War Crimes Against Euclid.

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Set Theory and Analysis Basics

2.1 Real Numbers

Let $\mathbb{R}$ be a set equipped with operations $+, \cdot : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ called addition and multiplication satisfying the following axioms:

1. $\forall a, b, c \in \mathbb{R} \quad a + (b + c) = (a + b) + c$
2. $\exists 0 \in \mathbb{R}\, \forall a \in \mathbb{R} \quad a + 0 = a$
3. $\forall a \in \mathbb{R}\, \exists b \in \mathbb{R} \quad a + b = 0$
4. $\forall a, b \in \mathbb{R} \quad a + b = b + a$
5. $\forall a, b, c \in \mathbb{R} \quad a \cdot (b \cdot c) = (a \cdot b) \cdot c$
6. $\exists 1 \in \mathbb{R}\, \forall a \in \mathbb{R} \quad a \cdot 1 = a$
7. $\forall a \in \mathbb{R}\, \exists b \in \mathbb{R} \quad a \cdot b = 1$
8. $\forall a, b \in \mathbb{R} \quad a \cdot b = b \cdot a$
9. $\forall a, b, c \in \mathbb{R} \quad a \cdot (b + c) = a \cdot b + a \cdot c$

Moreover $\mathbb{R}$ needs to satisfy the **least upper bound property**:

$$\forall S \subseteq \mathbb{R}\, \exists m \in \mathbb{R}\, \forall x \in S \quad x \leq m \implies (\exists u \in \mathbb{R}\, \forall v \in \mathbb{R}\, \forall x \in S \quad x \leq v \implies u \leq v)\,.$$

In words it simply means that every bounded set has a least upper bound. We can easily see that it has to be unique. We denoted it as $\sup(S)$. Similarly we denote the least lower bound of a set $\inf(S)$.

Chapter 3

Lines

Definition 1. ***Plane*** *is the set* $\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$.

Definition 2. ***Point*** *is any element of the plane.*

Definition 3. ***Origin*** *is the point* $(0, 0)$. *We denote it as* $\mathcal{O}$.

Definition 4. *Let* $P, Q \in \mathbb{R}^2$ *be points with* $P = (x_1, y_1)$, $Q = (x_2, y_2)$. ***Vector*** *from* P *to* Q *is* $\vec{PQ} = [x_2 - x_1, y_2 - y_1]$. *Without loss of generality me might take* $P = \mathcal{O}$ *and put vector to be* $\mathbf{v} = [v_1, v_2]$ *for* $v_1, v_2 \in \mathbb{R}$.

Definition 5. *Vector* $[0, 0]$ *is called the* ***zero vector***. *We denote it as* $\mathbf{0}$.

Definition 6. *Let* $\mathbf{v} \in \mathbb{R}^2$ *be a vector with* $\mathbf{v} = [v_1, v_2]$ *and* $\lambda \in \mathbb{R}$ *be a number. We define* ***scalar multiplication*** *as* $\lambda \cdot \mathbf{v} = [\lambda \cdot v_1, \lambda \cdot v_2]$.

Definition 7. *Let* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$ *be vectors with* $\mathbf{u} = [u_1, u_2]$, $\mathbf{v} = [v_1, v_2]$. *We define* ***vector addition*** *as* $\mathbf{u} + \mathbf{v} = [u_1 + v_1, u_2 + v_2]$.

Definition 8. *Let* $\mathbf{v} \in \mathbb{R}^2$ *be a vector with* $\mathbf{v} = [v_1, v_2]$ *and* $P \in \mathbb{R}^2$ *be a point with* $P = (x, y)$. *We define* ***point translation*** *as* $P + \mathbf{v} = (x + v_1, y + v_2)$.

Definition 9. *Let* $\mathbf{v} \in \mathbb{R}^2$ *be a vector with* $\mathbf{v} = [v_1, v_2]$. *We define it's* ***length*** *as* $\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2}$.

Definition 10. *Vector* $\mathbf{v} = [v_1, v_2]$ *is called a* ***unit vector*** *if* $\|\mathbf{v}\| = 1$.

Definition 11. *Let* $P, Q \in \mathbb{R}^2$ *be points.* ***Line segment*** *between* P *and* Q *is the set* $PQ = \{(1 - t) \cdot P + t \cdot Q : t \in [0; 1]\}$.

