Lid-Driven Cavity Flow Formulation

Yang Yang
Deparetment of Mechanics
Peking Univiersity, Beijing Haidian
-December 2015-

1.Vorticity and Stream function and Pressure

Considering imcompressible Navier-Stokes equations

$$\nabla \cdot \mathbf{u} = 0 \tag{1-1-1}$$

$$(\partial_t + \mathbf{u} \cdot \nabla)\, \mathbf{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u} \tag{1-1-2}$$

(1) Stream function:

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \tag{1-2}$$

and at the same time (1-1-1) is satisfied automatically.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial}{\partial x}\frac{\partial \psi}{\partial y} - \frac{\partial}{\partial y}\frac{\partial \psi}{\partial x} \tag{1-3}$$

and if $\psi \in C^2$

$$\frac{\partial}{\partial x}\frac{\partial \psi}{\partial y} - \frac{\partial}{\partial y}\frac{\partial \psi}{\partial x} = 0 \tag{1-4}$$

$$\Rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

So as if we introduce stream function then mass conservation law is satisfied automatically.

(2) Voritcity equation
Vorticity was defined as

$$\omega = \nabla \times \mathbf{u} \tag{1-5}$$

in planar case

$$\begin{aligned} \omega &= \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \\ &= -\frac{\partial}{\partial x}\frac{\partial \psi}{\partial x} - \frac{\partial}{\partial y}\frac{\partial \psi}{\partial y} \\ &= -\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right) \\ &= -\nabla^2 \psi \end{aligned} \tag{1-6}$$

and Derive from Navier-Stokes equation $\nabla\times$(1-1-2):

$$\nabla \times \left[(\partial_t + \mathbf{u} \cdot \nabla)\, \mathbf{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u}\right] \tag{1-7}$$

$$\partial_t (\nabla \times \mathbf{u}) + \nabla\times (\mathbf{u} \cdot \nabla)\, \mathbf{u} = \nu \nabla \times \nabla^2 \mathbf{u}$$

$$\partial_t (\nabla \times \mathbf{u}) + (\mathbf{u} \cdot \nabla)(\nabla \times \mathbf{u}) = \nu \nabla^2 (\nabla \times \mathbf{u})$$

Then we obtain worticity equation

$$(\partial_t + \mathbf{u} \cdot \nabla)\, \omega = \nu \nabla^2 \omega \tag{1-8}$$

rewite it in conservation format

$$\partial_t \omega + \nabla \cdot (\mathbf{u} \otimes \omega) = \nu \nabla^2 \omega \tag{1-9}$$

at this step we can solve N-S equation by using vorticity and stream function method but if we want to know the pressure we still have to solve pressure Poission equation.

$$\partial_t \omega + \nabla \cdot (\mathbf{u} \otimes \omega) = \nu \nabla^2 \omega \tag{1-10}$$

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}$$

(3) Pressure Poisson equation

Derive from Navier-Stokes equation: $\nabla \times$(1-1-2)

$$\nabla \cdot \left[(\partial_t + \mathbf{u} \cdot \nabla)\, \mathbf{u} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{u} \right] \tag{1-11}$$

$$\nabla \cdot \partial_t \mathbf{u} + \nabla \cdot (\mathbf{u} \cdot \nabla)\, \mathbf{u} = -\nabla \cdot \nabla p + \nu \nabla \cdot \nabla^2 \mathbf{u}$$

$$\partial_t \nabla \cdot \mathbf{u} + \nabla \cdot (\mathbf{u} \cdot \nabla)\, \mathbf{u} = -\nabla^2 p + \nu \nabla^2 \nabla \cdot \mathbf{u}$$

Considering mass conservation equation (1-1-1)

$$\partial_t \nabla \cdot \mathbf{u} + \nabla \cdot (\mathbf{u} \cdot \nabla)\, \mathbf{u} = -\frac{1}{\rho} \nabla^2 p + \nu \nabla^2 \nabla \cdot \mathbf{u} \tag{1-12}$$

$$\nabla \cdot (\mathbf{u} \cdot \nabla)\, \mathbf{u} = -\frac{1}{\rho} \nabla^2 p$$

index form:

$$\frac{\partial}{\partial x_i} \left(u_j \frac{\partial}{\partial x_j} \right) u_i = -\frac{1}{\rho} \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_k} p \tag{1-13}$$

