D.5}$$

where v is the vacuum expectation value of the ϕ field. Now this term will give us an extra contribution to the free energy due to the constant magnetic field.

When the magnetic field is a constant, the gauge potential A_i will increase proportional to r, the distance from some arbitrary reference point. It is a result from $\boldsymbol{B} = \nabla \times \boldsymbol{A}$. For example, recall that for $\boldsymbol{B} = (0, 0, B)$, one choice for the vector potential is $\boldsymbol{A} = (0, Bx, 0)$. Then the term $(2ev)^2 A_i^2$ will scale as $\sim (2ev)^2 r^2$. When integrated over the space, it has a contribution increasing faster than the volume V. It means that if there is a constant magnetic field, the corresponding free energy from $\boldsymbol{B}$ alone will increase much faster than $\mathcal{F} \sim \boldsymbol{B}^2 V$ as the volume increases, making it highly unfavored. This is what happens in a superconductor, when the temperature drop below a critical value, the medium goes into superconductor phase, and any magnetic fields permeating it are repelled, because the existence of a const magnetic field will increase the free energy, thus greatly reduce the probability for that to happen. This phenomenon is known as the Meissner effect, a hallmark of superconductivity.

Appendix E.

Basic Extra Dimensions

E.1. Basics of Kaluza Klein Theories

E.1.1. Scalar Field

First consider a massless real scalar theory in 5D $\phi(x^M)$ where $M = 0, 1, 2, 3, 4$. The action is

$$S_{5D} = \int d^5x \partial_M \phi \partial^M \phi. \tag{E.1}$$

Set the extra dimension $x^4 \equiv y$ defining a circle of radius r with $y \equiv y + 2\pi r$.

The space is now $\mathbb{R}^4 \times S^1$. Periodicity in y means the Fourier expansion is discrete

$$\phi(x^\mu, y) = \sum_{n=-\infty}^{\infty} \phi_n(x^\mu) e^{iny/r}. \tag{E.2}$$

The Fourier coefficients can be regarded as scalar fields on 4D space. Introducing a S^1 extra dimension changes one scalar fields to infinite ones. The equation of motion for them reads

$$\boxed{\partial_M \partial^M \phi = 0 \implies \partial_\mu \partial^\mu \phi_n(x^\mu) - \frac{n^2}{r^2}\phi_n(x^\mu) = 0} \tag{E.3}$$

$\phi_n(x^\mu)$ satisfy the massive Klein-Gordon equation with mass $m_n = n/r$. The only massless field is the zero mode $\phi_0(x^\mu)$. $\phi_n(x^\mu)$ can be visualized as a tower of states with different masses, which is called Kaluza Klein tower, and the massive states are called Kaluza Klein- or momentum- states.

If we want to get a 4D effective theory, we can integrate out the y dimension. Taking the mode expansion for ϕ to the original action, we get

$$S_{5D} = \int d^4x \int dy \sum_n \left(\partial_\mu \phi_n(x^\mu) \partial^\mu \phi_n(x^\mu) - \frac{n^2}{r^2}\phi_n^2 \right) \tag{E.4}$$

$$= 2\pi r \int d^4x [\partial_\mu \phi_0(x^\mu) \partial^\mu \phi_0(x^\mu) + \cdots] \equiv s_{4D} + \cdots \tag{E.5}$$

where the dots includes the massive modes(scalars).

Dimension reduction: If we only keep the zero mode.

Compactification: If we keep all the modes. In this case the extra dimension is compact and its existence is taken into account.

