Integrals IV

May 16, 2022

Theorem 0.1. *Volume by Disc for Rotation About the x-Axis*

$$V = \int_a^b \pi R^2(x)\mathrm{d}x,$$

where $R(x)$ is the distance from the axis of revolution to the planar regions boundary.

Problem 1. Find the volumes of the solid generated by revolving the regions bounded by the lines and curves about the x-axis.

(1) $y = x^2$, $y = 0$, $x = 2$
(2) $y = x^3$, $y = 0$ and $x = 2$
(3) $y = \sqrt{9 - x^2}$ and $y = 0$
(4) $y = x - x^2$ and $y = 0$
(5) $y = \sqrt{\cos x}$, $0 \le x \le \pi/2$, $y = 0$, $x = 0$
(6) $y = \sec x$, $y = 0$, $x = -\pi/4$, $x = \pi/4$.

Theorem 0.2. *If f' is continuous on $[a, b]$ then the **length (arc length)** of the curve $y = f(x)$ from point $A = (a, f(a))$ to the point $B = (b, f(b))$ is the value of the integral*

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\mathrm{d}x = \int_a^b \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\mathrm{d}x.$$

Problem 2. Find the length of the curves

(1) $y = \frac{(x^2+2)^{\frac{3}{2}}}{3}$, from $x = 0$ to $x = 3$
(2) $y = x^{3/2}$ from $x = 0$ to $x = 4$
(3) $x = \frac{y^3}{3} + \frac{1}{4y}$, from $y = 1$ to $y = 3$
(4) $x = \frac{y^{3/2}}{3} - y^{1/2}$, from $y = 1$ to $y = 9$
(5) $x = \frac{y^4}{4} + \frac{1}{8y^2}$, from $y = 1$ to $y = 2$

Theorem 0.3. *If the function $f(x) \ge 0$ is continuously differentiable on $[a, b]$, the **area of the surface** generated by revolving the graph of $y = f(x)$ about the x-axis is*

$$S = \int_a^b 2\pi y\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}$$

Problem 3. Find the areas of the surfaces generated by revolving the curves about the indicated axes.

(1) $y = \frac{x^3}{9}$, $0 \le x \le 2$; x-axis

(2) $y = \sqrt{x}$, $3/4 \leq x \leq 15/4$; x-axis
(3) $y = \sqrt{2x - x^2}$, $0.5 \leq x \leq 1.5$; x-axis
(4) $y = \sqrt{x+1}$, $1 \leq x \leq 5$; x-axis
(5) $x = \frac{x^3}{3}$, $0 \leq y \leq 1$; y-axis

Answers to Even-numbered exercise only:

Problem 1. (2) $\dfrac{182\pi}{7}$
(4) $\dfrac{\pi}{30}$
(6) 2π

Problem 2. (2) $\dfrac{8}{27}(10\sqrt{10} - 1)$
(4) $\dfrac{32}{3}$

Problem 3. (2) $\dfrac{28\pi}{3}$
(4) $\dfrac{49\pi}{3}$