PDE

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0$$

First-order upwind scheme

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{u_j^n - u_{j-1}^n}{\Delta x}$$

$$u_j^{n+1} = u_j^n - c\left(u_j^n - u_{j-1}^n\right)$$

$$c = \frac{\Delta t}{\Delta x}$$

Implicit scheme

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{u_{j+1}^{n+1} - u_{j-1}^{n+1}}{2\Delta x}$$

$$-\frac{c}{2}u_{j-1}^{n+1} + u_j^{n+1} + \frac{c}{2}u_{j+1}^{n+1} = u_j^n$$

$$c = \frac{\Delta t}{\Delta x}$$

Lax scheme

$$\frac{u_j^{n+1} - \frac{1}{2}\left(u_{j+1}^n + u_{j-1}^n\right)}{\Delta t} = \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}$$

$$u_j^{n+1} = \frac{1}{2}\left(1-c\right)u_{j+1}^n + \frac{1}{2}\left(1+c\right)u_{j-1}^n$$

Lax-wndroff scheme

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} = \frac{1}{2}\Delta t\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}$$

$$\begin{aligned}
u_j^{n+1} &= u_j^n - \frac{1}{2}c\left(u_{j+1}^n - u_{j-1}^n\right) + \frac{1}{2}c^2\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right) \\
&= \left(1-c^2\right)u_j^n + \frac{1}{2}c^2u_{j+1}^n - \frac{1}{2}cu_{j+1}^n + \frac{1}{2}c^2u_{j-1}^n + \frac{1}{2}cu_{j-1}^n \\
&= \left(1-c^2\right)u_j^n + \frac{1}{2}c\left(c-1\right)u_{j+1}^n + \frac{1}{2}c\left(c+1\right)u_{j-1}^n
\end{aligned}$$

Leap-frog scheme

$$\frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} = \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}$$

$$u_j^{n+1} = u_j^{n-1} - c\left(u_{j+1}^n - u_{j-1}^n\right)$$