$$\frac{\partial u_j}{\partial x_i} \frac{\partial u_i}{\partial x_j} + u_j \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} u_i = -\frac{1}{\rho} \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_k} p$$

$$\frac{\partial u_j}{\partial x_i} \frac{\partial u_i}{\partial x_j} + u_j \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_i} u_i = -\frac{1}{\rho} \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_k} p$$

$$\frac{\partial u_j}{\partial x_i} \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_k} p \tag{1-14}$$

$$\frac{\partial u_j}{\partial x_i} \frac{\partial u_i}{\partial x_j} = \frac{\partial u}{\partial x} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \frac{\partial v}{\partial y} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} + \frac{\partial v}{\partial x} \frac{\partial u}{\partial y} \tag{1-15}$$

introduce mass conservation equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1-16}$$

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}$$

$$\frac{\partial u_j}{\partial x_i}\frac{\partial u_i}{\partial x_j} = -\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + \frac{\partial v}{\partial x}\frac{\partial u}{\partial y} \tag{1-17}$$
$$= -2\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right)$$

Pressure poission equation in 2-D case

$$2\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right) = \frac{1}{\rho}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)p \tag{1-18}$$

(4) Summary
We obtain governing equations for this flow simulation, next list them in dimensionless format below
Vorticity evolution euqaiton

$$\frac{\partial \omega}{\partial t} + \frac{\partial (u\omega)}{\partial x} + \frac{\partial (v\omega)}{\partial y} = \frac{1}{Re}\nabla^2\omega$$

Stream function Poission equation

$$\nabla^2\psi = -\omega$$

Velocity formulas

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}$$

Pressure Poission equation

$$\nabla^2 p = 2\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right)$$

Laplace operator

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

---

2. Numerical Scheme

we already derive govering equations for this problem, then we list it below.
Vorticity evolution euqaiton:

$$\frac{\partial \omega}{\partial t} + \frac{\partial (u\omega)}{\partial x} + \frac{\partial (v\omega)}{\partial y} = \nu\nabla^2\omega$$

Stream function Poission equation:

$$\nabla^2\psi = -\omega$$

Velocity formulas:

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}$$

Pressure Poission equation:

$$\nabla^2 p = 2\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right)$$

Laplace operator:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

(1) Numerical Scheme for Vorticity Transport equation
Simple FTCS Scheme:

$$\begin{aligned} &\frac{\omega_{i,j}^{n+1} - \omega_{i,j}^{n}}{\Delta t} + \frac{(u\omega)_{i+1,j}^{n} - (u\omega)_{i-1,j}^{n}}{2\Delta x} + \frac{(v\omega)_{i,j+1}^{n} - (v\omega)_{i,j-1}^{n}}{2\Delta y} \\ &= \nu \left(\frac{\omega_{i+1,j}^{n} - 2\omega_{i,j}^{n} + \omega_{i-1,j}^{n}}{\Delta x^2} + \frac{\omega_{i,j+1}^{n} - 2\omega_{i,j}^{n} + \omega_{i,j-1}^{n}}{\Delta y^2} \right) \end{aligned} \tag{2-1}$$

$$\begin{aligned} \omega_{i,j}^{n+1} = \omega_{i,j}^{n} &- \Delta t \frac{(u\omega)_{i+1,j}^{n} - (u\omega)_{i-1,j}^{n}}{2\Delta x} - \Delta t \frac{(v\omega)_{i,j+1}^{n} - (v\omega)_{i,j-1}^{n}}{2\Delta y} \\ &+ \Delta t \nu \left(\frac{\omega_{i+1,j}^{n} - 2\omega_{i,j}^{n} + \omega_{i-1,j}^{n}}{\Delta x^2} + \frac{\omega_{i,j+1}^{n} - 2\omega_{i,j}^{n} + \omega_{i,j-1}^{n}}{\Delta y^2} \right) \end{aligned} \tag{2-2}$$

To avoid useless caculation we introduce parameters:

$$\begin{aligned} c_x = \frac{\Delta t}{2\Delta x}, \quad c_y = \frac{\Delta t}{2\Delta y} \\ Fo_x = \frac{\nu \Delta t}{\Delta x^2}, \quad Fo_y = \frac{\nu \Delta t}{\Delta y^2} \end{aligned} \tag{2-3}$$