Definition 12. *Let* $P, Q \in \mathbb{R}^2$ *be points with* $P = (x_1, y_1)$, $Q = (x_2, y_2)$. ***Distance*** *between* P *and* Q *is* $|PQ| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

Fact 1. *Let $P, Q, R \in \mathbb{R}^2$ be points. If $R \in PQ$ then $|PQ| = |PR| + |RQ|$.*

Proof. Let $P = (x_1, y_1),\ Q = (x_2, y_2),\ R = (x_3, y_3)$. Since $R \in PQ$ we have $R = (1-t)\cdot P + t \cdot Q$ for some $t \in [0; 1]$ and so

$$x_3 = (1-t)\cdot x_1 + tx_2, \quad y_3 = (1-t)\cdot y_1 + ty_2\ .$$

Plugging in we get

$$|PR| = \sqrt{(x_3-x_1)^2 + (y_3-y_1)^2} = \sqrt{t^2 \cdot ((x_2-x_1)^2 + (y_2-y_1)^2)} = t\cdot|PQ|\ .$$

$$|RQ| = \sqrt{(x_3-x_2)^2 + (y_3-y_2)^2} = \sqrt{(1-t)^2 \cdot ((x_2-x_1)^2 + (y_2-y_1)^2)} = (1-t)\cdot|PQ|\ .$$

Hence $|PR| + |RQ| = |PQ|$. $\square$

Definition 13. *Let $P \in \mathbb{R}^2$ be a point and $\mathbf{v} \in \mathbb{R}^2$ be a nonzero vector.* ***Ray*** *starting at P and with direction $\mathbf{v}$ is $\{P + t\cdot\mathbf{v} : t \in [0; +\infty)\}$.*

Definition 14. *Let $P \in \mathbb{R}^2$ be a point and $\mathbf{v} \in \mathbb{R}^2$ be a nonzero vector.* ***Line*** *passing through P and with direction $\mathbf{v}$ is $\{P + t\cdot\mathbf{v} : t \in \mathbb{R}\}$.*

Theorem 1. *Set $\ell \subseteq \mathbb{R}^2$ is a line if and only if there are $A, B, C \in \mathbb{R}$ with $A \neq 0$ or $B \neq 0$ so that*

$$\ell = \{(x, y) \in \mathbb{R}^2 : Ax + By + C = 0\}\ .$$

We write this as $\ell : Ax + By + C = 0$.

Proof. $\Longrightarrow$.
Let $\ell \subseteq \mathbb{R}^2$ be a line passing through point $P = (x_0, y_0)$ and with direction vector $\mathbf{v} = [v_1, v_2]$. Take $(x, y) \in \ell$. We have $(x, y) = (x_0, y_0) + t\cdot[v_1, v_2]$ for some $t \in \mathbb{R}$. Hence $x = x_0 + tv_1,\ y = y_0 + tv_2$. Also $v_2x - v_1y + v_2x_0 - v_1y_0 = v_2\cdot(x_0 + tv_1) - v_1\cdot(y_0 + tv_2) + v_2x_0 - v_1y_0 = 0$. Thus all points on ℓ satisfy the equation $Ax + By + C = 0$ with

$$A = v_2,\ B = -v_1,\ C = v_2x_0 - v_1y_0\ .$$

Moreover since $\mathbf{v} \neq \mathbf{0}$ we know that $A \neq 0$ or $B \neq 0$.
$\Longleftarrow$. Let $\ell : Ax + By + C = 0$ be a set with $A \neq 0$ or $B \neq 0$. Without loss of generality assume $A \neq 0$. Now $(-\frac{C}{A}, 0) \in \ell$ so $\ell \neq \varnothing$ and thus we can fix $(x_0, y_0) \in \ell$. Now take $(x, y) \in \ell$. We have $Ax + By + C = 0$ and so $x = -\frac{By+C}{A}$. Take $t \in \mathbb{R}$ such that $y = y_0 - tA$, preciously $t = -\frac{y-y_0}{A}$. Then we have

$$x = -\frac{B\cdot(y_0 - tA) + C}{A} = -\frac{By_0 + C}{A} + tB = x_0 + tB$$

where $x_0 = -\frac{By_0+C}{A}$ because $(x_0, y_0) \in \ell$. Hence we have $x = x_0 + tB,\ y = y_0 - tA$ and so $\ell = \{(x_0, y_0) + t \cdot [B, -A] : t \in \mathbb{R}\}$. Thus ℓ is a line.