E.1.2. Gauge Field

Next we consider an abelian vector fields in 5D.

$$A_M(x^M) = \begin{cases} A_\mu(x^M) & \text{vector field} \\ A_4(x^M) \equiv \rho & \text{scalar field} \end{cases}. \tag{E.6}$$

Again they can be separated into discrete Fourier modes in y direction

$$A_\mu = \sum_n A_\mu^n \exp\left(\frac{iny}{r}\right), \tag{E.7}$$

$$\rho = \sum_n \rho_n \exp\left(\frac{iny}{r}\right). \tag{E.8}$$

Consider the action for 5D gauge field

$$S_{5D} = \int d^5x \frac{1}{g_{5D}^2} F_{MN}F^{MN}, \quad F_{MN} \equiv \partial_M A_N - \partial_N A_M \tag{E.9}$$

The equation of motion is

$$\partial_M \partial^M A_N - \partial^M \partial_N A_M = 0, \tag{E.10}$$

Next we can fix a gauge. For example, if we choose the transverse gauge $A^0 = 0$, $\partial_M A^M = 0$, we have 5 Klein-Gordon equation

$$\partial_M \partial^M A_N = 0. \tag{E.11}$$

Each gauge field A_N results in an Kaluza Klein tower. In order to find a 4D effective theory, we once again plug the decomposed gauge field into the original action

$$S_{5D} = \int d^4x \left(\frac{2\pi r}{g_{5D}^2} \overline{F}_{\mu\nu}\overline{F}^{\mu\nu} + \frac{2\pi r}{g_{5D}^2} \partial_\mu \overline{\rho} \partial^\mu \overline{\rho} + \cdots \right) \tag{E.12}$$

where $\overline{F}$ means the zero mode of F, and the dots include the non-zero modes. We can read out the 4D effective coupling constant

$$\boxed{g_4^2 \equiv \frac{g_{5D}^2}{2\pi r}, \quad g_4^2 \equiv \frac{g_D^2}{V_{D-4}}} \tag{E.13}$$

where the second equality is a generalization to D dimension, V_{D-4} is the volume of the $D-4$ dimensional sphere with radius r.

Electric Potential

The Gaussian Law in D-Dimensional spacetime says

$$\oint_{S^{D-1}} \boldsymbol{E}\cdot d\boldsymbol{S} = Q \implies E \propto \frac{1}{r^{D-2}},\ \Phi \propto \frac{1}{r^{D-3}} \tag{E.14}$$

for example, in 4D we have $E \propto 1/r^2$, $\Phi \propto 1/r$ and in 5D we have $E \propto 1/r^3$, $\Phi \propto 1/r^2$.

If one dimension is compactified with radius r, if $r \to \infty$ limit the extra dimension is just another flat dimension, while if $r \to 0$ the extra dimension does not exist.

DoF Counting

We know that gauge field in 4D spacetime has 2 degrees of freedom, how about in D-dimensional spacetime? We can start from the Lorentz group commutation relations:

$$[M^{\mu\nu}, M^{\rho\sigma}] = (ig^{\mu\sigma}M^{\nu\rho} - (\mu \leftrightarrow \nu)) - (\rho \leftrightarrow \sigma) \tag{E.15}$$

Spin operator corresponds to M^{ij}. For the gauge field, since they are massless, the little group is the transformation that keep the four momentum $p = (E, 0, 0, E)$ invariant, namely O(2). Generalizing this to D-dimension, the little group is O(D-2) that leaves $p = (E, \underbrace{0, \cdots, 0}_{\text{D-2 zeros}}, E)$ invariant, the Lorentz generator $M^{\mu\nu}$ become M^{MN}.

The Degree of freedom for a gauge field in D-dimension is $D-2$, since a transverse wave in $D-1$ dimensional space has that degree of freedom.

E.1.3. Duality

There exists duality that connecting fields with different antisymmetric indices.

In 4D, we have the dual field strength

$$\tilde{F}_{\mu\nu} \equiv \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma} \tag{E.16}$$

The field equations in the vacuum are

$$\partial_\mu F^{\mu\nu} = 0 \quad \text{Field equation of motion} \tag{E.17}$$

$$\partial_\mu \tilde{F}^{\mu\nu} = 0 \quad \text{Bianchi identity} \tag{E.18}$$

Such duality can be generalized to field with arbitrary indices in D dimension.