Computing format:

$$\begin{aligned} \omega_{i,j}^{n+1} = \omega_{i,j}^{n} &- c_x \left[(u\omega)_{i+1,j}^{n} - (u\omega)_{i-1,j}^{n} \right] - c_y \left[(v\omega)_{i,j+1}^{n} - (v\omega)_{i,j-1}^{n} \right] \\ &+ Fo_x \left[\omega_{i+1,j}^{n} - 2\omega_{i,j}^{n} + \omega_{i-1,j}^{n} \right] + Fo_y \left[\omega_{i,j+1}^{n} - 2\omega_{i,j}^{n} + \omega_{i,j-1}^{n} \right] \end{aligned} \tag{2-4}$$

Predict and correct (2nd-order Runge-Kutta) Scheme:

$$\begin{aligned} &\frac{\mathring{\omega}_{i,j}^{n+1} - \omega_{i,j}^{n}}{\Delta t} + \frac{(u\omega)_{i+1,j}^{n} - (u\omega)_{i-1,j}^{n}}{2\Delta x} + \frac{(v\omega)_{i,j+1}^{n} - (v\omega)_{i,j-1}^{n}}{2\Delta y} \\ &= \nu \left(\frac{\omega_{i+1,j}^{n} - 2\omega_{i,j}^{n} + \omega_{i-1,j}^{n}}{\Delta x^2} + \frac{\omega_{i,j+1}^{n} - 2\omega_{i,j}^{n} + \omega_{i,j-1}^{n}}{\Delta y^2} \right) \end{aligned} \tag{2-5}$$

$$\begin{aligned} &\frac{\hat{\omega}_{i,j}^{n+1} - \omega_{i,j}^{n}}{\Delta t} + \frac{(u\mathring{\omega}^{n+1})_{i+1,j} - (u\mathring{\omega}^{n+1})_{i-1,j}}{2\Delta x} + \frac{(v\mathring{\omega}^{n+1})_{i,j+1} - (v\mathring{\omega}^{n+1})_{i,j-1}}{2\Delta y} \\ &= \nu \left(\frac{\mathring{\omega}_{i+1,j}^{n+1} - 2\mathring{\omega}_{i,j}^{n+1} + \omega_{i-1,j}^{n+1}}{\Delta x^2} + \frac{\mathring{\omega}_{i,j+1}^{n+1} - 2\mathring{\omega}_{i,j}^{n+1} + \mathring{\omega}_{i,j-1}^{n+1}}{\Delta y^2} \right) \end{aligned} \tag{2-6}$$

$$\omega_{i,j}^{n+1} = \frac{1}{2} \left(\mathring{\omega}_{i,j}^{n} + \hat{\omega}_{i,j}^{n} \right) \tag{2-7}$$

Computing format

$$\begin{aligned} \mathring{\omega}_{i,j}^{n+1} = \omega_{i,j}^{n} &- c_x \left[(u\omega)_{i+1,j}^{n} - (u\omega)_{i-1,j}^{n} \right] - c_y \left[(v\omega)_{i,j+1}^{n} - (v\omega)_{i,j-1}^{n} \right] \\ &+ Fo_x \left[\omega_{i+1,j}^{n} - 2\omega_{i,j}^{n} + \omega_{i-1,j}^{n} \right] + Fo_y \left[\omega_{i,j+1}^{n} - 2\omega_{i,j}^{n} + \omega_{i,j-1}^{n} \right] \end{aligned} \tag{2-8}$$

$$\hat{\omega}_{i,j}^{n+1} = \omega_{i,j}^{n} - c_x\left[\left(u\mathring{\omega}^{n+1}\right)_{i+1,j} - \left(u\mathring{\omega}^{n+1}\right)_{i-1,j}\right] - c_y\left[\left(v\mathring{\omega}^{n+1}\right)_{i,j+1} - \left(v\mathring{\omega}^{n+1}\right)_{i,j-1}\right] + Fo_x\left[\mathring{\omega}_{i+1,j}^{n+1} - 2\mathring{\omega}_{i,j}^{n+1} + \mathring{\omega}_{i-1,j}^{n+1}\right] + Fo_y\left[\mathring{\omega}_{i,j+1}^{n+1} - 2\mathring{\omega}_{i,j}^{n+1} + \mathring{\omega}_{i,j-1}^{n+1}\right] \tag{2-9}$$

$$\omega_{i,j}^{n+1} = \frac{1}{2}\left(\mathring{\omega}_{i,j}^{n} + \hat{\omega}_{i,j}^{n}\right) \tag{2-10}$$

