Solve ODE BVP problem by Weighted Residual Method

Yang Yang
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Define a ODE BVP problem:

$$\begin{aligned} &L\left(u\right)=\frac{d^2u}{dx^2}+u+x=0, \quad (0\le x\le 1) \\ &u\left(0\right)=u\left(1\right)=0 \end{aligned}$$

in fact the problem has a exact solution

$$u\left(x\right)=\frac{\sin\left(x\right)}{\sin\left(1\right)}-x$$

Next we find a test function satisfied boundary condition:

$$u=x\left(1-x\right)\left(a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+...+a_nx^n\right)$$

Some derive
Residual of equation

$$R=Lu-f$$

and we want the integral of residual equal to zero:

$$\langle R,\omega_i\rangle=\int_\Omega R\omega_i d\Omega=0$$

$$\int_\Omega\left(Lu-f\right)\omega_i d\Omega=0$$

1.Compute by hand
test function:

$$u=x\left(1-x\right)\left(a_0+a_1x\right)$$

here a_0 and a_1 are constant which need to be worked out.
(1) Collocation method:
Points:

$$x_1=\frac{1}{3},\quad x_2=\frac{2}{3}$$

$$\int_\Omega\left(Lu-f\right)\omega_i d\Omega=0$$

$$Weight:\omega_i=\delta\left(x-x_i\right)$$

$$\begin{aligned} u&=x\left(1-x\right)\left(a_0+a_1x\right) \\ \left(Lu-f\right)&=-4\,a_1\,x-2\,a_0+2\,\left(1-x\right)a_1+x\left(1-x\right)\left(a_1\,x+a_0\right)+x \end{aligned}$$

$$\int_{\Omega} (Lu - f)\, \omega_i d\Omega$$
$$= \int_0^1 \left[-4\, a_1\, x - 2\, a_0 + 2\, (1 - x)\, a_1 + x\, (1 - x)\, (a_1\, x + a_0) + x\right] \delta\, (x - x_i)\, dx$$

$$\int_{\Omega} (Lu - f)\, \omega_i d\Omega = 0$$

So we obtain linear algebra equations:

$$\left[-4\, a_1\, x - 2\, a_0 + 2\, (1 - x)\, a_1 + x\, (1 - x)\, (a_1\, x + a_0) + x\right]\Big|_{x=x_1} = 0$$
$$\left[-4\, a_1\, x - 2\, a_0 + 2\, (1 - x)\, a_1 + x\, (1 - x)\, (a_1\, x + a_0) + x\right]\Big|_{x=x_2} = 0$$

$$\frac{2}{27} a_1 - \frac{16}{9} a_0 + \frac{1}{3} = 0$$
$$-\frac{50}{27} a_1 - \frac{16}{9} a_0 + \frac{2}{3} = 0$$

Solve these two equation

$$a_0 = \frac{81}{416}, \quad a_1 = \frac{9}{52}$$

Approximation Solution:

$$u = x\, (1 - x)\, (a_0 + a_1 x)$$
$$u = x\, (1 - x) \left(\frac{81}{416} + \frac{9}{52} x\right)$$

(2) Sub-domain method:
Sub-domains:

$$D_1 : 0 < x < \frac{1}{2}$$
$$D_2 : 0 < x < 1$$

$$\int_{\Omega} (Lu - f)\, \omega_i d\Omega = 0$$
$$Weight : \omega_i = \begin{cases} 1, & x \in D_i \\ 0, & x \notin D_i \end{cases}$$

$$\int_{\Omega} (Lu - f)\, \omega_i d\Omega$$
$$= \int_0^1 \left[-4\, a_1\, x - 2\, a_0 + 2\, (1 - x)\, a_1 + x\, (1 - x)\, (a_1\, x + a_0) + x\right] \omega_i dx$$

$$\int_{\Omega} (Lu - f)\, \omega_i d\Omega = 0$$

$$\int_0^{\frac{1}{2}} \left[-4\, a_1\, x - 2\, a_0 + 2\, (1-x)\, a_1 + x\, (1-x)\, (a_1\, x + a_0) + x\right] \omega_i dx = 0$$
$$\int_0^{1} \left[-4\, a_1\, x - 2\, a_0 + 2\, (1-x)\, a_1 + x\, (1-x)\, (a_1\, x + a_0) + x\right] \omega_i dx = 0$$