$\square$

Theorem 2. *Let $\ell : Ax + By + C = 0$ and $k : Dx + Ey + F = 0$ be lines. The following are equivalent:*

1. $\ell = k$
2. $AE = BD$ *and* $AF = CD$
3. $\exists \lambda \in \mathbb{R} \setminus \{0\} \quad D = \lambda A,\ E = \lambda B,\ F = \lambda C$

Proof. $1 \implies 2$. Suppose $\ell = k$ and without loss of generality assume $A, D \neq 0$. If $(x, y) \in \ell$ then $x = -\frac{By+C}{A}$ and if $(x, y) \in k$ then $x = -\frac{Ey+F}{D}$. Since $\ell = k$ we get $\frac{By+C}{A} = \frac{Ey+F}{D}$. This equality holds for any $y \in \mathbb{R}$ so we must have $\frac{B}{A} = \frac{E}{D}$, $\frac{C}{A} = \frac{F}{D}$ which implies that $AE = BD$ and $AF = CD$.
$2 \implies 3$. Suppose that $AE = BD$ and $AF = CD$ and without loss of generality assume $A, D \neq 0$. We can write

$$D = \frac{D}{A} \cdot A,\ E = \frac{D}{A} \cdot B,\ F = \frac{D}{A} \cdot C\ .$$

Since $D \neq 0$ it means that $\frac{D}{A} \neq 0$.
$3 \implies 1$. Suppose $D = \lambda A,\ E = \lambda B,\ F = \lambda C$ for some $\lambda \in \mathbb{R}$ with $\lambda \neq 0$. Take $(x, y) \in \ell$. Then $Ax + By + C = 0$ but also $\lambda \cdot (Ax + By + C) = 0$ and so $Dx + Ey + F = 0$. Hence $(x, y) \in k$ and $\ell \subseteq k$. Take $(x, y) \in k$. Then $Dx+Ey+F = 0$ but that means $\lambda \cdot (Ax+By+C) = 0$. Since $\lambda \neq 0$ we have $Ax + By + C = 0$. Hence $(x, y) \in \ell$ and $k \subseteq \ell$. All in all we get $\ell = k$. $\square$

Theorem 3. *Let $\ell : Ax + By + C = 0$ and $k : Dx + Ey + F = 0$ be lines. Then $|\ell \cap k| \in \{0, 1, \infty\}$ and the value of $|\ell \cap k|$ can be determined as follows: Put*

$$\Delta = AE - BD,\ \Delta_x = CE - BF,\ \Delta_y = AF - CD\ .$$

Now

1. $|\ell \cap k| = 1$ *if and only if* $\Delta \neq 0$
2. $|\ell \cap k| = 0$ *if and only if* $\Delta = 0$ *and* $\Delta_x \neq 0$ *or* $\Delta_y \neq 0$
3. $|\ell \cap k| = \infty$ *if and only if* $\Delta = 0$ *and* $\Delta_x = 0$ *and* $\Delta_y = 0$

Moreover if $|\ell \cap k| = 1$ then the solution is given by

$$x = -\frac{\Delta_x}{\Delta},\ y = -\frac{\Delta_y}{\Delta}\ .$$

Proof. Point $(x, y) \in \ell \cap k$ solves the following system of equations:

$$\begin{cases} Ax + By + C = 0 \\ Dx + Ey + F = 0 \end{cases}$$

Without loss of generality assume $A \neq 0$. Then $x = -\frac{By+C}{A}$. Substituting it into the second equation we get $-D \cdot \frac{By+C}{A} + Ey + F = 0$ which simplifies to $\frac{AE-BD}{A}y + \frac{AF-CD}{A} = 0$ and further to $\Delta \cdot y + \Delta_y = 0$. If $\Delta \neq 0$ we have $y = -\frac{\Delta_y}{\Delta}$. Plugging back we get $x = -\frac{B \cdot \frac{CD-AF}{AE-BD} + C}{A} = -\frac{\Delta_x}{\Delta}$. However if $\Delta = 0$ we need to consider two cases:

Case 1:

Suppose $\Delta_y \neq 0$. Then the second equation is equivalent to $\Delta_y = 0$ which has no solutions.

Case 2:

Suppose $\Delta_y = 0$. By Theorem 2 we get $\ell = k$ and so $|\ell \cap k| = \infty$. All of the steps are reversible, and hence these conditions are necessary and sufficient. □

Fact 2. *Let $P, Q \in \mathbb{R}^2$ be points with $P \neq Q$. There is a unique line $\ell \subseteq \mathbb{R}^2$ such that $P, Q \in \ell$.*

Proof. Let $P = (x_1, y_1),\ Q = (x_2, y_2)$. Put $\ell = \{P + t \cdot \vec{PQ} : t \in \mathbb{R}\}$. We have $P \in \ell$ for $t = 0$ and $Q \in \ell$ for $t = 1$. Moreover suppose there is another line k such that $P, Q \in k$. But then $|\ell \cap k| \geq 2$ and by Theorem 3 $|\ell \cap k| = \infty$ and so $\ell = k$. □