(2) Numerical Scheme for Stream function Poission equation
SOR Iterative Scheme:

$$\psi_{i,j}^{n+1} = (1-\sigma)\,\psi_{i,j}^{n} + \frac{\sigma}{2\,(A+B)}\left[A\left(\psi_{i+1,j}^{n} + \psi_{i-1,j}^{n+1}\right) + B\left(\psi_{i,j+1}^{n} + \psi_{i,j-1}^{n+1}\right) - \omega_{i,j}^{n+1}\right] \tag{2-11}$$

$$A = \frac{1}{\Delta x^2}, \quad B = \frac{1}{\Delta y^2} \tag{2-12}$$

(3) Numerical Scheme for Velocity formula
Centered diffrential scheme:

$$u_{i,j}^{n+1} = \frac{\psi_{i,j+1}^{n+1} - \psi_{i,j-1}^{n+1}}{2\Delta y} \tag{2-13-1}$$

$$v_{i,j}^{n+1} = -\frac{\psi_{i+1,j}^{n+1} - \psi_{i-1,j}^{n+1}}{2\Delta x} \tag{2-13-2}$$

Computational format

$$A = -\frac{1}{2\Delta x}, \quad B = \frac{1}{2\Delta y} \tag{2-14}$$

$$u_{i,j}^{n+1} = B\left(\psi_{i,j+1}^{n+1} - \psi_{i,j+1}^{n+1}\right) \tag{2-15-1}$$

$$v_{i,j}^{n+1} = A\left(\psi_{i+1,j}^{n+1} - \psi_{i-1,j}^{n+1}\right) \tag{2-15-2}$$

(4) Numerical Scheme for pressure Poission equation
SOR Iterative Scheme:

$$p_{i,j}^{n+1} = (1-\sigma)\,p_{i,j}^{n} + \frac{\sigma}{2\,(A+B)}\left[A\left(p_{i+1,j}^{n} + p_{i-1,j}^{n+1}\right) + B\left(p_{i,j+1}^{n} + p_{i,j-1}^{n+1}\right) - (S_p)_{i,j}^{n+1}\right] \tag{2-16}$$

$$(S_p)_{i,j}^{n+1} = 2\left(\frac{u_{i+1,j}^{n+1} - u_{i-1,j}^{n+1}}{2\Delta x}\frac{u_{i,j+1}^{n+1} - u_{i,j+1}^{n+1}}{2\Delta y} - \frac{u_{i,j+1}^{n+1} - u_{i,j+1}^{n+1}}{2\Delta y}\frac{u_{i+1,j}^{n+1} - u_{i-1,j}^{n+1}}{2\Delta x}\right) \tag{2-17}$$

$$S_p = 2\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right) \tag{2-18}$$

Computational format

$$C = \frac{1}{2\Delta x\Delta y} \tag{2-19}$$

$$(S_p)_{i,j}^{n+1} = C\left[\left(u_{i+1,j}^{n+1} - u_{i-1,j}^{n+1}\right)\left(v_{i,j+1}^{n+1} - v_{i,j+1}^{n+1}\right) - \left(u_{i,j+1}^{n+1} - u_{i,j+1}^{n+1}\right)\left(v_{i+1,j}^{n+1} - v_{i-1,j}^{n+1}\right)\right] \tag{2-20}$$

3. Boundary Treatment

(1) Vorticity boundary treatment

Derive Wall Vorticity boundary treatment formula

we need stream function Poission eqation to reconstruct vorticity in wall boundary.

$$\left.\omega\right|_w = -\nabla^2 \left.\psi\right|_w \tag{3-1}$$

rewrite it in boundary coordinate

$$\frac{\partial^2\psi}{\partial n^2} + \frac{\partial^2\psi}{\partial \tau^2} = -\omega \tag{3-2}$$