Then we obtain two linear algebra equations:

$$\frac{53}{192} a_1 - \frac{11}{12} a_0 + \frac{1}{8} = 0$$
$$-\frac{11}{12} a_1 - \frac{11}{6} a_0 + \frac{1}{2} = 0$$

Solve these equations

$$a_0 = \frac{97}{517}$$
$$a_1 = \frac{8}{47}$$

Approximation solution

$$u = x\,(1-x)\,(a_0 + a_1 x)$$
$$u = x\,(1-x)\left(\frac{97}{517} + \frac{8}{47} x\right)$$

(3) Moment method:
Weight function:

$$\omega_i = |\mathbf{r}|^{i-1}$$

and in our case

$$\omega_1 (x) = 1$$
$$\omega_2 (x) = x$$

$$\int_{\Omega} (Lu - f)\, \omega_i d\Omega = 0$$

$$\int_0^{\frac{1}{2}} \left[-4\, a_1\, x - 2\, a_0 + 2\, (1-x)\, a_1 + x\, (1-x)\, (a_1\, x + a_0) + x\right] dx = 0$$
$$\int_0^{1} \left[-4\, a_1\, x - 2\, a_0 + 2\, (1-x)\, a_1 + x\, (1-x)\, (a_1\, x + a_0) + x\right] x dx = 0$$

we obtain two linearalgebra equations

$$-\frac{11}{12}a_1-\frac{11}{6}a_0+\frac{1}{2}=0$$
$$-\frac{19}{20}a_1-\frac{11}{12}a_0+\frac{1}{3}=0$$

Solve these linearalgebra equations we obtain

$$a_0=\frac{122}{649},\quad a_1=\frac{10}{59}$$

So the approximation solution is

$$u\left(x\right)=x\left(1-x\right)\left(\frac{122}{649}x+\frac{10}{59}\right)$$

(4) Least-Square method:
Target functional:

$$J=\int_\Omega R^2d\Omega$$

minimum Target functional

$$\begin{aligned}\frac{\partial J}{\partial a_i}&=\frac{\partial}{\partial a_i}\int_\Omega R^2d\Omega\\&=\int_\Omega\frac{\partial}{\partial a_i}R^2d\Omega\\&=\int_\Omega 2R\frac{\partial R}{\partial a_i}d\Omega=0\end{aligned}$$

So we know weight function is

$$\omega_i=\frac{\partial R}{\partial a_i}$$

$$\begin{aligned}\omega_0&=-2+x\left(1-x\right)\\\omega_1&=-6x+2+x^2\left(1-x\right)\end{aligned}$$

$$\int_\Omega\left(Lu-f\right)\omega_i d\Omega=0$$

$$\int_0^{\frac{1}{2}}\left[-4\,a_1\,x-2\,a_0+2\,\left(1-x\right)a_1+x\left(1-x\right)\left(a_1\,x+a_0\right)+x\right]\omega_0dx=0$$
$$\int_0^{1}\left[-4\,a_1\,x-2\,a_0+2\,\left(1-x\right)a_1+x\left(1-x\right)\left(a_1\,x+a_0\right)+x\right]\omega_1dx=0$$

Then we obtain two linear algebra equations:

$$\frac{101}{60}a_1 + \frac{101}{30}a_0 - \frac{11}{12} = 0$$
$$\frac{131}{35}a_1 + \frac{101}{60}a_0 - \frac{19}{20} = 0$$

Solve these equations:

$$a_0 = \frac{46161}{246137}$$
$$a_1 = \frac{413}{2437}$$

Approximation result

$$u = x\left(1-x\right)\left(a_0 + a_1 x\right)$$
$$u = x\left(1-x\right)\left(\frac{46161}{246137} + \frac{413}{2437}x\right)$$