For example, in 5D suppose we have a antisymmetric tensor field F_{MN}, we can define a dual field

$$\tilde{F}^{ABC} \equiv \epsilon^{ABCMN}F_{MN}. \tag{E.19}$$

Given a antisymmetric (p+1) tensor $A_{M_1\cdots M_{p+1}}$, we can define a field strength

$$F_{M_1\cdots M_{p+2}} \equiv \partial_{[M_1}A_{M_2\cdots M_{p+2}]} \tag{E.20}$$

where $[\cdots]$ is the anti-symmetrization operation. The dual field strength would be

$$\tilde{F}_{M_1\cdots M_{D-p-2}} \equiv \epsilon_{M_1\cdots M_{D-p-2}N_1\cdots N_{p+2}}F^{N_1\cdots N_{p+2}}. \tag{E.21}$$

Some Examples:

- $D = 4$

$$F_{\mu\nu\rho} = \partial_{[\mu} B_{\nu\rho]} \implies \tilde{F}_\sigma = \epsilon_{\sigma\mu\nu\rho} F^{\mu\nu\rho} = \partial_\sigma a$$

here $B_{\nu\rho}$ is a two tensor, its dual is a scalar.

- $D = 6$

$$F_{ABC} = \partial_{[A} B_{BC]} \implies \tilde{F}_{ABC} = \epsilon_{ABCDEF} F^{DEF} \equiv -\partial_{[A} \tilde{B}_{BC]}$$

here the potentials who are dual to each other have same number of indices.

Note that dual changes the coupling g to $1/g$

$$\mathcal{L} = \frac{1}{g^2} F^2 \Longleftrightarrow g^2 (\tilde{F})^2 \tag{E.22}$$

$$\boxed{\text{Gauge field, potential} \to \text{Field strength} \to \text{Dual Strength} \to \text{Dual field}}$$

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p Branes

Electric field couples to the current, and the current of a point charge is given by its world line, thus the electric field is coupled to the world line of a particle

$$S \sim \int A_\mu dx^\mu . \tag{E.23}$$

To be specific, suppose the world line of a charged point particle is given by

$$\xi^\mu(\tau) : \tau \to \mathbb{R}^4$$

where τ is the proper time, the conserved four current j^μ is

$$J^\mu = q \int d\tau \frac{d\xi^\mu}{d\tau} \delta^4(x - \xi(\tau)), \tag{E.24}$$

which resembles $j^\mu = (\gamma\rho) u^\mu, \gamma = 1/\sqrt{1 - v^2}$ where $\gamma\rho$ can be regarded as the contracted charge density and u^μ is the four velocity $d\xi^\mu/d\tau$. Eq. (E.24) is the same as

$$j^0(\boldsymbol{x}, t) = q\delta^3(\boldsymbol{x} - \boldsymbol{\xi}(\tau(t))), \tag{E.25}$$

$$\boldsymbol{j}(\boldsymbol{x}, t) = q\delta^3(\boldsymbol{x} - \boldsymbol{\xi}(\tau(t))) \frac{d\boldsymbol{\xi}}{dt}. \tag{E.26}$$

Then the couple $\int dx^4 A_\mu j^\mu$ becomes $q \int d\xi^\mu A_\mu$.

The world line of a point particle is a one dimensional line, it couples to fields with one Lorentz index. We can generalize this to coupling with tensor fields in higher dimension. For a potential $B_{[\mu\nu]}$, the analogue is

$$\int B_{\mu\nu} dx^\mu \wedge dx^\nu \tag{E.27}$$

i.e. a world sheet couples to fields with two Lorentz indices. Further generalizations are

membrane:

$$\int B_{\mu\nu\rho} dx^{\mu} \wedge dx^{\nu} \wedge dx^{\rho}$$

p brane:

$$\int B_{M_1 \cdots M_{p+1}} dx^{M_1} \wedge \cdots \wedge dx^{M_{p+1}}$$

E.2. Brane World Scenario

Appendix F.

Roots and Weights

Here is a very short note on roots and weights concerning compact Lie groups, algebraic approach.

A group is said to be simple if the generators cannot be divided into two mutually commuting sets, and the group cannot be written as the direct product of two smaller groups, not even locally. A group is said to be semi-simple if it can at least locally be written as the direct product of two smaller groups, but there is no $U(1)$ factor. For example, $SO(4) = [SU(2) \times SU(2)]/\mathbb{Z}_2$, so it is not simple but semi-simple.