According to defination of stream function this term $\frac{\partial^2\psi}{\partial \tau^2}$ can be write as:

$$\frac{\partial^2\psi}{\partial \tau^2} = \frac{\partial u_n}{\partial \tau} \tag{3-3}$$

to compute term $\frac{\partial^2\psi}{\partial n^2}$ in approximation we need taylor expanssion:

$$\psi(\tau, n) - \psi(\tau, 0) = \left.\frac{\partial\psi}{\partial n}\right|_w \Delta n + \frac{1}{2!}\left.\frac{\partial^2\psi}{\partial n^2}\right|_w (\Delta n)^2 + \frac{1}{3!}\left.\frac{\partial^3\psi}{\partial n^3}\right|_w (\Delta n)^3 + O\left(\tau, \Delta n^4\right) \tag{3-4}$$

cutoff

$$\psi(\tau, n) - \psi(\tau, 0) = \left.\frac{\partial\psi}{\partial n}\right|_w \Delta n + \frac{1}{2!}\left.\frac{\partial^2\psi}{\partial n^2}\right|_w (\Delta n)^2 + O\left(\tau, \Delta n^3\right) \tag{3-5}$$

here we know

$$\frac{\partial\psi}{\partial n} = -u_\tau \tag{3-6}$$

Consequently

$$\psi(\tau, n) - \psi(\tau, 0) = \left.u_\tau\right|_w \Delta n + \frac{1}{2!}\left.\frac{\partial^2\psi}{\partial n^2}\right|_w (\Delta n)^2 + O\left(\tau, \Delta n^3\right) \tag{3-7}$$

$$\left.\frac{\partial^2\psi}{\partial n^2}\right|_w = \frac{2}{\Delta n^2}\left[\psi(\tau, n) - \psi(\tau, 0)\right] + \frac{2}{\Delta n}u_\tau \tag{3-8}$$

So we obtain 1st-order accurate wall vorticity formula

$$\begin{aligned} \left.\omega\right|_w &= -\left.\left(\frac{\partial^2\psi}{\partial n^2} + \frac{\partial^2\psi}{\partial \tau^2}\right)\right|_w \\ &= -\frac{2}{\Delta n^2}\left[\psi(\tau, n) - \psi(\tau, 0)\right] - \frac{2}{\Delta n}u_\tau - \left.\frac{\partial u_n}{\partial \tau}\right|_w \\ &= -\frac{2}{\Delta n^2}\left[\psi(\tau, n) - \psi(\tau, 0) + u_\tau \Delta n\right] - \left.\frac{\partial u_n}{\partial \tau}\right|_w \end{aligned} \tag{3-9}$$

Thom formula:

$$\left.\omega\right|_w = -\frac{2}{\Delta n^2}\left[\left.\psi\right|_{w+1} - \left.\psi\right|_w + u_\tau \Delta n\right] \tag{3-10}$$

compute format

$$\left.\omega\right|_w = -\frac{2}{\Delta n^2}\left(\left.\psi\right|_{w+1} - \left.\psi\right|_w\right) - \frac{2}{\Delta n}u_\tau \tag{3-11}$$

$$
\begin{aligned}
\omega|_w &= A\left(\psi|_{w+1} - \psi|_w\right) + Bu_\tau \\
A &= -\frac{2}{\Delta n^2}, \quad B = -\frac{2}{\Delta n}
\end{aligned} \tag{3-12}
$$

(2) Stream Function's treatment
because of wall is a stream line as a consequence stream function should be

$$
\psi|_w = 0 \tag{3-13}
$$

What's more?
Stream function and Vorticity maybe singular in these coners.

---

4. Summary
(1) Vorticity evolution equation numerical scheme:

$$
\begin{aligned}
c_x &= \frac{\Delta t}{2\Delta x}, \quad c_y = \frac{\Delta t}{2\Delta y} \\
Fo_x &= \frac{\nu \Delta t}{\Delta x^2}, \quad Fo_y = \frac{\nu \Delta t}{\Delta y^2}
\end{aligned} \tag{2-3}
$$

$$
\begin{aligned}
\mathring{\omega}_{i,j}^{n+1} = \omega_{i,j}^n &- c_x\left[(u\omega)_{i+1,j}^n - (u\omega)_{i-1,j}^n\right] - c_y\left[(v\omega)_{i,j+1}^n - (v\omega)_{i,j-1}^n\right] \\
&+ Fo_x\left[\omega_{i+1,j}^n - 2\omega_{i,j}^n + \omega_{i-1,j}^n\right] + Fo_y\left[\omega_{i,j+1}^n - 2\omega_{i,j}^n + \omega_{i,j-1}^n\right]
\end{aligned} \tag{2-8}
$$

