Counting Equivalence Classes of Matrices under Row Permutations

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Problem

Let $n, m \in \mathbb{N}_+$, and let $\mathrm{Mat}_{n\times n}(\mathbb{Z}_m)$ denote the set of all $n \times n$ matrices with entries in $\mathbb{Z}_m = \{0, 1, \ldots, m-1\}$. Given $\mathbf{A}, \mathbf{B} \in \mathrm{Mat}_{n\times n}(\mathbb{Z}_m)$, we say that they are **equivalent** if we can obtain $\mathbf{B}$ by permuting the rows of $\mathbf{A}$. We wish to find the number of equivalence classes of matrices under this relation.

Solution

Let S_n denote the symmetric group on n elements. For $\sigma \in \mathrm{S}_n$, we define its **permutation matrix $\mathbf{P}_\sigma$** as follows:

$$[\mathbf{P}_\sigma]_{i,j} = \begin{cases} 1 & \text{if } i = \sigma(j), \\ 0 & \text{otherwise.} \end{cases}$$

Then we have $\mathbf{A} \sim \mathbf{B}$ if and only if $\mathbf{P}_\sigma \mathbf{A} = \mathbf{B}$ for some $\sigma \in \mathrm{S}_n$.

Next, let us define a group action $\theta : \mathrm{S}_n \times \mathrm{Mat}_{n\times n}(\mathbb{Z}_m) \to \mathrm{Mat}_{n\times n}(\mathbb{Z}_m)$ by $\theta(\sigma, \mathbf{A}) = \mathbf{P}_\sigma \mathbf{A}$.

Verification of Group Action

Let us verify that this is indeed a group action. Take any $\mathbf{A} \in \mathrm{Mat}_{n\times n}(\mathbb{Z}_m)$.

Identity: If id is the identity permutation, then $\mathbf{P}_{\mathrm{id}} = \mathcal{I}_n$, where $\mathcal{I}_n$ is the $n \times n$ identity matrix. Thus

$$\theta(\mathrm{id}, \mathbf{A}) = \mathbf{P}_{\mathrm{id}}\mathbf{A} = \mathcal{I}_n \mathbf{A} = \mathbf{A}.$$

Compatibility: To verify compatibility with the group operation, take $\sigma, \tau \in \mathrm{S}_n$. Notice that

$$[\mathbf{P}_\sigma \mathbf{P}_\tau]_{i,j} = \sum_{k=1}^{n} [\mathbf{P}_\sigma]_{i,k} [\mathbf{P}_\tau]_{k,j} = \sum_{k=1}^{n} \mathbf{1}_{i=\sigma(k)} \mathbf{1}_{k=\tau(j)} = \mathbf{1}_{i=\sigma(\tau(j))} = [\mathbf{P}_{\sigma\circ\tau}]_{i,j}.$$

We then have

$$\theta(\sigma, \theta(\tau, \mathbf{A})) = \theta(\sigma, \mathbf{P}_\tau \mathbf{A}) = \mathbf{P}_\sigma \mathbf{P}_\tau \mathbf{A} = \mathbf{P}_{\sigma\circ\tau}\mathbf{A} = \theta(\sigma \circ \tau, \mathbf{A}).$$

Therefore, θ is a group action.

Orbits and Fixed Points

Recall that the **orbit** of $\mathbf{A}$ under the action of S_n is

$$\mathrm{Orb}_{\mathrm{S}_n}(\mathbf{A}) = \{\mathbf{B} \in \mathrm{Mat}_{n\times n}(\mathbb{Z}_m) : \exists \sigma \in \mathrm{S}_n \quad \mathbf{B} = \theta(\sigma, \mathbf{A})\},$$

the set of **fixed points** of σ is

$$\mathrm{Fix}_{\mathrm{S}_n}(\sigma) = \{\mathbf{A} \in \mathrm{Mat}_{n\times n}(\mathbb{Z}_m) : \theta(\sigma, \mathbf{A}) = \mathbf{A}\},$$

and the **set of orbits** is

$$\mathrm{Mat}_{n\times n}(\mathbb{Z}_m)/\mathrm{S}_n = \{\mathrm{Orb}_{\mathrm{S}_n}(\mathbf{A}) : \mathbf{A} \in \mathrm{Mat}_{n\times n}(\mathbb{Z}_m)\}.$$

Now $\mathbf{A} \sim \mathbf{B}$ if and only if $\mathbf{B} \in \mathrm{Orb}_{\mathrm{S}_n}(\mathbf{A})$. Thus we wish to find the number of orbits of θ.

Application of Burnside's Lemma

By Burnside's lemma, we have

$$|\mathrm{Mat}_{n\times n}(\mathbb{Z}_m)/\mathrm{S}_n| = \frac{1}{|\mathrm{S}_n|} \sum_{\sigma \in \mathrm{S}_n} |\mathrm{Fix}_{\mathrm{S}_n}(\sigma)|\,.$$

Let us compute $|\,\mathrm{Fix}_{\mathrm{S}_n}(\sigma)|$. Suppose σ has k disjoint cycles in its cycle decomposition. For $\mathbf{A}$ to be fixed by σ, each row belonging to the same cycle must be identical. Since there are m^n ways to choose each row of a matrix $\mathbf{A}$, and we have k independent cycles, we have

$$|\mathrm{Fix}_{\mathrm{S}_n}(\sigma)| = (m^n)^k.$$

Recall that there are exactly $\begin{bmatrix} n \\ k \end{bmatrix}$ permutations in S_n having k disjoint cycles, where $\begin{bmatrix} n \\ k \end{bmatrix}$ is the (unsigned) Stirling number of the first kind. Let us write $x^{\overline{n}} = x(x+1)\cdots(x+n-1)$ for the rising factorial. Recall the identity

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k = x^{\overline{n}}.$$

Finally, we have

$$\frac{1}{|\mathrm{S}_n|} \sum_{\sigma \in \mathrm{S}_n} |\mathrm{Fix}_{\mathrm{S}_n}(\sigma)| = \frac{1}{n!} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (m^n)^k = \frac{1}{n!} (m^n)^{\overline{n}}.$$

Hence the number of equivalence classes of matrices under row permutations is

$$\boxed{\frac{1}{n!}(m^n)^{\overline{n}}}.$$