(5) Galerkin method:

$$u = x\left(1-x\right)\left(a_0 + a_1 x\right)$$

$$u = a_0 x\left(1-x\right) + a_1 x^2\left(1-x\right)$$

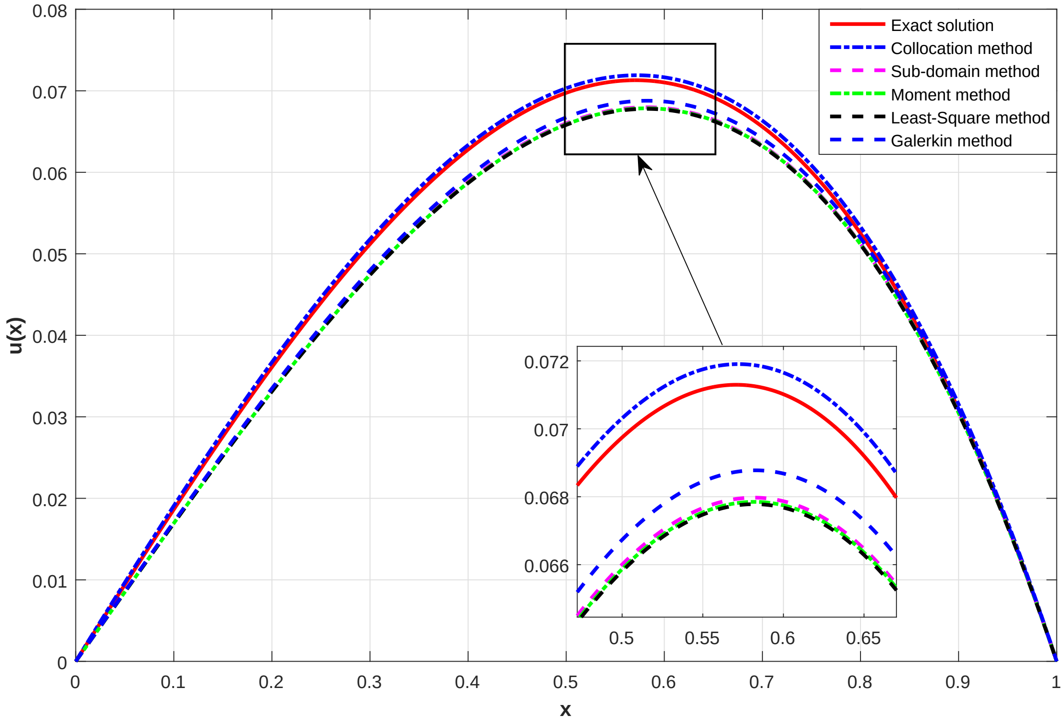
$$\varphi_0 = x\left(1-x\right)$$
$$\varphi_1 = x^2\left(1-x\right)$$

$$\int_\Omega \left(Lu - f\right)\omega_i d\Omega = 0$$
$$Wieght \quad function : \omega_i = \varphi_i$$

$$\int_0^{\frac{1}{2}} \left[-4\,a_1\,x - 2\,a_0 + 2\,\left(1-x\right)a_1 + x\left(1-x\right)\left(a_1\,x + a_0\right) + x\right]\omega_0 dx = 0$$
$$\int_0^{1} \left[-4\,a_1\,x - 2\,a_0 + 2\,\left(1-x\right)a_1 + x\left(1-x\right)\left(a_1\,x + a_0\right) + x\right]\omega_1 dx = 0$$

So we obtain two linear algebra equations

$$-\frac{3}{20}a_1 - \frac{3}{10}a_0 + \frac{1}{12} = 0$$
$$-\frac{13}{105}a_1 - \frac{3}{20}a_0 + \frac{1}{20} = 0$$



solve these equations

$$a_0 = \frac{71}{369}$$
$$a_1 = \frac{7}{41}$$
$$u = x\left(1-x\right)\left(\frac{71}{369}x + \frac{7}{41}\right)$$

Compare plot of these weighted residual method:

2.Galerkin method for any given n
approximation polynomial

$$u = x\left(1-x\right)\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + ... + a_nx^n\right)$$

and base functions

$$\begin{aligned} \varphi_0(x) &= x(1-x) \\ \varphi_1(x) &= x^2(1-x) \\ \varphi_2(x) &= x^3(1-x) \\ &... \\ \varphi_n(x) &= x^{n+1}(1-x) \end{aligned}$$

and approximation polynomial

$$u(x) = \sum_{i=0}^{n} a_i \varphi_i(x)$$

basic function

$$\omega_i(x) = \varphi_i(x)$$

Weighted residual method

$$\begin{gathered} \int_\Omega (Lu - f)\,\omega_i d\Omega = 0 \\ Wieght \quad function : \omega_i = \varphi_i \end{gathered}$$

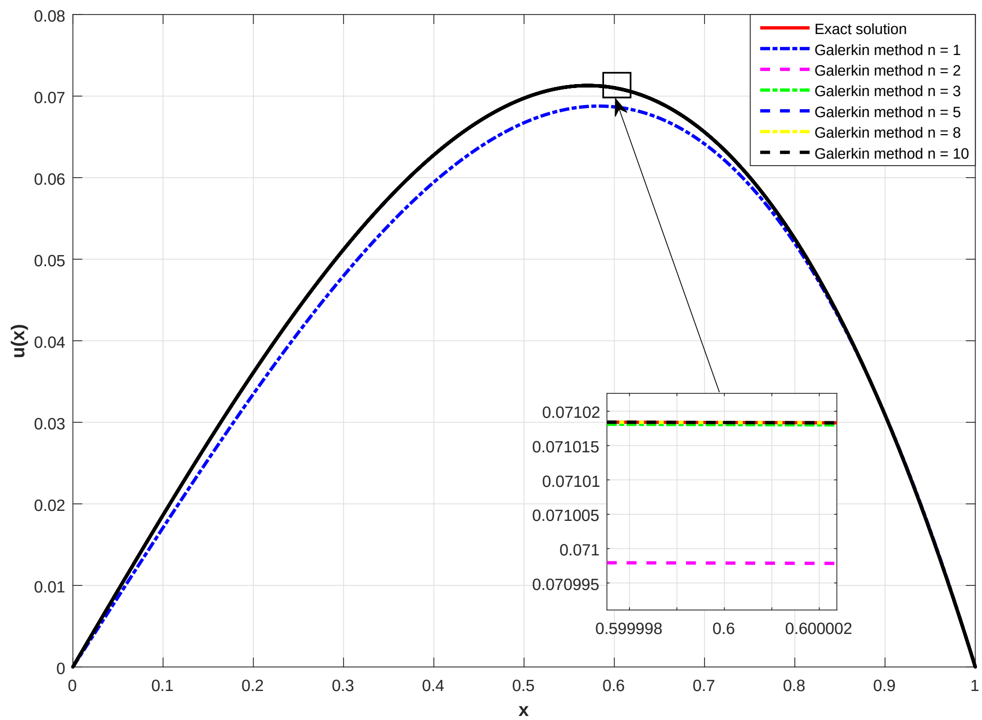
Then we'll get linear Algebra equations:

$$\begin{aligned} f_1(a_0, a_1, ..., a_n) &= 0 \\ f_2(a_0, a_1, ..., a_n) &= 0 \\ f_3(a_0, a_1, ..., a_n) &= 0 \\ &... \\ f_n(a_0, a_1, ..., a_n) &= 0 \end{aligned}$$

Solve these linear algebra equations, we obtain these coefficients in the approxiamtion polynomial.

$$\begin{aligned} & a_0, a_1, ..., a_n \\ & u = x(1-x)\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + ... + a_nx^n\right) \end{aligned}$$

We using Maple to realize this procedure.


Exact solution
Galerkin method n = 1
Galerkin method n = 2
Galerkin method n = 3
Galerkin method n = 5
Galerkin method n = 8
Galerkin method n = 10
u(x)
x
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0.07102
0.071015
0.07101
0.071005
0.071
0.070995
0.599998
0.6
0.600002