$$
\begin{aligned}
\hat{\omega}_{i,j}^{n+1} = \omega_{i,j}^n &- c_x\left[\left(u\mathring{\omega}^{n+1}\right)_{i+1,j} - \left(u\mathring{\omega}^{n+1}\right)_{i-1,j}\right] - c_y\left[\left(v\mathring{\omega}^{n+1}\right)_{i,j+1} - \left(v\mathring{\omega}^{n+1}\right)_{i,j-1}\right] \\
&+ Fo_x\left[\mathring{\omega}_{i+1,j}^{n+1} - 2\mathring{\omega}_{i,j}^{n+1} + \mathring{\omega}_{i-1,j}^{n+1}\right] + Fo_y\left[\mathring{\omega}_{i,j+1}^{n+1} - 2\mathring{\omega}_{i,j}^{n+1} + \mathring{\omega}_{i,j-1}^{n+1}\right]
\end{aligned} \tag{2-9}
$$

$$
\omega_{i,j}^{n+1} = \frac{1}{2}\left(\mathring{\omega}_{i,j}^n + \hat{\omega}_{i,j}^n\right) \tag{2-10}
$$

Stability condition:

$$
d = \nu\left(\frac{\Delta t}{\Delta x^2} + \frac{\Delta t}{\Delta y^2}\right) \leq \frac{1}{2} \tag{4-1}
$$

$$
\Delta t \leq \frac{1}{2\nu} \frac{1}{\left(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}\right)} \tag{4-2}
$$

(2) Stream function Poission equation numerical scheme:

$$
\psi_{i,j}^{n+1} = (1-\sigma)\,\psi_{i,j}^n + \frac{\sigma}{2\,(A+B)}\left[A\left(\psi_{i+1,j}^n + \psi_{i-1,j}^{n+1}\right) + B\left(\psi_{i,j+1}^n + \psi_{i,j-1}^{n+1}\right) - \omega_{i,j}^{n+1}\right] \tag{2-11}
$$

$$
A = \frac{1}{\Delta x^2}, \quad B = \frac{1}{\Delta y^2} \tag{2-12}
$$

(3) Velocity numerical scheme:

$$A=\frac{1}{2\Delta x},\quad B=\frac{1}{2\Delta y} \tag{2-14}$$

$$u_{i,j}^{n+1}=B\left(\psi_{i,j+1}^{n+1}-\psi_{i,j-1}^{n+1}\right) \tag{2-15-1}$$
$$v_{i,j}^{n+1}=A\left(\psi_{i+1,j}^{n+1}-\psi_{i-1,j}^{n+1}\right) \tag{2-15-2}$$

(4) Pressure Poission eqaution numerical scheme:

$$C=\frac{1}{2\Delta x\Delta y} \tag{2-19}$$

$$(S_p)_{i,j}^{n+1}=C\left[\left(u_{i+1,j}^{n+1}-u_{i-1,j}^{n+1}\right)\left(v_{i,j+1}^{n+1}-v_{i,j+1}^{n+1}\right)-\left(u_{i,j+1}^{n+1}-u_{i,j+1}^{n+1}\right)\left(v_{i+1,j}^{n+1}-v_{i-1,j}^{n+1}\right)\right] \tag{2-20}$$

$$p_{i,j}^{n+1}=(1-\sigma)\,p_{i,j}^{n}+\frac{\sigma}{2\,(A+B)}\left[A\left(p_{i+1,j}^{n}+p_{i-1,j}^{n+1}\right)+B\left(p_{i,j+1}^{n}+p_{i,j-1}^{n+1}\right)-(S_p)_{i,j}^{n+1}\right] \tag{2-16}$$

(5) Initial condition:

$$\rho=1 \tag{4-3}$$
$$u=v=0 \tag{4-4}$$
$$p=1 \tag{4-5}$$

(6) Boundary condition:
Up

$$\psi|_w=0$$

$$\omega|_w=A\left(\psi|_{w+1}-\psi|_w\right)+Bu_\tau \tag{3-12}$$
$$A=-\frac{2}{\Delta n^2},\quad B=-\frac{2}{\Delta n}$$
$$u_\tau=-16x^2\left(1-x^2\right)$$