Definition 15. *Lines $\ell, k \subseteq \mathbb{R}^2$ are said to be* ***parallel****, written $\ell \parallel k$, if $\ell = k$ or $\ell \cap k = \varnothing$.*

Fact 3. *Let $\ell : Ax + By + C = 0$ and $k : Dx + Ey + F = 0$ be lines. We have $\ell \parallel k$ if and only if $AE - BD = 0$.*

Proof. By Theorem 3 if $\ell \parallel k$ then $|\ell \cap k| \in \{0, \infty\}$ and so $AE - BD = 0$. □

Definition 16. *Points $P, Q, R \in \mathbb{R}^2$ are said to be* ***colinear*** *if there is a line $\ell \subseteq \mathbb{R}^2$ such that $P, Q, R \in \ell$.*

Theorem 4. *Let $P, Q, R \in \mathbb{R}^2$ be points with $P = (x_1, y_1),\ Q = (x_2, y_2),\ R = (x_3, y_3)$. Then P, Q, R are colinear if and only if*

$$det \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_3 \\ 1 & x_3 & y_3 \end{bmatrix} = 0$$

Proof. $\Longrightarrow$. Suppose $\ell : Ax + By + C = 0$ is a line with $P, Q, R \in \ell$. There are $s, t \in \mathbb{R}$ so that $Q = P + t \cdot [B, -A]$ and $R = P + t \cdot [B, -A]$. Thus we get

$$x_2 = x_1 + tB,\ y_2 = y_1 - tA,\ x_3 = x_1 + sB,\ y_3 = y_1 - sA\ .$$

Also note that

$$\det \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_3 \\ 1 & x_3 & y_3 \end{bmatrix} = x_2y_3 - x_3y_2 - x_1y_3 + x_3y_1 + x_1y_2 - x_2y_1\ .$$

Plugging in we get

$$\begin{aligned} &(x_1 + tB) \cdot (y_1 - sA) - (x_1 + sB) \cdot (y_1 - tA) - x_1 \cdot (y_1 - sA) \\ &\quad +(x_1 + sB) \cdot y_1 + x_1 \cdot (y_1 - tA) - (x_1 + tB) \cdot y_1 = 0\ . \end{aligned}$$

$\Longleftarrow$. Let $\ell : Ax + By + C = 0$ and $k : Dx + Ey + F = 0$ be lines with $P, Q \in \ell$ and $P, R \in k$. They are unique by **Fact 2**. Also we have $Q = P + t \cdot [B, -A]$ and $R = P + s \cdot [E, -D]$ for some $s, t \in \mathbb{R}$. Thus we get

$$x_2 = x_1 + tB,\ y_2 = y_1 - tA,\ x_3 = x_1 + sE,\ y_3 = y_1 - sD\ .$$

Plugging in we get

$$\begin{aligned} &(x_1 + tB) \cdot (y_1 - sD) - (x_1 + sE) \cdot (y_1 - tA) - x_1 \cdot (y_1 - sD) \\ &+(x_1 + sE) \cdot y_1 + x_1 \cdot (y_1 - tA) - (x_1 + tB) \cdot y_1 = st \cdot (AE - BD)\ . \end{aligned}$$

By assumption this expression is zero. Since $P \neq Q$ and $P \neq R$ we know that $s, t \neq 0$. Hence we get that $AE - BD$. By Fact 3 we know that $\ell \parallel k$. However since $P \in \ell \cap k$ by theorem 3 we get that $\ell = k$ and so P, Q, R are colinear. $\square$

Chapter 4

Circles

Definition 17. *Let $r > 0$ and $(p, q) \in \mathbb{R}^2$. Then **Circle** of center (p, q) and radius r is $\{(x, y) \in \mathbb{R}^2 : (x-p)^2 + (y-q)^2 = r^2\}$. We write this as $\mathcal{C} : (x-p)^2 + (y-q)^2 = r^2$.*

Definition 18. *Let $\mathcal{C} : (x-p)^2 + (y-q)^2 = r^2$ be a circle. **Diameter** of $\mathcal{C}$ is any line segment AB such that $A, B \in \mathcal{C}$ and $(p, q) \in AB$.*

Fact 4. *Let $\mathcal{C} : (x-p)^2 + (y-q)^2 = r^2$ be a circle and AB be it's diameter. Then $|AB| = 2r$.*

Proof. Since $(p, q) \in AB$ by Fact 1 we have $|AB| = |A(p, q)| + |(p, q)B| = r + r = 2r$. $\square$