Probability of Real Roots in Random Quadratic Equations

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Problem

Let $ax^2 + bx + c = 0$ be a quadratic equation with $a, b, c \in \mathbb{R}$.

We want to calculate the probability that it has real roots. Recall that a quadratic equation either has all real roots or all complex (non-real) roots, and the roots are real if and only if the discriminant satisfies $b^2 - 4ac \geq 0$.

We will assume that a, b, c are chosen uniformly and independently from a symmetric interval $[-M; M]$, and then take the limit as $M \to \infty$.

Solution

Let $M > 0$ and let A, B, C be independent random variables with uniform distribution:

$$A, B, C \sim \mathcal{U}([-M; M]).$$

Put $D = B^2 - 4AC$. We want to calculate

$$\lim_{M \to \infty} \mathbb{P}[D \geq 0].$$

The desired probability is

$$\mathbb{P}[D \geq 0] = \iiint_{[-M;M]^3} f_A(x) \cdot f_B(y) \cdot f_C(z) \cdot \mathbf{1}_{y^2 \geq 4xz} \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z,$$

where f_A, f_B, and f_C are the probability density functions. Recall that the probability density function for $\mathcal{U}([-M; M])$ is given by $f(t) = \frac{1}{2M}$.

Calculating the Integrals

We split the integration region into two pieces based on the sign of $4xz$:

- If $4xz < 0$, then $y^2 \geq 4xz$ for all $y \in [-M; M]$.
- If $4xz \geq 0$, then $y^2 \geq 4xz$ if and only if $|y| \geq 2\sqrt{xz}$, so $y \in [-M; -2\sqrt{xz}] \cup [2\sqrt{xz}, M]$.

Since $f_A = f_B = f_C = \frac{1}{2M}$, we have $f_A(x) \cdot f_B(y) \cdot f_C(z) = \left(\frac{1}{2M}\right)^3 = \frac{1}{8M^3}$. The desired integral now becomes $\frac{1}{8M^3} \cdot (I + J)$, where

$$I = \iint_{\substack{xz<0 \\ x,z \in [-M;M]}} \int_{-M}^{M} 1 \, \mathrm{d}y \, \mathrm{d}x \, \mathrm{d}z$$

and

$$J = \iint_{\substack{xz\geq 0\\ x,z\in[-M;M]}} \int_{[-M;-2\sqrt{xz}]\cup[2\sqrt{xz};M]} 1 \,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}z.$$

Computing I:

The innermost integral evaluates to $2M$. Then

$$\begin{aligned} I &= \iint_{\substack{xz<0\\ x,z\in[-M;M]}} 2M \,\mathrm{d}x\,\mathrm{d}z \\ &= \int_0^M \int_{-M}^0 2M \,\mathrm{d}x\,\mathrm{d}z + \int_{-M}^0 \int_0^M 2M \,\mathrm{d}x\,\mathrm{d}z. \end{aligned}$$

Both of these are integrals of a constant function over a region of area M^2, so each evaluates to $2M^3$. Hence $I = 4M^3$.

Computing J:

The innermost integral is the length of the union $[-M;-2\sqrt{xz}]\cup[2\sqrt{xz};M]$, which is $2M-4\sqrt{xz}$. Thus

$$\begin{aligned} J &= \iint_{\substack{xz\geq 0\\ x,z\in[-M;M]}} (2M-4\sqrt{xz}) \,\mathrm{d}x\,\mathrm{d}z \\ &= \int_0^M \int_0^M (2M-4\sqrt{xz}) \,\mathrm{d}x\,\mathrm{d}z + \int_{-M}^0 \int_{-M}^0 (2M-4\sqrt{xz}) \,\mathrm{d}x\,\mathrm{d}z. \end{aligned}$$

Note that both integrals have the same value by the substitution $x\mapsto -x$, $z\mapsto -z$. We get

$$\begin{aligned} J &= 2\int_0^M \int_0^M (2M-4\sqrt{xz}) \,\mathrm{d}x\,\mathrm{d}z = 2\left(M\cdot M\cdot 2M - 4\int_0^M \int_0^M \sqrt{xz}\,\mathrm{d}x\,\mathrm{d}z\right) \\ &= 4M^3 - 8\left(\int_0^M \sqrt{t}\,\mathrm{d}t\right)^2 = 4M^3 - 8\left(\frac{2}{3}t^{3/2}\Big|_0^M\right)^2 = 4M^3 - 8\left(\frac{2}{3}M^{3/2}\right)^2 \\ &= 4M^3 - 8\cdot\frac{4}{9}M^3 = 4M^3 - \frac{32}{9}M^3 = \frac{4}{9}M^3. \end{aligned}$$

Final Answer

Finally, we obtain

$$\mathbb{P}[D\geq 0] = \frac{1}{8M^3}\cdot(I+J) = \frac{1}{8M^3}\cdot\left(4M^3+\frac{4}{9}M^3\right) = \frac{1}{8M^3}\cdot\frac{40}{9}M^3 = \frac{5}{9}.$$

Since this probability does not depend on M, we have

$$\lim_{M\to\infty}\mathbb{P}[D\geq 0] = \boxed{\frac{5}{9}}.$$