Bottom

$$\psi|_w=0$$

$$\omega|_w=A\left(\psi|_{w+1}-\psi|_w\right)+Bu_\tau \tag{3-12}$$
$$A=-\frac{2}{\Delta n^2},\quad B=-\frac{2}{\Delta n}$$
$$u_\tau=0$$

Left

$$\psi|_w = 0$$

$$\omega|_w = A\left(\psi|_{w+1} - \psi|_w\right) + Bu_\tau \tag{3-12}$$
$$A = -\frac{2}{\Delta n^2}, \quad B = -\frac{2}{\Delta n}$$
$$u_\tau = 0$$

Right

$$\psi|_w = 0$$

$$\omega|_w = A\left(\psi|_{w+1} - \psi|_w\right) + Bu_\tau \tag{3-12}$$
$$A = -\frac{2}{\Delta n^2}, \quad B = -\frac{2}{\Delta n}$$

$$u_\tau = 0$$

Preesure Boundary condition for Pressure Poission equation:

To save computer resource, we only compute pressure at steady state. Pressure Poission equation is a elliptic PDE which showing the elliptic nature of pressure incompressible flows. Yet the boundary condition is really matter in the problem. To solve PPE, boundary conditions for pressure are required. On a solid wall boundary, boundary value of pressure obtained by tangengtial momentum equation to the tangential momentum equation to the fluid adjacent to the wall surface. For a wall located at $y = 0$ in Cartesian coordinate system, the tangential momentum equation (x-momentum euqation) reduces to

$$\left.\frac{\partial p}{\partial x}\right|_w = \mu \left.\frac{\partial^2 u}{\partial y^2}\right|_w = -\mu \left.\frac{\partial \omega}{\partial y}\right|_w \tag{4-6}$$

and $\frac{\partial v}{\partial x} = 0$ along the wall. (4-6) can be discretized as

$$\frac{p_{i+1,1} - p_{i-1,1}}{2\Delta x} = -\mu \frac{-3\omega_{i,1} + 4\omega_{i,2} - \omega_{i,3}}{2\Delta y} \tag{4-7}$$

Here we use second-order centered differential scheme for pressure gradient $\frac{\partial p}{\partial x}$. However, on the coner we need first-order differential scheme.

$$\frac{p_{i+1,1} - p_{i,1}}{\Delta x} = -\mu \frac{-3\omega_{i+1,1} + 4\omega_{i+1,2} - \omega_{i+1,3}}{2\Delta y} \tag{4-8-1}$$

$$\frac{p_{2,1} - p_{1,1}}{\Delta x} = -\mu \frac{-3\omega_{2,1} + 4\omega_{2,2} - \omega_{2,3}}{2\Delta y} \tag{4-8-2}$$

Thereafter, eqation (4-7) can be used to find the pressure at all other wall points. Note that, in order to apply these wall pressure formula, the pressure must be known for at least on point in th wall boundary, and we called it as reference pressure. Because of pressure difference is important in imcompressible flow in stead of pressure value.

6. Dimensionless
Mesh elements

$$N_x, \quad N_y$$

Length

$$\Delta x = \Delta y = 1$$

$$L_x = N_x, \quad L_y = N_y$$

Reynold number

$$Re = \frac{U_{\max} L_x}{\nu}$$

So viscosity can be obtained

$$\nu = \frac{U_{\max} L_x}{Re}$$

and time step size:

$$\Delta t \leq \frac{1}{2\nu} \frac{1}{\left(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}\right)}$$

In compution we select

$$\Delta t = \frac{1}{4\nu} \frac{1}{\left(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}\right)}$$

$$\Delta t = \frac{1}{8\nu}$$

7. Algorithm in brief

(1) Initialze the velocity field and compute the associated vorticity field and stream function field.

(2) Compute boundary conditions for vorticity.

(3) Solve the vorticity transport equation to compute the vorticity at a new time step; any standard time marching scheme may be used for this purpose.

(4) Solve streamfunction Poission equation to compute streamfunction field at new time step; any iterative scheme for elliptic equations may be used.

(5) Compute the velocity field at a new time step; prefering second-order centered diffrence scheme.

(6) Cheak: is this field satisfied steady condition? Yes, then break. No, Continue and return to step (2) and repeat the computation for another time step. Output Meso-results.

(7) Set up pressure boundary condition and solve Pressure Poission equation to get the pressure disturbution.

(8) Output results and report computing log, program end and exit.