Weak-form for poisson equation

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Poisson equation

$$\nabla^2 u = f \tag{1-1}$$

weak-form

$$\int_\Omega \left(\nabla^2 u - f\right) \delta u dV = 0 \tag{1-2}$$

$$\int_\Omega \left(\nabla^2 u - f\right) \delta u dV = \int_\Omega \nabla^2 u \delta u dV - \int_\Omega f \delta u dV \tag{1-3}$$

To get the functional we need to dal with this term:

$$\int_\Omega \nabla \cdot (\nabla u \delta u)\, dV = \int_\Omega \nabla^2 u \delta u dV + \int_\Omega (\nabla u \cdot \nabla \delta u)\, dV \tag{1-4}$$

Gauss Theory

$$\int_\Omega \nabla \cdot (\nabla u \delta u)\, dV = \int_{\partial\Omega} \nabla u \delta u \cdot dS \tag{1-5}$$

So we rewrite (1-3) as

$$\begin{aligned}\int_\Omega \left(\nabla^2 u - f\right) \delta u dV &= \int_\Omega \nabla^2 u \delta u dV - \int_\Omega f \delta u dV \\ &= \int_{\partial\Omega} \nabla u \delta u \cdot dS - \int_\Omega (\nabla u \cdot \nabla \delta u)\, dV - \int_\Omega f \delta u dV \\ &= \int_{\partial\Omega} \delta u \nabla u \cdot dS - \int_\Omega (\nabla u \cdot \nabla \delta u)\, dV - \int_\Omega \delta u f dV \\ &= \int_{\partial\Omega} \delta u \nabla u \cdot dS - \int_\Omega \frac{1}{2} \delta (\nabla u \cdot \nabla u)\, dV - \int_\Omega \delta u f dV \\ &= \int_{\partial\Omega} \delta u \nabla u \cdot dS - \frac{1}{2} \int_\Omega \delta \left|\nabla u\right|^2 dV - \int_\Omega \delta u f dV\end{aligned} \tag{1-6}$$

So we found the functional

$$I[u] = \int_{\partial\Omega} u \nabla u \cdot dS - \frac{1}{2} \int_\Omega \left|\nabla u\right|^2 dV - \int_\Omega f u dV \tag{1-7-1}$$

$$I[u] = -\int_{\partial\Omega} u \nabla u \cdot dS + \frac{1}{2} \int_\Omega \left|\nabla u\right|^2 dV + \int_\Omega f u dV \tag{1-7-2}$$

and variation of the functional is

$$\begin{aligned}\delta I[u] &= \int_{\partial\Omega} \delta u \nabla u \cdot dS - \frac{1}{2} \int_\Omega \delta \left|\nabla u\right|^2 dV - \int_\Omega \delta u f dV \\ &= \int_\Omega \left(\nabla^2 u - f\right) \delta u dV\end{aligned}$$

1.For homogenous Direchlet boundary condition:
Dirichelet BCs:

$$u = 0, \quad \partial\Omega$$

So variational of u must equal zero on the boundary.

$$\delta u = 0, \quad \partial\Omega$$

$$I[u] = -\frac{1}{2}\int_{\Omega} |\nabla u|^2 \, dV - \int_{\Omega} fudV$$

$$\begin{aligned}\delta I[u] &= \int_{\partial\Omega} \delta u \nabla u \cdot dS - \frac{1}{2}\int_{\Omega} \delta \, |\nabla u|^2 \, dV - \int_{\Omega} \delta u f dV \\ &= \int_{\Omega} \left(\nabla^2 u - f\right) \delta u dV\end{aligned}$$

2.For Neuman boundary condition:
Neuman BCs:

$$\frac{\partial u}{\partial n} = g, \quad \partial\Omega$$

Functional

$$\begin{aligned}I[u] &= \int_{\partial\Omega} u \nabla u \cdot dS - \frac{1}{2}\int_{\Omega} |\nabla u|^2 \, dV - \int_{\Omega} fudV \\ &= \int_{\partial\Omega} ugdS - \frac{1}{2}\int_{\Omega} |\nabla u|^2 \, dV - \int_{\Omega} fudV\end{aligned}$$

Variation

$$\begin{aligned}\delta I[u] &= \int_{\partial\Omega} \delta ugdS - \frac{1}{2}\int_{\Omega} \delta \, |\nabla u|^2 \, dV - \int_{\Omega} \delta u f dV \\ &= \int_{\Omega} \left(\nabla^2 u - f\right) \delta u dV - \int_{\partial\Omega} \delta ugdS\end{aligned}$$

3.For Mix Boundary condition:

$$\frac{\partial u}{\partial n} + ku = g, \quad \partial\Omega$$

Functional

$$\begin{aligned}I[u] &= \int_{\partial\Omega} u \left(g - ku\right) dS - \frac{1}{2}\int_{\Omega} |\nabla u|^2 \, dV - \int_{\Omega} fudV \\ &= \int_{\partial\Omega} u \left(g - ku\right) dS - \frac{1}{2}\int_{\Omega} |\nabla u|^2 \, dV - \int_{\Omega} fudV\end{aligned}$$

Variation

$$\begin{aligned}\delta I[u] &= \int_{\partial\Omega} \delta u \left(g - ku\right) dS - \frac{1}{2}\int_{\Omega} \delta \, |\nabla u|^2 \, dV - \int_{\Omega} \delta u f dV \\ &= \int_{\Omega} \left(\nabla^2 u - f\right) \delta u dV - \int_{\partial\Omega} \delta u \left(g - ku\right) dS\end{aligned}$$