The generators satisfy

$$\mathrm{tr}\left\{T^a T^b\right\} = \delta_{ab} T(R) \tag{F.1}$$

where $T(R)$ is a constant that depends on the representation, $T(R) = 1/2$ in the fundamental representation. From all the generators one can choose the maximal set of commuting generators H_i, $i = 1, 2, \cdots, r$. They span the $r-$dimensional Cartan subalgebra, and r is the rank of the Lie group. The remaining generators can be linearly combined to form the eigenvectors for the cartan generators, the generalized ladder operators $E_{\boldsymbol{\alpha}}$, satisfy

$$H_i \left|E_{\boldsymbol{\alpha}}\right\rangle \equiv [H_i, E_{\boldsymbol{\alpha}}] = \alpha_i E_{\boldsymbol{\alpha}}, \quad [E_{\boldsymbol{\alpha}}, E_{-\boldsymbol{\alpha}}] = \boldsymbol{\alpha} \cdot \boldsymbol{H}. \tag{F.2}$$

with

$$E_{-\boldsymbol{\alpha}} = E_{\boldsymbol{\alpha}}^{\dagger}. \tag{F.3}$$

The $r-$dimensional vector $\boldsymbol{\alpha}$ is called the root.

There are strong constraints on the root system. For one, each Lie algebra is actually comprised of a lot of $SU(2)$ algebras, since for each root, the generators

$$t_1(\boldsymbol{\alpha}) = \frac{1}{\sqrt{2\boldsymbol{\alpha}^2}}(E_{\boldsymbol{\alpha}} + E_{-\boldsymbol{\alpha}}), \tag{F.4}$$

$$t_2(\boldsymbol{\alpha}) = \frac{-i}{\sqrt{2\boldsymbol{\alpha}^2}}(E_{\boldsymbol{\alpha}} - E_{-\boldsymbol{\alpha}}), \tag{F.5}$$

$$t_3(\boldsymbol{\alpha}) = \frac{1}{\boldsymbol{\alpha}^2}\boldsymbol{\alpha} \cdot \boldsymbol{H}. \tag{F.6}$$

$$\tag{F.7}$$

They corresponds to the usual J_1, J_2, J_3 in $SU(2)$. All the other roots $\left|E_{\boldsymbol{\beta}}\right\rangle$ fall into the irreducible representations of this $SU(2)$. As you can verify yourself, $\left|E_{\boldsymbol{\beta}}\right\rangle$ is the eigenstate of t_3, with eigenvalues integer or half-integer. This means

$$\frac{\boldsymbol{\alpha} \cdot \boldsymbol{\beta}}{\boldsymbol{\alpha}^2} = \frac{p}{2}, \quad p \in \mathbb{Z}. \tag{F.8}$$

Exchange the role of $\boldsymbol{\alpha}, \boldsymbol{\beta}$ there is a dual relation,

$$\frac{\boldsymbol{\alpha} \cdot \boldsymbol{\beta}}{\boldsymbol{\beta}^2} = \frac{q}{2}, \quad q \in \mathbb{Z}. \tag{F.9}$$

Multiplying and dividing these two relations we get

$$\frac{4(\boldsymbol{\alpha} \cdot \boldsymbol{\beta})^2}{\boldsymbol{\alpha}^2 \boldsymbol{\beta}^2} \equiv 4\cos^2\theta = pq, \tag{F.10}$$

$$\frac{\boldsymbol{\beta}^2}{\boldsymbol{\alpha}^2} = \frac{p}{q}. \tag{F.11}$$

Since $-1 < \cos\theta < 1$, the possible choices for pq is also limited. We will not enumerate them here.

Given any root, one can define a dual root

$$\boldsymbol{\alpha}^* = \frac{\boldsymbol{\alpha}}{\boldsymbol{\alpha}^2}. \tag{F.12}$$

The duals of a root system also forms a acceptable root system, meaning the satisfies the constraints of a root system. A_N, the Lie algebra of $SU(N+1)$, is self-dual.